Algebraic and Combinatorial Properties of Minimal Border Strip Tableaux

by

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Abstract

Motivated by results and conjectures of Stanley concerning minimal border strip tableaux of partitions, we present three results.

First we generalize the rank of a partition λ to the rank of a shifted partition $S(\lambda)$. We show that the number of bars required in a minimal bar tableau of $S(\lambda)$ is $\max(o, e + (\ell(\lambda) \bmod 2))$, where o and e are the number of odd and even rows of λ. As a consequence we show that the irreducible negative characters of $\tilde{S}_n$ vanish on certain conjugacy classes. Another corollary is a lower bound on the degree of the terms in the expansion of Schur's Q_λ symmetric functions in terms of the power sum symmetric functions.

The second result gives a basis for the space spanned by the lowest degree terms in the expansion of the Schur symmetric functions in terms of the power sum symmetric functions. These lowest degree terms studied by Stanley correspond to minimal border strip tableaux of λ. The Hilbert series of these spaces is the generating function giving the number of partitions of n into parts differing by at least 2. Applying the Rogers-Ramanujan identity, the generating function also counts the number of partitions of n into parts $5k + 1$ and $5k - 1$.

Finally for each λ we give a relation between the power sum symmetric functions and the monomial symmetric functions; the terms are indexed by the types of minimal border strip tableaux of λ.

Thesis Supervisor: Richard P. Stanley
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Go raibh míle maith agaibh go leir.

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Notation

$\mathbb{P}$	positive integers
$\mathbb{Q}$	rational numbers
$\mathcal{P}(n)$	partitions of n
$\mathcal{D}(n)$	partitions of n into distinct parts
$\mathcal{P}^0(n)$	partitions of n into odd parts
$\ell(\lambda)$	the number of parts in the partition λ
$m_i(\lambda)$	the number of parts of λ equal to i
z_λ	$1^{m_1(\lambda)}m_1(\lambda)!2^{m_2(\lambda)}m_2(\lambda)!\dots$
$S(\lambda)$	the shifted diagram of the partition λ
$\chi^\lambda(\pi)$	the character indexed by λ evaluated at π
$\langle\lambda\rangle(g)$	the projective character indexed by λ evaluated at g
$c(\mathcal{I})$	the number of crossings in the interval set $\mathcal{I}$
$z(\lambda)$	the height of a greedy border strip tableau of λ

Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of the integer n, i.e., $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ and $\sum \lambda_i = n$. The *length* $\ell(\lambda)$ of a partition λ is the number of nonzero parts of λ. Partition theory is of fundamental importance in the representation theory of the symmetric group; it is a classical result of Frobenius that the irreducible characters are indexed by partitions λ. Murnaghan and Nakayama gave a combinatorial formula for the irreducible character indexed by λ evaluated at a conjugacy class of cycle type π, the *Murnaghan-Nakayama rule*:

$$\chi^\lambda(\pi) = \sum_T (-1)^{\mathrm{ht}(T)},$$

where the sum is over all border strip tableaux of shape λ and type π.

The (Durfee or Frobenius) *rank* of λ, denoted $\mathrm{rank}(\lambda)$, is the length of the main diagonal of the diagram of λ, or equivalently, the largest integer i for which $\lambda_i \geqslant i$. The rank of λ is the least integer r such that λ is a disjoint union of r border strips. Hence if $\ell(\pi) < \mathrm{rank}(\lambda)$, we must have $\chi^\lambda(\pi) = 0$.

The machinery of symmetric functions is very applicable to the representation theory of the symmetric group. For if χ is any character of S_n, the problem of decomposing χ into irreducibles is equivalent to expanding the Frobenius characteristic of χ into Schur functions s_λ. Frobenius showed that

$$s_\lambda = \sum_\pi \chi^\lambda(\pi) \frac{p_\pi}{z_\pi}.$$

This formula is the symmetric function analogue of the Murnaghan-Nakayama rule.

Above we saw that $\chi^\lambda(\pi) = 0$ when $\ell(\pi) < \text{rank}(\lambda)$; therefore the only terms $\chi^\lambda(\pi)\frac{p_\pi}{z_\pi}$ arising in the Murnaghan-Nakayama rule must have $\ell(\pi) \geqslant \text{rank}\ (\lambda)$.

