ANALYSIS AND ERROR MODEL FOR A TWO-DEGREE-OF-FREEDOM ELASTICALLY SUPPORTED TUNED GYRO

by

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ABSTRACT

The spinning rotor is supported by torsional elements in an elastically supported tuned gyro. These torsional elements create a positive spring restraint on the rotor. By spinning the entire support structure of the rotor at a "tuned" speed, the positive spring restraints are cancelled by negative spring restraints created from the dynamics of the spinning rotor and support structure. The result is a freely supported rotor uncoupled from any case motion inputs.

How the balancing of spring restraints is performed is shown for an Oscillogyro, a Hooke's Joint Gyro, and a multigimbaled gyro.

An error analysis for the multigimballed gyro is then given with its error model for testing and a general description of a test procedure for testing the instrument.

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LIST OF SYMBOLS

A, B, C		Principal moments of inertia of rotor about rotor x, y, z axes respectively.
A_n B_n C_n		Principal moments of inertia of the nth gimbal about x_n, y_n, z_n axes respectively.
a_X, a_Y, a_Z		Rotor linear accelerations resolved along case fixed coordinate set.
a_x, a_x, a_z		Rotor linear accelerations resolved along rotor fixed coordinate set.
D_{xn}, D_{yn}, D		Damping coefficients associated with the torsional flexures of the nth gimbal about x_n, y_n axes.
D_D		Damping coefficient due to rotor drag
F_m		Gyro figure of merit.
j	=	$\sqrt{-1}$
K_{xn}, K_{yn}		Torsional stiffness of flexures for the nth gimbal about gimbal x_n and y_n axes respectively.
k_x, k_y, k_z		Translational stiffness of rotor to shaft suspension along rotor fixed axes.
M		Mass of rotor
m_n		Mass of the n^{th} gimbal.

M_x, M_y, M_z	Externally applied moment to rotor resolved along shaft fixed coordinate set.
M_X, M_Y, M_Z	Externally applied moment to rotor resolved along case fixed coordinate set.
N	Speed of shaft rotation relative to case.
N_o	Gyro tuned speed.
n	Number of gimbals
q_x, q_y	Quadrature coefficients resolved along rotor fixed axes.
s	LaPlacian operator.
t	Time.
T_D	Rotor to case drag torque.
T_{xn}	Moment applied to the nth gimbal by the rotor about the gimbal x_n axis.
T_{yn}	Moment applied to the nth gimbal by the rotor about the gimbal y_n axis.
T_{xs}	Shaft reaction moment exerted on the nth gimbal about the gimbal x_n axis.
T_{ys}	Shaft reaction moment exerted on the nth gimbal about the gimbal y_n axis.

α_n	Angle between x_n axis of the nth gimbal and x' axis of the rotor.
$\dot{\theta}_x, \dot{\theta}_y$	Angular velocity of rotor relative to shaft resolved along rotor fixed coordinate set.
$\dot{\Theta}_X, \dot{\Theta}_Y$	Angular velocity of rotor relative to case resolved along case fixed coordinate set.
ϕ_X, ϕ_Y	Absolute angular rates of gyro case resolved along case fixed coordinate set.
$\omega_x, \omega_y, \omega_z$	Absolute angular rates of the shaft resolved along shaft fixed coordinate set.
$\omega_{x'}, \omega_{y'}, \omega_{z'}$	Absolute angular rates of rotor resolved along rotor fixed coordinate set.
$\omega_{xn}, \omega_{yn}, \omega_{zn}$	Absolute angular rate of the nth gimbal resolved along the coordinate set fixed to the nth gimbal.
β	Phase angle of vibration in space domain.
γ	Phase angle of vibration in time domain.
ω	Angular frequency of vibration.
ω_n	Angular frequency of nutation.
τ_n	Time constant at nutation frequency.
τ	Gyro time constant.

Introduction

A freely supported gyro is a spinning mass mounted on its supported and protecting case so that the case can be turned and translated in any direction without causing torques that disturb the angular orientation of the rotor axis relative to inertial space.

One design approach has been to find a substitute for the rotating wheel. Thus there are gyros designed with various kinds of spinning fluids, vibrating masses, nuclear and atomic inertia effects, and laser frequency difference effects.[7, 8, 12] These designs have had some good performance but thus far none have been operationally available in practical sizes and power ranges.

The other approach has been to refine the output axis support to the rotating mass. In this class of gyros we have the fluid, electrostatic, electromagnetic and gas bearing suspensions.[4, 9] The rotating suspension of the elastically supported tuned gyro that this thesis talks about is in this class of instruments. Over the last decade the elastically supported tuned gyro has been developed with simplicity, low cost (25% of conventional floated gyro)[2] and long term drift rates of 0.01 degree per hour, to give a promising approach for future low cost inertial quality gyros.

This thesis will present the basic concepts of elastically supported tuned gyros with particular emphasis on the two-degree-of-freedom elastically-supported tuned gyro. It will also develop the instrument's error model and present a test procedure for testing that instrument in a dynamic environment.

Basic Concept

If a gyro with a rotating mass is to keep a fixed direction in inertial space (it is then a free gyro), the support structure for this rotating mass must apply no torques to the rotating mass when the gyro is moved relative to inertial space.

Conventional approaches to achieve this freely supported rotating mass have been to mount a rotating mass through a number of gimbals to give it the required degrees of freedom. For a two-degree-of-freedom gyro the support is as shown in Figure (1). This type of approach creates the following problems: (1) The gimbal bearing friction is a distrubance torque to the rotating mass, (2) Mass unbalance torques on the rotating mass occur from the spin motor mounted inside the gimbals. The stability of that unbalance is directly related to the gyro acceleration drift stability. (3) The electrical leads needed to carry wheel current from one gimbal to another can create elastic restraint torques on the spinning rotor.

An alternate approach used in the elastically supported tuned gyros is to support the spinning rotor from the inside out and rotate the whole suspension by a spin motor connected to the shaft as shown in Figure (2). Some advantages in this approach are that: (1) no mass unbalance torques occur from the spin motor, (2) no electrical power is transferred through the gimbals, (3) no bearings are needed to give the rotor its degrees of freedom. Thus with this method of motor suspension, only two restraints occur on the rotating mass. One of these restraints is due to the torsion elements and creates

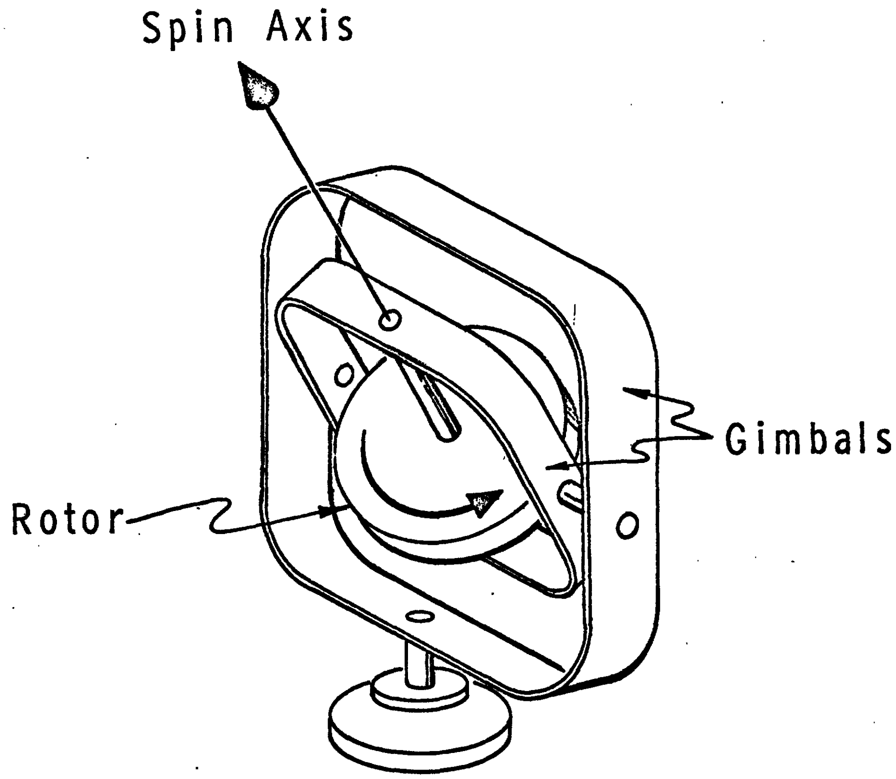


Figure 1 - Two-Degree-of-Freedom Conventional Support

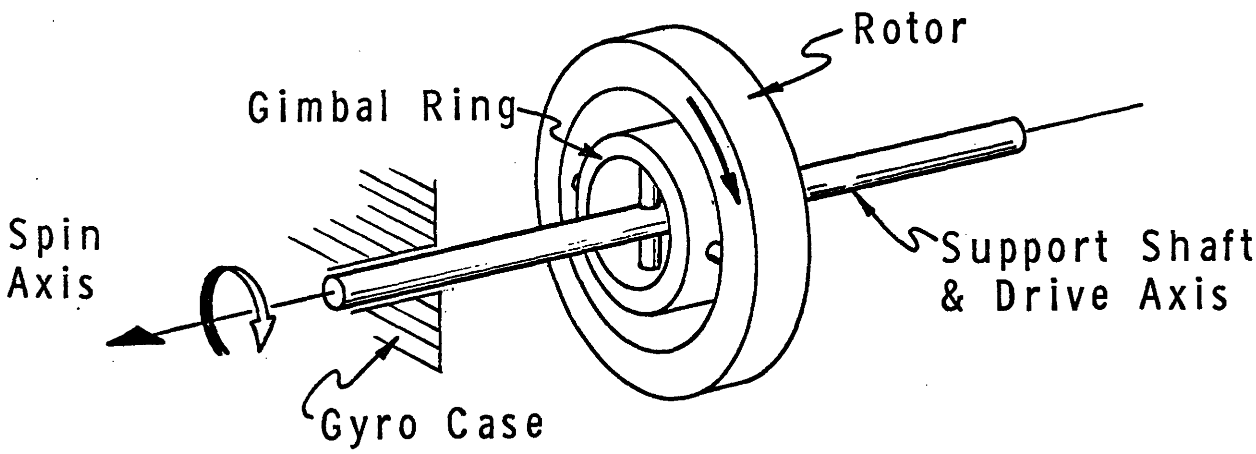


Figure 2 - Rotating Suspension Configuration

an expected positive spring torque between the rotor and shaft. The second restraint, and the key point in the basic concept of these gyros, is that a negative spring torque is created between the rotor and shaft due to the dynamic behavior of the rotating mass. This negative spring restraint is proportional to the rotation speed squared and can therefore be used to compensate for the positive spring torque, produced by the restraint of the torsion element, by rotating the entire system at a selected speed ("tuned" speed) that balances the two restraints. Perfect balancing of these two restraints on the spinning mass results in a freely supported spinning mass -- the condition we are trying to achieve.

A better understanding of this concept will be developed in the following sections after considering how this condition of balancing spring restraints is performed for some specific gyros.

The Oscillogyro

3.1 Ideal Operating Condition

In the oscillogyro, as shown in Figure (3), a single-degree-of-freedom structure supports the rotating mass.[1] In this instrument, the plane of rotation of the sensitive element should ideally remain fixed in inertial space for small angular rotations of the instrument about axes perpendicular to the shaft. This motion between the oscillobar and a surrounding case is sensed by pick-offs and used by some torquing mechanism to realign the oscillobar to a nulled position. The pick-offs can be arranged in two pairs in each of four quadrants to measure the proximity of the bar to a reference plane on the instrument. One pair of pick-offs measures displacement of the bar about an axis and a quarter of a revolution later the other pair measures displacement of the bar about an axis at right angles to it. The instrument can then be regarded as a two-degree-of-freedom gyro by working on a time sharing-basis.

To achieve this ideal operating condition the resultant torques on the sensitive element must be zero when the shaft is rotated through this small offset angle about an axis perpendicular to itself. It will be shown in the derivation of the rotor's equation of motion how the dynamic torque acting about the suspension axis will be used to modify the torsion torque of the suspension element to achieve the ideal operating condition.

