

**A new framework of iterated forcing along a gap
one morass at ω_1**

by

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Submitted to the Department of Mathematics
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Abstract

A new framework of iterated forcing is developed by attaching forcing notions to the points of a gap -1 morass at ω_1 , thereby determining an \aleph_2 long iteration. The forcing is shown to be $\omega_2 - cc$ and ω - distributive.

Thesis Supervisor: Sy D. Friedman
Title: Professor

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Introduction

The technique of finite support iterated forcing was developed by Solovay, Martin and Tennenbaum in 1969. This method has been used to establish many consistency results, most notably that $CON(ZFC + GCH) \implies CON(ZFC + MA + 2^{\aleph_0} = \kappa)$, where $\kappa \geq \aleph_2$. In that construction, the value of the continuum is made arbitrarily large by adding a huge number of reals. In 1974 Laver introduced the idea of countable iteration and used it to prove that $CON(ZFC + GCH) \implies CON(ZFC + 2^{\aleph_0} = \aleph_2 + \textit{(All sets of strong measure zero are countable)})$. Later on, Shelah, again using a countable support iteration of length \aleph_2 constructed a generic extension in which $2^{\aleph_0} = \aleph_2$ and there are no p -points. Both Laver's and Shelah's constructions use a countable support iteration of length \aleph_2 . The natural approach to establishing the consistency of *Borel Conjecture* + $2^{\aleph_0} = \aleph_3$ and *there are no p -points* + $2^{\aleph_0} = \aleph_3$ would be to try an countable support iteration of length \aleph_3 . Unfortunately this will not work. The problem with such a construction is that \aleph_2 collapses onto \aleph_1 resulting in the value of the continuum being \aleph_2 . This is a recurring theme, not limited to the previous two problems but is rather intrinsic to the method of linear iterated forcing itself. There are a number of propositions whose consistency is known with the value of the continuum being \aleph_2 but are open for $2^{\aleph_0} = \aleph_3$. The failure of countable support iteration to establish consistency results with the continuum large has become known as the continuum problem.

The theory of gap-1 morasses was originally developed by R. Jensen. Morasses are fairly complicated combinatorial objects which pack immense combinatorial power. Intuitively speaking, a gap-1 morass at κ , where $\kappa \geq \omega$, is used to build an "object" of size κ^+ in κ steps from "subobjects" of size less than κ . This suggests that the theory of morasses may provide us with a strategy suitable to the continuum problem.

In this thesis I examine a new framework of iterated forcing, where modified ω_1 - Cohen forcing notions are attached at the points of a gap-1 morass at ω_1 . More precisely, forcing notions $P(\sigma)$ are attached at each morass point σ satisfying certain requirements so that the top level is completely determined by the morass construction from below. The morass maps $f_{\sigma\tau}$, where $\sigma <_1 \tau$ are used to tightly control the support of conditions in $P(\omega_2)$. The construction is arranged to guarantee that generic sets fit together along the morass branches and that extendibility and amalgamation hold.

Chapter 1

Morasses and Forcing

1.1 Gap -1 morasses

The theory of Gap -1 morasses was developed by Jensen in the 1970's in his work on cardinal transfer theorems. The gap -2 transfer property at a regular uncountable cardinal κ is the statement that if G is a structure of type (κ^{++}, κ) , then there exists an elementarily equivalent structure H of type (ω_2, ω) . In unpublished notes, Jensen proved the following theorem:

Theorem 1 Suppose that κ is a regular uncountable cardinal such that $2^\kappa = \kappa^+$. Then,

1. If there exists a gap -1 morass at ω_1 , then the gap -2 transfer property at κ holds.
2. If $V = L$, then there exists a gap -1 morass at ω_1 .

Jensen also formulated the theory of higher gap morasses and applied it to obtain higher gap cardinal transfer theorems.

In section 1.1.2 we will present a sketchy account of the second part of the foregoing theorem. The proof draws heavily upon the fine structure of L . Morasses can also be added by forcing, thus avoiding the intricacies of fine structure theory. The gap

-1 case of that statement was proven by R. Jensen. The higher gap one is due to Stanley and Jensen.

In what follows, we state the definition of a gap-1 morass followed by some explanatory remarks aimed at clarifying the many clauses of the definition and a proof of theorem 1. Definition 1.1 will closely follow Devlin's exposition in [De].

In order to state the definition of a gap-1 morass we need to introduce some terminology. Let κ be a fixed regular cardinal and let S be a set of ordered pairs of pr closed ordinals satisfying the following two conditions:

1. If $(\alpha, \nu) \in S$ then $\alpha < \nu < \kappa^+$ and $\alpha \leq \kappa$.
2. For $(\alpha_1, \nu_1), (\alpha_2, \nu_2) \in S$, we have $\alpha_1 < \alpha_2 \rightarrow \nu_1 < \alpha_2$.

We then define:

1. $S^0 = \{\alpha \in \kappa + 1 \mid (\alpha, \nu) \in S \text{ for some } \nu\}$.
2. $S^1 = \{\nu \in \kappa^+ \mid (\alpha, \nu) \in S \text{ for some } \alpha\}$.
3. $S_\alpha = \{\nu \mid (\alpha, \nu) \in S\}$.

In the sequel, we refer to S_α as the α th morass level. We will reserve the letters ν, μ, τ and α, β, γ to represent elements of S^1 and S^0 respectively. If $(\alpha_1, \nu) \in S$ then $\alpha(\nu) = \alpha_1$. Also, we define $\nu <_0 \tau$ if $\alpha(\nu) = \alpha(\tau)$ and $\nu < \tau$. Suppose that $<_1$ is a tree order on S^1 such that whenever $\nu_1 <_1 \nu_2$ we have $\alpha(\nu_1) < \alpha(\nu_2)$ and let $\{\pi_{\nu_1 \nu_2} \mid \pi_{\nu_1 \nu_2} : \nu_1 + 1 \rightarrow \nu_2 + 1, \nu_1 <_1 \nu_2\}$ be a commutative system of maps. The notation $\nu <_0^* \tau$ and $\mu <_1^* \tau$ stands for the immediate level and tree predecessor of ν and μ respectively.

It is standard practice to represent a gap-1 morass at κ by a bent line at κ , as depicted below. We use horizontal lines to draw the levels S_α and slanted lines to

connect morass tree related points. It is a consequence of one of the morass axioms the tree and morass levels do not coincide. As a result, morass constructions can not be viewed as being monotonically increasing.

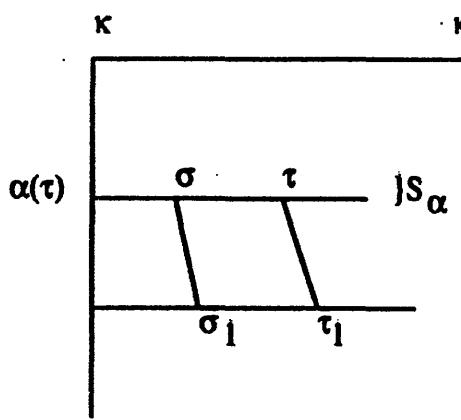


Figure 1.1

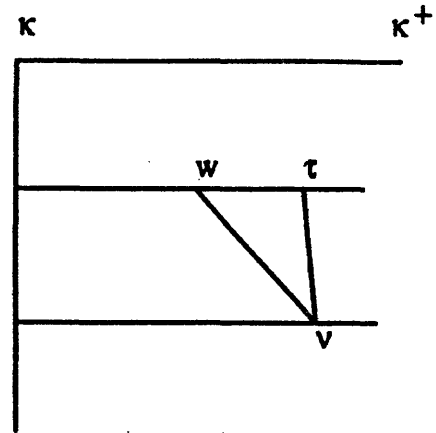


Figure 1.2

Definition 1.1 $(S^0, S^1, \{\pi_{\nu\tau} \mid \nu <_1 \tau\}, S)$ is a gap-1 morass at κ if it satisfies the following axioms:

Axiom 1 $\kappa = \max S^0 = \sup(S^0 \cap \kappa)$, $\kappa = \sup S^1$, S_α is closed in $\sup S_\alpha$ for all $\alpha \in S^0$ and $\sup S_\alpha \in S_\alpha$, for $\alpha < \kappa$.

Axiom 2 Let $\nu <_1 \tau$. Then, we have:

- (1) $\pi_{\nu\tau} \mid \alpha(\nu) = id_{\alpha(\nu)}$, $\pi_{\nu\tau}(\alpha(\nu)) = \alpha(\tau)$, $\pi_{\nu\tau}(\nu) = \tau$.
- (2) If ν_1 is a limit point of $S_{\alpha(\nu)} \cap (\nu + 1)$ then $\pi_{\nu\tau}(\nu_1)$ is a limit point in $S_{\alpha(\tau)}$.
- (3) If ν_1 is the immediate successor of τ_1 in $S_{\alpha(\nu)} \cap (\nu + 1)$ $\pi_{\nu\tau}(\nu_1)$ is the immediate successor of $\pi_{\nu\tau}(\tau_1)$ in $S_{\alpha(\tau)}$.
- (4) If ν_1 is the first element of $S_{\alpha(\nu)}$ then $\pi_{\nu\tau}(\nu_1)$ is the first element of the set $S_{\alpha(\tau)}$.

Axiom 3 (The Coherence Property) Suppose $\nu <_1 \tau$, $\nu_1 \in S_{\alpha(\nu)} \cap \nu$ and let $\tau_1 = \pi_{\nu\tau}(\nu_1)$. Then, $\tau_1 <_1 \nu_1$ and $\pi_{\nu\tau} \mid \nu_1 = \pi_{\nu_1\tau_1} \mid \nu_1$.

Axiom 4 (Vertical Continuity) For all $\tau \in S^1$ the set $\{\alpha(\nu) \mid \nu <_1 \tau\}$ is closed in $\alpha(\tau)$.

