DIMENSIONAL PHASE TRANSITIONS

by

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ABSTRACT

We define a dimensional phase transition as one in which the dimensionality of space-time changes. Two possible scenarios for such a phase transition are presented.

One of the scenarios may be thought of as a generalization of the random lattice techniques while the other may be interpreted as a dynamical realization of the spontaneous compactification in a Kaluza-Klein theory.

Most of the concrete results presented are about the second scenario. Specifically, we investigate the dimensional phase transitions through collison of space-time boundaries.

The classical dynamics of space-time boundaries is established. The dynamics is uniquely defined when gravity is present. It is shown that for a certain choice of boundary conditions on metric fluctuations and the matter fields the coupling of space-time boundary proceeds in a universal manner analogous to the case of gravity, i.e. the back reaction of the matter and gravity is represented by the matter stress-energy tensor only. The boundary condition for the gravity is derived and discussed in connection with the conservation of energy.

Quantum properties of the matter-gravity-boundary system are discussed. It is shown that the measured Casimir effect of attraction between two conductors can be calculated using the framework of boundary dynamics, i.e. in terms of local quantities as opposed to conventional calculations based on global quantities like total renormalized energy.

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To the memory of Professor Felix Berezin

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DIMENSIONAL PHASE TRANSITIONS, AN INTRODUCTION

It is widely believed that if one looks back into the history of the universe one will observe a number of phase transitions. The chiral and confinement transitions are expected to occur at temperatures around 0.5 GeV, the Weinberg-Salam transition around 10^2 GeV, the GUT transition somewhere between 10^{14}--10^{16} GeV and effects of quantum gravity will come into play when the energy scale of the particles reaches 10^{19} GeV.

In this thesis we speculate about and obtain some concrete results concerning another phase transition. We explore the idea that at certain energy, temperature or density scale our present-day vacuum becomes unstable with respect to change of space-time dimensionality.

First of all, why one should expect such a change in dimensionality, or dimensional phase transition? It is a simple fact that in a flat m-dimensional Euclidean space R^m one can draw a unique line through any two points, a unique plane through any three points and so on. That is, generically, $n < m$ points in R^m belong to a n-1 dimensional hyperplane $R^n \subset R^m$. If we were to take this argument seriously and apply it to the physical world, then one cannot but wonder why about 10^{80} protons or electrons or any other kind of stable particles that are estimated to inhabit the observable universe fit so nicely into the three-dimensional space at each point in time, instead of some space of dimensionality of $10^{80}-1$, where they truly belong. Indeed, putting aside cosmologically short-scale inhomogeneities one could easily imagine our universe populated by rather weakly interacting stable elementary particles. Why then the observed matter moves along trajectories that lie in a three-dimensional subspace of $10^{80}-1$ dimensional space? Or, in other words, why is the dimensionality of space-time so low?

The answer to these questions obviously lies in properties of the vacuum. Even if one observes a one-particle state, without reference to any other particle, one still assumes that the momentum is four-dimensional. This assumption states implicitly that it is the vacuum, that serves as a reference frame for measuring of dimensionality. It is apparent, looking from this angle, that it is through interaction with the vacuum that particles "know" which dimension they are in, even if they do not interact or interact very weakly.

One is led, consequently, to the notion that the dimensionality is a property of the vacuum. Furthermore, we know that the vacuum is not an empty abstraction but is a rather complex physical system, whose properties can and do change, the simplest example being a spontaneous decay of "false" vacuum into the"true" vacuum when a symmetry is spontaneously broken. It is natural to assume, therefore, and it constitutes the basis for this research, that dimensionality is one of the _physical_ properties of the vacuum. Thus it is natural to ask the following questions: can the dimensionality change? has it ever changed in the history of the universe? There certainly is no experimental evidence to support (or disprove, for that matter) that the second question has a positive answer. If however we can convince ourselves that it can change, by building , for example, an aesthetically pleasing formalism which will be natural in some way, then one will have some ground to predict that it did change at a certain point during the universe's evolution, since if anything can happen in this world, apparently it does. It is the first question we are concerned with.

An idea that there may exist some extra dimensions to the world we live in is far from new. Kaluza and Klein were the first to apply the idea of higher dimensions to physics [1]. And although they did not succeed in presenting a realistic model when treating four-dimensional gravity plus electromagnetism as a manifestation of pure five-dimensional gravity, the idea has been pursued with varying degrees of interest ever since.

More recently, in conjunction with the higher-dimensional supergravity theories, the interest has been revived again [2]. It is nesessary to point out, however, that the question of dimensionality in the Kaluza-Klein approach is largely ignored. Apart from some recent attempts to make the central concept of a Kaluza-Klein theory, spontaneous compactification, dynamical [3], the question of why eleven or ten is as unaddressed as why four.

Both scenarios for the dimensional phase transitions which we present here are devoid of the idea of spontaneous compactification and the initial dimensionality of space-time is not determined and is not importnat. Furthermore, the author believes that the initial singularity which the classical General Relativity predicts may be avoided as a concept if, instead of a picture of the Big Bang , one can develop a consistent picture of an infinite series of dimensional phase transitions that take place as the typical energy scale of the universe decreases.

Of course the final justification for such a scheme could only be an experiment confirming the existence of the dimensional phase transitions. We cannot claim that we can devise a determinate experiment nor can we present the reader with a consistent formalism as yet. However even if the ultimate goal of this research is far away, the by-products of it seem to be interesting in themselves. Although at the present moment the idea of the dimensional phase transition in the universe seems to be no more than a guiding light, the results we obtained in general relativity and theory of the Casimir effect provide an interesting outlook on these subjects.

Finally, even if as mentioned earlier, the evidence for a dimensional phase transition is absent, phase transitions of a similar kind do occur in solid state physics. The most obvious example is the foam formation in soapy water, where the ratio of area to volume goes to infinity under certain conditions. A less trivial example would be stratification in liquid crystals, where a three-dimensional volume of a specific crystal becomes, under certain conditions, stratified into an array of parallel, thin, two-dimensional layers. Another area of contact with the solid state physics would be the roughening phase transitions, also yet to be found experimentally [4].

We now present the two scenarios mentioned above and list the mathematical problems whose solution is necessary in order to develop the scenarios into full-fledged consistent formalisms. The rest of this chapter is on the question of the boundary conditions for the gravitational radiation and sections II and III of the thesis constitute partial solutions of some problems for the second scenario.