3.2 Equation of Motion

A beam, called the oscillobar, is supported about the torsion element axis, y', and free to rotate about that axis through a small

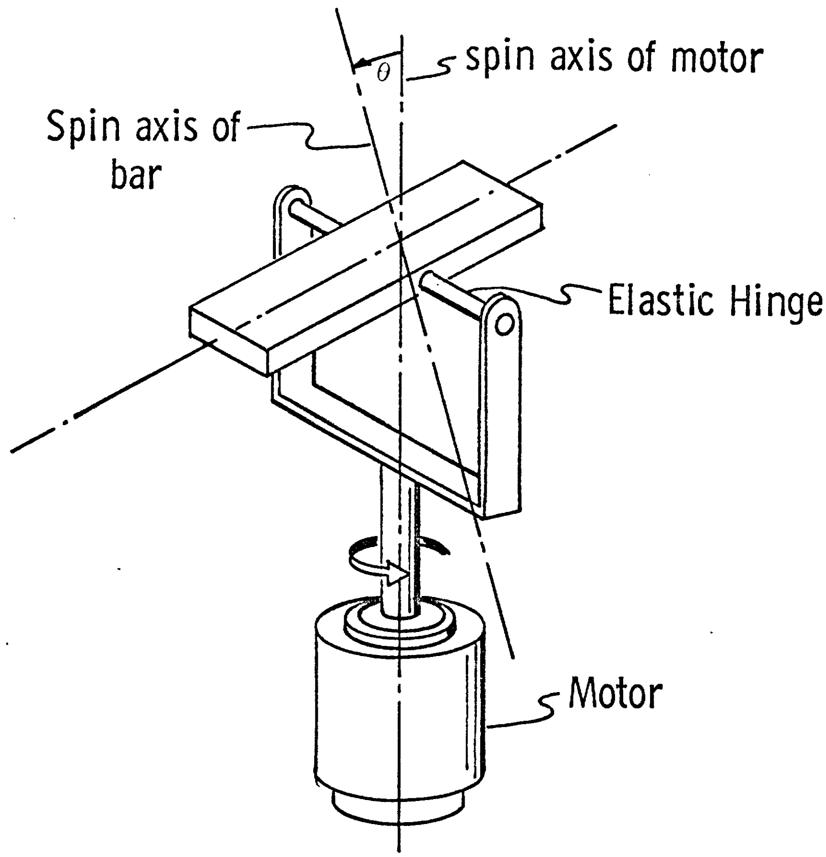


Figure 3 - Features of Oscillobar

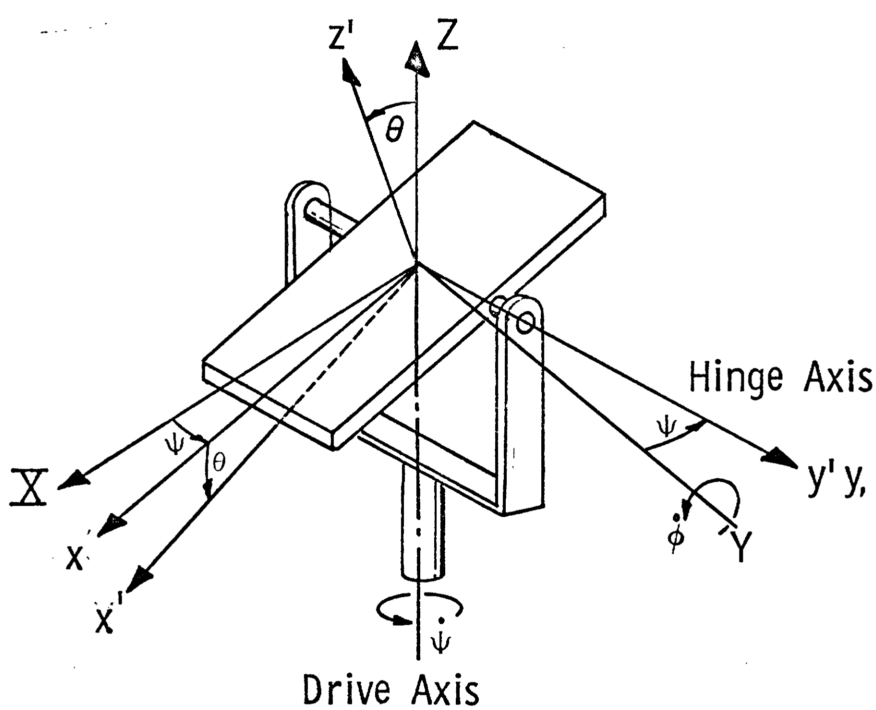


Figure 4 - System Axes

angle, θ. The entire support structure is rotated at a rate $\dot{\psi}$ about the drive axis, Z, as shown in Figure (4). The entire shaft and support structure is then rotated at a rate $\dot{\phi}$ about an axis Y which is perpendicular to the drive shaft and fixed in the gyro case surrounding the entire system. The axes x y z are fixed in the shaft and rotate with respect to the case fixed set X Y Z at a rate N about the case Z axis. The x'y'z' set is fixed in the oscillobar along its principal axes of inertia.

The angular velocities of the oscillobar with respect to inertial space along oscillobar axes are

$$\begin{aligned} \omega_{x'} &= \dot{\phi} \sin\psi \cos\theta - \dot{\psi} \sin\theta \\ \omega_{y'} &= \dot{\phi} \cos\psi + \dot{\theta} \\ \omega_{z'} &= \dot{\phi} \sin\psi \sin\theta + \dot{\psi} \cos\theta \end{aligned} \tag{3.1}$$

where $w_{x'}, w_{y'}, w_{z'}$ are angular velocities of the oscillobar with respect to inertial space.

Using Euler's equations of motion for a rigid body to describe the motion of the oscillobar about the y' axis, one obtains

$$T_2 = B\dot{\omega}_{y'} - (C-A)\,\omega_{z'}\,\omega_{x'} \tag{3.2}$$

where T_2 is the torque applied about the y' axis and A, B, C are the principal moments of inertia of the oscillobar.

Substituting equation (3.1) into (3.2) gives

$$\begin{aligned} B\ddot{\theta} + (C-A)(\dot{\psi}^2 - \dot{\phi}^2 \sin^2\psi)\sin\theta\cos\theta &= T_2 - B\ddot{\phi}\cos\psi \\ &\quad + \left[B + (C-A)\cos 2\theta\right]\dot{\phi}\dot{\psi}\sin\psi \end{aligned} \tag{3.3}$$

The torque T_2 contains a contribution from the spring torque, the windage torque, and a torquing mechanism torque. Writing

$$T_2 = -K\theta - C\dot{\theta} + T_2' \tag{3.4}$$

where T_2' is the torquing mechanism torque, $K\theta$ is the spring torque, $C\dot{\theta}$ is the windage torque, and substituting into (3.2) assuming that $\dot{\psi} \gg \dot{\phi}$, $\sin\theta \simeq \theta$, and $\cos\theta \simeq 1$, one obtains

$$B\ddot{\theta} + C\dot{\theta} + \left[K + (C-A)\dot{\psi}^2\right]\theta = T_2' - B\ddot{\phi}\cos\psi + (B+C-A)\dot{\phi}\dot{\psi}\sin\psi \tag{3.5}$$

Finally assuming that $\dot{\psi}$ is constant $(\dot{\psi} = N)$, the equation of motion for the oscillobar becomes

$$B\ddot{\theta} + C\dot{\theta} + \left[K + (C-A)N^2\right]\theta = T_2' - B\ddot{\phi}\cos Nt + (B+C-A)\dot{\phi}N\sin Nt \tag{3.6}$$

where T_2' may have any designated variation in time.

3.3 Tuning Condition

The response of the gyro to a steady rate of turn, $\dot{\phi} = \Omega$ and no control torque input, is

$$B\ddot{\theta} + C\dot{\theta} + \left[K + (C-A)N^2\right]\theta = (B+C-A)\Omega N\sin Nt \tag{3.7}$$

The desired response of this system to the steady rate input is obtained by making $K = (A + B - C)N^2$ (i.e. balancing the torsional restraint by the dynamic restraint).

The equation of motion for the undamped case becomes

$$\ddot{\theta} + N^2\theta = \frac{(B+C-A)}{B}\Omega N\sin Nt \tag{3.8}$$

with θ expressed as (Θ_o , ϵ as initial conditions)

$$\Theta = \Theta_o \sin(Nt + \epsilon) - \frac{(B+C-A)}{2B}\Omega t \cos Nt \qquad (3.9)$$

The first term is small compared to the second term for $A \ll B$, C, and $\theta_0 \leq 0.5$ degrees. For this condition, the amplitude of the deflection θ is proportional to the input rotation, Ωt, thereby permitting a sensing element (pick-off) to detect inertial rotations of the gyro case.

It can also be seen that the phase of the θ response (again neglecting the first term) is exactly 90° out of phase with $(B + C - A)\Omega N \sin Nt$, the dynamic driving torque, causing the oscillobar motion. When $\frac{(B + C - A)}{2B} = 1$, the plane of rotation of x' would lag the plane XY by exactly the rotation of XY, thus x' or the oscillobar, rotates in a fixed plane in inertial space. This is assuming that $(A + B) = C$ and $K = 0$. This condition can only be achieved approximately by concentrating the mass of the oscillobar near the axis x' so that $A \ll B$ and $B \simeq C$ and keeping the value of K as small as possible. The absolute movement of the plane of rotation is then minimized and satisfies the condition for a free elastically supported tuned gyro. Note again the tuning condition was found by making the torsional restraint equal to the dynamic restraint or by tuning the undamped natural frequency of the system to the system's rotation speed.

A physical picture of how the oscillobar stays fixed in inertial space is seen in Figure (5).

The plane of rotation of the bar stays fixed in inertial space, but has a cyclic rotation (with amplitude equal to the shaft's offset angle) with respect to the shaft fixed coordinate set.

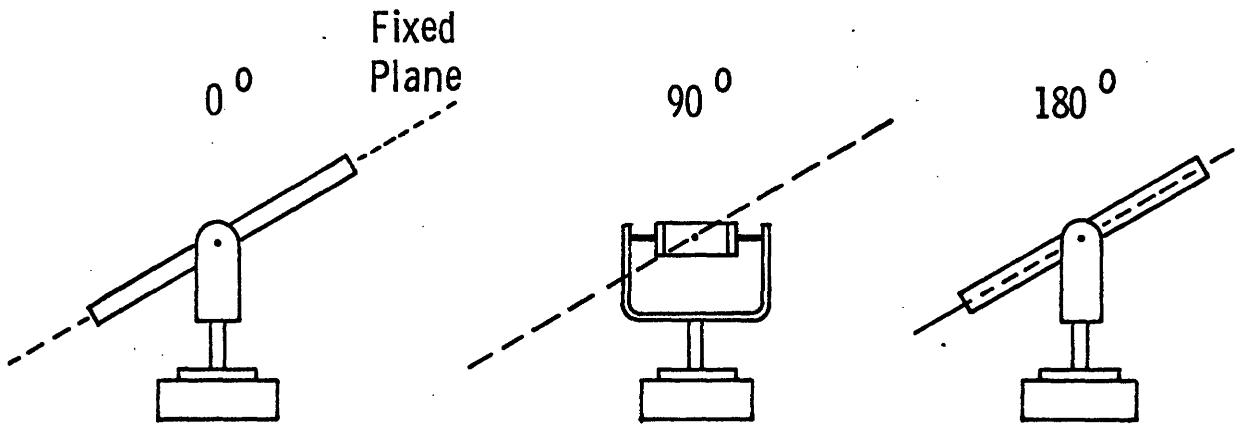


Figure 5 - Fixed Plane of Oscillobar Oscillation

Hooke's Tuned-Joint Gyro

4.1 Ideal Operating Condition

This gyro's suspension gives the sensitive element two degrees of freedom plus increasing the gyro's angular momentum by adding an extra ring outside the oscillobar as shown in Figure (6). This arrangement can be simplified to the symmetric representation in Figure (7).[3]

For this instrument, the ideal mode of operation is to have the rotor spin axis remain fixed in inertial space when the shaft is rotated through some small angle about an axis perpendicular to the shaft. This small rotation of the shaft with respect to the inertially fixed rotor, is then a direct measure of the case's motion with respect to inertial space. This ideal mode of operation is achieved by correct "tuning" of the positive spring restraints of the torsion members to the negative spring restraints caused by the rotor dynamics.

4.2 Equations of Motion

For ideal gyro performance the torques that will be transferred to the rotor when the case is rotated through small angles Θ_X or Θ_Y with respect to the rotor must equal zero. As a simple model of this system, assume the torsion elements are replaced by perfect frictionless pivot bearings.

