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A FREQUENCY-DOMAIN ESTIMATOR FOR USE IN ADAPTIVE CONTROL SYSTEMS

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ABSTRACT

The paper presents a frequency-domain estimator which can identify both a nominal model of a plant as well as a frequency-domain bounding function on the modeling error associated with this nominal model. This estimator, which we call a robust estimator, can be used in conjunction with a robust control-law redesign algorithm to form a robust adaptive controller.

1. INTRODUCTION AND MOTIVATION

The use of feedback control in systems having large amounts of uncertainty requires the use of algorithms that learn or adapt in an on-line situation. A control system that is designed using only a priori knowledge results in a relatively low bandwidth closed-loop system so as to guarantee stable operation in the face of large uncertainty. An adaptive control algorithm, which can identify the plant on-line, thereby decreasing the amount of uncertainty, can yield a closed-loop system that has a higher bandwidth and thus better performance than a non-adaptive algorithm. There are many problems with the adaptive control algorithms which have been developed, to date. In particular, most adaptive control algorithms are not robust to unmodeled dynamics and an unmeasurable disturbance, particularly in the absence of a persistently-exciting input signal.

In this section, we will motivate the robust estimation problem by first discussing the adaptive control problem, in general, and then presenting a perspective on the robust adaptive control problem. Further, we justify the choice of an infrequent adaptation strategy before discussing the main focus of the paper, the development of a robust estimator.

Stability of Adaptive Control Algorithms

The use of adaptive control yields systems that are nonlinear and time-varying. Thus, the stability of these systems depends on the inputs and disturbances, as well as the plant (including any unmodeled dynamics) and the compensator. However, the stability properties of a linear time-invariant (LTI) feedback system depend only on the plant and compensator, not the inputs and disturbances. Because of this fact, we take the point of view that it is desirable to make the system 'as LTI as possible'. Of course, our motivation for using adaptive control is to achieve a performance improvement (increased bandwidth) over the best non-adaptive LTI compensator. So, there is the ever present tradeoff between performance and robustness.

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The preceding argument can be used to justify an infrequent control-law redesign strategy. It is envisioned that a discrete-time estimator will be used to continually update the frequency-domain estimate of the plant as long as there is useful information in the input/output data of the plant. The plant is in a closed-loop that is controlled by a discrete-time compensator that is only infrequently updated (redesigned). It can be shown that if the compensator is redesigned sufficiently infrequently, then the LTI stability of the 'frozen' system at every point in time guarantees the exponential stability of the time-varying system. In this way, the control system looks nearly LTI and consequently is more robust to disturbances, than a highly nonlinear adaptive controller. It is emphasized here that a robust adaptive controller that slowly learns and produces successively better LTI compensators is the end product envisioned in this paper. The paper aims to develop only the estimation part of this robust adaptive controller. On the other end of the adaptive control spectrum are algorithms that quickly adapt to a changing system. However, these systems have poor robustness properties in that they are highly sensitive to unmodeled dynamics and unmeasurable disturbances, particularly in the absence of persistent excitation.

A Perspective on the Robust Adaptive Control Problem

With the solution of the adaptive control problem for the ideal case, that is, when there are no unmodeled dynamics nor unmeasurable disturbances, the problem of robustness has become a focus of current research. Recently, a new perspective on the robust adaptive control problem has appeared in the literature [1]. Briefly, a 'robust' adaptive controller is viewed as a combination of a 'robust' estimator and a 'robust' control law. This is an appealing point of view. For example, if the robust estimator is not getting any useful information and consequently, is not able to improve on the current knowledge of the plant, then the adaptation aspect of the algorithm can be disabled and the adaptive controller reduces to a robust control law. That is, in a situation where the adaptive algorithm is not learning, the adaptive controller becomes simply the best robust LTI control law that one could design based only on a priori information and any additional information learned since the algorithm began.

Brief Statement of the Robust Estimation Problem

The main focus of this paper is the development of a robust estimator for use in an adaptive controller. In non-adaptive robust control, the designer must first obtain a nominal model along with some measure of its goodness. A practical measure of goodness is a bounding function on the magnitude of the modeling errors in the frequency-domain. Since non-adaptive robust control requires these steps, the same steps must implicitly, or explicitly, be present in a robust adaptive control scheme, the difference being that the steps are carried out on-line rather than off-line. Thus, we assume that our robust estimator must supply:

1) a nominal plant model,
2) a frequency-domain bounding function on the magnitude of the modeling uncertainty between the true plant and this nominal model.

So, the robust estimator must provide an estimate of the parameters for the structure of the nominal model, as well as a frequency-domain uncertainty bounding function corresponding to this nominal model. Given this information, several robust control-law design methodologies could be used, including the LQG/LTR design methodology [2]. The envisioned adaptive control system is illustrated in Figure 1. In this paper, we will use a discrete-time model of a sampled-data control system.