Axiom 5 If ν is not maximal in $S_{\alpha(\nu)}$, then $\{\alpha(\tau) \mid \tau <_1 \nu\}$ is unbounded in $\alpha(\nu)$.

Axiom 6 If $\{\alpha(\tau) \mid \tau <_1 \nu\}$ is unbounded in $\alpha(\nu)$, then $\nu = \sup\{Rng(\pi_{\tau\nu}) \mid \tau <_1 \nu\}$.

Axiom 7 (Horizontal Continuity) Let ν be a level limit point of $S_{\alpha(\nu)}$ and let $\nu <_1^* \tau$. Let $w = \bigcup \pi_{\nu\tau}[w]$, then $\nu <_1 w$ and $\pi_{\nu w} \mid \nu = \pi_{\nu\tau} \mid \nu$. We depict that situation in the picture 1.2.

In order to state Property (M8) we first need to define the notion of τ being “diagonally below” σ .

Definition 1.2 Let $\tau, \sigma \in S^1$. We define :

$$\tau \dashv \sigma \leftrightarrow \exists \tau_1 \exists \sigma_1 [\tau_1 <_1^* \tau \wedge \sigma_1 <_1^* \sigma \wedge \tau_1 <_0 \sigma_1 \wedge \tau <_1 \pi_{\sigma_1\sigma}(\tau_1)]$$

Using Definition 1.2 we may now state the diagonal continuity property:

Axiom 8 (Diagonal Continuity Property) Let $\nu_1 <_1^* \nu$, and let ν_1 be a limit point of $S_{\alpha(\nu_1)}$, and let $\nu = \sup \pi''_{\nu_1\nu}(\nu_1)$. Then,

$$\alpha(\nu) = \bigcup \{\alpha(\tau) \mid \tau \dashv \nu\}$$

In the next section we proceed to make a series of comments aimed at shedding some light in the definition of a gap-1 morass.

1.1.1 Comments on the definition of a Gap-1 morass at κ

To simplify matters in this and the following section we proceed on the additional assumption that we are dealing with a gap-1 morass at ω_1 . The theory of gap-1 morasses for an arbitrary regular cardinal $\kappa \neq \omega_1$, is essentially the same.

The most important application of a gap-1 morass is the gap-2 cardinal transfer theorem. Note the respective statement for the gap-1 case is a theorem of *ZFC*. The proof of the gap-1 cardinal transfer theorem is a union of chains argument. It entails the construction of a structure of size ω_1 , using countable substructures in ω_1 steps. An easy cardinality argument shows that the union of chains technique is not a suitable approach to establishing the Gap-2 cardinal theorem. A new kind of limit, a tree-like one, is needed if one is to succeed in putting together such a construction. A Gap-1 morass provides us with the combinatorial apparatus needed to construct an object of size ω_2 in ω_1 steps from countable subobjects. Note that the defining property of morasses is reminiscent of \diamond , another versatile and more well known combinatorial object used in the construction of a Souslin Tree. L possesses enough condensation some of which is embodied in the statement of \diamond to kill ω_2 antichains in ω_1 steps.

We now proceed to make some comments on the defining axioms of a gap-1 morass.

1. Axioms 2 and 3 ensure that the maps embed “nicely”, sending limit, successor and minimal points in level $\alpha(\sigma)$ to like elements in level $\alpha(\tau)$. Axiom 2 of the definition is a coherence property which ensures that different constructions mesh together; if we follow the construction along branch $T_{\nu\tau}$ it agrees with the construction along branch $T_{\tau_1\nu_1}$.
2. Axiom 4 is a “vertical continuity” principle assuring that limit points are on the right place and there are no holes in the construction.
3. The effect of axiom 5 is to introduce as many tree limits in the construction as possible. In essence, if τ is not the right-most point on level $\alpha(\tau)$, then τ is a tree limit. A consequence of this axiom is that the α -th morass level does not coincide with the α -th tree level, for if α is a successor point in S^0 , $S_\alpha = \emptyset$.

4. Axiom 6 ensures that if τ is a limit of the morass tree, then the construction up to τ is determined as a direct limit. This will be of importance in later constructions.
5. Axioms 5 and 6 show that in a morass construction, the only place where something new is possibly constructed is the last level point of a morass level which is not a successor point in S^0 .
6. Axioms 7 and 8 refer to the range of the map $\pi_{\nu, \nu}$. Axiom 8 is a diagonal continuity property ensuring that if the map $\pi_{\nu, \nu}$ is cofinal in ν , the “diagonal” limits are exactly where they should be. Property 7 expresses continuity along level $\alpha(\nu)$.

1.1.2 Construction of a Gap-1 Morass in L

In this section, we outline the construction of a $(\kappa, 1)$ morass in $V = L$. The account we give here is fairly compact. For more details the reader is referred to [De]. For the fine structural matters see [Jen]. In order to simplify notation we only consider the case $\kappa = \omega_1$.

Let $S = \{(\alpha, \nu) \mid \alpha < \nu < \omega_2, \text{ where } \nu \text{ is a pr closed ordinal and } J_\nu \models (\alpha = \omega_1 \text{ and } \alpha \text{ is the largest cardinal})\}$.

Since $\nu \in S^1$ implies that $\nu \neq \omega_1$, we may define:

- **1** $\beta(\nu)$ to be the least ordinal β such that ν is singular in $J_{\beta+1}$.
- **2** $n(\nu)$ to be the least $\nu < \omega$ such that there exists a $\Sigma_n(J_{\beta(\nu)})$ cofinal partial map $f: \gamma \mapsto \nu$, where γ is a bounded subset of ν .
- **3** Let $Q(\nu) = (J_{\rho(\nu)}, A(\nu))$ and let $\rho(\nu) = \rho_{n(\nu)-1}^{\beta(\nu)}$ and $A(\nu) = A_{n(\nu)-1}^{\beta(\nu)}$ denote the $n(\nu) - 1$ projectum and master code, respectively, of the ordinal $\beta(\nu)$.

The following proposition is easy to prove:

Proposition 1

1. $\rho(\nu) \geq \nu$. If $\rho(\nu) > \nu$ then $J_{\rho(\nu)} \models \nu$ is regular.
2. If $\nu \in S_{\alpha(\nu)}$ then $J_{\beta(\nu)} \models \alpha(\nu)$ is regular.
3. If $\tau \in S_{\alpha(\nu)} \cap \nu$ then $\rho(\tau) \leq \beta(\tau) < \nu \leq \rho(\nu) \leq \beta(\nu)$. Hence, $Q(\tau) \in L_\nu$.
4. $\rho_n^{\beta(\nu)} \leq \alpha(\nu)$. +

The goal is to introduce a special parameter $\mathbf{p}(\nu)$, which in turn will enable us to define on S^1 the relation $<_1$. We obtain the special parameter as follows. Using 1(iv) we define a parameter $\mathbf{p}_0(\nu)$ such that

$$J_{\rho(\nu)} = \Sigma_1 - SH_{Q(\nu)}[\alpha(\nu) \cup \{\mathbf{p}_0(\nu)\}] \quad (1.1)$$

Without loss of generality, we may assume that $\mathbf{p}_0(\nu)$ is least such 1.1 holds. We then define,

$$\mathbf{p}(\nu) = \begin{cases} (\mathbf{p}_0(\nu), \nu_1, \alpha(\nu)) & \text{if } \nu_1 <_0^* \nu \\ (\mathbf{p}_0(\nu), \nu, \alpha(\nu)) & \text{if } \nu < \rho(\nu) \\ (\mathbf{p}_0(\nu), \alpha(\nu)) & \text{if } \nu = \rho(\nu), \nu \text{ a level limit} \end{cases}$$

Having defined $\mathbf{p}(\nu)$ we now let $\bar{\nu} <_1 \nu$ if $\alpha(\bar{\nu}) < \alpha(\nu)$ and there exists a Σ_1 elementary embedding $\sigma_{\bar{\nu}\nu} : Q(\bar{\nu}) \rightarrow Q(\nu)$ such that $\sigma_{\bar{\nu}\nu}(\mathbf{p}(\bar{\nu})) = \mathbf{p}(\nu)$, $\sigma \upharpoonright \alpha(\bar{\nu}) = id_{\alpha(\bar{\nu})}$ and $\sigma_{\bar{\nu}\nu} \upharpoonright J_{\bar{\nu}} \prec_Q J_\nu$.

The following three propositions stated without proof summarize the most important properties of the special parameter which are instrumental in verifying (M0)-(M7).

Proposition 2 *Suppose $\nu, \tau \in S^1$ and let $\nu <_1 \tau$.*

1. $n(\nu) = n(\tau)$.

2. $\rho(\nu) = \nu \iff \rho(\tau) = \tau$.

3. If $\rho(\nu) > \nu$ then $\sigma_{\nu\tau}(\nu) = \tau$. †

Proposition 3 *Let $\sigma : Q(\nu_1) \longrightarrow_{\Sigma_1} Q(\nu)$, where σ is not the identity on $Q(\nu_1)$ and let $\sigma \upharpoonright \alpha(\nu_1) = id_{\alpha(\nu_1)}$ and $\mathfrak{p}(\nu) \in Rng(\sigma)$, then $\sigma(\mathfrak{p}(\nu_1)) = \mathfrak{p}(\nu)$ and thus $\nu_1 < \nu$. †*

Proposition 4 *Let \tilde{Q} be transitive and assume that $\sigma : \tilde{Q} \longrightarrow_{\Sigma_1} Q(\nu)$, is not the identity. Suppose that $\mathfrak{p}(\nu) \in Rng(\sigma)$, then there exists a unique $\bar{\nu} < \nu$ such that $\tilde{Q} = Q(\bar{\nu})$, $\sigma(\mathfrak{p}(\bar{\nu})) = \mathfrak{p}(\nu)$ and $\sigma \upharpoonright \alpha(\bar{\nu}) = id$. †*

1.2 Iterated Forcing

The method of forcing was invented by Cohen in 1963 and was applied to show that $Con(ZF) \longrightarrow Con(ZF + \neg AC)$ and $Con(ZFC) \longrightarrow Con(ZFC + 2^{\aleph_0} = \aleph_2)$. Earlier on Gödel had established, in his work on the constructible universe, that $Con(ZFC) \longrightarrow Con(ZFC + V = L)$. Since $(V = L) + ZFC \vdash GCH$, Cohen's discovery coupled with Gödel's results show CH is independent of the Axioms of set theory. The Cohen construction opened a new era in set theory leading to a multitude of consistency results. The method of forcing was greatly enhanced when Solovay, Martin and Tennenbaum developed the method of finite support iterated forcing. In 1972, Laver introduced countable support iterated forcing, a further refinement of the method.