DIMENSIONAL PHASE TRANSITIONS THROUGH ROUGHENING OF SPACE-TIME

In this scenario we consider a lattice approximation to the continuum field theory as a basic concept. However instead of taking the lattice to be a triangulation of some smooth manifold we start with the premise that the sites of the lattice do not "know" beforehand any global information about the structure obtained by linking some of the sites. Thus we start with some fixed number of sites N (although it must ultimately vary). These sites are linked in a random manner by $M \leqslant \frac{N(N-1)}{2}$ links, or one-dimensional simplices. We call any three sites to be in common position if their links form a boundary of a triangle, or 2-simplex. We call a n-simplex elementary if it contains no sub-simplices. Elementary 2-simplices are attached in a random manner to the set of triads in common position without intersections. We proceed with this prosess until all elementary n-simpleces are joined in some random manner up to the largest dimension posssible. The result of this construction is what mathematicians call a simplicial complex. We bring physics into the picture by associating a statistical weight to each complex in such a manner that complexes that are "close" to those which are triangulations of some smooth ,"simple" manifold -- the base manifold X -- have the highest weight W , while complexes that do not approximate any smooth, "simple" manifold have the least weight. We then redefine the partition function for, say, the Euclidean formulation of a lattice field theory by introduction of additional averaging over all complexes with some suitable boundary conditions.

Thus instead of

$$Z(J) = \sum_{\substack{\text{field} \\ \text{configurations} \\ \text{for a given lattice}}} M(\phi_A) \exp\left[-\mathcal{L}(\phi_A) + \phi^A J_A\right] \tag{1}$$

where ϕ_A are some fields , $M(\phi_A)$ is the measure weight and J_A is the current, one writes

$$Z(J,K) = \sum_{\text{complexes}, C} \; \sum_{\substack{\text{field conf.} \\ \text{for a given} \\ \text{complex } C}} M(\phi_A, C) \tag{2}$$

$$\exp\left[-\mathcal{L}(\phi_A) + W(C) + \phi^A J_A + C^i K_i\right]$$

where C^i , K_i are parameters of the complex and corresponding currents, $W(C)$ is the statistical weight of the complex C and $M(\phi_A, C)$ is the summation weight.

The summation in (2) includes summation over all different triangulations of the same smooth base manifold X . In this sense our approach resembles the random lattice approach [5]. However we cannot attach physical difference to two different triangulations of X and , consequently, require

$$W(C) = W(C')$$

when C and C' triangulate the same manifold X . In this respect our approach is different from that of [5].

We now list the main problems one has to resolve in order to reach the calculational stage for a realistic theory.

1. Extension of a lattice field configuration from a lattice to an arbitrary complex.

This problem is difficult for any field with spin higher than zero and is especially difficult for fermions. The extension problem stems from the fact that spinors and boson fields with non-zero spin have the notion of dimensionality built into their definition. The problem may be avoided for boson fields by using singular exterior and symmetric form formalism, while use of"doubling"of fermion species may help solving the problem for fermions.

2. Determination of $W(C)$ and $M(\phi_A,C)$.

The expression (1) can be regarded as a dicrete approximation for the functional integral and the measure weight $M(\phi_A)$ ensures that $Z(J)$ has a sensible limit when the lattice size goes to zero. In the same sense $Z(J,K)$ in (2) can be thought of as a finite-dimensional approximation for a functional integral over fields and complexes. Part of the problem is to find $M(\phi_A,C)$ such that $Z(J,K)$ has a sensible limit when the sizes of elementary simplices go to zero.

The second part, more physical one, is the determination of the statistical weight $W(C)$, the weight which suppresses "bizarre" complexes and does not suppress "nice" complexes. If we were to consider at most two-dimensional space-times there would be a natural candidate for $W(C)$. It is a well-known fact that all compact two-dimensional manifolds are uniquely charachterized by their Euler number χ [6]. It was also shown by Rota [6] that the Euler characteristic can be written for an arbitrary abstract simplicial complex. The Euler number takes high bounded values for "nice" manifolds, while it monotonically decreases to minus infinity as the complexity of the manifold increases. Thus weighting complexes with the factor

$$\exp \lambda \chi \qquad , \quad \lambda \gg 1$$

would pick out only "nice" ones. Another non-trivial candidate is

$$W(C) = \frac{\hbar c}{a}\Big[\sum_{i,j \in C}(m_i - m_j)^2 + \lambda \sum_{j \in C} m_j\Big], \quad a = const$$
$$\lambda = const$$

where m_j is the number of nearest neighbours to the site j. $W(C)$ as above ensures existence of phase transition at $T_c \simeq \frac{\hbar c}{a}$ [4].

Having resolved the two main problems mentioned above it would be easy to pose a correct mathematical problem about existence of a dimensional phase transition in a particular system. One will have to prove existence of a complex which in the distant past is a triangulation of a space-time of some dimension N and in the distant future is a triangulation of a space-time of dimension $M \neq N$. Such a complex by definition cannot be a triangulation of any smooth manifold and may be called a rough instanton (borrowing terminology from the solid state physics). Actually one can imagine a rough instanton to connect space-times of the same dimensionality but different topological structure. It is rather hard to imagine making more dimensions from less, globally, and it is rather easy to imagine the reversed process. Thus it will be natural for a system to undergo a sequence of phase transitions with dimensionality decreasing. A possible answer to why four may be then suggested. Namely, our universe is four dimensional because it is near the end of its evolution Whether or not a phase transition to three space-time dimensions is possible in our universe is another interesting question to consider.

DIMENSIONAL TRANSITIONS THROUGH SPACE-TIME BOUNDARY COLLISION

In this scenario , which in its present form would work only when quantum gravity is not dominant, one considers spontaneous formation of a space-time boundary ∂X in some D-dimensional space-time X . If, during their evolution, two or more disjoint components of ∂X collide and "stick" together i.e. if an infinite area of contact is energetically favorable and if the thickness of the layer between two boundaries with, possibly, some matter trapped inside, is small enough then the "squashed" observer will not notice the extra dimension in the direction of the normal to the layer, just as the extra dimensions are unobservable in a Kaluza-Klein theory.

Since this scenario is discussed in greater detail in part II and III of the thesis, we now list the problems one has to solve in order to make the scenario work.

1. Developement of formalism of classical dynamics of space-time boundary.
2. Semiclassical quantization for the boundary dynamics and renormalization.
3. Developement of a formalism to describe spontaneous boundary formation as a tunneling process.

4. Late evolution of the layer of the collision.

The first and the second problems are largely solved and some of the results are presented below. Additional results will appear in forthcoming publications (see ref [13] in chapter III). At the same time there are some indications that the third problem is tractable and that in the late stage of its evolution the layer of the collision decouples from the matter and gravity inside the layer. This is largely due to the buundary conditions on the fields on the boundary. They are such that certain types of boundaries decouple from interactions inside. It is a plausible hypothesis that such configurations of the boundaries are in fact in some sense stable and consequently are the end points of boundary evolution.

We now would like to elaborate on the boundary condition for gravity proposed in chapter II.

The boundary condition in question is (all notations are on page 23 of chapter II)

$$(\chi^{\mu\nu} - \chi h^{\mu\nu})\epsilon_{\mu\nu}\big|_{\partial X} = 0 \tag{3}$$

for any boundary ∂X , where $\epsilon_{\mu\nu}$ is the metric variation. There is no boundary condition on the background metric itself.