Consider a system of Cartesian axes fixed relative to the gyro case such that the Z axis coincides with the driving shaft and the XY

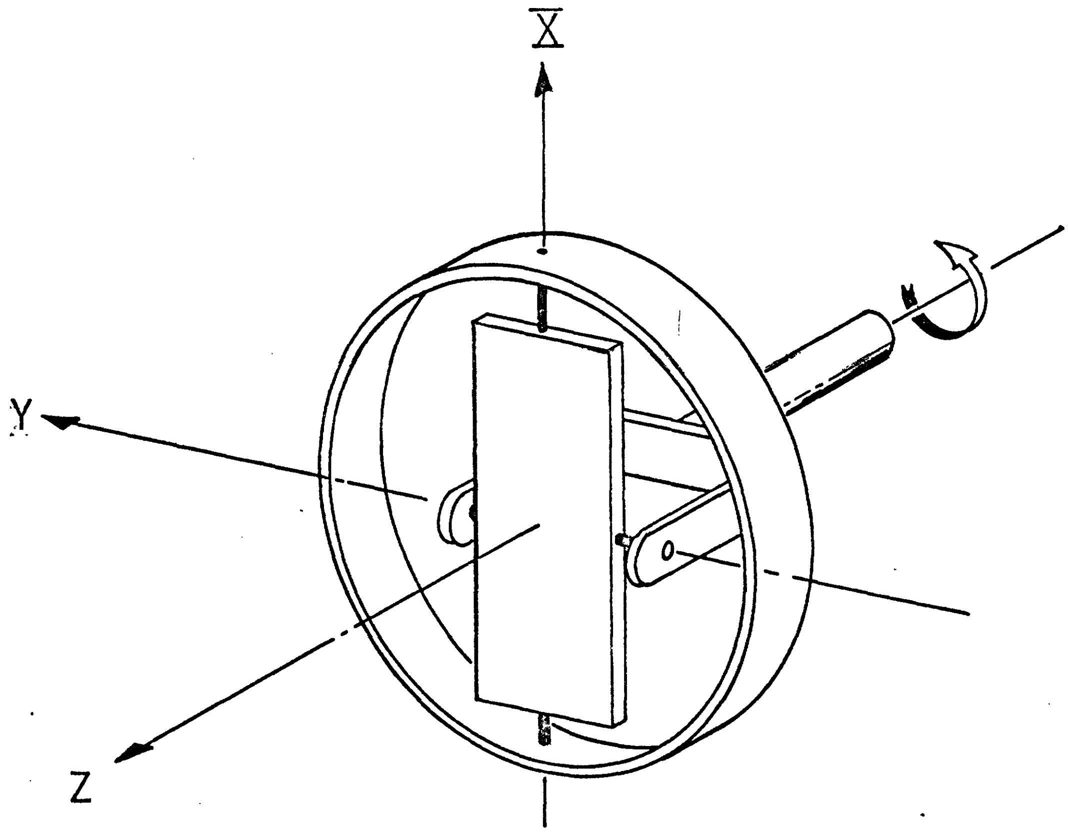


Figure 6 - Unsymmetric Two- Degree-of-Freedom Gyro

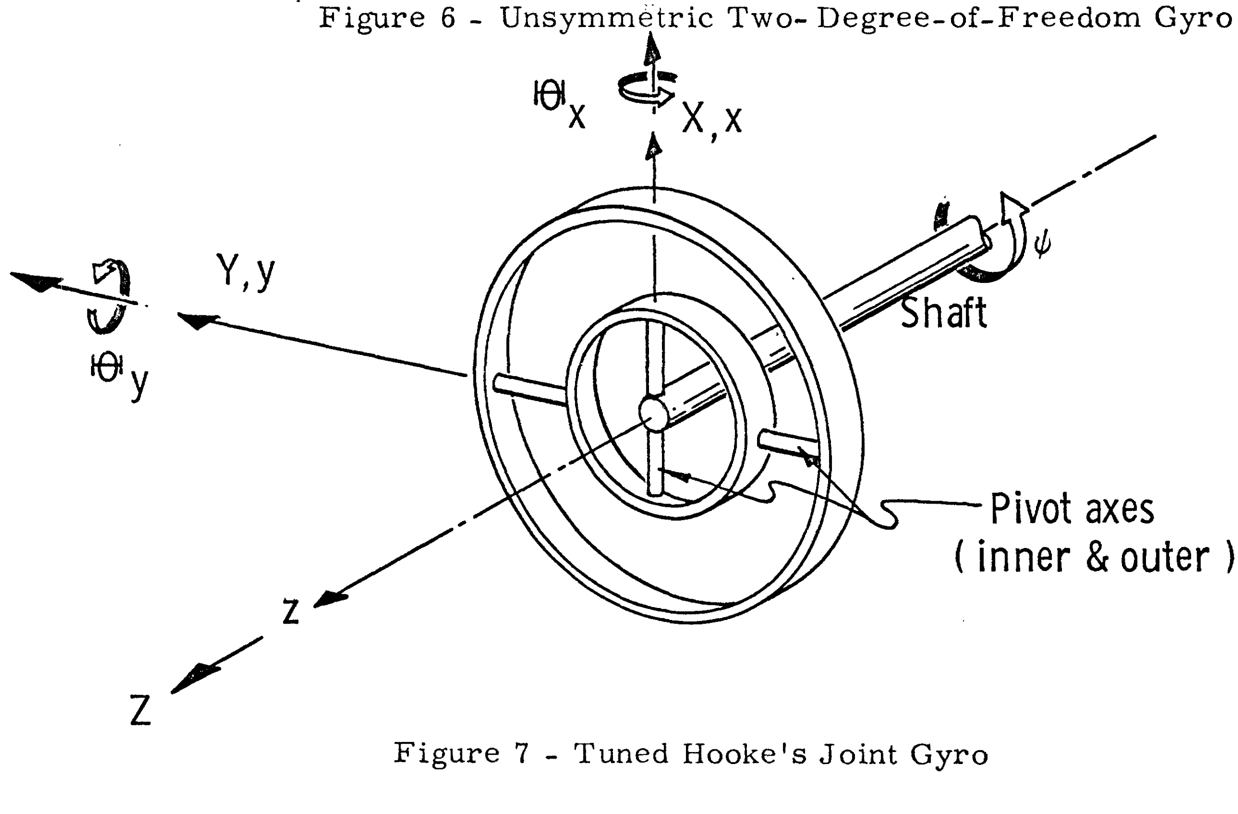


Figure 7 - Tuned Hooke's Joint Gyro

plane includes the gimbal pivot intersection point. For a simple analysis[2] (a more complete analysis is given in Section 5 for the non-symmetric gyro with "n" gimbals), consider the rotor deflections from the case fixed axes X and Y to be Θ_X and Θ_Y respectively (Figure 7).

If θ_{xn} and θ_{yn} represent small deflections of the gimbal about the pivot axes, then

$$\begin{aligned} \theta_{xn} &= \Theta_X \cos\psi + \Theta_Y \sin\psi \quad &\text{(inner pivot axis)} \\ \theta_{yn} &= 0 \quad &\text{(outer pivot axis)} \end{aligned} \tag{4.1}$$

where ψ is the shaft rotation angle. θ_{yn} is zero since the gimbal is only capable of rotating about the inner pivot joint. Transforming these rotations to case fixed coordinates XYZ, one obtains the gimbal ring deflections as

$$\begin{aligned} \Theta_{Xn} &= \theta_{xn} \cos\psi = \Theta_X \cos^2\psi + \Theta_Y \sin\psi \cos\psi \\ \Theta_{Yn} &= \theta_{yn} \sin\psi = \Theta_X \sin\psi \cos\psi + \Theta_Y \sin^2\psi \end{aligned} \tag{4.2}$$

The torque about the X and Y axes to accelerate the gimbal inertias are

$$\begin{aligned} T_{Xn} &= A_n \ddot{\Theta}_{Xn} + C\dot{\psi}\dot{\Theta}_{Yn} \\ T_{Yn} &= B_n \ddot{\Theta}_{Yn} - C\dot{\psi}\dot{\Theta}_{Xn} \end{aligned} \tag{4.3}$$

where A_n and B_n are moments of inertia of the gimbal about the torsion element axes, and C_n is the gimbal moment of inertia about the shaft axis. T_{xn} and T_{Yn} are external torques applied to the gimbal.

Since these torques cannot be transferred from the gimbal to the rotor by way of the outer pivot joint, only the components of the torques in (4. 3) along the inner pivot joint axis are transmissable. Therefore

$$T_i = T_{Xn}\cos\psi + T_{Yn}\sin\psi \qquad (T_i \text{ along inner torsion element})$$

$$T_o = 0 \qquad (T_0 \text{ along outer torsional element}) \tag{4.4}$$

Transforming these torques to case fixed coordinates,

$$\begin{aligned} T_{Xr} &= T_i\cos\psi = T_{Xn}\cos^2\psi + T_{Yn}\sin\psi\cos\psi \\ T_{Yr} &= T_i\sin\psi = T_{Xn}\cos\psi\sin\psi + T_{Yn}\sin^2\psi \end{aligned} \tag{4.5}$$

which, after substitution of equations (4. 2) and (4. 3), become

$$\left.\begin{aligned} T_{Xr} &= \frac{A_n\ddot{\Theta}_X}{2} + A_n N\dot{\Theta}_Y - \left(A_n - \frac{C_n}{2}\right)N^2\Theta_X \\ T_{Yr} &= \frac{A_n\ddot{\Theta}_Y}{2} - A_n N\dot{\Theta}_X - \left(A_n - \frac{C_n}{2}\right)N^2\Theta_Y \end{aligned}\right\} + \text{terms in } \begin{matrix}\cos 2\Theta_X \\ \sin 2\Theta_Y\end{matrix} \tag{4.6}$$

where $A_n = B_n$ and $\dot{\psi} = N$. Note, these equations describe torques on the rotor in a cased fixed coordinate system as a function of gimbal dynamics for perfect pivot bearings.

In this equation (4. 6) there exists a negative spring torque on the rotor due to the dynamics of the gimbal. This negative torque appears in the third term on the right hand side. It is caused by the combined effects of the gimbal swiveling and the torque restraint $T_0 = 0$ (the gimbal swivels as shown in Figure 8). A spring restraint in any gyro's output axis suspension results in the familiar rate gyro characteristics. Specifically, the float is offset when the gyro is given an input rate. The offset produces a torque (via the spring) that precesses the gyro to follow its input. The net motion of the float is the familiar conical precession of the rotor's spin axis. The above negative spring torque then creates a coning of the rotor's spin axis.

The important feature of this negative spring restraint, however, is that it can be used to cancel a positive spring restraint. This cancellation is accomplished by replacing the pivot joints with torsional elements. The positive spring restraints created by the torsion elements causes the coning of the spin axis to be in a direction opposite to the coning caused by the negative spring restraint. Since the dynamic (negative) spring restraint is a function of the rotor spin speed, spinning the rotor at a specific speed that makes the dynamic restraint equal to the spring restraint will cause the two restraints to cancel one another. If one calculates the coning due to each of the spring restraints, it can be shown that when these two restraints become equal, the period of coning becomes infinite, or the rotor becomes effectively uncoupled from its suspension (Figure 9).

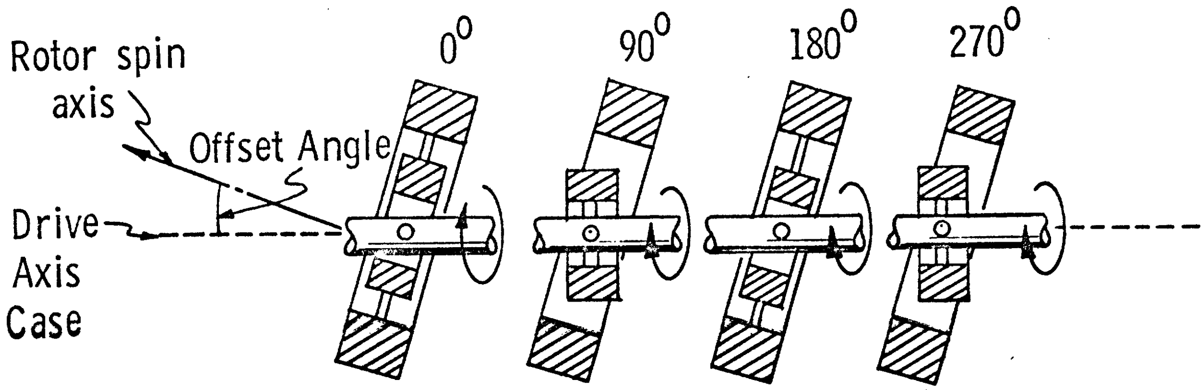


Figure 8 - Rotating Gimbal Ring Oscillations In Space

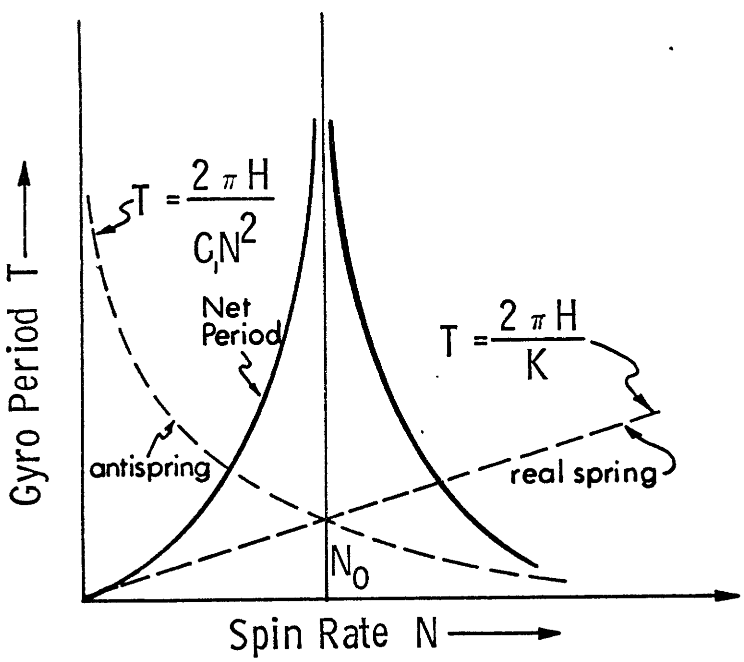


Figure 9 - Tuning Condition

4.3 Tuning Condition

The general open loop equation of motion (5.10) for a two degree of freedom instrument (derived in Section 5) can be used to describe the equations of motion for this instrument. Assume the gyro is symmetric and has no external torques or case motion inputs. Also, consider the shaft has been rotated through some small angle perpendicular to the shaft. The gyro's equation of motion becomes

$$(A+A_n)\ddot{\theta}_x - N\dot{\theta}_y(A+B-C) + (C-B+C_n-B_n)N^2\theta_x + K_x\theta_x = 0$$

$$B\ddot{\theta}_y + N\dot{\theta}_x(A+B-C) + N^2(C-A)\theta_y + K_y\theta_y = 0 \qquad (4.7)$$

where

N - shaft rotation speed

A, B, C - inertias of rotor about principal axes x'y'z'

A_n, B_n, C_n - inertias of gimbal about principal axes $x_n y_n z_n$

K_x, K_y - spring constant of inner and outer torsional elements respectively

θ_x, θ_y - rotations of the rotor with respect to the shaft about rotating shaft coordinates

Remember it is desired to have no torques transferred through the support structure to the rotor for ideal gyro operation. This ideal condition is met when the above equations (4.7) are satisfied for a specific rotation speed N and torsional element spring constants K_x and K_y. At this rotation speed, the gyro is said to be "tuned".