In this section we present the technique of iterated forcing and describe what has been known as the "Continuum problem". The terminology we follow is fairly

standard, and is briefly reviewed here. We assume familiarity with the rudiments of Cohen forcing as developed in [Ku]. We consider forcing to take place over a ctm M of ZFC. For a given partial order (P, \leq) a filter $G \subseteq P$ is P -Generic if $G \cap D \neq \emptyset$, for all dense sets D in M . We reserve the greek letter κ to denote an infinite regular cardinal.

The following definition of iterated forcing, in its full generality, is taken from [Bau 1].

Definition 1.3 (Iterated Forcing) *Suppose that $\alpha \geq 1$ and let P_α be a set of α -sequences. P_α is an iteration of length α if the following conditions are satisfied:*

1. *If $\alpha = 1$ then for some partial order Q_0 , $P_1 = \{\langle p(0) \rangle \mid p(0) \in Q_0\}$. The order on P_1 is defined by $p \leq q$ if $p(0) \leq_0 q(0)$.*

2. *If $\alpha = \beta + 1$, then $P_\beta = \{p|\beta \mid p \in P_\alpha\}$ is an iteration of length β and there exists a \dot{Q}_β such that $\Vdash_\beta \dot{Q}_\beta$ is a p.o. and $p \in P_\alpha$ if $p|\beta \in P_\beta$ and $\Vdash_\beta p(\beta) \in \dot{Q}_\beta$. The ordering on P_α is defined by $p \leq q$ iff $p|\beta \leq_\beta q|\beta$ and $p|\beta \Vdash_\beta p(\beta) \leq q(\beta)$.*

3. *If $\text{Lim}(\alpha)$ then :*

a. *For all $\beta < \alpha$, $P_\alpha|\beta$ is an iteration of length β ,*

b. *$1_\alpha \in P_\alpha$, where $1_\alpha(\gamma) = 1_{P_\gamma}$ for all $\gamma < \alpha$ and*

c. *If $p \in P_\alpha$, $\beta < \alpha$, and $q \leq p|\beta$ then $r \in P_\alpha$, where $r(\delta) = p(\delta)$ if $\beta \leq \delta < \alpha$ and $r(\delta) = q(\delta)$ otherwise.*

The order on P_α is defined by: $p \leq_\alpha q \leftrightarrow \forall \beta < \alpha p|\beta \leq_\beta q|\beta$.

It is easy to see that if α is a limit ordinal, P_α is not completely determined by $\langle P_\beta \mid \beta < \alpha \rangle$. We need to specify the kind of limit to be taken at α . The most commonly used limits are either direct or inverse . Recall that the support of a condition $p \in P_\alpha$ is defined by $\text{supp}(p) = \{\beta < \alpha \mid \neg \Vdash_\beta p(\beta) = 1\}$.

We distinguish two cases:

Finite Support Iteration If Direct limits are taken at all limit ordinals $\gamma \leq \alpha$ then P_α is an α -stage iteration with conditions of finite support.

Countable Support Iteration If Direct limits are taken at limit ordinals γ with $cf(\gamma) > \omega$ and inverse limits elsewhere, then P_α is an α -stage iteration with conditions of countable support.

The preservation of cardinals in forcing constructions is shown by means of chain conditions, distributivity conditions and fusion arguments . If P has the κ -cc then all cardinals $\lambda \geq \kappa$ are preserved in a generic extension. If P is κ -distributive then all cardinals $\leq \kappa$ are preserved. Preservation of ω_1 for Axiom A forcings, a wide class of forcing notions, which includes Sacks and Laver's forcing is accomplished by the technique of fusion. Let P_α be an iteration of length α . Let us note the following:

1. Finite support iterations preserve chain conditions. In other words, if P_α is the finite support iteration of $\{\dot{Q}_\beta \mid \beta < \alpha\}$ and for every $\beta < \alpha \Vdash_\beta \dot{Q}_\beta$ has the κ -cc, then P_α has the κ -cc . The respective statement for countable support iterated forcing need not hold. However, if in addition we assume that whenever $cf(\alpha) = \kappa$ then $\{\beta < \alpha \mid P_\beta = \text{Dirlim}_{\gamma < \alpha} P_\gamma\}$ is stationary in α , then P_α has the κ -cc. For most practical purposes, if the iteration has length κ , where $\kappa \geq \omega_2$, it suffices to show that for each $\alpha < \omega$, there exists a dense subset W_α of P_α such that $|W_\alpha| < \kappa$.

2. The preservation of \aleph_1 in most forcing constructions is accomplished as follows:

2a. For the case of finite support iterated forcing, if the partial orders being iterated are ccc, then the preservation of \aleph_1 follows from 1. However, if the iteration contains cofinally many partial orders not satisfying the ccc, then \aleph_1 is not preserved. Moreover, many of the forcing notions we wish to iterate, such as Sacks and Laver forcing fail to satisfy the ccc.

2b. Countable support iteration preserves Axiom A. In other words, if for all $\beta < \alpha$ we have $\Vdash_{\beta} \dot{Q}_{\beta}$ satisfies Axiom A, then P_{α} satisfies Axiom A. Thus \aleph_1 is preserved.

Shelah developed his theory of proper forcing in his search to find a property which implies preservation of ω_1 and which is itself preserved under countable support iteration.

The most notable application of finite support iterated forcing is the relative consistency of Martin's Axiom with the continuum large.

Theorem 2 (MST) Let $\kappa \geq \aleph_2$. Then,

$$\text{Con}(\text{ZFC} + \text{GCH}) \longrightarrow \text{Con}(\text{ZFC} + \text{MA} + 2^{\aleph_0} = \kappa)$$

The proof of theorem 2 can be found in [MS]. In a finite support iteration Cohen reals are added at each limit point. This is a problem in constructions where we wish to preserve CH or avoid adding Cohen reals such as the construction of the Laver model.

A set $X \subseteq [0, 1]$ has strong measure zero if for every sequence $\langle \epsilon_n \mid n < \omega \rangle$ in \mathbb{R}^+ there exists $\langle I_n \mid n < \omega \rangle$, a sequence of intervals such that $lh(I_n) \leq \epsilon_n$ and $X \subseteq \bigcup_{n < \omega} I_n$. In 1919, Borel conjectured that all set of strong measure zero are countable. Subsequently, Lusin established in [Lu] that

$$CH + \text{ZFC} \vdash \text{there exist uncountable strong measure zero set.}$$

Thus, the following question was raised: Is Borel's Conjecture consistent with ZFC? In [La] Laver answered that question by proving the following theorem:

Theorem 3 $Con(ZFC) \longrightarrow Con(ZFC + 2^{\aleph_0} = \aleph_2 + \text{Borel Conjecture})$

In the proof of theorem 3, Laver introduces the method of countable support iterated forcing. The construction uses an \aleph_2 long iteration of Laver forcing notions with conditions of countable support. Preservation of cardinals is handled by 1 and 2b.

1.2.1 The Continuum Problem

The natural approach to establishing the consistency of Borel's Conjecture with the continuum being \aleph_3 would be an \aleph_3 long countable support iteration of Laver forcing. Unfortunately, this approach will not produce the desired consistency result. The following proposition, due to Roitman, and cited in [Bau] shows that for non-trivial forcings, the value of the continuum will be \aleph_2 , even in a countable support iteration of length \aleph_3 ; the problem is that 2^{\aleph_0} gets collapsed onto \aleph_1 after iterating ω_1 steps past a stage where $2^{\aleph_0} > \aleph_1$.

Proposition 1 *Let $2^{\aleph_0} > \aleph_1$ and let $P(\omega_1)$ be a countable support iteration of length ω_1 such that for all $\beta < \omega_1$ the following condition is satisfied*

$$\Vdash_{\beta} \exists p, q \in \dot{Q}_{\beta} [\neg \exists r \in \dot{Q}_{\beta} (r \leq p, q)]$$

Then, if G is $P(\omega_1)$ - generic over M , $V[G] \vdash 2^{\aleph_0} = \aleph_1$

Proof: A density argument. ⊣

The problem of obtaining the consistency of Borel's conjecture with the continuum large was solved by H. Woodin 1981. Woodin showed that assuming \diamond_{\aleph_1} , adding random reals to the model obtained by Laver's countable support iterated forcing,

produces a model satisfying $2^{\aleph_0} = \aleph_3$ in which there are no uncountable sets of strong measure zero. A different proof by Shelah can be found in [Sh].

There are many propositions whose relative consistency is known with the value of the continuum \aleph_2 but not known to hold with the continuum \aleph_3 . A representative problem of that type is the p -points problem.

