

ASYMPTOTIC APPROXIMATIONS TO THE ERROR PROBABILITY
FOR DETECTING GAUSSIAN SIGNALS

by

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ABSTRACT

The optimum detector for Gaussian signals in Gaussian noise has been known for many years. However, due to the nonlinear nature of the receiver, it is extremely difficult to calculate the probability of making decision errors. Over the years, a number of alternative performance measures have been prepared, none of which are of universal applicability. A new technique is described which combines "tilting" of probability densities with the Edgeworth expansion to obtain an asymptotic expansion for the error probabilities.

The unifying thread throughout this discussion of performance is the semi-invariant moment generating function $\mu(s)$. For the problem of detecting Gaussian signals, $\mu(s)$ can be expressed in terms of the Fredholm determinant. Several methods for evaluating the Fredholm determinant are discussed. For the important class of Gaussian random processes which can be modeled via state variables, a straightforward technique for evaluating the Fredholm determinant is presented.

A number of examples are given illustrating application of the error approximations to all three levels in the hierarchy of detection problems, with the emphasis being on random process problems. The approximation to the error probability is used as a performance index for optimal signal design for Doppler-spread channels subject to an energy constraint. Pontryagin's minimum principle is used to demonstrate that there is no waveform which achieves the best performance. However, a set of signals is exhibited which is capable of near optimum performance. The thesis concludes with a list of topics for future research.

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TABLE OF CONTENTS

| | <u>Page</u> |
|--|-------------|
| ABSTRACT | |
| ACKNOWLEDGMENT | 3 |
| TABLE OF CONTENTS | 4 |
| LIST OF ILLUSTRATIONS | 8 |
| CHAPTER I INTRODUCTION | 11 |
| A. The Detection Problem | 11 |
| B. The Gaussian Model | 12 |
| C. The Optimum Receiver | 13 |
| 1. The Estimator-Correlator | 16 |
| 2. Eigenfunction Diversity | 18 |
| 3. Optimum Realizable Filter Realization | 20 |
| D. Performance | 22 |
| CHAPTER II APPROXIMATIONS TO THE ERROR PROBABILITIES FOR BINARY HYPOTHESIS TESTING | 26 |
| A. Tilting of Probability Distributions | 26 |
| B. Expansion in Terms of Edgeworth Series | 34 |
| C. Application to Suboptimum Receivers | 40 |
| D. Summary | 45 |
| CHAPTER III GAUSSIAN SIGNALS IN GAUSSIAN NOISE | 46 |
| A. Calculation of $\mu(s)$ for Finite- Dimensional Gaussian Signals | 46 |

| | <u>Page</u> |
|--|-------------|
| B. Transition from Finite to Infinite Set of Observables | 48 |
| C. Special Cases | 53 |
| 1. Simple Binary Hypothesis, Zero Mean | 53 |
| 2. Symmetric Binary Problem | 56 |
| 3. Stationary Bandpass Signals | 57 |
| D. Simplified Evaluation for Finite State Random Processes | 61 |
| E. Semi-Invariant Moment Generating Function for Suboptimum Receivers | 68 |
| F. Summary | 76 |
| CHAPTER IV APPROXIMATIONS TO THE PROBABILITY OF ERROR FOR M-ARY ORTHOGONAL DIGITAL COMMUNICATION | 77 |
| A. Model | 77 |
| B. Bounds on the Probability of Error | 79 |
| C. Exponent-Rate Curves | 80 |
| CHAPTER V EXAMPLES | 85 |
| A. General Binary Problem, Known Signals, Colored Noise | 86 |
| B. Slow Rayleigh Fading with Diversity | 88 |
| 1. Simple Binary Problem, Bandpass Signals | 88 |
| 2. Symmetric Binary Problem, Bandpass Signals | 92 |
| 3. The Effect of Neglecting Higher Order Terms in the Asymptotic Expansion | 95 |
| 4. Optimum Diversity | 95 |

| | <u>Page</u> |
|--|-------------|
| 5. The Inadequacy of the Bhattacharyya Distance | 99 |
| 6. The Inadequacy of the Kullbach- Leibler Information Number as a Performance Measure | 101 |
| C. Random Phase Angle | 105 |
| 1. Simple Binary, No Diversity | 105 |
| 2. Simple Binary, With Diversity | 110 |
| D. Random Process Detection Problems | 114 |
| 1. A simple Binary (Radar) Problem, Symmetric Bandpass Spectrum | 117 |
| 2. A Symmetrical (Communication) Problem, Symmetric Bandpass Spectra | 120 |
| 3. Optimum Time-Bandwidth Product | 124 |
| 4. Long Time Intervals, Stationary Processes (Bandpass Signals, Symmetric Hypotheses) | 124 |
| 5. Pierce's Upper and Lower Bounds (Bandpass, Binary Symmetric) | 131 |
| 6. Bounds on the Accuracy of Our Asymptotic Approximation for Symmetric Hypotheses and Bandpass Spectra | 140 |
| 7. A Suboptimum Receiver for Single- Pole Fading | 142 |
| 8. A Class of Dispersive Channels | 146 |
| 9. The Low Energy Coherence Case | 149 |
| E. Summary | 152 |

| | <u>Page</u> |
|--|-------------|
| CHAPTER VI | |
| OPTIMUM SIGNAL DESIGN FOR SINGLY- SPREAD TARGETS (CHANNELS) | 154 |
| A. A Lower Bound on $\mu(\frac{1}{2})$ | 156 |
| B. Formulation of a Typical Optimization Problem | 158 |
| C. Application of Pontryagin's Minimum Principle | 162 |
| D. A Nearly Optimum Set of Signals for Binary Communication | 170 |
| CHAPTER VII | |
| TOPICS FOR FURTHER RESEARCH AND SUMMARY | 178 |
| A. Error Analysis | 178 |
| B. More than Two Hypotheses | 178 |
| C. Markov (Non-Gaussian) Detection Problems | 179 |
| D. Delay-Spread Targets | 180 |
| E. Doubly-Spread Targets | 181 |
| F. Suboptimum Receivers | 181 |
| G. Bandpass Signals | 181 |
| H. Summary | 181 |
| APPENDIX A | |
| ERROR PROBABILITIES IN TERMS OF TILTED RANDOM VARIABLES | 185 |
| APPENDIX B | |
| $\mu(s)$ FOR OPTIMAL DETECTION OF GAUSSIAN RANDOM VECTORS | 192 |
| APPENDIX C | |
| $\mu(s)$ FOR OPTIMAL DETECTION OF GAUSSIAN RANDOM PROCESSES | 198 |
| APPENDIX D | |
| REALIZABLE WHITENING FILTERS | 205 |
| REFERENCES | 213 |
| BIOGRAPHY | 221 |

LIST OF ILLUSTRATIONS

| | <u>Page</u> | |
|-----|---|-----|
| 1.1 | Form of the Optimum Receiver | 14 |
| 1.2 | Estimator-Correlator Realization | 19 |
| 1.3 | Eigenfunction Diversity Realization | 19 |
| 1.4 | Optimum Realizable Filter Realization | 21 |
| 2.1 | Graphical Interpretation of Exponents | 33 |
| 3.1 | State-Variable Random Process Generation | 62 |
| 3.2 | A Class of Suboptimum Receivers | 69 |
| 3.3 | Suboptimum Receiver: State-Variable Model | 73 |
| 4.1 | Exponent-Rate Curve for Fading Channel | 83 |
| 5.1 | Bounds and Approximations for Known Signals | 89 |
| 5.2 | Approximate Receiver Operating Characteristic, Slow Rayleigh Fading | 93 |
| 5.3 | Error Probability for Slow Rayleigh Fading with Diversity | 96 |
| 5.4 | Optimum Signal-to-Noise Ratio per Channel, Slow Rayleigh Fading | 98 |
| 5.5 | $\mu(s)$ Curves for Two Diversity Channels | 102 |
| 5.6 | Receiver Operating Characteristic, Random Phase Channel | 111 |
| 5.7 | Receiver Operating Characteristic, Random Phase Channel, $N = 1$ | 115 |
| 5.8 | Receiver Operating Characteristic, Random Phase Channel, $N = 2$. | 116 |

| | <u>Page</u> | |
|------|--|-----|
| 5.9 | Approximate Receiver Operating Characteristic, Single-Pole Rayleigh Fading | 121 |
| 5.10 | Probability of Error, Single-Pole Rayleigh Fading | 123 |
| 5.11 | Optimum Time-Bandwidth Product, Single-Pole Rayleigh Fading | 125 |
| 5.12 | Optimum Exponent vs. Signal-to-Noise Ratio, Single Square Pulse, Single-Pole Rayleigh Fading | 126 |
| 5.13 | Optimum Signal-to-Noise Ratio vs. Order of Butterworth Spectrum | 129 |
| 5.14 | Optimum Exponent vs. Order of Butterworth Spectrum | 130 |
| 5.15 | Single-Pole Fading, Suboptimum Receiver | 143 |
| 5.16 | Tapped Delay Line Channel Model | 147 |
| 5.17 | $\mu(s)$ for Three Tap Channel | 150 |
| 6.1 | Communication System Model | 155 |
| 6.2 | General Behavior of Optimum Signal | 167 |
| 6.3 | A Nearly Optimum Set of Signals | 172 |
| 6.4 | Optimum Exponent vs. kT , N Samples, Single- Pole Fading | 174 |
| 6.5 | Optimum Exponent vs. kT , Sequence of Square Pulses, Single-Pole Fading | 176 |
| D-1 | Realizable Whitening Filter | 206 |
| D-2 | State-Variable Realizable Whitening Filter | 212 |

DEDICATION

This thesis is dedicated to

my parents.

I. INTRODUCTION

In this chapter, we give a brief discussion of the detection problem. We briefly review the form of the optimum receiver and motivate the discussion of performance which occupies the remainder of this thesis.

A. The Detection Problem

We shall be concerned with a subclass of the general problem which statisticians for a couple of centuries have called "hypothesis testing" [1-3]. Since World War II this mathematical framework has been applied by engineers to a wide variety of problems in the design and analysis of radar, sonar, and communication systems [4-7]. More recently it has been applied to other problems as well, such as seismic detection.

The general mathematical model for the class of problems that we shall consider is as follows: We assume there are M hypotheses H_1, H_2, \dots, H_M which occur with a priori probabilities p_1, p_2, \dots, p_M . On the j^{th} hypothesis the received waveform is:

$$H_j: r(t) = s_j(t) + m_j(t) + w(t), \quad T_i \leq t \leq T_f, \quad (1.1)$$

where $s_j(t)$ is a sample function from a zero mean random process having finite power, $m_j(t)$ is a known waveform, and $w(t)$ is a sample function of zero mean white noise.

Our goal as engineers is to design and build efficient detection systems. We shall be concerned with the issue of how well detection systems work. The performance criteria which we shall apply

are all related to the error probabilities:

$$\Pr[\varepsilon|H_j] \triangleq \Pr[\text{an incorrect decision}|H_j] \quad (1.2)$$

The bulk of this thesis is concerned with the issue of computing these probabilities.

Detection problems may be logically divided into three broad categories in order of increasing complexity. The simplest category is detecting the presence of a known waveform in additive noise. The next level of difficulty treats those problems in which the signal waveform is known, except for a finite number of random (unwanted) parameters. Finally, there is the problem of detecting sample functions of random processes. Here the "signal" is an unknown waveform about which our only knowledge is statistical in nature. Equivalently, we may think of it as a signal with an infinity of unknown parameters. In all that follows we shall concentrate on this last category because the performance problem is unsolved except for a few special cases.

B. The Gaussian Model

In much of what follows we choose to model the signal and noise waveform as sample functions from Gaussian random processes. The first reason for this is purely for mathematical convenience since this assumption enables us to completely characterize the random processes without an undue amount of complexity. The second reason, being a physical one, is of greater consequence from an engineering point of view. First of all, many of the additive disturbances which corrupt real-world detection systems indeed do have a Gaussian

distribution. For example, the shot and thermal noise arising in an amplifier or the solar and galactic noise picked up by an antenna fit the Gaussian model very well [8-9]. Secondly, the random signals which we are interested in detecting are often well modeled as sample functions from Gaussian random processes. For example, the signals received over several types of fading radio channels have been observed in many independent experiments to obey Gaussian statistics [10-11].

C. The Optimum Receiver

The structure of the optimum receiver is well-known [12-13] so we shall not dwell upon it. For a number of detection criteria the optimum receiver computes the likelihood ratio or some monotone function of it, such as the logarithm. This number is then compared with a threshold whose value depends upon the criterion chosen. For the M-hypothesis problem, the optimum receiver consists of M-1 branches each of which computes a likelihood ratio or some equivalent test statistic.

For most of this thesis, we restrict ourselves to the binary (i.e., M = 2) problem in the interest of simplicity. In Chapter IV we shall discuss the generalization of our results to the more general M-ary problem.

The most general form of the optimum processor under our Gaussian assumption consists of a linear and a quadratic branch as indicated in Fig. 1.1. In this figure, the output random variable is the logarithm of the likelihood ratio $\Lambda (r(t))$.

$$l = \ln \Lambda (r(t)) \quad (1.3)$$

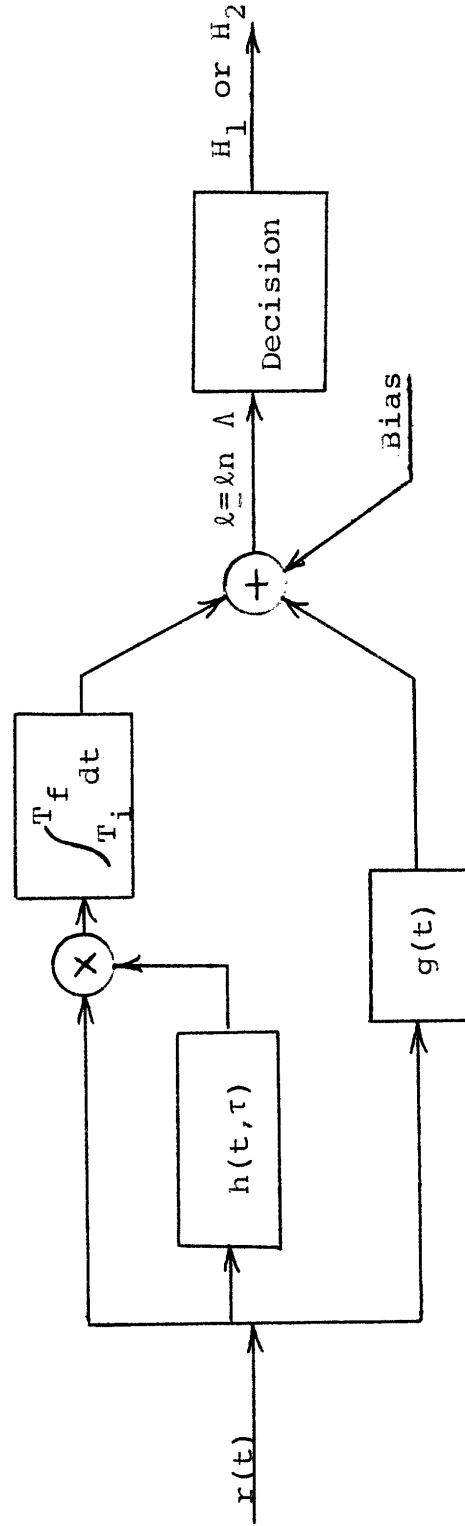


Figure 1.1. Form of the Optimum Receiver

There are several equivalent forms of the optimum processor, some of which we shall encounter as we proceed with our work. Here we briefly review these optimum receiver realizations. For derivation and more detailed discussion, see Van Trees [13]. For simplicity in our brief discussion, we shall discuss only the simple binary detection problem:

$$H_2: r(t) = s(t) + w(t) \quad T_i \leq t \leq T_f \quad (1.4)$$

$$H_1: r(t) = w(t)$$

where $s(t)$ is a sample function from a zero mean Gaussian random process with covariance $K_s(t, \tau)$, and $w(t)$ is a sample function of zero-mean white Gaussian noise with spectral density $N_0/2$.

For the problem that we have outlined, the log-likelihood ratio may conveniently be broken into two parts, a random variable and a constant term (the "bias") that is independent of the received data.

$$\ell = \ell_R + \ell_B \quad (1.5)$$

ℓ_R depends on the received data and

$$\ell_B = \sum_{i=1}^{\infty} \ln(1 + 2\lambda_i/N_0) \quad (1.6a)$$

$$= \ln \prod_{i=1}^{\infty} (1 + 2\lambda_i/N_0) \quad (1.6b)$$

where $\{\lambda_i\}$ are the eigenvalues of the homogenous Fredholm integral equation,

$$\lambda_i \phi_i(t) = \int_{T_i}^{T_f} K_s(t, \tau) \phi_i(\tau) d\tau, \quad T_i \leq t \leq T_f. \quad (1.7)$$

In integral equation theory, infinite products of the form in Eq. 1.6b are called Fredholm determinants. We shall define the Fredholm determinant to be,

$$D_s(z) = \prod_{i=1}^{\infty} (1+z\lambda_i). \quad (1.8)$$

This function occurs repeatedly in our performance calculations and we shall develop closed-form techniques for evaluating it in Chapter III. We now shall discuss techniques for generating ℓ_R .

1. The Estimator-Correlator. Price [12] originally derived the intuitively pleasing realization of Fig. 1.2,

$$\ell_R = \frac{1}{N_0} \int_{T_i}^{T_f} \int_{T_i}^{T_f} r(t) h_I(t, u) r(u) dt du \quad (1.9a)$$

$$= \frac{1}{N_0} \int_{T_i}^{T_f} r(t) \left[\int_{T_i}^{T_f} h_I(t, u) r(u) du \right] dt \quad (1.9b)$$

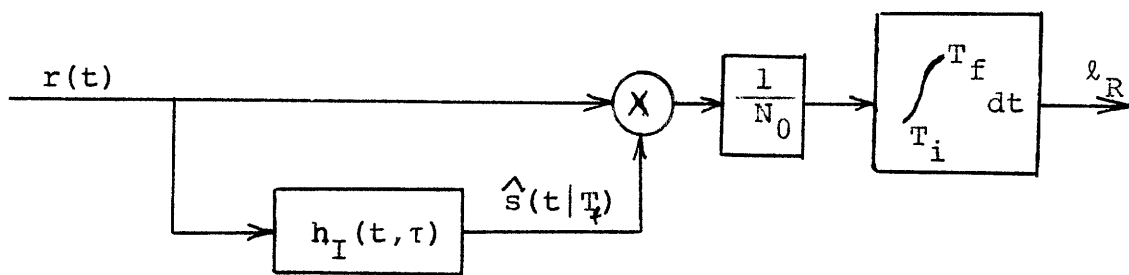


Figure 1.2. Estimator-Correlator Realization

where $h_I(t, u)$ is specified by an integral equation of the Wiener-Hopf type

$$K_S(t, u) = \frac{N_0}{2} h_I(t, u) + \int_{T_i}^{T_f} K_S(t, \tau) h_I(\tau, u) d\tau \quad T_i \leq t, u \leq T_f. \quad (1.10)$$

We immediately recognize that $h_I(t, u)$ is the impulse response of the optimum MMSE unrealizable linear filter for estimating $s(t)$, given $r(t) = s(t) + u(t)$, $T_i \leq t \leq T_f$. Hence, from Eq. 1.9b

$$\ell_R = \frac{1}{N_0} \int_{T_i}^{T_f} r(t) \hat{s}(t|T_f) dt \quad (1.11)$$

where $\hat{s}(t|T_f)$ denotes the unrealizable estimate of $s(t)$. Thus, the receiver performs the familiar correlation operation, where we correlate with the best (MMSE) estimate of the signal.

2. Eigenfunction Diversity. A second realization which is useful from a conceptual point of view is that shown in Fig. 1.3. It follows immediately from the Estimator-Correlator if we expand $r(t)$ and $h_I(t, u)$ in the eigenfunctions of $s(t)$.

$$\ell_R = \frac{1}{N_0} \sum_{i=1}^{\infty} \frac{\lambda_i r_i^2}{\lambda_i + \frac{N_0}{2}} \quad (1.12)$$

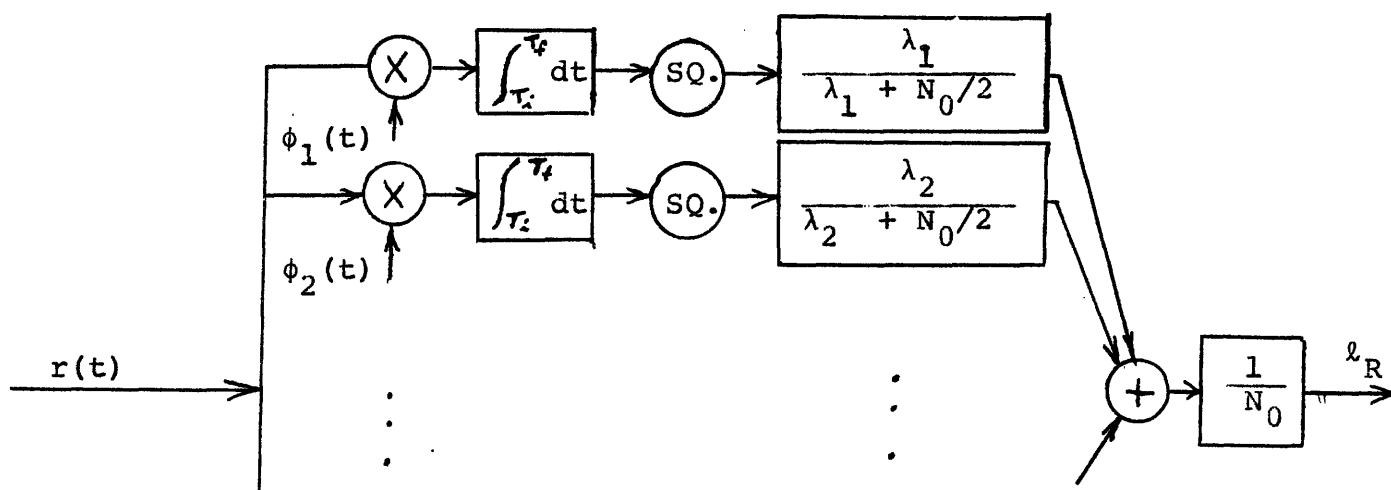


Figure 1.3. Eigenfunction Diversity Realization

Equivalently, the estimator-correlator can be (and usually is) derived from Eq. 1.12. We shall find it useful to make use of the observation from this realization that ℓ_R is the sum of squares of statistically independent Gaussian random variables.

3. Optimum Realizable Filter Realization. A third realization due to Scheppe [14] is shown in Fig. 1.4.

$$\ell_R = \frac{1}{N_0} \int_{T_i}^{T_f} [2r(t)\hat{s}(t|t) - \hat{s}^2(t|t)] dt \quad (1.13)$$

where

$$\hat{s}(t|t) = \int_{T_i}^t h_o(t,\tau)r(\tau) d\tau \quad (1.14)$$

This realization has the advantage that the linear filter is realizable. It satisfies the linear integral equation,

$$K_s(t,u) = \frac{N_0}{2} h_o(t,u) + \int_{T_i}^t K_s(t,\tau)h_o(\tau,u)d\tau, \quad T_i \leq u \leq t \leq T_f, \quad (1.15)$$

which is frequently easier to solve than Eq. 1.10.

This realization is of particular importance when $s(t)$ can be modeled as the output of a linear dynamic system which is driven with white Gaussian noise. Then the techniques of Kalman-Bucy

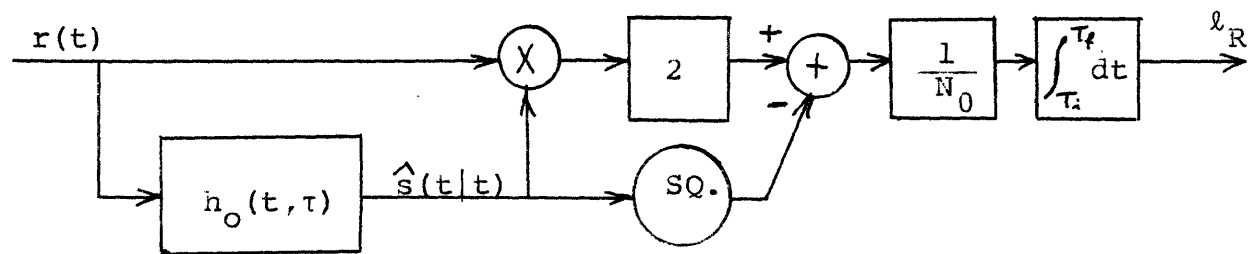


Figure 1.4. Optimum Realizable Filter Realization

filtering [15] enable us to find $\hat{S}(t|t)$ as the output of a second dynamic system. We shall return to this class of problem in Chapter III.

D. Performance

A problem which is closely related to that of determining the optimum receiver is the question of evaluating its performance. We would like to be able to do this for two reasons: (1) to enable us to pick the setting of the threshold on which the decision is based; and (2) to evaluate and compare various detectors.

In designing a digital communication system, one most frequently desires to minimize the total probability of error

$$\Pr[\epsilon] = \Pr[\epsilon|H_1]\Pr[H_1] + \Pr[\epsilon|H_2]\Pr[H_2]. \quad (1.16)$$

On the other hand, in radar and sonar problems it is common practice to employ a Neyman-Pearson criterion [2] in which the probability of a false alarm, P_F , is constrained; and we desire to minimize the probability of a miss, P_M ,

$$P_F = \Pr[\epsilon|H_1] \quad (1.17)$$

$$P_M = \Pr[\epsilon|H_2]. \quad (1.18)$$

Another performance measure of this same type is the Bayes cost,

$$C = C_F P_F + C_M P_M, \quad (1.19)$$

where C_F and C_M denote the costs assigned to a false alarm and miss, respectively.

The problem of computing the output probability distribution of a nonlinear detector, such as shown in Fig. 1.1, has been studied for over twenty years [16-20]. The basic technique is to compute the characteristic function of the output random variable in terms of the eigenvalues of a suitable eigenfunction expansion. Only recently has a satisfactory technique for finding the eigenvalues become available [21]. We thus can approximate the characteristic function by using the most significant eigenvalues. However, we then are faced with the computational problem of evaluating (numerically) an inverse Fourier transform. Although highly efficient algorithms exist [22], the constraint of computer memory size makes it difficult to obtain sufficient accuracy on the tail of the probability density.

Perhaps the most widely used measure of detector performance is the "output signal-to-noise ratio." This terminology certainly has a physical, intuitive appeal from an engineering point of view. As commonly defined, the output signal to noise ratio depends only on the first and second moments of the receiver output,

$$d^2 = \frac{[E(L|H_1) - E(L|H_2)]^2}{\sqrt{\text{Var}[L|H_1] \text{Var}[L|H_2]}} \quad (1.20)$$

Therefore, the ease of computing d^2 contributes to its popularity. However, since two moments do not completely characterize the probability distribution of the test statistic in most cases, this performance measure is not of universal applicability.

Two important examples when d^2 is an adequate performance measure are the known signal detection problem and the "low energy coherence" or "threshold" case in which the signal energy is spread among a large number of coordinates [23]. Then the error probabilities can be expressed in terms of the Gaussian error function.

$$P_F = 1 - \Phi\left(\frac{d}{2} + \frac{\gamma}{d}\right) \quad (1.21a)$$

$$P_M = 1 - \Phi\left(\frac{d}{2} - \frac{\gamma}{d}\right) \quad (1.21b)$$

$$\text{where } \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy. \quad (1.22)$$

The simplest distance measures are the Kullback-Leibler information numbers [24].

$$I(H_1:H_2) = -E[\ell|H_1] \quad (1.23a)$$

$$I(H_2:H_1) = E[\ell|H_2] \quad (1.23b)$$

The divergence [25] is related to the Kullback-Leibler information numbers and may also be regarded as an unnormalized output signal-to-noise ratio.

$$J = I(H_2:H_1) + I(H_1:H_2) \quad (1.24a)$$

$$= E[\ell|H_2] - E[\ell|H_1] \quad (1.24b)$$

The Bhattacharyya distance [26] has been proposed as an alternative performance measure which is of wider applicability than d^2 or the divergence.

$$\begin{aligned} B &= -\ln \int \dots \int [p_{\underline{r}|H_1}(\underline{R}) p_{\underline{r}|H_2}(\underline{R})]^{\frac{1}{2}} d\underline{R} \\ &= -\ln E[\Lambda^{\frac{1}{2}}|H_1] \\ &= -\ln E[\Lambda^{-\frac{1}{2}}|H_2]. \end{aligned} \quad (1.25)$$

This distance measure is sometimes referred to as the Kakutani [27] or Hellinger [28] distance.

Kailath recently discussed these last two performance measures [29]. In Chapter V, we shall give simple counterexamples where these performance measures give misleading results because the ordering of the values of the performance measure doesn't always correspond to the same ordering of the error probabilities. Therefore, we shall conclude that these performance measures should be applied with caution.

II. APPROXIMATIONS TO THE ERROR PROBABILITIES FOR BINARY HYPOTHESIS TESTING

In this chapter we develop asymptotic approximations to the error probabilities for detecting random signals. We first use the technique of tilted probability distributions to write the error integrals in a form well suited to our subsequent application of the Edgeworth expansion.

A. Tilting of Probability Distributions

Since exact calculation of error probabilities is in most cases difficult, we shall consider the possibility of approximating the error probabilities in some fashion which is computationally attractive and which is of sufficient accuracy to be of real value. One possibility is to find upper and lower bounds for the error probabilities. This approach has been successfully employed by the information theorists in proving coding theorems, for example [30]. A second possibility is to make an asymptotic approximation [31] to the error probability which is valid over a range of some parameter such as the signal-to-noise ratio.

The technique that we employ is usually called "tilting" of probability distributions. It was introduced into information and coding theory by Shannon [32] and has been employed with great success. Earlier applications in the field of mathematical statistics are due to Chernoff [33] and Esscher [34]. In this section, we summarize the notation and results. In the interest of readability, the mathematical details are relegated to Appendix A.

Since the log-likelihood ratio may be considered as the weighted sum of statistically independent random variables, a logical approach to computing its probability density is to consider the characteristic function, which is the product of the characteristic functions of the individual random variables in the sum. Actually, we use the semi-invariant moment-generating function which is the logarithm of the conditional moment generating function $M_{\ell|H_1}(s)$.*

$$\begin{aligned} \mu(s) &= \ln M_{\ell|H_1}(s) \\ &= \ln \int p_{\underline{r}|H_2}^s(\underline{R}) p_{\underline{r}|H_1}^{1-s}(\underline{R}) d(\underline{R}). \end{aligned} \quad (2.1)$$

For our purposes, it suffices to consider s to be real. Furthermore, Equation 2.1 is valid only over some range of s , say, $s_1 \leq s \leq s_2$, which is the familiar "region of convergence" associated with a Laplace transform. We shall have more to say about the range of s as we proceed.

It is sufficient to consider only one of the two conditional characteristic functions since the two are simply related for the optimum receiver which makes its decision from the logarithm of the likelihood ratio.

* Note that we have chosen to use the symbol commonly used for the characteristic function to denote the moment-generating function.

$$M_{\lambda|H_2}(s) = M_{\lambda|H_1}(s+1). \quad (2.2)$$

In the last section of this chapter, we shall present the necessary modifications of our development which enable us to treat suboptimum receivers.

Straightforward differentiation of Eq. 2.1 yields

$$\dot{\mu}(s) = \int p_{\underline{r}|s}(\underline{R}) \lambda(\underline{R}) d\underline{R}, \quad (2.3)$$

$$\ddot{\mu}(s) = \int p_{\underline{r}|s}(\underline{R}) \lambda^2(\underline{R}) d\underline{R} - [\dot{\mu}(s)]^2 \quad (2.4)$$

which are the mean and variance of $\lambda(\underline{R})$ with respect to the probability density

$$p_{\underline{r}|s}(\underline{R}) = \frac{p_{\underline{r}|H_1}^{1-s}(\underline{R}) p_{\underline{r}|H_2}^s(\underline{R})}{\int p_{\underline{r}|H_1}^{1-s}(\underline{R}') p_{\underline{r}|H_2}^s(\underline{R}') d\underline{R}'} \quad (2.5a)$$

$$= \exp[-\mu(s)] p_{\underline{r}|H_1}^{1-s}(\underline{R}) p_{\underline{r}|H_2}^s(\underline{R}). \quad (2.5b)$$

Hence $\ddot{\mu}(s) \geq 0$, with equality occurring only in the uninteresting case when λ is a constant with probability one. Thus, $\mu(s)$ is strictly convex downward in all cases of interest to us. In addition,

$$\mu(0) = \mu(1) = 0, \quad (2.6)$$

so that

$$\mu(s) < 0 \text{ for } 0 < s < 1. \quad (2.7)$$

We refer to $p_{\underline{r}_s}(\underline{R})$ as the "tilted" density. The amount of "tilting" depends on the value of the parameter s . It follows from our definition that,

$$p_{\underline{r}|H_1}(\underline{R}) = p_{\underline{r}_s}(\underline{R}) \exp[\mu(s) - s\ell(\underline{R})], \quad (2.8)$$

$$p_{\underline{r}|H_2}(\underline{R}) = p_{\underline{r}_s}(\underline{R}) \exp[\mu(s) + (1-s)\ell(\underline{R})]. \quad (2.9)$$

Hence, the error probabilities may be expressed in terms of $\mu(s)$ and the tilted density $p_{\underline{r}_s}(\underline{R})$.

$$\begin{aligned} \Pr[\varepsilon|H_1] &= \int_{\{\underline{R}: \ell(\underline{R}) > \gamma\}} p_{\underline{r}|H_1}(\underline{R}) d\underline{R} \\ &= \int_{\gamma}^{\infty} p_{\underline{r}_s}(L) \exp[\mu(s) - sL] dL \end{aligned} \quad (2.10)$$

$$\begin{aligned} \Pr[\varepsilon|H_2] &= \int_{\{\underline{R}: \ell(\underline{R}) < \gamma\}} p_{\underline{r}|H_2}(\underline{R}) d\underline{R} \\ &= \int_{-\infty}^{\gamma} p_{\underline{r}_s}(L) \exp[\mu(s) + (1-s)L] dL, \end{aligned} \quad (2.11)$$

where γ denotes the threshold level, and $p_{\ell_s}(L)$ is the tilted probability density for the log-likelihood ratio corresponding to the nonlinear transformation $\ell_s = \ell(\frac{r}{s})$.

At this point, we should perhaps comment on the range of validity of Eqs. 2.10 and 2.11. In general, we can only say that Eq. 2.1 is valid for $0 \leq s \leq 1$, since, if either of the conditional probability densities should vanish on some set of finite measure the integral would not converge for $s < 0$ or $s > 1$. Of course, Eq. 2.1 may be valid for values of s outside of $[0,1]$.

Furthermore, it is only for $0 \leq s \leq 1$ that the integrands of Eqs. 2.10 and 2.11 both contain a factor which decays exponentially in the range of integration. Presently, we shall see how this behavior enables us to approximate these integrals.

We perhaps should emphasize that up to this point, we have made no approximations. We have just rewritten the error probabilities in a different form in which we have introduced an arbitrary parameter s . Although it would appear that we have traded a difficult problem for one that is even more complex, we shall see that this added complexity leads to some useful approximations.

A simple, well-known upper bound that follows immediately from Eqs. 2.10 and 2.11 is the Chernoff bound [33,35]. For example, if in Eq. 2.10 we bound $\exp[-sL]$ by $\exp[-s\gamma]$, then bound

$$\int_{\gamma}^{\infty} p_{\ell_s}(L) dL$$

by unity, we obtain the upper bounds

$$\Pr[\varepsilon|H_1] \leq \exp[\mu(s)-s\gamma] \quad (2.12)$$

and

$$\Pr[\varepsilon|H_2] \leq \exp[\mu(s)+(1-s)\gamma]. \quad (2.13)$$

We can minimize these bounds by proper choice of the parameter s .

$$\dot{\mu}(s) = \gamma. \quad (2.14)$$

Observing from Eq. 2.3 that

$$\dot{\mu}(0) = E[\ell|H_1] \quad (2.15a)$$

and

$$\dot{\mu}(1) = E[\ell|H_2] \quad (2.15b)$$

then the range $0 \leq s \leq 1$ corresponds to

$$E[\ell|H_1] \leq \gamma \leq E[\ell|H_2]. \quad (2.16)$$

Moreover, from Eq. 2.4, $\ddot{\mu}(s) \geq 0$, so that $\dot{\mu}(s)$ is a monotone function.

Hence, a solution to Eq. 2.14 exists and is unique provided the

threshold γ lies between the means of the conditional densities

$p_{\ell|H_1}(L)$ and $p_{\ell|H_2}(L)$. This condition is usually satisfied in the

applications, since when the threshold lies above or below both

conditional densities, one of the error probabilities will be very

large (greater than one-half in many cases).

As pointed out by Shannon, et.al. [36], the exponents in the Chernoff bounds, Eqs. 2.12 and 2.13, have a simple graphical interpretation. We draw a tangent to the $\mu(s)$ curve at $s = s^*$ where $\mu(s^*) = \gamma$. This tangent line intersects the lines $s = 0$ and $s = 1$ at $\mu(s^*) - s^* \dot{\mu}(s^*)$ and $\mu(s^*) + (1-s^*) \dot{\mu}(s^*)$, respectively. See Fig. 2.1. Therefore, the intersection of the tangent line with the vertical lines $s = 0$ and $s = 1$ are the exponents in the Chernoff bounds.

Let us digress briefly to point out that all the commonly used "distance measures" can be calculated from the semi-invariant moment generating function $\mu(s)$.

From Eqs. 1.23 and 2.15, the Kullback-Leibler information numbers are

$$I(H_1:H_2) = -\dot{\mu}(0), \quad (2.17)$$

and

$$I(H_2:H_1) = \dot{\mu}(1). \quad (2.18)$$

The divergence follows immediately from Eq. 1.24.

$$J = \dot{\mu}(1) - \dot{\mu}(0) \quad (2.19)$$

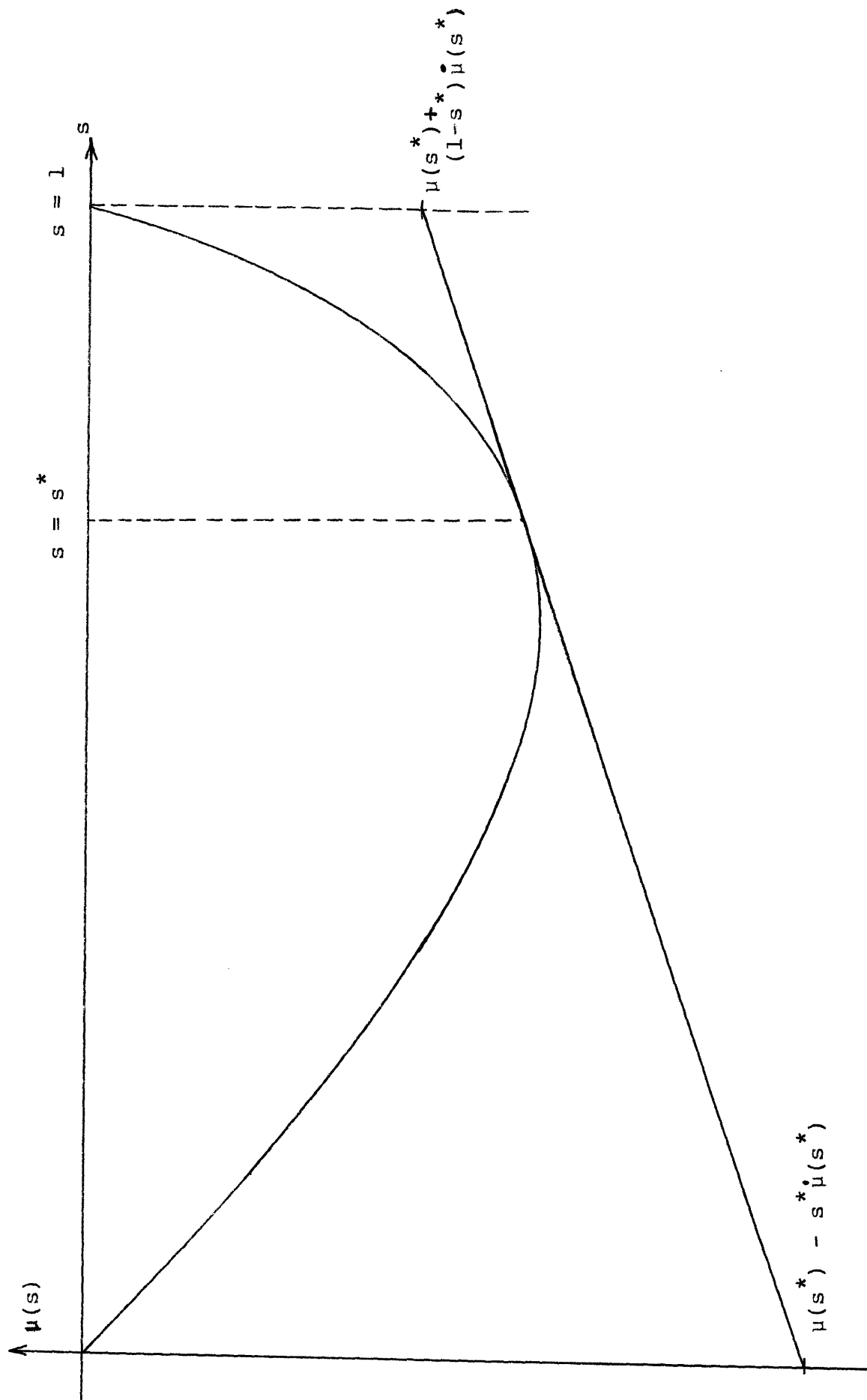


Figure 2.1. Graphical Interpretation of Exponents

The output signal-to-noise ratio is only slightly more complex. Using Eqs. 2.3 and 2.4 in 1.20, we have

$$d^2 = \frac{[\dot{\mu}(1) - \dot{\mu}(0)]^2}{\sqrt{\ddot{\mu}(1)\ddot{\mu}(0)}} \quad (2.20)$$

Finally, from Eq. 2.1, we see that the Bhattacharyya distance is just one point on the $\mu(s)$ curve.

$$\begin{aligned} B &\stackrel{\Delta}{=} -\ln \int \dots \int \sqrt{p_{\underline{r}|H_2}(\underline{R}) p_{\underline{r}|H_1}(\underline{R})} d\underline{R} \\ &= -\mu\left(\frac{1}{2}\right) \end{aligned} \quad (2.21)$$

B. Expansion in Terms of Edgeworth Series

A simple example of an asymptotic expansion for error probabilities is provided by the known signal in Gaussian noise problem. A well-known asymptotic power series for the Gaussian error function Eq. 1.22 is

$$1 - \Phi(x) \sim \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \left[\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \dots \right]. \quad (2.22)$$

The first term of this series is an upper bound which becomes extremely tight for moderately large values of x .

$$1 - \Phi(x) \leq \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right). \quad (2.23)$$

Either of the above results can be used in Eqs. 1.21a and 1.21b to obtain approximations to P_F and P_M . In this section we generalize these notions to our more general problem. We now derive an asymptotic expansion for the error probabilities in the general case. For simplicity in all that follows we shall treat only $\Pr[\epsilon|H_1]$. The derivation for $\Pr[\epsilon|H_2]$ is very similar.