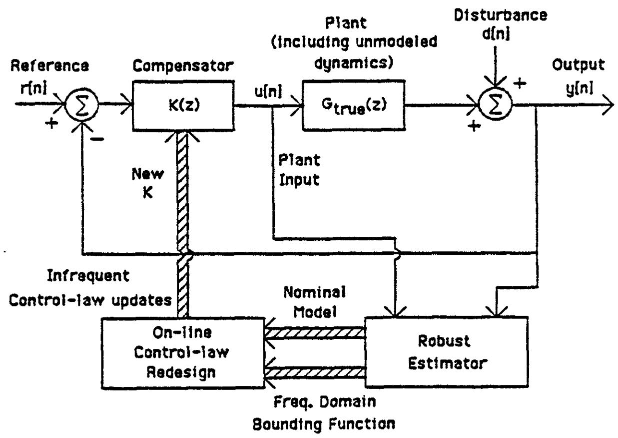

Figure 1. A Robust Adaptive Control System

The robust estimator presented in this paper is the first of its kind in that it provides guarantees concerning the current estimate of the nominal model of the plant. This requirement is essential if the estimator is to be used in a robust adaptive control situation. If the estimator cannot provide guarantees about the model it provides to the control-law redesign algorithm, then the redesign algorithm cannot guarantee stability of the closed-loop system. We will use a deterministic framework throughout the paper, since guarantees of stability are sought.

Related Literature

In Ljung [3,4] the 'empirical transfer function estimate' (ETFE) is introduced. This ETFE is computed using the Fourier transforms of finite-length input/output data of the plant. In [4], Ljung finds bounds on the effects of using finite-length data to compute the ETFE, for strictly stable plants. This work provides the background for the development of the frequency-domain estimation techniques of this paper.

2. MATHEMATICAL PRELIMINARIES

In this section, we will present the notation and definitions that will be used in the paper, as well as some results and theorems that will be useful later on. We denote a discrete-time signal by $x[n]=x(nT)$ where $x(t)$ denotes the sampled continuous-time signal and where n is an integer and T is the sampling period. The z-transform of x[n] on the unit circle is called the discrete-time Fourier transform (DTFT) and is defined as follows

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega T)n}. \qquad (2.1)$$

We define the N-point discrete Fourier transform (DFT) of x[n] at the N frequency points, $\omega_k=(k/N)\omega_s$, for k=0,..,N-1, where $\omega_s=2\pi/T$ is the sampling frequency,

$$X_N(\omega_k) = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \text{ for } k=0,\ldots,N-1. \qquad (2.2)$$

and where $W_N=e^{-j(2\pi/N)}$. (2.3)

Further, we define the inverse N-point discrete Fourier transform of $X_N(\omega_k)$ as follows,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_N(\omega_k)\, W_N^{-kn}, \text{ for } n=0,\ldots,N-1 \qquad (2.4)$$

Since we will not always be working with N-point sequences that begin at 0, we define the following versions of the DFT and inverse DFT for a sequence of N points ending with time index n.

$$X_N^n(\omega_k) = \sum_{m=n-N+1}^{n} x[m]\, W_N^{km} \quad \text{for } k=0,\ldots,N-1 \qquad (2.5)$$

$$x[m] = \frac{1}{N} \sum_{k=0}^{N-1} X_N^n(\omega_k)\, W_N^{-km} \quad \text{for } m=n-N+1,\ldots,n. \qquad (2.6)$$

A useful recursive equation for computing $X_N^n(\omega_k)$ from $X_N^{n-1}(\omega_k)$ can be derived from the above definitions and is given as follows

$$X_N^n(\omega_k) = X_N^{n-1}(\omega_k) + (x[n] - x[n-N])\, W_N^{kn}, \quad \text{for } k=0,\ldots,N-1. \qquad (2.7)$$

If x[n] is of finite duration, for example if $x[n]\neq 0$ only for n=0,..,N-1, then the N-point DFT of x[n] and the DTFT of x[n] are equal at ω_k,

$$X_N(\omega_k) = X(e^{j\omega T})\Big|_{\omega=\omega_k}, \text{ for } k=0,\ldots,N-1. \qquad (2.8)$$

Signal Processing Theorems

In this subsection, we will develop results that can be used to bound the effects of using finite-length data to compute frequency-domain quantities. In the later parts of this paper, the frequency-domain estimate of a stable, causal, transfer function $H(e^{j\omega T})$ will be computed based on the N-point DFTs of the transfer function's input and output signals. We will now state a theorem that bounds the error in the frequency domain between this DFT derived frequency-domain estimate and the true transfer function.

Theorem 2.1: Let $y[m]=h[m]*u[m]$, where h[m] is an infinite-length, causal, impulse response with all its poles in the open unit disk. We denote the DTFT of h[m] by $H(e^{j\omega T})$, and the DFTs of the N-points of u[m] and y[m] ending with time index n, by $U_N^n(\omega_k)$ and $Y_N^n(\omega_k)$, respectively. Then,

$$Y_N^n(\omega_k) = H(e^{j\omega_k T})\, U_N^n(\omega_k) + E_N^n(\omega_k), \quad \text{for } k=0,\ldots,N-1, \qquad (2.9)$$

where the discrete function $E_N^n(\omega_k)$ is given by

$$E_N^n(\omega_k) = \sum_{p=1}^{\infty} h[p]\, W_N^{kp} \left(U_N^{n-p}(\omega_k) - U_N^n(\omega_k) \right), \quad \text{for } k=0,\ldots,N-1, \tag{2.10}$$

where W_N is defined in Eqn. (2.3).

Proof: See [5].