An ultrafilter D on ω is called a p -point if for every countable sequence $\langle X_n \mid n < \omega \rangle$ of elements of $P(\omega)$ there exists an $X \in D$ with the property that $X - X_n$ is finite, for all $n < \omega$. Note that in the presence of CH , it is a theorem of ZFC that there exist p -points. The consistency, relative to ZFC , of the non-existence of p -points with the continuum being \aleph_2 remained open till 1982, when Shelah proved the following theorem, using a countable support iteration of length \aleph_2 :

Theorem 4 $Con(ZFC) \longrightarrow Con(ZFC + 2^{\aleph_0} = \aleph_2 + \text{there are no } p\text{-points})$.

The respective statement with the value of the continuum being \aleph_3 remains open. Another relevant statement which is again not known to be consistent with the continuum large is the statement that there are no p or q points. Taylor showed that $2^{\aleph_0} = \aleph_2$ implies that there exist either a p or a q point.

Chapter 2

The Construction

2.1 Introduction

In light of the failure of the method of countable support iteration to establish consistency results with the continuum large, it appears that a new framework of generalized iterated forcing is needed to surpass these difficulties.

T. Jech and M. Groszek have done some relevant work in the direction of nonlinear iterated forcing. They have devised a generalized iteration of forcing along a well founded partial order. Their technique has been used to show the relative consistency of $FP(C, \omega_2)$, an MA type principle stating that if P is a forcing notion on a class C of forcing notions called “Perfect Tree Forcing”- which contains Cohen, Laver and Sacks forcing among others- and $D = \langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a family of ω_1 -dense subsets of P , then there exists a filter G on P such that $G \cap D_\alpha \neq \emptyset$, for all $\alpha < \omega_1$:

Theorem 5 $Con(ZF) \longrightarrow Con(ZFC + 2^{\aleph_0} = \kappa + FP(C, \omega_1))$

In this thesis we develop a new framework of iterated forcing by attaching forcing notions to the points of a gap-1 morass at ω_1 in, such a way as to have the top level being an ω_2 iteration determined by the morass tree. We look at a special case,

the ω_1 - Cohen case. The definition proceeds by morass induction on $\tau \in S^1$ and is comprised of several subcases.

Before commencing with the definition of $\{P(\tau) \mid \tau \in S^1\}$ we list some of the requirements governing our construction. We proceed inductively and define for each $\tau \in S^1$ the sets $P(\tau)$, $P^*(\tau)$ such that the following conditions are met:

- **1** For each $\tau \in S^1$ $P^*(\tau) \subseteq P(\tau) \subseteq L_\tau$.
- **2** A condition $p \in P(\tau)$ is a function $p : Pred_0(\tau) \cup \{\alpha(\tau)\} \longrightarrow L_\tau$, where $Pred_0(\tau)$ is the set of level predecessors of τ .
- **3** $P^*(\tau)$ is dense in $P(\tau)$.
- **4** Suppose that σ is not a level limit and let $\sigma <_1 \tau$. We want a set “generic at τ ” to pull back to a “generic at σ ”, for $\sigma <_1 \tau$, via the map $f_{\sigma\tau} : L_\sigma \longrightarrow L_\tau$. The concept of “generic at τ ” is formulated in definition 1.
- **5** Intuitively the dense sets $P^*(\tau)$ defined at morass point τ will contain the conditions which are generic with respect to a tree predecessor. $P(\tau)$ will consist of conditions which are extended by conditions in $P^*(\tau)$.
- **6** The definition is monotone along the levels, that is for all $\sigma, \tau \in S^1$, if $\sigma <_0 \tau$ then , there exist a natural embedding $i_{\sigma\tau} : P(\sigma) \longrightarrow P(\tau)$.
- **7** For $p \in P(\tau)$ the support of p is controlled by the morass maps from below. More precisely, we require that if $\tau \notin Min_{<_0} S^1$, then there exists τ_1, σ such that $\sigma <_1 \tau_1 <_0 \tau$, and

$$supp(p) \subseteq Rng f_{\sigma\tau_1} \cup \{\tau_1\}$$

- **8** $P = Dirlim\{P(\sigma) \mid \alpha(\sigma) = \omega_1\}$ is ω_2 -cc and ω -distributive.

Note $P(\tau)$ will not be a (linear) iteration in the standard sense.

We first define the notion of a set $G \subseteq P(\sigma)$ being σ -generic:

Definition 2.1 Let $\mathbf{P}(\sigma)$ be the forcing notion inductively defined at σ and let σ^* denote the last morass point on level $\alpha(\sigma)$.

$G \subseteq P(\sigma)$ is σ -generic if $G \cap D \neq \emptyset$ for all $D \in L_{\sigma^*}$ which are dense in $\mathbf{P}(\sigma)$.

The goal is to prove that there exists a morass indexed system of forcing notions satisfying the foregoing requirements. In section 2.2 we define a new framework of iterated forcing followed by some remarks aimed at clarifying the many clauses of the definition. In section 2.3 we prove the key lemmas related to this construction.

Let us first dispose of some terminology which will be used in the definition of the forcing.

- **1** Let $\tau \in S^1$. The notation $Min_{<_0}(\tau)$, $Succ_{<_0}(\tau)$, $Lim_{<_0}(\tau)$ is used to signify that τ is a level minimal, level successor, level limit respectively. Similar notation for the respective $<_1$ notions. We denote the set of level predecessors of τ by $Pred_0(\tau)$. Also, let $Min_{<_0}S^1 = \{\tau \in S^1 \mid \tau \text{ is level minimal}\}$. $Lim_{<_0}S^1$ is defined similarly.
- **2** Let $p \in P(\tau)$ and let $G \subseteq P(\tau)$. Suppose that $\sigma <_1 \tau$. We define $Gen_{\sigma\tau}(p)$ and $Gen_{\sigma\tau}(G)$ by:

$$Gen_{\sigma\tau}(p) = \{q \in P(\sigma) \mid f_{\sigma\tau}(q) \geq p\}, Gen_{\sigma\tau}(G) = \{q \in P(\sigma) \mid f_{\sigma\tau}(q) \in G\}$$
- **3** When we say that p pulls back to a generic at σ we mean that $Gen_{\sigma\tau}(p)$ is a σ generic set. Similarly, G pulls back to a generic at σ if $Gen_{\sigma\tau}(G)$ is a σ -generic set. Also,

$$Gen_{\tau}(\sigma) = \{p \in P(\tau) \mid Gen_{\sigma\tau}(p) \text{ is } \sigma - \text{generic}\}$$

- 4 In the definition of our forcing it will not necessarily be the case that if $p \in P(\tau)$ and $\sigma <_0 \tau$ then $p|_\sigma \in P(\sigma)$. Since we will be taking inverse limits at certain morass points, we need to use the following modified definition of an inverse limit. Let σ be a level limit and let us assume that $P(\sigma_1)$ has been defined for each $\sigma_1 <_0 \sigma$. Then,

$$\text{Invlim}_{\sigma_1 <_0 \sigma}^* P(\sigma_1) = \{p \in L_{\sigma^*} \mid \forall \sigma_1 <_0 \sigma [\sigma_1 \notin \text{Lim}_{<_0} S^1 \rightarrow p|_{\sigma_1} \in P(\sigma_1)]\}$$

2.2 Definition of the Forcing

In this section we present the definition of the generalized iterated forcing along a gap-1 morass at ω_1 . The definition and all subsequent proofs will be a morass induction.

Definition 2.2 Let $\tau \in S^1$. We define $P^*(\tau) \subseteq P(\tau) \subset L_\tau$ using morass induction on S^1 :

CASE A: τ tree minimal point

Subcase 2.2.1 τ is level minimal. Let $P(\tau) = (\omega_1 - \text{Cohen})^{L(\tau)}$ and let $P^*(\tau) = P^*(\tau)$.

Subcase 2.2.2 τ is level successor to a $\text{Min}_{<_0} S^1$ point or τ is a level double successor. Let $\sigma <_0^* \tau$ and let $\sigma \notin \text{Lim}_{<_0} S^1$. If G is τ -generic, let $G_\sigma = \{p|_\sigma \mid p \in G\}$. We will show in proposition 4 that G_σ is σ -generic.

We define:

$$P^*(\tau) = \{(p, \theta) \mid p \in P^*(\sigma) \wedge p \Vdash_\sigma \theta \in (\omega_1 - \text{Cohen})^{L_\sigma}\} \quad (2.1)$$

$$P(\tau) = P^*(\tau) \quad (2.2)$$

Subcase 2.2.3 τ is a level successor to a level limit. Let σ be a level limit and let $\sigma <_0^* \tau$. Define $P^*(\tau)$ and $P(\tau)$ as follows:

$$P^*(\tau) = \{p \in \text{Invlim}_{\sigma_1 <_0 \sigma}^* P(\sigma_1) \cap L_\tau \mid (|\text{supp}(p)| \leq \omega)^{L_\tau}\} \quad (2.3)$$

$$P^*(\tau) = P(\tau) \quad (2.4)$$

Subcase 2.2.4 τ is a level limit. Define

$$P(\tau) = \text{Dirlimit}_{\sigma <_0 \tau} P(\sigma) \quad (2.5)$$

$$P^*(\tau) = P(\tau) \quad (2.6)$$

CASE B : τ is a tree successor

Subcase 2.2.5 τ is level minimal. Suppose $\bar{\tau} <_1^* \tau$; then $\text{Min}_{<_0}(\bar{\tau})$. Define:

$$P^*(\tau) = \{p \in (\omega_1 - \text{Cohen})^{L_\tau} \mid p \in \text{Gen}_\tau(\bar{\tau})\} \quad (2.7)$$

$$P(\tau) = \{p \in (\omega_1 - \text{Cohen})^{L_\tau} \mid \exists q \in P^*(\tau)(q \leq p)\} \quad (2.8)$$

Subcase 2.2.6 τ is level successor to a $\text{Min}_{<_0} S^1$ point or τ is a level double successor. Suppose that $\bar{\tau} <_1^* \tau$, $\sigma <_0^* \tau$, and $\text{Min}_{<_0}(\sigma)$ or $\text{Succ}_{<_0}(\sigma)$. Let $\bar{\sigma} = f_{\bar{\tau}\tau}^{-1}(\sigma) <_1 \sigma$. By induction, it is dense on $P(\sigma)$ for $p \in P(\sigma)$ to pull back to a $\bar{\sigma}$ -generic set. See lemma 2.1.