First, we introduce a normalized, zero-mean random variable

$$z_s = \frac{\ell_s - \dot{\mu}(s)}{\sqrt{\ddot{\mu}(s)}}. \quad (2.24)$$

Then,

$$\Pr[\epsilon|H_1] = \exp[\mu(s) - s\dot{\mu}(s)] \int_0^{\infty} e^{-s\sqrt{\ddot{\mu}(s)}Z} p_{z_s}(Z) dZ. \quad (2.25)$$

Before proceeding from Eq. 2.25, let us point out the motivation for introducing the tilted random variable ℓ_s (and subsequently z_s).

One of the serious practical problems which we encounter in the straightforward evaluation of the error probability is that we are generally interested in the behavior far out on the tail of the probability density. Since the test statistic is made up of a large number of statistically independent components, we would like to apply the Central Limit Theorem. However, this theorem is of little use when the region of interest is the tail of the probability density.

But observe that in our alternative error expression, Eqs. 2.10 and 2.11, we no longer are integrating under the tail of a

probability density, but rather we start integrating at the mean of the tilted variable l_s . Furthermore, the integrand contains a decaying exponential factor which results in the value of the integral being determined primarily by the behavior of $p_{l_s}(L)$ near the mean rather than on the tail. Thus, we expect, at least heuristically, that the Central Limit Theorem may be used in approximating the error probabilities.

Unfortunately, in most cases of interest $p_{l_s}(L)$ does not tend to a Gaussian distribution in the limit as the number of independent components goes to infinity. This is a consequence of our covariance functions being positive definite, square integrable functions from which it immediately follows that the variance of l_s remains finite as the number of independent components goes to infinity. In this case, a necessary and sufficient condition that $p_{l_s}(L)$ approaches the Gaussian density is that each component random variable in the sum be Gaussian [38], which is not the case except in the known-signal detection problem. However, experience has shown us that the limiting distribution, while not converging to the Gaussian distribution, does not differ a great deal from the Gaussian. Therefore, it is fruitful to make an expansion of $p_{z_s}(Z)$ which is related to the Gaussian distribution. Such an expansion is the Edgeworth expansion, the first few terms of which are given below [39].

$$p_{z_s}(Z) = \phi(Z) - \left[\frac{\gamma_3}{6} \phi^{(3)}(Z) \right] + \left[\frac{\gamma_4}{24} \phi^{(4)}(Z) + \frac{\gamma_3^2}{72} \phi^{(6)}(Z) \right]$$

$$\begin{aligned}
& - \left[\frac{\gamma_5}{120} \phi^{(5)}(z) + \frac{\gamma_3 \gamma_4}{144} \phi^{(7)}(z) + \frac{\gamma_3^2}{1296} \phi^{(4)}(z) \right] \\
& + \left[\frac{\gamma_6}{720} \phi^{(6)}(z) + \frac{\gamma_4^2}{1152} \phi^{(8)}(z) + \frac{\gamma_3 \gamma_5}{720} \phi^{(8)}(z) \right. \\
& \quad \left. + \frac{\gamma_3^2 \gamma_4}{1728} \phi^{(10)}(z) + \frac{\gamma_3^4}{31104} \phi^{(12)}(z) \right] \\
& - \dots \tag{2.26}
\end{aligned}$$

where

$$\gamma_n = \frac{1}{[\mu'(s)]^{n/2}} \frac{d^n}{ds^n} \mu(s) \tag{2.27}$$

and

$$\phi^{(k)}(z) = \frac{d^k}{dz^k} \left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \right], \quad k = 0, 1, 2, \dots \tag{2.28}$$

This expansion may be obtained from a Hermite expansion of $p_{z_s}(z)$ (also sometimes called a Gram-Charlier series) upon reordering the terms. It has the further advantage for our purposes that the coefficients are expressed in terms of the semi-invariants of the random variable z_s , which are readily computed from $\mu(s)$.

We now substitute the Edgeworth expansion into the integral in Eq. 2.25 and interchange orders of integration and summation. We then are faced with the task of evaluating integrals of the form

$$I_k(\alpha) = \int_0^{\infty} \phi^{(k)}(Z) \exp(-\alpha Z) dZ. \quad (2.29)$$

Repeated integrations by parts enable us to express these integrals in terms of the Gaussian error function

$$\phi(X) = \int_{-\infty}^X \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z^2}{2}\right) dZ. \quad (2.30)$$

The first few integrals are:

$$I_0(\alpha) = \phi(-\alpha) \exp\left(\frac{\alpha^2}{2}\right) \quad (2.31a)$$

$$I_1(\alpha) = \alpha I_0(\alpha) - \frac{1}{\sqrt{2\pi}} \quad (2.31b)$$

$$I_2(\alpha) = \alpha^2 I_0(\alpha) - \frac{\alpha}{\sqrt{2\pi}} \quad (2.31c)$$

$$I_3(\alpha) = \alpha^3 I_0(\alpha) + \frac{1}{\sqrt{2\pi}} (1 - \alpha^2) \quad (2.31d)$$

$$I_4(\alpha) = \alpha^4 I_0(\alpha) + \frac{1}{\sqrt{2\pi}} (\alpha - \alpha^3) \quad (2.31e)$$

$$I_5(\alpha) = \alpha^5 I_0(\alpha) + \frac{1}{\sqrt{2\pi}} (-3 + \alpha^2 - \alpha^4) \quad (2.31f)$$

$$I_6(\alpha) = \alpha^6 I_0(\alpha) + \frac{1}{\sqrt{2\pi}} (-3\alpha + \alpha^3 - \alpha^5). \quad (2.31g)$$

Thus, we have our approximation to $\Pr[\varepsilon|H_1]$. We simply evaluate $\mu(s)$ and its derivatives then substitute. This procedure is far too complex to be used analytically, but if we use a digital computer to obtain $\mu(s)$ as we must in many problems then there is no disadvantage in having the computer evaluate our error approximation as well.

We shall frequently retain only the first term in Eq. 2.26 in the asymptotic expansion of the integral in Eq. 2.25. Then,

$$\Pr[\varepsilon|H_1] \approx \frac{1}{\sqrt{2\pi}} \phi(-s\sqrt{\ddot{\mu}(s)}) \exp[\mu(s) - s\dot{\mu}(s) + \frac{s^2}{2} \ddot{\mu}(s)]. \quad (2.32)$$

Similarly, we obtain

$$\Pr[\varepsilon|H_2] \approx \frac{1}{\sqrt{2\pi}} \phi(-(1-s)\sqrt{\ddot{\mu}(s)}) \exp \left[\mu(s) + (1-s)\dot{\mu}(s) + \frac{(1-s)^2}{2} \ddot{\mu}(s) \right] \quad (2.33)$$

Higher order approximations to the error probabilities are obtained in the same fashion. For example, the second order approximation to $\Pr[\varepsilon|H_1]$ follows from retaining the first two terms in Eq. 2.26 and using Eqs. 2.31a and 2.31d. The result is,

$$\Pr[\varepsilon|H_1] \approx \exp[\mu(s) - s\dot{\mu}(s)] \left[\frac{1}{\sqrt{2\pi}} \phi(-s\sqrt{\ddot{\mu}(s)}) \exp\left(\frac{s^2 \ddot{\mu}(s)}{2}\right) \left(1 - \frac{1}{6} s^3 \mu^{(3)}(s)\right) - \frac{1}{6\sqrt{2\pi}} \frac{\mu^{(3)}(s)}{(\ddot{\mu}(s))^{3/2}} (1-s^2 \ddot{\mu}(s)) \right] \quad (2.34)$$

A further approximation results from using the asymptotic power series, Eq. 2.22, for the Gaussian error function in the above equation. Then Eq. 2.32 becomes,

$$\Pr[\varepsilon|H_1] \approx \frac{1}{\sqrt{2\pi s^2 \ddot{\mu}(s)}} \exp[\mu(s) - s\dot{\mu}(s)], \quad (2.35)$$

and after making some cancellation, Eq. 2.34 becomes

$$\Pr[\varepsilon|H_1] \approx \frac{\exp[\mu(s) - s\dot{\mu}(s)]}{\sqrt{2\pi s^2 \ddot{\mu}(s)}} \left[1 - \frac{\mu^{(3)}(s)}{3! [\ddot{\mu}(s)]^{3/2}} \frac{3}{s\sqrt{\ddot{\mu}(s)}} \right] \quad (2.36)$$

Similar expressions result for $\Pr[\varepsilon|H_2]$.

It should be pointed out here that we have not used the Gaussian assumption on the received signals up to this point. In the next chapter, when we actually compute $\mu(s)$, we shall make use of this assumption to obtain closed-form expressions. Furthermore, observe that Eqs. 2.32 and 2.33 are the same approximations which would result if we were (in most cases incorrectly) to apply the Central Limit Theorem to Eq. 2.25.

C. Application to Suboptimum Receivers

In many situations it is convenient for either mathematical or physical reasons to use a suboptimum receiver. Although our techniques are best suited to the analysis of optimum receivers because of the property given in Eq. 2.2, we frequently may want to

consider suboptimum receivers. For example, instead of building a time-varying linear estimator followed by a correlator, we may choose to use a time-invariant filter followed by an envelope detector, as is frequently done in practice. We would like to investigate how our approximation can be adapted to such problems. On the basis of previous experience, we suspect that suboptimum systems can be found which work very well. In this section, we develop the necessary modifications to allow us to approximate the error performance of such suboptimum receivers. Since this development differs somewhat from the usual case where the test statistic is the logarithm of the likelihood ratio, we shall include the details.

Since Eq. 2.2 no longer holds, we must treat the two hypotheses separately. First, consider H_1 . Our development closely parallels that for the logarithm of the likelihood ratio which was outlined in Section A of this chapter.

$$\mu_1(s) \triangleq \ln M_{\ell|H_1}(s) \quad (2.37)$$

where we no longer require that

$$\ell = \ln \Lambda(\underline{r}) = \ln \frac{p_{\underline{r}|H_2}(\underline{R})}{p_{\underline{r}|H_1}(\underline{R})} \quad (2.38)$$

Now we define a new tilted random variable ℓ_{1s} to have the probability density

$$p_{\ell_{1s}}(L) = e^{sL - \mu_1(s)} p_{\ell|H_1}(L) \quad (2.39)$$

Then,

$$\Pr[\varepsilon|H_1] = \int_{\gamma}^{\infty} P_{\ell_{1s}}(L) \exp[\mu_1(s) - sL] dL \quad (2.40)$$

where once again γ denotes the threshold level. Just as before, we expand $p_{\ell_{1s}}(L)$ in an Edgeworth expansion. Therefore, we shall need the semi-invariants of the titled random variable ℓ_{1s}

$$\begin{aligned} \ln M_{\ell_{1s}}(t) &= \ln E[e^{t\ell_{1s}}] \\ &= \ln \int_{-\infty}^{\infty} e^{(s+t)L} - \mu_1(s) \\ &= \mu_1(s+t) - \mu_1(s) \end{aligned} \quad (2.41)$$

Therefore,

$$\frac{d^k}{ds^k} \mu_1(s) = k^{\text{th}} \text{ semi-invariant of } \ell_{1s}, \quad (2.42)$$

and the coefficients in the Edgeworth expansion of $p_{\ell_1}(L)$ are obtained from the derivatives of $\mu_1(s)$ just as before.

Similarly, under H_2 ,

$$\mu_2(s) \stackrel{\Delta}{=} \ln M_{\ell|H_2}(s). \quad (2.43)$$

We define,

$$p_{\ell_{2s}}(L) = e^{sL - \mu_2(s)} p_{\ell|H_2}(s) \quad (2.44)$$

Then,

$$\Pr[\varepsilon|H_2] = \int_{-\infty}^{\gamma} e^{\mu_2(s) - sL} p_{\ell_{2s}}(L) dL \quad (2.45)$$

As before,

$$\ln M_{\ell_{2s}}(t) = \mu_2(t+s) - \mu_2(s) \quad (2.46)$$

and the semi-invariants of ℓ_{2s} are,

$$\frac{d^k}{ds^k} \mu_2(s)$$

which appear in the Edgeworth expansion of $p_{\ell_{2s}}(L)$.

Our bounds and approximations follow immediately from Eqs. 2.10 and 2.11. The Chernoff bounds are:

$$\Pr[\varepsilon|H_1] \leq \exp[\mu_1(s_1) - s_1\gamma] \quad \text{for } s_1 > 0 \quad (2.47)$$

$$\Pr[\varepsilon|H_2] \leq \exp[\mu_2(s_2) - s_2\gamma] \quad \text{for } s_2 < 0. \quad (2.48)$$

The first-order approximations are,

$$\Pr[\varepsilon|H_1] \approx \phi(-s_1 \sqrt{\dot{\mu}_1(s_1)}) \exp[\mu_1(s_1) - s_1 \dot{\mu}_1(s_1) + \frac{s_1^2}{2} \ddot{\mu}_1(s_1)] \quad (2.49)$$

$$\Pr[\varepsilon|H_2] \approx \phi(+s_2 \sqrt{\dot{\mu}_2(s_2)}) \exp[\mu_2(s_2) - s_2 \dot{\mu}_2(s_2) + \frac{s_2^2}{2} \ddot{\mu}_2(s_2)] \quad (2.50)$$

where

$$\dot{\mu}_1(s_1) = \gamma \quad (2.51)$$

and

$$\dot{\mu}_2(s_2) = \gamma \quad (2.52)$$

Since,

$$\ddot{\mu}_1(s) = \text{Var}(l_{1s}) \geq 0, \quad (2.53)$$

$$\ddot{\mu}_2(s) = \text{Var}(l_{2s}) \geq 0, \quad (2.54)$$

$\dot{\mu}_1(s)$ and $\dot{\mu}_2(s)$ are monotone increasing functions of s . Then, Eq. 2.51 has a unique solution for $s \geq 0$ if,

$$\gamma \geq \dot{\mu}_1(0) = E[l|H_1] \quad (2.55)$$

and Eq. 2.52 has a unique solution for $s \leq 0$ if

$$\gamma \leq \dot{\mu}_2(0) = E[l|H_2]. \quad (2.56)$$

Thus, just as before, we require

$$E[\lambda|H_1] \leq \gamma \leq E[\lambda|H_2] \quad (2.57)$$

in order to be able to solve Eqs. 2.51 and 2.52.

D. Summary

In this chapter, we have developed several bounds on and approximations to the error probabilities for a rather general binary detection problem. The semi-invariant moment generating function $\mu(s)$ played a central role in all our results. These results were obtained without making any assumptions on the conditional statistics of the received signals. In the next chapter we evaluate $\mu(s)$ for the specific class of Gaussian random processes.

III. GAUSSIAN SIGNALS IN GAUSSIAN NOISE

In this chapter, we make use of the Gaussian assumption on the received signals to obtain closed-form expressions for the semi-invariant moment generating function $\mu(s)$ in terms of minimum mean-square estimation errors. In the important case when we can model our random processes via state variables, this calculation is particularly straightforward.

A. Calculation of $\mu(s)$ for Finite-Dimensional Gaussian Signals

As a first step in obtaining $\mu(s)$ for the general Gaussian binary detection problem, we consider a related problem from the second level in our hierarchy of detection problems.

$$\begin{aligned} H_1: \underline{r} &= \underline{s}_1 + \underline{m}_1 + \underline{w} \\ H_2: \underline{r} &= \underline{s}_2 + \underline{m}_2 + \underline{w}, \end{aligned} \quad (3.1)$$

where \underline{s}_1 and \underline{s}_2 are N-dimensional zero-mean Gaussian random vectors with covariance matrices $\underline{K}_{\underline{s}_1}$ and $\underline{K}_{\underline{s}_2}$, respectively; \underline{m}_1 and \underline{m}_2 are known N-vectors; and \underline{w} is a Gaussian random N-vector with statistically independent components each having variance σ_w^2 . Thus,

$$\underline{K}_{\underline{r}|H_1} = \underline{K}_{\underline{s}_1} + \sigma_w^2 \underline{I} \triangleq \underline{K}_1 \quad (3.2a)$$

$$\underline{K}_{\underline{r}|H_2} = \underline{K}_{\underline{s}_2} + \sigma_w^2 \underline{I} \triangleq \underline{K}_2 \quad (3.2b)$$

$$E\{\underline{r}|H_1\} \triangleq \underline{m}_1 \quad (3.2c)$$

$$E[\underline{r}|H_2] = \underline{m}_2 \quad (3.2d)$$

$$p_{\underline{r}|H_1}(\underline{R}) = \frac{1}{(2\pi)^{N/2} |\underline{K}_1|^{1/2}} \exp \left[-\frac{1}{2} (\underline{R} - \underline{m}_1)^T \underline{K}_1^{-1} (\underline{R} - \underline{m}_1) \right] \quad (3.3a)$$

$$p_{\underline{r}|H_2}(\underline{R}) = \frac{1}{(2\pi)^{N/2} |\underline{K}_2|^{1/2}} \exp \left[-\frac{1}{2} (\underline{R} - \underline{m}_2)^T \underline{K}_2^{-1} (\underline{R} - \underline{m}_2) \right] \quad (3.3b)$$

We now substitute Equations 3.3a and 3.3b in Equation 2.1 for $\mu(s)$. We make use of the existence of an orthogonal matrix (i.e., $\underline{Q}^T = \underline{Q}^{-1}$) which diagonalizes the quadratic term in the exponent of the integrand. This enables us to complete the square in the exponent, making the evaluation of the N-fold integral straightforward. The tedious algebraic details of these calculations appear in Appendix B. The result is

$$\begin{aligned} \mu(s) = & \frac{s}{2} \ln |\underline{K}_1| + \frac{1-s}{2} \ln |\underline{K}_2| \\ & - \frac{1}{2} \ln |(1-s)\underline{K}_2 + s\underline{K}_1| \\ & - \frac{s(1-s)}{2} [\underline{m}_2 - \underline{m}_1]^T [(1-s)\underline{K}_2 + s\underline{K}_1]^{-1} [\underline{m}_2 - \underline{m}_1] \end{aligned} \quad (3.4a)$$

or

$$\begin{aligned} \mu(s) = & \frac{s}{2} \ln |\underline{I} + \frac{1}{\sigma_w^2} \underline{K}_{s_1}| + \frac{1-s}{2} \ln |\underline{I} + \frac{1}{\sigma_w^2} \underline{K}_{s_2}| \\ & - \frac{1}{2} \ln |\underline{I} + \frac{1}{\sigma_w^2} ((1-s)\underline{K}_{s_2} + s\underline{K}_{s_1})| \\ & - \frac{s(1-s)}{2} [\underline{m}_2 - \underline{m}_1]^T [\sigma_w^2 \underline{I} + (1-s)\underline{K}_{s_2} + s\underline{K}_{s_1}]^{-1} [\underline{m}_2 - \underline{m}_1]. \end{aligned} \quad (3.4b)$$

B. Transition from Finite to Infinite Set of Observables

We now make use of the results of the previous section to evaluate $\mu(s)$ for the general Gaussian binary detection problem.

$$H_1: r(t) = s_1(t) + m_1(t) + w(t),$$

$$T_i \leq t \leq T_f, \quad (3.5)$$

$$H_2: r(t) = s_2(t) + m_2(t) + w(t),$$

where $s_1(t)$ and $s_2(t)$ are sample functions from zero-mean Gaussian random processes with known covariance functions $K_1(t, \tau)$ and $K_2(t, \tau)$, respectively; $m_1(t)$ and $m_2(t)$ are known waveforms; and $w(t)$ is a sample function of white Gaussian noise with spectral density $N_0/2$.

We choose the components of \underline{r} to be N uniformly spaced time samples of the random process $r(t)$. We then let the sampling become dense; that is, we let $N \rightarrow \infty$. From Fredholm integral

equation theory we are able to express the first three terms in Eq. 3.4b as Fredholm determinants of appropriate covariance functions. The last term in Eq. 3.4b in the limit can be expressed in terms of the solution to a non-homogeneous Fredholm integral equation. We relegate the details to Appendix B and state the results here.

$$\begin{aligned} \mu(s) = & \frac{s}{2} \sum_{i=1}^{\infty} \ln\left(1 + \frac{2}{N_0} \lambda_{i1}\right) + \frac{1-s}{2} \sum_{i=1}^{\infty} \ln\left(1 + \frac{2}{N_0} \lambda_{i2}\right) \\ & - \frac{1}{2} \sum_{i=1}^{\infty} \ln\left(1 + \frac{2}{N_0} \lambda_{i\text{comp}}\right) \\ & - \frac{s(1-s)}{2} \sum_{i=1}^{\infty} \frac{(m_{i2} - m_{i1})^2}{\lambda_{i\text{comp}} + \frac{N_0}{2}} \end{aligned} \quad (3.6)$$

where $\{\lambda_{i1}\}$ and $\{\lambda_{i2}\}$ are the eigenvalues of $K_1(t, \tau)$ and $K_2(t, \tau)$, respectively; and $\{m_{i1}\}$ and $\{m_{i2}\}$ are the coefficients of $m_1(t)$ and $m_2(t)$ when expanded in terms of the eigenfunctions of the composite random process $s_{\text{comp}}(t)$.

$$K_{\text{comp}}(t, \tau; s) \triangleq (1-s)K_2(t, \tau) + sK_1(t, \tau). \quad (3.7)$$

The first three terms in Eq. 3.6 are all Fredholm determinants of the form

$$D_{\mathcal{F}}(Z) = \prod_{i=1}^{\infty} (1 + \lambda_i Z) \quad (3.8)$$

Several closed-form expressions for the Fredholm determinant have appeared in the literature. Siegert [20, 40] related it to the solution of integral equations of the Wiener-Hopf type.

$$\ln D_{\mathcal{F}}\left(\frac{2}{N_0}\right) = \sum_{i=1}^{\infty} \ln\left(1 + \frac{2}{N_0} \lambda_i\right) \quad (3.9)$$

$$= \int_0^{\frac{2}{N_0}} dz \frac{1}{z} \int_{T_i}^{T_f} h_I(t, t: \frac{1}{z}) dt \quad (3.10)$$

$$= \int_{T_i}^{T_f} h_o(t, t: \frac{N_0}{2}) dt \quad (3.11)$$

where $h_I(t, \tau: \frac{N_0}{2})$ and $h_o(t, \tau: \frac{N_0}{2})$ are impulse responses for optimum minimum mean-square error linear filters (unrealizable and realizable, respectively) for estimating the random process $y(t)$ when the observations have been corrupted by additive white noise with spectral height $N_0/2$.

Since the minimum mean-square estimation error is related to the impulse response of the optimum linear filter [41],

$$\xi_y(t|T_f:\frac{N_0}{2}) = \frac{N_0}{2} h_I(t, t:\frac{N_0}{2}) \quad (3.12)$$

$$\xi_y(t|t:\frac{N_0}{2}) = \frac{N_0}{2} h_o(t, t:\frac{N_0}{2}) \quad (3.13)$$

we have two alternate expressions for the Fredholm determinant.

$$\ln D_{\frac{2}{N_0}} = \int_0^{\frac{2}{N_0}} dz \int_{T_i}^{T_f} \xi_y(t|T_f:\frac{1}{z}) dt \quad (3.14)$$

$$= \frac{2}{N_0} \int_{T_i}^{T_f} \xi_y(t|t:\frac{N_0}{2}) dt \quad (3.15)$$

where $\xi_y(t|t:\frac{N_0}{2})$ denotes the minimum mean-square point estimation error for estimating $y(t)$ using observations over the interval $[T_i, \tau]$ which are corrupted by additive white noise with spectral density $N_0/2$.

Thus, we have our expression for $\mu(s)$ in terms of quantities which can be readily computed.

$$\mu(s) = \frac{2}{N_0} \left[\frac{s}{2} \int_{T_i}^{T_f} \xi_{s_1}(t|t:\frac{N_0}{2}) dt \right]$$

$$\begin{aligned}
& + \frac{1-s}{2} \int_{T_i}^{T_f} \xi_{s_2} \left(t | t: \frac{N_0}{2} \right) dt \\
& - \frac{1}{2} \int_{T_i}^{T_f} \xi_{\text{comp}} \left(t | t: \frac{N_0}{2}, s \right) dt \\
& - \frac{s(1-s)}{2} \int_{T_i}^{T_f} \left[(m_2(t) - m_1(t)) - \right. \\
& \left. \int_{T_i}^{T_f} h_{\text{comp}}(t, \tau: s) [m_2(\tau) - m_1(\tau)] d\tau \right]^2 dt \quad (3.16)
\end{aligned}$$

ξ_{s_1} , ξ_{s_2} , and ξ_{comp} denote the minimum mean-square linear estimation error for estimating $s_1(t)$, $s_2(t)$, and $s_{\text{comp}}(t) \triangleq \sqrt{s} s_1(t) + \sqrt{1-s} s_2(t)$, respectively, when observed over $[T_i, t]$ in additive white noise of spectral height $N_0/2$. $h_{\text{comp}}(t, \tau: s)$ denotes the minimum mean-square linear estimator for estimating the composite process $s_{\text{comp}}(t)$. The last term in Eq. 3.16 can also be interpreted as $d_{\text{comp}}^2(s)$, the output signal-to-noise ratio for the problem of deciding which of the known signals $m_1(t)$ or $m_2(t)$ was sent when observed in colored Gaussian noise with covariance

$$\begin{aligned}
K_n(t, \tau; s) &= K_{\text{comp}}(t, \tau; s) + \frac{N_0}{2} \delta(t - \tau) \\
&= s K_1(t, \tau) + (1-s) K_2(t, \tau) + \frac{N_0}{2} \delta(t - \tau). \quad (3.17)
\end{aligned}$$

An alternate way of writing Eq. 3.16 results from using Eq. 3.15 and the observation that the last term in Eq. 3.16 is an appropriately defined "signal-to-noise ratio".

$$\begin{aligned}
\mu(s) &= \frac{s}{2} \ln D_{\mathcal{F}s_1} \left(\frac{2}{N_0} \right) + \frac{1-s}{2} \ln D_{\mathcal{F}s_2} \left(\frac{2}{N_0} \right) \\
&\quad - \frac{1}{2} \ln D_{\mathcal{F}\text{comp}} \left(\frac{2}{N_0}; s \right) - \frac{s(1-s)}{2} d_{\text{comp}}^2(s) \quad (3.18)
\end{aligned}$$

C. Special Cases

There are several special cases of Eq. 3.16 which deserve a few additional comments, particularly since many of the applications fall into these cases. For these cases, one frequently can derive the expression for $\mu(s)$ in much a simpler fashion. We do not include these simplified derivations here.

1. Simple Binary Hypotheses, Zero Mean. For this problem, only white noise is present on H_1 .

$$\begin{aligned}
H_2: r(t) &= s(t) + w(t) \\
T_i &\leq t \leq T_f \quad (3.19)
\end{aligned}$$

$$H_1: r(t) = w(t)$$

The "composite" random process is now the same as the "signal" random process, except for the power level. Then,

$$\mu(s) = \frac{2}{N_0} \left[\frac{1-s}{2} \int_{T_i}^{T_f} \xi_s(t|t: \frac{N_0}{2}) dt - \int_{T_i}^{T_f} \xi_{\sqrt{1-s} s}(t|t: \frac{N_0}{2}) dt \right] \quad (3.20a)$$

$$= \frac{1-s}{2} \frac{2}{N_0} \int_{T_i}^{T_f} \xi_s(t|t: \frac{N_0}{2}) dt - \frac{1}{2} \frac{2(1-s)}{N_0} \int_{T_i}^{T_f} \xi_s(t|t: \frac{N_0}{2(1-s)}) dt \quad (3.20b)$$

$$= \frac{1}{2} [(1-s) \ln D_{\frac{2}{N_0}} - \ln D_{\frac{2(1-s)}{N_0}}] \quad (3.20c)$$

In terms of the eigenvalues of $s(t)$,

$$\mu(s) = \frac{1}{2} \left[(1-s) \sum_{i=1}^{\infty} \ln(1 + \frac{2\lambda_i}{N_0}) - \sum_{i=1}^{\infty} \ln(1 + \frac{2(1-s)\lambda_i}{N_0}) \right]. \quad (3.21)$$

We can also obtain closed-form expressions for $\dot{\mu}(s)$ for this case. Differentiating Eq. 3.21, we obtain

$$\dot{\mu}(s) = \frac{1}{2} \left[- \sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i}{N_0} \right) + \sum_{i=1}^{\infty} \frac{\frac{2\lambda_i}{N_0}}{1 + \frac{2(1-s)\lambda_i}{N_0}} \right] \quad (3.22a)$$

$$= \frac{1}{2} \left[- \frac{2}{N_0} \int_{T_i}^{T_f} \xi_s \left(t \mid t: \frac{N_0}{2} \right) dt + \frac{2}{N_0} \int_{T_i}^{T_f} \xi_s \left(t \mid T_f: \frac{N_0}{2(1-s)} \right) dt \right].$$

(3.22b)

The second term in Eq. 3.22b is an unrealizable minimum-mean-square filtering error.

Thus, we have closed-form expressions for the exponents in our bounds and approximation.

$$\mu(s) + (1-s)\dot{\mu}(s)$$

$$= \frac{2}{N_0} (1-s) \int_{T_i}^{T_f} \left[\xi_s \left(t \mid T_f: \frac{N_0}{2} \right) - \xi_s \left(t \mid t: \frac{N_0}{2} \right) \right] dt \quad (3.23)$$

$$\mu(s) - s\dot{\mu}(s) = \frac{2}{N_0} \left[\int_{T_i}^{T_f} [\xi_s(t|t:\frac{N_0}{2}) - \xi_s(t|t:\frac{N_0}{2(1-s)})] dt \right. \\ \left. + s \int_{T_i}^{T_f} [\xi_s(t|t:\frac{N_0}{2(1-s)}) - \xi_s(t|t:\frac{N_0}{2(1-s)})] dt \right] \quad (3.24)$$

2. Symmetric Binary Problem. Now consider the symmetric binary problem where

$$H_1: r(t) = s_1(t) + w(t) \quad T_i \leq t \leq T_f \quad (3.25)$$

$$H_2: r(t) = s_2(t) + w(t)$$

We assume that $s_1(t)$ and $s_2(t)$ have the same eigenvalues, with the corresponding eigenfunctions being orthogonal. This situation can occur in practice, for example, when $s_2(t)$ is a frequency-shifted version of $s_1(t)$, as in an FSK system operating over a fading channel.

From Eq. 3.16

$$\mu_{sy}(s) = \frac{1}{2} \left\{ \ln D_{\mathcal{F}}\left(\frac{2}{N_0}\right) - \ln D_{\mathcal{F}}\left(\frac{2s}{N_0}\right) - \ln D_{\mathcal{F}}\left(\frac{2(1-s)}{N_0}\right) \right\} \quad (3.26)$$

where the subscript "SY" denotes "symmetric".

Recall that

$$\mu_{SI}(s) = \frac{1}{2} \left\{ (1-s) \ln D_{\frac{2}{N_0}} - \ln D_{\frac{2(1-s)}{N_0}} \right\} \quad (3.27)$$

where "SI" denotes "simple binary". Thus,

$$\mu_{SY}(s) = \mu_{SI}(s) + \mu_{SI}(1-s) \quad (3.28)$$

This equation enables us to compute $\mu(s)$ and all its derivatives for the symmetric problem from those for the simple binary problem. In particular,

$$\mu_{SY}\left(\frac{1}{2}\right) = 2\mu_{SI}\left(\frac{1}{2}\right) \quad (3.29a)$$

$$\dot{\mu}_{SY}\left(\frac{1}{2}\right) = 0 \quad (3.29b)$$

$$\ddot{\mu}_{SY}\left(\frac{1}{2}\right) = 2\ddot{\mu}_{SI}\left(\frac{1}{2}\right). \quad (3.29c)$$

3. Stationary Bandpass Signals. In many applications, the signals of interest are narrowband around some carrier frequency ω_c . That is,

$$s(t) = \sqrt{2} A(t) \cos(\omega_c t + \theta(t)), \quad (3.30)$$

where the envelope and phase waveforms, $A(t)$ and $\theta(t)$, have negligible energy at frequencies comparable to ω_c . Equivalently, we can write $s(t)$ in terms of its quadrature components.

$$s(t) = \sqrt{2} s_c(t) \cos \omega_c t + \sqrt{2} s_s(t) \sin \omega_c t. \quad (3.31)$$

In some of our examples in Chapter V, we shall want to consider stationary bandpass random processes which can be modeled as in Eq. 3.31 over some interval $T_i \leq t \leq T_f$, where $s_c(t)$ and $s_s(t)$ are statistically independent stationary random processes with identical statistics. For this case, the eigenvalues of the bandpass random process $s(t)$ are equal to the eigenvalues of the quadrature components. Each eigenvalue of $s_c(t)$ or of $s_s(t)$ of multiplicity N is an eigenvalue of $s(t)$ with multiplicity $2N$.

We denote the eigenvalues of $s(t)$ by $\{\lambda_i^{BP}\}$ and those of $s_c(t)$ and $s_s(t)$ by $\{\lambda_i^{LP}\}$. Then from Eq. 3.9, the Fredholm determinant for $s(t)$ is

$$\begin{aligned} \ln D_{\mathcal{F}, s(t)} \left(\frac{2}{N_0} \right) &= \sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i^{BP} \right) \\ &= 2 \sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i^{LP} \right) \end{aligned}$$

$$= 2 \ln D_{\sigma, s_c}(t) \left(\frac{2}{N_0}\right) \quad (3.32a)$$

$$= 2 \ln D_{\sigma, s_s}(t) \left(\frac{2}{N_0}\right) \quad (3.32b)$$

Further, observe that

$$\begin{aligned} \bar{E} \triangleq \bar{E}_{s(t)} &= \sum_{i=1}^{\infty} \lambda_i^{BP} \\ &= 2 \sum_{i=1}^{\infty} \lambda_i^{LP} \end{aligned} \quad (3.33)$$

Then,

$$\begin{aligned} \bar{E}_{LP} \triangleq \bar{E}_{s_c}(t) &= \bar{E}_{s_s}(t) \\ &= \sum_{i=1}^{\infty} \lambda_i^{LP} \\ &= \bar{E}/2 \end{aligned} \quad (3.34)$$

or

$$\frac{2\bar{E}_{LP}}{N_0} = \frac{\bar{E}}{N_0} \quad (3.35)$$

For the zero mean case, it follows from Eq. 3.18 that

$$\begin{aligned}\mu_{\text{BP}}\left(s: \frac{2\bar{E}}{N_0}\right) &= 2\mu_{\text{LP}}\left(s: \frac{2\bar{E}_{\text{LP}}}{N_0}\right) \\ &= 2\mu_{\text{LP}}\left(s: \frac{\bar{E}}{N_0}\right)\end{aligned}\tag{3.36}$$

where we have explicitly indicated the signal-to-noise ratio in each term.

We comment that the results in this section are not the most general that can be obtained for bandpass processes, but they suffice for the examples we shall consider. A more detailed discussion of properties and representations for bandpass random processes would take us too far afield. Two appropriate references are [43,44].

D. Simplified Evaluation for Finite State Random Processes

An important class of problems in which we can readily compute the various terms which comprise $\mu(s)$ is those problems in which the random processes can be modeled as the output of linear state-variable systems which are driven by white Gaussian noise. This model includes all stationary processes with rational spectra as one important subclass of interesting problems.

The state variable model allows us to use the results of the Kalman-Bucy formulation of the optimum linear filtering problem to determine the optimum receiver structure as well as to calculate $\mu(s)$ [14-15]. The straightforward way to compute the first three terms in Eq. 3.16 is to solve the appropriate matrix Riccati equation and then integrate the result over the time interval $[T_1, T_f]$ as indicated.

We assume that the random process $y(t)$ ($s_1(t)$, $s_2(t)$ or $s_{\text{comp}}(t)$ for the first, second, and third terms in Eq. 3.16, respectively) is the output of a linear system which has a state-variable representation and which is excited by zero mean white noise, $\underline{u}(t)$; that is

$$\frac{d}{dt} \underline{x}(t) = \underline{F}(t) \underline{x}(t) + \underline{G}(t) \underline{u}(t) \quad (3.37a)$$

$$y(t) = \underline{C}(t) \underline{x}(t) \quad (3.37b)$$

$$E[\underline{u}(t)\underline{u}^T(\tau)] = \underline{Q}(t)\delta(t-\tau) \quad (3.37c)$$

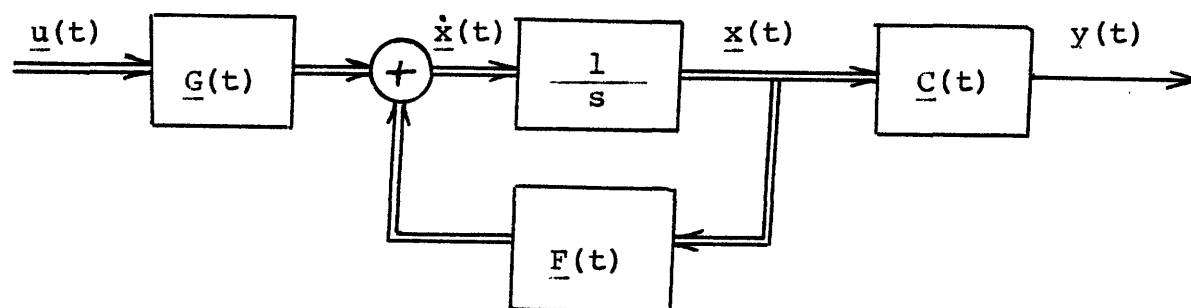


Figure 3.1. State-variable Random Process Generation

$$E[\underline{x}(T_1)\underline{x}^T(T_1)] = \underline{\Sigma}_0. \quad (3.37d)$$

Eq. 3.15 provides a convenient way of computing the Fredholm determinant when we realize the detector or estimator, for $\xi_y(t|t: \frac{N_0}{2})$ appears explicitly in the realization [13]. For the problems at hand, the techniques of Kalman-Bucy filtering give $\xi_y(t|t: \frac{N_0}{2})$ in terms of the solution to a matrix Riccati equation [15].

$$\xi_y(t|t: \frac{N_0}{2}) = \underline{C}(t)\underline{\Sigma}(t)\underline{C}^T(t) \quad (3.38)$$

where

$$\begin{aligned} \frac{d}{dt} \underline{\Sigma}(t) = & \underline{F}(t)\underline{\Sigma}(t) + \underline{\Sigma}(t)\underline{F}^T(t) + \underline{G}(t)\underline{Q}(t)\underline{G}^T(t) \\ & - \underline{\Sigma}(t)\underline{C}^T(t) \frac{2}{N_0} \underline{C}(t)\underline{\Sigma}(t) \end{aligned} \quad (3.39a)$$

with

$$\underline{\Sigma}(T_1) = \underline{\Sigma}_0. \quad (3.39b)$$

We now derive a new expression for the Fredholm determinant which does not require integrating a time function over the interval $[T_1, T_f]$. Moreover, we obtain this result in terms of the solution to a set of linear differential equations, as contrasted with the nonlinear set in Eq. 3.39a. These advantages are particularly useful from an

analytical and computational point of view. In particular, for the important case in which Eqs. 3.37a-3.37b are time invariant, which includes all stationary processes with rational spectra, the solution to this set of linear differential equations can be written in closed-form in terms of a matrix exponential. This matrix exponential is frequently easier to compute than integrating the differential equation directly.