The above equations are second order differential equations having cross coupling terms in θ_y and θ_x. In general, solutions for θ_x and θ_y will contain two sinusoidal terms of differing frequencies.[3]

If the gyro is to be a "free rotor" gyro (the rotor being effectively uncoupled from its suspension), then after each cycle of revolution of the gyro spin motor, the rotor should find itself in the same position with respect to inertial space, therefore, θ_x and θ_y should contain terms in sin Nt and cos Nt.

Assuming,

$$\theta_x = \cos Nt + \sin Nt$$

$$\theta_y = \cos(Nt+\gamma) + \sin(Nt+\gamma) \quad .$$

then

$$\ddot{\theta}_x = -N^2\theta_x$$

$$\ddot{\theta}_y = -N^2\theta_y$$

The equations of motion (4.7) can be combined to give

$$\left\{\left[(A+A_n)s^2+(C-B+C_n-B_n)N^2+K_{xn}\right]\left[Bs^2+N(\dot{C}-A)+K_{yn}\right]-N^2(A+B-C)^2s^2\right\}\begin{matrix}\theta_x\\ \theta_y\end{matrix}=0 \qquad (4.8)$$

Substituting for $\ddot{\theta}_x$ and $\ddot{\theta}_y$ from the above

$$\left\{\left[K_{xn}-N^2(A+B-C)-N^2(A_n+B_n-C_n)\right]\left[K_{yn}-N^2(A+B-C)\right]-N^4(A+B-C)^2\right\}\begin{matrix}\theta_x\\ \theta_y\end{matrix}=0 \quad .$$

This equation describes clearly the balance that must exist between the positive spring torques K_{xn} and K_{yn} and the negative spring torques $N^2(A+B-C)$ and $N^2(A_n+B_n-C_n)$.

A "tuned" gyro is one that is so designed by varying the inertias, spin speed, and torsional stiffness such that the above equation is satisfied.[3] This requires the values of K_{xn} and K_{yn} to satisfy the following equation

$$K_{xn} K_{yn} - K_{yn}(\alpha N^2 + \beta N^2) - K_{xn}\alpha N^2 + N^4 \alpha\beta = 0 \qquad (4.9)$$

where $\alpha = (A + B - C)$ and $\beta = (A_n + B_n - C_n)$.

If, for design simplicity, $K_{xn} = K_{yn} = K$, the above equation requires that (assuming $\alpha \gg \beta$)

$$K = \frac{1}{2} N^2 (A_n + B_n - C_n)$$

or

$$N = \left[\frac{2K}{(A_n+B_n-C_n)}\right]^{1/2} \qquad (4.10)$$

which is the "tuned" speed for this gyro. Note again, as in the oscillogyro, we have effectively "tuned" one mode of oscillation of the system to the rotation frequency by requiring any wobbling of the rotor to return to its initial position with respect to inertial space after one revolution of the shaft. Since, as was mentioned earlier, the general solution is to contain two sinusoidal terms in differing frequencies, there exists a second frequency of oscillation of the system. This second mode of oscillation[3] is found by dividing the combined equations of motion (4.8) by $(s^2 + N^2)$ providing the tuning condition is met.

The second frequency becomes

$$\Omega^2 = \frac{\left[(C-A)N^2 + K_{yn}\right]\left[(C-B+C_n-B_n)N^2 + K_{xn}\right]}{(A+A_n)BN^2} \tag{4.11}$$

For $(A + B + C) \gg (A_n + B_n - C_n)$, which is a good approximation for this gyro, and K_{xn}, K_{yn} on the order of $N^2(A_n + B_n - C_n)$,

$$\Omega = \frac{(C-B)N}{B}$$

This second frequency is always less than N and only approaches N as C approaches 2A (i.e. the rotor approaches disk shape). As an inertial observer, the gyro's spin axis would appear to be performing a complex nutation motion about its spin axis. If the gyro were not tuned, an inertial observer would see the above mentioned nutation modes superimposed on a steady precessional motion of the spin axis.

Two-Degree-of-Freedom Tuned Gyro with "n" Gimbals

5.1 Ideal Operating Condition

For a more general case of Hooke's Joint Gyro presented in section 4, consider a gyro with n gimbals supporting the rotor (Figure 10). The additional gimbals increase the gyro's angular momentum by increasing the mass of the rotating structure. It will be seen in Section 6 that the increased angular momentum is desireable to reduce mass unbalance errors. Also, the extra gimbals reduce errors due to twice spin frequency angular oscillations of the gyro case about an axis perpendicular to the shaft.

The ideal operating condition for a gyro with n gimbals is to have the rotor remain fixed in inertial space for any type of case motion input. This is the same condition required in the Hooke's Tuned-Joint Gyro.

5.2 Equations of Motion

The general open loop equation of motion derived below will account for nonsymmetries, damping forces, and torsion element stiffness differences in an n gimbal gyro.

The following four orthogonal reference frames to be defined have origins coincident with the center of torsion established by the rotor to shaft torsion elements (Figure 11).

XYZ	- fixed in gyro case
xyz	- fixed in the shaft which rotates with respect to the case fixed set with angular velocity N about the Z axis

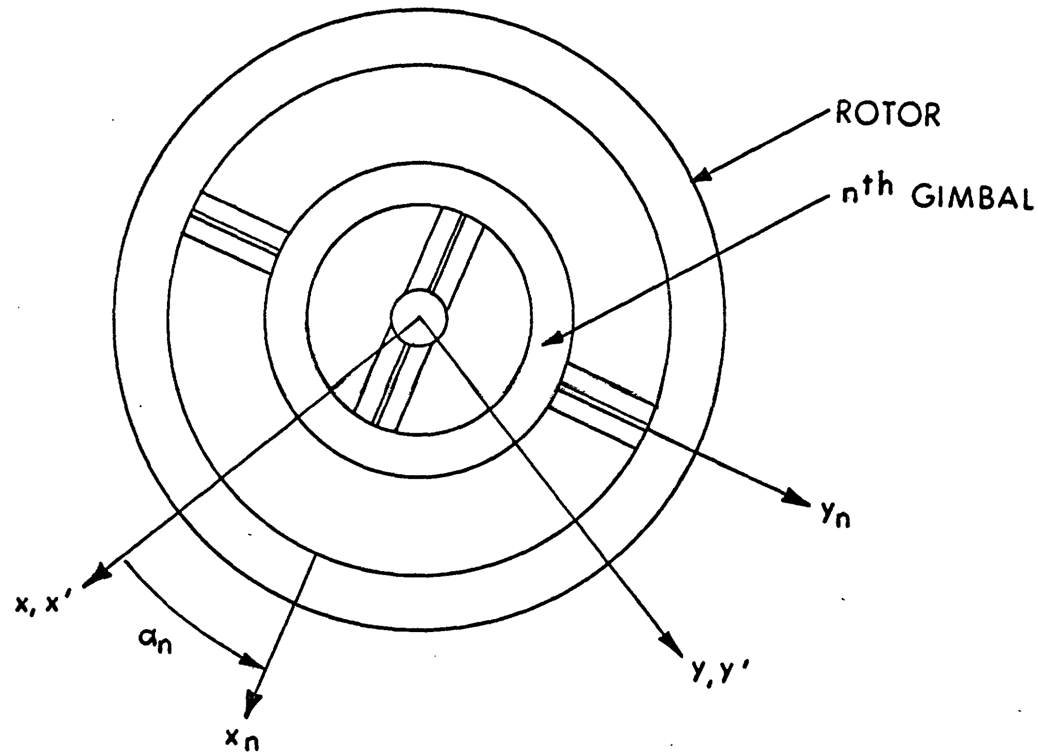


Figure 10 - Rotor and n^{th} Gimbal

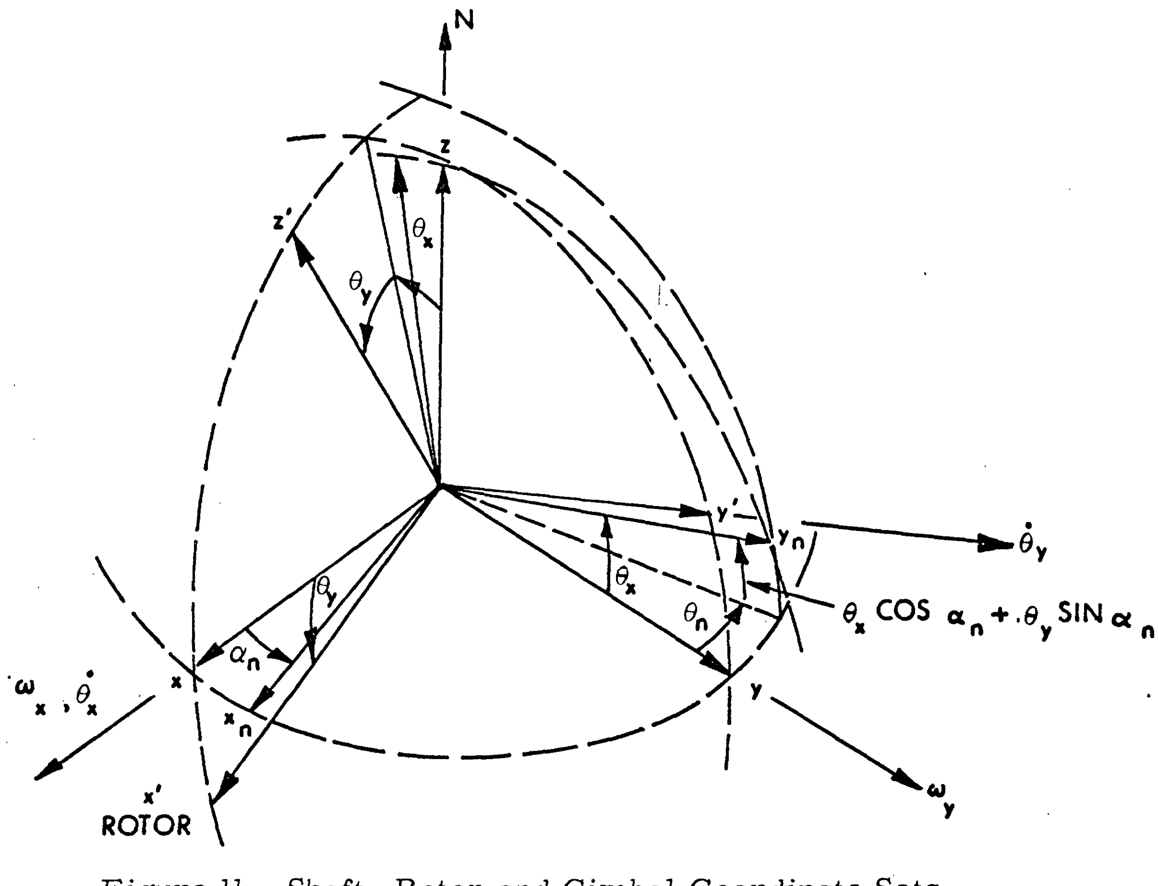


Figure 11 - Shaft, Rotor and Gimbal Coordinate Sets

$x_n y_n z_n$ - fixed in n^{th} gimbal with x_n axis along the inner torsion element and rotated from the $x'y'z'$ set by α_n ($\alpha_1 = 0^o$) about the z axis when the rotor is perpendicular to the shaft

$x'y'z'$ - fixed in the rotor with the y' axis along the outer torsion element axes

The equations of motion for the rotor have been derived using Euler's equations of motion for a rigid body.[5] A summary of this derivation is given below.

Angular Velocities:

Shaft

The angular velocity of the shaft with respect to inertial space in shaft coordinates is

$$\begin{aligned} \omega_x &= \dot{\phi}_X \cos Nt + \dot{\phi}_Y \sin Nt \\ \omega_y &= -\dot{\phi}_X \sin Nt + \dot{\phi}_Y \cos Nt \\ \omega_z &= N \end{aligned} \tag{5.1}$$

Case rotation speeds with respect to inertial space have been assumed negligible compared to the shaft spin speed N.

Rotor

The angular velocity of the rotor with respect to inertial space in rotor coordinates is

$$\begin{aligned} \omega_{x'} &= \omega_x + \dot{\theta}_x - N\theta_y \\ \omega_{y'} &= \omega_y + \dot{\theta}_y + N\theta_x \\ \omega_{z'} &= N \end{aligned} \tag{5.2}$$

where θ_x and θ_y are angular rotations of the rotor about the shaft fixed coordinate set. Small angle approximations have been made, i.e. $\sin\theta = \theta$ and $\cos\theta = 1$; for most gyros θ_x and θ_y rarely exceed 0.5°. The angular velocity terms along the z component, $\theta_y w_x$, $\theta_x w_y$, $\theta_x \dot{\theta}_y$, $\theta_y \dot{\theta}_x$, have been neglected since they are small compared to the spin speed N.