On ∂X one can decompose $\epsilon_{\mu\nu}$ in tangential and normal directions

$$\epsilon_{\mu\nu} = \tilde{\epsilon}_{\mu\nu} + A_{(\mu} n_{\nu)} + B\, n_\mu n_\nu \tag{4}$$

$$A_\mu n^\mu = 0 \tag{5}$$

$$n^\mu \tilde{\epsilon}_{\mu\nu} = 0 \tag{6}$$

We now would like to enforce the transverse-traceless gauge for $\epsilon_{\mu\nu}$ since we are most interested in gravitational radiation which can carry energy and momentum. This means that

$$\epsilon^\mu{}_\mu = 0 \tag{7}$$

$$D^\mu \epsilon_{\mu\nu} = 0 \tag{8}$$

Applying constraint (7) to (4) we get

$$B = -\tilde{\epsilon}^{\mu}{}_{\mu} \tag{9}$$

Differentiatiog (6) we obtain

$$\chi^{\mu\nu}\tilde{\epsilon}_{\mu\nu} + n^{\mu} D^{\nu} \tilde{\epsilon}_{\mu\nu} = 0 \tag{10}$$

Applying (8) and (10) to (4) and projecting along and perpen- dicular to the normal we get on ∂X

$$n_{\mu} D^{\mu} \tilde{\epsilon}^{\mu}{}_{\mu} + 2\chi \tilde{\epsilon}^{\lambda}{}_{\lambda} = 0 \tag{11}$$

$$n_{\mu} F^{\mu}{}_{\nu} + \chi A_{\nu} + 2 h_{\nu}{}^{\rho} D^{\mu} \tilde{\epsilon}_{\mu\rho} = 0 \tag{12}$$

$$F_{\mu\nu} = D_{\mu} A_{\nu} - D_{\nu} A_{\mu} \tag{13}$$

These are Neumann-type boundary conditions on $\tilde{\epsilon}^{\mu}{}_{\mu}$ and A_{μ} provided they do not vanish. They vanish only when a gravitational wave falls on ∂X along the normal n_{μ}. In this case

$$k_{\mu} \sim n_{\mu}$$

where K_{μ} is the direction of the wave propagation. This leads us to considering the boundary condition on the traceless part of $\tilde{\epsilon}_{\mu\nu}$:

$$\tilde{\epsilon}^{T}_{\mu\nu} = \tilde{\epsilon}_{\mu\nu} - \frac{1}{D-1} h_{\mu\nu} \tilde{\epsilon}^{\lambda}{}_{\lambda}$$

Eliminating the remaining freedom by requiring

$$\epsilon_{\mu 0} = 0$$

we find that the boundary condition (3) applied to an arbitrary boundary is equivalent to the Dirichlet boundary condition

$$\tilde{\epsilon}^{T}_{\mu\nu} = 0 \tag{14}$$

which mean that for a plane gravitational wave with

$$\epsilon_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{12} & -g_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} ; \quad g_{ij} = g_{ij}(z-t)$$

$$g_{ij}\Big|_{\partial X} = 0$$

The boundary condition (14) is fulfilled by sending in the opposite direction a wave with the same amplitude and thus is of purely reflecting type. The boundary conditions (11) and (12) are not, in general of reflecting type unless $\chi_{\mu\nu} = 0$ i.e. the boundary ∂X is a so-called totally geodesic submanifold of X. This mean that if $\chi_{\mu\nu} \neq 0$ then even a static boundary can absorb energy from gravitational radaition. The absorbed energy becomes then the energy of the boundary exitations, which propogate along the classical background boundary ∂X.

Boundaries with $\chi_{\mu\nu} = 0$ have non-compact spacial sections and thus cannot be created spontaneously. However boundaries at infinity or conformal boundaries can be made to obey $\chi_{\mu\nu}=0$. An example of such boundary can be found in the work by Hawking [7] done in connection with earlier work by Breitenlohner and Freedman [8]. Hawking showed that that there is only one choice of supersymmetric reflecting boundary conditions for gravity in the case of $X = CAdS$ -- the covering of anti-de-Sitter space-time -- and not two as found in [8].

It was essential for Hawking's argument to have $\chi_{\mu\nu}=0$. Our boundary conditions (11), (12) and (14) appear to be unique, apply for any boundary and do become reflecting when $\chi_{\mu\nu}=0$. This suggests that our boundary conditiond are in fact generalization of Hawking's. This question will be addressed in detail elsewhere.

Finally, having derived the boundary conditions for the gravitational radiation it is interesting to speculate whether one can build a resonant cavity to amplify the gravitational waves for the purpose of their detection and, possibly, generation. Presumably this would involve calculation of propagation of the gravitational radiation in real media and it appears unlikely that such a resonant device could be constructed. However if a resonant cavity with sufficient degree of amplification can be built on Earth or in its vicinity on may try to detect the radiation from exploding nuclear devices or earthquackes.

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ON THE COUPLING OF MATTER AND GRAVITY TO THE BOUNDARY OF SPACE-TIME

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Abstract

We show that under certain boundary conditions on the matter fields and on the fluctuations of the background metric the gravity-matter system can be coupled to the boundary of space-time through the stress-energy tensor. The connection of the formalism developed to the Casimir effect is discussed.

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Introduction

Relativistic extended objects -- membranes and strings -- with the "surface" action proportional to the "area" the object "sweeps" during its motion, have been studied extensively from classical and quantum points of view [1].

In this letter we treat the boundary of D-dimensional space-time as a D-1-dimensional membrane and show that there exists a natural coupling of the boundary to the gravity-matter system.

The boundaries of vacuum phases were treated dynamically in the Bag models of hadrons [2], the boundary of space-time as a dynamical object was considered by Witten [3], who discussed a vacuum decay mechanism which involves a spontaneous creation and expansion of the boundary. Finally, since the discovery of the Casimir effect, much work has been done on the boundary effects in quantum field theory [4].

However, in the work cited above the motion of the boundary-like objects is either predetermined or the coupling between the matter and the boundary is non-dynamical. Furthermore, the Cazimir forces which one obtains by varying the total renormalized vacuum energy can only serve as a non-relativistic approximation of the back reaction of the matter fields on the boundary. The boundary of space-time with gravity, to my knowledge, has not been discussed as a dynamical object in the literature.

It is interesting, therefore, to derive covariant equations of motion for the boundary from the action principle and to see whether the Casimir forces appear in the non-relativistic limit.