It will later be useful to be able to find a magnitude bounding function on $E_N^n(\omega_k)$. The following theorem provides such a bounding function by using only a finite summation and therefore can be implemented in practice.

Theorem 2.2: Under the assumptions of Theorem 2.1 we find that given some finite integer M, the magnitude of $E_N^n(\omega_k)$ is bounded for each k as follows,

$$|E_N^n(\omega_k)| \le \sum_{p=1}^{M-1} |h[p]|\, |U_N^{n-p}(\omega_k) - U_N^n(\omega_k)| + 2\, u_{max} \sum_{p=M}^{\infty} p\, |h[p]|, \quad \text{for } k=0,\ldots,N-1, \tag{2.11}$$

where $u_{max} = \sup_m |u[m]|$.

Proof: See [5].

3. ROBUST ESTIMATOR PROBLEM STATEMENT

In this section, we list assumptions about the plant and the disturbance in preparation for the statement of the robust estimation problem. Consider the system of Figure 1 where the discrete-time plant $G_{true}(z)$ has input u[n] and an output y[n] that is corrupted by an additive output disturbance d[n].

A1) Plant Assumptions. We assume a structure for the nominal model of $G_{true}(z)$ and a magnitude bounding function on the unstructured uncertainty. That is, we assume that

$$G_{true}(z) = G(z,\theta_0)\, [1 + \delta_u(z)] \tag{3.1}$$

where $G(z,\theta_0)$ is a nominal model, $\delta_u(z)$ denotes the unstructured uncertainty of the plant, θ_0 is a vector of plant parameters and we assume,

A1.1) $G(z,\theta_0) = B(z) / A(z)$, (3.2)

where the polynomials B(z) and A(z) are,

$$B(z) = b_0\, z^{(m_1-n_1)} + b_1\, z^{(m_1-n_1-1)} + \ldots + b_{m1}\, z^{-n_1}, \tag{3.3}$$

$$A(z) = 1 - a_1 z^{-1} + \ldots - a_{n1}\, z^{-n_1}, \quad n_1 > m_1, \tag{3.4}$$

and where the parameter vector is,

$$\theta_0 = [\, a_1 \ldots a_{n1}\ b_0\ b_1 \ldots b_{m1}\,]^T. \tag{3.5}$$

A1.2) $\theta_0 \in \Theta$, where Θ is a known bounded set. (3.6)

A1.3) $|\delta_u(e^{j\omega T})| \le \Delta_u(e^{j\omega T}),\ \forall\omega.$ (3.7)

A1.4) $|d\delta_u(e^{j\omega T}) / d\omega| \le \nabla_u(e^{j\omega T}),\ \forall\omega.$ (3.8)

A1.5) $G_{true}(z)$ and $G(z,\theta_0)$ have all their poles in the open unit disk, for all $\theta_0 \in \Theta$.

A1.6) A coarse bounding function on the magnitude of the impulse response of the true plant, denoted by $g_{true}[n]$, is known such that

$$|g_{true}[n]| \le \sum_{i=1}^{I_0} g_i\, n^{(r_i)}\, p_i^n, \tag{3.9}$$

where r_i is a positive integer, and $g_i > 0$, $0 < p_i < 1$ (i.e. all the poles of $g_{true}[n]$ are in the open unit disk), and r_i are known for $i=1,\ldots,I_0$. $g_{true}[n]$ is assumed to be causal.

A1.7) zero initial conditions.

Thus, our a priori assumptions are that we know m_1 and n_1, the degrees of B(z) and A(z), respectively, and the bounding functions $\Delta_u(e^{j\omega T})$ and $\nabla_u(e^{j\omega T})$. Further, we assume that the parameter vector θ_0 is in some known bounded set, Θ, which is only a coarse, and hence large, a priori estimate of the parameter space. The parameter vector θ_0 is not required to be unique.

A2) Disturbance Assumption. We assume that the N-point DFT of the disturbance signal d[n], whose DFT is denoted by $D_N^n(\omega_k)$, satisfies

$$|D_N^n(\omega_k)| \le \overline{D}_N(\omega_k), \quad \text{for } k=0,\ldots,N-1,\ \forall n. \tag{3.10}$$

A3) Input Signal Assumption. We assume that the input signal u[n] is bounded and that we know u_{max} where

$$|u[n]| \le u_{max},\ \forall n. \tag{3.11}$$

Remark: Based on input/output measurements alone we cannot determine a specific θ_0 for the nominal model because of the unstructured uncertainty. That is, if we assume the structure of A1.1 above and assume only that $\delta_u(z) \in \mathbf{S}$ where

$$\mathbf{S} = \{\, \delta(z) \mid |\delta(e^{j\omega T})| \le \Delta_u(e^{j\omega T}),\ \forall\omega \,\}, \tag{3.12}$$

then we can define a smallest set

$$\Theta^* = \{\, \theta \mid G_{true}(z) = G(z,\theta)[1 + \delta_u(z)] \text{ and } \delta_u(z) \in \mathbf{S} \,\} \tag{3.13}$$

in which θ_0 lies. Thus, $\theta_0 \in \Theta^* \subset \Theta$ where only Θ is known a priori. Note that, in general, Θ^* will be a point only when $\Delta_u(e^{j\omega T}) = 0$ for all ω.