Hence the following definition is justified:

$$P^*(\tau) = \{(p, \theta) \in P^*(\sigma) * (\omega_1 - \text{Cohen})^{L_\sigma} \mid \{(\bar{p}, \bar{\theta}) \in P^*(\bar{\tau}) \mid \bar{p} \in \text{Gen}_{\bar{\sigma}\sigma}(p) \wedge p \Vdash \bar{\theta}^{\text{G}_{\bar{\sigma}\sigma}(p)} \geq \theta\} \text{ is } \bar{\tau}\text{-generic}\} \quad (2.9)$$

Note that $P^*(\tau)$ is of the form: $P^*(\tau) = P^{**}(\sigma) * Q(\sigma)$, where $P^{**}(\sigma)$ is dense in $P^*(\sigma)$. We may then define $P(\tau)$ to consist of weakenings of $P^*(\tau)$:

$$P(\tau) = \{(p, \theta) \in P(\sigma) * (\omega_1 - \text{Cohen})^{L_\tau} \mid \exists (p_1, \theta_1) \in P^*(\tau) \text{ such that } p_1 \leq p \text{ and } p_1 \Vdash \theta_1 \leq \theta\}.$$

Subcase 2.2.7 τ is a level successor to a level limit. Suppose that $\sigma <_0^* \tau$, $\text{Lim}_{<_0}(\sigma)$, $\bar{\tau} <_1^* \tau$.

We define $P^*(\tau)$ and then let $P(\tau)$ to be the upward closure of $P^*(\tau)$:

$$P^*(\tau) = \{p \in \text{Invlim}_{\bar{\sigma} <_0 \sigma}^* P(\bar{\sigma}) \cap L_\tau \mid (|\text{supp}(p)| \leq \omega)^{L_\tau} \wedge p \in \text{Gen}_\tau(\bar{\tau})\} \quad (2.10)$$

$$P(\tau) = \{p \in \text{Invlim}_{\bar{\sigma} <_0 \sigma}^* P(\bar{\sigma}) \cap L_\tau \mid (|\text{supp}(p)| \leq \omega)^{L_\tau} \wedge \exists p^* \in P^*(\tau) \text{ such that } p^* \leq p\}. \quad (2.11)$$

Subcase 2.2.8 τ is a level limit. Define:

$$P(\tau) = \text{Dirlimit}_{\sigma <_0 \tau} P(\sigma) \quad (2.12)$$

$$P^*(\tau) = \{p \in P(\tau) \mid \exists \bar{p} \in P^*(\tau_1), \tau_1 <_1 \tau, \pi_{\tau_1 \tau}(\bar{p}) = p\} \quad (2.13)$$

Case C : τ is a Tree Limit

In this case we define:

$$P(\tau) = \text{Dirlim}\{P(\tau_1) \mid \tau_1 <_1 \tau\} \quad (2.14)$$

$$P^*(\tau) = P(\tau) \quad (2.15)$$

Let us note the following with regard to definition 2.2:

1. We have assumed that if $G \subseteq P(\tau)$ is τ -generic and $\sigma <_0 \tau$, where $\sigma \notin \text{Lim}_{<_0} S^1$, then $G|_\sigma = \{p|_\sigma \mid p \in G\}$ is σ -generic. We prove that claim in proposition 4.

2 We have to establish that the above construction is well defined and nontrivial; for instance we need to show that $P^*(\tau) \neq \emptyset$, for all $\tau \in S^1$. This is a consequence of the extendibility lemma.

3. It will not necessarily be the case that a restriction of a condition $p \in P(\tau)$ to a level predecessor σ of τ will be a condition in $P(\sigma)$. We examine that in lemma 2.2.

4. The notions of support and length have to be defined. This is done next.

5. In the case of $\tau \in \text{Lim}_{<0} S^1 \cap \text{Lim}_{<1} S^1$, we need to prove that the definitions of the forcing corresponding to Case C and 2.2.8 coincide. This is proven in proposition 7.

We now clarify the notions of support and length of a condition $p \in P(\tau)$.

Definition 2.3 We proceed by morass induction on $\tau \in S^1$ and define $\text{supp}(p)$, $lh(p) \geq \alpha$, for $p \in P(\tau)$, $\alpha \in ORD$.

Subcase 2.2.1 τ is a level minimal point.

Since $P(\tau) \subseteq (\omega_1 - \text{Cohen})^{L_\tau}$, $lh(p) \geq \alpha$ is well defined. For the support of p we let:

$$\text{supp}(p) = \begin{cases} \emptyset & \text{if } p = \emptyset \\ \{\alpha(\tau)\} & \text{otherwise} \end{cases}$$

Subcase 2.2.2 τ is a level successor to a level minimal point or a double successor.

Let $\sigma <_0 \tau$. Suppose that $p = (q, \theta)$. Define:

$$\text{supp}(p) = \begin{cases} \text{supp}(q) \cup \{\theta\} & \text{if } \neg q \Vdash \theta = 1 \\ \text{supp}(q) & \text{otherwise} \end{cases}$$

For $lh(p) \geq \alpha$, since $lh(q) \geq \alpha$ has been inductively defined for q , let

$$lh(p) \geq \alpha \Leftrightarrow lh(q) \geq \alpha \wedge q \Vdash lh(\theta) \geq \alpha$$

Subcase 2.2.3 τ is a level limit.

Since $P(\tau) = \text{Dir}\lim_{\sigma <_0 \tau} P(\sigma)$, $lh(p) \geq \alpha$ and support have been inductively defined.

Subcase 2.2.4 τ is a level successor to a level limit.

Define

$$\text{supp}(p) = \bigcup \{ \text{supp}(p|_{\bar{\sigma}}) \mid \text{Succ}_{<_0}(\bar{\sigma}) \wedge \bar{\sigma} <_0 \tau \}$$

and

$$lh(p) \geq \alpha \leftrightarrow \forall \bar{\sigma} <_0 \tau [\bar{\sigma} \in \text{Succ}_{<_0} S^1 \rightarrow lh(p|_{\bar{\sigma}}) \geq \alpha]$$

The foregoing completely specifies the support of a condition $p \in P(\tau)$. If τ is a tree limit, the support of a condition $p \in P(\tau)$ may also be characterized as follows. Let $p \in P(\tau)$. There exists $\bar{\tau} <_1 \tau$ and $\bar{p} \in P(\bar{\tau})$ such that $p = f_{\bar{\tau}\tau}(\bar{p})$. Inductively, $lh(p) \geq \alpha$ has been defined at stage $\bar{\tau}$. Then, $lh(p) \geq \alpha \Leftrightarrow lh(f_{\bar{\tau}\tau}(\bar{p})) \geq \alpha$ and thus $\text{supp}(p) = f_{\bar{\tau}\tau}(\text{supp}(\bar{p}))$.

2.3 Key Lemmas

In this section we establish several lemmas key to the definition and show that the above construction satisfies the requirements set forth on page 22. The key lemmas are the Extendibility and Amalgamation Property. Both properties will be proved in a simultaneous morass induction. The following two propositions are easy to prove and will be used extensively in subsequent proofs.

Proposition 2

(a) Let $q \leq p, \bar{\tau} <_1 \tau$ and let $p \in \text{Gen}_{\tau}(\bar{\tau})$. Then $q \in \text{Gen}_{\tau}(\bar{\tau})$ and $\text{Gen}_{\bar{\tau}\tau}(p) = \text{Gen}_{\bar{\tau}\tau}(q)$.

(b) Suppose that τ is a tree limit and let $\sigma <_0 \tau$. There exists $\bar{\tau}, \bar{\sigma} \in S^1$ such that $\bar{\tau} <_1 \tau$, $\bar{\sigma} <_0 \bar{\tau}$ and $\sigma = f_{\bar{\tau}\tau}(\bar{\sigma})$.

Proof

(b) Since τ is a tree limit point $\{\alpha(\nu) \mid \nu <_1 \tau\}$ is unbounded in $\alpha(\tau)$ and thus not maximal in $S_{\alpha(\tau)}$. By axiom 6, there exists a $\bar{\tau} <_1 \tau$ and $\bar{\sigma} \in \bar{\tau}$ such that $\sigma = f_{\bar{\tau}\tau}(\bar{\sigma})$. By Axioms 1 and 2 we have $\bar{\sigma} <_0 \bar{\tau}$.

(a) *Easy.* ⊣

Proposition 3 Suppose that $\sigma <_0 \tau, X \in L_\tau$ and let $L_\tau \models |X| \leq \aleph_0$. Suppose that $X \subseteq ORD$ and $\sup X < \sigma$. Then $X \in L_\sigma$ and $L_\sigma \models |X| \leq \aleph_0$.

Proof Let $\bar{\sigma} = \sup X$. Working in L_σ , since $\omega_1^{L_\sigma} = \alpha(\bar{\sigma})$, let us choose a bijection $f : \bar{\sigma} \rightarrow \alpha(\bar{\sigma})$. Then letting $\bar{X} = Rng f \upharpoonright X$, we have $\bar{X} \subseteq \eta < \alpha(\bar{\sigma})$, for otherwise in L_τ , we have $\text{cof}(\alpha(\bar{\sigma})) = \omega$ which is a contradiction, since $\alpha(\bar{\sigma}) = \omega_1^{L_\tau}$. Then, $\bar{X} \in L_{\alpha(\bar{\sigma})}$. But since $L_{\alpha(\bar{\sigma})} \subseteq L_\sigma$ we then have $f^{-1} \upharpoonright \bar{X} \in L_\sigma$. Since $f^{-1} \upharpoonright \bar{X} = X$, it follows that $X \in L_\sigma$ and obviously $L_\sigma \models |X| \leq \aleph_0$. ⊣

Some of the key properties of the definition are contained in extendibility and amalgamation lemmas, which are proved by a simultaneous morass induction. The amalgamation lemma will be applied to show that $P(\omega_2)$ has the ω_2 -cc. The extendibility lemma is used to establish the nontriviality and ω -distributivity of the forcing. Lemma 1(ii) states a natural closure condition of the forcing notions $P(\tau)$ which is needed to carry on the induction.

