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OPTIMAL RECURSIVE MAXIMUM LIKELIHOOD ESTIMATION

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Abstract: In this paper we derive stochastic differential equations for recursive maximum-likelihood estimates for the joint filtering-parameter estimation problem.

Keywords: Maximum likelihood estimate; stochastic differential equation; Hamilton-Jacobi Equation; Nonlinear Filtering

1. INTRODUCTION

In this paper we would like to consider the joint states and parameter estimation problem for the following non-linear stochastic differential system:

$$dx(t) = f(x(t),\theta)dt + g(x(t),\theta)dw(t),\ 0\le t\le T \quad (1.1)$$

with the observation system

$$dy(t) = h(x(t),\theta)dt + dv(t),\ 0\le t\le T. \quad (1.2)$$

In the above, $w(t)$ and $v(t)$ are standard independent Brownian motions, f,g,h are at least thrice continuously differentiable with bounded derivatives with respect to $x\epsilon R$ and $\theta\epsilon R$ and $g(x,\theta) \geq \alpha > 0,\ \forall x,\ \theta\epsilon R$.

In addition we assume

$$E\int_0^T |h(x(t),\theta)|^2 dt < \infty, \quad (1.3)$$

and the initial state satisfies

Either i) $x(0) = x_0 \epsilon R$ (1.4)

ii) $x(0) = x_0$, a random variable with density $p_0(x)\ \epsilon\ C_b^2(R;R)$, $p_0(x)>0$.

Let $\phi_{s,t}(x)$ denote the solution of the stochastic differential equation (1.1) starting at $x_s=x$. Then from a result of Kunita [2], we know that $\phi_{s,t}$ is a C^k-diffeomorphism, and the inverse map $\phi_{s,t}^{-1}$ satisfies a backward stochastic differential equation.

Let

$$\Lambda(\theta,t) = \exp(\int_0^t h(x(s),\theta)dy(s) - \frac{1}{2}\int_0^t h^2(x(s),\theta)ds)$$

$$= \exp[\int_0^t \tilde{h}(\phi_{s,t}^{-1}(x(t)),\theta)d\tilde{y}(s) \quad (1.5)$$

$$- \frac{1}{2}\int_0^t \tilde{h}(\phi_{s,t}^{-1}(x(t),\theta))^2 dt],$$

where ~ denotes a backway stochastic differential (and backward Ito integral respectively).

Let

$$L(\theta,t) = E(\Lambda(\theta,t)|x(t)=z), \quad (1.6)$$

where E denotes expectation with respect to the path space measure of $x(\cdot)$.

As a criterion, we choose as an estimate

$$\hat{z}(t) = \operatorname*{Arg\,Max}_{z,\theta} L(z,\theta,t).$$

which is a maximum likelihood criterion.

2. STOCHASTIC HAMILTON-JACOBI BELLMAN EQUATION FOR L(z,θ,t)

Using the work of Fleming-Mitter [1] and the theory of backway stochastic differential equations [cf. Kunita, loc.cit] one can show that

$$S(z,\theta,t) = -\ln L(z,\theta,t) \quad (2.1)$$

satisfies the stochastic Bellman Hamilton-Jacobi equation:

$$dS(x,\theta,t) = \sigma(x,\theta)(S_{xx}-S_x^2)dt + \alpha(x,\theta)S_x dt \quad (2.2)$$

$$+ h(x,\theta)^2 dt - h(x,\theta).dy(t)$$

where

$$\sigma(x,\theta) = \frac{1}{2}g(x,\theta)^2$$

$$\alpha(x,\theta) = 2\sigma_x - 2\sigma(x,\theta)\bar{S}_x - f(x,\theta)$$

$$\bar{S} = -\ln p^\theta(x,t)$$

where under our assumptions $p^\theta(x,t)$, the density corresponding to the $x(\cdot)$ process exists and is positive for all x,t.

We now define a recursive maximum likelihood estimate. By applying the Generalized Ito Differential Rule [cf. Kunita, loc.cit], we get

$$\partial_t VS + V^2 S d\xi(t) + \frac{1}{2} V^3 S d\langle \xi, \xi \rangle_t - V(Vh) d\langle y, \xi \rangle_t = 0 \tag{2.3}$$

where

$$V = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial \theta} \end{pmatrix}, \quad \xi(t) = \begin{pmatrix} \hat{x}(t) \\ \hat{\theta}(t) \end{pmatrix}$$

which is obtained from the stationarity condition

$$VS = 0. \tag{2.4}$$

In (2.3) all partial derivatives are computed along $(\hat{x}, \hat{\theta})$ which is obtained from the stationarity condition (2.4).

Assuming $V^2 S$ is invertible, we obtain a maximum likelihood trajectory for $\xi(t)$ from (2.2), (2.3) and (2.4) and using $\partial_t VS = VdS$.

A rigorous derivation of these results will be presented elsewhere.

REFERENCES

Fleming, W.H. and S.K. Mitter: Optimal Control and Nonlinear Filtering for Nondegenerate Diffusion Processes, Stochastics, 8, 1982, pp. 63-77.

Kunita, H.: Stochastic Differential Equations and Stochastic Flows of Diffeomorphism, E'cole de Probabilites de Saint-Flour XII - 1982.

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