The study of the projective representations of the symmetric group and their associated combinatorial and algebraic structure began with Schur, who published degree and character formulas in 1911 [8]. He showed that the characters of the irreducible negative representations of $\tilde{S}_n$ are indexed by partitions λ of n with distinct parts and proved degree and character formulae. He defined the Schur Q-functions which (analogously to the Schur functions in the linear representation case) algebraically encode the structure of the negative characters. Later Morris [4] gave a projective analogue of the Murnaghan-Nakayama rule.

In Chapter 1 we generalize rank to shifted diagrams $S(\lambda)$ of a partition with distinct parts. This allows us to show that the irreducible negative characters vanish on certain conjugacy classes. This enables us to give a lower bound on the length of the μ which appear in the expansion of the Schur Q-functions in terms of the p_μ.

Nazarov and Tarasov [7, Sect. 1], in connection with tensor products of Yangian modules, defined a generalization of rank to skew partitions (or skew diagrams) λ/μ. In [11, Proposition 2.2] Stanley gave several simple equivalent definitions of $\text{rank}(\lambda/\mu)$. One of the definitions is (as expected) that $\text{rank}(\lambda/\mu)$ is the least integer r such that λ/μ is a disjoint union of r border strips. He developed a general theory of minimal border strip tableaux of skew shapes, introducing the concepts of the snake sequence of a skew shape and the interval set of a skew shape λ/μ. These tools are used to count the number of minimal border strip decompositions and minimal border strip tableaux of λ/μ. In particular, he gave an explicit combinatorial formula for the coefficients of the p_ν, where $\ell(\nu) = \text{rank}(\lambda/\mu)$, which appear in the expansion of $s_{\lambda/\mu}$.

Stanley considered a degree operator $\deg(p_\nu) = \ell(\nu)$ and defined the *bottom Schur functions* to be the terms of lowest degree which appear in the expansion of $s_{\lambda/\mu}$ as a linear combination of the p_ν. We study the bottom Schur functions in detail when $\mu = \emptyset$. In particular, in Chapter 2 we give a basis for the vector space they span.

Finally in Chapter 3 we prove relations between the monomial symmetric functions and the power sum symmetric functions. The relations are linear; we get an independent one for every λ such that $\ell(\lambda) = \text{rank}(\lambda)$. The monomials arising are indexed by those ν where $\ell(\nu) = \text{rank}(\lambda)$ and the coefficients occur in the combinatorial expression for the bottom Schur functions.

Chapter 1

Shifted Shapes

After introducing the necessary background, we formulate a preprocessing operation on minimal bar tableaux which preserves the number of bars, and prove some Lemmas about tableaux resulting from this operation. We apply these results to counting how many bars are in a minimal bar tableau. We discuss the connection between minimal bar tableaux and Schur Q-functions, and give some Corollaries of our previous results about the terms of the Q-functions. Finally we discuss the application of our machinery to the skew shifted case.

1.1 Definitions

Let $\mathcal{D}(n)$ be the set of all partitions of n into distinct parts. The *shifted diagram*, $S(\lambda)$, of shape λ, is obtained by forming l rows of nodes, with λ_i nodes in the ith row such that, for all $i > 1$, the first node in row i is placed underneath the second node in row $(i-1)$. For instance Figure 1-1 shows the shifted diagram of the shape 97631.

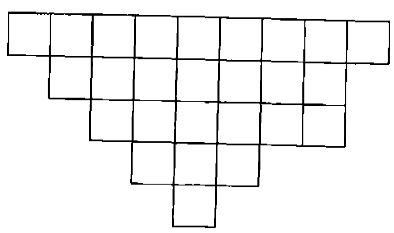


Figure 1-1: The shifted diagram of the shape 97631

We follow the treatment of Hoffman and Humphreys [1] to define bar tableaux. These occur in the inductive formula for the projective characters of S_n, first proved by Morris [4]. Let r be an odd positive integer, and let $\lambda \in \mathcal{D}(n)$ have length l. Below we define:

(a) a subset, $I_+ \cup I_0 \cup I_- = I(\lambda, r)$, of integers between 1 and l; and

(b) for each $i \in I(\lambda, r)$, a strict partition $\lambda(i, r)$ in $\mathcal{D}(n-r)$ (despite the notation, $\lambda(i, r)$ is a function of λ, as well as of (i, r)).