We start with Eq. 3.15 and then make use of the linear system of equations which is equivalent to Eq. 3.39a [15].

$$\frac{d}{dt} \underline{\phi}_1(t) = \underline{F}(t)\underline{\phi}_1(t) + \underline{G}(t)\underline{Q}(t)\underline{G}^T(t)\underline{\phi}_2(t) \quad (3.40a)$$

$$\frac{d}{dt} \underline{\phi}_2(t) = \underline{C}^T(t) \frac{2}{N_0} \underline{C}(t)\underline{\phi}_1(t) - \underline{F}^T(t)\underline{\phi}_2(t) \quad (3.40b)$$

$$\underline{\phi}_1(T_i) = \underline{\Sigma}_0 \quad (3.40c)$$

$$\underline{\phi}_2(T_i) = \underline{I} \quad (3.40d)$$

$$\underline{\Sigma}(t) = \underline{\phi}_1(t)\underline{\phi}_2^{-1}(t). \quad (3.41)$$

Then using Eqs. 3.15, 3.38, and 3.41,

$$\ln D_{\frac{2}{N_0}} = \frac{2}{N_0} \int_{T_i}^{T_f} \xi_y(t|t: \frac{N_0}{2}) dt$$

$$\begin{aligned}
&= \frac{2}{N_0} \int_{T_i}^{T_f} \underline{C}(t) \underline{\Sigma}(t) \underline{C}^T(t) dt \\
&= \frac{2}{N_0} \int_{T_i}^{T_f} \underline{C}(t) \underline{\phi}_1(t) \underline{\phi}_2^{-1}(t) \underline{C}^T(t) dt \\
&= \int_{T_i}^{T_f} \text{Tr} \left[\left(\underline{C}^T(t) \frac{2}{N_0} \underline{C}(t) \underline{\phi}_1(t) \right) \underline{\phi}_2^{-1}(t) \right] dt \quad (3.42)
\end{aligned}$$

where $\text{Tr}[\cdot]$ denotes the trace of a square matrix and where we have used the result that

$$\underline{x}^T \underline{A} \underline{x} = \text{Tr}[(\underline{x} \underline{x}^T) \underline{A}] \quad (3.43)$$

for any column vector \underline{x} and square matrix \underline{A} . Substituting Eq. 3.40b we have

$$\begin{aligned}
\ln D_{\underline{F}} \left(\frac{2}{N_0} \right) &= \int_{T_i}^{T_f} \text{Tr} \left[\left(\frac{d\underline{\phi}_2(t)}{dt} + \underline{F}^T(t) \underline{\phi}_2(t) \right) \underline{\phi}_2^{-1}(t) \right] dt \\
&= \int_{T_i}^{T_f} \text{Tr}[\underline{\phi}_2^{-1}(t) d\underline{\phi}_2(t)] + \int_{T_i}^{T_f} \text{Tr}[\underline{F}(t)] dt \quad (3.44)
\end{aligned}$$

We make use of Eq. 9.31 of Ref. [45].

$$\ln D_{\frac{2}{N_0}}(T_f) = \int_{T_i}^{T_f} d[\ln \det \underline{\phi}_2(t)] + \int_{T_i}^{T_f} \text{Tr}[\underline{F}(t)] dt. \quad (3.45)$$

The final result follows from integrating the first term in Eq. 3.45 and using Eq. 3.40d

$$\ln D_{\frac{2}{N_0}}(T_f) = \ln \det \underline{\phi}_2(T_f) + \int_{T_i}^{T_f} \text{Tr}[\underline{F}(t)] dt. \quad (3.46)$$

It is important to observe that the second term depends only on the system matrix, $\underline{F}(t)$, and is independent of the signal and noise levels and the modulation, $\underline{C}(t)$. Then, in many problems of interest, such as signal design, we need only be concerned with the first term in Eq. 3.46.

It further should be noted that the result derived here was first obtained via Baggeroer's technique for solving Fredholm integral equations for state-variable processes [21].

The first term in Eq. 3.46 is readily computed in terms of the transition matrix of the canonical system

$$\frac{d}{dt} \underline{\theta}(t, T_i) = \left[\begin{array}{c|c} \underline{F}(t) & \underline{G}(t)\underline{Q}(t)\underline{G}^T(t) \\ \hline \underline{C}^T(t) \frac{2}{N_0} \underline{C}(t) & -\underline{F}^T(t) \end{array} \right] \underline{\theta}(t, T_i) \quad (3.47a)$$

$$\underline{\theta}(T_i, T_i) = \underline{I}. \quad (3.47b)$$

If we partition the transition matrix

$$\underline{\theta}(t, T_i) = \left[\begin{array}{c|c} \underline{\theta}_{11}(t, T_i) & \underline{\theta}_{12}(t, T_i) \\ \hline \underline{\theta}_{21}(t, T_i) & \underline{\theta}_{22}(t, T_i) \end{array} \right], \quad (3.48)$$

then

$$\underline{\phi}_2(T_f) = \underline{\theta}_{21}(T_f, T_i) \underline{\Sigma}_0 + \underline{\theta}_{22}(T_f, T_i). \quad (3.49)$$

In the case of constant parameter systems, it is particularly easy to compute the transition matrix in terms of a matrix exponential [46].

It is interesting and useful to observe that $\mu(s:t)$, where we have explicitly indicated the dependence on the time t , may be regarded as a state-variable of a (realizable) dynamic system.

Straightforward differentiation of Eq. 3.16 yields

$$\begin{aligned} \frac{\partial \mu(s:t)}{\partial t} &= \frac{s}{N_0} \xi_{s_1} \left(t | t: \frac{N_0}{2} \right) + \frac{1-s}{N_0} \xi_{s_2} \left(t | t: \frac{N_0}{2} \right) \\ &\quad - \frac{1}{N_0} \xi_{\text{comp}} \left(t | t: \frac{N_0}{2}, s \right) \\ &\quad - \frac{s(1-s)}{N_0} \left[(m_2(t) - m_1(t)) - \int_{T_i}^t h_{\text{comp}}(t, \tau; s) (m_2(\tau) - m_1(\tau)) d\tau \right]^2 \end{aligned} \quad (3.50a)$$

with the initial condition

$$\mu(s:T_1) = 0. \quad (3.50b)$$

Since we can compute each term on the right-hand side of Eq. 3.50a in real time, that is, as the outputs of realizable dynamic systems, $\mu(s:t)$ can also be computed as the output of such a system.

E. Semi-Invariant Moment-Generating Function for Suboptimum Receivers^{*}

In Chapter II, we saw that our "tilting" development could be modified to allow us to evaluate the performance of suboptimum receivers. Recall that we obtained bounds on and approximations to the probability of error in terms of two functions, $\mu_1(s)$ and $\mu_2(s)$. These functions were the logarithms of the moment-generating function of the receiver output conditioned on H_1 and H_2 respectively. However, the closed-form expressions we have obtained for $\mu(s)$ hold only when the receiver is optimum. In this section, we obtain $\mu_1(s)$ and $\mu_2(s)$ for an important class of suboptimum receivers.

The class of suboptimum receivers which we shall consider are indicated in Fig. 3.2. They consist of a linear filter (possibly time-varying) followed by a squarer and an integrator. This is an accurate model for many of the receivers which are used in practice. A number of authors have considered this problem over the years. A good discussion and list of references appears in [47].

* The results in this section were developed in discussions with R. R. Kurth [48].

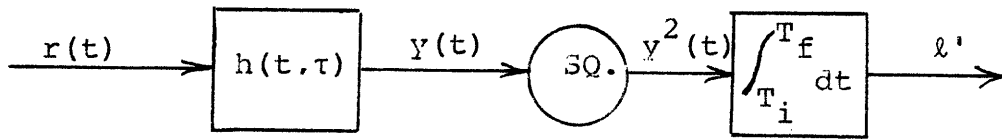


Figure 3.2. A Class of Suboptimum Receivers

For the class of problems we are considering, $r(t)$ and $y(t)$ are Gaussian random processes. Expanding $y(t)$ in its Karhunen-Loève expansion

$$y(t) = \sum_{i=1}^{\infty} y_i \phi_i(t), \quad (3.51)$$

$$\ell = \int_0^T y^2(t) dt \quad (3.52a)$$

$$= \sum_{i=1}^{\infty} y_i^2 \quad (3.52b)$$

Hence, ℓ is the sum of squares of statistically independent Gaussian random variables and all of our previous discussion about the inapplicability of the Central Limit Theorem to $p_{\ell}(L)$ still holds. However, it is still very natural to proceed from a characteristic function or moment generating function point of view.

$$\begin{aligned} \mu_j(s) &= \ln E[e^{s\ell} | H_j] \\ &= \ln E[e^{s \sum_{i=1}^{\infty} y_i^2} | H_j] \\ &= \sum_{i=1}^{\infty} \ln E[e^{s y_i^2} | H_j] \end{aligned}$$

$$= -\frac{1}{2} \sum_{i=1}^{\infty} \ln(1-2s \lambda_i^j) \quad (3.53)$$

$$\text{for } \text{Re}(s) < \frac{1}{2\lambda_i^j}$$

where $\{\lambda_i^j\}$ denotes the eigenvalues of $y(t)$, $T_i \leq t \leq T_f$ conditioned on H_j , for $j = 1, 2$ and where we have used Eq. 7.67 from Ref. [49].

The previous approaches to this problem started with an expression for the characteristic function analogous to Eq. 3.53 and after finding the significant eigenvalues, performed an inverse Fourier transform to obtain the conditional probability densities. In Section D of Chapter I, we discussed the difficulties inherent in such an approach.

Instead, we use the moment generating function (with real values for its argument) to obtain bounds and approximations as discussed in Chapter II. All that remains is to obtain closed-form expression for $\mu(s)$. From Eq. 3.53

$$\mu_j(s) = -\frac{1}{2} \ln D_{\mathcal{F}, H_j}(-2s) \quad (3.54)$$

$$\text{for } \text{Re}(s) < \frac{1}{2\lambda_i^j}$$

The technique discussed in the previous sections of this chapter for evaluating Fredholm determinants are applicable here as well. Here the pertinent random process is the input to the square-law device

in the receiver, conditioned on the two hypotheses. Again, in the case where the random process generation and the receiver filter can be modeled via state variables, we can readily evaluate the Fredholm determinant. The model, conditioned on one of the hypotheses, is shown in Fig. 3.3.

The equations specifying the message generation are:

$$\dot{\underline{x}}_1(t) = \underline{F}_1(t)\underline{x}_1(t) + \underline{G}_1(t)\underline{u}_1(t) \quad (3.55a)$$

$$r(t) = \underline{G}_1(t)\underline{x}_1(t) + w(t) \quad (3.55b)$$

and those specifying the receiver are

$$\dot{\underline{x}}_2(t) = \underline{F}_2(t)\underline{x}_2(t) + \underline{G}_2(t)r(t) \quad (3.56a)$$

$$y(t) = \underline{G}_2(t)\underline{x}_2(t). \quad (3.56b)$$

The initial conditions are zero-mean Gaussian random variables.

$$E[\underline{x}_1(T_1)\underline{x}_1^T(T_1)] = \underline{P}_0 \quad (3.57a)$$

$$E[\underline{x}_2(T_1)\underline{x}_2^T(T_1)] = \underline{0} \quad (3.57b)$$

The entire system can be rewritten in canonical state-variable form.

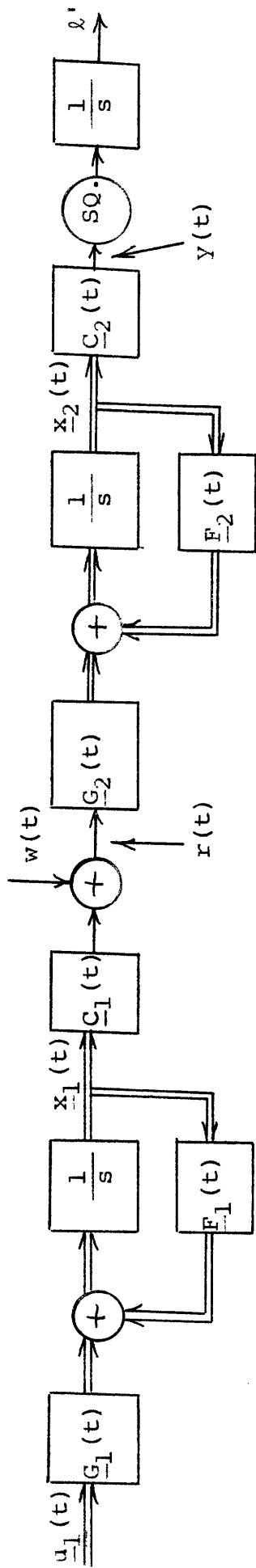


Figure 3.3, Suboptimum Receiver: State-Variable Model

$$\frac{d}{dt} \begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix} = \begin{bmatrix} \underline{F}_1(t) & \underline{0} \\ \underline{G}_2(t)\underline{G}_1(t) & \underline{F}_2(t) \end{bmatrix} \begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix} + \begin{bmatrix} \underline{G}_1(t) & \underline{0} \\ \underline{0} & \underline{G}_2(t) \end{bmatrix} \begin{bmatrix} \underline{u}_1(t) \\ w(t) \end{bmatrix} \quad (3.58a)$$

$$E \left\{ \begin{bmatrix} \underline{x}_1(T_1) \\ \underline{x}_2(T_1) \end{bmatrix} \quad [\underline{x}_1^T(T_1), \underline{x}_2^T(T_1)] \right\} = \begin{bmatrix} \underline{P}_0 & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \quad (3.58b)$$

$$\text{Defining } \underline{x}(t) = \begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix} \quad (3.59a)$$

$$\underline{F}(t) = \begin{bmatrix} \underline{F}_1(t) & \underline{0} \\ \underline{G}_2(t)\underline{G}_1(t) & \underline{F}_2(t) \end{bmatrix} \quad (3.59b)$$

$$\underline{G}(t) = \begin{bmatrix} \underline{G}_1(t) & \underline{0} \\ \underline{0} & \underline{G}_2(t) \end{bmatrix} \quad (3.59c)$$

$$\underline{c}(t) = [\underline{0}, \underline{c}_2(t)] \quad (3.59d)$$

$$\underline{u}(t) = \begin{bmatrix} \underline{u}_1(t) \\ \underline{w}(t) \end{bmatrix} \quad (3.59e)$$

$$\underline{\Sigma}_0 = \begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \cdot \quad (3.59f)$$

Then,

$$\dot{\underline{x}}(t) = \underline{F}(t)\underline{x}(t) + \underline{G}(t)\underline{u}(t) \quad (3.60a)$$

$$\underline{y}(t) = \underline{C}(t)\underline{x}(t) \quad (3.60b)$$

$$E[\underline{x}(t)\underline{x}^T(t)] = \underline{\Sigma}_0 \quad (3.60c)$$

and the evaluation of the Fredholm determinants is carried out exactly as before, using Eq. 3.46.

F. Summary

In this chapter, we calculated the semi-invariant moment generating function $\mu(s)$ for the problem of detecting Gaussian signals in Gaussian noise. We considered both optimum and sub-optimum receivers. We expressed $\mu(s)$ in terms of the Fredholm determinant, which can be obtained from the solution to a related optimum linear filtering problem. For the important class of random processes which can be modeled via state variables, we obtained a straightforward technique for computing $\mu(s)$.

IV. APPROXIMATIONS TO THE ERROR PROBABILITY FOR M-ARY ORTHOGONAL DIGITAL COMMUNICATION

In the previous chapters we have been concerned with the performance of binary detection systems on a bit-by-bit basis. This model is appropriate to many radar and sonar problems. However, many digital communication systems have more than two hypotheses. In this chapter, we shall discuss M-ary orthogonal communication systems. By restricting ourselves to this class of problems, we shall be able to relate the performance of the M hypothesis problem to the function $\mu(s)$ for a related simple binary problem. Therefore, the techniques of Chapter III are directly applicable to performance calculations for M-ary digital communication.

A. Model

The problem which we shall consider is the M hypothesis generalization of the symmetric binary problem discussed in Chapter III, Section C-2. We receive a sample function from one of M Gaussian random processes in additive white Gaussian noise,

$$\begin{aligned} H_i: r(t) &= s_i(t) + w(t), & 0 \leq t \leq T \\ & & i = 1, 2, \dots, M \end{aligned} \quad (4.1)$$

We assume that the eigenvalues of the processes $s_i(t)$ are identical and that their eigenfunctions are orthogonal. We also assume that we have an infinite bandwidth available.

One common system which fits into this model is a M-ary frequency shift system operating over a non-dispersive fading channel.

The transmitted signal is

$$f_i(t) = \sqrt{2} A(t) \cos[(\omega_c + i\omega_\Delta)t + \theta(t)]$$

$$0 \leq t \leq T$$

$$i = 1, 2, \dots, M \quad (4.2)$$

where $A(t)$ and $\theta(t)$ are known amplitude and phase modulations.

This signal is transmitted over a time-selective fading channel so that the received signal on hypothesis H_i is

$$r(t) = \sqrt{2} B(t) \cos[(\omega_i + (i-1)\omega_\Delta)t + \theta(t) + \phi(t)] + w(t),$$

$$0 \leq t \leq T, \quad (4.3)$$

or

$$r(t) = \sqrt{2} a_c(t) b_c(t) \cos[(\omega_c + (i-1)\omega_\Delta)t]$$

$$+ \sqrt{2} a_s(t) b_s(t) \sin[(\omega_c + (i-1)\omega_\Delta)t] + w(t),$$

$$0 \leq t \leq T, \quad (4.4)$$

where $b_c(t)$ and $b_s(t)$ are zero-mean, statistically independent Gaussian random processes. Hence, $B(t)$ is a Rayleigh distributed random process and $\theta(t)$ is a uniformly distributed random process. We assume that the frequency shift ω_Δ is chosen significantly large so that the orthogonality assumption of the previous paragraph is

essentially satisfied.

The related simple binary problem which we consider is,

$$\begin{aligned} H_1: r(t) &= s_1(t) + w(t) \\ & 0 \leq t \leq T \\ H_0: r(t) &= w(t) \end{aligned} \quad (4.5)$$

Note that due to the symmetry of the problem, we could equally well have used any one of the M signals for H_1 . The techniques of Chapter III enable us to compute $\mu(s)$ for this simple binary problem. The performance then follows upon using the bounds and approximations in Chapter II.

B. Bounds on the Probability of Error

There are several different bounds on the error probability. The key to all these performance calculations is the "exponent-rate" curve or "system reliability function." Space does not permit a detailed development of this topic here. Instead, we refer the reader to derivations which are available elsewhere [30, 49-51].

The simplest upper bound, analogous to the Chernoff bound of Eqs. 2.12 and 2.13, was originally derived by Yudkin [49]. A more readily available reference is Viterbi [51]. Their result is,

$$\Pr[\epsilon] \leq \exp[-T C E(R/C)] \quad (4.6)$$

where

$$R = \frac{\log_2 M}{T} \quad (4.7)$$

and

$$C = \frac{P}{N_0 \ln 2} \quad (4.8a)$$

$$= \frac{E}{N_0 T \ln 2} \quad (4.8b)$$

C is the infinite bandwidth capacity of the additive Gaussian noise channel.

Kennedy obtains both upper and lower bounds [50] of the form

$$K_1 \exp[-T C E(R/C)] \leq P_n[\epsilon] \leq K_2 \exp[-T C E(R/C)]. \quad (4.9)$$

A number of alternate expressions for the coefficients have been obtained. The important feature for many problems is that the behavior of the error probability is dominated by the exponential factor involving the exponent-rate curve $E(R/C)$. We devote the next section to a discussion of this function.

C. Exponent-Rate Curves

Kennedy has derived expressions for the exponent-rate curve for the problem which we have described. We state his results, then relate them to $\mu(s)$ for the simple binary problem given in Eq. 4.5. Both forms of the results are expressed in terms of infinite sums similar to those we encountered in Section B of Chapter III, or equivalently, in terms of Fredholm determinants.

Kennedy derived his results in terms of the function,

$$\gamma(s) \triangleq -\frac{N_0}{E} \sum_{i=1}^{\infty} [\ln(1-s\lambda_i/N_0) + s \ln(1+\lambda_i/N_0)]$$

$$s < 0 \quad (4.10)$$

Comparing Eq. 4.10 with Eq. 3.21, we observe that

$$\gamma(s) = \frac{N_0}{E} \mu(1+s), \quad (4.11)$$

where the function $\mu(s)$ in Eq. 4.11 is the semi-invariant moment generating function for the simple binary problem of Eq. 4.5

Making this substitution, Kennedy's Eqs. 5.12 and 5.13 become

$$R_{\text{crit}}/C = \frac{N_0}{E} \left[\frac{1}{2} \mu' \left(\frac{1}{2} \right) - \mu \left(\frac{1}{2} \right) \right] \quad (4.12)$$

$$E(R/C) = -\frac{N_0}{E} 2\mu \left(\frac{1}{2} \right) - R/C \quad (4.13)$$

for $0 \leq R \leq R_{\text{crit}}$

$$E(R/C) = -\frac{N_0}{E} [\mu(s) + (1-s)\mu'(s)] \quad (4.14)$$

where s is implicitly defined by,

$$R/C = - \frac{N_0}{E} [\dot{\mu}(s) - s\ddot{\mu}(s)] \quad (4.15a)$$

$$\text{for } R_{\text{crit}} \leq R \leq \frac{N_0}{E} (1) C$$

$$\text{or equivalently for } \frac{1}{2} \leq s \leq 1$$

In passing, we note the similarity between Eqs. 4.15 and 4.14 and the exponents in Eqs. 2.12 and 2.13.

We can immediately sketch the exponent rate curve and label the important points (see Fig. 4.1). We also can obtain numerical results for the same classes of problem that we discussed in Chapter III.

$\mu(s)$ depends on the channel and modulation. The capacity of the overall channel which includes the transmitter, fading channel, and optimum receiver is given by,

$$C_{\text{system}}/C = \frac{N_0}{E} \dot{\mu}(1). \quad (4.15b)$$

We now give an alternate parametric expression for the exponent-rate curve which is closely analogous to the expressions that arise in Gallager's random coding theorem [30]. We include it for the benefit of those readers who may be more familiar with this formulation of the exponent-rate curve.

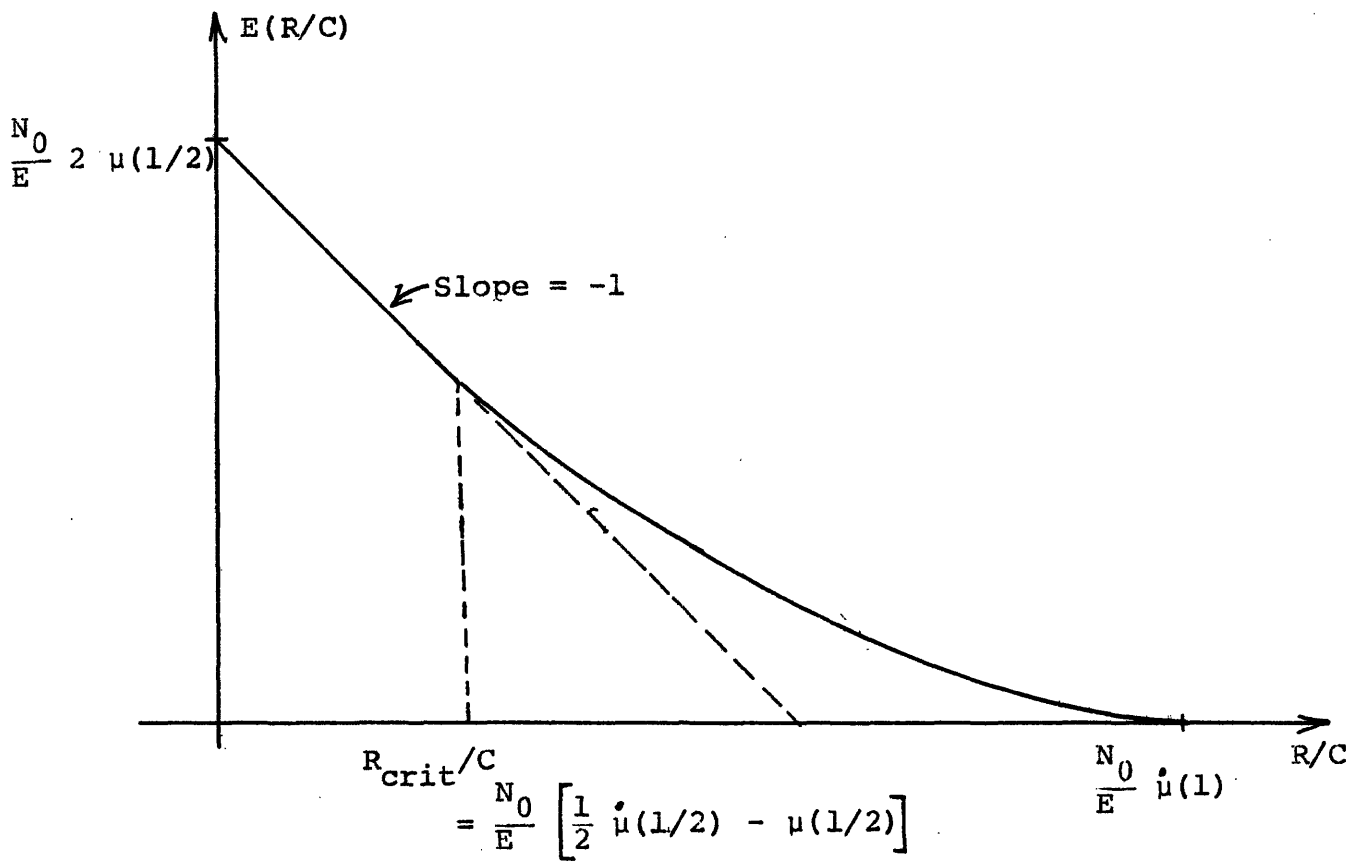


Figure 4.1. Exponent-Rate Curve for Fading Channel

$$E(R/C) = \max_{0 \leq \rho \leq 1} [E_o(\rho) - \rho R/C], \quad (4.16)$$

where in Kennedy's notation

$$E_o(\rho) = -(1+\rho)\gamma\left(\frac{-\rho}{1+\rho}\right). \quad (4.17)$$

Thus, using Eq. 4.11,

$$E_o(\rho) = -(1+\rho)\frac{N_0}{E} \left(\frac{1}{1+\rho}\right) \quad (4.18)$$

Using Eq. 3.20c, Eq. 4.18 becomes

$$E_o(\rho) = -\frac{N_0}{E} \left[\rho \ln D_{\mathcal{F}}\left(\frac{1}{N_0}\right) - (1+\rho) \ln D_{\mathcal{F}}\left(\frac{\rho}{1+\rho} \frac{1}{N_0}\right) \right] \quad (4.19)$$

which is Viterbi's Eq. 25 rewritten slightly.

Viterbi proceeded from Eq. 4.19 and calculated exponent-rate curves assuming long time intervals and stationary processes. The techniques we have developed in Chapter III enable us to treat the much wider class of problems where the processes may be non-stationary and where the time intervals are arbitrary.

V. EXAMPLES

In this chapter, we shall consider a number of examples illustrating the application of the techniques we have developed to the calculation of error probabilities for some simple detection problems. We shall choose examples from all three levels of detection problems: known signal, signal with a finite number of random parameters, and random signals. Whenever possible, we include the results of exact calculations for comparison purposes. However, the real power of our technique lies in those problems for which we do not have other techniques available, and we shall devote most of this chapter to examples taken from this category.

As an introductory exercise, we apply our error approximations to the known signal problem. We then consider several slow Rayleigh fading diversity examples. For these problems, we compare our results with those obtained by others and demonstrate simple examples where other performance criteria lead to misleading results. As an example of the analysis of a suboptimum receiver, we analyze a square-law receiver for the random phase channel. Finally, we consider a number of simple binary and symmetric binary random process detection problems. We consider a simple signal optimization problem, illustrate how our results can be simplified for the case of long time intervals, and show how our results can be applied to dispersive channels which can be modeled by means of a tapped delay line. For the symmetric binary, bandpass problem we obtain upper and lower bounds on our approximations.

A. General Binary Problem, Known Signals, Colored Noise

This problem is the simplest of the waveform detection problems. It has been studied for over twenty-five years. For a detailed discussion and list of references, see Van Trees [42]. The receiver is linear and the test statistic is Gaussian. Thus the calculation of error probabilities is straightforward, so that the only reason for applying our approximation techniques to this problem is as an exercise.

The mathematical model is as follows:

$$\begin{aligned}
 H_1: r(t) &= m_1(t) + n_c(t) + w(t) \\
 & \qquad \qquad \qquad T_i \leq t \leq T_f \\
 H_2: r(t) &= m_2(t) + n_c(t) + w(t)
 \end{aligned} \tag{5.1}$$

Here $m_1(t)$ and $m_2(t)$ are the known signals, the random "signal" components $n_c(t)$ are identical and represent the colored component of the additive Gaussian noise, and $w(t)$ is white Gaussian noise of spectral density $N_0/2$.

Substituting into Eq. 3.16, we find that

$$\mu(s) = - \frac{s(1-s)}{2} d^2 \tag{5.2}$$

where

$$d^2 = \frac{2}{N_0} \int_{T_i}^{T_f} \left[[m_2(t) - m_1(t)] - \int_{T_i}^{T_f} h_o(t,u) [m_2(u) - m_1(u)] du \right]^2 dt. \tag{5.3}$$

$h_o(t,u)$ is the MMSE linear estimator for estimating $n_c(t)$ given $r(u) = n_c(u) + w(u)$, $T_i \leq u \leq t$. Alternate expressions for d^2 follow from our realizable whitening filter results in Appendix D.

$$d^2 = \int_{T_i}^{T_f} \left\{ \int_{T_i}^{T_f} h_w(t,u) [m_2(u) - m_1(u)] du \right\}^2 dt \quad (5.4)$$

where

$$h_w(t,u) = \delta(t-u) - h_o(t,u) \quad (5.5)$$

is the realizable whitening filter.

$$\dot{u}(s) = d^2 \left(s - \frac{1}{2} \right) = \gamma \quad (5.6)$$

$$\ddot{u}(s) = d^2 \quad (5.7)$$

$$u^{(n)}(s) = 0 \text{ for } n \geq 3. \quad (5.8)$$

From Eq. 5.6, we can explicitly solve for s in terms of γ . Substituting Eq. 2.26 in Eqs. 2.10 and 2.11 and simplifying, we have

$$\begin{aligned} \Pr[\epsilon | H_1] &= \phi \left[- \left(\frac{d}{2} + \frac{\gamma}{d} \right) \right] \\ &= 1 - \phi \left(\frac{d}{2} + \frac{\gamma}{d} \right) \end{aligned} \quad (5.9)$$

$$\Pr[\varepsilon|H_2] = 1 - \Phi\left(\frac{d}{2} - \frac{Y}{d}\right). \quad (5.10)$$

Thus, in this case, our approximations, in fact, yield the exact error expression, since for this problem the tilted probability density $p_{\ell_s}(L)$ is Gaussian. Fig. 5.1 illustrates the exact expression for the symmetric case where $\gamma = 0$, as well as the asymptotic approximation Eq. 2.23 and the bound Eq. 2.12 or Eq. 2.13. This curve has appeared previously as Fig. 2.36 of [35] and Fig. 2.10 of [52].

B. Slow Rayleigh Fading With Diversity

This is another case in which analytical results are available for comparison. It represents an important example of the second level of difficulty in the hierarchy of detection problems.

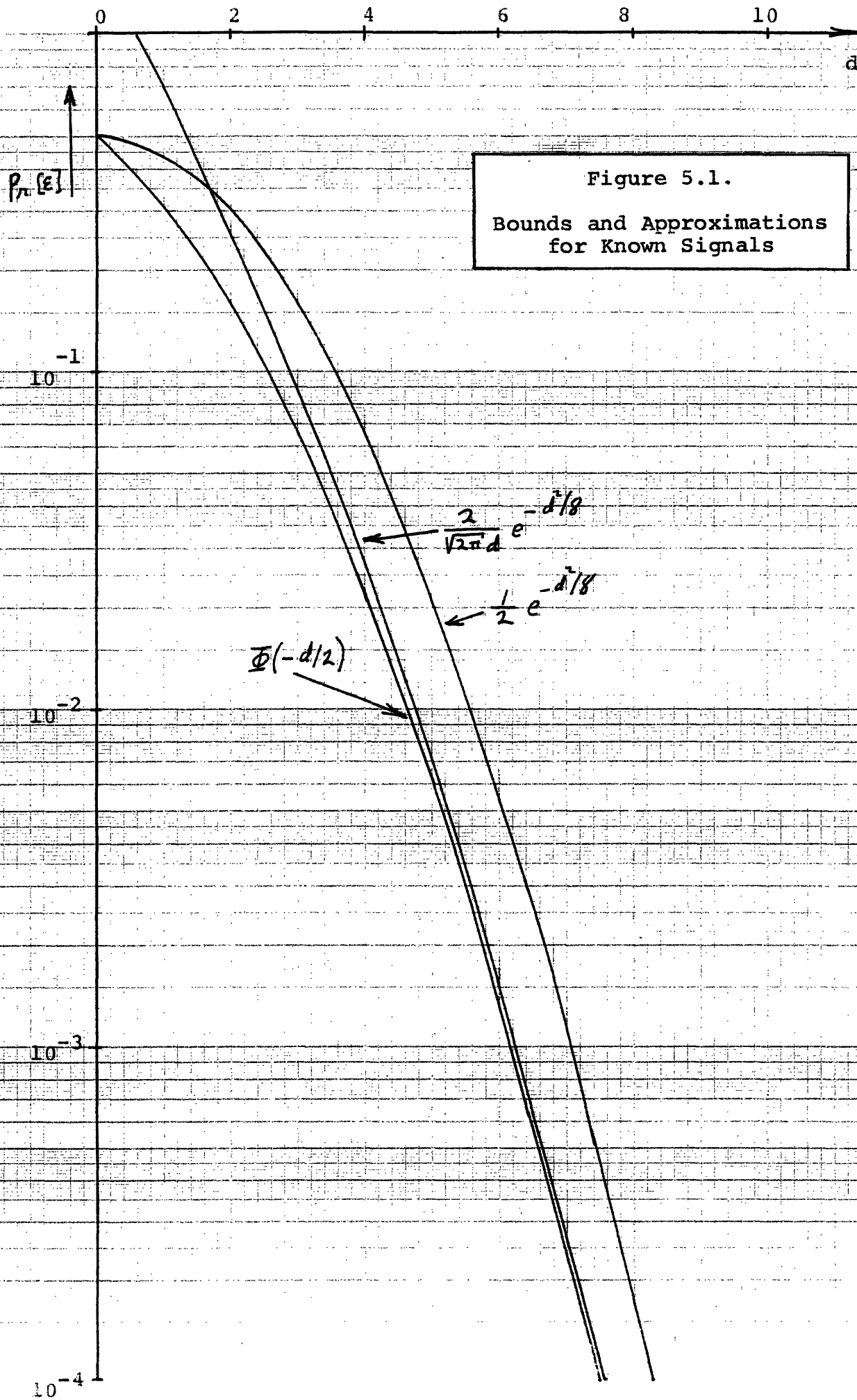
1. Simple Binary Problem, Bandpass Signals. On hypothesis H_2 , a known bandpass signal

$$s(t) = \sqrt{2} f(t) \cos[\omega_c t + \phi(t)], \quad 0 \leq t \leq T \quad (5.11)$$

of energy

$$E_t = \int_0^T s^2(t) dt = \int_0^T f^2(t) dt \quad (5.12)$$

is transmitted over each of N statistically independent fading channels. The received signal in each channel is of the form



$$H_2: r(t) = v\sqrt{2} f(t)\cos[w_c t + \phi(t) + \theta] + w(t), \quad 0 \leq t \leq T, \quad (5.13)$$

where v is a Rayleigh random variable with

$$E[v^2] = 2\sigma^2, \quad (5.14)$$

θ is a uniform random variable, $0 \leq \theta < 2\pi$, and $w(t)$ is white Gaussian noise of spectral density $N_0/2$. On hypothesis H_1 , no signal is sent, hence

$$H_1: r(t) = w(t) \quad 0 \leq t \leq T. \quad (5.15)$$

Therefore, the average received energy in each channel is

$$\bar{E}_r = 2\sigma^2 E_t \quad (5.16)$$

Writing Eq. 5.13 in terms of quadrature components, we have

$$H_1: r(t) = a_c \sqrt{2} f(t)\cos[w_c t + \phi(t)] \\ + a_s \sqrt{2} f(t)\sin[w_c t + \phi(t)] + w(t), \quad 0 \leq t \leq T, \quad (5.17)$$

where a_c and a_s are statistically independent zero-mean Gaussian random variables with variance σ^2 . A suitable orthogonal expansion for each channel output is given by Eq. 5.17. Therefore, this problem is equivalent to a $2N$ dimensional vector problem.

$$\begin{aligned} H_2: r_i &= s_i + w_i \\ H_1: r_i &= w_i \quad i = 1, 2, \dots, 2N \end{aligned} \quad (5.18)$$

where

$$E[s_i] = E[w_i] = 0, \quad (5.19a)$$

$$\text{Var}[s_i] = \bar{E}_r/2 = \sigma^2 E_t \quad (5.19b)$$

$$\text{Var}[w_i] = N_0/2 \quad (5.19c)$$

Using Eq. 3.21, we obtain,

$$\begin{aligned} \mu(s) &= N[(1-s)\ln(1+\bar{E}_r/N_0) \\ &\quad - \ln(1+(1-s)\bar{E}_r/N_0)], \end{aligned} \quad (5.20)$$

$$\dot{\mu}(s) = -N \ln(1+\bar{E}_r/N_0) + \frac{\bar{E}_r/N_0}{1 + (1-s)\bar{E}_r/N_0} \quad (5.21)$$

$$\mu(s) = N \left[\frac{\bar{E}_r/N_0}{1 + (1-s)\bar{E}_r/N_0} \right]^2. \quad (5.22)$$

Substitution of Eq. 5.20 through Eq. 5.22 into Eqs. 2.32 and 2.33 yields the approximate receiver operating characteristic (ROC) of Fig. 5.2*. For larger values of N , the exact and approximate curves are practically identical.

An equivalent way to work this problem is to consider the low pass problem.

$$H_2: r'(t) = a_c f(t) + w(t)$$

$$H_1: r'(t) = w(t), \quad 0 \leq t \leq T, \quad (5.23)$$

and then use Eq. 3.36.

2. Symmetric Binary Problem, Bandpass Signals

In the communications context, we assume that one of two equally probable, orthogonal bandpass waveforms of energy E_t is transmitted over the fading channel modeled in the previous example. A frequency-shift keying system is representative of the physical systems that fit into this model. We assume that there are N channels operating in space, time, or frequency diversity. $\mu(s)$ follows immediately from Eq. 3.28 and Eq. 5.20.

*The calculations for Fig. 5.2 were performed by A. B. Baggeroer and were used to plot Fig. 2.42 in Ref. [52].