Gimbal

The angular velocity of the gimbal with respect to inertial space in gimbal coordinates is

$$
\begin{aligned}
\omega_{xn} &= (\omega_x + \dot{\theta}_x)\cos\alpha_n + (\omega_y + \dot{\theta}_y)\sin\alpha_n \\
\omega_{yn} &= -\omega_x \sin\alpha_n + \omega_y \cos\alpha_n + N(\theta_x \cos\alpha_n + \theta_y \sin\alpha_n) \\
\omega_{zn} &= N
\end{aligned}
\tag{5.3}
$$

The angular velocity terms, $\theta_{xn}\omega_x \sin\alpha_n$, $\theta_{xn}\omega_y \cos\alpha_n$ along the z_n axis have been neglected since they are small compared to the spin speed N.

Moments:

Gimbal

Applying Euler's equations of motion for a rigid body to the gimbal

$$
\begin{aligned}
T_{xn} - T_{xs} &= A_n \dot{\omega}_{xn} + (C_n - B_n)\omega_{yn}\omega_{zn} \\
T_{yn} - T_{yz} &= B_n \dot{\omega}_{yn} - (C_n - A_n)\omega_{xn}\omega_{zn}
\end{aligned}
\tag{5.4}
$$

where T_{xn} and T_{yn} are the moments applied to the gimbal by the rotor along the gimbal x_n and y_n axes respectively. T_{xs} and T_{ys}, the shaft reaction moments exerted on the gimbal along the same axes, are given by

$$T_{xs} = K_{xn}\left(\theta_x \cos\alpha_n + \theta_y \sin\alpha_n\right) + D_{xn}\left(\dot\theta_x \cos\alpha_n + \dot\theta_y \sin\alpha_n\right)$$

$$T_{yn} = K_{yn}\left(-\theta_x \sin\alpha_n + \theta_y \cos\alpha_n\right) + D_{yn}\left(-\dot\theta_x \sin\alpha_n + \dot\theta_y \cos\alpha_n\right) \tag{5.5}$$

Substituting (5. 5) into (5. 4) yields

$$T_{xn} = K_{xn}\left(\theta_x \cos\alpha_n + \theta_y \sin\alpha_n\right) + D_{xn}\left(\dot\theta_x \cos\alpha_n + \dot\theta_y \sin\alpha_n\right) + A_n \dot\omega_{xn} + \left(C_n - B_n\right)\omega_{yn}\,\omega_{zn}$$

$$T_{ys} = -K_{yn}\left(-\theta_x \sin\alpha_n + \theta_y \cos\alpha_n\right) - D_{yn}\left(-\dot\theta_x \sin\alpha_n + \dot\theta_y \cos\alpha_n\right) - B_n \dot\omega_{yn} + \left(C_n - A_n\right)\omega_{xn}\,\omega_{zn} \tag{5.6}$$

Rotor:

Applying Euler's equations of motion for a rigid body to the rotor, where T_D is a rotor to case rotational damping torque whose vector lies along z',

$$M_x - \left(T_{xn}\cos\alpha_n - T_{yn}\sin\alpha_n\right) - T_D \cdot \theta_y = A\dot\omega_{x'} + (C-B)\,\omega_{y'}\,\omega_{z'}$$

$$M_y - \left(T_{xn}\sin\alpha_n + T_{yn}\cos\alpha_n\right) + T_D\,\theta_x = B\dot\omega_{y'} - (C-A)\,\omega_{x'}\,\omega_{z'} \tag{5.7}$$

$$M_z + T_D = 0$$

where M_x, M_y, M_z are externally applied moments resolved along the rotor fixed coordinate set.

The case fixed moments applied to the rotor are

$$M_X = (M_x - D_R\Theta_x)\cos Nt + (M_y - D_R\Theta_y)\sin Nt$$
$$M_Y = -(M_x - D_R\Theta_x)\sin Nt + (M_y - D_R\Theta_y)\cos Nt \qquad (5.8)$$

Substituting equations (5.1), (5.2), (5.3), (5.4) and (5.5) into equation (5.7) one obtains

$$\begin{aligned}
&\ddot{\Theta}_x\left[A + \sum_1^n A_n\cos^2\alpha_n\right] + \dot{\Theta}_x[nD] \\
&+ \Theta_x\left[N^2(C-B) + N^2\sum_1^n (C_n - B_n)\cos^2\alpha_n + \sum_1^n K_{xn}\cos^2\alpha_n + \sum_1^n K_{yn}\sin^2\alpha_n\right] \\
&+ \Theta_y\left[\tfrac{1}{2}N^2\sum_1^n (C_n - B_n)\sin 2\alpha_n + \tfrac{1}{2}\sum_1^n K_{xn}\sin 2\alpha_n - \tfrac{1}{2}\sum_1^n K_{yn}\sin 2\alpha_n + T_D\right] \\
&+ \ddot{\Theta}_y\left[\tfrac{1}{2}\sum_1^n A_n\sin 2\alpha_n\right] + \dot{\Theta}_y[(C-A-B)N] \\
&= G_x(t)
\end{aligned} \qquad (5.9)$$

and

$$\begin{aligned}
&\ddot{\Theta}_y\left[B + \sum_1^n A_n\sin^2\alpha_n\right] + \dot{\Theta}_y[nD] \\
&+ \Theta_y\left[(C-A)N^2 + N^2\sum_1^n (C_n - B_n)\sin^2\alpha_n + \sum_1^n K_{xn}\sin^2\alpha_n + \sum_1^n K_{yn}\cos^2\alpha_n\right] \\
&+ \ddot{\Theta}_x\left[\tfrac{1}{2}\sum_1^n A_n\sin 2\alpha_n\right] + \dot{\Theta}_x\left[-(C-A-B)N\right] \\
&+ \Theta_x\left[\tfrac{1}{2}N^2\sum_1^n (C_n - B_n)\sin 2\alpha_n + \tfrac{1}{2}\sum_1^n K_{xn}\sin 2\alpha_n - \tfrac{1}{2}\sum_1^n K_{yn}\sin 2\alpha_n - T_D\right] \\
&= G_y(t)
\end{aligned} \qquad (5.10)$$

where

$$-G_x(t) = \left[A + \sum_1^n A_n \cos^2\alpha_n\right]\left[\ddot{\Phi}_X \cos Nt + \ddot{\Phi}_Y \sin Nt\right]$$
$$+\left[-\tfrac{1}{2}N\sum_1^n (C_n - B_n + A_n)\sin 2\alpha_n\right]\left[\dot{\Phi}_X \cos Nt + \dot{\Phi}_Y \sin Nt\right]$$
$$+\left[\tfrac{1}{2}\sum_1^n A_n \sin 2\alpha_n\right]\left[-\ddot{\Phi}_X \sin Nt + \ddot{\Phi}_Y \cos Nt\right] \tag{5.11}$$
$$+\left[(C-B+A)N + N\sum_1^n (C_n - B_n + A_n)\cos^2\alpha_n\right]\left[-\dot{\Phi}_X \sin Nt + \dot{\Phi}_Y \cos Nt\right]$$
$$-(M_x - D_R\Theta_x)\cos Nt - (M_y - D_R\Theta_y)\sin Nt$$

and

$$-G_y(t) = \left[B + \sum_1^n A_n \sin^2\alpha_n\right]\left[-\ddot{\Phi}_X \sin Nt + \ddot{\Phi}_Y \cos Nt\right]$$
$$+\left[\tfrac{1}{2}N\sum_1^n (C_n - B_n + A_n)\sin 2\alpha_n\right]\left[-\dot{\Phi}_X \sin Nt + \dot{\Phi}_Y \cos Nt\right]$$
$$+\left[\tfrac{1}{2}\sum_1^n A_n \sin 2\alpha_n\right]\left[\ddot{\Phi}_X \cos Nt + \ddot{\Phi}_Y \sin Nt\right] \tag{5.12}$$
$$+\left[-(C-A+B)N - N\sum_1^n (C_n - B_n + A_n)\sin^2\alpha_n\right]\left[\dot{\Phi}_X \cos Nt + \dot{\Phi}_Y \sin Nt\right]$$
$$+(M_x - D_R\Theta_x)\sin Nt - (M_y - D_R\Theta_y)\cos Nt$$

In these equations, $D_{xn} = D_{yn} = D$, and the x axis of each gimbal is displaced from the rotor reference x' axis by angles $\alpha_1, \alpha_2, \alpha_3 \cdots\cdot \alpha_n$ (Figure 12).

Multiplying equation (5.10) by $j = \sqrt{-1}$ and adding to equation (5.9) noting the definitions

$$\Theta_{xy} = \Theta_x + j\Theta_y$$

$$\bar{\Theta}_{xy} = \Theta_x - j\Theta_y$$

$$G_{xy}(t) = G_x(t) + jG_y(t)$$

The basic equation of motion for the rotor in rotating shaft coordinates becomes

$$\begin{aligned} I\ddot{\Theta}_{xy} &+ nD\dot{\Theta}_{xy} + \left[K - jT_D + N^2(C+I_s)\right]\Theta_{xy} \\ &- j(C-A-B)N\dot{\Theta}_{xy} + \Delta I\ddot{\bar{\Theta}}_{xy} + \left[\Delta K + N^2\Delta I_s\right]\bar{\Theta}_{xy} \\ &+ jI_R\ddot{\bar{\Theta}}_{xy} + jK_p\bar{\Theta}_{xy} = G_{xy}(t) \end{aligned} \tag{5.13}$$

Since case fixed pickoffs and torquers are generally used in these instruments, one needs to determine the motion of the rotor with respect to case fixed coordinates. Transforming then the above equation into case fixed coordinates[5] gives

$$\Theta_{XY} = \frac{\begin{array}{l} -\Phi_{XY}(s)\left[F_1(s)\bar{T}_1(s-2jN) - \bar{F}_2(s-2jN)\,T_2(s)\right] \\ -\Phi_{XY}(s-2jN)\left[F_2(s)\bar{T}_1(s-2jN) - \bar{F}_1(s-2jN)\,T_2(s)\right] \\ +M_{XY}(s)\bar{T}_1(s-2jN) - M_{XY}(s-2jN)\,T_2(s) \end{array}}{T_1(s)\bar{T}_1(s-2jN) - T_2(s)\bar{T}_2(s-2jN)} \tag{5.14}$$

where

$$T_1(s) = Is^2 + \left[nD + D_R + jN(A+B-C-2I\right]s + \left[K - jT_D - jnND - N^2J\right]$$

$$T_2(s) = (\Delta I + jI_R)s^2 - 2jN(\Delta I + jI_R)s + \left[\Delta K + j\,\Delta k + N^2\Delta J\right]$$

$$\bar{T}_1(s-2jN) = Is^2 + \left[nD - jN(A+B-C+2I)\right]s + K - jT_D$$
$$- jnND - N^2\left[2(A+B-C) + J\right]$$

$$\bar{T}_2(s-2jN) = (\Delta I - jI_R)s^2 - 2jN(\Delta I - jI_R)s + \Delta K - j\Delta k + N^2\Delta J$$

$$F_1(s) = Is^2 - jN(C + I_s + I)s$$

$$F_2(s) = (\Delta I + jI_R)s^2 + \left[(3I_R - I_g) - j(3\Delta I - \Delta I_s)\right]Ns$$
$$+ 2\Delta J N^2$$

$$\bar{F}_1(s-2jN) = Is^2 + jN(C+I_s-3I)s + 2N^2(C+I_s-I)$$

$$\bar{F}_2(s-2jN) = (\Delta I + jI_R)s^2 - \left[(I_g+I_R) - j(\Delta I_s + \Delta I)\right]Ns$$

$$I = \frac{1}{2}\left[A+B+\sum_1^n A_n\right]$$

$$\Delta I = \frac{1}{2}\left[A-B+\sum_1^n A_n \cos 2\alpha_n\right]$$

$$K = \frac{1}{2}\sum_1^n (K_{zn}+K_{yn})$$

$$\Delta K = \frac{1}{2}\sum_1^n (K_{xn}-K_{yn})\cos 2\alpha_n$$

$$I_s = \frac{1}{2}\left[-A-B+\sum_1^n (C_n-B_n)\cos 2\alpha_n\right.$$

$$\Delta I_s = \frac{1}{2}\left[A-B+\sum_1^n (C_n-B_n)\right]$$

$$I_R = \frac{1}{2}\sum_1^n A_n \sin^2\alpha_n$$

$$I_g = \frac{1}{2}\sum_1^n (C_n-B_n)\sin^2\alpha_n$$

$$K_p = \frac{1}{2} N^2 \sum_1^n (C_n - B_n) \sin 2\alpha_n + \frac{1}{2} \sum_1^n K_{xn} \sin 2\alpha_n - \frac{1}{2} \sum_1^n K_{yn} \sin 2\alpha_n$$

$$\Delta k = \frac{1}{2} \sum_1^n (K_{xn} - K_{yn}) \sin 2\alpha_n$$

$$J = \frac{1}{2} \sum_1^n (A_n + B_n - C_n)$$

$$\Delta J = -\frac{1}{2} \sum_1^n (A_n + B_n - C_n)\, e^{j2\alpha_n}$$

This equation of motion of the rotor is completely general and describes the performance of an asymmetric gyro in response to any form of angular or torque input. Note that $F_2(s)$ and $T_2(s)$ result from nonsymmetries and torsion element stiffness differences in the gyro.