Let

$$I_+ = \{i : \lambda_{j+1} < \lambda_i - r < \lambda_j \text{ for some } j \leqslant l, \text{ taking } \lambda_{l+1} = 0\}.$$

In other words I_+ is the set of all rows of λ which we can remove r squares from and still leave a composition with distinct parts. For example, if $r = 5$ and $\lambda = 97631$, then $I_+ = \{1, 2\}$. If $i \in I_+$, then $\lambda_i > r$, and we define $\lambda(i, r)$ to be the partition obtained from λ by removing λ_i and inserting $\lambda_i - r$ between λ_j and λ_{j+1}. Continuing our example above, $\lambda(2, 5) = 96321$. Let

$$I_0 = \{i : \lambda_i = r\},$$

which is empty or a singleton. For $i \in I_0$, remove λ_i from λ to obain $\lambda(i, r)$. Let

$$I_- = \{i : r - \lambda_i = \lambda_j \text{ for some } j \text{ with } i < j \leqslant l\}.$$

Equivalently I_- is the set of all rows of λ for which there is some shorter row of λ such that the total number of squares in both rows is r. For example, if $r = 7$ and $\lambda = 97631$, then $I_- = \{3\}$. If $i \in I_-$, then $\lambda_i < r$, and $\lambda(i, r)$ is formed by removing both λ_i and λ_j from λ.

For each $i \in I(\lambda, r)$ the associated *r-bar* is given as follows. If i is in I_+ or I_0, the r-bar consists of the rightmost r nodes in the ith row of $S(\lambda)$. We say the r-bar is of *type 1* or *type 2* respectively. For example, the squares in Figure 1-2 labelled by 6 are a 7-bar of type 1. The squares labelled by 4 are a 3-bar of type 2. If i is in I_-, the r-bar consists of all the nodes in both the ith and jth rows, a total of r nodes. We say the r-bar is of *type 3.* The squares in Figure 1-2 labelled by 3 are a 7-bar of type 3.

Define a *bar tableau* of shape λ to be an assignment of positive integers to the squares of $S(\lambda)$ such that

(a) the set of squares occupied by the biggest integer is an r-bar B, and

(b) if we remove the r-bar B and reorder the rows, the result is a bar tableau.

1	1	6	6	6	6	6	6	6
	1	2	2	2	5	5	5	
		3	3	3	3	3	3	
			4	4	4			
				3				

Figure 1-2: A bar tableau of the shape 97631

Equivalently we can define a *bar tableau* of shape λ to be an assignment of positive integers to the squares of $S(\lambda)$ such that

(a) the entries are weakly increasing across rows,

(b) each integer i appears an odd number of times,

(c) i can appear in at most two rows; if it does, it must begin both rows (equivalent to the bar being of type 3),

(d) the composition remaining if we remove all squares labelled by integers larger than some i has distinct parts.

For example, Figure 1-3 shows the chain of partitions remaining if we remove all squares labelled by integers larger than some i from the tableau in Figure 1-2. This demonstrates the legality of that tableau.