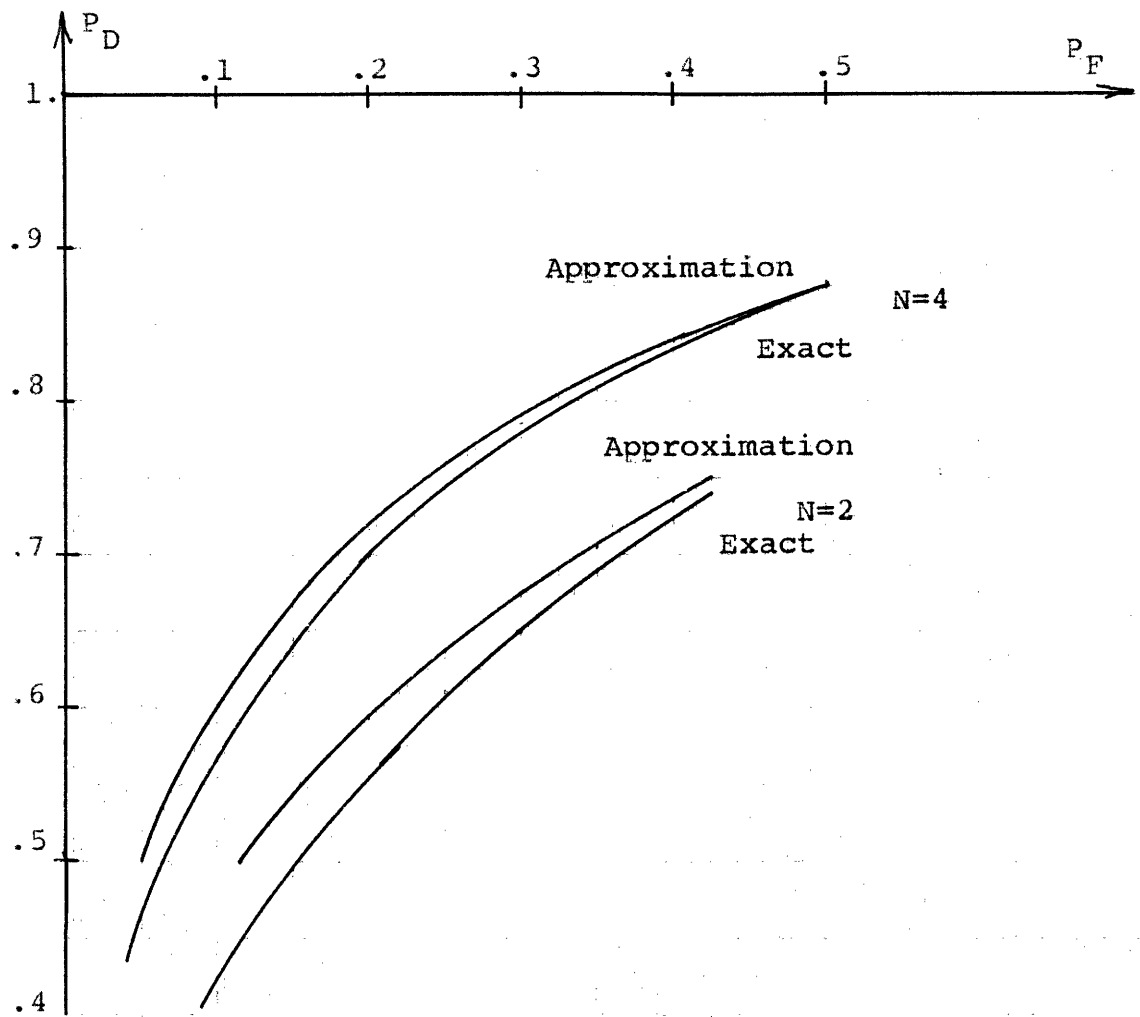


Figure 5.2.
Approximate Receiver Operating Characteristic,
Slow Rayleigh Fading

$$\begin{aligned}
\mu_{SY}(s) &= \mu_{SI}(s) = \mu_{SI}(1-s) \\
&= N[\ln(1 + \bar{E}_r/N_0) - \ln(1+s \bar{E}_r/N_0) \\
&\quad - \ln(1 + (1-s)\bar{E}_r/N_0)] \tag{5.24}
\end{aligned}$$

If the threshold is chosen to yield minimum total probability of error, then $\gamma = 0$. From Eq. 3.29b, this implies $s = \frac{1}{2}$. Then,

$$\mu\left(\frac{1}{2}\right) = N[\ln(1 + \bar{E}_r/N_0) - 2\ln(1 + \frac{1}{2} \bar{E}_r/N_0)], \tag{5.25a}$$

$$\dot{\mu}\left(\frac{1}{2}\right) = 0, \tag{5.25b}$$

and

$$\ddot{\mu}\left(\frac{1}{2}\right) = \frac{N}{\left(\frac{1}{2} + N_0/\bar{E}_r\right)^2}. \tag{5.25c}$$

The first order approximation to the probability of error is

$$\Pr[\epsilon] \approx \phi\left(-\frac{1}{2} \sqrt{\ddot{\mu}\left(\frac{1}{2}\right)}\right) \exp\left(\mu\left(\frac{1}{2}\right) + \frac{1}{8}\ddot{\mu}\left(\frac{1}{2}\right)\right). \tag{5.26}$$

The exact expression for the error probability is given by [53].

$$P_r[\epsilon] = \frac{\bar{E}_r/N_0}{2(2+\bar{E}_r/N_0)} \sum_{m=N}^{\infty} \binom{2m}{m} \left[\frac{1 + \bar{E}_r/N_0}{2 + \bar{E}_r/N_0} \right]^m \quad (5.27)$$

Eqs. 5.26 and 5.27 are plotted in Fig. 5.3. We observe there is very little difference between the exact expression and the first order approximation. In fact, for $N = 16$ and larger, we are unable to see any difference in their graphs. In Section D-5 of this chapter, we shall show that our approximation is in fact a lower bound on $P_r[\epsilon]$ for this problem, and in Section D-6 we shall make a quantitative statement about its accuracy.

3. The Effect of Neglecting Higher Order Terms in the Asymptotic Expansion. In addition to computing the exact expression for the error probability and the first order approximation given in Eq. 5.26, we have made a calculation retaining the first four terms of the asymptotic approximation (all the terms given in Eq. 2.26). Even in the extreme case when $N = 1$, the difference between the exact and approximate expressions is imperceptible when presented graphically on the scale of Fig. 5.3.

4. Optimum Diversity. A further extension of this example is the question of optimum diversity. That is, given a fixed total transmitted energy,

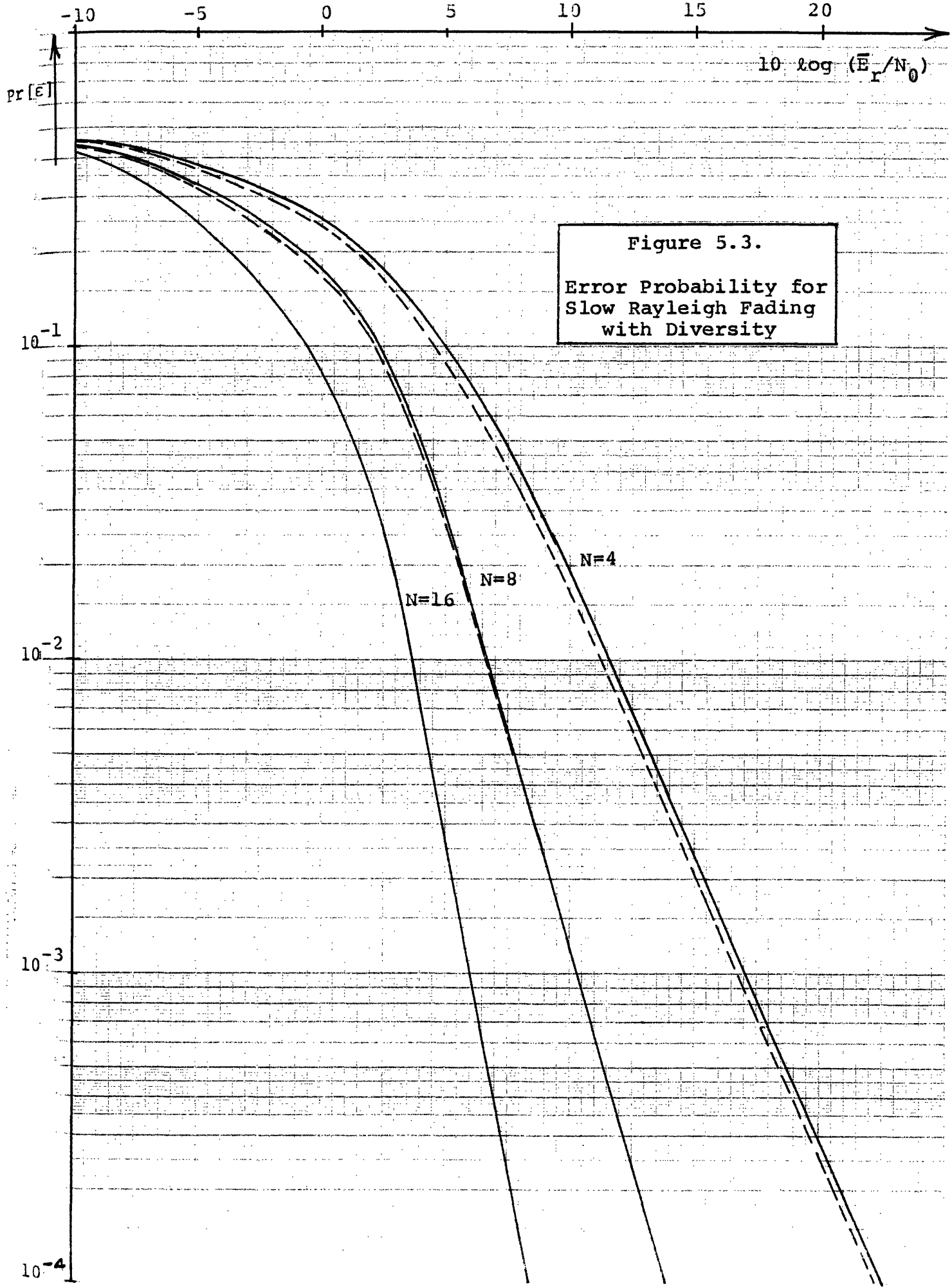


Figure 5.3.
Error Probability for
Slow Rayleigh Fading
with Diversity

$$E_{tT} = N E_t, \quad (5.28)$$

what should N be to minimize the probability of error? We assume as before that the strengths of all the diversity channels is the same. Then the condition given by Eq. 5.28 is equivalent to the constraint,

$$\begin{aligned} \bar{E}_{rT} &= 2N \sigma^2 E_t \\ &= N \bar{E}_r \end{aligned} \quad (5.29)$$

Pierce originally solved this problem numerically using Eq. 5.52 [53-55]. He also solved it by minimizing the Chernoff bound [56]. More recently, Wozencraft and Jacobs actually attached Chernoff's name to this solution [35].

Using the exact expression of Eq. 5.27, Pierce found that for large total signal-to-noise ratio, the optimum signal-to-noise ratio in each diversity channel approaches 3.0700. Minimizing the Chernoff bound given by Eq. 2.12 or Eq. 2.13,

$$\Pr[\epsilon] \leq \exp[\mu(\frac{1}{2})], \quad (5.30)$$

leads to the conclusion that the optimum signal-to-noise ratio per channel is 3.0700. for all values of total signal-to-noise ratio. This is equivalent to minimizing $\mu(\frac{1}{2})$, the dominant term in our approximate error expression.

We have determined the optimum value of N using the approximate error expression given by Eq. 5.26. In Fig. 5.4 we plot

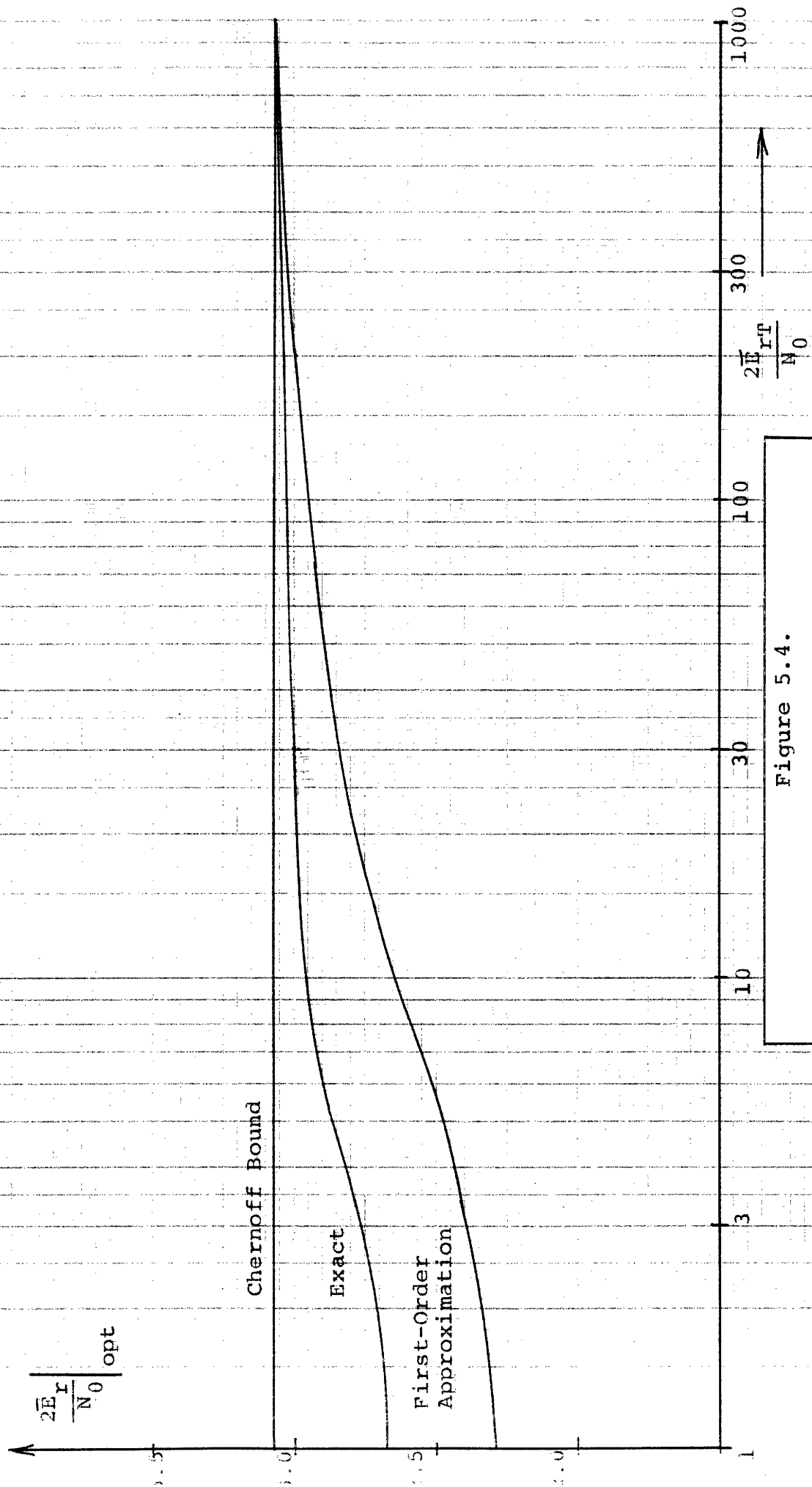


Figure 5.4.
Optimum Signal-to-Noise Ratio per Channel
Slow Rayleigh Fading

Figure 5.4. Optimum Signal-to-Noise Ratio per Channel
Slow Rayleigh Fading

the optimum average received signal-to-noise energy ratio per diversity channel as a function of the total average received signal-to-noise ratio for these three different criteria.

A couple of comments are in order regarding Fig. 5.4. First, we can achieve the optimum solution only for discrete values of total signal-to-noise ratio. In particular, the optimum can be achieved only for those values of $2\bar{E}_{rT}/N_0$ which are integer multiples of the optimum ratio per channel. The curves in Fig. 5.4 are smooth curves drawn through these points.

Second, the minimum is not particularly sensitive to the exact value of signal-to-noise ratio per channel. This is demonstrated graphically in Fig. 7.44 of Ref. [35]. Therefore, the probability of error which results from using any one of the three criteria is practically the same. The only difference in the two approximate methods is that the probability of error computed from the Chernoff bound is not as good an approximation to Eq. 5.27 as is Eq. 5.26.

It is also instructive to compare these results with those obtained using output signal-to-noise ratio d^2 as the performance criterion. Straightforward calculations yields

$$d^2 = \frac{1}{2N} \left(\frac{2\bar{E}_{rT}}{N_0} \right)^2 \quad (5.31)$$

Hence, the maximum d^2 occurs for $N = 1$, which does not yield the minimum probability of error, when $2\bar{E}_{rT}/N_0 > 3$.

5. The Inadequacy of the Bhattacharyya Distance. In Eq. 2.21, we saw that $-\mu(\frac{1}{2}) = B$, the Bhattacharyya Distance. Thus,

the above example is one for which using the Bhattacharyya distance as a performance measure leads to an optimum diversity system, while the "divergence" or d^2 does not. However, a slight modification of this example produces a situation in which the Bhattacharyya distance is not a good performance measure.

We assume that we have available two independent slow Rayleigh fading channels, which are modeled as described at the beginning of this section. On hypothesis H_2 , a signal of energy E is transmitted; on H_1 no signal is transmitted. Again, there is additive white Gaussian noise in each channel. We wish to investigate the consequences of dividing the available energy between the two channels. For example, we might like to achieve the optimum division of energy.

We assume the total average received energy is \bar{E}_r , with average received energy \bar{E}_1 and \bar{E}_2 in the two channels, respectively.

$$\bar{E}_1 = \alpha \bar{E}_r \quad (5.32a)$$

$$\bar{E}_2 = (1-\alpha)\bar{E}_r, \quad \text{for } 0 \leq \alpha \leq 1. \quad (5.32b)$$

Note that this problem is symmetric about $\alpha = 0.5$ so that it is sufficient to consider the range $0 \leq \alpha \leq \frac{1}{2}$.

The noise power density in each channel is $N_0/2$. Then,

$$\mu(s) = (1-s) \ln \left[1 + \frac{\bar{E}_r}{N_0} \left(1 + \alpha \left(1 - \alpha \frac{\bar{E}_r}{N_0} \right) \right) \right]$$

$$- \ln \left[1 + (1-s) \frac{\bar{E}_r}{N_0} \left\{ 1 + (1-s)\alpha(1-\alpha) \frac{\bar{E}_r}{N_0} \right\} \right] . \quad (5.33)$$

Figs. 5.5a, 5.5b, and 5.5c show Eq. 5.33 for several values of $2\bar{E}_r/N_0$, with α as a parameter. For $2\bar{E}_r/N_0 = 5$ and 10 , the $\mu(s)$ curves cross, and, hence, $\mu(\frac{1}{2})$ is not representative of the ordering of the curves for all other values of s . Therefore, it appears to be worth the extra effort to use $\mu(s)$ as our performance index for non-symmetrical problems.

6. Inadequacy of the Kullback-Leibler Information Number as a Performance Measure. Recently, the Kullback-Leibler information number

$$I(H_1:H_2) = -\dot{\mu}(0) \quad (5.34)$$

has been advocated as a performance measure of more universal applicability than the divergence for the simple binary problem, particularly when it is desired to use a Neyman-Pearson criterion [57]. For a simple counterexample, refer to Fig. 5.5b.

Recall from Chapter II that the tangent line to the $\mu(s)$ curve at $s = s^*$ intersects the line $s = 0$ at the value $\mu(s^*) - s^* \dot{\mu}(s^*)$ which is the exponent of our error approximations and bounds on P_F . Similarly, the tangent line intersects the line $s = 1$ at $\mu(s^*) + (1-s^*) \dot{\mu}(s^*)$ which is the exponent for P_M . Consider the case $2\bar{E}_r/N_0 = 5$. If we require the exponent for P_F to be -0.30 , for example, then the proper choice of α to minimize P_M is $\alpha = 0$. However, the choice which maximizes $-\dot{\mu}(0)$ is $\alpha = \frac{1}{2}$. Thus, again

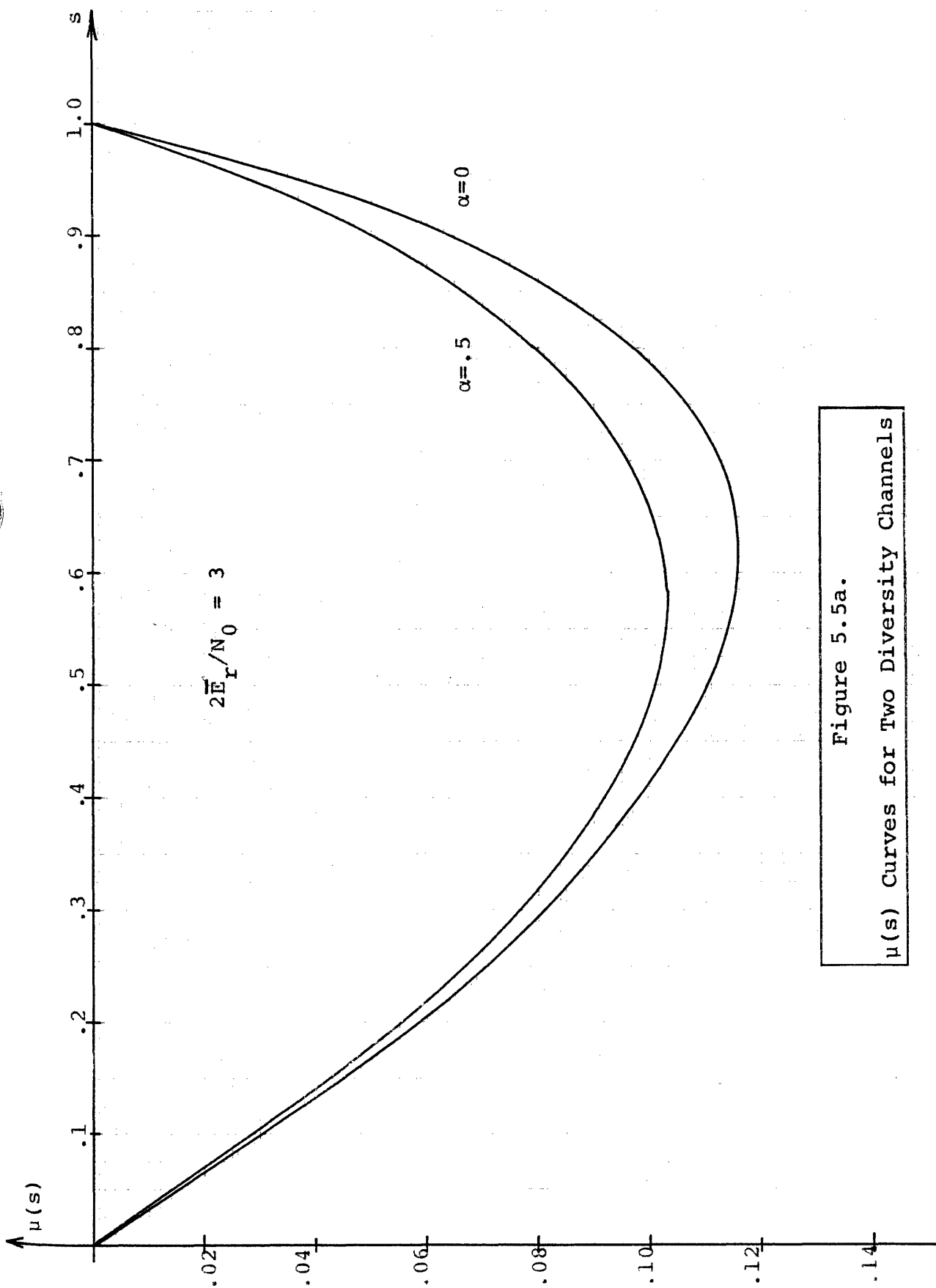


Figure 5.5a.
 $\mu(s)$ Curves for Two Diversity Channels

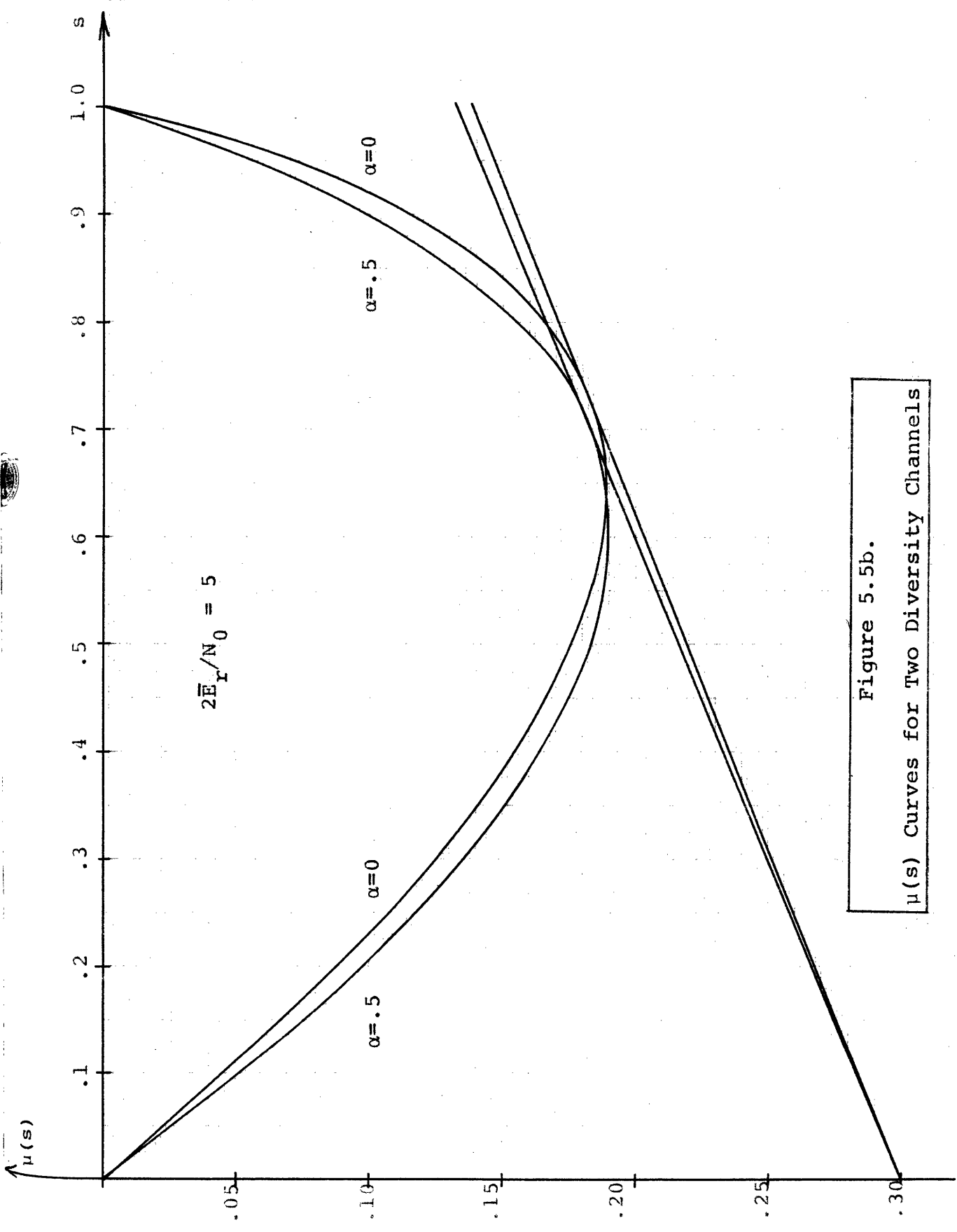


Figure 5.5b.
 $\mu(s)$ Curves for Two Diversity Channels

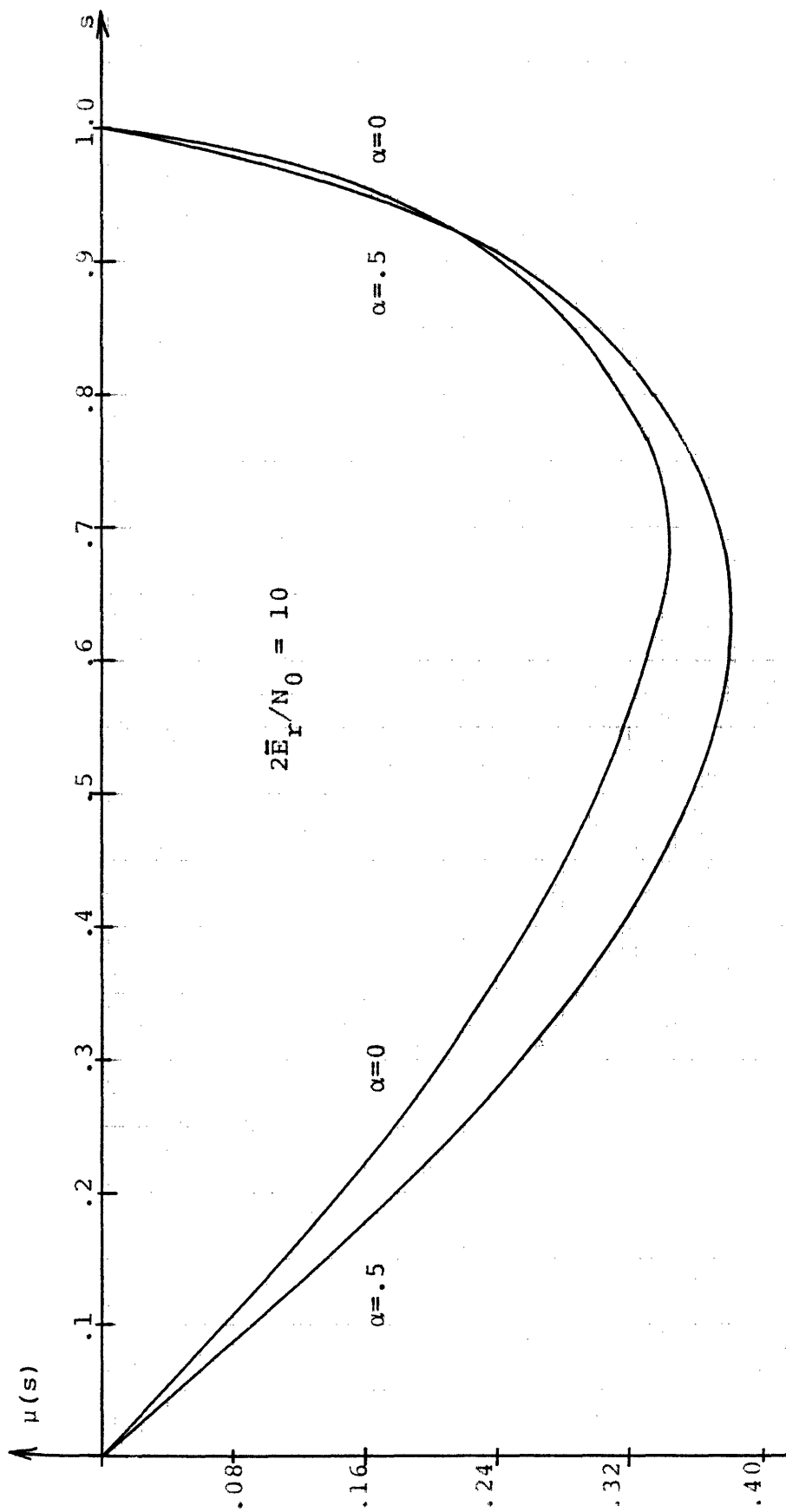


Figure 5.5c.
 $\mu(s)$ Curves for Two Diversity Channels

the extra effort required for our performance technique is justified.

C. Random Phase Angle

In this section, we consider a second example where there is a finite set of random parameters. We assume that the received signal is a band-pass signal about some carrier frequency. The random parameter is the phase of the carrier. This phase uncertainty might arise in practice from instability in the carrier oscillator at either the transmitter or receiver. This problem has been studied in great detail [42,58] and the error probabilities have been computed numerically.

1. Simple Binary, No Diversity. The mathematical model for the problem is

$$\begin{aligned} H_2: r(t) &= \sqrt{2E_r} f(t) \cos(\omega_c t + \phi(t) + \theta) + w(t) \\ H_1: r(t) &= w(t) \end{aligned} \quad (5.35)$$

$$0 \leq t \leq T$$

$f(t)$ and $\phi(t)$ are known waveforms which contain no frequencies comparable to ω_c ; $w(t)$ is white Gaussian noise of spectral height $N_0/2$; and θ is a uniform random variable $0 \leq \theta < 2\pi$. The log-likelihood ratio is

$$\lambda = \ln \Lambda(r(t)) = \ln I_0 \left(\frac{2\sqrt{E_r}}{N_0} \sqrt{L_c^2 + L_s^2} \right) - \frac{E_r}{N_0} \quad (5.36)$$

where $I_0(\cdot)$ is a modified Bessel function of the first kind [59],
and

$$L_c = \sqrt{2} \int_0^T r(t)f(t)\cos[\omega_c t + \phi(t)]dt \quad (5.37a)$$

$$L_s = \sqrt{2} \int_0^T r(t)f(t)\sin[\omega_c t + \phi(t)]dt. \quad (5.37b)$$

An equivalent test statistic is

$$\ell' = L_c^2 + L_s^2 \quad (5.38)$$

since both the square root and $\ln I_0(\cdot)$ are monotone functions.
However, since ℓ' is not the logarithm of the likelihood ratio,
we must treat Eq. 5.38 as a "suboptimum" receiver for purposes of
analyzing the performance via our techniques.

For this problem we can compute $P_F = \Pr[\epsilon|H_1]$ exactly
since on hypothesis H_1 , L_c and L_s are statistically independent
Gaussian random variables.

$$E[L_c|H_1] = E[L_s|H_1] = 0 \quad (5.39)$$

$$\text{Var}[L_c|H_1] = \text{Var}[L_s|H_1] = \frac{N_0}{2} \quad (5.40)$$

Therefore, $\ell' = L_c^2 + L_s^2$ has an exponential distribution, and

$$\begin{aligned}
 P_F &= \Pr[\ell' > \gamma | H_1] \\
 &= \exp\left[-\frac{\gamma}{N_0}\right], \quad 0 \leq \gamma \leq \infty.
 \end{aligned} \tag{5.41}$$

Note that $\ell' \geq 0$, hence we restrict γ to be positive. However, $P_M = \Pr[\varepsilon | H_2]$ is not so easy, for on H_2 , L_c and L_s are Gaussian random variables only when they are conditioned on θ . Then,

$$E[L_c | H_2, \theta] = \sqrt{E_r} \cos \theta \tag{5.42a}$$

$$E[L_s | H_2, \theta] = \sqrt{E_r} \sin \theta \tag{5.42b}$$

$$\text{Var}[L_c | H_2] = \text{Var}[L_s | H_2] = \frac{N_0}{2} \tag{5.43}$$

The resulting probability of error is given in terms of an integral which cannot be evaluated analytically.

$$P_M = \int_0^{\gamma \sqrt{2/N_0}} z \exp\left(-\frac{z^2 + 2E_r/N_0}{2}\right) I_0\left(z \sqrt{\frac{2E_r}{N_0}}\right) dz \tag{5.44}$$

This integral was first tabulated by Marcum [56] and is usually referred to in the detection theory literature as Marcum's Q-function.

We now shall proceed to use our approximation techniques for suboptimum receivers to approximate Eq. 5.44. First, we

compute the semi-invariant moment generating function conditioned on hypothesis H_2 . Since L_c and L_s are Gaussian random variables when conditioned on θ , we first compute $\mu_2(s|\theta)$.

$$\begin{aligned}\mu_2(s|\theta) &= \ln E[e^{sL'}|\theta] \\ &= \ln E[e^{sL_c^2}|\theta] + \ln E[e^{sL_s^2}|\theta]\end{aligned}\quad (5.45)$$

This is a straightforward exercise in computing integrals of exponentials of quadratic forms.

$$\begin{aligned}\mu_2(s|\theta) &= \frac{s E_r \cos^2\theta}{1 - 2s \frac{N_0}{2}} - \frac{1}{2} \ln(1 - 2s \frac{N_0}{2}) \\ &\quad + \frac{s E_r \sin^2\theta}{1 - 2s \frac{N_0}{2}} - \frac{1}{2} \ln(1 - 2s \frac{N_0}{2}) \\ &= \frac{s E_r}{1 - sN_0} - \ln(1 - sN_0)\end{aligned}$$

for $\text{Re}[s] \leq \frac{1}{N_0}$ (5.46)

The condition on s below Eq. 5.46 is a necessary condition for convergence of the integrals. Recall from Chapter III that we are only interested in real s . Hence, this condition can be replaced by,

$$s < \frac{1}{N_0}. \quad (5.47)$$

Now observe that the right-hand side of Eq. 5.46 is not a function of θ , hence,

$$\mu_2(s) = \mu_2(s|\theta) \quad (5.48)$$

To evaluate the first-order error approximation, we shall need the first two derivatives of $\mu_2(s)$.

$$\dot{\mu}_2(s) = \frac{N_0}{1 - sN_0} + \frac{E_r}{(1 - sN_0)^2} = \gamma \quad (5.49)$$

$$\ddot{\mu}_2(s) = \frac{N_0^2}{(1 - sN_0)^2} + \frac{2E_r N_0}{(1 - sN_0)^3} \quad (5.50)$$

Recall from Chapter II that when approximating the error probability on H_2 , we require that s be negative, in order that the integrand in Eq. 2.45 contains a decaying exponential and that Eq. 5.49 has a solution. Thus, this condition is more stringent than the one given in Eq. 5.47. This condition is equivalent to requiring,

$$\gamma = \dot{\mu}_2(s) < \dot{\mu}_2(0) = E[\ell' | H_2] \quad (5.51)$$

Solving Eq. 5.41 for γ in terms of P_F , substituting into Eq. 5.49 and solving for s , we obtain

$$s = \frac{1}{N_0} \left[1 + \frac{1}{2 \ln P_F} \left(1 + \sqrt{1 - \frac{4E_r}{N_0} \ln P_F} \right) \right] \quad (5.52)$$

We use the approximation,

$$P_M \approx \phi(s \sqrt{\ddot{u}(s)}) \exp[\mu_2(s) - s \dot{\mu}_2(s) + \frac{s^2}{2} \ddot{u}(s)], \quad (5.53)$$

where s is given by Eq. 5.52, subject to the additional condition

$$s < 0 \quad (5.54)$$

to obtain the approximate ROC of Fig. 5.6. We have included for comparison results from Fig. 4.59 of Ref. [42] which were obtained from a numerical evaluation of Eq. 5.44. We observe that the two sets of curves compare very favorably.