Robust Estimation Problem Statement

We rewrite the true discrete-time plant of Eqn. (3.1) as

$$G_{true}(z) = G(z,\hat\theta)\, [1 + \delta_{su}(z,\hat\theta)] \tag{3.14}$$

where again $G(z,\hat\theta)$ is the nominal model using an estimate $\hat\theta$ of the parameter vector θ_0 in the structure of assumption A1.1, and $\delta_{su}(z,\hat\theta)$ denotes the modeling error due to both structured and unstructured uncertainty. That is, since a priori we only know that $\hat\theta \in \Theta$, where $\hat\theta$ is not necessarily in Θ^*, there is structured uncertainty associated with this choice of $\hat\theta$ as well as the ever present unstructured uncertainty.

Problem Statement: The robust estimator must provide:

1) a parameter estimate $\hat{\theta}$, and hence a nominal model $G(z, \hat{\theta})$,

2) a corresponding bounding function, $\Delta_{su}^n(e^{j\omega T}, \hat{\theta})$, such that

$$|\delta_{su}(e^{j\omega T}, \hat{\theta})| \leq \Delta_{su}^n(e^{j\omega T}, \hat{\theta}), \ \forall \omega. \quad (3.15)$$

That is, at a given sample time n we want to generate a new nominal model $G(z, \hat{\theta})$, and a corresponding bounding function $\Delta_{su}^n(e^{j\omega T}, \hat{\theta})$ in the frequency domain indicating how good the current nominal model is. Given 1 and 2 above and a compensator we can use discrete-time versions of the stability robustness tests of [6] to guarantee stability in the face of bounded modeling uncertainty.

The goal of the robust estimator is to find a $\hat{\theta}$ in Θ^* and to have $\Delta_{su}^n(e^{j\omega T}, \hat{\theta})$ approach $\Delta_u(e^{j\omega T})$. The viewpoint taken here is that the unstructured uncertainty $\Delta_u(e^{j\omega T})$ is the best we can do given the structure of our nominal model. Thus, even though $\Delta_{su}^n(e^{j\omega T}, \hat{\theta})$ can conceivably become smaller than our a priori assumed bound $\Delta_u(e^{j\omega T})$ we will not let this occur and will instead view the function $\Delta_u(e^{j\omega T})$ as the desirable lower bound of the function $\Delta_{su}^n(e^{j\omega T}, \hat{\theta})$.

The problem that we have described in this subsection will be referred to as the robust estimation problem. An algorithm which satisfies this problem will be referred to as a robust estimator since it provides a nominal model of the plant as well as a guaranteed frequency-domain bounding function on the accuracy of this nominal model.

Outline of Problem Solution

In the following three sections of this paper, we will develop a solution to the robust estimation problem stated above. First, in Section 4, we will develop a method for computing a frequency-domain estimate of the true plant along with a bounding function on the additive error in the frequency domain. Then, in Section 5, the frequency-domain estimate of Section 4 will be used to find parameter estimates for the nominal model. Lastly, in Section 6, we will compute a frequency-domain bounding function on the magnitude of the uncertainty $\delta_{su}(e^{j\omega T}, \hat{\theta})$.

4. FREQUENCY-DOMAIN ESTIMATION AND ERROR BOUNDING

In this section, we will develop the basic methodology for finding a frequency-domain estimate of the true plant and a corresponding error bounding function on the frequency-domain modeling error.

Development of the Basic Methodology

Consider the true discrete-time plant $g_{true}[n]$ whose input is u[n], and whose disturbance-corrupted output is y[n]. Assuming zero initial conditions, we know that

$$y[n] = g_{true}[n] * u[n] + d[n]. \quad (4.1)$$

Then, using the notation of Section 2 and Theorem 2.1, we find that for some time index n,

$$Y_N^n(\omega_k) = G_{true}(e^{j\omega_k T})\, U_N^n(\omega_k) + E_N^n(\omega_k) + D_N^n(\omega_k) \quad \text{for } k=0, \ldots, N-1, \quad (4.2)$$

where from Theorem 2.2 we know that for some M,

$$|E_N^n(\omega_k)| \leq \overline{E}_N^n(\omega_k), \ \text{ for } k=0, \ldots, N-1 \quad (4.3)$$

with

$$\overline{E}_N^n(\omega_k) = \sum_{i=1}^{M-1} |g_{true}[i]|\, |U_N^{n-i}(\omega_k) - U_N^n(\omega_k)| + 2\, u_{max} \sum_{i=M}^{\infty} i\, |g_{true}[i]|, \ \text{ for } k=0, \ldots, N-1, \quad (4.4)$$

where we know u_{max} from assumption A3. Assume, for example, that

$$|g_{true}[n]| \leq g_2\, p_2^n, \ \text{ for } n=0, 1, \ldots \quad (4.5)$$

In this case, we can find a closed-form expression for the infinite summation term, so using Eqn. (4.4) we find

$$\overline{E}_N^n(\omega_k) \leq \sum_{i=1}^{M-1} g_2\, p_2^i\, |U_N^{n-i}(\omega_k) - U_N^n(\omega_k)| + 2\, u_{max}\, g_2\, p_2^M (M - M p_2 + p_2) / (1 - p_2)^2, \ \text{ for } k=0, \ldots, N-1. \quad (4.6)$$

The bounding function of Eqn. (4.6) can be computed on-line by using the current N-point DFT of u[n] along with M-1 old N-point DFTs of u[n]. We note that the second line of the previous equation can be made arbitrarily small by choosing M to be sufficiently large.