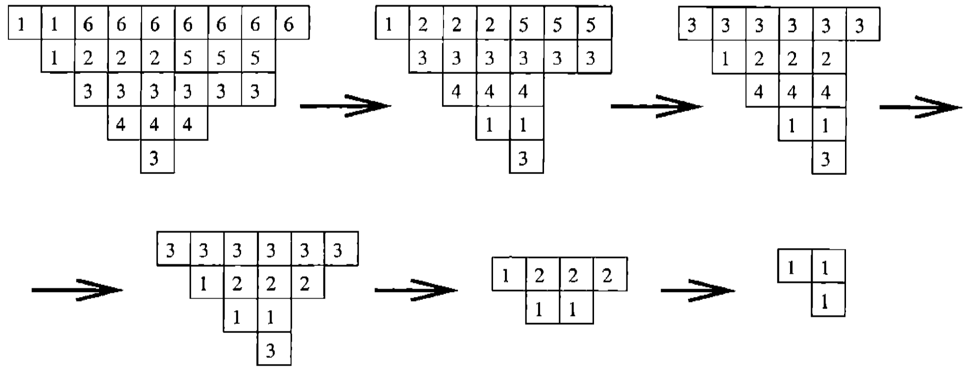


Figure 1-3: Checking legality of the bar tableau of the shape 97631

1.2 Minimal Bar Tableau

In this section we introduce a preprocessing operation on minimal bar tableau which preserves the number of bars, and prove some facts about tableaux resulting from this operation. A bar tableau of λ is *minimal* if the number of bars is minimized, i.e. there does not exist a bar tableau with fewer bars.

Lemma 1.2.1. *There exists a minimal bar tableau T^* such that there is no bar boundary an even number of squares along any row.*

For example Figure 1-4 shows a minimal bar tableau T of shape 97631 and a minimal bar tableau T^* of the same shape with no bar boundaries an even number of squares along any row (we will see later why these tableaux are minimal).

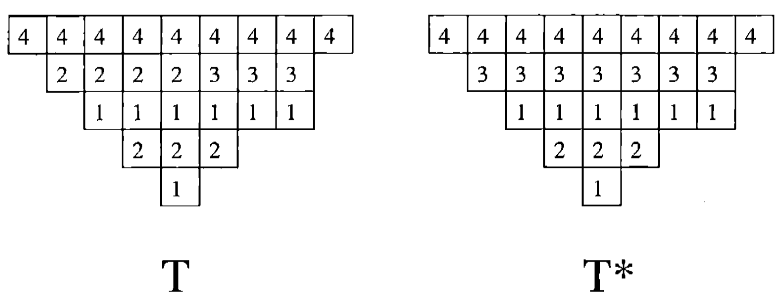


Figure 1-4: Two minimal bar tableaux of shape 97631

Proof. Let T be a minimal bar tableau of λ. In each row r_k of T, at the last bar boundary an even number of squares along a row, let b be the bar which begins to the right of the boundary. Say that b is labelled by j. Relabel the squares to the left of the boundary with j. This preserves the ordering on labels and the parity of b. The partitions remaining if we remove all squares labelled by integers larger than $i(> j)$ will be the same as before and have distinct parts. The partitions remaining if we remove all squares labelled by $i(< j)$ will not contain row r_k but will otherwise have the same (distinct) parts as before. □

Lemma 1.2.2. *Let T^* be a minimal bar tableau of λ such that there is no bar boundary an even number of squares along any row. Then if row r_k is odd, it is labelled entirely by one label j. If row r_k is even, it is labelled entirely by one label j or it has exactly two labels each occurring an odd number of times.*

Proof. If row r_k is odd and has more than one label, the second bar must be of type 1 and so must be odd, forcing the first bar to be even which is a contradiction.

If row r_k is even and has more than two labels, the final two bars are both of type 1 and so must be odd, forcing there to be a bar boundary an even number of squares along the row, a contradiction. □

1.3 Number of strips in a Minimal Tableau

In this section we use the results from the previous section to give a count of how many bars are needed in a minimal bar tableau. Define the *shifted rank* of a shape λ, denoted $\operatorname{srank}(\lambda)$, to be the number of bars in a minimal bar tableau of λ. Given an integer a, define a mod 2 to be 1 if a is odd and 0 if a is even.

Theorem 1.3.1. *Given a shape λ, let o be the number of odd rows of λ and e be the number of even rows. Then* $\operatorname{srank}(\lambda) = \max(o, e + (\ell(\lambda) \bmod 2))$.

For example, if $\lambda = 97631$, we have $o = 4, \ell(\lambda) = 5$ and $e = 1$. So $\operatorname{srank}(\lambda) = \max(4, 1 + 1) = 4$ which verifies that the tableaux shown in Figure 1-4 are indeed minimal. If $\lambda = 432$, we have $o = 1, \ell(\lambda) = 3$ and $e = 2$. So $\operatorname{srank}(\lambda) = \max(1, 3) = 3$. Such a tableau is illustrated below.