2. Simple Binary, With Diversity. A logical extension of this example is to N random phase channels operating in space, time, or frequency diversity. We assume the phase angles are statistically independent. The optimum receiver consists of N parallel channels with their outputs added. In practice, square-law weighting is used instead of $\ln I_0(\cdot)$. We shall analyze this

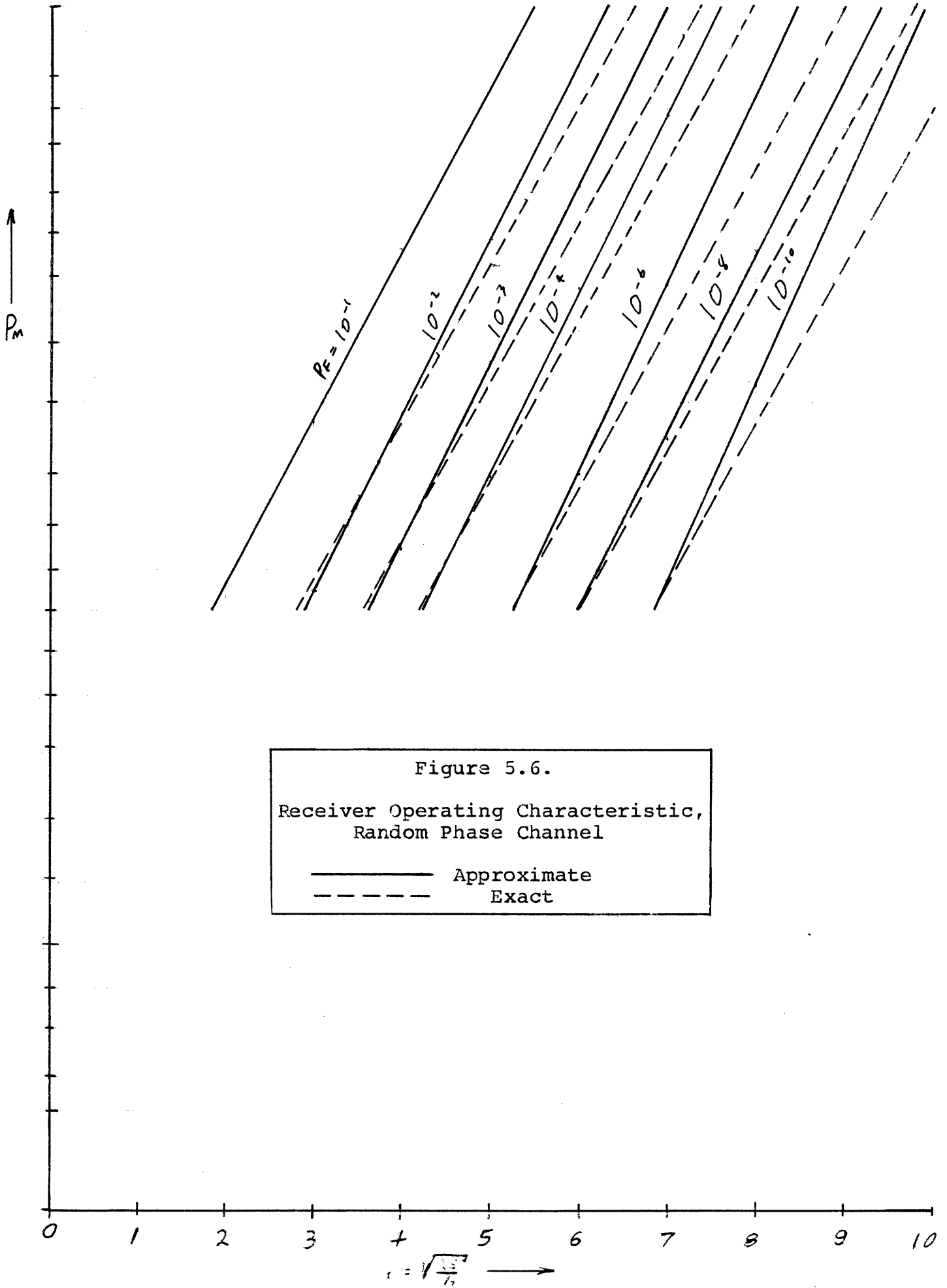


Figure 5.6.
Receiver Operating Characteristic,
Random Phase Channel

suboptimum receiver.

$$\begin{aligned} \ell' &= \sum_{i=1}^N \ell'_i \\ &= \sum_{i=1}^N (L_c^2 + L_s^2) \end{aligned} \quad (5.55)$$

On H_1 , ℓ' has a Chi-square probability density with $2N$ degrees of freedom.

$$P_{\ell|H_1}(L) = \frac{1}{(N_0)^N (N-1)!} L^{N-1} \exp\left(-\frac{L}{N_0}\right) \quad (5.56)$$

Except for the case $N = 1$, which we have already treated, there is no simple expression for P_F . Hence, we use our approximations on both hypotheses. We, therefore, compute two semi-invariant moment generating functions.

$$\mu_1(s) = -N \ln(1-sN_0) \quad (5.57a)$$

$$\begin{aligned} \mu_2(s) &= \sum_{i=1}^N \frac{s E_i}{1 - sN_0} - N \ln(1-sN_0) \\ &= \frac{s E_r}{1 - sN_0} - N \ln(1-sN_0) \end{aligned} \quad (5.57b)$$

where

$$\sum_{i=1}^N E_i = E_r$$

For convergence, we require $s < \frac{1}{N_0}$ for both functions. However, we must have $s > 0$ in Eq. 5.57a and $s < 0$ in Eq. 5.57b. Therefore, the appropriate limits on s are

$$0 < s < \frac{1}{N_0} \quad \text{for Eq. 5.57a} \quad (5.58a)$$

$$s < 0 \quad \text{for Eq. 5.57b} \quad (5.58b)$$

Again, we need the first two derivatives of the $\mu(s)$ function,

$$\dot{\mu}_1(s_1) = \frac{NN_0}{1 - s_1 N_0} = \gamma \quad (5.59a)$$

$$\dot{\mu}_2(s_2) = \frac{NN_0}{1 - s_2 N_0} + \frac{E_r}{(1 - s_2 N_0)^2} = \gamma \quad (5.59b)$$

$$\ddot{\mu}_1(s) = \frac{NN_0^2}{(1 - sN_0)^2} \quad (5.60a)$$

$$\ddot{\mu}_2(s) = \frac{NN_0^2}{(1 - sN_0)^2} + \frac{2E_r N_0}{(1 - sN_0)^3} \quad (5.60b)$$

Solving Eqs. 5.59a and 5.59b for s_1 and s_2 ,

$$s_1 = \frac{1}{N_0} \left[1 - \frac{NN_0}{\gamma} \right] \quad (5.61a)$$

$$s_2 = \frac{1}{N_0} \left[1 - \frac{NN_0}{2} \left(1 + \sqrt{1 + \frac{4\gamma E_r}{N^2 N_0^2}} \right) \right] \quad (5.61b)$$

We now substitute these equations into the approximate error expressions of Eqs. 2.49 and 2.50.

$$P_F \approx \phi(-s_1 \sqrt{\ddot{u}_1(s_1)}) \exp[\mu_1(s_1) - s_1 \dot{\mu}_1(s_1) + \frac{s_1^2}{2} \ddot{\mu}_1(s_1)] \quad (5.62a)$$

$$P_M \approx \phi(+s_2 \sqrt{\ddot{u}(s_2)}) \exp[\mu_2(s_2) - s_2 \dot{\mu}_2(s_2) + \frac{s_2^2}{2} \ddot{\mu}_2(s_2)] \quad (5.62b)$$

We then vary γ to construct the approximate ROC's of Fig. 5.7 and Fig. 5.8. The difference between Figs. 5.7 and 5.6 is that we used the exact expression for P_F in Fig. 5.6.

D. Random Process Detection Problems

For our third set of examples, we consider problems with an infinite set of unknown parameters. It is for this class of problems (the third level in our hierarchy) that our asymptotic approximations are the most useful.

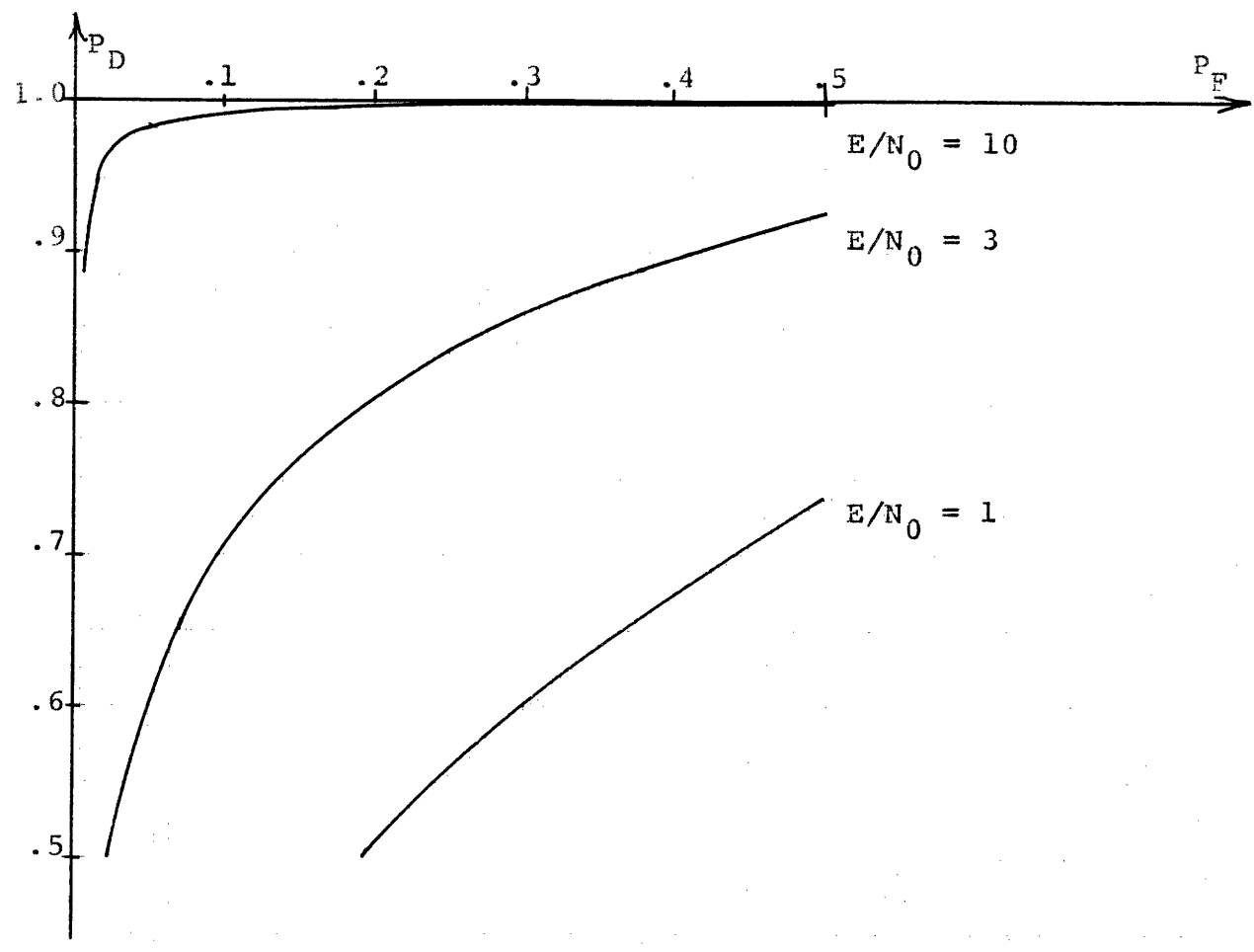


Figure 5.7
Receiver Operating Characteristic,
Random Phase Channel
 $N = 1$

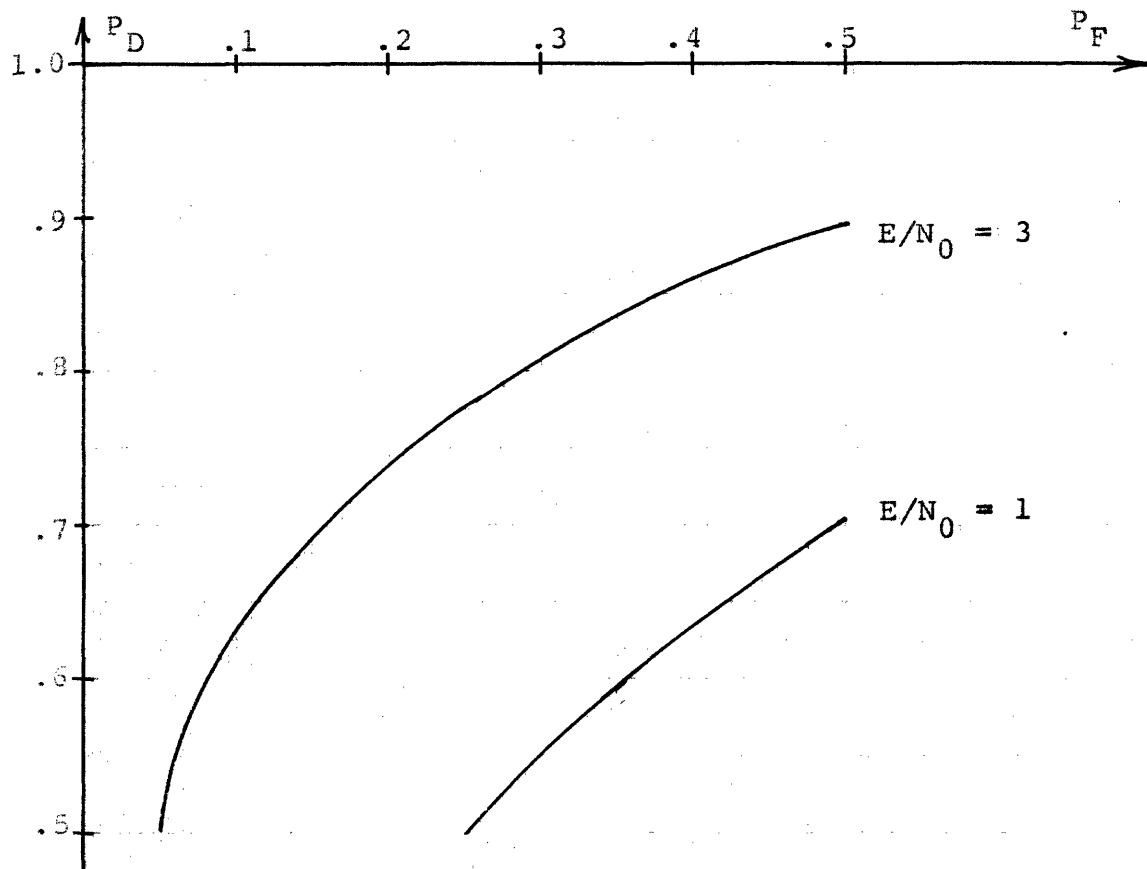


Figure 5.8
Receiver Operating Characteristic,
Random Phase Channel
 $N = 2$

1. A Simple Binary (Radar) Problem, Symmetric Bandpass Spectrum. The model here is the same as that in Example B except that now we remove the assumption of slow fading. That is, we allow the channel attenuation and phase shift to be a sample function from a random process. For simplicity, we assume only one diversity channel. We also assume that the fading spectrum is symmetric about the carrier frequency. This allows us to use the results for bandpass signals in Chapter II. The details of the channel model we are using are developed numerous places. Ref.[43] gives a good discussion. Using Eqs. 3.36 and 3.20c, we obtain,

$$\mu(s) = (1-s) \ln D_{\frac{1}{3}}\left(\frac{1}{N_0}\right) - \ln D_{\frac{1}{3}}\left(\frac{1-s}{N_0}\right) \quad (5.63)$$

In Eq. 5.63, we have suppressed the obvious subscripts for "simple binary" and "bandpass".

We assume that the quadrature components of the fading have single-pole spectra.

$$S_y(\omega) = \frac{2kP}{\omega^2 + k^2} \quad (5.64)$$

We have seen in Chapter III, Section C that the calculation of $\mu(s)$ is particularly easy when the random processes have a state-variable representation. The appropriate state-variable representation for the spectrum of Eq. 5.64 is

$$\underline{F}(t) = -k \quad (5.65a)$$

$$\underline{G}(t) \quad \underline{Q}(t) \quad \underline{G}^T(t) = 2kP \quad (5.65b)$$

$$\underline{C}(t) = 1 \quad (5.65c)$$

$$\underline{\Sigma}_0 = P. \quad (5.65d)$$

First, we find the transition matrix for the canonical system,

Eq. 3.47.

$$\frac{d}{dt} \underline{\theta}(t, T_1) = \begin{bmatrix} -k & | & 2kP \\ \hline \frac{2}{N_0} & | & k \end{bmatrix} \underline{\theta}(t, T_1) \quad (5.66a)$$

$$\underline{\theta}(T_1, T_1) = \underline{I} \quad (5.66b)$$

The solution is readily obtained by using Laplace transforms.

$$\underline{\theta}(t+T_1, T_1) = \begin{bmatrix} \cosh k\beta t - \frac{1}{\beta} \sinh k\beta t & | & \frac{2P}{\beta} \sinh k\beta t \\ \hline \frac{2}{N_0 k\beta} \sinh k\beta t & | & \cosh k\beta t + \frac{1}{\beta} \sinh k\beta t \end{bmatrix} \quad (5.67)$$

where

$$\beta \triangleq \sqrt{1 + \frac{4P}{kN_0}}$$

Then,

$$\begin{aligned} \phi_2(T_f) &= \theta_{21}(T_f, T_i) \phi_1(T_i) + \theta_{22}(T_f, T_i) \phi_2(T_i) \\ &= \cosh k\beta T + \frac{1}{\beta} \left[\frac{2P}{kN_0} + 1 \right] \sinh k\beta T \end{aligned} \quad (5.68)$$

where

$$T \triangleq T_f - T_i.$$

We can rewrite Eq. 5.68 in a more useful form by making use of the definition of β .

$$\begin{aligned} \phi_2(T_f) &= \cosh(k\beta T) + \left(\frac{\beta^2 + 1}{2\beta} \right) \sinh(k\beta T) \\ &= e^{k\beta T} \left[\frac{1 - \left(\frac{\beta-1}{\beta+1} \right)^2 e^{-2k\beta T}}{1 - \left(\frac{\beta-1}{\beta+1} \right)^2} \right] \end{aligned} \quad (5.69)$$

Then,

$$\ln D_{\mathcal{F}}\left(\frac{2}{N_0}\right) = \ln \left[\frac{1 - \left(\frac{\beta-1}{\beta+1}\right)^2 e^{-2k\beta T}}{1 - \left(\frac{\beta-1}{\beta+1}\right)^2} \right] + (\beta-1)kT \quad (5.70)$$

We use $\mu(s)$ as given by Eq. 2.26 in the first order approximations to Eqs. 2.10 and 2.11 which are obtained by retaining only the first term in the Edgeworth series, Ex. 2.26, to construct the approximate receiver operating characteristic shown in Fig. 5.9. We are unable to construct the entire ROC because the threshold must lie between the conditional means for our approximation techniques to work. The reason for this was discussed in connection with Eq. 2.16. This is not a serious limitation for rarely do we operate in the portion of the ROC where P_F or P_M is greater than 0.5.

2. A Symmetrical (Communication) Problem, Symmetric Bandpass Spectra. We assume the same channel model as in the previous example. The transmitted signals are chosen so that the received random processes are essentially orthogonal. We assume the transmitted signals are equally likely and have equal energies. For the symmetric problem the pertinent value is $s = \frac{1}{2}$. Then, using Eq. 3.29a, the $\mu(s)$ function evaluated at $s = \frac{1}{2}$ is just twice Eq. 5.63.

$$\mu\left(\frac{1}{2}\right) = \ln D_{\mathcal{F}}\left(\frac{1}{N_0}\right) - 2 \ln D_{\mathcal{F}}\left(\frac{1}{2N_0}\right). \quad (5.71)$$

Again, we have suppressed the obvious subscripts for "symmetric binary" and "bandpass spectra". We use the same approximation that

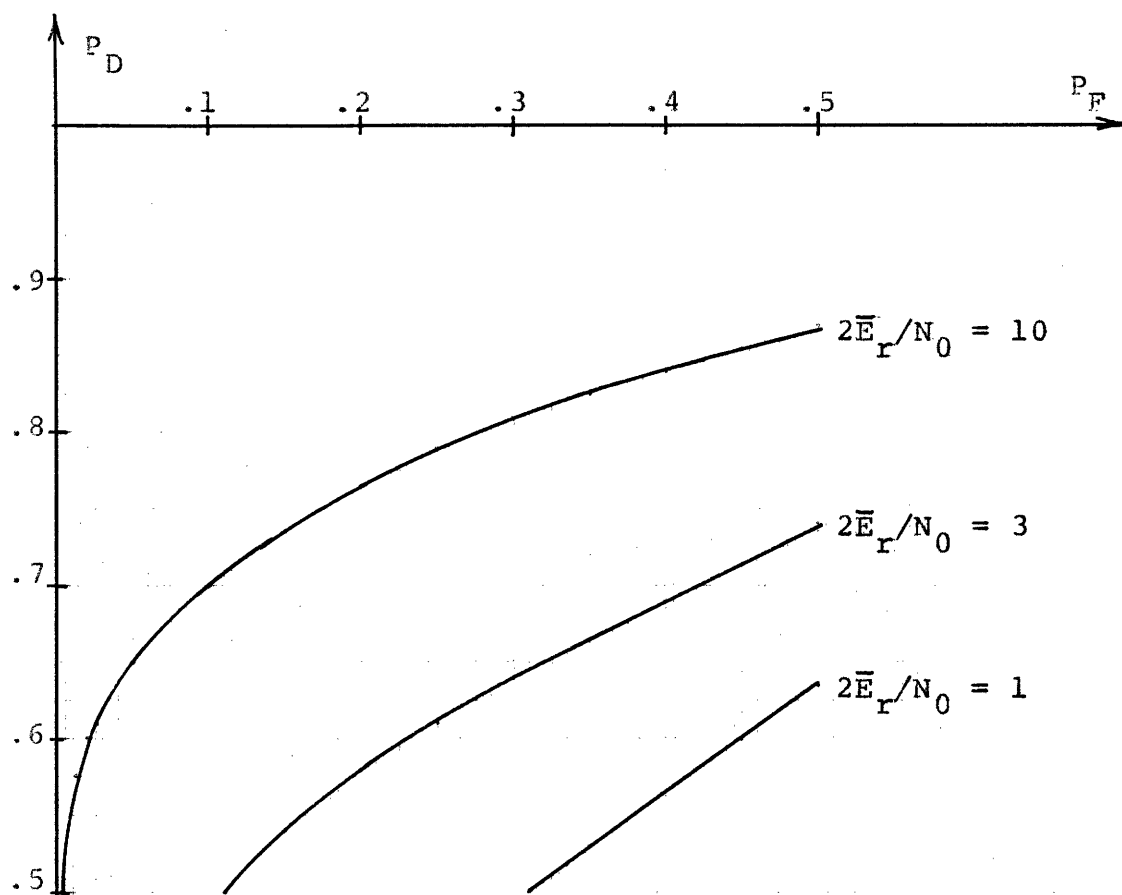


Figure 5.9.

Approximate Receiver Operating Characteristic,
Single-Pole Rayleigh Fading

we did in the slow fading case (Eq. 5.26).

$$P_r[\epsilon] \approx \phi\left(-\frac{1}{2}\sqrt{\ddot{u}(1/2)}\right) \exp\left[u\left(\frac{1}{2}\right) + \frac{1}{8}\ddot{u}\left(\frac{1}{2}\right)\right] \quad (5.72)$$

The approximation of Eq. 5.72 is plotted in Fig. 5.10 for several values of kT . Several results are apparent:

1. There is an implicit diversity due to the time-selective fading, so that for moderate kT values the error probability is noticeably less than for the case of slow fading, provided $2E/N_0$ is sufficiently large.
2. Eq. 5.70 for the Fredholm determinant consists of three terms: One vanishes as T becomes large, one remains constant, and one increases linearly with T . Thus, for large T , the third term dominates. Recalling from optimum linear filtering theory that this last term is just the steady-state MMSE estimation error [41], suggests a simple approximation to the Fredholm determinant for long time intervals. We shall exploit this simplification in Subsection 4 below.
3. A suboptimum detector for this problem has previously been analyzed, and it was found to exhibit an irreducible error probability [60]. That is, the $\text{Pr}[\epsilon]$ curve saturated with increasing E/N_0 . This is not the case for the optimum detector.

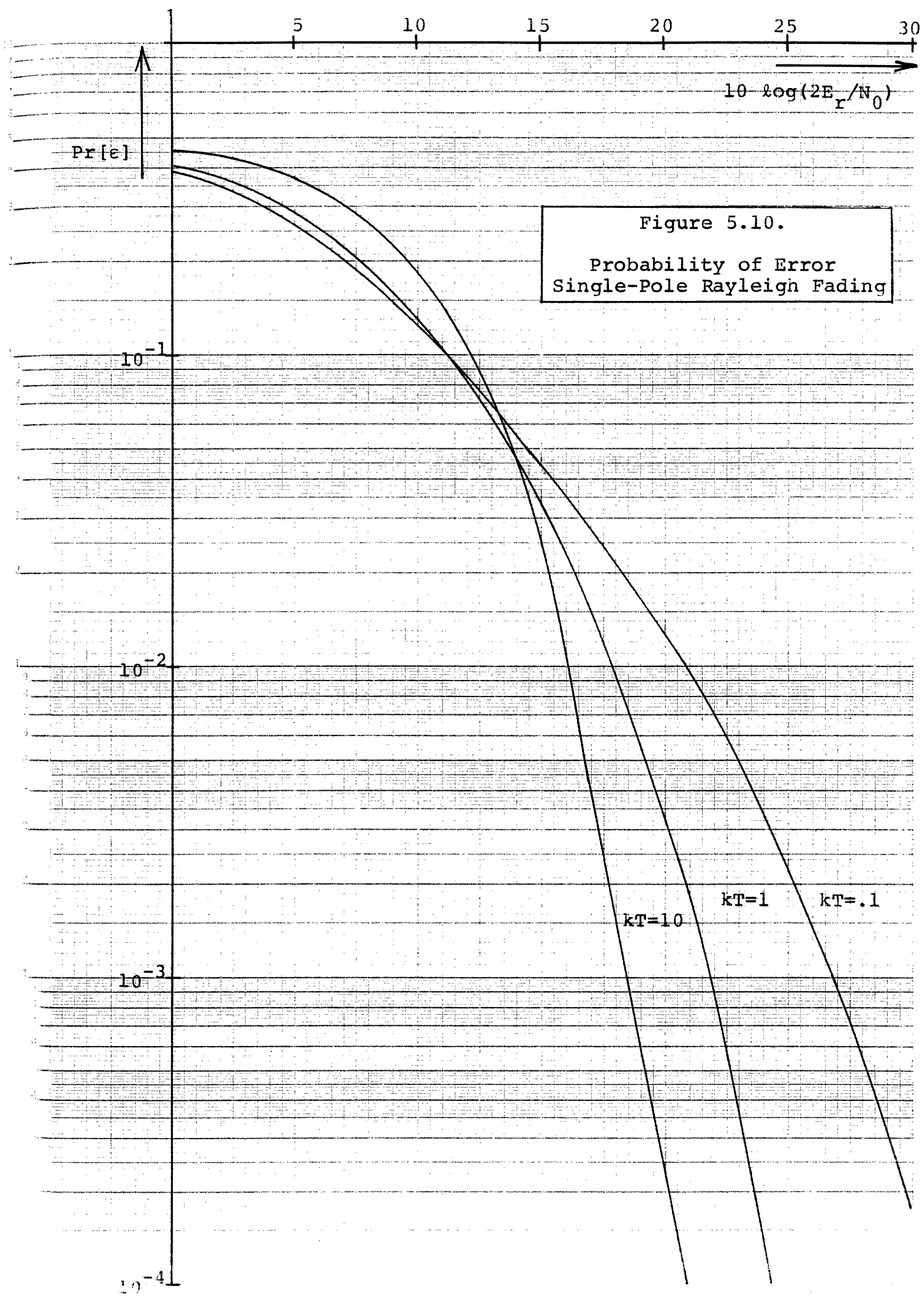


Figure 5.10.
Probability of Error
Single-Pole Rayleigh Fading

3. Optimum Time-Bandwidth Product. As we have observed, the time-selective fading results in a form of diversity. It is straightforward to determine the value of kT which minimizes any one of our approximations to the error probability. For example, the optimum value of kT is plotted vs. signal-to-noise ratio in Fig. 5.11 where the criterion is minimum $\mu(1/2)$. This criterion is equivalent to minimizing the Chernoff bound. The resulting minimum value for $\mu(1/2)$ is plotted in Fig. 5.12.

We saw in our slow fading example that the results obtained from minimization of the probability of error, the Chernoff bound, and our first order approximation gave essentially the same performance.

4. Long Time Intervals, Stationary Processes (Bandpass Signals, Symmetric Hypotheses). In many problems of interest to us, the processes are stationary and the time interval is sufficiently long that the pertinent Fredholm determinants can be approximated by using the steady-state filtering error in Eq. 3.15. Then,

$$\ln D_{\frac{2}{N_0}} \approx \frac{2}{N_0} T \xi_{y^\infty} \left(\frac{N_0}{2} \right) \quad (5.73)$$

where the subscript ∞ denotes the steady-state MMSE error.

For the first order example discussed in Section C-1, this approximation is good whenever the first term in Eq. 5.70 is small compared to the second. This can be achieved if T is sufficiently large. Computationally, this approximation has the advantage that closed-form error expressions have been derived for

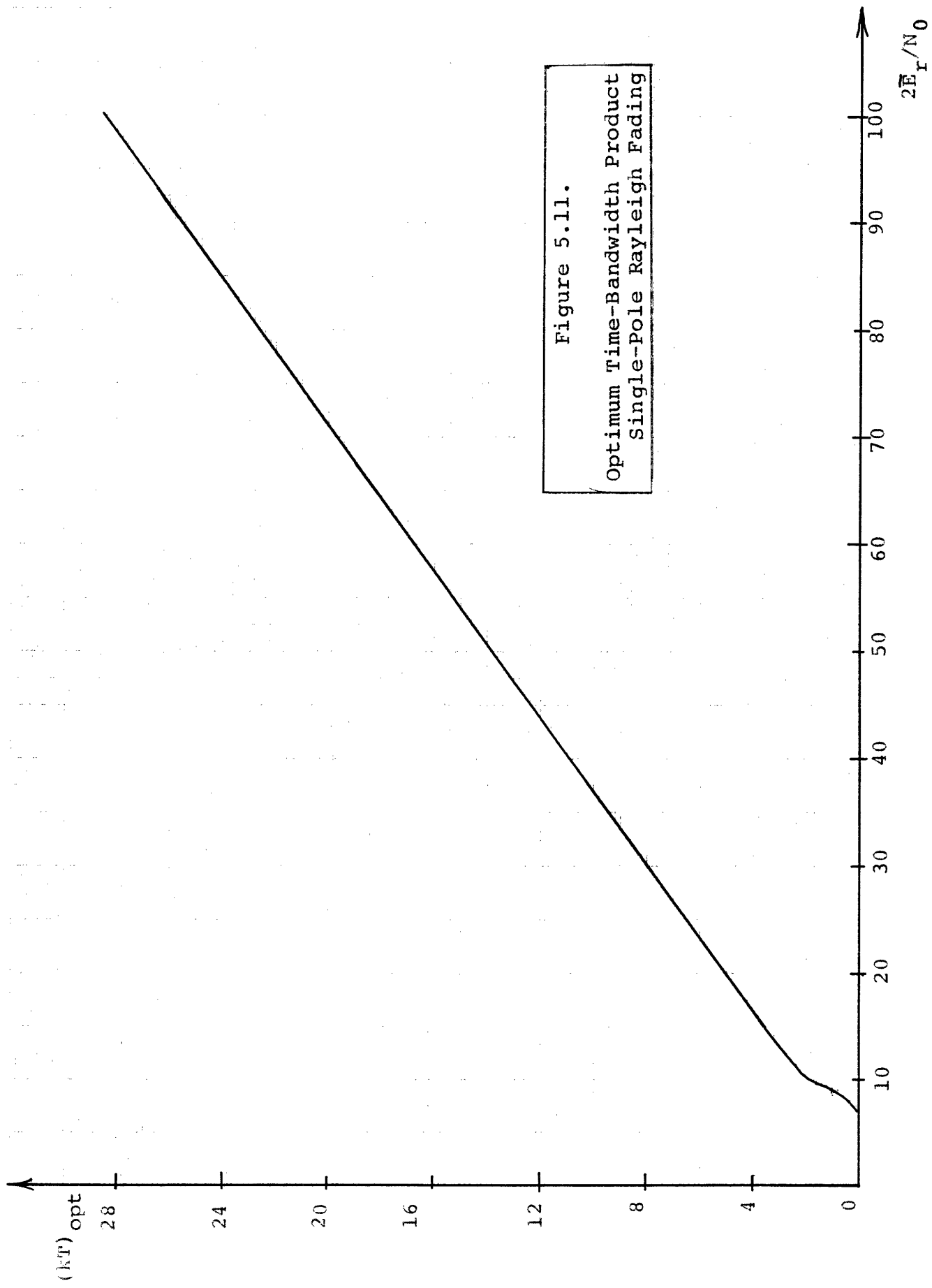


Figure 5.11.
Optimum Time-Bandwidth Product
Single-Pole Rayleigh Fading

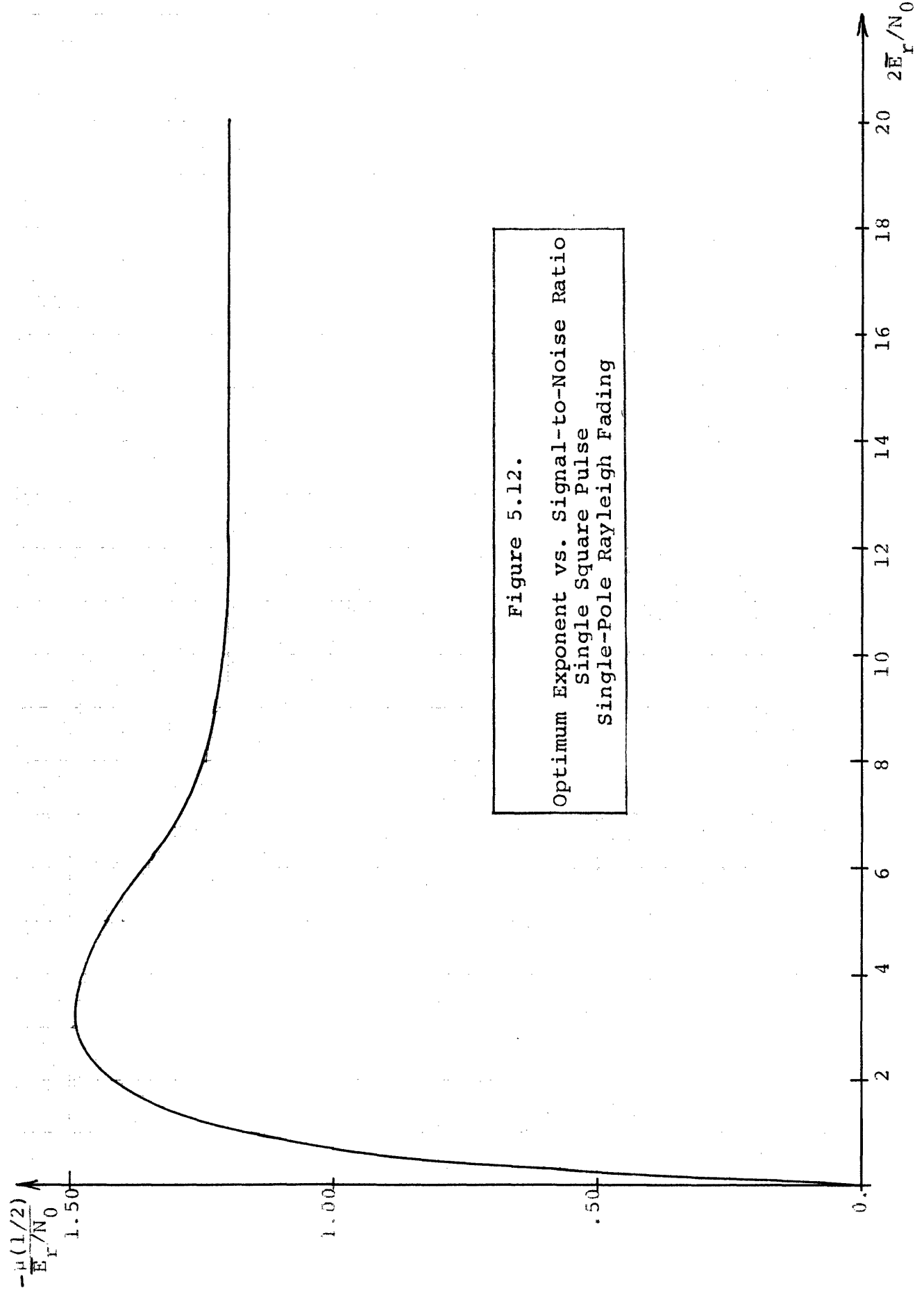


Figure 5.12.
Optimum Exponent vs. Signal-to-Noise Ratio
Single Square Pulse
Single-Pole Rayleigh Fading

the steady-state filtering problem [41],

$$\xi_{y_{\infty}}\left(\frac{2}{N_0}\right) = \frac{N_0}{2} \int_{-\infty}^{\infty} \ln\left[1 + \frac{2S_y(\omega)}{N_0}\right] \frac{d\omega}{2\pi} \quad (5.74)$$

For example, for the family of Butterworth spectra,

$$S_{y_n}(\omega) = \frac{nP}{k} \frac{\sin\left(\frac{\pi}{2n}\right)}{1 + (\omega/k)^{2n}}, \quad (5.75)$$

we have,

$$\xi_{y_n} \left(\frac{N_0}{2} \right) = \frac{nk N_0}{2P a_n} \left[\left(1 + \frac{4P}{kN_0} a_n \right)^{\frac{1}{2n}} - 1 \right] \quad (5.76)$$

$$\text{where } a_n = n \sin\left(\frac{\pi}{2n}\right).$$

We assume that the signals are bandpass with Eq. 5.75 specifying the spectrum of the quadrature components. We then use Eq. 3.36 to obtain $\mu(1/2)$ for the bandpass problem in terms of the quadrature components.

$$\mu_{\infty}\left(\frac{1}{2}\right) = \frac{2nD}{a_n} \left[\begin{array}{c} \left(1 + \frac{Y}{D} a_n \right)^{\frac{1}{2n}} \\ - 2 \left(1 + \frac{Y}{D} \frac{a_n}{2} \right)^{\frac{1}{2n}} + 1 \end{array} \right] \quad (5.77)$$

where

$$\gamma = \frac{\Delta}{2} \frac{kT}{\bar{E}_r}$$

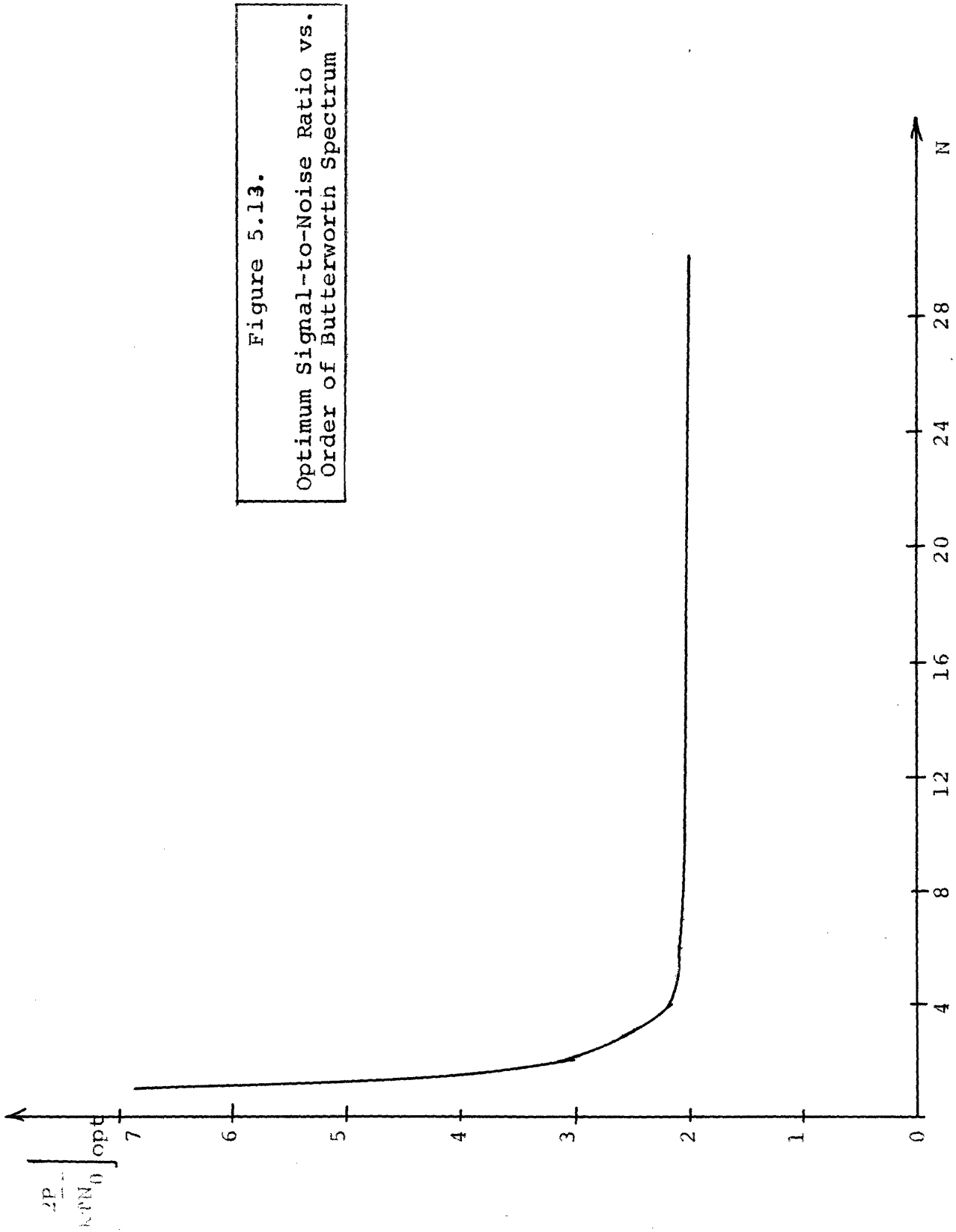
$$\frac{\Delta}{\bar{E}_r} = \frac{P}{TN_0}$$

$$= \frac{P}{TN_0}$$

Now we can minimize $\mu_\infty(\frac{1}{2})$ with respect to D as a function of γ . Differentiating with respect to D and equating the result to zero yields a nonlinear equation in γ/D .

$$0 = \frac{a_n}{2n} \left(\frac{\gamma}{D} \right) \left[\left(1 + a_n \left(\frac{\gamma}{D} \right) \right)^{\frac{1}{2n} - 1} - \left(1 + \frac{a_n}{2} \left(\frac{\gamma}{D} \right) \right)^{\frac{1}{2n} - 1} \right] - \left[\left(1 + a_n \left(\frac{\gamma}{D} \right) \right)^{\frac{1}{2n}} - 2 \left(1 + \frac{a_n}{2} \left(\frac{\gamma}{D} \right) \right)^{\frac{1}{2n}} + 1 \right] \quad (5.78)$$

We can solve (numerically) for the value of γ/D which minimizes $\mu_\infty(\frac{1}{2})$ for a given n . The results for $n = 1, 2, \dots, 30$ are shown in Fig. 5.13 and the resulting exponent is plotted in Fig. 5.14.