Now, we define the frequency-domain estimate $G_{f,N}^n(\omega_k)$ and the corresponding frequency-domain error $E_{f,N}^n(\omega_k)$.

$$G_{f,N}^n(\omega_k) = Y_N^n(\omega_k) / U_N^n(\omega_k) \quad (4.7)$$

$$E_{f,N}^n(\omega_k) = G_{f,N}^n(\omega_k) - G_{true}(e^{j\omega_k T}), \quad \text{for } k=0, \ldots, N-1. \quad (4.8)$$

From Eqn. (4.2),

$$E_{f,N}^n(\omega_k) = (\, E_N^n(\omega_k) + D_N^n(\omega_k)\,) / U_N^n(\omega_k) \quad (4.9)$$

and using the triangle inequality we find,

$$|E_{f,N}^n(\omega_k)| \leq \overline{E}_{f,N}^n(\omega_k) \quad (4.10)$$

where

$$\overline{E}_{f,N}^n(\omega_k) = (\, \overline{E}_N^n(\omega_k) + \overline{D}_N(\omega_k)\,) / |U_N^n(\omega_k)|, \quad (4.11)$$

and where $\overline{E}_N^n(\omega_k)$ is given by Eqn. (4.4). We will refer to $G_{f,N}^n(\omega_k)$ as our frequency-domain estimate of the true plant at time index n. Note that $G_{f,N}^n(\omega_k)$ is the set of N complex numbers computed using the N-point DFTs of u[n] and y[n], which are computed on-line. Further, we will refer to $\overline{E}_{f,N}^n(\omega_k)$ as the frequency-domain error bounding function at time index n. In Eqn. (4.11), the bounding function $\overline{E}_N^n(\omega_k)$ and $|U_N^n(\omega_k)|$ are computed on-line at each time index n, while the function $\overline{D}_N(\omega_k)$ is known from assumption A2.

The Cumulative Frequency-domain Estimate and Error Bounding Function

In this subsection, we will discuss a straight-forward technique for combining the frequency-domain estimates and the corresponding error bounding functions from different time intervals. That is, we show how to combine all of the past frequency-domain information into a cumulative estimate and cumulative error bounding function. The basic idea is that at a given frequency point ω_k we use the value of $G_{f,N}^n(\omega_k)$ that has the smallest corresponding error bounding function $\overline{E}_{f,N}{}^{n}(\omega_k)$, at that frequency. To formalize this we define the cumulative error bounding function at ω_k,

$$\overline{E}_{cumf,N}{}^{n}(\omega_k) = \min_{p \le n} \{ \overline{E}_{f,N}{}^{p}(\omega_k) \}, \quad (4.12)$$

and the cumulative frequency-domain estimate at ω_k,

$$G_{cumf,N}{}^{n}(\omega_k) = \{G_{f,N}{}^{m}(\omega_k) \mid \overline{E}_{f,N}{}^{m}(\omega_k)=\overline{E}_{cumf,N}{}^{n}(\omega_k)\}. \quad (4.13)$$

The subscript 'cumf' in Eqns. (4.12-13) denotes the fact that they are the 'cumulative frequency-domain' estimate and error bounding function. We define, for time index n,

$$E_{cumf,N}{}^{n}(\omega_k) = G_{cumf,N}{}^{n}(\omega_k) - G_{true}(e^{j\omega_k T}), \quad \text{for } k=0,\ldots,N-1. \quad (4.14)$$

Then, Eqn. (4.10) ensures that at time index n,

$$|E_{cumf,N}{}^{n}(\omega_k)| \le \overline{E}_{cumf,N}{}^{n}(\omega_k), \text{ for } k=0,.., N-1. \quad (4.15)$$

In practice, the following simple recursive algorithm will be used to compute $G_{cumf,N}{}^{n}(\omega_k)$ and $\overline{E}_{cumf,N}{}^{n}(\omega_k)$ at a given frequency ω_k.

Algorithm:

If $\overline{E}_{f,N}{}^{n}(\omega_k) < \overline{E}_{cumf,N}{}^{n-1}(\omega_k)$ then set

$$\overline{E}_{cumf,N}{}^{n}(\omega_k) = \overline{E}_{f,N}{}^{n}(\omega_k), \text{ and}$$

$$G_{cumf,N}{}^{n}(\omega_k) = G_{f,N}{}^{n}(\omega_k). \quad (4.16)$$

else set

$$\overline{E}_{cumf,N}{}^{n}(\omega_k) = \overline{E}_{cumf,N}{}^{n-1}(\omega_k), \text{ and}$$

$$G_{cumf,N}{}^{n}(\omega_k) = G_{cumf,N}{}^{n-1}(\omega_k).$$

Thus, our algorithm only updates the cumulative frequency-domain estimate and the corresponding cumulative error bounding function when useful information is learned, at a given frequency.