Proof. Let T be a minimal bar tableau of λ. Preprocess T into T^* so that there are no bar boundaries an even number of squares along any row. This must preserve the

2	3	3	3
	1	1	1
		1	1

number of bars. Bars of type 3 consist of one even initial bar and one odd initial bar, and so by Lemma 1.2.2 must be an entire even row and an entire odd row, or an entire even row and the initial odd bar of some other even row.

First assume that $o \geqslant e$. Note that if $o = e$, then $\ell(\lambda) \bmod 2 = 0$. So when $o \geqslant e$, $\max(o, e + (\ell(\lambda) \bmod 2))) = o$. We claim that the bars of type 3 all consist of entire even row and entire odd row pairs, and that there are exactly e of them.

From the observations above the number of bars of type 3 cannot be larger than e. Suppose that there is a bar of type 3 consisting of an entire even row and the initial odd bar of some other even row. Since $o \geqslant e$, there must also be two other odd rows, not parts of bars of type 3, each labelled entirely by some label (by Lemma 1.2.2). The total number of bars in these 4 rows is 4. So if we relabel (with new large labels) these four rows as two bars of type 3, we save two bars, contradicting the minimality of T^*. We illustrate this (impossible) situation below, and show the more economical version. Thus there are no such bars of type 3.

4	4	4	4	4
	1	1	1	1
		1	2	
			3	

8	8	8	8	8
	8	8	8	8
		9	9	
			9	

Now suppose that the number of bars of type 3 is smaller than e; thus there is some even row r_1 of the tableau which is not part of a bar of type 3. Also there is an odd row r_2 which is not part of a type 3 bar (since $o \geqslant e$). But we could relabel both these rows with some new large label saving at least one bar and contradict the minimality of T^*.

So there are exactly e bars of type 3, filling $2e$ rows of λ. The remaining $o - e$ rows are odd and so must each be completely filled by a unique label. So the total number of bars is o as required.

Now assume that $e \geqslant o$.

Claim. We can relabel so that every odd row is part of a bar of type 3.

Proof of claim. Suppose r_3 is an odd row which is not part of a bar of type 3.

Subclaim. There is an even row r_4, completely filled by a label, which is part of a bar of type 3 with the initial part of some other even row r_5.

Proof of subclaim. Assume by way of contradiction that there is not, i.e. that the completely filled even rows are all parts of bars of type 3 with complete odd rows. But there must be at least one even row r_4 (since $e \geqslant o$ and r_3 is not part of a bar of type 3) which is not part of a bar of type 3 with a complete odd row. So r_4 must not be part of a bar of type 3 at all (by our subclaim assumption). But then we could relabel r_4 and r_3 entirely with some new large label and save a bar, a contradiction. This proves our subclaim.

Relabel r_4 and r_3 with some new large label. This leaves an odd number of squares in the initial part of row r_5, and so preserves legality. These two rows are now a valid bar of type 3, and this process did not cost us any bars. We illustrate one step of this process below. Simply iterate this process until there are no odd rows which are not part of bars of type 3. This proves our claim.

2	2	2	2
	1	1	1
		2	4

9	9	9	9
	9	9	9
		2	4

So every odd row is part of a bar of type 3, filling $2o$ rows of λ. All but one of the even rows remaining must be paired up with another remaining even row, and each pair must contain one bar of type 3 (filling one entire row and the odd initial part of the other row) and one bar of type 1 (filling the odd final part of the other row). If they were not, we would have two even rows costing 4 strips, and could reduce the number of strips by relabelling as above with large new numbers. The extra row exists only when $\ell(\lambda)$ is odd, and costs two bars (i.e. one extra). This situation is illustrated on the right hand side of the above figure. So we have $o + e - o + (\ell(\lambda) \bmod 2)$ strips as required. □

1.4 Projective Representations of the Symmetric Group

Here we recall some facts about the projective representations of the symmetric group. We follow the treatment of Stembridge [12].