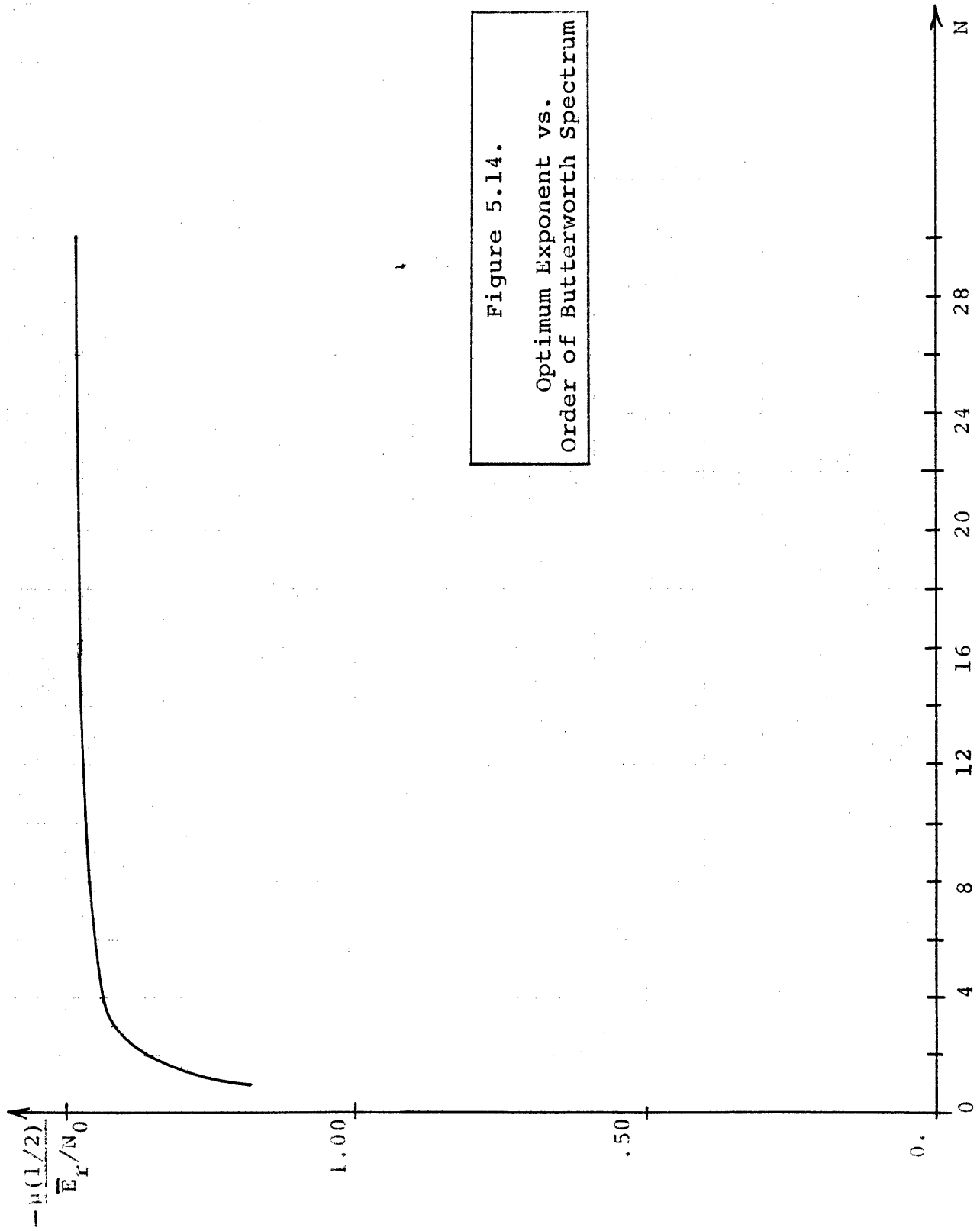


Figure 5.14.
Optimum Exponent vs.
Order of Butterworth Spectrum

A limiting case of the family of Butterworth spectra is the bandlimited spectrum

$$S_{y_{\infty}}(\omega) = \begin{cases} \frac{P}{4W} & |\omega| \leq 2\pi W \\ 0 & \text{elsewhere,} \end{cases} \quad (5.79)$$

for which

$$\mu_{\infty}(s) = D \ln \left[\frac{1 - \gamma/D}{(1 + (1-s)\gamma/D)(1+\gamma/D)} \right] \quad (5.80)$$

where now $D \triangleq 2WT$.

Proceeding as in Example B-4, we find,

$$\left. \frac{2E}{kTN_0} \right|_{\text{opt.}} = 3.0700 \pm 10^{-4} \quad (5.81)$$

and

$$\left. \mu_{\infty}\left(\frac{1}{2}\right) \right|_{\text{opt.}} \approx -.1488E/N_0 \quad (5.82)$$

5. Pierce's Upper and Lower Bounds (Bandpass, Binary Symmetric). Up to this point, we have used the semi-invariant moment generating function, $\mu(s)$, to obtain asymptotic expansions of

and bounds on the error probabilities. In addition, we expressed all of the commonly used distance measures in terms of this function. In this section we shall use $\mu(s)$ to obtain an error expression originally derived by Turin [61] for the symmetric hypothesis binary problem. Our method has the advantage of being appreciably simpler than the original derivation. Pierce [56] then used Turin's expression to obtain upper and lower bounds on the error probability. We shall express these bounds in terms of $\mu(s)$, which enables us to use the techniques of Chapter III to evaluate these bounds without first having to find the eigenvalues of the random processes.

The model was discussed in detail in Example D-2; Example C-2 is a special case. Since throughout this section we will be discussing only the bandpass, symmetric binary problem we shall omit the pertinent subscripts from $\mu(s)$.

Recall that in Chapter II, we considered only real values of s , the argument of the semi-invariant moment generating function, when we derived our various bounds and approximations on the probability of error. However, in Chapter III when we actually computed $\mu(s)$ for Gaussian signals, we could have just as well let s be complex, with a suitable restriction on its real part, up to the point where we derived closed-form expression for $\mu(s)$. Analogous to our previous condition on s , we now can show that it suffices to consider,

$$0 \leq \text{Re}[s] \leq 1 \quad (5.83)$$

Since we have assumed equally likely hypotheses and a minimum total probability of error criterion, we have

$$\begin{aligned}
 \Pr\{\varepsilon\} &= \Pr\{e|H_0\} \\
 &= \int_0^{\infty} p_{2|H_0}(L) dL \\
 &= \int_0^{\infty} \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} M_{2|H_0}(s) e^{-sL} ds dL \quad \text{for } 0 \leq \sigma \leq 1 \quad (5.84)
 \end{aligned}$$

where $s = \sigma + j\omega$. Now interchanging the order of integration,

$$\begin{aligned}
 \Pr\{\varepsilon\} &= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} M_{2|H_0}(s) \int_0^{\infty} e^{-sL} dL ds \\
 &= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{s} M_{2|H_0}(s) ds \quad \text{for } 0 \leq \sigma \leq 1 \text{ and } \sigma > 0 \\
 &= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{e^{u(s)}}{s} ds \quad \text{for } 0 < \sigma \leq 1. \quad (5.85)
 \end{aligned}$$

For our specific problem, $u(s)$ follows from Eq. 3.6 where now s is allowed to be complex.

$$\mu(s) = \sum_{n=1}^{N/2} [\ln(1+2\lambda_n/N_0) - \ln(1+2s\lambda_n/N_0) - \ln(1+2(1-s)\lambda_n/N_0)] \quad (5.36a)$$

or

$$e^{\mu(s)} = \prod_{n=1}^{N/2} \frac{(1+2\lambda_n/N_0)}{(1+2s\lambda_n/N_0)(1+2(1-s)\lambda_n/N_0)} \quad (5.36b)$$

Thus,

$$\Pr\{\epsilon\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{s} \prod_{n=1}^{N/2} \frac{(1+2\lambda_n/N_0)}{(1+2s\lambda_n/N_0)(1+2(1-s)\lambda_n/N_0)} ds \quad (5.87)$$

for $0 < \sigma \leq 1$.

This is Turin's [27] or Pierce's [5]. From this equation, in special cases, we can evaluate $\Pr\{\epsilon\}$ by performing the indicated integration. Furthermore, Pierce has derived tight upper and lower bounds on $\Pr\{\epsilon\}$ from this equation. We include his derivation here for completeness, expressing his final result in terms of $\mu(s)$.

For ease in writing, define

$$B_n = \frac{2\lambda_n}{N_0} \quad (5.88)$$

We choose $\sigma = 1/2$. Thus,

$$\Pr\{\varepsilon\} = \frac{1}{2\pi j} \int_{1/2-j\infty}^{1/2+j\infty} \frac{1}{s} \prod_{n=1}^{N/2} \frac{1 + B_n}{(1+sB_n)(1+(1-s)B_n)} ds. \quad (5.89)$$

Now let $s = 1/2 + jx$. Then,

$$\begin{aligned} \Pr\{\varepsilon\} &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1/2+jx} \prod_{n=1}^{N/2} \frac{1 + B_n}{((1+1/2 B_n)+jxB_n)((1+1/2 B_n)-jxB_n)} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1/2-jx}{1/4-x^2} \prod_{n=1}^{N/2} \frac{1 + B_n}{(1+1/2B_n)^2+B_n^2x^2} dx \end{aligned} \quad (5.90)$$

The imaginary part of this integral vanishes. Next, we let

$2x = \tan \theta$, then

$$\begin{aligned} \Pr\{\varepsilon\} &= \frac{1}{\pi} \int_0^{\pi/2} d\theta \prod_{n=1}^{N/2} \frac{1 + B_n}{(1+B_n/2)^2+1/4(B_n \tan \theta)^2} \\ &= \prod_{n=1}^{N/2} \frac{1 + B_n}{(1+B_n/2)^2} \frac{1}{\pi} \int_0^{\pi/2} d\theta \prod_{n=1}^{N/2} \frac{1}{1 + \left(\frac{B_n \tan \theta}{2+B_n}\right)^2} \\ &= e^{\mu(1/2)} \frac{1}{\pi} \int_0^{\pi/2} d\theta \prod_{n=1}^{N/2} \frac{1}{1 + \left(\frac{B_n \tan \theta}{2 + B_n}\right)^2}. \end{aligned} \quad (5.91)$$

Now we derive bounds on the integrand in Eq. 5.91. Using the inequality $\ln(1+x) \leq x$ for $x \geq 0$,

$$\begin{aligned}
 & \prod_{n=1}^{N/2} \left[1 + \left(\frac{B_n}{2+B_n} \tan \theta \right)^2 \right] \\
 &= \exp \sum_{n=1}^{N/2} \ln \left[1 + \left(\frac{B_n}{2+B_n} \tan \theta \right)^2 \right] \\
 &\leq \exp \left[\sum_{n=1}^{N/2} \left(\frac{B_n}{2+B_n} \tan \theta \right)^2 \right] \\
 &= \exp \left[\tan^2 \theta \sum_{n=1}^{N/2} \left(\frac{B_n}{2+B_n} \right)^2 \right] \tag{5.92}
 \end{aligned}$$

To evaluate the sum in Eq. 5.92, we differentiate $\mu(s)$ twice.

$$\dot{\mu}(s) = \sum_{n=1}^{N/2} \left[-\frac{B_n}{1+sB_n} + \frac{B_n}{1+(1-s)B_n} \right] \tag{5.93}$$

$$\ddot{\mu}(s) = \sum_{n=1}^{N/2} \left[\left(\frac{B_n}{1+sB_n} \right)^2 + \left(\frac{B_n}{1+(1-s)B_n} \right)^2 \right] \tag{5.94}$$

$$\begin{aligned}
\ddot{u}(1/2) &= 2 \sum_{n=1}^{N/2} \left(\frac{B_n}{1+1/2 B_n} \right)^2 \\
&= 8 \sum_{n=1}^{N/2} \left(\frac{B_n}{2+B_n} \right)^2.
\end{aligned} \tag{5.95}$$

Therefore,

$$\prod_{n=1}^{N/2} \left[1 + \left(\frac{B_n}{2+B_n} \tan \theta \right)^2 \right] \leq \exp\left(\frac{1}{8} \ddot{u}(1/2) \tan^2 \theta\right) \tag{5.96}$$

A lower bound on the product in Eq. 5.96 follows from the observation that expanding the product yields a sum of positive terms. Thus,

$$\begin{aligned}
\prod_{n=1}^{N/2} \left[1 + \left(\frac{B_n}{2+B_n} \tan \theta \right)^2 \right] &\geq 1 + \sum_{n=1}^{N/2} \left(\frac{B_n}{2+B_n} \right)^2 \tan^2 \theta \\
&= 1 + \frac{1}{8} \ddot{u}(1/2) \tan^2 \theta.
\end{aligned} \tag{5.97}$$

The desired upper and lower bounds follow from evaluating two integrals.

$$\begin{aligned}
I_1 &\triangleq \frac{1}{\pi} \int_0^{\pi/2} d\theta \frac{1}{1 + \frac{\ddot{u}(1/2)}{8} \tan^2 \theta} \\
&= \frac{1}{2 \left(1 + \sqrt{\frac{\ddot{u}(1/2)}{8}} \right)}
\end{aligned} \tag{5.98}$$

$$I_2 = \frac{1}{\pi} \int_0^{\pi/2} d\theta \exp\left[-\frac{\ddot{\mu}(1/2)}{8} \tan^2\theta\right]. \quad (5.99)$$

Letting $x = \tan^2\theta$,

$$I_2 = \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-\frac{1}{8} \ddot{\mu}(1/2)x}}{\sqrt{x}(1+x)} dx. \quad (5.100)$$

We recognize this as a Laplace transform. Using CRC #111 on Page 334 [62], we obtain,

$$I_2 = \exp\left[\frac{\ddot{\mu}(1/2)}{8}\right] \Phi\left(-\sqrt{1/4\ddot{\mu}(1/2)}\right). \quad (5.101)$$

Thus, we have

$$\Phi\left(-\sqrt{1/4\ddot{\mu}(1/2)}\right) \exp\left[\mu(1/2) + \frac{\ddot{\mu}(1/2)}{8}\right] \leq \Pr[\epsilon] \leq \frac{\exp[\mu(1/2)]}{2 \left(1 + \sqrt{\frac{\ddot{\mu}(1/2)}{8}}\right)}$$

(5.102)

The lower bound may be relaxed to yield*

$$\frac{\exp \mu(\frac{1}{2})}{2 \left(1 + \sqrt{\frac{\pi}{8}} \ddot{\mu}(\frac{1}{2}) \right)} \leq \Pr[\varepsilon] \leq \frac{\exp \mu(\frac{1}{2})}{2 \left(1 + \sqrt{\frac{1}{8}} \ddot{\mu}(\frac{1}{2}) \right)}$$

(5.103)

*One way of doing this is by showing that the function

$$f(x) = \phi(-\sqrt{2} x) - \frac{e^{-x^2}}{2(1+\sqrt{\pi} x)} \geq 0$$

This follows immediately from

$$f(0) = f(\infty) = 0, \quad f'(0) = \frac{\pi - 2}{2\sqrt{\pi}} > 0,$$

and

$$f'(x) = 0 \text{ only at } x = \frac{\pi - 2}{2\sqrt{\pi}}$$

Eq. 5.103 is Pierce's result in our notation. The upper and lower bounds in Eq. 5.103 differ at most by $\sqrt{\pi}$.

6. Bounds on the Accuracy of Our Asymptotic Approximation for Symmetric Hypotheses and Bandpass Spectra. A topic of prime importance is the analysis of the accuracy of our probability of error approximations. We would like to be able to make some quantitative statements about the accuracy which results from retaining a given number of terms in Eq. 2.26. For several examples where the tilted density was far from being Gaussian (e.g. slow Rayleigh fading with no diversity) the first order approximation differs from the exact expressions by only about 10 percent. In some other examples where the "exact" error probability is computed from the significant eigenvalues of typical random processes, even smaller errors result from our first-order approximation.

We have made several attempts at using techniques similar to various Central Limit Theorem proofs [38, 63-65] to estimate the accuracy of the first-order approximation, e.g., Eqs. 2.32 and 2.33. However, the estimates we obtained are far too conservative to be useful.

In this section, we obtain upper and lower bounds on our first-order error approximations for an important subclass of problems, namely, the symmetric hypotheses (communication) problem with equally likely hypotheses, bandpass signals, and a minimum total probability of error criterion. For this problem $\mu(\frac{1}{2})$ (the Bhattacharyya distance) in the appropriate performance measure, and several of our examples in the previous section were in this class. Furthermore, this is the problem which Turin and Pierce treated, as discussed in the previous section. Pierce's upper and lower bounds are

$$\frac{\exp \mu(\frac{1}{2})}{2 \left(1 + \sqrt{\frac{\pi}{8} \ddot{\mu}(\frac{1}{2})} \right)} \leq \phi\left(-\sqrt{\frac{1}{4} \ddot{\mu}(\frac{1}{2})}\right) \exp\left[\mu(\frac{1}{2}) + \frac{1}{8} \ddot{\mu}(\frac{1}{2})\right] \leq \text{Pr}[\epsilon] \quad (5.104a)$$

and

$$\text{Pr}[\epsilon] \leq \frac{\exp \mu(\frac{1}{2})}{2 \left(1 + \sqrt{\frac{1}{8} \ddot{\mu}(\frac{1}{2})} \right)} \quad (5.104b)$$

We observe from Eq. 5.26 that our asymptotic approximation is identical with the lower bound in Eq. 5.104a. More important, Pierce's upper and lower bounds are also upper and lower bounds on our first-order asymptotic approximation. Therefore, this approximation is within a factor of $\sqrt{\pi}$ of the correct value. This is far better than the factor by which the Chernoff bound is in error, as shown in Fig. 7.43 of Ref.[35].

Pierce's bounds could also be used as approximations to the probability of error, and we could make the same statement about their accuracy. However, we have seen examples (slow Rayleigh fading with diversity, for example) where our first-order asymptotic approximation converges to the exact probability of error.

We have chosen to include this discussion of accuracy at this point, rather than in one of the earlier chapters, because it only applies to the bandpass, binary symmetric problem. The technique used to obtain these bounds does not appear to generalize to the simple binary or the general binary problem.

7. A Suboptimum Receiver for Single-Pole Fading. We now demonstrate how the techniques of Chapter II, Section C, are used to analyze the system shown in Fig. 5.15. This is the problem for which we analyzed the optimum receiver in Example D-1. For simplicity, since we only intend to illustrate the method here, we consider a simple binary, lowpass problem. The numerical results for this example will appear elsewhere [66,67], along with the results for more complicated suboptimum receivers. We just set up the problem here.

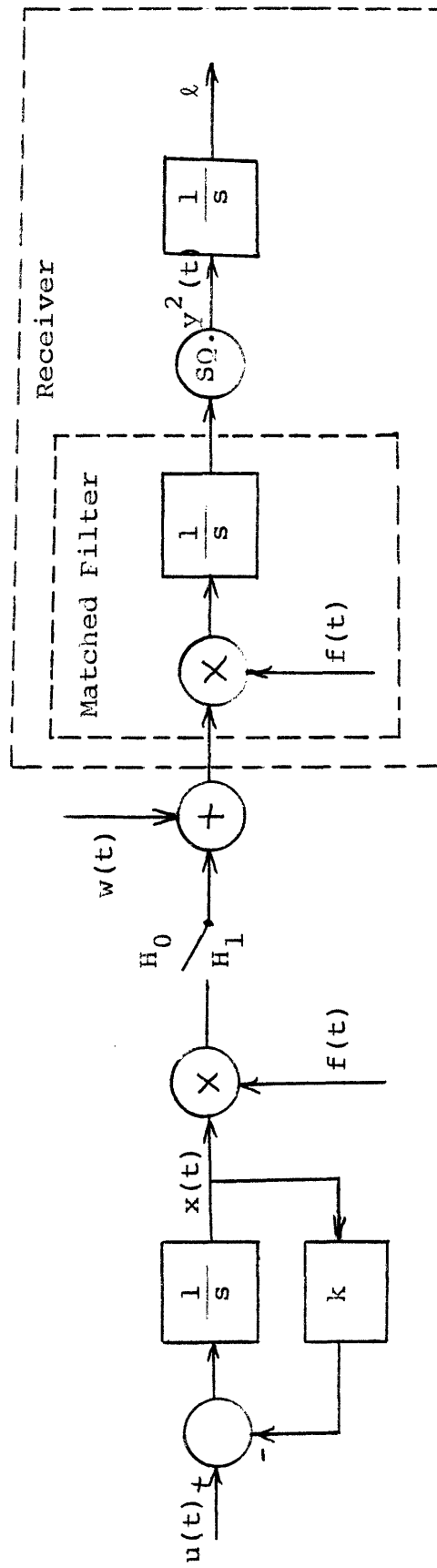


Figure 5.15. Single-Pole Fading, Suboptimum Receiver

On H_1 , we have only a single state system to consider,

$$\dot{y}(t) = f(t)w(t) \quad (5.105a)$$

$$E[w(t)w(u)] = \frac{N_0}{2} \delta(t-u) \quad (5.105b)$$

$$E[y(t)y(t)] = 0 \quad (5.105c)$$

In order to obtain the Fredholm determinant, we must solve the "canonical" system of Eq. 3.40.

$$\frac{d}{dt} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} 0 & f(t) \\ -2s & 0 \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}. \quad (5.106)$$

$$\begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (5.107)$$

On H_2 , the equations are considerably more complex.

$$\underline{F}(t) = \begin{bmatrix} -k & 0 \\ f^2(t) & 0 \end{bmatrix} \quad (5.108a)$$

$$\underline{G}(t) = \begin{bmatrix} 1 & 0 \\ 0 & f(t) \end{bmatrix} \quad (5.108b)$$

$$\underline{C}(t) = [0, 1] \quad (5.108c)$$

$$\underline{\Sigma}_0(t) = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \quad (5.108d)$$

We, therefore, want to solve the system

$$\frac{d}{dt} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} -k & 0 & 2kP & 0 \\ f^2(t) & 0 & 0 & \frac{N_0}{2} f^2(t) \\ \hline 0 & 0 & k & -f^2(t) \\ 0 & -2s & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \quad (5.109)$$

with $\phi_1(0) = \underline{\Sigma}_0$,

$\phi_2(0) = \underline{I}$

for $s < 0$.

Since s is negative, this corresponds to solving a linear filtering problem where the observation noise has spectral density

$$S_n(\omega) = \frac{1}{2|s|} . \quad (5.110)$$

8. A Class of Dispersive Channels. One model for dispersive channels (or deep, fluctuating radar/sonar targets) is the tapped-delay line model [68], as indicated in Fig. 5.16. For purposes of illustration, we consider a simple binary, lowpass problem. If we assume the transmitted signal $s(t)$ is a known waveform and the tap-gain random processes can be modeled via state-variables, we can readily compute $\mu(s)$ using the techniques of Chapter III, Section D.

For example, if we assume that the tap gains are uncorrelated with the state representations,

$$\dot{\underline{x}}_i(t) = \underline{F}_i(t)\underline{x}_i(t) + \underline{G}_i(t)\underline{u}(t) \quad (5.111a)$$

$$B_i(t) = s(t-i\Delta)\underline{C}_i(t)\underline{x}_i(t), \quad i = 0, 1, \dots, N, \quad (5.111b)$$

then the appropriate state variable model is

$$\dot{\underline{x}}(t) = \underline{F}(t)\underline{x}(t) + \underline{G}(t)\underline{u}(t) \quad (5.112a)$$

$$y(t) = \underline{C}(t)\underline{x}(t) \quad (5.112b)$$

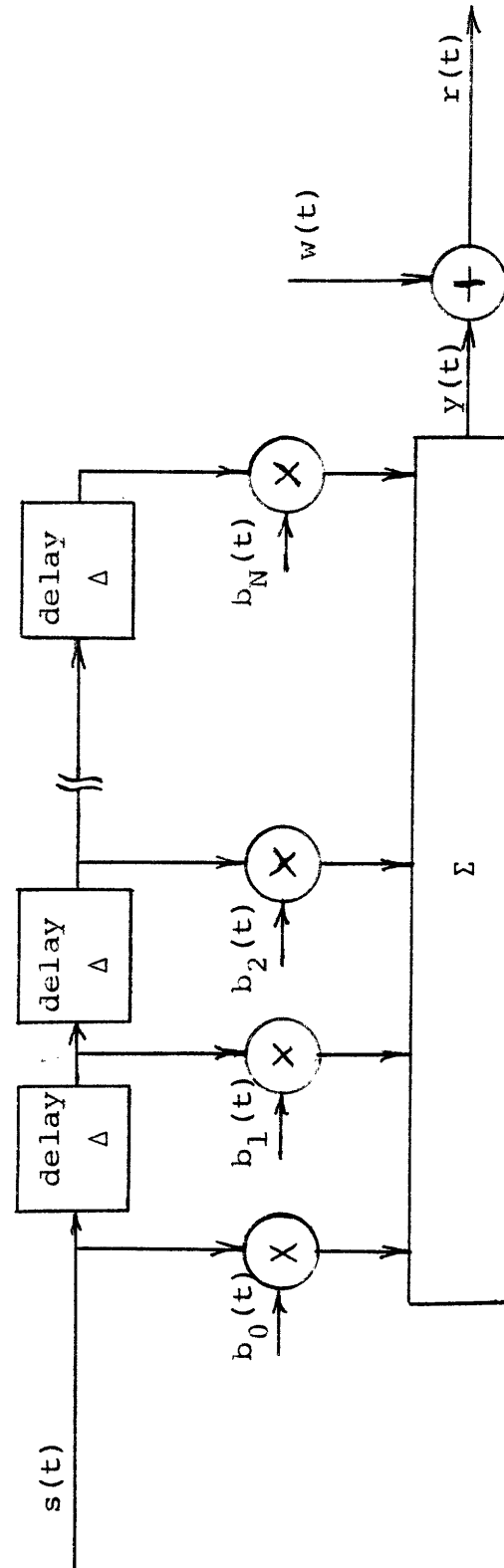


Figure 5.16. Tapped Delay Line Channel Model

where

$$\underline{x}(t) = \begin{bmatrix} \underline{x}_0(t) \\ \underline{x}_1(t) \\ \vdots \\ \underline{x}_N(t) \end{bmatrix} \quad (5.113a)$$

$$\underline{F}(t) = \begin{bmatrix} \underline{F}_0(t) & & & \underline{0} \\ & \underline{F}_1(t) & & \\ & & \ddots & \\ \underline{0} & & & \underline{F}_N(t) \end{bmatrix} \quad (5.113b)$$

$$\underline{G}(t) = \begin{bmatrix} \underline{G}_0(t) & & & \underline{0} \\ & \underline{G}_1(t) & & \\ & & \ddots & \\ \underline{0} & & & \underline{G}_N(t) \end{bmatrix} \quad (5.113c)$$

$$\underline{C}(t) = [s(t)\underline{C}_0(t), s(t-\Delta)\underline{C}_1(t), \dots, s(t-N\Delta)\underline{C}_N(t)] \quad (5.113d)$$

To illustrate the feasibility of such a calculation, we consider a simple example. We assume the tap gains are sample functions from stationary single-pole Gaussian random processes with spectra

$$S_{b_i}(\omega) = \frac{2}{\omega^2 + 1} \quad i = 0, 1, 2 \quad (5.114)$$

The transmitted signal on hypotheses H_1 is a square pulse

$$s(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (5.115)$$

On H_0 no signal is sent. $w(t)$ is white Gaussian noise of spectral height $N_0/2 = 1$. Thus, the average received energy-to-noise ratio is 3 for our example. In Fig. 5.17, we plot $\mu(s)$ for three different values of Δ .

9. The Low Energy Coherence Case. In Chapter I, we briefly mentioned the "low energy coherence" or "threshold" case, Ref. [69]. In this example, we calculate $\mu(s)$ for the zero-mean, simple binary problem assuming low energy coherence. Then applying our asymptotic approximation, we obtain the previously quoted results for this problem, Eqs. 1.21.

The simplifying assumption is that all of the eigenvalues of the signal random process are small when compared with the white noise level.

$$\lambda_i \ll N_0/2 \quad \text{for all } i. \quad (5.116)$$

We start with Eq. 3.21.

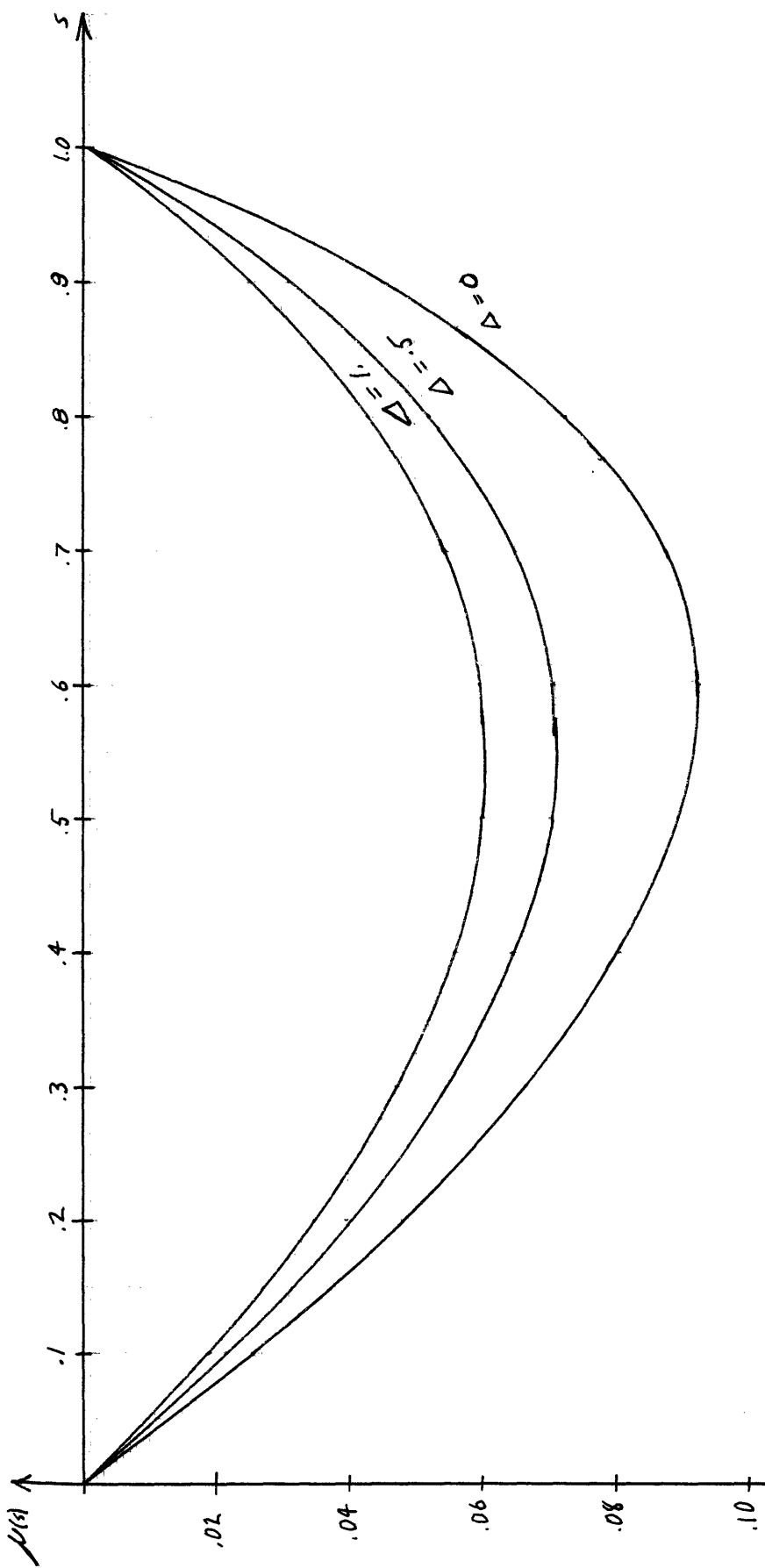


Figure 5.17.
 $\mu(s)$ for Three Tap Channel
 $2E_r/N_0 = 3$

$$\mu(s) = \frac{1}{2} \sum_{i=1}^{\infty} [(1-s) \ln(1+2\lambda_i/N_0) - \ln(1+(1-s)2\lambda_i/N_0)] \quad (5.117)$$

We expand the logarithms in a power series and retain the first two terms. The first-order terms cancel, and we obtain

$$\begin{aligned} \mu(s) &= -\frac{s(1-s)}{4} \left(\frac{2}{N_0}\right)^2 \sum_{i=1}^{\infty} \lambda_i^2 \\ &= -\frac{s(1-s)}{2} \left[\frac{1}{2} \left(\frac{2}{N_0}\right)^2 \int_{T_i}^{T_f} \int K_s^2(t,u) dt du \right] \end{aligned} \quad (5.118)$$

We observe that Eq. 5.118 is of the form of Eq. 5.2, which was the result for the known signal problem.

$$\mu(s) = -\frac{s(1-s)}{2} d^2 \quad (5.119)$$

Therefore, the appropriate approximations to the error probabilities are given by Eqs. 5.9 and 5.10.

$$\Pr[\epsilon|H_1] \approx 1 - \phi\left(\frac{d}{2} + \frac{Y}{d}\right) \quad (5.120a)$$

$$\Pr[\epsilon|H_2] \approx 1 - \phi\left(\frac{d}{2} - \frac{Y}{d}\right), \quad (5.120b)$$

where

$$d^2 \triangleq \frac{1}{2} \left(\frac{2}{N_0} \right)^2 \int_{T_i}^{T_f} \int_{T_i}^{T_f} K_s^2(t, u) dt du \quad (5.121)$$

All that remains is to show that d^2 as defined in Eq. 5.121 is a suitably defined "output signal-to-noise ratio". This is a straightforward calculation. The steps are indicated in Ref. [13]. This is also discussed in Ref. [23].

E. Summary

In this rather lengthy chapter, we have presented examples illustrating the application of our approximation techniques to performance calculations for a number of detection problems. Our examples were chosen from all three levels of detection problems. In those few cases where exact answers were available for comparison, our approximation compared favorably with the exact results.

As an introductory exercise, we applied our error approximations to the known signal problem. We then considered several slow Rayleigh fading diversity examples. As an example of the analysis of a suboptimum receiver, we considered a square-law receiver for the random phase channel. Finally, we considered a number of simple binary and symmetric binary random process detection problems. We considered a simple signal optimization problem, we illustrated how our results could be simplified for the case of long time intervals and stationary processes, and we showed how our approximation could be applied to dispersive channels which could be

modeled by means of a tapped delay line. For the symmetric binary, bandpass problem we obtained upper and lower bounds on our first order approximation. For the simple binary, low energy coherence case we demonstrated that our approximations are identical with the results usually used for this special case.

VI. OPTIMUM SIGNAL DESIGN FOR SINGLY-SPREAD TARGETS (CHANNELS)

In the previous chapters, we saw how our approximation techniques can be used to analyze the error probability for several different types of radar and communication systems. We also considered several simple optimization problems where we optimized the performance as a function of some parameter such as number of diversity channels and duration of the signal pulse.

In this chapter, we shall study a more general problem in system design. The problem of interest (see Fig. 6.1) is essentially that which was analyzed in Example D-1 in the previous chapter. It is closely related to the more realistic problem of communicating over a Rayleigh fading channel, when the fading spectrum is symmetrical about the carrier and is stationary. Then, using Eq. 3.36,

$$\mu_{BP}(s:2E/N_0) = 2\mu_{LP}(s:E/N_0), \quad (6.1)$$

where the subscripts BP and LP denote "low-pass" and "bandpass".

We wish to pick $s(t)$, subject to an energy constraint to maximize the performance of the optimum detector. The performance measure which we would like to minimize is the error probability. However, the complexity of the error expressions is not very encouraging. Instead, we minimize $\mu(s)$ which appears as the dominant term in our error approximations. Recall from our diversity examples in Chapter V that the diversity systems designed to minimize the exact

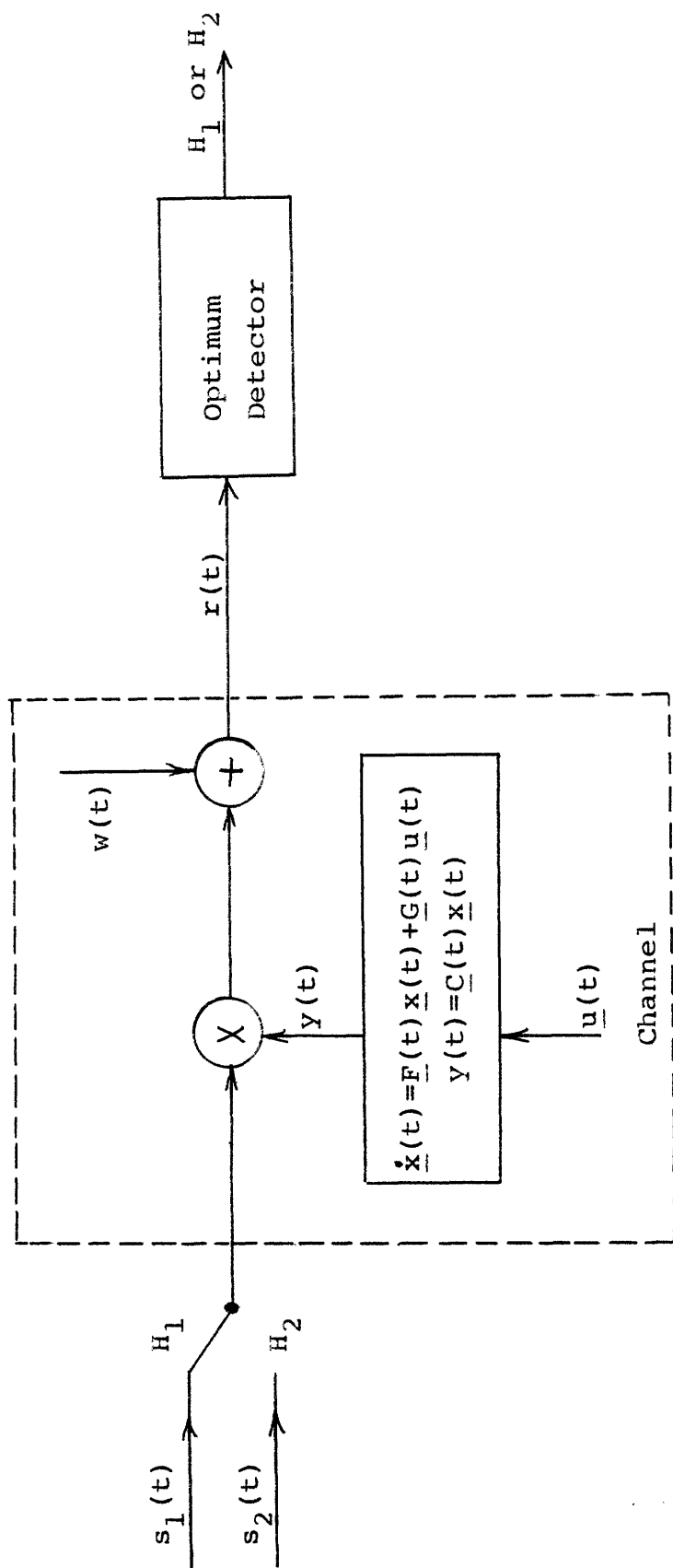


Figure 6.1. Communication System Model

error expression, the Chernoff bound, and our first order approximations gave essentially the same performance. See Fig. 5.4.

A. A Lower Bound on $\mu(1/2)$

Kennedy [50] has obtained a lower bound on $\mu(\frac{1}{2})$ for the symmetric hypothesis problem, assuming that we can achieve the optimum energy distribution for the channel output subject to a constraint on its average energy. For stationary channels, this is equivalent to constraining the transmitted energy. His derivation does not tell us how to achieve this optimum energy distribution when we only have the transmitted signal at our disposal, nor does it tell us if the lower bound can be attained. However, it does provide us with a useful "yardstick" for gaging the performance of suboptimum signals. Since Kennedy's derivation is very short, we include it here.