As a final note, we observe that since we are working with real-valued time-domain signals, the properties of the DFTs of real-valued signals can be used to show that,

$$G_{cumf,N}{}^{n}(\omega_k) = G^*{}_{cumf,N}{}^{n}(\omega_{N-k}), \quad (4.17)$$

$$\overline{E}_{cumf,N}{}^{n}(\omega_k) = \overline{E}_{cumf,N}{}^{n}(\omega_{N-k}), \quad (4.18)$$

$$\text{for } k=1,\ldots,(N/2)-1,$$

where '*' denotes complex conjugate and where we have assumed that N is even. This means that the information for the frequency points k=0,..,N-1 is contained in the information for the frequency points k=0,..,N/2.

5. COMPUTING PARAMETER ESTIMATES

In this section, we will show how the cumulative frequency-domain estimate of the previous section can be used to find parameter estimates for the nominal model. We use the structure of the nominal model, which was assumed in A1.1, and a type of weighted least-squares fit to the frequency-domain estimate $G_{cumf,N}{}^{n}(\omega_k)$. The procedure is best illustrated by an example. Consider the nominal model,

$$G(z,\theta_0) = b_0 / (z - a_1), \text{ where} \quad (5.1)$$

$$\theta_0 = [\, a_1 \;\; b_0 \,]^T. \quad (5.2)$$

Using this nominal model structure we can write

$$(z - a_1)\, G(z,\theta_0) = b_0, \quad (5.3)$$

$$\text{or } \; z\,G(z,\theta_0) = [\, G(z,\theta_0) \;\; 1 \,]\, \theta_0. \quad (5.4)$$

Since the parameters are assumed to be real-valued, we find

$$\mathrm{Re}\{z\,G(z,\theta_0)\} = [\, \mathrm{Re}\{G(z,\theta_0)\} \;\; 1 \,]\, \theta_0, \quad (5.5)$$

$$\mathrm{Im}\{z\,G(z,\theta_0)\} = [\, \mathrm{Im}\{G(z,\theta_0)\} \;\; 0 \,]\, \theta_0. \quad (5.6)$$

Thus, if we know the complex value of $G(z,\theta_0)$ for some known z, we can find two linear equations in the parameters. Our frequency-domain estimation method yields an estimate of the plant at frequencies ω_k for k=0,..,N/2. So, letting $z=e^{j\omega_k T}$ for k=0,..,N/2 we define the (N+2)×2 matrix,

$$A(\, G(e^{j\omega_k T},\theta_0)\,) = \begin{bmatrix} \mathrm{Re}\{G(e^{j\omega_0 T},\theta_0)\} & 1 \\ \vdots & \\ \mathrm{Re}\{G(e^{j\omega_{(N/2)} T},\theta_0)\} & 1 \\ \mathrm{Im}\{G(e^{j\omega_0 T},\theta_0)\} & 0 \\ \vdots & \\ \mathrm{Im}\{G(e^{j\omega_{(N/2)} T},\theta_0)\} & 0 \end{bmatrix} \quad (5.7)$$

and the (N+2) vector,

$$B(\, G(e^{j\omega_k T},\theta_0)\,) = \begin{bmatrix} \mathrm{Re}\{e^{j\omega_0 T} G(e^{j\omega_0 T},\theta_0)\} \\ \vdots \\ \mathrm{Re}\{e^{j\omega_{(N/2)} T} G(e^{j\omega_{(N/2)} T},\theta_0)\} \\ \mathrm{Im}\{e^{j\omega_0 T} G(e^{j\omega_0 T},\theta_0)\} \\ \vdots \\ \mathrm{Im}\{e^{j\omega_{(N/2)} T} G(e^{j\omega_{(N/2)} T},\theta_0)\} \end{bmatrix} \quad (5.8)$$

Using Eqns. (5.2) and (5.7-8) we can write,

$$A(\, G(e^{j\omega_k T},\theta_0)\,)\, \theta_0 = B(\, G(e^{j\omega_k T},\theta_0)\,). \quad (5.9)$$

In summary, we have shown how knowledge of the complex values of $G(e^{j\omega_k T},\theta_0)$ at the (N/2)+1 frequencies $\omega_0,..,\omega_{(N/2)}$ can be used to write N+2 linear equations in the parameters. In the ideal situation where one could exactly find $G(e^{j\omega_k T},\theta_0)$ for k=0,..,(N/2), the matrix equation (5.9) will have a solution. That is, given the matrices A and B, we could solve for the true parameter vector using any 2 linear equations. However, in practice we will only have our cumulative frequency-domain estimate $G_{cumf,N}{}^{n}(\omega_k)$ with which to estimate the parameters. If we use $G_{cumf,N}{}^{n}(\omega_k)$ instead of $G(e^{j\omega_k T},\theta_0)$ in Eqns. (5.7-8), then the equation,

$$A(\, G_{cumf,N}{}^{n}(\omega_k)\,)\, \hat{\theta} = B(\, G_{cumf,N}{}^{n}(\omega_k)\,) \quad (5.10)$$

will not, in general, have a solution. Eqn. (5.10) is in the form of the standard least-squares problem, which is discussed in Strang [7].