A projective representation of a group G on a vector space V is a map $P : G \to GL(V)$ such that

$$P(x)P(y) = c_{x,y}P(xy) \quad (x, y \in G)$$

for suitable (nonzero) scalars $c_{x,y}$. For the symmetric group, the associated Coxeter presentation shows that a representation P amounts to a collection of linear transformations $\sigma_1, \ldots, \sigma_{n-1} \in GL(V)$ (representing the adjacent transpositions) such that $\sigma_j^2, (\sigma_j\sigma_{j+1})^3$, and $(\sigma_j\sigma_k)^2$ (for $|j-k| \geqslant 2$) are all scalars. The possible scalars that arise in this fashion are limited. Of course, one possibility is that the scalars are trivial; this occurs in any ordinary linear representation of S_n. According to a result of Schur [8], there is only one other possibility (occurring only when $n \geqslant 4$); namely

$$\sigma_j^2 = -1; \quad (\sigma_j\sigma_k)^2 = -1 \text{ (for } |j-k| \geqslant 2); \quad (\sigma_j\sigma_{j+1})^3 = -1. \tag{1.4.1}$$

All other possibilities can be reduced to this case or the trivial case by a change of scale. See [2], [13] for details.

It is convenient to regard $\sigma_1, \ldots, \sigma_{n-1}$ as elements of an abstract group, and to take 1.4.1 as a set of defining relations. More precisely, for $n \geqslant 1$ let us define $\tilde{S}_n$ to be the group of order $2 \cdot n!$ generated by $\sigma_1, \ldots, \sigma_{n-1}$ (and -1), subject to the relations 1.4.1, along with the obvious relations $(-1)^2 = 1, (-1)\sigma_j = \sigma_j(-1)$ which force -1 to be a central involution. By Schur's Lemma, an irreducible linear representation of $\tilde{S}_n$ must represent -1 by either of the scalars $+1$ or -1. A representation of the former type is a linear representation of S_n, whereas one of the latter type corresponds to a projective representation of S_n as in 1.4.1. We will refer to any representation of $\tilde{S}_n$ in which the group element -1 is represented by the scalar -1 as a *negative representation* of $\tilde{S}_n$.

Next we review the characters of the irreducible negative representations of $\tilde{S}_n$. Define $\mathcal{P}(n)$ to be the set of all partitions of n. We say that a partition λ is *odd* if and only if the number of even parts in λ is odd, and is *even* if and only if it is not odd. Thus, the parity of a permutation agrees with the parity of its cycle type. The parity of λ is also the parity of the integer $|\lambda| + \ell(\lambda)$. Schur showed that the irreducible negative representations are indexed by partitions λ with distinct parts. Recall that if P is an irreducible negative representation indexed by λ that the character $\langle\lambda\rangle$ is a class function $\langle\lambda\rangle : \tilde{S}_n \to \mathbb{Q}$ defined by $\langle\lambda\rangle(g) = \mathrm{trace}(P(g))$.

If $g = \pm\sigma_{i_1}\sigma_{i_2}\cdots$, let $\pi \in \mathcal{P}(n)$ be the cycle type (in S_n) of $\sigma_{i_1}\sigma_{i_2}\cdots$. In the sequel we will evaluate $\langle\lambda\rangle(\pi)$ instead of $\langle\lambda\rangle(g)$. Define $\mathcal{P}^0(n)$ to be all partitions of n such that all parts are odd.

Theorem 1.4.1 (Schur 1911 [8]). *Let $\lambda \in \mathcal{D}(n)$ have length ℓ, and let $\pi \in \mathcal{P}(n)$.*

(a) Suppose that λ is odd. If π is neither in $\mathcal{P}^0(n)$ nor equal to λ then $\langle\lambda\rangle(\pi) = 0$.

(b) Suppose that λ is odd. If π equals λ then

$$\langle\lambda\rangle(\pi) = \pm i^{(n-\ell+1)/2}(\lambda_1\lambda_2\cdots\lambda_\ell/2)^{1/2}.$$

(c) Suppose that λ is even. If π is not in $\mathcal{P}^0(n)$ then $\langle\lambda\rangle(\pi) = 0$.