Recall that throughout this chapter we are considering a low-pass problem. Then from Eqs. 3.21 and 3.28,

$$\mu\left(\frac{1}{2}\right) = \frac{1}{2} \sum_{i=1}^{\infty} \left[\ln\left(1 + \frac{2\lambda_i}{N_0}\right) - 2\ln\left(1 + \frac{\lambda_i}{N_0}\right) \right] \quad (6.2)$$

with

$$\sum_{i=1}^N \lambda_i = \bar{E}_r$$

Define,

$$g(x) = -\frac{1}{x} \left[\frac{1}{2} \ln(1+x) - \ln(1+x/2) \right] \quad (6.3)$$

Then,

$$\mu(1/2) = - \sum_{i=1}^{\infty} \frac{2\lambda_i}{N_0} g\left(\frac{2\lambda_i}{N_0}\right) \quad (6.4)$$

The function $g(x)$ is plotted in Fig. 7.44 of Ref. [35]. It is a positive function with a unique minimum which is the solution to

$$g(x_0) = \frac{1}{4(1+x_0)(1+x_0/2)} \quad (6.5)$$

Solving numerically, we find

$$x_0 = 3.0700 \pm 10^{-4} \quad (6.6)$$

Then,

$$\begin{aligned} \mu(1/2) &\geq - \sum_{i=1}^{\infty} \frac{2\lambda_i}{N_0} g(x_0) \\ &= - \frac{2\bar{E}}{N_0} g(x_0) \\ &= -0.1488 \bar{E}_z / N_0 \end{aligned} \quad (6.7)$$

When \bar{E}_r/N_0 is an integral multiple of x_0 , we can achieve the bound by letting

$$\lambda_i = \begin{cases} \bar{E}_r/x_0 & i = 1, 2, \dots, \frac{\bar{E}_r}{N_0 x_0} \\ 0 & \text{otherwise} \end{cases} \quad (6.8)$$

Therefore, the most desirable situation is to have an equal number of non-zero eigenvalues in the received random process, with the number of components determined by the available signal-to-noise ratio. For the important class of state-variable random processes, it follows from Baggeroer's technique for solving homogeneous Fredholm integral equations, Ref. [21] that a random process having N states can have at most N equal eigenvalues. Hence, we immediately conclude that for most problems of interest, we can never exactly achieve the bound of Eq. 6.7. However, when the fading spectrum is bandlimited, we saw in Example D-4 that we could approach this bound. Recall that the expressions for $\mu(s)$ used in that example were approximations based on the long time interval approximation. The actual minimum value of $\mu(1/2)$ for that example will be slightly larger.

B. Formulation of a Typical Optimization Problem

We now shall investigate whether we can find a transmitted signal $s(t)$ which minimizes $\mu(1/2)$, subject to a constraint on its energy. We shall use Pontryagin's minimum principle to obtain a set of differential equations and boundary conditions which the

optimum signal must satisfy. An intelligent reading of this section presumes a working knowledge of the minimum principle, such as given in Chapter V of Ref. [46]. The necessary conditions which we derive could also be obtained from the classical calculus of variations with a little more effort.

The conclusion that we shall reach is that there is no waveform that achieves a minimum value of $\mu(1/2)$. Therefore, the reader who is not facile with the minimum principle can skip this section. We include it for the benefit of those persons who are using, or who plan to use similar optimization techniques for other problems.

Our cost functional* is

$$J = \mu\left(\frac{1}{2}\right) = \frac{1}{N_0} \int_{T_i}^{T_f} \xi_1(t) dt - \frac{2}{N_0} \int_{T_i}^T \xi_2(t) dt \quad (6.9)$$

where

$$\xi_1(t) = s^2(t) \underline{C}(t) \underline{\Sigma}_1(t) \underline{C}^T(t) \quad (6.10a)$$

$$\xi_2(t) = s^2(t) \underline{C}(t) \underline{\Sigma}_2(t) \underline{C}^T(t) \quad (6.10b)$$

$$\frac{d}{dt} \underline{\Sigma}_1(t) = \underline{F}(t) \underline{\Sigma}_1(t) + \underline{\Sigma}_1(t) \underline{F}^T(t) + \underline{G}(t) \underline{Q}(t) \underline{G}^T(t)$$

$$-s^2(t) \frac{2}{N_0} \underline{\Sigma}_1(t) \underline{C}^T(t) \underline{C}(t) \underline{\Sigma}_1(t) \quad (6.11a)$$

*The symbol J should not be confused with the same symbol which we used for the divergence in Chapters I and II. We use J here to be consistent with the control theory literature.

$$\frac{d}{dt} \underline{\Sigma}_2(t) = \underline{F}(t)\underline{\Sigma}_2(t) + \underline{\Sigma}_2(t)\underline{F}^T(t) + 1/2\underline{G}(t)\underline{Q}(t)\underline{G}^T(t) - s^2(t)\frac{2}{N_0}\underline{\Sigma}_2(t)\underline{C}^T(t)\underline{C}(t)\underline{\Sigma}_2(t) \quad (6.11b)$$

$$\underline{\Sigma}_1(T_1) = \underline{\Sigma}_0 \quad (6.12a)$$

$$\underline{\Sigma}_2(T_1) = \frac{1}{2}\underline{\Sigma}_0 \quad (6.12b)$$

By the results of Chapter III, Section D, this can be rewritten in terms of the solution to a linear system of equations.

$$\begin{aligned} J &= \frac{1}{2} \left(\ln \det \underline{X}_2(T_f) + \int_{T_i}^{T_f} \text{Tr}[\underline{F}(t)] dt \right) \\ &\quad - \left(\ln \det \underline{X}_4(T_f) + \int_{T_i}^{T_f} \text{Tr}[\underline{F}(t)] dt \right) \\ &= 1/2 \ln \det \underline{X}_2(T_f) - \ln \det \underline{X}_4(T_f) - 1/2 \int_{T_i}^{T_f} \text{Tr}[\underline{F}(t)] dt \end{aligned} \quad (6.13)$$

The last term in Eq. 6.13 is independent of the modulation and may be ignored. The first two terms are obtained from the solution of a system of linear differential equations:

$$\frac{d}{dt} \underline{X}_1(t) = \underline{F}(t)\underline{X}_1(t) + \underline{G}(t)\underline{Q}(t)\underline{G}^T(t)\underline{X}_2(t) \quad (6.14a)$$

$$\frac{d}{dt} \underline{X}_2(t) = s^2(t)\underline{C}^T(t)\frac{2}{N_0}\underline{C}(t)\underline{X}_1(t) - \underline{F}^T(t)\underline{X}_2(t) \quad (6.14b)$$

$$\frac{d}{dt} \underline{X}_3(t) = \underline{F}(t)\underline{X}_3(t) + \frac{1}{2}\underline{G}(t)\underline{Q}(t)\underline{G}^T(t)\underline{X}_4(t) \quad (6.14c)$$

$$\frac{d}{dt} \underline{X}_4(t) = s^2(t)\underline{C}^T(t)\frac{2}{N_0}\underline{C}(t)\underline{X}_3(t) - \underline{F}^T(t)\underline{X}_4(t) \quad (6.14d)$$

with the boundary conditions

$$\underline{X}_1(T_1) = \underline{\Sigma}_0 \quad (6.15a)$$

$$\underline{X}_2(T_1) = \underline{I} \quad (6.15b)$$

$$\underline{X}_3(T_1) = \frac{1}{2}\underline{\Sigma}_0 \quad (6.15c)$$

$$\underline{X}_4(T_1) = \underline{I}. \quad (6.15d)$$

It is more convenient for our present purposes to use the nonlinear system of equations in Eqs. 6.11 a and b. Then,

$$\begin{aligned} J = & \frac{1}{N_0} \int_{T_1}^{T_f} s(t) \text{Tr}[\underline{C}^T(t)\underline{C}(t)\underline{\Sigma}_1(t)] dt \\ & - \frac{2}{N_0} \int_{T_1}^{T_f} s^2(t) \text{Tr}[\underline{C}^T(t)\underline{C}(t)\underline{\Sigma}_2(t)] dt \end{aligned} \quad (6.16)$$

We incorporate our energy constraint

$$E = \int_{T_i}^{T_f} s^2(t) dt \quad (6.17)$$

by introducing a third state variable

$$\dot{\sigma}_3(t) = s^2(t) \quad (6.18)$$

The constraint then becomes a target value (i.e. boundary condition) for $\sigma_3(t)$.

$$\sigma_3(T_f) = E \quad (6.19)$$

C. Application of Pontryagin's Minimum Principle

First, we form the Hamiltonian:

$$\begin{aligned} H &= p_0 \dot{J}(t) + \text{Tr}[\dot{\underline{\Sigma}}_1(t) \underline{P}_1^T(t) + \dot{\underline{\Sigma}}_2(t) \underline{P}_2^T(t)] + p_s(t) \dot{\sigma}_3(t) \\ &= \frac{p_0}{N_0} s^2(t) \text{Tr}[\underline{C}^T(t) \underline{C}(t) (\underline{\Sigma}_1(t) - 2\underline{\Sigma}_2(t))] \\ &\quad + \text{Tr}[\underline{F}(t) \underline{\Sigma}_1(t) \underline{P}_1^T(t) + \underline{\Sigma}_1(t) \underline{F}^T(t) \underline{P}_1^T(t) + \underline{G}(t) \underline{Q}(t) \underline{G}^T(t) \underline{P}_1^T(t)] \\ &\quad - \frac{2}{N_0} s^2(t) \text{Tr}[\underline{\Sigma}_1(t) \underline{C}^T(t) \underline{C}(t) \underline{\Sigma}_1(t) \underline{P}_1^T(t)] \end{aligned}$$

$$\begin{aligned}
& + \text{Tr}[\underline{F}(t)\underline{\Sigma}_2(t)\underline{P}_2^T(t) + \underline{\Sigma}_2(t)\underline{F}^T(t)\underline{P}_2^T(t) + \frac{1}{2}\underline{G}(t)\underline{Q}(t)\underline{G}^T(t)\underline{P}_2^T(t)] \\
& - \frac{2}{N_0} s^2(t)\text{Tr}[\underline{\Sigma}_2(t)\underline{C}^T(t)\underline{C}(t)\underline{\Sigma}_2(t)\underline{P}_2^T(t)] \\
& + p_3(t)s^2(t) \tag{6.20}
\end{aligned}$$

where $\underline{P}_1(t)$, $\underline{P}_2(t)$, and $p_3(t)$ are the Lagrange multiplier (costate) functions. We now obtain the differential equations for the costates.

$$\begin{aligned}
\frac{d}{dt} \underline{P}_1(t) &= - \frac{\partial H}{\partial \underline{X}_1} \\
&= [-\underline{F}^T(t) + \frac{2}{N_0} s^2(t)\underline{C}^T(t)\underline{C}(t)\underline{\Sigma}_1(t)]\underline{P}_1(t) \\
&\quad + \underline{P}_1(t)[- \underline{F}^T(t) + \frac{2}{N_0} s^2(t)\underline{\Sigma}_1(t)\underline{C}^T(t)\underline{C}(t)] \\
&\quad - \frac{P_0}{N_0} s^2(t)\underline{C}^T(t)\underline{C}(t) \tag{6.21}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \underline{P}_2(t) &= - \frac{\partial H}{\partial \underline{X}_2} \\
&= [-\underline{F}^T(t) + \frac{2}{N_0} s^2(t)\underline{C}^T(t)\underline{C}(t)\underline{\Sigma}_2(t)]\underline{P}_2(t)
\end{aligned}$$

$$\begin{aligned}
& + \underline{P}_2(t) \{-\underline{F}^T(t) + \frac{2}{N_0} s^2(t) \underline{\Sigma}_2(t) \underline{C}^T(t) \underline{C}(t)\} \\
& + \frac{2p_0}{N_0} s^2(t) \underline{C}^T(t) \underline{C}(t)
\end{aligned} \tag{6.21b}$$

$$\frac{dp_3}{dt} = 0 \rightarrow p_3(t) = \text{constant} \stackrel{\Delta}{=} p_3 \tag{6.21c}$$

Since we have no terminal cost, the boundary conditions on the costates are:

$$\underline{P}_1(T_f) = \underline{0} \tag{6.22a}$$

$$\underline{P}_2(T_f) = \underline{0} \tag{6.22b}$$

Let us define,

$$\begin{aligned}
\alpha(t) = & \frac{p_0}{2} \text{Tr}[\underline{C}^T(t) \underline{C}(t) (\underline{\Sigma}_1(t) - 2\underline{\Sigma}_2(t))] \\
& - \text{Tr}[\underline{\Sigma}_1(t) \underline{C}^T(t) \underline{C}(t) \underline{\Sigma}_1(t) \underline{P}_1^T(t)] \\
& - \text{Tr}[\underline{\Sigma}_2(t) \underline{C}^T(t) \underline{C}(t) \underline{\Sigma}_2(t) \underline{P}_2^T(t)] \\
& + p_3 N_0/2, \text{ and}
\end{aligned} \tag{6.23}$$

$$\begin{aligned}
\beta(t) = & \text{Tr}[\underline{F}(t)\underline{\Sigma}_1(t)\underline{P}_1^T(t) + \underline{\Sigma}_1(t)\underline{F}^T(t)\underline{P}_1^T(t) \\
& + \underline{G}(t)\underline{Q}(t)\underline{G}^T(t)\underline{P}_1^T(t) \\
& + \underline{F}(t)\underline{\Sigma}_2(t)\underline{P}_2^T(t) + \underline{\Sigma}_2(t)\underline{F}^T(t)\underline{P}_2^T(t) \\
& + \frac{1}{2}\underline{G}(t)\underline{Q}(t)\underline{G}^T(t)]. \tag{6.24}
\end{aligned}$$

Then,

$$H = \frac{2}{N_0} s^2(t)\alpha(t) + \beta(t) \tag{6.25}$$

The necessary condition that the Hamiltonian be minimized as a function of $s^2(t)$ implies,

$$\text{i.} \quad \text{If } \beta(t) > 0, s^2(t) = 0 \tag{6.26a}$$

$$\text{ii.} \quad \text{If } \beta(t) = 0, \text{ this condition implies nothing about } s(t) \tag{6.26b}$$

$$\text{iii.} \quad \text{If } \beta(t) < 0, s^2(t) \text{ is unbounded.} \tag{6.26c}$$

Since we have an energy constraint on $s(t)$, the third alternative implies the use of an arbitrarily short pulse of the required energy. This gives rise to only one non-zero eigenvalue in our observation.

We have seen that for signal-to-noise ratios greater than about three, this is not optimum. Hence, we discard the third possibility. Thus, either $s(t) = 0$ or $\beta(t) = 0$.

The general behavior of the optimum signal is indicated in Fig. 6.2. Denoting the set of times where the solution switches from (i) to (ii) or vice-versa by $\{T_j, j = 1, 2, \dots, N\}$ we then conclude from the differential equations and boundary conditions for $\underline{P}_1(t)$ and $\underline{P}_2(t)$ that

$$\underline{P}_1(T_N) = \underline{P}_2(T_N) = \underline{0} \quad (6.27)$$

regardless of whether $T_N = T_f$.

We now shall consider the first-order case in the interest of algebraic simplicity, and we shall attempt to obtain additional information about $s(t)$ in the interval $T_{N-1} \leq t \leq T_N$. In this interval,

$$\alpha(t) = 0$$

$$= \frac{p_0}{2} [\Sigma_1(t) - 2\Sigma_2(t)]$$

$$- \Sigma_1^2(t)p_1(t) - \Sigma_2^2(t)p_2(t) + \frac{N}{2} p_3 \quad (6.28)$$

Thus, all the derivatives of $\alpha(t)$ must also vanish on the interval

$$T_{N-1} \leq t \leq T_N.$$

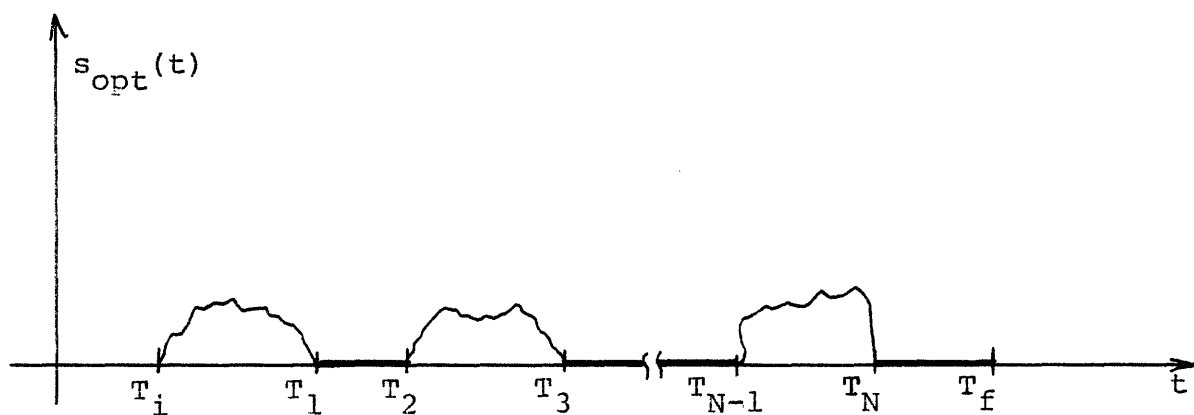


Figure 6.2. General Behavior of Optimum Signal

$$\begin{aligned}
\dot{\alpha}(t) &= \frac{p_0}{2} [\dot{\Sigma}_1(t) - 2\dot{\Sigma}_2(t)] \\
&\quad - 2\dot{\Sigma}_1(t)\Sigma_1(t)p_1(t) - 2\dot{\Sigma}_2(t)\Sigma_2(t)p_2(t) \\
&\quad - \Sigma_1^2(t)\dot{p}_1(t) - \Sigma_2^2(t)\dot{p}_2(t) \tag{6.29}
\end{aligned}$$

Now substitute the differential equations for $\dot{\Sigma}_1(t)$, $\dot{\Sigma}_2(t)$, $\dot{p}_1(t)$ and $\dot{p}_2(t)$.

$$\begin{aligned}
\dot{\alpha}(t) &= \left[\frac{p_0}{2} - 2\Sigma_1(t)p_1(t) \right] \left[-2k\Sigma_1(t) + 2kP - \frac{2}{N_0} s^2(t)\Sigma_1^2(t) \right] \\
&\quad + \left[-p_0 - 2\Sigma_2(t)p_2(t) \right] \left[-2k\Sigma_2(t) + kP - \frac{2}{N_0} s^2(t)\Sigma_2^2(t) \right] \\
&\quad - \Sigma_1^2(t) \left[\left(2k + \frac{4}{N_0} s^2(t)\Sigma_1(t) \right) p_1(t) - \frac{p_0}{N_0} s^2(t) \right] \\
&\quad - \Sigma_2^2(t) \left[\left(2k + \frac{4}{N_0} s^2(t)\Sigma_2(t) \right) p_2(t) + \frac{2p_0}{N_0} s^2(t) \right] \tag{6.30}
\end{aligned}$$

Making all possible cancellations, we obtain

$$\begin{aligned}
\dot{\alpha}(t) &= -2k \left[\beta(t) + \frac{N}{2} p_3 \right] \\
&\quad - 4kP \left[\Sigma_1(t)p_1(t) + \frac{1}{2}\Sigma_2(t)p_2(t) \right] \tag{6.31}
\end{aligned}$$

Therefore,

$$\frac{N}{2} p_3 = 2P\{\Sigma_1(t)p_1(t) + \frac{1}{2}\Sigma_2(t)p_2(t)\} \quad (6.32)$$

Applying boundary conditions on

$$p_1(T_N) \text{ and } p_2(T_N),$$

$$p_1(T_N) = p_2(T_N) = 0, \quad (6.33)$$

then,

$$p_3 = 0. \quad (6.34)$$

This is impossible, since this says that the solution is independent of the energy constraint. Therefore, there is no optimum waveform $s(t)$. Since we have found a lower bound on $\mu(\frac{1}{2})$, we knew there is a greatest lower bound. Thus, we have demonstrated that this greatest lower bound can never be achieved. In the next section we shall see that this result is not of much practical importance for we can find signals which are capable of performing nearly as well as the bound of Section A.

D. A Nearly Optimum Signals for Binary Communication

Having not succeeded at an analytical solution to the signal design problem which was formulated earlier in this chapter, we now proceed to analyze a set of nearly optimum signals. First, we indicate the line of reasoning that led us to consider this signal set.

In Section A of this chapter, we demonstrated that the smallest error exponent was obtained when we had N equal eigenvalues, where N depended on the available average signal-to-noise energy ratio.

In Example C-4 of Chapter V, we saw that for a band-limited fading spectrum, we could achieve this lower bound by transmitting a square pulse whose duration T is properly chosen. Furthermore, we know that samples of a bandlimited random process taken every $1/(2W)$ seconds are uncorrelated, where W denotes the single-sided bandwidth of the process. Therefore, we can take $N = 2WT$ uncorrelated samples in T seconds. Hence, these N time samples are a suitable set of coordinates for detecting a bandlimited spectrum. In practice, of course, we observe N segments of the random process since our sampling pulses must have finite duration. Therefore, our observations are only approximately uncorrelated. The height of these sampling pulses are determined in practice by the practical consideration of a peak-power constraint, such as at the transmitter.

The above discussion suggests that for an arbitrary channel fading process, a good signal to use might be N identical pulses,

separated sufficiently so that the correlation between the various segments of the observed process is small, as indicated in Fig. 6.3. If we do not constrain the duration of the signal, the problem is trivial, for by taking our samples far enough apart the correlation between samples can be made arbitrarily small. So, the meaningful problem is to constrain both the energy and duration of the signal, then pick the optimum number of pulses. We then have a N -th order diversity system, where the diversity branches are correlated, and we want to determine the optimum diversity.

It is straightforward to evaluate the performance of this set of "near optimum" signals by using the techniques we have developed. Recall from Example C-4 of Chapter V that for a single-pole fading spectrum the optimum value of $\mu(1/2)$ was the farthest from the lower bound of $-.1488 \bar{E}_r/N_0$. Therefore, we treat this "worst case" spectrum in our example.

It is interesting from a theoretical point of view to consider the limiting case as the width of the pulses vanishes, for this will give us a bound on how well we can do with a set of finite width pulses. We then have a slowly-fluctuating correlated diversity problem. We observe N Gaussian random variables, so that we don't need any of our sophisticated results for random processes. From Eq. 3.4b we have,

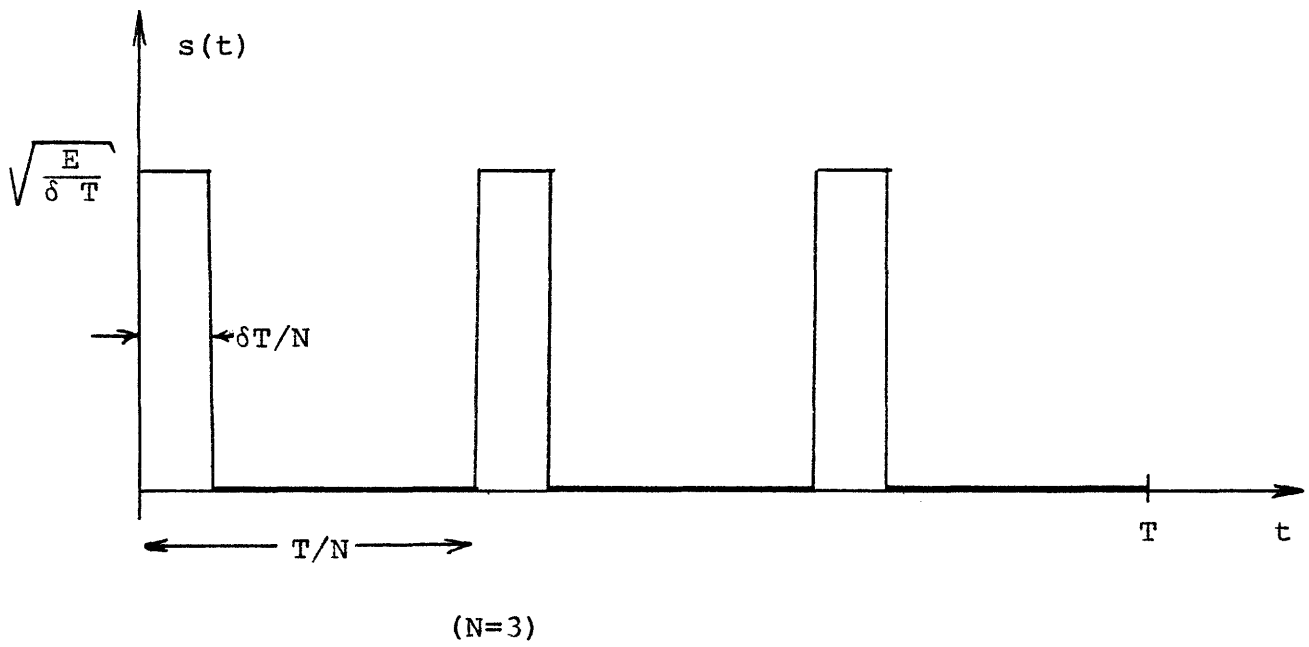


Figure 6.3. A Nearly Optimum Set of Signals

$$\begin{aligned} \mu\left(\frac{1}{2}\right) &= \frac{1}{2} \ln \det\left(\underline{I} + \frac{2\bar{E}}{NN_0} \underline{P}\right) \\ &\quad - \ln \det\left(\underline{I} + \frac{\bar{E}}{NN_0} \underline{P}\right) \end{aligned} \quad (6.35)$$

where

$$\underline{P} = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & & \\ \rho^2 & \rho & 1 & & \\ & & & \ddots & \rho \\ & & & & 1 \end{bmatrix} \quad (6.36)$$

and

$$\rho = e^{-kT/N} \quad (6.37)$$

The evaluation of these determinants is straightforward. We can either proceed numerically or we can use a difference equation derived by Kurth [70,71].

In Fig. 6.4, we plot the optimum exponent as a function of kT , with $2\bar{E}/N_0$ as a parameter. We observe that when kT is approximately the optimum value for a single square pulse (see Example C-3), we can obtain an exponent very near the bound of $-.1488 \bar{E}/N_0$ by using N short pulses, as compared with the exponent $-.12\bar{E}/N_0$ which results from using a single square pulse. Specifically, we achieved an exponent of $-.143 \bar{E}/N_0$ which is a degradation of less than 0.2 db.

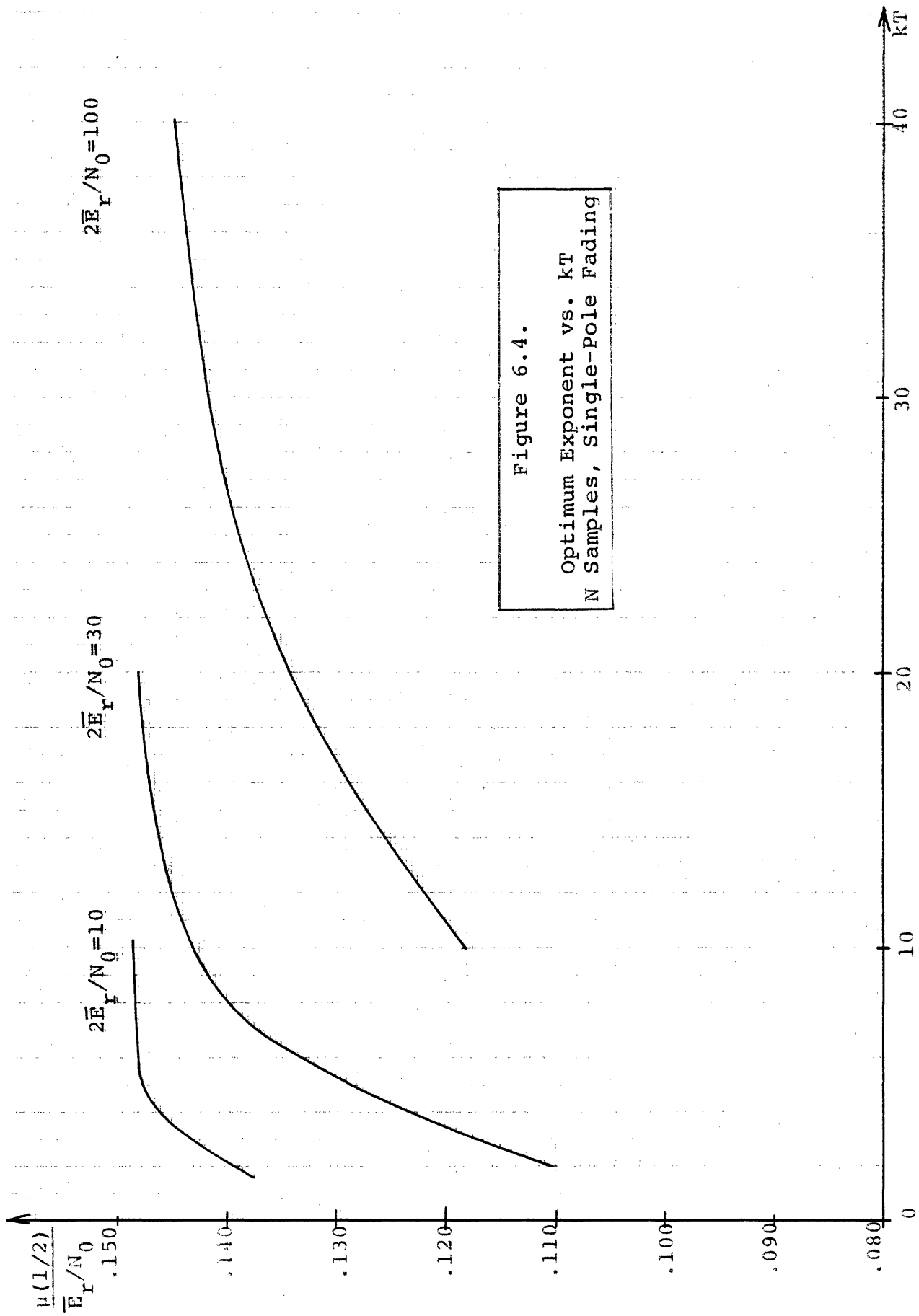


Figure 6.4.
Optimum Exponent vs. kT
 N Samples, Single-Pole Fading

In Fig. 6.5a and b we plot similar curves for finite-width pulses. For these signals the details of the calculations are considerably more complex, since we must use the general results for calculating $\mu(s)$ for random processes that was developed in Chapter III. The parameter δ is the "duty-factor" of the waveform; it is the ratio of average to peak power. Then for a given $2\bar{E}/N_0$, choosing a value for d corresponds to a particular peak power constraint. We observe that for $\delta = .1$ (certainly a rather stringent peak power constraint) we can achieve an exponent of $-.140 \bar{E}/N_0$. Furthermore, for $\delta = .01$, the results are essentially the same as those for the limiting case of zero-width pulses.

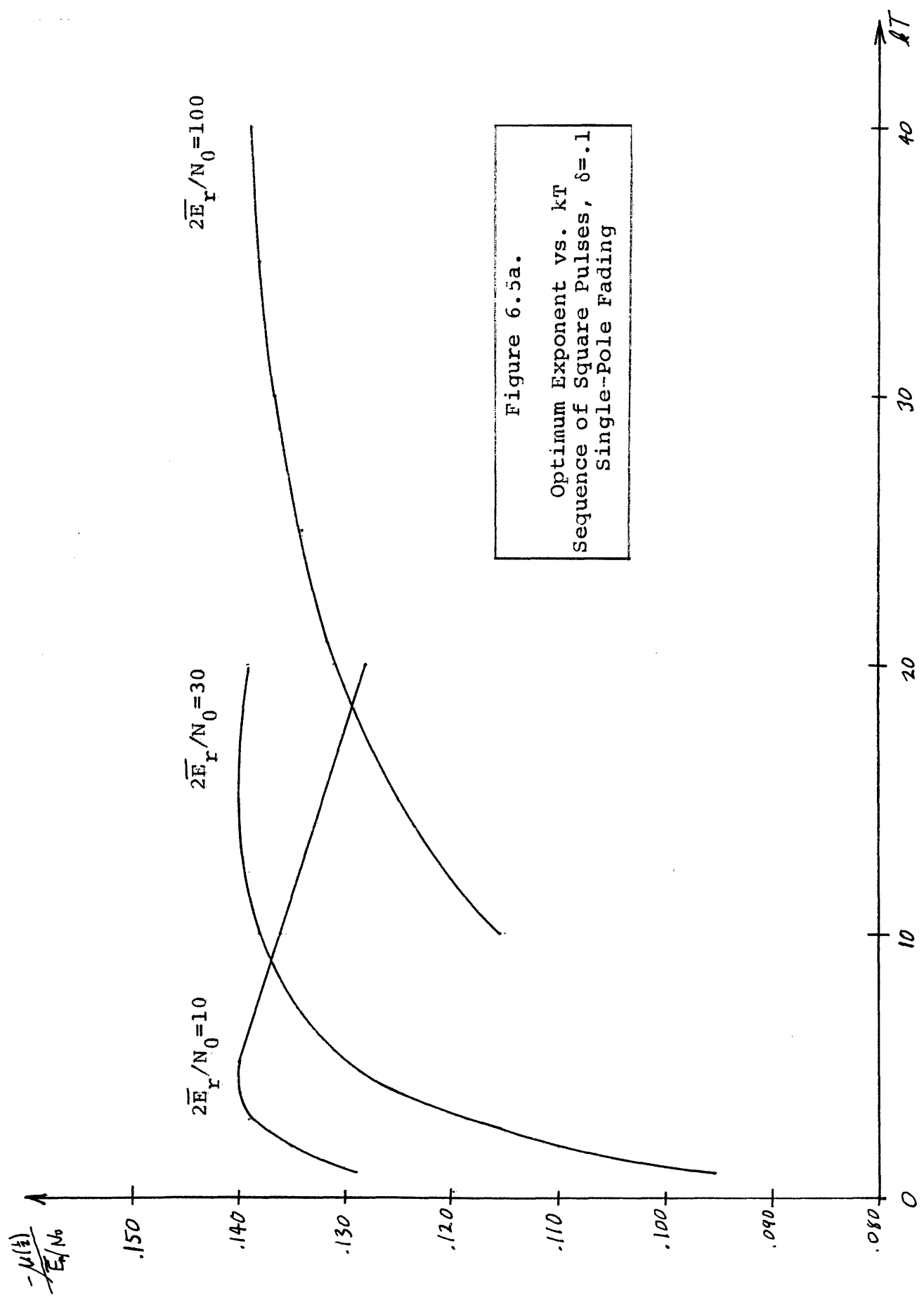


Figure 6.5a.
Optimum Exponent vs. kT
Sequence of Square Pulses, $\delta=.1$
Single-Pole Fading

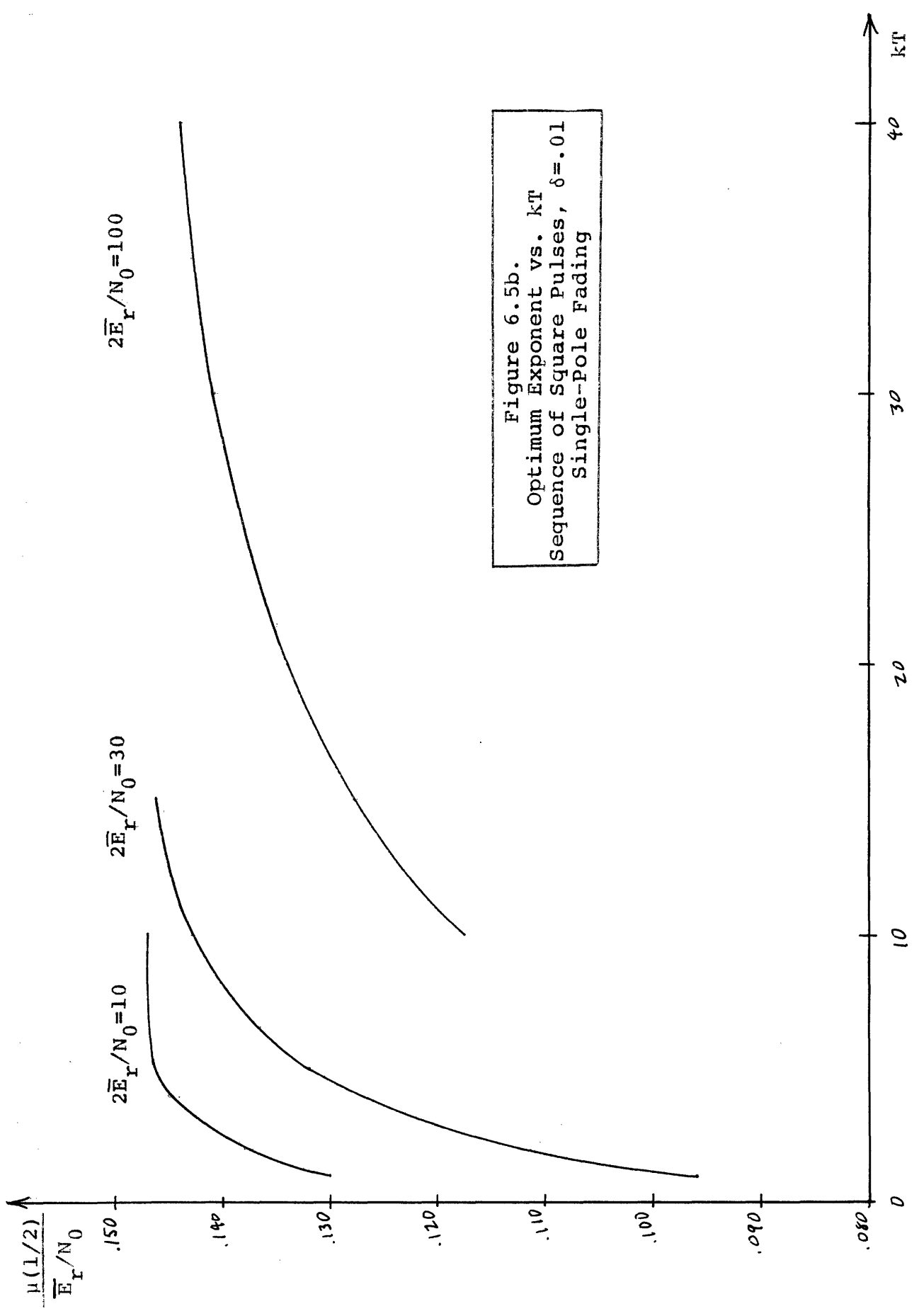


Figure 6.5b.
Optimum Exponent vs. kT
Sequence of Square Pulses, $\delta=.01$
Single-Pole Fading

VII. TOPICS FOR FURTHER RESEARCH AND SUMMARY

In this chapter we first outline several topics for further research. They represent refinements and logical extensions of our error approximations and applications to several important physical problems. We then conclude with a summary of the results contained in this thesis.

A. Error Analysis

A topic of prime importance which has not been treated to our satisfaction is the analysis of the accuracy of our probability of error approximations. In Chapter V, we obtained upper and lower bounds on our first-order asymptotic approximation for the symmetric binary, bandpass problem. We would like to be able to make further quantitative statements about the accuracy which results from retaining a given number of terms in Eq. 2.26. Preliminary results indicate errors of less than ten percent when using only the first term approximations for several examples where the tilted density is far from being Gaussian. Other examples where probabilities were computed from the eigenvalues of typical random processes indicate even smaller errors.

B. More than Two Hypotheses

For the most of this thesis, we have been concerned with the binary detection problem. An obvious extension is to the case of M hypotheses. In addition to allowing us to consider problems with more than two hypotheses, we also can consider transmitting sequences of binary digits so that our estimation of the multiplicative channel

disturbances can exploit the correlation from bit to bit. That is, if the bit duration is less than the correlation time, we would like to be able to predict the improvement in performance which results from exploiting this correlation.

In Chapter IV, we saw that the performance for a particular class of M-ary problem could be obtained from the binary results of Chapters II and III. Furthermore, in Chapter VI, we obtained a set of nearly optimum signals for the binary problem. An interesting problem will be the evaluation of the performance which can be obtained by using a similar set of signals in an M-ary system. This straightforward extension will be included in a forthcoming report [72].

C. Markov (Non-Gaussian) Detection Problems

In all our work up to this point, we relied heavily on the Gaussian assumption which enabled us to compute the semi-invariant moment generating function $\mu(s)$ that played a central role in our error approximations. We also found that our calculations were particularly straightforward when the random processes of interest could be modeled via state variables. We, thus, have been treating a class of finite-dimensional Gaussian, Markov processes.

A more general class of Markov random processes can be generated by allowing the random processes to be generated by nonlinear (compare Eqs. 3.37a and b) dynamic systems excited by white (not necessarily Gaussian) noise, $\underline{u}(t)$.

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), t) + \underline{G}(\underline{x}(t), t)\underline{u}(t) \quad (7.1)$$

$$y(t) = h(\underline{x}(t)). \quad (7.2)$$

The optimum detector (likelihood function computer) for this problem has been studied, and bears a striking similarity to that for the Gaussian problem we have been studying, since it consists of an optimum MMSE (nonlinear) estimator and a correlator [73]. The structure and performance of the MMSE estimator has been studied previously in the context of analog communication [74]. It may be possible to express $\mu(s)$ in terms of MMSE estimation error just as in the Gaussian case of Chapter II. This would enable us to apply the probability approximations to a much wider class of problems.

D. Delay-Spread Targets

We have seen how our error probability approximations coupled with state-variable models for random processes lead to straightforward performance evaluation when our signals are propagated over a channel which is spread in Doppler, but not in range. The concept of time-frequency duality enables us to treat a class of delay-spread targets (channels) [75]. For example, the dual of a channel whose Doppler scattering function is rational in ω^2 is a channel whose delay scattering function is a rational function of t^2 . Such problems readily fit into the state-variable formulation. Moreover, we can approximate a number of delay scattering functions by this class of functions, just as the network designer is able to approximate a wide class of frequency responses by rational functions in ω^2 .

E. Doubly-Spread Targets

The next logical extension of our results is to the case of targets spread in both range and Doppler. This model corresponds more closely to some physical channels than do our previous models. Clearly, no lumped state-space model is appropriate in general, since we are modeling effects which take place in a distributed system. One model which is widely used is the tapped-delay line which was discussed briefly in one of the examples in Chapter V.