An operational instrument is constructed as symmetric as possible and operated at its tuned speed. For a symmetric and tuned gyro, the general open loop equation above reduces to

$$\Theta_{XY}(s) = \frac{\Phi_{XY} F_1(s) + M_{XY}(s)}{\Upsilon(s)} \qquad (5.15)$$

Neglecting damping effects and operating the instrument at tuned speed, this equation reduces to

$$\Theta_X(s) = -\phi_X(s) + \frac{sM_X(s) - M_Y(s)\frac{NC}{A}}{As\left[s^2 + \left(\frac{NC}{A}\right)^2\right]}$$

$$\Theta_Y(s) = -\phi_Y(s) + \frac{M_X(s)\frac{NC}{A} + sM_Y(s)}{As\left[s^2 + \left(\frac{NC}{A}\right)^2\right]} \tag{5.16}$$

These are the classical equations of motion for a two axis, symmetrical, free rotor gyro.

Note that for no externally applied moments on the rotor, the case motion inputs ϕ_X and ϕ_Y are equal and opposite to the rotor motion with respect to case fixed coordinates. This is the desired relationship for an ideal free gyro.

5.3 Tuning Condition

The tuning condition necessary for this gyro to operate as an ideal gyro is found using the general open loop equation of motion. For a step angular input to the case, a constant rotor to case angle is desired for ideal operation considering all other disturbances are zero. From the general open loop equation this requires that $\Theta_{XY_{ss}}$ be a constant, or, from equation (5.10)

$$\Theta_{XY_{ss}} = \lim_{s \to 0} \frac{s - \frac{\phi_{XYo}}{s}\left[F_1(s)\,T_1(s-2jN) - \bar{F}_2(s-2jN)\,T_2(s)\right]}{T_1(s)\,T_1(s-2jN) - T_2(s)\,\bar{T}_2(s-2jN)} \tag{5.17}$$

(The second term in the numerator of equation (5.14) has been neglected in the above expression since it equals zero in the limit as $s \rightarrow 0$). The functions $F_1(s)$ and $F_2(s-2jN)$ both have roots equal to zero, therefore, for $\Theta_{XY_{SS}}$ to be constant in the limit, the denominator must have one root equal to zero or the characteristic equation must equal zero, that is

$$\left[K - j(T_D + nND) - N^2 J\right]\left[K - j(T_D + nND) - N^2(2A+2B-2C+I)\right] - \left[\Delta K + N^2 \Delta J + j\Delta k\right]\left[\Delta K + N^2 \Delta J - j\Delta k\right] = 0 \tag{5.18}$$

The value of N that satisfies this equation (neglecting damping) is then the gyro "tuned" speed. It can be shown[5] that the value of N which satisfies this equation is

$$N = \sqrt{\frac{K}{J}}\left\{1 + \frac{J}{4(A+B-C)}\left[\left(\frac{\Delta K}{K} + \frac{\Delta J}{J}\right)^2 + \left(\frac{\Delta k}{K}\right)^2\right]\right\} \tag{5.19}$$

For a practical design

$$\frac{J}{4(A+B-C)}\left[\left(\frac{\Delta K}{K} + \frac{\Delta J}{J}\right)^2 + \left(\frac{\Delta k}{K}\right)^2\right] \ll 1$$

thus, the angular velocity of spin necessary for unrestrained operation is

$$N_o = \sqrt{\frac{K}{J}} = \left\{\frac{\sum_1^n (K_{xn} + K_{yn})}{\sum_1^n (A_n + B_n - C_n)}\right\}^{1/2} \tag{5.20}$$

The tuned condition has balanced the negative and positive spring restraints such that the gyro operates as free rotor gyro. This is the same tuning condition derived for the one gimbal Hooke's Tuned-Joint Gyro.

Errors for Multigimbal Elastically Supported Tuned Gyros

Errors for the elastically supported tuned instruments usually occur from the following sources: (1) mistuning (2) windage friction forces (3) mass and quadrature unbalance (4) twice spin frequency angular oscillations (5) anisoelasticity (6) and rotor angular offset. The errors produced by these sources differ for each type of instrument, therefore, one particular instrument (a Two-Degree-of-Freedom Tuned Gyro with three gimbals),[5] will be analyzed to illustrate how the above error terms can be derived. The instrument can be modeled by the general open loop equation (5.14) for the case of n=3 gimbals.

6.1 Mistuning

For the Tuned Hooke's Joint Gyro, it is shown (Section 4) that the rotor is unrestrained in space if the positive spring restraint from the torsion elements cancel the negative spring restraint from the rotor dynamics. Since the negative spring restraint is a function of the rotor's spin speed, it can balance the positive spring restraint when the rotor is rotated at a specific speed, i.e. the tuned speed.

If the rotor is not rotating at its tuned speed, the torsional and dynamic spring restraints will not cancel, resulting in a net gyro spring restraint. This restraint will result in a conical precession of the rotor's spin axis about an inertial space fixed axis.

In the three gimbal, two-degree-of-freedom gyro, a similar balance between torsional and dynamic restraint must exist as in the single gimbal Hooke's Tuned-Joint Gyro. This unwanted conical precession resulting from incorrectly setting the rotor speed, i.e. mistuning the gyro, is called drift.

The precessional period of this drift is somewhat analogous to a conventional two-degree-of-freedom gyro.[4] The precessional period is

$$T = \frac{2\pi H}{\sum_1^n (K_{xn} + K_{yn}) - N^2 \sum_1^n (A_n + B_n - C_n)}$$

where the denominator is the gyro net spring restraint and H is the rotor angular momentum.

The inverse of the gyro precessional period times 2π is then the drift rate (D in radians/second) due to the uncancelled spring restraint given by

$$D = \frac{K_1 - C_1 N^2}{H}$$

where

$$K_1 = \sum_1^n (K_{xn} + K_{yn})$$

$$C_1 = \sum_1^n (A_n + B_n - C_n)$$

For H = AN (assuming gimbal inertias are much smaller than rotor inertias),

$$D = \frac{K_1 - C_1 N^2}{AN}$$

$$= \frac{K_1}{AN} - \frac{C_1 N}{A} \qquad (6.1)$$

At the tuned speed, N_0, the drift equals zero, thus

$$\frac{K_1}{AN_0} = \frac{C_1 N_0}{A}$$

or, (6.2)

$$K_1 = C_1 N_0^2$$

Substituting equation (6.2) into equation (6.1), the gyro drift is given by

$$D = \frac{C_1}{A}\left[\frac{N_0^2 - N^2}{N}\right]$$

or, normalizing the drift,

$$\frac{D}{C_1 N_0 / A} = \frac{1}{N/N_0} - \frac{N}{N_0}.$$

This drift is plotted as a function of spin rate in Figure 13. For spin rates below N_0, the inherent suspension spring rate creates a positive direction gyro drift (i.e. in a direction expected for a gyro with positive spring restraint). For spin speeds above N_0, the anti-spring nature of the dynamic coupling force controls, resulting in a negative gyro drift. At N_0 the two drifts cancel one another.

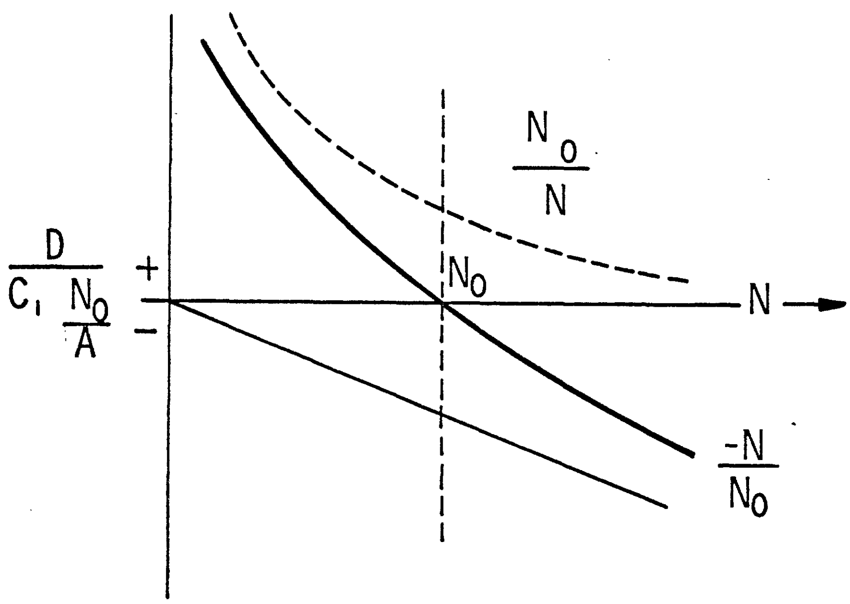


Figure 13 - Sensitivity of Gyro to Changes in N is indicated by the slope of this curve

The change in the gyro drift near its tuned condition with spin speed change is the slope of the curve at N_0. As C_1/A is decreased, the slope of the curve at N_0 becomes smaller and thereby the sensitivity of the gyro to changes in its spin speed is reduced.

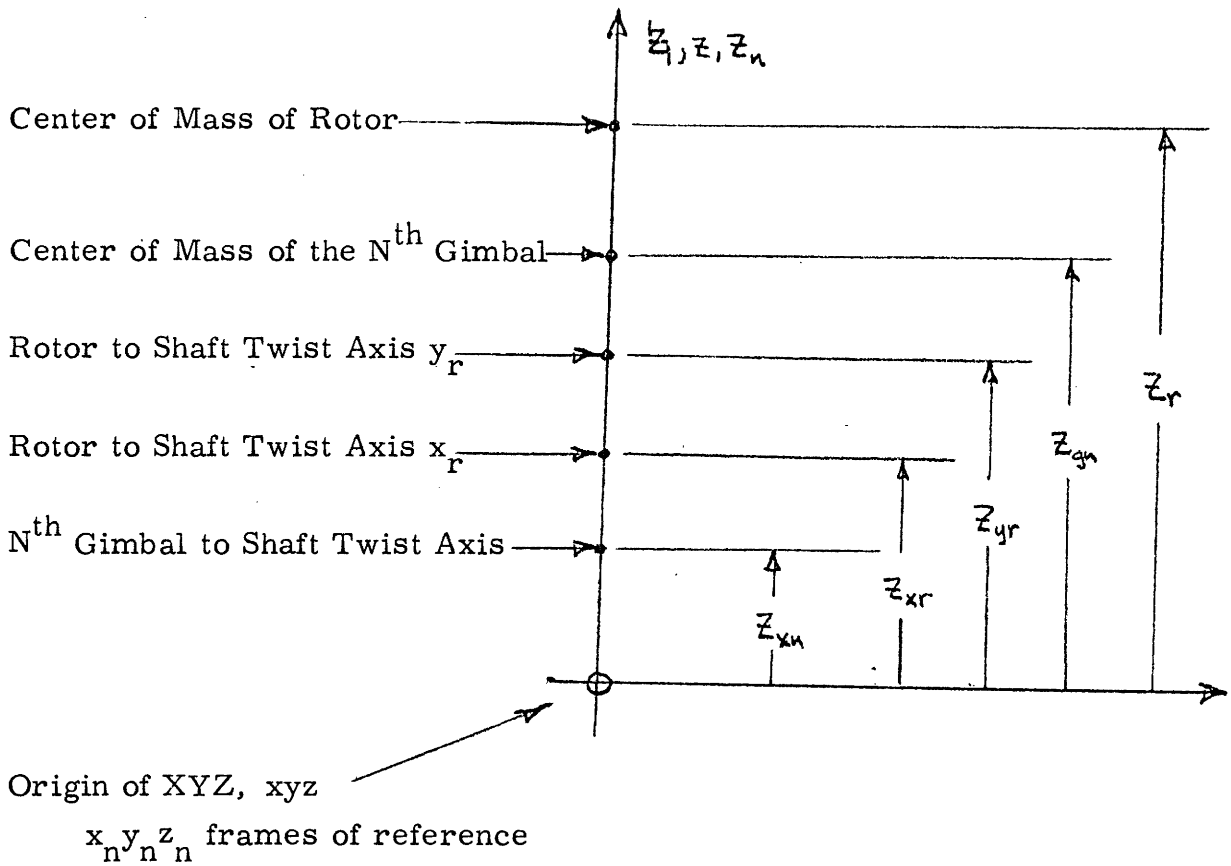
6.2 Mass and Quadrature Unbalance Errors

Refer to the coordinate frames defined for the Two-Degree-of-Freedom Gyro with "n" gimbals (Section 5.2). The coordinates of the center of mass of the n^{th} gimbal are x_{gn}, y_{gn}, z_{gn}. The coordinates of the center of mass of the rotor are x_r, y_r, z_r. Assume that the center of rotation of the torsional elements can not be perfectly aligned with gimbal and rotor coordinate frames. Since mass unbalances in the radial direction (along any direction perpendicular to the shaft) create a zero net torque on the rotor for one rotor revolution, only mass unbalances along the spin axis need to be considered. The torsion element axes are displaced from the rotor and gimbal reference frames only in the z direction by z_{xr} and z_{yr} (Figure 14).