Equations of Motion

Our conventions are: X denotes a D-dimensional manifold with Lorentzian metric $g_{\mu\nu}$, which has positive signature; ∂X is the boundary of X; $z^{\mu}(y^{a})$ are the local coordinates of ∂X in X; n^{μ} is the outward pointing space-like normal $n_{\mu}=\lambda\varepsilon_{\mu\nu\ldots\rho}\varepsilon^{a\ldots b}$ $\partial_a z^{\nu} \ldots \partial_b z^{\rho}$ where λ is chosen such that $n_{\mu}n^{\mu}=1$; $h_{ij}=\frac{\partial z^{\mu}}{\partial yi}\frac{\partial z^{\nu}}{\partial yj}\, g_{\mu\nu}$ is the induced metric on ∂X; $h_{\mu}^{\nu}=\delta_{\mu}^{\nu}-n_{\mu}n^{\nu}$ is the projection operator from the tensor bundle of X on that of ∂X; $\chi_{\mu\rho}=D_{\nu}n_{\mu}h_{\rho}^{\nu}$ is the second fundamental form. Greek indices run 1 to D=dimX, while Latin ones 1 to D-1. The coordinates x^{μ} always refer to X and y^{a} to ∂X. With our conventions $\int dx g^{1/2} D_{\mu}A^{\mu} = \int dy h^{1/2} n_{\mu}A^{\mu}$. All quantities are made dimensionless by a choice of units. All fundamental constants are put to one.

First consider an arbitrary matter system on a manifold with a boundary and a fixed background metric. The action for the system is:

$$S = S_M + S_{\partial X} \tag{1}$$

$$S_M = \int dV \mathcal{L}(\phi_A) \quad ; \quad dV = g^{1/2}\, dx \tag{2}$$

$$S_{\partial X} = a\int dS \quad ; \quad dS = h^{1/2}\, dy \;;\; a=const \tag{3}$$

Before deriving the equations of motion let us consider coordinate invariance of the action (1). The requirement that (1) is invariant w.r.t. the transformations that map ∂X into ∂X identically: $y^{a}\to\tilde{y}^{a} = y^{a}$, gives us the usual covariant conservation law for the stress-energy tensor

$$D_\mu T^{\mu\nu} = 0 \tag{4}$$

where by definition $T^{\mu\nu} = g^{-1/2} \frac{\delta S_M}{\delta g_{\mu\nu}}$.

If, however, under the coordinate transformation ∂X maps into ∂X not identically: $y^a \to \tilde{y}^a \neq y^a$, the situation is slightly different since, in general, additional boundary terms will appear in δS due to the non-vanishing of the boundary variations of fields and metric

$$\delta g_{\mu\nu} = L_\xi g_{\mu\nu} \tag{5a}$$

$$\delta \phi_A = L_\xi \phi_A \tag{5b}$$

Here the index A combines external and internal indices and L_ξ is the Lie derivative w.r.t. the vector field ξ^μ which is generated by an infinitesimal coordinate transformation.

Indeed a vector field ξ^μ which is non-zero at the boundary and is tangential to it -- $\xi^\mu n_\mu|_{\partial X} = 0$ -- still induces a coordinate transformation according to (5a-b).

In this case, performing such a transformation, we obtain

$$\delta S_M = \int dV \left\{ \left[-D_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \right) + \frac{\partial \mathcal{L}}{\partial \phi_A} \right] L_\xi \phi_A + D^\nu T_{\nu\mu} \xi^\mu \right\}$$
$$+ \int dS \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu g_{\lambda\nu})} L_\xi g_{\lambda\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} L_\xi \phi_A \right\} n_\mu$$

$$+ \int ds\, n^{\mu} T_{\mu\nu} \xi^{\nu} \tag{8a}$$

and separately

$$\delta S_{\partial X} = a \int ds\, h^{\mu\nu} D_{\mu} \xi_{\nu} \tag{8b}$$

Note that additional terms will appear in (8a) if the action (1) depends on higher than first derivatives of the metric or the n-bein.

Using the equations of motion for ϕ_A and (4) we obtain that we must have

$$\int ds\, n_{\mu} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} g_{\lambda\nu})} L_{\xi} g_{\lambda\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \phi_A)} L_{\xi} \phi_A \right\} + \int ds\, n^{\mu} T_{\mu\nu} \xi^{\nu} = 0 \tag{9}$$

and

$$\int ds\, h^{\mu\nu} D_{\mu} \xi_{\nu} = 0 \tag{10}$$

when $\xi^{\mu} \cdot n_{\mu} |_{\partial X} = 0$.

We now require that the boundary conditions on the matter fields cancel identically the first integral in the LHS of (9) for all $\xi^{\mu} = h^{\mu}_{\rho} \tilde{\xi}^{\rho}$. Then for the matter fields we have

$$n^{\mu} T_{\mu\nu} h^{\nu}_{\rho} = 0 \tag{11}$$

The condition (10) is trivially satisfied since $\int dS h^{\mu\nu} D_{\mu} \xi_{\nu} = \int dS \chi n^{\mu} \xi_{\nu}$, where $\chi = h^{\mu\nu} \chi_{\mu\nu}$. Thus, under the boundary conditions we required, (11) supplements the condition (4). For a static boundary and Minkowski metric on X, (11) takes the form

$$\begin{aligned} n_i T_{i0} &= 0 \\ n_i T_{ij} &= (n_i T_{ik} n_k) n_j \\ n_{\mu} &= (0, n_i) \end{aligned} \tag{12}$$

This just means that there is no energy flow through the boundary and that the force exerted on the boundary always points along the normal.

The main justification for the restriction on possible boundary conditions comes from consideration of equations of motion for the boundary. These are obtained by varying the range of integration in (2) and varying (3) w.r.t. $z^{\mu}(y^{a})$. However, if our theory is generally covariant then the range variation is equivalent to the variation (5a-b) such that $\xi^{\mu} n_{\mu}|_{\partial X} \neq 0$. Indeed in this case the variation of the volume is, for instance,

$$\delta V = \delta \int dx\, g^{1/2} = \int dx\, g^{1/2} D_{\mu} \xi^{\mu} = \int dy\, h^{1/2} h^{\mu} \xi_{\mu}$$

i.e. the same result as we would obtain by the range variation with $\delta Z^{\mu}(y^{a}) = -\xi^{\mu}|_{\partial X}$.

The transformations (5a-b) are coordinate transformations for any internal point of X, thus only surface terms could appear in variation of all quantities.

Taking into account the boundary conditions and (4) we obtain for the boundary

$$a\chi n^{\mu} + T^{\mu\nu} n_{\nu} = 0 \tag{13}$$

or

$$-a\chi = T^{\mu\nu} n_{\mu} n_{\nu} \tag{14}$$

We see that the restriction on the boundary condition makes the equation of motion to have a particularly simple form. It means that the normal-normal component of the stress-energy tensor provides the source term for the "free" boundary, described by the equation $\chi = 0$.

The stress-energy tensor in (14) is symmetric. This is due to the general covariance of the action (1). Relativistic covariance inplies the canonical stress-energy tensor in (14) and, in general, different dynamics for the boundary.

For the electromagnetic field the RHS of (14) reduces to the familiar Casimir pressure in the case of the static boundary $R \times S^2$. As noted in [5] for the ideal conductor boundary -- but, in fact, valid for any -- the pressure on the R x S^2 boundary given by the principle of virtual work coincides with that obtained from the normal-normal component of the renormalized stress-energy tensor.