F. Sub-Optimum Receivers

In Chapters II and III we developed all the results necessary to analyze a relatively large class of suboptimum receivers. There are a number of interesting applications of these results. Some preliminary results along these lines will be presented elsewhere [67].

G. Bandpass Signals

In many of the examples that we considered, the signals were narrow-band about some carrier frequency. However, in the interest of simplicity we restricted ourselves to stationary processes with symmetric spectra. The techniques of complex state variables [76] will enable us to consider a much wider class of problems.

H. Summary

This thesis has been concerned with the problem of evaluating the error probabilities for detecting random signals. We concentrated on the development and application of an asymptotic

approximation to the error probabilities which combined the technique of "tilted" probability densities with the Edgeworth expansion. The unifying thread throughout our discussion of performance was the semi-invariant moment generating function $\mu(s)$.

In Chapter I, we gave a brief discussion of the detection problem. We reviewed the form of the optimum receiver and motivated the discussion of performance which occupied the remainder of the thesis.

In Chapter II, we developed several bounds on and approximations to the error probabilities for a rather general binary detection problem. $\mu(s)$ played a central role in all our results. These results were obtained without making any assumptions on the conditional statistics of the received signals.

In Chapter III, we calculated the semi-invariant moment generating function $\mu(s)$ for the problem of detecting Gaussian signals in Gaussian noise. For both optimum and suboptimum receivers, we expressed $\mu(s)$ in terms of the Fredholm determinant. The Fredholm determinant can be obtained from the solution to a related minimum mean square error optimum linear filtering problem. For the important class of random processes which can be modeled via state variables, we obtained a straightforward technique for computing $\mu(s)$.

In Chapter IV, we demonstrated how the results of Chapter III could be used to obtain the exponent-rate curve for M -ary orthogonal, symmetric communication systems.

In Chapter V, we considered a number of examples illustrating the application of the techniques we have developed to the calculation

of error probabilities for some simple detection problems. As an introductory exercise, we applied our error approximations to the known signal problem. We then considered several slow Rayleigh fading diversity examples. For these problems, we compared our results with those obtained by others, and we demonstrated simple examples where other performance criteria lead to erroneous results. As an example of the analysis of a suboptimum receiver, we analyzed a square-law receiver for the random phase channel. Finally, we considered a number of simple binary and symmetric binary random process detection problems. We considered a simple signal optimization problem, we illustrated how our results could be simplified in the case of long time intervals and rational spectra, and we showed how our results could be applied to dispersive channels which can be modeled by means of a tapped delay line. We showed that our approximations agreed with the output signal-to-noise ratio criterion for the low energy coherence case. For the symmetric binary, bandpass problem we obtained upper and lower bounds on our first-order asymptotic approximation.

In Chapter VI, we considered a simple signal optimization problem. We chose as our performance criterion $\mu(1/2)$ for purposes of illustration. After obtaining a lower bound on $\mu(1/2)$, we applied Pontryagin's minimum principle to obtain a set of necessary conditions which the optimum signal must satisfy. We then demonstrated that no signal satisfied these conditions. We then demonstrated a class of waveforms whose performance closely approached

the lower bound.

The present chapter has given a brief discussion of topics for further research and a summary.

APPENDIX A ERROR PROBABILITIES IN TERMS OF TILTED RANDOM VARIABLES

In this Appendix, we derive the expression for the error probabilities in terms of the semi-invariant moment generating function $\mu(s)$. Furthermore, we demonstrate some useful properties of this function. Much of this material has appeared previously [33, 36, 52, 77, 78, 79]. However, the details of the proof are our own.

Throughout this Appendix, we assume \underline{r} is a N-dimensional random vector with conditional probability densities

$$p_{\underline{r}|H_1}(\underline{R}) \quad \text{and} \quad p_{\underline{r}|H_2}(\underline{R})$$

on hypotheses H_1 and H_2 respectively.

The function $\mu(s)$ is defined as the semi-invariant moment generating function for the log-likelihood ratio conditioned on hypothesis H_1 . We consider only real values of the argument s .

$$\begin{aligned} \mu(s) &= \ln M_{\lambda|H_1}(s) \\ &= \ln \int_{-\infty}^{\infty} e^{sL} p_{\lambda|H_1}(L) dL \\ &= \ln \int \dots \int e^{s\lambda(\underline{R})} p_{\underline{r}|H_1}(\underline{R}) d\underline{R} \\ &= \ln \int \dots \int \Lambda^s(\underline{R}) p_{\underline{r}|H_1}(\underline{R}) d\underline{R} \end{aligned}$$

$$= \ln \int \dots \int p_{\underline{r}|H_2}^s(\underline{R}) p_{\underline{r}|H_1}^{1-s}(\underline{R}) d\underline{R} \quad (\text{A-1})$$

where

$$\lambda(\underline{R}) = \ln \Lambda(\underline{R}) = \ln \left[\frac{p_{\underline{r}|H_2}(\underline{R})}{p_{\underline{r}|H_1}(\underline{R})} \right] \quad (\text{A-2})$$

We assume that s is restricted to be in the region where the integrals in (A-1) converge.

We do not need a second function conditioned on hypothesis H_2 because the moment generating functions are simply related.

$$\begin{aligned} M_{\lambda|H_2}(s) &= E[e^{s\lambda}|H_2] \\ &= E[\Lambda^s(\underline{R})|H_2] \\ &= \int \dots \int p_{\underline{r}|H_2}^{1+s}(\underline{R}) p_{\underline{r}|H_1}^{-s}(\underline{R}) d\underline{R} \\ &= e^{\mu(s+1)} \\ &= M_{\lambda|H_1}(s+1) \end{aligned} \quad (\text{A-3})$$

We should emphasize that this simple relation is a consequence of our use of the log of the likelihood ratio for the test statistic.

Now recall that the semi-invariants γ_k of a random variable x are the coefficients in the power series expansion of the logarithm of the moment generating function,

$$\ln M_x(s) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} s^k, \quad (\text{A-4})$$

or

$$\gamma_k = \left. \frac{d^k}{ds^k} (\ln M_x(s)) \right|_{s=0} \quad (\text{A-5})$$

Thus,

$$\left. \frac{d^k}{ds^k} \mu(s) \right|_{s=0} = k\text{-th semi-invariant of } \mathcal{L}|H_1 \quad (\text{A-6})$$

$$\left. \frac{d^k}{ds^k} \mu(s) \right|_{s=1} = k\text{-th semi-invariant of } \mathcal{L}|H_2. \quad (\text{A-7})$$

For example,

$$\dot{\mu}(0) = E[\mathcal{L}|H_1] \quad (\text{A-8a})$$

$$\dot{\mu}(1) = E[\mathcal{L}|H_2] \quad (\text{A-8b})$$

$$\ddot{\mu}(0) = \text{Var}[\ell|H_1] \quad (\text{A-9a})$$

$$\ddot{\mu}(1) = \text{Var}[\ell|H_2]. \quad (\text{A-9b})$$

Now we shall define the "tilted" random variable \underline{r}_s to have the probability density

$$p_{\underline{r}_s}(\underline{R}) = \frac{p_{\underline{r}|H_1}^{1-s}(\underline{R}) p_{\underline{r}|H_2}^s(\underline{R})}{\int \dots \int p_{\underline{r}|H_1}^{1-s}(\underline{R}') p_{\underline{r}|H_2}^s(\underline{R}') d\underline{R}'} \quad (\text{A-10})$$

$$= \frac{p_{\underline{r}|H_1}^{1-s}(\underline{R}) p_{\underline{r}|H_2}^s(\underline{R})}{\exp[\mu(s)]}, \quad 0 \leq s \leq 1. \quad (\text{A-11})$$

This is a legitimate probability density since it is non-negative and integrates to unity.

We can define a second "tilted" random variable by the transformation

$$\ell_s = \ell(\underline{r}_s) \quad (\text{A-12})$$

where $\ell(\underline{R})$ is given by Eq. A-2.

We now compute the semi-invariant moment generating function for ℓ_s .

$$\begin{aligned}
\ln M_{\lambda_s}(t) &= \ln E[e^{\lambda_s t}] \\
&= \ln E[\Lambda^t(\underline{r}_s)] \\
&= \ln \int \dots \int p_{\underline{r}|H_1}^{1-(s+t)}(\underline{R}) p_{\underline{r}|H_1}^{(s+t)}(\underline{R}) d\underline{R} - \mu(s) \\
&= \mu(t+s) - \mu(s). \tag{A-13}
\end{aligned}$$

Thus,

$$\frac{d^k}{ds^k} \mu(s) = k\text{-th semi-invariant of } \lambda_s \tag{A-14}$$

and

$$\ddot{\mu}(s) = \text{Var}[\lambda_s] \geq 0. \tag{A-15}$$

Hence, $\mu(s)$ is a convex (upward) function of s .

Furthermore, $\ddot{\mu}(s) = 0$ only in the uninteresting case when λ_s equals a constant with probability one, so that in all cases of interest to us $\ddot{\mu}(s) > 0$.

Then, $\dot{\mu}(s)$ is a monotone increasing function of s .

Furthermore,

$$\mu(0) = \mu(1) = 0 \tag{A-16}$$

so that $\mu(s) < 0$ for $0 < s < 1$. From the definition of the "tilted" density in Eq. A-11, we have

$$\begin{aligned}
 p_{\underline{r}_s}(\underline{R}) &= e^{-\mu(s)} \left[\frac{p_{\underline{r}|H_2}(\underline{R})}{p_{\underline{r}|H_1}(\underline{R})} \right]^s p_{\underline{r}|H_1}(\underline{R}) \\
 &= e^{-\mu(s)} \Lambda^s(\underline{R}) p_{\underline{r}|H_1}(\underline{R}) \\
 &= \exp[s\lambda(\underline{R}) - \mu(s)] p_{\underline{r}|H_1}(\underline{R}).
 \end{aligned} \tag{A-17}$$

Thus,

$$p_{\underline{r}|H_1}(\underline{R}) = p_{\underline{r}_s}(\underline{R}) \exp[\mu(s) - s\lambda(\underline{R})]. \tag{A-18}$$

Similarly,

$$\begin{aligned}
 p_{\underline{r}_s}(\underline{R}) &= e^{-\mu(s)} \left[\frac{p_{\underline{r}|H_2}(\underline{R})}{p_{\underline{r}|H_1}(\underline{R})} \right]^{s-1} p_{\underline{r}|H_2}(\underline{R}) \\
 &= \exp[(s-1)\lambda(\underline{R}) - \mu(s)] p_{\underline{r}|H_2}(\underline{R}) \\
 p_{\underline{r}|H_2}(\underline{R}) &= p_{\underline{r}_s}(\underline{R}) \exp[\mu(s) + (1-s)\lambda(\underline{R})].
 \end{aligned} \tag{A-20}$$

The alternate expressions for the error probabilities follow immediately:

$$\begin{aligned}
 \Pr[\varepsilon|H_1] &= \int_{\{\underline{R}: \lambda(\underline{R}) > \gamma\}} \dots \int p_{\underline{r}|H_1}(\underline{R}) d\underline{R} \\
 &= \int \dots \int p_{\underline{r}_s}(\underline{R}) \exp[\mu(s) - s\lambda(\underline{R})] d\underline{R} \\
 &= \int_{\gamma}^{\infty} p_{\lambda_s}(L) \exp[\mu(s) - sL] dL \quad (A-21)
 \end{aligned}$$

$$\begin{aligned}
 \Pr[\varepsilon|H_2] &= \int_{\{\underline{R}: \lambda(\underline{R}) < \gamma\}} \dots \int p_{\underline{r}|H_2}(\underline{R}) d\underline{R} \\
 &= \int \dots \int p_{\underline{r}_s}(\underline{R}) \exp[\mu(s) + (1-s)\lambda(\underline{R})] d\underline{R} \\
 &= \int_{-\infty}^{\gamma} p_{\lambda_s}(L) \exp[\mu(s) + (1-s)L] dL. \quad (A-22)
 \end{aligned}$$

APPENDIX B $\mu(s)$ FOR OPTIMAL DETECTION OF GAUSSIAN RANDOM VECTORS

In this Appendix, we obtain a simple expression for $\mu(s)$, assuming that $p_{\underline{r}|H_1}(\underline{R})$ and $p_{\underline{r}|H_2}(\underline{R})$ are Gaussian probability densities. Once again, similar calculations have previously appeared (see the references to Appendix A).

$$p_{\underline{r}|H_1}(\underline{R}) = \frac{1}{\sqrt{(2\pi)^N |\underline{K}_1|}} \exp[-1/2(\underline{R}-\underline{m}_1)^T \underline{K}_1^{-1} (\underline{R}-\underline{m}_1)] \quad (B-1)$$

$$p_{\underline{r}|H_2}(\underline{R}) = \frac{1}{\sqrt{(2\pi)^N |\underline{K}_2|}} \exp[-1/2(\underline{R}-\underline{m}_2)^T \underline{K}_2^{-1} (\underline{R}-\underline{m}_2)] \quad (B-2)$$

Substituting into Eq. 2.1 and expanding, we have

$$\mu(s) = \ln \left\{ \frac{\exp\left(-1/2[\underline{m}_1^T(1-s)\underline{K}_1^{-1}\underline{m}_1 + \underline{m}_2^T s\underline{K}_2^{-1}\underline{m}_2]\right)}{\sqrt{(2\pi)^N |\underline{K}_1|^{1-s} |\underline{K}_2|^s}} \int \dots \int \exp\left[-1/2 \left\{ \begin{aligned} &\underline{R}^T [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}] \underline{R} \\ &- [\underline{m}_1^T(1-s)\underline{K}_1^{-1} + \underline{m}_2^T s\underline{K}_2^{-1}] \underline{R} \\ &- \underline{R}^T (1-s)\underline{K}_1^{-1}\underline{m}_1 + s\underline{K}_2^{-1}\underline{m}_2 \end{aligned} \right\} \right] \right\} \quad (B-3)$$

We know there exists an orthogonal matrix \underline{Q} (i.e., $\underline{Q}^T = \underline{Q}^{-1}$) such that

$$\underline{Q}^T [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}] \underline{Q} = \underline{\Lambda} \quad (\text{B-4})$$

where

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \cdot \\ & & & & & & \lambda_N \\ 0 & & & & & & & \end{bmatrix}$$

is a diagonal matrix whose entries are the eigenvalues of $[(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}]$. Now let,

$$\underline{R} = \underline{Q} \underline{X} \quad (\text{B-5})$$

and

$$(1-s)\underline{K}_1^{-1} \underline{m}_1 + s\underline{K}_2^{-1} \underline{m}_2 = \underline{Q} \underline{A}. \quad (\text{B-6})$$

Then, the integral in Eq. B-3 becomes

$$\begin{aligned} \underline{I} &= \int \dots \int \exp -1/2 [\underline{X}^T \underline{\Lambda} \underline{X} - \underline{A}^T \underline{X} - \underline{X}^T \underline{A}] |\underline{Q}| d\underline{X} \\ &= \int \dots \int \exp -1/2 [\underline{X}^T \underline{\Lambda} \underline{X} - 2\underline{A}^T \underline{X}] d\underline{X} \end{aligned}$$

since

$$\underline{A}^T \underline{X} = \underline{X}^T \underline{A}$$

and

$$|Q| = +1,$$

$$\begin{aligned} I &= \int \dots \int \exp\left[-\frac{1}{2} \sum_{i=1}^N (\lambda_i x_i^2 - 2a_i x_i)\right] dx_1 dx_2 \dots dx_N \\ &= \prod_{i=1}^N \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(\lambda_i x_i^2 - 2a_i x_i)\right] dx_i. \end{aligned}$$

Now we can complete the square in the exponent and integrate

$$\begin{aligned} I &= \exp\left[\frac{1}{2}(\underline{A}^T \underline{\Lambda}^{-1} \underline{A})\right] \prod_{i=1}^N \left(\frac{2\pi}{\lambda_i}\right)^{\frac{1}{2}} \\ &= \exp\left[\frac{1}{2} \underline{A}^T \underline{\Lambda}^{-1} \underline{A}\right] \frac{(2\pi)^{\frac{N}{2}}}{|\underline{\Lambda}|^{\frac{1}{2}}} \\ &= \frac{(2\pi)^{\frac{N}{2}}}{|(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}|^{\frac{1}{2}}} \left\{ \exp\left[\frac{1}{2}[(1-s)\underline{K}_1^{-1} \underline{m}_1 + s\underline{K}_2^{-1} \underline{m}_2]^T \right] \right\} \end{aligned}$$

$$\left. \left\{ [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}]^{-1} [(1-s)\underline{K}_1^{-1} \underline{m}_1 + s\underline{K}_2^{-1} \underline{m}_2] \right\} \right. \quad (B-7)$$

$$\mu(s) = \frac{1}{2} \left\{ \begin{aligned} & [(1-s)\underline{K}_1^{-1} \underline{m}_1 + s\underline{K}_2^{-1} \underline{m}_2]^T [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}]^{-1} \\ & [(1-s)\underline{K}_1^{-1} \underline{m}_1 + s\underline{K}_2^{-1} \underline{m}_2] \\ & - [\underline{m}_1^T (1-s)\underline{K}_1^{-1} \underline{m}_1 + \underline{m}_2^T s\underline{K}_2^{-1} \underline{m}_2] \end{aligned} \right\} \\ - 1/2 \ln \{ |\underline{K}_1|^{1-s} |\underline{K}_2|^s |(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}| \} \quad (B-8)$$

Each of these terms may be simplified somewhat. Expanding the first term we have

$$\begin{aligned} & \underline{m}_1^T \{ (1-s)\underline{K}_1^{-1} [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}]^{-1} (1-s)\underline{K}_1^{-1} - (1-s)\underline{K}_1^{-1} \} \underline{m}_1 \\ & + \underline{m}_1^T (1-s)\underline{K}_1^{-1} [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}]^{-1} s\underline{K}_2^{-1} \underline{m}_2 \\ & + \underline{m}_2^T s\underline{K}_2^{-1} [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}]^{-1} (1-s)\underline{K}_1^{-1} \underline{m}_1 \\ & + \underline{m}_2^T \{ s\underline{K}_2^{-1} [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}]^{-1} s\underline{K}_2^{-1} - s\underline{K}_2^{-1} \} \underline{m}_2 \end{aligned}$$

$$\begin{aligned}
&= - \underline{m}_1^T \left[\frac{\underline{K}_1}{1-s} + \frac{\underline{K}_2}{s} \right]^{-1} \underline{m}_1 \\
&\quad + \underline{m}_1^T \left[\frac{\underline{K}_1}{1-s} + \frac{\underline{K}_2}{s} \right]^{-1} \underline{m}_2 \\
&\quad + \underline{m}_2^T \left[\frac{\underline{K}_1}{1-s} + \frac{\underline{K}_2}{s} \right]^{-1} \underline{m}_1 \\
&\quad - \underline{m}_2^T \left[\frac{\underline{K}_1}{1-s} + \frac{\underline{K}_2}{s} \right]^{-1} \underline{m}_2 \\
&= -s(1-s) [\underline{m}_2 - \underline{m}_1]^T [s\underline{K}_1 + (1-s)\underline{K}_2]^{-1} [\underline{m}_2 - \underline{m}_1] \tag{B-9}
\end{aligned}$$

where we have made use of the identities

$$(\underline{A} + \underline{B})^{-1} = \underline{A}^{-1} - \underline{A}^{-1} (\underline{A}^{-1} + \underline{B}^{-1})^{-1} \underline{A}^{-1}$$

and

$$(\underline{A} \ \underline{B} \ \underline{C})^{-1} = \underline{C}^{-1} \ \underline{B}^{-1} \ \underline{A}^{-1}.$$

We rewrite the second term in Eq. B-3.

$$\frac{1}{2} \ln \left[\frac{|\underline{K}_1|^{s-1} |\underline{K}_2|^{-s}}{|(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}|} \right]$$

$$\begin{aligned}
&= \frac{1}{2} \ln \left[\frac{|\underline{K}_1|^s |\underline{K}_2|^{1-s}}{|\underline{K}_1| |(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}| |\underline{K}_2|} \right] \\
&= \frac{1}{2} \ln \left[\frac{|\underline{K}_1|^s |\underline{K}_2|^{1-s}}{|(1-s)\underline{K}_2 + s\underline{K}_1|} \right] \\
&= \frac{s}{2} \ln |\underline{K}_1| + \frac{1-s}{2} \ln |\underline{K}_2| - \frac{1}{2} \ln |(1-s)\underline{K}_2 + s\underline{K}_1| \quad (\text{B-10})
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu(s) &= \frac{s}{2} \ln |\underline{K}_1| + \frac{1-s}{2} \ln |\underline{K}_2| - \frac{1}{2} \ln |(1-s)\underline{K}_2 + s\underline{K}_1| \\
&\quad - \frac{s(1-s)}{2} [\underline{m}_2 - \underline{m}_1]^T [(1-s)\underline{K}_2 + s\underline{K}_1]^{-1} [\underline{m}_2 - \underline{m}_1] \quad (\text{B-11})
\end{aligned}$$

A side result of our calculations is the observation that $p_{\underline{r}_s}(\underline{R})$ is a Gaussian probability density with covariance matrix

$$\underline{K}_{\underline{r}_s} = [(1-s)\underline{K}_1^{-1} + s\underline{K}_2^{-1}]^{-1}.$$

APPENDIX C $\mu(\underline{s})$ FOR OPTIMAL DETECTION OF GAUSSIAN RANDOM PROCESSES

In this Appendix, we calculate $\mu(\underline{s})$ for the case in which the received signals are sample functions from random processes, rather than random vectors. Thus, we are now considering an infinite number of dimensions.

We shall make use of the results of the previous appendices as follows. We choose a suitable set of observables. Two useful sets are the coefficients in a Karhunen-Loève expansion and the values of time samples taken arbitrarily close together. We first consider only a finite number of the observables, then apply the results of the previous appendix. Then we let the number of observables become arbitrarily large.

First, we consider the first three terms in Eq. B-11. They all are of the form

$$\ln \left| \underline{I} + \frac{2}{N_0} \underline{K}_N \right|$$

where \underline{K}_N is a $N \times N$ covariance matrix.

Now suppose that the N components of \underline{s} are obtained by taking uniformly spaced time samples from the (Gaussian) random process $s(t)$ over the interval $[T_i, T_f]$. Call the sampling points t_1, t_2, \dots, t_N , where

$$t_i = t_i + i \Delta t \tag{C-1}$$

$$\Delta t = \frac{T_f - T_i}{N} \tag{C-2}$$

Then,

$$\underline{K}_N = \begin{bmatrix} K_s(t_1, t_1) & K_s(t_1, t_2) & & & \\ K_s(t_2, t_1) & K_s(t_2, t_2) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & K_s(t_N, t_N) \end{bmatrix} \quad (C-3)$$

We define

$$\Delta_N(\lambda) = \det[\underline{I} + \frac{1}{\lambda} \Delta t \underline{K}_N] \quad (C-4)$$

Expanding this determinant

$$\begin{aligned} \Delta_N(\lambda) = & 1 + \frac{\Delta t}{\lambda} \sum_{i=1}^N K_{ii} \\ & + \frac{1}{2!} \left(\frac{\Delta t}{\lambda}\right)^2 \sum_{i,j=1}^N \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} \\ & + \dots + \frac{1}{N!} \left(\frac{\Delta t}{\lambda}\right)^N \begin{vmatrix} K_{11} & \dots & K_{1N} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ K_{N1} & \dots & K_{NN} \end{vmatrix} \end{aligned} \quad (C-5)$$

where we have written

$$K_{ij} \triangleq K_s(t_i, t_j) \text{ for brevity.}$$

Now letting $N \rightarrow \infty$, each term of this series has a limit.

(Recall how we define integrals in terms of Riemann sums.) Therefore,

at least formally,

$$D(\lambda) = \lim_{N \rightarrow \infty} \Delta_N(\lambda)$$

$$= 1 + \frac{1}{\lambda} \int_{T_i}^{T_f} K_s(t, t) dt$$

$$\frac{1}{2! \lambda^2} \int_{T_i}^{T_f} \int_{T_i}^{T_f} \begin{vmatrix} K_s(t_1, t_1) & K_s(t_1, t_2) \\ \text{---} & \text{---} \\ K_s(t_2, t_1) & K_s(t_2, t_2) \end{vmatrix} dt_1 dt_2$$

$$\frac{1}{3! \lambda^3} \int_{T_i}^{T_f} \int_{T_i}^{T_f} \int_{T_i}^{T_f} \begin{vmatrix} K_s(t_1, t_1) & K_s(t_1, t_2) & K_s(t_1, t_3) \\ \text{---} & \text{---} & \text{---} \\ \cdot & K_s(t_2, t_2) & K_s(t_2, t_3) \\ \text{---} & \text{---} & \text{---} \\ \cdot & \cdot & K_s(t_3, t_3) \end{vmatrix}$$

+ . . .

(C-6)

This is one definition of the Fredholm determinant.

We now cite two theorems from Fredholm integral equation theory which enable us to express $D(\lambda)$ in terms of the eigenvalues of $K_s(t, \tau)$ [80].

Theorem 1

The series $D(\lambda)$ is an absolutely and permanently converging power series in $\frac{1}{\lambda}$.

Remark

Therefore, Eq. C-6 is not just a purely formal result.

Theorem 2

If $\lambda = \lambda_0$ is a root of $D(\lambda) = 0$ of order m , then the homogeneous integral equation

$$\lambda_0 \phi(t) = \int_{T_i}^{T_f} K_s(t, \tau) \phi(\tau) d\tau \quad (C-7)$$

has m linearly independent solutions in terms of which every other solution is expressible linearly and homogeneously.

Remark

The roots of $D(\lambda)$ and the eigenvalues of the homogeneous integral equation are in one-one correspondence. Therefore,

$$D(\lambda) = \prod_{i=1}^{\infty} \left(1 + \frac{\lambda_i}{\lambda}\right) \triangleq D_{\frac{1}{\lambda}} \quad (C-8)$$

where the λ_i are counted according to their multiplicity. Therefore,

$$\begin{aligned} \lim_{\substack{N \rightarrow \infty \\ \Delta t \rightarrow 0}} \ln \left| \underline{I} + \frac{2}{N_0} \Delta t \underline{K}_N \right| &= \sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i \right) \\ &= \frac{2}{N_0} \int_{T_i}^{T_f} \xi_s \left(t \mid t: \frac{N_0}{2} \right) dt \end{aligned} \quad (C-9)$$

Now consider the last term in $\mu(s)$

$$\begin{aligned} &[\underline{m}_2 - \underline{m}_1]^T [(1-s)\underline{K}_2 + s\underline{K}_1]^{-1} [\underline{m}_2 - \underline{m}_1] \\ &= [\underline{m}_2 - \underline{m}_1]^T [\underline{K}_{\text{comp}}(s) + \frac{N_0}{2} \underline{I}]^{-1} [\underline{m}_2 - \underline{m}_1] \end{aligned} \quad (C-10)$$

where

$$\underline{K}_{\text{comp}}(s) = (1-s)\underline{K}_{s_2} + s\underline{K}_{s_1}$$

denotes the covariance of the composite random vector.

Now supposing that the observables have been chosen as the Karhunen-Loève components of the corresponding random process

$$\underline{K}_{\text{comp}}(t, \tau; s) = (1-s)\underline{K}_{s_2}(t, \tau) + s\underline{K}_{s_1}(t, \tau) \quad (C-12)$$

we have

$$\begin{aligned}
 & [\underline{m}_2 - \underline{m}_1]^T [(1-s)\underline{K}_2 + s\underline{K}_1]^{-1} [\underline{m}_2 - \underline{m}_1] \\
 &= \sum_{i=1}^N \frac{(m_{i2} - m_{i1})^2}{\lambda_i + \frac{N_0}{2}} \quad (C-13)
 \end{aligned}$$

Now

$$\begin{aligned}
 & \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{(m_{i2} - m_{i1})^2}{\lambda_i + \frac{N_0}{2}} \\
 &= \int_{T_i}^{T_f} \int_{T_i}^{T_f} [m_2(t) - m_1(t)] Q_{\text{comp}}(t, \tau; s) [m_2(\tau) - m_1(\tau)] dt d\tau \quad (C-14)
 \end{aligned}$$

This last term is familiar [42] as the performance index $d_{\text{comp}}^2(s)$ for detecting the known signals $m_1(t)$ and $m_2(t)$ in colored Gaussian noise with covariance

$$K_n(t, \tau; s) = \frac{N_0}{2} \delta(t - \tau) + K_{\text{comp}}(t, \tau; s) \quad (C-15)$$

Thus, we have

$$\begin{aligned}
 \mu(s) &= \frac{s}{2} \frac{2}{N_0} \int_{T_i}^{T_f} \xi_{s_1}(t | t: \frac{N_0}{2}) dt \\
 &+ \frac{1-s}{2} \frac{2}{N_0} \int_{T_i}^{T_f} \xi_{s_2}(t | t: \frac{N_0}{2}) dt
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{2} \frac{2}{N_0} \int_{T_i}^{T_f} \xi_{\text{comp}}(t | t: \frac{N_0}{2}, s) dt \\
 & - \frac{s(1-s)}{2} d_{\text{comp}}^2(s) \tag{C-16}
 \end{aligned}$$

where

$$\xi_{s_1}(t | t: \frac{N_0}{2}), \xi_{s_2}(t | t: \frac{N_0}{2}), \text{ and } \xi_{\text{comp}}(t | t: \frac{N_0}{2}, s)$$

are the minimum mean-square point estimation errors for estimating $s_1(t)$, $s_2(t)$ and the composite random process $s_{\text{comp}}(t) = \sqrt{1-s} s_2(t) + \sqrt{s} s_1(t)$, respectively, in additive white noise of spectral density $N_0/2$; and where $d_{\text{comp}}^2(s)$ is the performance index (output signal-to-noise ratio) for deciding which of the known signals $m_1(t)$ or $m_2(t)$ was sent when observed in colored Gaussian noise with covariance

$$K_n(t, \tau; s) = \frac{N_0}{2} \delta(t-\tau) + K_{\text{comp}}(t, \tau; s).$$

APPENDIX D REALIZABLE WHITENING FILTERS

A convenient conceptual and practical aid in solving a number of problems in detection theory is the whitening filter [81, 82]. Upon passing a sample function from a random process $r(t)$ through the whitening filter, the output $w_*(t)$ is a sample function from a white random process. For both mathematical and physical reasons, we require that the input contain a white component $w(t)$.

We shall demonstrate that

$$h_w(t,u) = \delta(t-u) - h_o(t,u) \quad (D-1)$$

is a realizable whitening filter, where $h_o(t,u)$ denotes the optimum realizable time-varying linear filter for estimating $s(t)$ over the observation interval $T_i \leq t \leq T_f$, assuming we observe $r(t) = s(t) + w(t)$. In the case of stationary processes with rational spectra where the observation interval is infinite, this is readily verified by taking Fourier transforms and substituting the expression for $H_o(j\omega)$. We now present the proof for non-stationary processes and finite time intervals.

The optimum linear filter satisfies the Wiener-Hopf equation

$$K_s(t,u) = \frac{N_0}{2} h_o(t,u) + \int_{T_i}^t h_o(t,\sigma) K_s(\sigma,u) d\sigma \quad (D-2)$$

$$T_i \leq u \leq t \leq T_f$$

We denote the output of the optimum linear filter by

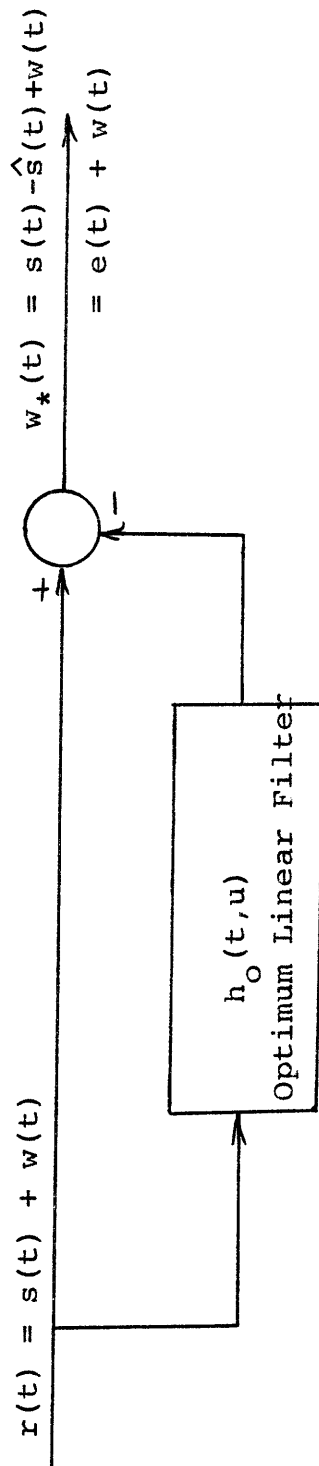


Figure D.1. Realizable Whitening Filter

$$\hat{s}(t) = \int_{T_i}^t h_o(t,u) [s(u) + w(u)] du. \quad (D-3)$$

We also define

$$e(t) = s(t) - \hat{s}(t) \quad (D-4)$$

and

$$w_*(t) = e(t) + w(t) \quad (D-5)$$

We now compute

$$\begin{aligned} K_{w_*}(t,u) &= K_w(t,u) + K_{we}(t,u) \\ &\quad + K_{ew}(t,u) + K_e(t,u) \end{aligned} \quad (D-6)$$

We consider each term separately.

$$K_w(t,u) = \frac{N_0}{2} \delta(t-u) \quad (D-7)$$

$$K_{we}(t,u) = -K_{w\hat{s}}(t,u) \quad (D-8)$$

$$K_{we}(t,u) = -E\{w(t) \int_{T_i}^u h_o(u,\sigma) [s(\sigma) + w(\sigma)] d\sigma\} \quad (D-9)$$

$$= \begin{cases} 0 & \text{for } t > u \\ -\frac{1}{2} \frac{N_0}{2} h_0(t, t) & \text{for } t = u \\ -\frac{N_0}{2} h_0(u, t) & \text{for } T_i < t < u \end{cases} \quad (\text{D-10})$$

Similarly,

$$K_{ew}(t, u) = \begin{cases} 0 & \text{for } u > t \\ -\frac{1}{2} \frac{N_0}{2} h_0(t, t) & \text{for } u = t \\ -\frac{N_0}{2} h_0(t, u) & \text{for } T_i < u < t \end{cases} \quad (\text{D-11})$$

$$K_e(t, u) = K_{es}(t, u) - K_{e\hat{s}}(t, u) \quad (\text{D-12})$$

First, we compute $K_{e\hat{s}}(t, u)$ for $t > u$.

$$\begin{aligned} K_{e\hat{s}}(t, u) &= K_{s\hat{s}}(t, u) - K_{\hat{s}}(t, u) \quad \text{for } T_i < u < t \\ &= E\left\{s(t) \int_{T_i}^u h_0(u, \sigma) [s(\sigma) + w(\sigma)] d\sigma\right\} \\ &\quad - E\left\{\int_{T_i}^t h_0(t, \sigma) [s(\sigma) + w(\sigma)] d\sigma \int_{T_i}^u h_0(u, \gamma) [s(\gamma) + w(\gamma)] d\gamma\right\} \end{aligned}$$

$$\begin{aligned}
&= \int_{T_i}^u h_o(u, \sigma) K_s(\sigma, t) d\sigma \\
&\quad - \int_{T_i}^u h_o(u, \gamma) \left\{ \int_{T_i}^t h_o(t, \sigma) [K_s(\sigma, \gamma) + \frac{N_0}{2} \delta(\sigma - \gamma)] d\sigma \right\} d\gamma
\end{aligned}$$

But the term in braces is just $K_s(t, \gamma)$ from Eq. D-2. Therefore,

$$K_{es}^{\wedge}(t, u) = 0 \quad (D-13)$$

Then for $t > u$

$$\begin{aligned}
K_e(t, u) &= K_{es}(t, u) \\
&= K_s(u, t) - K_{ss}^{\wedge}(u, t) \\
&= K_s(u, t) \\
&\quad - E\{s(u) \int_{T_i}^t h_o(t, \sigma) [s(\sigma) + w(\sigma)] d\sigma\} \\
&= K_s(t, u) - \int_{T_i}^t h_o(t, \sigma) K_s(\sigma, u) d\sigma \\
&= \frac{N_0}{2} h_o(t, u) \quad (D-14)
\end{aligned}$$

Similarly, for $t < u$

$$K_e(t, u) = \frac{N_0}{2} h_o(u, t) \quad (D-15)$$

and for $t = u$

$$K_e(t, t) = \xi_p(t) = \frac{N_0}{2} h_o(t, t) \quad (D-16)$$

Therefore,

$$\boxed{K_{w*}(t, u) = \frac{N_0}{2} \delta(t-u)} \quad (D-17)$$

When we assume that $s(t)$ may be obtained as the output of a linear (possibly time varying) dynamic system which is driven with white noise and which has a state variable representation,

$$\dot{\underline{x}}(t) = \underline{F}(t)\underline{x}(t) + \underline{C}(t)\underline{u}(t) \quad (D-18)$$

$$s(t) = \underline{C}(t)\underline{x}(t) \quad (D-19)$$

we can explicitly specify the system with impulse response $h_o(t, \tau)$ in terms of the Kalman-Bucy filter [15].

$$\begin{aligned} \dot{\hat{\underline{x}}}(t) &= [\underline{F}(t) - \xi_p(t)\underline{C}^T(t)\frac{2}{N_0}\underline{C}(t)]\hat{\underline{x}}(t) \\ &\quad + \xi_p(t)\underline{C}^T(t)\frac{2}{N_0}r(t) \end{aligned} \quad (D-20)$$

$$\hat{\underline{x}}(T_i) = \underline{0} \quad (D-21)$$

$$\hat{\underline{s}}(t) = \underline{C}(t)\hat{\underline{x}}(t) \quad (D-22)$$

$$\begin{aligned} \dot{\underline{\xi}}_p(t) = & \underline{F}(t)\underline{\xi}_p(t) + \underline{\xi}_p(t)\underline{F}^T(t) - \underline{\xi}_p(t)\underline{C}^T(t)\frac{2}{N_0}\underline{C}(t)\underline{\xi}_p(t) \\ & + \underline{G}(t)\underline{Q}(t)\underline{G}^T(t) \end{aligned} \quad (D-23)$$

$$\underline{\xi}_p(T_i) = \underline{K}_x(T_i, T_i) \quad (D-24)$$

The state-variable realizable whitening filter is shown in Fig. D-2. This result has also been obtained recently by several others, although to the best of our knowledge, we were the first to observe that the input to the loop filter in the Kalman-Bucy filter is white noise [83,41]. We also derived Eq. D-1 at about the same time using a different approach [84,42]. Other more recent references to this result are [85,86].

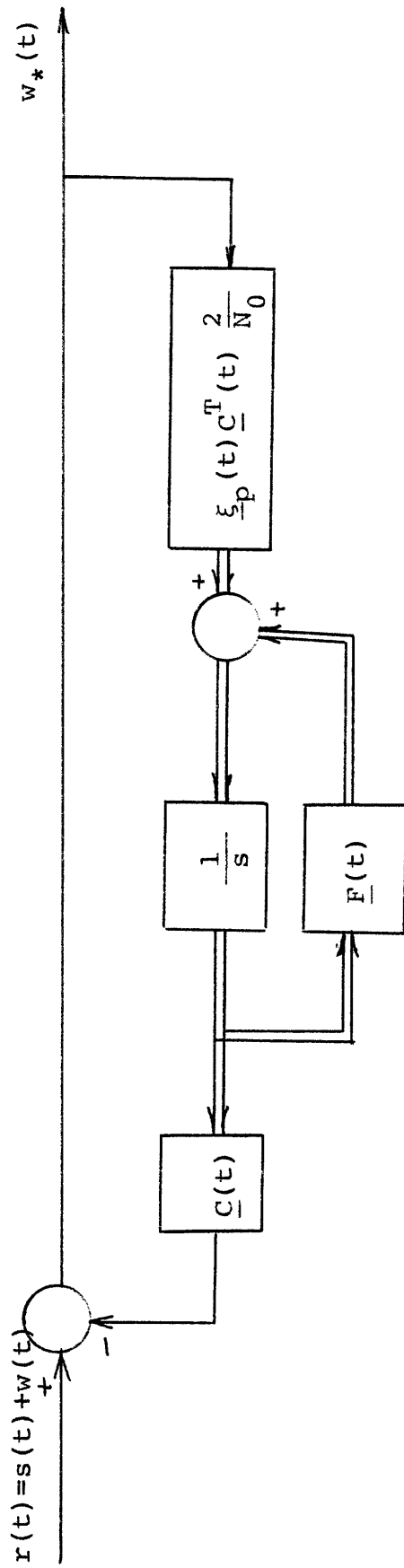


Figure D-2. State-Variable Realizable Whitening Filter

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