We will choose the parameter estimate $\hat{\theta}$ as the vector that minimizes the frequency weighted norm of the error vector,

$$A(\, G_{cumf,N}{}^{n}(\omega_k)\,)\, \hat{\theta} - B(\, G_{cumf,N}{}^{n}(\omega_k)\,). \quad (5.11)$$

We define, with reference to Eqns. (5.7-8), the diagonal frequency weighting matrix,

$$W = \mathrm{diag}[\, f(\omega_0) \ldots, f(\omega_{(N/2)})\;\; f(\omega_0) \ldots, f(\omega_{(N/2)})\,]. \quad (5.12)$$

where f(•) is the frequency weighting function. The parameter estimate that minimizes the norm of the error vector

$$W\,(\, A(\, G_{cumf,N}{}^{n}(\omega_k)\,)\, \hat{\theta} - B(\, G_{cumf,N}{}^{n}(\omega_k)\,)\,) \quad (5.13)$$

is given by the well-known result,

$$\hat{\theta} = (A^T W^T W A)^{-1} A^T W^T W B \quad (5.14)$$

where the A and B matrices in this equation depend on the values of the estimate $G_{cumf^{\prime}N}^{n}(\omega_k)$.

To gain insight as to what weighting function to choose, we examine Eqns. (5.3-4). Consider the use of the above methodology using the estimate $\hat{G}(z)$. Then, we find that the error

$$z\,\hat{G}(z) - [\,\hat{G}(z)\;\; 1\,]\,\theta_0 = (z - a_1)\,\hat{G}(z) - b_0 \qquad (5.15)$$

$$= (z - a_1)\,(\,\hat{G}(z) - G(z,\theta_0)\,) \qquad (5.16)$$

So, $|\hat{G}(z) - G(z,\theta_0)| = |\,z\hat{G}(z) - [\,\hat{G}(z)\;\; 1\,]\,\theta_0\,| \,/\, |z - a_1|$. (5.17)

From Eqn. (5.17) we see that if we want our parameter estimation method to be a least-squares fit in the frequency-domain, then we want to choose a weighting function that is one over the magnitude of the denominator of the nominal model. Of course, we do not know what the parameter a_1 really is, so one can only approximately choose this frequency weighting function.

6. COMPUTING A FREQUENCY-DOMAIN UNCERTAINTY BOUNDING FUNCTION

In this section, we discuss the computation of a frequency-domain uncertainty bounding function for the nominal model $G(e^{j\omega_k T},\hat{\theta})$. Specifically, we will compute a magnitude bounding function, $\Delta_{su}{}^{n}(e^{j\omega_k T},\hat{\theta})$, on $\delta_{su}(e^{j\omega_k T},\hat{\theta})$ at the frequency points corresponding to ω_k for k=0,..,N-1.

The nominal model at time index n is obtained by using the nominal model structure and the current parameter vector estimate $\hat{\theta}$ yielded by the parameter estimator described in Section 5. Thus, we can compute the value of the nominal model $G(e^{j\omega_k T},\hat{\theta})$ for k=0,..,N-1. Now, using the triangle inequality, we find that at time index n, and for frequency ω_k,

$$|G(e^{j\omega_k T},\hat{\theta}) - G_{true}(e^{j\omega_k T})| \le |G(e^{j\omega_k T},\hat{\theta}) - G_{cumf^{\prime}N}^{n}(\omega_k)| + |G_{cumf^{\prime}N}^{n}(\omega_k) - G_{true}(e^{j\omega_k T})|. \qquad (6.1)$$

and using Eqns. (4.14-15),

$$|G(e^{j\omega_k T},\hat{\theta}) - G_{true}(e^{j\omega_k T})| \le |G(e^{j\omega_k T},\hat{\theta}) - G_{cumf^{\prime}N}^{n}(\omega_k)| + \overline{E}_{cumf^{\prime}N}^{n}(\omega_k). \qquad (6.2)$$

We now can find a bound on $\delta_{su}(e^{j\omega_k T},\hat{\theta})$. Rewriting Eqn. (3.14),

$$G_{true}(e^{j\omega_k T}) = G(e^{j\omega_k T},\hat{\theta})\,[1 + \delta_{su}(e^{j\omega_k T},\hat{\theta})], \quad \text{for } k=0,\ldots,N-1. \qquad (6.3)$$

So, rearranging yields,

$$\delta_{su}(e^{j\omega_k T},\hat{\theta}) = [\,G_{true}(e^{j\omega_k T}) - G(e^{j\omega_k T},\hat{\theta})\,] \,/\, G(e^{j\omega_k T},\hat{\theta}). \qquad (6.4)$$

Thus, using Eqn. (6.2), we find the bounding function,

$$|\delta_{su}(e^{j\omega_k T},\hat{\theta})| \le \Delta_{su}{}^{n}(e^{j\omega_k T},\hat{\theta}), \qquad (6.5)$$

where

$$\Delta_{su}{}^{n}(e^{j\omega_k T},\hat{\theta}) = \{\,|G(e^{j\omega_k T},\hat{\theta}) - G_{cumf^{\prime}N}^{n}(\omega_k)| + \overline{E}_{cumf^{\prime}N}^{n}(\omega_k)\,\} \,/\, |G(e^{j\omega_k T},\hat{\theta})|, \quad \text{for } k=0,\ldots,N-1. \qquad (6.6)$$

and where we have included a superscript 'n' after the Δ_{su} to denote the fact that this bound on $|\delta_{su}(e^{j\omega_k T},\hat{\theta})|$ depends on the time index n, since $G_{cumf^{\prime}N}^{n}(\omega_k)$, $\overline{E}_{cumf^{\prime}N}^{n}(\omega_k)$ and also $\hat{\theta}$ depend on n.