There is nothing exotic about the restriction we imposed on the boundary conditions. Indeed in flat space-time for the canonical scalar field ϕ it is equivalent to the Neumann boundary condition. In this case

$$\mathcal{L} = \frac{1}{2} g^{1/2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi)$$

$$n^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bigg|_{\partial X} = n^\mu \partial_\mu \phi \Big|_{\partial X} = 0 \tag{15}$$

For the gauge fields it is satisfied by the Bag boundary condition

$$\mathcal{L} = -\frac{1}{4} g^{1/2} F_{\mu\nu} F^{\mu\nu}$$

$$n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \bigg|_{\partial X} = n_\mu F^{\mu\nu} \Big|_{\partial X} = 0 \tag{16}$$

The case of spin 1/2 fields is somewhat more complicated since the corresponding Lagrangian depends on n-bein derivatives (in the generalization of the tetrad formalism). For spinors

$$\mathcal{L} = \frac{i}{2} g^{1/2} \left(\bar{\Psi} \gamma^\mu \overleftrightarrow{\nabla} \Psi - V(\bar{\Psi}, \Psi)\right)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \quad ; \quad \{\gamma^a, \gamma^b\} = 2 \eta^{ab}$$

$$\nabla_\mu = \partial_\mu + \frac{1}{8}[\gamma^a, \gamma^b] V_a^{\ \lambda} \partial_\mu V_b^{\ \nu} g_{\lambda\nu}$$

$$\eta^{ab} = diag\,\underbrace{(-1, 1, \ldots, 1)}_{D \ times}$$

where V^a_λ is the n-bein with the upper world and lower space-time indices. After simple calculations we find that the well-known boundary condition

$$i\gamma_\mu n^\mu \psi = \psi \tag{17}$$

Note that due to (17) presence of a boundary breaks chiral invariance This is also true for supersymmetrically invariant theories.

We now consider inclusion of gravity into the system. It is well known [6] that in the presence of the boundary the gravity action must be modified to cancel surface terms that contain normal derivatives of the variation of the metric $g_{\mu\nu}$. The simplest way to make the modification is to take for the gravity action

$$S_G = \frac{1}{\kappa}\int dV\, R \quad - \frac{2}{\kappa}\int ds\, \chi \tag{18}$$

where κ is the Newton's constant and R is the curvature scalar. Variation of the action (18) vanishes when the Einstein equations of motion are satisfied and when

$$[\chi^{\mu\nu} - \chi h^{\mu\nu}]\, \delta g_{\mu\nu}\Big|_{\partial X} = 0$$

We consider this as a boundary condition on the metric fluctuations, but not on the metric itself. This interpretation is in accordance with the approach taken in [7].

We thus arrive to the total action for the system

$$S = S_G + S_M + S_{\partial X} \tag{19}$$

Variation of the action (19) w.r.t. the boundary gives

$$\delta S = \kappa^{-1}\int dV\left(R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu} - \kappa T_{\mu\nu}\right)\delta g^{\mu\nu} + \kappa^{-1}\int ds\left[\chi^{\mu\nu} - \chi\, h^{\mu\nu}\right]\delta g_{\mu\nu} - a\int ds\, \chi\, n^{\mu}\xi_{\mu}$$

where $\delta g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu$.

Using Bianci identity and (4) and when $\xi^\nu = n^\nu \cdot F(y)$ we obtain the boundary equation of motion

$$\frac{1}{\kappa}\left(\chi^{\mu\nu}\chi_{\mu\nu} - \chi^2\right) - \frac{1}{2}a\chi = \frac{1}{\kappa}n^{\mu}\left(R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu} - \kappa T_{\mu\nu}\right)n^{\nu} \tag{20}$$

where we used the unique extension of n^μ off ∂X in its neighborhood in X defined by $n^\lambda D_\lambda n_\mu = 0$. (In the case when $\xi^\nu = h^\nu_\rho \tilde{\xi}^\rho$, the Codacci equation ensures $\delta S = 0$.) When the Einstein equations are satisfied, it appears that in this equation the effects of matter are cancelled by the effects of gravity. Remarkably, this is not true. The equation

$$-\frac{1}{2}a\chi + \kappa^{-1}(\chi^{\mu\nu}\chi_{\mu\nu} - \chi^2) = 0 \tag{21}$$

does contain the stress-energy tensor. Indeed, using the expression for the curvature scalar R' of the induced metric

$$R' = R - 2R_{\mu\nu}n^{\mu}n^{\nu} + \chi^{\mu\nu}\chi_{\mu\nu} - \chi^2 \tag{22}$$

and using Einstein's equations once more we obtain

$$-\frac{1}{2}a\chi + \kappa^{-1}R' + 2T_{\mu\nu}n^{\mu}n^{\nu} = 0 \tag{23}$$

The equation (23) contains only one free parameter: the "surface tension" constant a. It is tempting, in view of the fact that R' contains only second derivatives of the small fluctuations of $z^{\mu}(y^{a})$, to remove the ambiguity by putting a=0. Then we would have

$$R' + 2\kappa T_{\mu\nu}n^{\mu}n^{\nu} = 0 \tag{24}$$

Since κ is very small we see that the strong coupling of (14) gave way for the weak gravitationally induced coupling. When $T_{\mu\nu}=0$ R'=0 according to (24) which, for example when D=3, means that the curvature in time direction is zero.

The problem with (23) and (24) is to prove that the equations are equations of the propagating type. This question will be considered elsewhere.

Note that one cannot take the limit of the flat space-time by by putting $g_{\mu\nu}=\eta_{\mu\nu}$ in (23). One has to consider $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$ and retain the first order terms in κ in R'. This just means that gravitons also contribute to the force on the boundary.

Note that if $T_{\nu\mu}n^{\mu}n^{\nu} < 0$, R' is positive. That means that the boundary whose spatial curvature is everywhere positive, i.e. that looks, for example, like a distorted sphere without "dents", tends to expand with non-zero acceleration. This fact suggests that space-times with negative "pressure" may be unstable w.r.t. the spontaneous boundary formation.

Remarks on Quantization

The formalism presented here was developed for the description of the boundary dynamics under the influence of the Casimir forces. It is hoped that two spontaneously created boundaries would like to develop an infinite area of the surface of their contact when they collide due to their expansion. This would produce an effective Kaluza-Klein compactification for the region between the two boundaries. It is known that a spherical ideal conductor boundary tends to expand infinitely while two parallel conductor planes tend to reduce distance between them to zero [5]. So the scenario described above is not that far-fetched.