Lemma 2.1 (Extendibility Lemma) Suppose that $X \subseteq \text{Pred}_{<_0}(\tau) = \{\sigma \in S^1 \mid \sigma <_0 \tau\}$, where $X \in L_\tau$ and $L_\tau \models |X| \leq \aleph_0$. Let $p \in F(\tau)$ and let $\alpha < \alpha(\tau)$. Then:

(i). There exists $q \in P(\tau)$ such that $q \leq p$, $\text{lh}(q) \geq \alpha$ and $X \subseteq \text{supp}(q)$. Furthermore, if $\bar{\tau} <_1 \tau, \tau \notin \text{Lim}_{<_0} S^1$, then q may be chosen to be $\bar{\tau}$ -generic.

- (ii) Let $\bar{\tau} <_1 \tau$, and let $\bar{p} \in P(\bar{\tau})$. Then $f_{\bar{\tau}\tau}(\bar{p}) \in P(\tau)$.
- (iii) Suppose $\bar{\sigma}_0 <_1 \sigma_0 <_0 \tau$, $\bar{\tau}_1 <_1 \tau$ where $\sigma_0 \notin \text{Lim}_{<_0} S^1$, and let $\bar{p} \in P(\bar{\tau}_1)$. Suppose that $p_0 \in \text{Gen}_{\sigma_0}(\bar{\sigma}_0)$. Then, there exists a p in $P(\tau)$ such that:
- (a) $p|_{\sigma_0} = p_0$, if $\tau \notin \text{Lim}_{<_0} S^1$ then $p \in \text{Gen}_{\tau}(\bar{\tau}_1)$.
- (b) $\bar{p} \in \{q \in P(\bar{\tau}_1) \mid f_{\bar{\tau}_1\tau}(q) \geq p\} = \text{Gen}_{\bar{\tau}_1}(p)$.

Lemma 2.2 (The Amalgamation Property) Suppose that $\sigma <_0 \tau$, and let $\sigma \notin \text{Lim}_{<_0} S^1$.

- (a) For every $p \in P(\tau)$ we have $p|_{\sigma} \in P(\sigma)$.
- (b) Suppose that $q \in P(\sigma)$, $p \in P^*(\tau)$ and let $q \leq p|_{\sigma}$. Consider the function $p_{\sigma}^u : \{\bar{\sigma} \mid \sigma \leq_0 \bar{\sigma} <_0 \tau\} \rightarrow L_{\tau}$ defined by $p_{\sigma}^u(\bar{\sigma}) = p(\bar{\sigma})$. Then, $p^* \in P(\tau)$, where $p^* : \text{Pred}_{<_0}(\tau) \rightarrow L_{\tau}$ is defined by:

$$p^*(\bar{\sigma}) = \begin{cases} q(\bar{\sigma}) & \text{if } \bar{\sigma} <_0 \sigma \\ p_{\sigma}^u(\bar{\sigma}) & \text{otherwise} \end{cases}$$

Proof: We use the notation $p^* = q \sqcup p_{\sigma}^u$ to refer to the amalgam of q and p_{σ}^u , in Lemma 2(b). We prove lemma 2.1 and 2.2 by a simultaneous induction on τ . The proof splits into 10 cases, corresponding to the definition of the forcing.

CASE A : τ is a tree minimal point

Subcase 2.3.1 τ is level minimal. Parts 2.1(ii), (iii) of lemma 2.1 and lemma 2.2 are vacuously satisfied. For 2.1(i) we merely take $q \leq p$ with $lh(q) \geq \alpha$.

Subcase 2.3.1 τ is level successor to a level minimal point. Suppose that $\sigma <_0^* \tau$, $\text{Min}_{<_0}(\sigma)$.

To show lemma 2.2 let $p = (w, \theta) \in P(\tau)$. By definition, $p|_{\sigma} \in P(\sigma)$. If $q \leq w$, then $(q, \theta) \in P(\tau)$. To prove the extendibility property, by induction hypothesis

applied to σ , we have that there exists $q_1 \leq p|_\sigma$ such that $q_1 \in P(\sigma)$, $X \cap \sigma \subseteq \text{supp}(q_1)$ and $lh(q_1) \geq \alpha$. If $p = (p|_\sigma, \theta)$ let $\bar{\theta} \in \text{Term}_{L_\sigma[G]}(\omega_1 - \text{Cohen})^{L_\sigma}$ be such that $lh(\bar{\theta}) \geq \alpha$ and $\theta \geq \bar{\theta}$. The condition $(q_1, \bar{\theta})$ satisfies the conclusion of the lemma. Parts (ii) and (iii) of lemma 2.1 are vacuous.

Subcase 2.3.2 τ is a level double successor. Let $\tau_2 <_0^* \tau_1 <_0^* \tau$. In lemma 2.2, if $\tau_1 = \sigma$ or $\sigma = \tau_2$ the result is immediate as in the previous subcase. Write $p = (w, \theta)$. If $\sigma \neq \tau_1, \tau_2$, as $w \in P(\tau_1)$ and $w|_\sigma = p|_\sigma$, by inductive hypothesis $w|_\sigma \in P(\sigma)$. Also, inductively (b) is established for τ_1, σ, w ; hence, $q \sqcup w_\sigma^u \in P(\tau_1)$. Then $(q \sqcup w_\sigma^u, \theta) \in P(\tau)$. If $\sigma = \tau_2$, then it is also immediate as above. This establishes lemma 2.2.

For the extendibility property select $q_1 \in P(\sigma_1)$ satisfying 1(i) and $\bar{\theta} \in \text{Term}_{L_\sigma[G]}(\omega_1 - \text{Cohen})^{L_\tau}$ as in the previous subcase.

Subcase 2.3.3 τ is a level limit. For lemma 2.2, let $p \in P(\tau)$ and let $\sigma <_0 \tau$. We have $\text{supp}(p) \subseteq \sigma_1 <_0 \tau$ for some $\sigma_1 \in \text{Succ}_{<_0} S^1$. If $\sigma_1 \leq \sigma$, the result is immediate; if $\sigma <_0 \sigma_1$, it follows by induction hypothesis. For the extendibility claim, apply induction hypothesis to σ_1 . There exists $q \leq p$ such that $X \cap \sigma_1 \subseteq \text{supp}(q)$ and $lh(q) \geq \alpha$. By definition of support and length, q is the required condition.

Subcase 2.3.4 τ is a level successor to a level limit. Let $\text{Lim}_{<_0}(\tau_1)$ and $\tau_1 <_0^* \tau$. Since $\sigma \neq \tau_1$, $p|_\sigma \in P(\sigma)$, by definition of inverse limit. Also, if $q \leq p|_\sigma$, then, for $\nu \in \text{Succ}_{<_0} S^1$ we have:

$$p^*|_\nu = \begin{cases} p|_\nu & \text{if } \sigma <_0 \nu <_0 \tau_1 \\ q|_\nu & \text{if } \nu <_0 \sigma \end{cases}$$

Therefore,

$$p^* \in \text{Invlim}_{\nu <_0 \sigma}^* P(\nu) \cap L_\tau$$

This establishes lemma 2.2 For lemma 2.1, let

$$\sigma_0 <_0 \sigma_1 <_0 \sigma_2 <_0 \dots$$

be a countable sequence with $\sup_{n < \omega} \tau_n = \sigma$, such that for all $n < \omega$, $\tau_n \notin \text{Lim}_{<_0} S^1$. Since $\sigma_0 \notin \text{Lim}_{<_0} S^1$, we have $p|_{\sigma_0} \in P(\sigma_0)$ by 2.2(i) and thus, applying inductive hypothesis to $X \cap \sigma_0$, there exists a $w_0 \leq p|_{\sigma_0}$ in $P(\sigma_0)$ such that $X \cap \sigma_0 \subseteq \text{supp}(w_0)$. Consider

$$w_1 = w_0 \sqcup (p|_{w_0})_\sigma^u$$

Since the amalgamation property has been established for τ we have that $w_1 \in P(\tau)$. Hence, we may now induce on $w_1|_{\sigma_1}$ and $X \cap \sigma_1$, etc.

CASE B : τ is a tree successor point

Subcase 2.3.5 τ is level minimal. Suppose that $\bar{\tau} <_1^* \tau$. Lemma 2.2 and 2.1(iii) are vacuous. To establish the extendibility lemma we only need show that we can extend p

to a $\bar{\tau}$ - generic condition. (Note that (ii) is then immediate since $f_{\bar{\tau}\tau} = \text{id}_{L_\tau}$). To show that there exists $q \leq p$ with $q \in \text{Gen}_\tau(\bar{\tau})$, we proceed as follows. In L_τ , let $\langle D_n \mid n < \omega \rangle$ be a countable listing of the dense open sets of $P(\bar{\tau})$ in $L_{\bar{\tau}}$. Using the density of the D_n we construct

$$\bar{q}_1 \geq_0 \bar{q}_2 \geq_0 \bar{q}_3 \geq ..$$

a countable sequence such that $\bar{q}_{n+1} \in D_n$ and let $\bar{q} = \bigcup_{n < \omega} \bar{q}_n$. Since $\bar{q}_n \in (\omega_1 - \text{Cohen})^{L_\tau}$, it follows that $\bar{q} \in (\omega_1 - \text{Cohen})^{L_\tau}$. Hence we have that $q \in \text{Gen}_\tau(\bar{\tau})$.