In summary, we have shown how to compute a discrete function $\Delta_{su}{}^{n}(e^{j\omega_k T},\hat{\theta})$ that bounds the net effect of structured and unstructured uncertainty of the current nominal model $G(e^{j\omega_k T},\hat{\theta})$ relative to the true plant, at the frequencies, $\omega_0,..,\omega_{N-1}$. We used the nominal model structure of A1.1, the current parameter estimate $\hat{\theta}$; and the cumulative frequency-domain estimate $G_{cumf^{\prime}N}^{n}(\omega_k)$ and corresponding cumulative error bounding function $\overline{E}_{cumf^{\prime}N}^{n}(\omega_k)$, which were developed in Section 4.

A Smoothed Uncertainty Bounding Function

In this subsection, we discuss the computation of a smoothed, magnitude bounding function on $|\delta_{su}|$. This development is motivated by the observation that, depending upon the spectrum of the input signal, one may have a very jagged bounding function on the modeling uncertainty $|\delta_{su}(e^{j\omega_k T},\hat{\theta})|$. That is, at the frequency point ω_k the bound $\Delta_{su}{}^{n}(e^{j\omega_k T})$ may be very tight, however, at an adjacent frequency point ω_{k+1} the bound $\Delta_{su}{}^{n}(e^{j\omega_{k+1} T})$ may be very poor. In [5], it is shown how the assumptions of Section 3 can be used to find a derivative bounding function, $\nabla_{su}{}^{n}(e^{j\omega T})$, satisfying

$$|d\delta_{su}(e^{j\omega_k T},\hat{\theta}) \,/\, d\omega| \le \nabla_{su}{}^{n}(e^{j\omega T}), \;\; \forall\omega. \qquad (6.7)$$

Assuming the analyticity of δ_{su}, it is shown in [5] that

$$|\delta_{su}(e^{j\omega T},\hat{\theta})| \le |\delta_{su}(e^{j\omega_k T},\hat{\theta})| + |\omega-\omega_k|\,\nabla_{su,i}{}^{n}(\omega_k,\omega_{k+1}) \qquad (6.8)$$

and

$$|\delta_{su}(e^{j\omega T},\hat{\theta})| \le |\delta_{su}(e^{j\omega_{k+1} T},\hat{\theta})| + |\omega_{k+1}-\omega|\,\nabla_{su,i}{}^{n}(\omega_k,\omega_{k+1}) \qquad (6.9)$$

for $\omega \in [\omega_k,\omega_{k+1}]$ where

$$\nabla_{su,i}{}^{n}(\omega_k,\omega_{k+1}) = \sup_{\omega\in[\omega_k,\omega_{k+1}]} \{\,\nabla_{su}{}^{n}(e^{j\omega T})\,\}. \qquad (6.10)$$

From these equations we see that it may be possible to obtain a tighter bound on $|\delta_{su}(e^{j\omega_k T},\hat{\theta})|$ than $\Delta_{su}{}^{n}(e^{j\omega_k T},\hat{\theta})$, by using the bound at an adjacent frequency point, $\Delta_{su}{}^{n}(e^{j\omega_{k-1} T},\hat{\theta})$ or $\Delta_{su}{}^{n}(e^{j\omega_{k+1} T},\hat{\theta})$, along with the smoothness information of $\nabla_{su,i}$.

Bounding Inter-sample Variations

In this brief subsection, we discuss the computation of a safety factor that must be added to the discrete bounding function $\Delta_{su}{}^{n}(e^{j\omega_k T},\hat{\theta})$ to account for inter-sample variations. Ultimately, the uncertainty bounding function at discrete frequency points will be used in stability-robustness tests to design a new robust compensator. These stability-robustness tests are meant to be used with continuous functions of frequency. Since the actual computations will be performed with an uncertainty bounding function that is a discrete function of frequency, we must add the aforementioned safety factor to the discrete function to account for the worst possible peaks that may occur between frequency samples ω_k. In [5], it is shown how Eqns. (6.8-9) can be used to choose this additive safety factor in such a way that the largest inter-sample variations lie below a line drawn between the values of the final uncertainty bounding function (including the safety factor) at two adjacent frequency samples.

7. CONCLUDING REMARKS

In this paper, we presented a new estimation (identification) methodology that can be used in a robust adaptive controller to provide stability-robustness guarantees. The key feature of the robust estimator is the frequency-domain bounding function on the modeling uncertainty. In [5], our simulation results revealed that the use of the robust estimator can yield improved closed-loop perfomance, that is, increased bandwidth as compared with the best LTI compensator that could have been designed using only a priori knowledge of the plant. In some situations, the plant input signal is not rich enough to allow identification. In these cases, an external probing signal must be introduced. This issue of adding probing signals is discussed further in [5]. As a final remark, we note that the robust estimator provides guarantees that no other methodology can; however, the price is the extensive frequency-domain calculations.

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