However, the naive substitution of $\langle T_{\mu\nu}\rangle_{ren}$ instead of $T_{\mu\nu}$ in the boundary equation of motion (14) or (27) does not make much sense, since it is well-known that generically $\langle T_{\mu\nu}\rangle_{ren}$ diverges on the boundary. Indeed, for the canonical scalar field (15), according to Deutsch and Candelas [4], $\langle T_{\mu\nu}\rangle_{ren}$ has the following asymptotic form in four dimensions and for Minkowski space-time

$$
\begin{aligned}
\langle T_{\mu\nu}\rangle_{ren} \underset{\epsilon\to 0}{\sim} & \ \epsilon^{-4}\left(-\frac{1}{16\pi^2}\right) h_{\mu\nu} \\
& + \epsilon^{-3}\left[\frac{1}{60\pi^2}\chi_{\mu\nu} - \frac{13}{240\pi^2}\chi h_{\mu\nu} - \frac{1}{48\pi^2}\chi n_\mu n_\nu\right] \\
& + \epsilon^{-2}\Big[\frac{1}{48\pi^2}\chi_{\mu\lambda}\chi^{\lambda}{}_{\nu} - \frac{11}{480\pi^2}\chi_{\rho\sigma}\chi^{\rho\sigma} h_{\mu\nu} - \\
& - \frac{3}{80\pi^2} n_{(\mu} h^{\alpha}_{\nu)} D_\alpha \chi + \frac{1}{60\pi^2}\chi^2 n_\mu n_\nu - \frac{11}{480\pi^2}\chi^{\alpha\beta}\chi_{\alpha\beta} n_\mu n_\nu \\
& - \frac{11}{480\pi^2}\chi^2 h_{\mu\nu} + \frac{1}{120\pi^2}\chi\chi_{\mu\nu}\Big] + O(\epsilon^{-2})
\end{aligned}
\tag{25}
$$

where the Neumann boundary condition is assumed and with ε being the distance to the boundary.

On the second examination, however, we see that the normal-normal component of $\langle T_{\mu\nu}\rangle_{ren}$ is much less divergent.

$$
n^\mu n^\nu \langle T_{\mu\nu}\rangle_{ren} \underset{\epsilon\to 0}{\sim} -\frac{\epsilon^{-3}}{48\pi^2}\chi + \frac{\epsilon^{-2}}{60\pi^2}\chi^2 - \frac{11\epsilon^{-2}}{480\pi^2}\chi^{\mu\nu}\chi_{\mu\nu} + O(\epsilon^{-2}). \tag{26}
$$

We find that most of the divergent terms are insignificant for the boundary dynamics while the remaining divergent terms have the same functional form as the terms of boundary equation of motion (21). Thus there is hope that, when properly renormalized, eq. (21) will somehow absorb at least some of the infinities in (26).

The naive substitution of $\langle T_{\mu\nu}\rangle_{ren}$ in (23) might not be justified. Indeed on one hand we expect the condition $n^{\mu}T_{\mu\nu}h^{\nu}_{\rho} = 0$ to hold, together with $D^{\mu}T_{\mu\nu} = 0$, even after renormalization since both are due to the coordinate invariance of the matter-gravity-boundary system. On the other hand, substitution of (25) into (11) yields

$$n^{\mu}\langle T_{\mu\nu}\rangle_{ren}\, h^{\nu}{}_{\rho} \underset{\epsilon\to 0}{\sim} -\epsilon^{-2}\frac{3}{80\pi^2}\, h_{\rho}{}^{\alpha} D_{\alpha}\chi + O(\epsilon^{-2}) \qquad (27)$$

In general this is not zero. Taken literally, (27) means a breakdown of the coordinate invariance of the quantized system. This is unacceptable on general grounds. One possible explanation for (27) is that quantum fluctuations of the boundary must be taken into account to ensure that the RHS of (27) is zero.

It is also not obvious that the naively expected expressions

$$\langle n^{\mu} T_{\mu\nu}\big|_{\partial X} h^{\nu}{}_{\rho}\rangle_{ren} = n^{\mu}\langle T_{\mu\nu}\rangle_{ren}\big|_{\partial X} h^{\nu}{}_{\rho}$$

and

$$\langle n^{\mu} n^{\nu} T_{\mu\nu}\big|_{\partial X}\rangle_{ren} = n^{\mu} n^{\nu}\langle T_{\mu\nu}\rangle_{ren}\big|_{\partial X}$$

are consistent.

Apparently more work on quantization of the interacting boundary-gravity-matter system is needed before one can have a definite answer to whether or not one can describe the motion of the boundary under the "pressure" from the Casimir "forces". If one can find a consistent formalism, however, it may help resolving the mystery of spontaneous compactification in Kaluze-Klein theories by providing another possible mechanism for its dynamical realization, in addition to that currently discussed in the literature [8].

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A CALCULATION OF THE OBSERVED CASIMIR EFFECT BASED ON A LOCAL APPROACH*

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Abstract

We present a calculation of the observed Casimir effect of attraction between two conducting plates. The calculation is based on the recently introduced space-time boundary dynamics.

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Existence of forces acting on conducting surfaces in vacuum [1] is a remarkable consequence of quantum nature of the vacuum. A generalization to the case of non-perfect conductors has been done by Lifshitz [2]. For the real materials the force F per square unit has two asymptotes. The first is dominant when the separation a between, say, two parallel flat conductors is much less than a characteristic frequency of the absorption spectrum of the plate's material λ_c. Then

$$F = \frac{\hbar c}{8\pi^2 a^3} \int_0^\infty \left(\frac{e-1}{e+1}\right)^2 d\rho \tag{1}$$

where

$$e(\rho) = 1 + \frac{2}{\pi} \int_0^\infty \frac{\omega}{\omega^2 + \rho^2} \operatorname{Im} \epsilon(\omega)\, d\omega$$

with Imε being the imaginary part of the dielectric constant of the plate's material. This force is due to the molecular structure of the plates.

When $a \gg \lambda_c$ then

$$F = -\frac{\hbar c}{a^4} \frac{\pi^2}{240} \left(\frac{\epsilon_0 - 1}{\epsilon_0 + 1}\right)^2 \mu(\epsilon_0) \tag{2}$$

where $\mu(\varepsilon_0)$ is some definite function [2]. The case of the perfect conductor [1] is recovered by putting $\varepsilon_0=\infty$ and $\mu(\infty)=1$.

then we have the famous

$$F = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} \tag{3}$$

This expression is independent of the material of the plates and is solely due to the quantum fluctuations of the vacuum.

The validity of the expressions (1) and (2) has been proven experimentally [3]. Here we rely on the paper by Tabor and Winterton. For the case of ε_o=1.59 and the refractive index parallel to the cleavage plane of the two perpendicular, covered with evaporated silver, mica cylinders about 1.56, they measured that in the case a>19nm, where the second asymptotic takes over, the force F is

$$F = (0.81 \pm 0.04)\cdot 10^9\left(\frac{1\,nm}{a}\right)^4 \frac{erg}{cm^3}$$

This should be compared with the theoretical prediction

$$F = 0.79 \cdot 10^9\left(\frac{1\,nm}{a}\right)^4 \frac{erg}{cm^3}$$

The agreement is remarkable considering the uncertainties in measuring various constants.