Subcase 2.3.6 τ is level successor to a level minimal point. Suppose that $\bar{\tau} <_1^* \tau$, $\sigma <_0^* \tau$ and $\bar{\sigma} = f_{\bar{\tau}\tau}^{-1}(\sigma)$. Let $p = (p|_\sigma, \theta)$.

For the amalgamation, since $p \in P^*(\tau)$ we have $p \in \text{Gen}_\tau(\bar{\tau})$ and thus, by induction, $p|\sigma \in \text{Gen}_\sigma(\bar{\sigma})$. Thus, since $q \leq p|\sigma$, $\text{Gen}_{\bar{\sigma}\sigma}(p|\sigma) = \text{Gen}_{\bar{\sigma}\sigma}(q|\sigma)$ and therefore $p^* \in P^*(\tau)$. The proof of the extendibility clause is the same as in the next subcase:

Subcase 2.3.7 τ is a level double successor. Let $\bar{\tau} <_1^* \tau$, $\sigma_1 <_0^* \sigma <_0^* \tau$ and let $\bar{\sigma} = f_{\bar{\tau}\tau}^{-1}(\sigma)$. Suppose that $p = (p|\sigma, \theta)$. We first show 2.1(i). Notice that we can assume without loss of generality that $\alpha = \alpha(\bar{\sigma}_1)$ for some $\bar{\sigma}_1 <_1 \sigma$. Furthermore, we may choose $p \in P^*(\tau)$. By proposition 2, $X \cap \sigma \in L_\sigma$ and $L_\sigma \models |X \cap \sigma| \leq \omega$. Hence, by applying inductive hypothesis to $X \cap \sigma$, $p|\sigma$ there is a $q_1 \leq p|\sigma$ such that $X \cap \sigma \subseteq \text{supp}(q_1)$ and $\text{lh}(q_1) \geq \alpha$. We consider a term $\theta^* \in \text{Term}_{L_{\bar{\sigma}|\sigma}(\omega_1\text{-Cohen})}^{L_\tau}$ with $\text{lh}(\theta^*) \geq \alpha$ such that $\theta^* \leq \theta$. The condition $q = (q_1, \theta^*)$ satisfies 2.1(i).

To prove the second clause of lemma 2.1, let us first assume that $\bar{\tau}_1 = \bar{\tau}$. Since $f_{\bar{\tau}\tau}(\bar{p}) \in P(\sigma) * (\omega_1 - \text{Cohen})^{L_\sigma}$ we only need show that we can extend $f_{\bar{\tau}\tau}(\bar{p}) = p$ to a condition $q \in \text{Gen}_\tau(\bar{\tau})$. Let $q^* \leq p|\sigma$, where $q^* \in \text{Gen}_\sigma(\bar{\sigma})$. Now, working in L_τ , let $\langle D_n \mid n < \omega \rangle$ be a listing (in $L_{\alpha(\tau)}$) of the dense open subsets of $P(\bar{\tau})$ in $L_{\bar{\tau}}$. We construct a sequence

$$\bar{p} = (\bar{q}_0, \bar{\theta}_0) \geq (\bar{q}_1, \bar{\theta}_1) \geq \dots$$

with $(\bar{q}_{n+1}, \bar{\theta}_{n+1}) \in D_n$ and $\bar{q}_{n+1} \in \text{Gen}_{\bar{\tau}}(q^*)$. Obviously, $\bigcup_{n \leq \omega} \bar{\theta}_n = \bar{\theta}$ is an ω_1 -Cohen term. Then, $(q^*, \theta), \theta = f_{\bar{\tau}\tau}(\bar{\theta})$ is the required extension of p . If $\bar{\tau}_1 \neq \bar{\tau}$, note that since inductive hypothesis implies that $f_{\bar{\tau}_1\bar{\tau}}(\bar{p}) \in P(\bar{\tau})$, the foregoing implies that $f_{\bar{\tau}\tau}(\bar{p}) = f_{\bar{\tau}\tau}(f_{\bar{\tau}_1\bar{\tau}}(\bar{p})) \in P(\tau)$.

For 2.1(iii), again it suffices to show the claim for $\bar{\tau}_1 = \bar{\tau}$, for having established that the case $\bar{\tau}_1 <_1 \bar{\tau}$ is handled by considering $f_{\bar{\tau}_1\bar{\tau}}(\bar{p})$. Suppose now that $\bar{p} \in P(\bar{\tau})$ and $p_0 \in P(\sigma_0)$, where $p_0 \in \text{Gen}_{\sigma_0}(\bar{\sigma})$. Since $p_0 \in \text{Gen}_{\sigma_0}(\bar{\sigma})$ we proceed as in 2.1(ii) to construct a $\bar{\tau}$ -generic condition $p \in P(\tau)$, by considering a sequence $\langle \bar{q}_n, \bar{\theta}_n \mid n < \omega \rangle$, where $(\bar{q}_{n+1}, \bar{\theta}_n) \in D_n$ and $\bar{q}_{n+1} \in \text{Gen}_{\bar{\sigma}_0}(p_0)$. Let $p = (p_0, \theta)$. The amalgamation

Property is shown as in the previous subcase.

Subcase 2.3.8 (τ is a level successor to a level limit) Suppose that $\sigma_1 <_0^* \tau$ and $\bar{\tau} <_1^* \tau$. For the amalgamation property, first note that by assumption $p \in Gen_\tau(\bar{\tau})$. Notice that $p^* = q \sqcup p_\sigma^u$ is a condition in the $Invlim_{\bar{\sigma} <_0 \sigma_1}^* P(\bar{\sigma})$ since for $\nu \in Succ_{<_0} S^1$ we have:

$$q^* \upharpoonright \nu = \begin{cases} q \upharpoonright \nu & \text{if } \nu <_0 \sigma \\ p \upharpoonright \nu & \text{otherwise} \end{cases}$$

We have that $p^* < p$. Thus, $p^* \in Gen_\tau(\bar{\tau})$.

For the extendibility property we first show 2.1(ii) assuming 2.1(iii). As in the previous subcase, we need only establish that the statement holds for $\bar{p} \in P(\bar{\tau})$. By inductive hypothesis and the elementarity of $f_{\bar{\tau}\tau}$, $f_{\bar{\tau}\tau}(\bar{p}) \in Invlim_{\bar{\sigma} <_0 \sigma_1}^* P(\bar{\sigma})$. By 2.1(iii), there exists a $p \in P(\tau)$ with $p \in Gen_\tau(\bar{\tau})$ and $f_{\bar{\tau}\tau}(\bar{p}) \geq p$. Hence, $f_{\bar{\tau}\tau}(\bar{p}) \in P(\tau)$.

To show 2.1(iii), it suffices to assume that $\bar{\tau}_1 = \bar{\tau}$. If $Rng(f_{\bar{\tau}\tau}) \cap \sigma_1$ is bounded in σ_1 apply induction hypothesis, otherwise let $\langle D_n \mid n < \omega \rangle$ be a listing of the dense open sets of $P(\bar{\tau})$ in $L_{\bar{\tau}^*}$. We construct

$$\bar{p}_0 \geq \bar{p}_1 \geq \dots$$

a sequence of conditions in $P(\bar{\tau})$ such that $\bar{p}_n \in D_{n+1}$ and a sequence of conditions in $P(\tau)$,

$$p_0 \geq p_1 \geq p_2 \dots$$

Let $\langle \lambda_n \mid n < \omega \rangle$ be a cofinal sequence in $Rng(f_{\bar{\tau}\tau}) \cap \sigma_1$, such that for all $n < \omega$ $\lambda_n \notin Lim_{<_0} S^1$ and $\sigma_0 <_0 \lambda_n$. Furthermore, let $\bar{\lambda}_n = f_{\bar{\tau}\tau}^{-1}(\lambda_n)$. We first prove the density of $D_1 \upharpoonright \bar{\sigma}_0$ in $P(\bar{\sigma}_0)$. Since the amalgamation property has been established for $\bar{\tau}, \bar{\sigma}_0$, if $\bar{w} \in P(\bar{\sigma}_0)$, then letting $\bar{q}_1 : Pred_{<_0}(\bar{\tau}) \rightarrow L_{\bar{\tau}}$ be defined

by $\bar{q}_1(\gamma) = \bar{w}(\gamma)$ if $\gamma <_0 \bar{\sigma}_0$ and $\bar{q}_1(\gamma) = 1$ otherwise, we have, by the density of D_1 , that $\exists \bar{q}_2 \in D_1(\bar{q}_2 \leq \bar{q}_1)$. Let $\bar{w} = \bar{q}_2 | \bar{\sigma}_0$. Then $\bar{w} \leq w \wedge \bar{w} \in D_1 | \bar{\sigma}_0$. Since $p_0 \in Gen_{\sigma_0}(\bar{\sigma}_0)$ using the density of $D_1 | \bar{\sigma}_0$, we select \bar{p}_1 such that $\bar{p}_1 \leq \bar{p}_0$, $\bar{p}_1 \in D_1$ and $\bar{p}_1 | \bar{\sigma}_0 \in Gen_{\bar{\sigma}_0}(p_0)$. By 2(i), $\bar{p}_1 | \bar{\lambda}_1 \in P(\bar{\lambda}_1)$. Applying induction hypothesis to $\bar{p}_1 | \bar{\lambda}_1$, λ_1 , p_0 , and σ_0 , there exists a condition $p_1 \in P(\lambda_1)$ such that p_1 is $\bar{\lambda}_1$ -generic, $p_1 | \sigma_0 = p_0$ and $\bar{p}_1 | \bar{\lambda}_1 \in \{q \in P(\bar{\lambda}_1) \mid f_{\bar{\lambda}_1, \lambda_1}(q) \geq p_1\}$. We select a condition $\bar{p}_2 \in P(\bar{\tau})$ such that $\bar{p}_2 \leq \bar{p}_1$, $\bar{p}_2 \in D_2$, $\bar{p}_2 | \bar{\lambda}_2 \in P(\bar{\lambda}_2)$ and $p | \bar{\lambda}_2 \in Gen_{\lambda_1}(p_1)$. As before we apply inductive hypothesis to $\bar{p}_2 | \lambda_2$, λ_1 , λ_2 and p_1 . This way we generate the sequences in (*) and (**).