Mass unbalance torques from shifts of the gimbal and rotor centers of mass are now derived accounting for misalignments in the torsion element axes from the reference coordinate sets.

Moments acting on the rotor along the rotor fixed coordinate set due to rotor mass and quadrature unbalance are[5]

$$\begin{aligned} T_{xr} &= M(z_r - z_{xr})a_y - My_r a_z + q_x a_x \\ T_{yr} &= -M(z_r - z_{yr})a_x + Mx_r a_z + q_y a_y \end{aligned} \qquad (6.3)$$



Position of Torsion Twist Axes

Figure 14

The source of quadrature unbalance torque is related to the characteristics of torsion bars when subjected to axial stress from an acceleration field.[6] The quadrature torque is in the same direction as the acceleration field.

The n^{th} gimbal will cause a torque on the rotor. This torque resolved along the gimbal fixed set is

$$T_{xn} = m_n \left(z_{gn} - z_{xn} \right) a_{yn} - m_n y_{gn} a_z \tag{6.4}$$

The gimbal is restrained from moving about the y_n axis, therefore no torques can be transmitted to the rotor about this axis, i.e. $T_{yn} = 0$.

Resolving gimbal accelerations into rotor accelerations one obtains

$$a_{yn} = - a_x \sin\alpha_n + a_y \cos\alpha_n$$

The total moment applied to the rotor will be equal to the sum of rotor mass unbalance torque and gimbal mass unbalance torque, or

$$\begin{aligned} T_x &= T_{xr} + \sum_1^n T_{xn} \cos\alpha_n \\ T_y &= T_{yr} + \sum_1^n T_{xn} \sin\alpha_n \end{aligned} \tag{6.5}$$

Substituting and combining terms from equations (6.3) and (6.4) yields

$$T_x = q_1 a_x + p_1 a_y + p_a a_z \tag{6.6}$$
$$T_y = q_2 a_y + p_2 a_x + p_b a_z$$

where

$$q_1 = q_x - \sum_1^n m_n (z_{gn} - z_{xn}) \sin\alpha_n \cos\alpha_n$$
$$q_2 = q_y + \sum_1^n m_n (z_{gn} - z_{xn}) \sin\alpha_n \cos\alpha_n$$
$$p_1 = M(z_r - z_{xr}) + \sum_1^n m_n (z_{gn} - z_{xn}) \cos^2\alpha_n$$
$$p_2 = -M(z_r - z_{yr}) - \sum_1^n m_n (z_{gn} - z_{xn}) \sin^2\alpha_n$$
$$p_a = -M y_r - \sum_1^n m_n y_{gn} \cos\alpha_n$$
$$p_b = M x_r - \sum_1^n m_n y_{gn} \sin\alpha_n$$

Rearranging the terms in equation (6.6),

$$T_x = (q + \Delta q) a_x + (p + \Delta p) a_y + p_a a_z \tag{6.7}$$
$$T_y = (q - \Delta q) a_y + (p - \Delta p) a_x + p_b a_z$$

where

$$q = \frac{q_1 + q_2}{2} \qquad \Delta q = \frac{q_1 - q_2}{2}$$

$$p = \frac{p_1 + p_2}{2} \qquad \Delta p = \frac{p_1 - p_2}{2}$$

Combining the torques in equation (6.7) and transforming the torques into case fixed coordinates, equation (6.7) becomes

$$T_{XY} = (q - j\Delta p)a_{XY} + (\Delta q + jp)a_{XY}e^{2jNt} + (p_a + jp_b)a_z e^{jNt} \tag{6.8}$$

where

$$\left.\begin{aligned} T_X &= T_x \cos Nt - T_y \sin Nt \\ T_Y &= T_x \sin Nt + T_y \cos Nt \\ T_{XY} &= T_X + jT_Y \\ a_{xy} &= a_{XY} e^{-jNt} \\ \bar{a}_{xy} &= a_{XY} e^{jNt} \end{aligned}\right\} \begin{array}{l}\text{Transformations} \\ \text{from rotating} \\ \text{frame to case} \\ \text{fixed frame}\end{array}$$

Equation (6.8) represents the torque acting on the rotor due to mass unbalance in case fixed coordinates. Note, linear accelerations at spin frequency, twice spin frequency, and constant magnitude cause time invariant torques on the rotor. These unwanted torques cause the instrument to drift.

For a_{XY} = constant

$$T_{XY} = (q - j\Delta p)\, a_{XY}$$

$$T_X + jT_Y = q a_X + \Delta p a_Y + j q a_Y - j \Delta p a_X$$

$$T_X = q a_X + \Delta p a_Y$$

$$T_Y = q a_Y - \Delta p a_X$$

$$T_X = \left[M\left(z_r - \frac{z_{xr} + z_{yr}}{2}\right) + \sum_1^n m_n (z_{gn} - z_{xn})\right] a_Y + \frac{q_x + q_y}{2}\, a_X$$

$$T_Y = -\left[M\left(z_r - \frac{z_{xr} + z_{yr}}{2}\right) + \sum_1^n m_n (z_{gn} - z_{xn})\right] a_X + \frac{q_x + q_y}{2}\, a_Y$$

Note that for no torsion element axis displacement

$$T_X = M z_r a_Y + \sum_1^n m_n z_{gn} a_Y + \frac{q_x + q_y}{2}\, a_X$$

$$T_Y = -M z_r a_X + \sum_1^n m_n z_{gn} a_X + \frac{q_x + q_y}{2}\, a_Y \qquad (6.9)$$

as one would expect.

How these unwanted torques affect the drift of the instrument can be seen by substituting into the classical equations of motion, equation (5.16), for a two axis, symmetric free rotor gyro.

$$\dot{\Theta}_X = \frac{sT_X(s) - T_Y(s)\frac{NC}{A}}{A\left[s^2 + \left(\frac{NC}{A}\right)^2\right]} \tag{6.10}$$

$$\dot{\Theta}_Y = \frac{T_X(s)\frac{NC}{A} + sT_Y(s)}{A\left[s^2 + \left(\frac{NC}{A}\right)^2\right]}$$

For constant values of torque T_x and T_Y

$$\dot{\Theta}_{X_{ss}} = -\frac{T_{Y_o}}{NC} \tag{6.11}$$

$$\dot{\Theta}_{Y_{ss}} = \frac{T_{X_o}}{NC}$$

To decrease these drift rates the Z axis inertia and spin speed must be increased as one would expect. Note here the two-degree-of-freedom instrument could be considered as two uncoupled single-degree-of-freedom instruments.

For a_{XY} = twice spin frequency

Errors produced by twice spin frequency accelerations along an axis perpendicular to the spin axis are found by substituting

$$a_{XY} = a_{XY_o} \sin(2Nt + \gamma)$$

into the general torque equation (6.8). The constant time invariant torque becomes

$$T_{XY} = (\Delta q + jp)\, a_{XY_0} \frac{j e^{-j\gamma}}{2} \tag{6.12}$$

or substituting for Δq and P

$$T_{XY} = -\left[M(z_{yr} - z_{xr}) - j(q_x - q_y) + \sum_1^n m_n (z_{gn} - z_{xn}) e^{2j\gamma}\right] \frac{e^{-j\gamma}}{4}\, a_{XY_0}$$

which reduces to

$$T_{XY} = -\left[\sum_1^n m_n z_{gn} e^{2j\alpha_n}\right] \frac{e^{-j\gamma}}{4}\, a_{XY_0} \tag{6.13}$$

for no torsion element axes displacement or quadrature unbalance. Note only gimbal pendulosities contribute to the torque.

Separating equation (6.13)

$$\begin{aligned} T_X &= \left(-\sum_1^n m_n z_{gn} \cos 2\alpha_n\right) \frac{a_{X_0}}{4} + \left(\sum_1^n m_n z_{gn} \sin 2\alpha_n\right) \frac{a_{Y_0}}{4} \\ T_Y &= \left(-\sum_1^n m_n z_{gn} \sin 2\alpha_n\right) \frac{a_{X_0}}{4} - \left(\sum_1^n m_n z_{gn} \cos 2\alpha_n\right) \frac{a_{Y_0}}{4} \end{aligned} \tag{6.14}$$

It is seen that the torques on the rotor are coupled for acceleration inputs at twice spin frequency.

For a_{XY} = spin frequency

Errors produced by vibrations at spin frequency along the shaft axis are found by substituting $a_z = a_{z_0} \sin(Nt + \gamma)$ into the general torque equation (6.8)

$$T_{XY} = -\left(p_a + j p_b\right) \frac{a_{z_0}}{2j} e^{-j\gamma}$$

substituting for p_a and p_b

$$T_{XY} = -\frac{a_{z_0}}{2} e^{-j\gamma} \left[M(x_r + j y_r) + j \sum_{1}^{n} m_n y_{gn} e^{j\alpha_n} \right] \qquad (6.15)$$

Again, as in the twice spin frequency acceleration input, the torques are coupled.

6.3 Twice Spin Frequency Angular Oscillations of the Case

A steady state drift of the rotor with respect to the case can exist when the case is oscillated at twice spin frequency about an axis perpendicular to the shaft.

To derive an expression for this drift rate,[5] consider case motion inputs of the following

$$\phi_X(t) = \phi_0 \sin(\omega t + \gamma) \cos\beta$$
$$\phi_Y(t) = \phi_0 \sin(\omega t + \gamma) \sin\beta$$

where β and γ are arbitrary phase angles in space and time domain respectively.

Multiplying the Y-input by $j = \sqrt{-1}$, combining the inputs, and substituting into the gyro's general open loop equation of motion (5.14), the gyro's drift becomes

$$\dot{\Theta}_{XY}(s) = -\phi_{XY}(s)\frac{s\,A(s)}{\Delta(s)} - \bar{\Phi}_{XY}(s-2jN)\frac{s\,B(s)}{\Delta(s)} \qquad (6.16)$$

where

$$\begin{aligned} A(s) &= F_1(s)\bar{T}_1(s-2jN) - F_2(s-2jN)T_2(s) \\ &= A_0 s^a + A_1 s^{a-1} + \cdots + A_{a-1}s + A_a \\ B(s) &= F_2(s)\bar{T}_1(s-2jN) - \bar{F}_1(s-2jN)T_2(s) \\ &= B_0 s^b + B_1 s^{b-1} + \cdots + B_{b-1}s + B_b \\ \Delta(s) &= T_1(s)\bar{T}_1(s-2jN) - T_2(s)\bar{T}_2(s-2jN) \\ &= \Delta_0 s^c + \Delta_1 s^{c-1} + \cdots + \Delta_{c-1}s + \Delta_c \end{aligned}$$

Applying the final value theorem to this expression, the steady state drift becomes

$$\dot{\Theta}_{XY_{ss}} = -\lim_{s\to 0}\phi_{XY}(s)\frac{s^2A(s)}{\Delta(s)} - \lim_{s\to 0}\bar{\Phi}_{XY}(s-2jN)\frac{s^2B(s)}{\Delta(s)} \qquad (6.17)$$

The first term of this expression vanishes as $s \to 0$.

The second term expanded becomes

$$\dot{\Theta}_{XY_{ss}} = -\lim_{s\to 0} s^2 \phi_0 e^{-j\beta} \frac{\omega\cos\gamma + (s-2jN)\sin\gamma}{(s-2jN-j\omega)(s-2jN+j\omega)}$$

$$\times \frac{B_0 s^b + B_1 s^{b-1} + \cdots + B_{b-1} s + B_b}{\Delta_0 s^c + \Delta_1 s^{c-1} + \cdots + \Delta_{c-1} s + \Delta_c} \qquad (6.18)$$

where

$$\Delta_c = \left[K - j(T_D + nND) - N^2 J\right]\left[K - j(T_D + nND) - N^2(2A+2B-2C+J)\right]$$
$$-\left[\Delta K + N^2 \Delta J + j\Delta k\right]\left[\Delta K + N^2\Delta J - j\Delta k\right]$$

If the instrument is operated at tuned speed the Δ_c term will vanish provided damping is neglected (i.e. $T_D + nND = 0$) and $K = N^2 J$.

When the gyro is oscillated about an axis perpendicular to the shaft at twice spin frequency (i.e. $\omega = 2N$), equation (6.18) reduces to

$$\dot{\Theta}_{XY_{ss}} = -j\,\phi_0 e^{-j(\beta+\gamma)} \frac{B_b}{\Delta_{c-1}} \qquad (6.19)$$

Assuming that,

$$\frac{J}{A+B+C} \ll 1$$

then

$$\dot{\Theta}_{XY_{ss}} = \frac{-N_0 \Phi_0 e^{-j(\beta+\gamma)}}{\Delta F_m} \left[\frac{\sum_1^n (K_{xn} - K_{yn}) e^{2j\alpha_n}}{\sum_1^n (K_{xn} + K_{yn})} + \frac{\sum_1^n (A_n + B_n - C_n) e^{2j\alpha_n}}{\sum_1^n (A_n + B_n - C_n)} \right] \tag{6.20}$$

Notice that angularly spacing three gimbals equally (i.e. n=1 at 0^o, n=2 at 120^o, n=3 at 240^o) about the spin axis, reduces the term in brackets to zero and therefore reduces the drift to zero. A three gimballed gyro has its three gimbals angularly displaced in this manner. The error sensitivity to twice spin frequency angular inputs can be reduced by making the ratio F_m/N_0 as large as possible.