The explanation of the physics of the effect is based on the fact that due to the quantum nature of the electromagnetic vacuum, the renormalized vacuum expectation value of the stress-energy tensor $T_{\mu\nu}$ acquires a nonvanishing value. Indeed, in the case of two infinite parallel ideal conductor planes it is easy to show [4] that because of requirements of covariant conservation, Lorentz and conformal invariance of the electromagnetic stress-energy tensor its renormalized value (neglecting gravity effects) is

$$\langle T_{\mu\nu}(x)\rangle_{ren} = \frac{\pi^2}{720}\frac{\hbar c}{a^4}\,\mathrm{diag}(-1,1,1,-3), \qquad (4)$$

where a is the separation between the planes and the direction of separation along z-axis corresponds to the last entry in the matrix in (4).

Varying the total renormalized energy per unit area $U=a\langle T^{0\,0}\rangle_{ren}$ w.r.t. the separation a one obtains the force F in (3)

$$F = -\frac{\partial U}{\partial a}.$$

This is the conventional derivation. Unfortunately it is inherently non-relativistic, not to speak of general covariance.

An attempt has been recently made [5] to consider a space-time boundary as a dynamical object. It was shown there that, provided certain boundary conditions hold, the space-time boundary is a rather well-defined classical object and couples to the matter and gravity in a universal manner, through the stress-energy tensor.

We now present an extention of the results for the ideal conductor boundary and show that the Casimir force (3) arises naturally in this formalism.

Let us first give a more direct derivation of the boundary dynamics for any boundary condition in arbitrary space-time dimension D. The total action for the matter-gravity-boundary system is taken as

$$S=\int d^D x\, g^{1/2}\mathcal{L}(\phi_A)-\frac{1}{16\pi G}\int d^D x\, g^{1/2}(R+2\Lambda)+\frac{1}{8\pi G}\int d^{D-1}y\, h^{1/2}\chi\,, \qquad (5)$$

where x^μ are some coordinates on a manifold X, $g=-\det g_{\mu\nu}$, $\mathcal{L}(\phi_A)$ is the Lagrangian gensity of the matter fields ϕ_A ($\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ for the case of electromagnetism), G is the Newton's constant, R and Λ are the curvature scalar and cosmological constant respectively, y^a are some coordinates on the boundary ∂X, $h=-\det h_{ab}$ where $h_{ab}=\frac{\partial z^\mu}{\partial y^a}\frac{\partial z^\nu}{\partial y^b}g_{\mu\nu}$ and $z^\mu=z^\mu(y^a)$ defines the boundary imbedding and, finally, $\chi=\chi^\mu{}_\mu$ where $\chi_{\mu\nu}=D_\mu n_\nu$ with n_ν being a vector field in a neighborbood of X such that $n^\mu n_\mu|_{\partial x}=1$ $n_j D^j n_\mu|_{\partial x}=0$ and $n_\mu|_{\partial x}\frac{\partial z^\mu}{\partial y^a}=0$, $D_\rho(n^j D_j n_\sigma)|_{\partial x}=0$.

The action S can be rewritten as a pure volume integral

$$S=\int d^D x\, g^{1/2}\left[\mathcal{L}(\phi_A)-\frac{1}{16\pi G}(R+2\Lambda)+\frac{1}{8\pi G}D^\mu(n_\mu\chi)\right] \qquad (6)$$

It is now easy to perform the boundary variation of the action using the fact that any vector field V_μ on X defines an infinitesimal boundary variation

$$\xi_\mu = \epsilon V_\mu \quad , \quad |\epsilon| \ll 1$$

and using

$$\delta_\xi \int d^D x\, g^{1/2} F = \int d^{D-1} y\, h^{1/2}\, \xi^\mu n_\mu F$$

Thus, in addition to E.o.M. for matter and gravity we obtain on ∂X

$$\mathcal{L}(\phi_A) - \frac{1}{16\pi G}(R + 2\Lambda) + \frac{1}{8\pi G} D^\mu (n_\mu \chi) = 0 \qquad (7)$$

After some manipulations with the last term in (7) and using the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G\, T_{\mu\nu}$$

$$T^{\mu\nu} = 2 g^{-1/2} \frac{\delta S_M}{\delta g_{\mu\nu}} \quad ; \quad S_M = \int d^D x\, g^{1/2} \mathcal{L}(\phi_A),$$

we obtain

$$\mathcal{L}(\phi_A) - \frac{1}{16\pi G}\left[-6 n^\mu n^\nu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) - 2R' + 2\Lambda\right] = 0$$

and, finally,

$$R' + 2\Lambda + 8\pi G\left(\mathcal{L}(\phi_A) - 3 n^\mu n^\nu T_{\mu\nu}\right) = 0 \tag{8}$$

where R' is the intrinsic curvature of the boundary. We now can connect this derivation with the one in [5]. Indeed, when second derivatives of metric do not enter the matter action

$$T^{\mu\nu} = 2\left[\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - g^{-1/2}\partial_\rho\left(\frac{\partial(g^{1/2}\mathcal{L})}{\partial(\partial_\rho g_{\mu\nu})}\right)\right] + g^{\mu\nu}\mathcal{L}$$

If we require

$$n^\mu\left[\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - g^{-1/2}\partial_\rho \frac{\partial(g^{1/2}\mathcal{L})}{\partial(\partial_\rho g_{\mu\nu})}\right]\Bigg|_{\partial \bar{X}} = 0$$

as a boundary condition, $n^\mu n^\nu T_{\mu\nu} = \mathcal{L}$ and putting Λ to zero we obtain the eq. (24) of [5] provided we identify κ in [5] as -8π G.

This equation is

$$R' - 16\pi G\, n^\mu n^\nu T_{\mu\nu} = 0 \tag{9a}$$

or alternatively

$$R' - 16\pi G\,\mathcal{L}(\phi_A) = 0 \tag{9b}$$

Note that when $\mathcal{L}=\phi^A D\phi_A$, where D is some differential operator then on solutions $\mathcal{L}$ is zero. For such matter actions, classically, the boundary decouples. This observation also provides a convinient way of determination of $\mathcal{L}$ on ∂X by reducing the matter action to a form with $\mathcal{L}=\phi^A D\phi_A$ plus surface terms.