We now prove 2.1(i). We may assume that $p \in P^*(\tau)$. Using the above notation, let $w \leq p | \sigma_0$ be such that $X \cap \sigma_0 \subseteq supp(w)$. By amalgamation, $w_1 = w \sqcup p_{\sigma_0}^u \in P(\tau)$. We apply induction hypothesis to $w_1 | \lambda_1$, etc. If τ is a level limit proceed as in the tree minimal case.

CASE C: τ is a tree limit

Both lemmas are shown by reflecting to an appropriate tree predecessor of τ . For the Extendibility lemma, clause (ii) is immediate from the definition of the forcing. To prove clause (i), let τ_1 be a tree predecessor of τ such that

$$\exists \bar{p} \in P(\tau_1) [p = f_{\tau_1 \tau}(\bar{p})] \wedge X \in Rng f_{\tau_1 \tau}$$

Inductively, let $\bar{q} \in P(\tau_1)$ be such that $\bar{q} \leq \bar{p}$, $lh(\bar{q}) \geq \alpha$ and $\bar{X} \subseteq supp(\bar{q})$, where $X = f_{\tau_1 \tau}(\bar{X})$. Then, letting $q = f_{\tau_1 \tau}(\bar{q})$, $lh(q) \geq \alpha$ and $supp(q) = f_{\tau_1 \tau}(supp(\bar{q})) \supseteq X$.

To prove the amalgamation property, let $\tau_1 <_1 \tau$ be such that

$$\exists \bar{p} \in P(\tau_1) [p = f_{\tau_1 \tau}(\bar{p})] \wedge (\sigma, q \in Rng_{\tau_1 \tau})$$

Letting $\sigma_1 = f_{\tau_1 \tau}^{-1}(\sigma)$ we may now apply inductive hypothesis to \bar{p} , \bar{q} and τ_1 . \dashv

The following two propositions examine genericity along the morass levels and along the branches of the morass tree.

Proposition 4 *Suppose that $G \subseteq P(\tau)$, G is τ -generic, $\sigma <_0 \tau$, $\sigma, \tau \notin \text{Lim}_{<_0} S^1$ and let $G|\sigma = \{p|\sigma \mid p \in G\}$. Then $G|\sigma$ is σ -generic.*

Proof: Let $D \subseteq P(\sigma)$ be dense in $P(\sigma)$ and let $D_1 = \{p \in P(\tau) \mid p|\sigma \in D\}$. We show that D_1 is dense in $P(\tau)$. Since $p|\sigma \in P(\sigma)$, by density of D there exists $q \leq p|\sigma$ with $q \in D$. Let $p^* = q \sqcup p_\sigma^u$. Then by amalgamation, $p^* \in P(\tau)$, $p^* \in D_1$ and $p^* \leq p$. Thus, D_1 is dense in $P(\tau)$. Now if $D \in L_{\sigma^*}$, since $L_{\sigma^*} = L_{\tau^*}$, we have $D_1 \in L_{\sigma^*}$. Since $D_1 \cap G \neq \emptyset$, we have that $G|\sigma$ meets D . \dashv

Proposition 5 *Let $G \subseteq P(\tau)$ be τ -generic, $\sigma <_1 \tau$ and let $\tau \notin \text{Lim}_{<_0} S^1$.*

$G^* = \{q \in P(\sigma) \mid f_{\sigma\tau}(q) \in G\}$ *is σ -generic.*

Proof: Let D be dense in $P(\sigma)$, $D \subseteq P(\sigma)$, $D \in L_{\sigma^*}$. It suffices to show that there exists a $p_0 \in G$ such that $\text{Gen}_{\sigma\tau}(p_0)$ is σ -generic for then, since $\text{Gen}_{\sigma\tau}(p_0) = \{q \in P(\sigma) \mid f_{\sigma\tau}(q) \geq p_0\} \cap D \neq \emptyset$, letting q_0 be such that $f_{\sigma\tau}(q_0) \geq p_0$, $q_0 \in D$ we have $q_0 \in \text{Gen}_{\sigma\tau}(G)$. The existence of p_0 follows by extendibility. \dashv

Proposition 6 (Monotonicity of The Definition) *Suppose that $\sigma, \tau \in S^1$ and $\sigma <_0 \tau$ and let $\sigma \notin \text{Lim}_{<_0} S^1$. There is an embedding $i_{\sigma\tau} : P(\sigma) \rightarrow P(\tau)$ such that if $\nu <_0 \sigma$, $i_{\nu\tau} = i_{\sigma\tau} \circ i_{\nu\sigma}$.*

Proof: Proceed by induction on τ . If τ is a double successor, we use the fact that $P(\tau)$ is a two step iteration and define the maps accordingly. Otherwise use the fact that $P(\tau)$ is either a direct or (type of) inverse limit. \dashv

The next proposition shows that if τ is a level and tree limit, then the direct limit along its tree branch coincides with the direct limit along its level.

Proposition 7 *Let $\tau \in \text{Lim}_{<_0}(S^1) \cap \text{Lim}_{<_1}(S^1)$. Then:*

$$\text{Dirlim}_{\bar{\tau} <_0 \tau} P(\bar{\tau}) = \text{Dirlim}_{\bar{\tau} <_1 \tau} P(\bar{\tau})$$

Proof: If $p \in \text{Dirlim}_{\bar{\tau} <_1 \tau} P(\bar{\tau})$, then there exists a $\tau_1 <_1 \tau$ and a $\bar{p} \in P(\tau_1)$ such that $f_{\tau_1 \tau}(\bar{p}) = p$. By Property M2 of the morass τ_1 is level limit ; thus there exists $\bar{\tau}_1 <_0 \tau_1$ such that $\bar{p} \in P(\bar{\tau}_1)$. Let $\sigma = f_{\tau_1 \tau}(\bar{\tau}_1)$. Then, $f_{\bar{\tau}_1 \sigma}(\bar{p}) \in P(\sigma)$, by the extendibility lemma. But $f_{\tau_1 \tau}(\bar{p}) = f_{\tau_1 \sigma}(\bar{p}) \in P(\sigma)$. Therefore, $p \in P(\sigma)$, and $\text{Dirlim}_{\bar{\sigma} <_0 \sigma} P(\bar{\sigma})$.

Conversely, if $p \in \text{Dirlim}_{\bar{\tau} <_0 \tau} P(\bar{\tau})$, let $p \in P(\sigma)$, $\sigma <_0 \tau$. Since τ is a tree limit , by proposition 1, there exists a $\tau_1 \in S^1$, with $\sigma \in \text{Rng} \pi_{\tau_1 \tau}$. Thus, letting $\bar{\tau}_1$ be such that $f_{\tau_1 \tau}(\bar{\tau}_1) = \sigma$, we may proceed as above. ⊣

2.4 Preservation of Cardinals

The next two propositions establish preservation of cardinals for $P(\omega_2)$.

Proposition 8 *$P(\omega_2)$ is ω_2 -cc*

Proof: Let Δ be a maximal antichain in $P(\omega_2)$. There exists a $\tau \in S^1$ such that $\Delta \cap L(\tau)$ is an antichain on $P(\tau)$, where $\text{cof}(\tau) = \omega_1$, building τ in ω_1 steps. Trivially, $\Delta \cap P(\tau)$ is a maximal antichain in $P(\tau)$. We show that it is a maximal antichain in $P(\omega_2)$. Since $\text{cof}(\tau) = \omega_1$, $p|_{\tau} \in P(\tau)$ if $p \in P(\omega_2)$. Thus, given $p \in P(\omega_2)$, let q be an extension of $p|_{\tau}$ in $X \cap P(\tau)$. By the amalgamation property of our conditions, $q^* = q \sqcup p|_{\tau}$ is a condition in $P(\omega_2)$. ⊣

We now prove that $P(\omega_2)$ is ω - distributive. The proof uses the extendibility lemma.

Proposition 9 $P(\omega_2)$ is ω -distributive.

Proof: In order to establish distributivity, let $\langle D_n \mid n < \omega \rangle$ be a countable sequence of dense open sets in $P = P(\omega_2)$ and let $p \in P$. Since $P(\omega_2)$ has the ω_2 -cc, we may choose $\tau \in S^1$ with $\alpha(\tau) = \omega_1, \text{cof}(\tau) = \omega_1$ such that $p, D_n \in L_\tau$. Now we may choose $\tau_1 <_1 \tau, \tau_1 \in S^1$ such that $D_i \in \text{Rng}f_{\tau_1, \tau}$. By the extendibility lemma $\exists q \leq p (q \in \text{Gen}_\tau(\tau_1))$. Letting $D_n^* = \pi_{\tau_1, \tau}^{-1}(D_n)$, we have that D_n^* is dense in $P(\tau_1)$, for all $n < \omega$. Then there exists a $\bar{p}_1 \in P(\tau)$, such that $\bar{p}_1 \in (\bigcap_{n < \omega} D_n^*) \cap \text{Gen}_{\tau_1, \tau}(q)$. Let $f_{\tau_1, \tau}(\bar{p}_1) = p_1$. Then $p_1 \in \bigcap_{n < \omega} D_n$ and $p_1 \leq p$. -1

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