This error term would be difficult to test for since table oscillations would have to be around 200 cps. This alone might justify not considering this error term in more detail for this thesis.

6.4 Damping forces due to windage

When the rotor is rotating in a fixed plane and the driving shaft is misaligned by some small angle α, the gimbals will oscillate about its torsion bar hinges $\pm \alpha$ in one revolution relative to the rotating shaft (Figure 8). This motion introduces a damping torque from the windage forces which precesses the gyro about the axis of misalignment in a direction to reduce the misalignment to zero.

Consider a gyro where rotor inertias are much larger than gimbal inertias (which is a good approximation for this instrument) and differences in inertias of the rotor about x' and y' axes and differences in torsion bar stiffnesses are negligible. The characteristic equation for the gyro becomes[5]

$$\Upsilon_1(s)\,\overline{\Upsilon}_1(s-2jN) \tag{6.21}$$

where

$$\Upsilon_1(s) = I\left(s+\frac{1}{\tau_n}-j\omega_n\right)\left(s+\frac{1}{\tau}+j\frac{\delta N}{F_m}\right)$$

$$\overline{\Upsilon}_1(s-2jN) = I\left[s+\frac{1}{\tau_n}-j(2N-\omega_n)\right]\left[s+\frac{1}{\tau}-2jN\right]$$

$$\delta N = \frac{N_o^2-N^2}{N}$$

$$D_D = \frac{T_D}{N}$$

$$\frac{1}{\tau_n} = \frac{D_R}{I}+\frac{nD}{I}\left(1-\frac{N}{\omega_n}\right)-\frac{D_D}{I}\left(\frac{N}{\omega_n}\right)$$

Notice from the gyros characteristic equation that the gyros time constant is reduced by adding the damping term D_R/I. For the gyro to operate ideally, the closed loop response time must be faster than the gyro's response to the damping force. The damping forces are then undesireable. If the closed loop null torquing response is much faster than the rot or response to the damping force, the windage forces can be completely neglected. For a three gimballed instrument this will be true.

6.5 Rotor Offset Angle

Drift rates can occur when the rotor spin axis is offset from the shaft Z axis. This offset angle is produced by a pick-off null shift, when operating the gyro in a torque to balance mode, or an angular tilt of the gyro case with respect to the rotor, if the gyro is operated in an open loop mode.

The errors that are produced by the rotor offset angle can be derived from the gyro's general open loop equation (5.14).

Assuming gimbal inertias are much smaller than rotor inertias, the general equation of motion reduces to

$$\Theta_{XY}(s) = \frac{-\Phi_{XY}(s)\, F_1(s)}{T_1(s)} \tag{6.22}$$

Substituting for $T_1(s)$, $F_1(s)$, and assuming a step input in $\Phi_{XY}(s) = \frac{\Phi_{XY}(0)}{s}$, the solution in the time domain is[5]

$$\Theta_{XY}(t) = -\Phi_{XY}(0)\, e^{-\left(\frac{1}{\tau} + j\frac{\delta N}{F_m}\right)t} \tag{6.23}$$

where

$$\frac{1}{\tau} = \frac{nD}{\frac{1}{2}(A+B)}$$

$$\delta N = \frac{N_o^2 - N^2}{N}$$

$$F_m = \frac{C + \sum_1^n A_n}{\sum_1^n (A_n + B_n - C_n)}$$

For the input $\phi_{XY}(0) = \phi_X(0) + j\,\phi_Y(0)$, the rotor deflections with respect to case fixed coordinates become

$$\Theta_X(t) = -\phi_X(0)\,e^{-t/\tau}\cos\frac{\delta N}{F_m}t + \phi_Y(0)\,e^{-t/\tau}\sin\frac{\delta N}{F_m}t$$

(6. 24)

$$\Theta_Y(t) = -\phi_Y(0)\,e^{-t/\tau}\cos\frac{\delta N}{F_m}t - \phi_X(0)\,e^{-t/\tau}\sin\frac{\delta N}{F_m}t$$

Expressions for the error rates due to rotor angular offsets are obtained by evaluation of the differentials of the above equations.

$$\dot{\Theta}_X = \frac{\phi_X(0)}{\tau} + \phi_Y(0)\frac{\delta N}{F_m}$$

(6. 25)

$$\dot{\Theta}_Y = \frac{\phi_Y(0)}{\tau} - \phi_X(0)\frac{\delta N}{F_m}$$

These drift rates can be reduced by minimizing the rotor offsets $\phi_X(0)$ and $\phi_Y(0)$, damping, and operating the gyro at a spin speed corresponding to its tuned speed.

6.6 Anisoelasticity

In this family of instruments, the anisoelastic effect is a function of mismatch between translational stiffness of the rotor to shaft suspension along the spin axis and along an axis perpendicular to the spin axis.

For a non symmetric rotor to shaft suspension, the anisoelastic coefficient is given by[5] (assuming rotor mass >> gimbal mass)

$$T = \frac{M}{\omega_z^2}\left[\frac{k_z - k}{k} + \frac{\left(\frac{\omega_s}{N}\right)^2}{\left(\frac{\omega_s}{N}\right)^2 - 4}\left(\frac{k_x - k_y}{2k}\right)^2\right] \qquad (6.26)$$

(moment per unit acceleration^2)

where

$$\omega_z^2 = \frac{k_z}{M}, \quad \omega_s^2 = \frac{k}{M}, \quad k = \frac{k_x + k_y}{2}$$

For the radially symmetric gyro, equation (6.26) reduces to

$$T = \frac{M}{\omega_z^2}\left(\frac{k_z}{k} - 1\right) \qquad (6.27)$$

(moment per unit acceleration^2)

Error Model for a Two-Degree-of-Freedom Tuned Gyro

One philosophy of inertial gyro testing is that the test program should provide accurate measurement of the coefficients of a modeling equation which mathematically defines the gyro performance in any environment.

This model should contain the errors which have been measured over a wide range of environments for a given gyro family. Requirements for a particular application are then included in the test and evaluation program to extend the model to more accurately describe its performance for that particular application. A "complete" model of the gyro is finally obtained which describes the gyro's performance in a given environment without having to test in that particular environment ("complete" is enclosed in quotation marks since an adequate model for today's applications will not necessarily be adequate for future applications).

To derive an error model for the family of two-degree-of-freedom elastically supported tuned gyros consider the type of previously derived error terms in Section 6. The error terms can be divided into terms independent of an acceleration field (mistuning, windage friction forces, rotor offset angle), proportional to an acceleration field (mass and quadrature unbalance), and proportional to an acceleration field squared (anisoelasticity).

The general torque equation about the Y axis is divided into these three torque categories as follows

$$M_Y = M_{IY} + M_{aY} + M_{a^2Y} \tag{7.1}$$

where

M_{IY} = acceleration insensitive torques about the Y axis

M_{aY} = acceleration sensitive torque about the Y axis

M_{a^2Y} = acceleration squared sensitive torques about the Y axis

In spite of the differences in this gyro from the single-degree-of-freedom gyro, the gyro is found to be susceptible to the same type of errors when subject to translational accelerations.

The acceleration sensitive torques can be represented from equation (6.8) as

$$M_{aY} = K_q a_Y + K_{PY} a_x \tag{7.2}$$

where

$$K_q = \frac{q_x + q_y}{2}$$

$$K_{PY} = -\left[M\left(z_r - \frac{z_{xr} + z_{yr}}{2}\right) + \sum_1^n m_n (z_{gn} - z_{xn})\right]$$

The acceleration squared sensitive torques occur from compliance effects. The compliance coefficients are defined as follows:

K'_{zz} = displacement along spin axis due to a unit force along Z

K'_{zY} = displacement along spin axis due to a unit force along Y

The displacement of the rotor center of mass due to the compliance of the support structure along the Z axis is then

$$d_K = K'_{ZZ}\, m(sf)_Z + K'_{ZY}\, m(sf)_Y + K'_{ZX}\, m(sf)_X \tag{7.3}$$

where

m = effective displaced mass

$(sf)_X, (sf)_Y, (sf)_Z$ = specific force with respect to inertial space along X, Y, Z axes respectively

The displacement of the center of mass along the x axis will not contribute to a steady state torque about Y for a constant a_Z input since the whole support structure is rotating about the Z axis (i. e., in one revolution of the rotor, the resultant torque on the rotor due to a_Z would equal zero.)

This displaced mass creates a torque on the rotor in an acceleration field a_X of

$$M_{a^2Y} = (sf)_X \left[K'_{ZZ}\, m^2 (sf)_Z + K'_{ZY}\, m^2 (sf)_Y + K'_{ZX}\, m^2 (sf)_X \right]$$

or

$$M_{a^2Y} = K_{ZZ}\, a_X a_Z + K_{ZY}\, a_X a_Y + K_{ZX}\, a_X^2 \tag{7.4}$$

where

$$K_{zz} = K'_{zz} m^2 \text{, etc.}$$

$$(sf)_X = -a_X$$

$$(sf)_Y = -a_Y$$

$$(sf)_Z = -a_Z$$

The total torque about the Y axis is the sum of the acceleration insensitive torque M_{IY}, the mass unbalance torque M_{aY}, and the compliance torques M_{a^2Y}, therefore,

$$M_Y = M_{IY} + K_q a_Y + K_{pY} a_X + K_{zz} a_X a_Z + K_{zY} a_X a_Y + K_{zx} a_X^2 \qquad (7.5)$$

Equation (7.) characterizes the error torques about the gyro's Y axis on the basis of linear theory.

With a similar analysis to the above, the X axis torque equation becomes

$$M_X = M_{IX} + K_q a_X + K_{pX} a_Y - K_{zz} a_Y a_Z - K_{zY} a_Y^2 - K_{zX} a_Y a_X \qquad (7.6)$$

where

M_{IX} = acceleration insensitive torque along X axis

$$K_q = \frac{q_x + q_y}{2}$$

$$K_{pX} = M\left(z_r - \frac{z_{xr} + z_{yr}}{2}\right) + \left(\sum_1^n m_n (z_{gn} - z_{xn})\right)$$

These error models take into account only errors terms that have been analyzed thus far in this thesis. It is likely that more error sources will become evident as the gyro is tested. In particular, error terms proportional to a_Z, a_Z^2, $a_Y a_Z$ (for the Y torque equation), and $a_X a_Z$ (for the X torque equation) might become apparent.

Test Procedure for a Two-Degree-of-Freedom Tuned Gyro

A general format for gyro testing should include the following:[13]

(1) Verification of assembly procedures (commonly called acceptance tests)

(2) Verification of design parameters (commonly called qualification or engineering evaluation tests)

(3) Development of the gyro error model equation

(4) Development of advanced test techniques

(5) Indication of the need for development of advanced test equipment

(6) Determination of the reasons for failure to perform as expected (commonly called diagnostic tests)

It was noted in Section 6 that for constant g inputs the two-degree-of-freedom instrument could be considered as two uncoupled single-degree-of-freedom gyros (for oscillatory linear and angular inputs, the instrument cannot be considered as two uncoupled single-degree-of-freedom gyros). The two-degree-of-freedom gyro can therefore be subject to all the tests for a single-degree-of-freedom gyro in the one g environment. By amplifying the output axis of the X axis pickoff, the instrument is reducd to a single-degree-of-freedom gyro, having the X axis as electrical input axis while the Y axis serves as pickoff output axis and mechanical input axis. The gyro coefficients in equation (7.5) (drift about the Y axis, or equivalently, the null torque applied to the X axis) can then be determined by conventional isolation techniques used in testing single-degree-of-freedom instruments.[14] The gyro is then reconnected so that pickoff Y drives torquer X to determine the coefficients in equation (7.6).

For tumble testing, it is more efficient to utilize the signals of both X and Y torquers simultaneously. Signal generator Y is connected to the X axis torquer and the signal generator X is connected to the Y torquer. By positioning the X and Y axis orthogonal to the table axis, (i.e. with the spin axis along the table axis), the coefficients in equations (7.5) and (7.6) can be attained most efficiently.

A typical test procedure to test this instrument after it has passed its acceptance tests would be to:

1) Identify gyro drift stability in fixed orientation
2) Identify gyro drift stability across a disturbance
3) Multiple position drift coefficient calibration
4) Repeat 3) for stability determination
5) Put in disturbance
    a) cooldown
    b) shut rotor off
    c) temperature
6) Repeat 3) and 4)

7) Evaluate torque generator characteristics
    a) dc linearity
    b) ac linearity at two frequencies and float angle

8) Evaluate instrument as a strapdown unit; constant rate input

9) Dynamic tests
    a) Vibrations at spin frequency along spin axis
    b) Vibrations at twice spin frequency (if test equipment permits)
    c) Angular oscillations at twice spin frequency (if test equipment permits)

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