Let us now turn to the case of the electromagnetic field. To do this we have to drop the boundary term in (5) since the conductor plates material is certainly transparent to gravitons and put $\Lambda=0$. We should also add a term to the action which would describe the "free" phenomenological boundary. Agruably, it could be put as $(M\frac{\dot{a}^2}{2} - \frac{1}{2}\omega^2 a^2)$ for the non-relativistic experiment in question. Here M is the cylinder's moment of inertia and ω depends on the elasticity of the cylinders. The classical E.o.M. we get is

$$M\ddot{a} + \omega^2 a + \int_{z=0} d^2y\, h^{1/2}\left(\mathcal{L}(\phi_A) - \frac{1}{16\pi G} R\right) = 0 \tag{10}$$

The conformal invariance of the EM field ensures that R=0 and we arrive to

$$M\ddot{a} + \omega^2 a = -\int d^2y\, \mathcal{L}(\phi_A)\, h^{1/2} \tag{11}$$

That is, the classical Casimir force F in this case is just

$$F = -S\mathcal{L}(\phi_A) = +\frac{1}{4} F^{\mu\nu} F_{\mu\nu} S \; ; \; S = \underbrace{\int d^2y}_{\text{area of the plates}} ;$$

were we put $h=1$ in the weak gravity approximation. For the classical vacuum $F_{\mu\nu}=0$ and, consequently, $F=0$. This is not true when quantum fluctuations are taken into account.

Let us now present a derivation of the Casimir force (3) from semiclassical approximation to eq. (10). In this approximation only EM field is treated as a quantum quantity while the space-time metric and the boundary are not quantized. Strictly speaking one also has to derive the effective action then to renormalize it [6]. This should be distinguished from the renormalization performed on the source of the boundary equation of motion (9b) or (10).

The renormalization of the matter action in presence of boundary has been extensively studied [6, 7] in conjunction with renormalization of stress-energy tensor. And while earlier work [7] revealed that the renormalized stress energy tensor (and the Lagrangian) generically diverges as one approaches the boundary, Kennedy et al. showed that the total renormalized energy, say, in a cavity, can be made finite due to the presence of a priori divergent boundary counterterms. We shall observe the same effect when renormalizing the EM Lagrangian.

To separate the largest contibution to the quantum force $\tilde{F}$ in the setting of the experiment described, one has to note that it will come from the Minkowsky metric since the trace anomaly is proportional to powers of curvature tensor. In this case one can disregard all effects of gravity. However, in general, this is not true for a non-conformally invariant matter, where $T^{\mu}{}_{\mu} \neq 0$ in classical dynamics. This observation may be relevant, for example, for some bag models.

It is easy to evaluate the leading contribution using the image solution for the photon's Green function given in [8]. Using the point splitting technique one obtains for $\tilde{F} = -\langle \mathcal{L}(x) \rangle_{\mathrm{ren}}|_{\partial X}$

$$\langle \mathcal{L}(x,x') \rangle = -\frac{1}{4} G^{\mu\nu}{}_{;\mu\nu}(x,x') \tag{12}$$

where

$$G^{\mu\nu;\lambda\kappa} = d^{\mu\nu;\lambda\kappa} \sum_{\ell=-\infty}^{\infty} D(x-x'-2a\ell n)$$

$$- \tilde{d}^{\mu\nu;\lambda\kappa} \sum_{\ell=-\infty}^{\infty} D(x-\tilde{x}'-2a\ell n)$$

$$n = (0,0,0,1)$$

and

$$d^{\mu\nu;\lambda\kappa} = \eta^{\nu\kappa}\partial^{\mu}\partial'^{\lambda} - \eta^{\mu\kappa}\partial^{\nu}\partial'^{\lambda} + \eta^{\lambda\mu}\partial^{\nu}\partial'^{\kappa} - \eta^{\nu\lambda}\partial^{\mu}\partial'^{\kappa},$$

$$\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1).$$

The quantity $\tilde{d}^{\mu\nu;\lambda\kappa}$ is obtained from $d^{\mu\nu;\lambda\kappa}$ by substitution $g_{\mu\nu} \to g_{\mu\nu} - 2n_{\mu}n_{\nu}$. The function $D(x-x')$ in (12) is a usual Minkowsky generalized function for a massless scalar

$$D(x) = \frac{i}{4\pi^2}\,\frac{1}{x^2 + i\epsilon}$$

and $\tilde{x}_{\mu}$ is the reflected x_{μ} i.e., $\tilde{x}_{\mu} = (x_0, x_1, x_2, -x_3)$. Renormalization of $< \mathcal{L}(x, x) >|_{\partial X}$ must be made separately for $x=x'=0$ and $x=x'=a$. In the former it is equivalent to removal of the zero mode from the sum (12), while in the latter the first mode must be removed. Thus we have

$$< \mathcal{L}(x,x') > = \sum_{\ell=-\infty}^{\infty} \Big[\frac{3}{2}\,\delta(x - x' - 2a\ell n) - \frac{1}{2}\,\delta(x - \tilde{x}' - 2a\ell n)$$

$$+ 2\,\partial_z^2 (-i)\, D(x - \tilde{x}' - 2a\ell n) \Big]$$

Discarding the $\ell=0$ mode as x, $x' \to 0$ we get $(S=1)$

$$\tilde{F} = -\langle \mathcal{L}(x) \rangle_{ren} = \lim_{z \to z'} \frac{-12}{4\pi^2} \sum_{\ell \neq 0} \frac{1}{(z-z'-2a\ell)^4} = -\frac{\pi^2}{240} \frac{1}{a^4}$$

This is precisely the Casimir force (3) which has been measured in experiments. The same result is obtained when $x \to x'$ near $z=z'=a$ by discarding the first mode in the sum (12). Note, that from a general point of view, the renormalization carried out above amounts to renormalization of the term in the bare surface action, which is proportional to the area of space-time boundary [7]. The fact that mere removal of divergent terms suffices to reach an agreement with the experiment suggests that the corresponding renormalized constant is zero. This is in accordance with analysis based on renormalization of total energy [9].

CONCLUSION

We have established here a crucial link connecting the formal boundary dynamics introduced in [5] with experiment. We have also pointed out that for conformally non-invariant theories gravitational contributions may be considerable even in the case of the weak gravitational field. This may have implications for the Bag models.

The two most important questions about the boundary dynamics are whether the space-time boundaries exist or can be spontaneously created and are there any physical processes in which the boundary dynamics can play an important role. (To generate a dimensional reduction, for example). Witten's reservation to inclusion of boundaries into the functional integral in gravity [9] may be circumvented by considering the boundary not as the boundary of classical background space-time but as a boundary for quantum excitations above the background. The possibility of such an approach, which induces obvious modifications of eq. (9a), was pointed out by Kennedy et al [6]. This would also ensure validity of the Positive Energy Theory [10]. The ambiguity of the choice of the boundary conditions is resolved by requirement of universality of coupling of the space-time boundary, i.e., coupling through stress-energy tensor $T_{\mu\nu}$ only. As for applications of the formalism, one which immediately comes to mind is the de-Sitter phase in the

Inflationary Universe scenario and its modifications [11] where the very high negative pressure density that drives the inflation at the same time creates instability w.r.t. spontaneous boundary formation. Discussions of these questions as well as more detailed analysis of renormalization of the boundary matter-gravity system will be presented in the forthcoming publications [12].

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BIOGRAPHICAL NOTE

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During the author's graduate studies at M.I.T. the following papers have been written.

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