For example we consider the situation when $n = 6$ and $\lambda = 321$. Then $\langle\lambda\rangle(\pi) = 0$ when π is $(6), (42), (411), (222), (2211)$ or (21111), as these partitions all have one even part. The second fact gives $\langle\lambda\rangle(\pi) = \sqrt{3}$ when $\pi = (321)$. If $\lambda = 51$ then $\langle\lambda\rangle(\pi) = 0$ when π is $(6), (42), (411), (321), (222), (2211)$ or (21111).

A combinatorial rule for calculating the characters not specified by Schur's theorem was given by Morris; it is the projective analogue of the Murnaghan-Nakayama rule.

Theorem 1.4.2 (Morris 1962 [4]). *Let $\lambda \in \mathcal{D}(n)$ have length ℓ. Suppose that $\pi \in \mathcal{P}^0(n)$ and that π contains r at least once. Define $\pi' \in \mathcal{P}^0(n-r)$ by removing a* *copy of r from π. Then*

$$\langle\lambda\rangle(\pi) = \sum_{i\in I(\lambda,r)} n_i\langle\lambda(i,r)\rangle(\pi'),$$

where

$$n_i = \begin{cases} (-1)^{j-i}2^{1-\varepsilon(\lambda)} & \text{if } i \in I_+; \\ (-1)^{\ell-i} & \text{if } i \in I_0; \\ (-1)^{j-i+\lambda_i}2^{1-\varepsilon(\lambda)} & \text{if } i \in I_-. \end{cases}$$

(The integer j is that occurring in the definitions of $I_\pm$, and $\varepsilon(\lambda)$ is the parity of λ; i.e. 0 or 1.)

For example if $n = 6$, $\lambda = (51)$ and $r = 1$, we have $\varepsilon(\lambda) = 0, I_+ = \{1\}, I_0 = \{2\}$ and $I_- = \emptyset$. So $I(\lambda, r) = \{1, 2\}$, and we have

$$\begin{aligned} \langle 51\rangle(1^6) &= (-1)^{1-1}2^{1-0}\langle 41\rangle(1^5) + (-1)^{2-2}\langle 5\rangle(1^5) \\ &= 2\langle 41\rangle(1^5) + \langle 5\rangle(1^5). \end{aligned}$$

We can expand this sum into a sum over all possible bar tableaux. Define the *weight* of a tableau $\text{wt}(T)$ to be the product of all the powers of -1 and 2 which appear. Then we have

$$\langle\lambda\rangle(\pi) = \sum_T wt(T),$$

summed over all bar tableaux of shape λ and type π. We know that the shifted rank of λ is the minimum number of bars needed in a bar tableau of shape λ. So we obtain the following result as a corollary to Theorem 1.3.1:

Corollary 1.4.3. *Given a shape λ of shifted rank k and a shape π such that $\ell(\pi) < k$, we have $\langle\lambda\rangle(\pi) = 0$.* □

1.5 Schur Q-Functions

We begin with Schur's original inductive definition of the Q_λ functions. First define symmetric functions $m_\lambda(x)$ by

$$m_\lambda = \sum_\alpha x^\alpha,$$

where the sum ranges over all *distinct* permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$. Define symmetric functions q_k of degree k by

$$q_k = \sum_{\lambda \in \mathcal{P}(k)} 2^{\ell(\lambda)} m_\lambda.$$

Now we can state the base cases for the inductive definition. Put $Q_{(a)} = q_a$ and

$$Q_{(a,b)} = q_a q_b + 2\sum_{n>0} (-1)^n q_{a+n} q_{b-n}.$$

Inductively we define

$$Q_{\lambda_1,\ldots,\lambda_{2k+1}} = \sum_{i=1}^{2k+1} (-1)^{i+1} q_{\lambda_i} Q_{\lambda_1,\ldots,\hat{\lambda}_i,\ldots,\lambda_{2k+1}}$$

and

$$Q_{\lambda_1,\ldots,\lambda_{2k}} = \sum_{i=2}^{2k} (-1)^i Q_{\lambda_1,\lambda_i} Q_{\lambda_2,\ldots,\hat{\lambda}_i,\ldots,\lambda_{2k}}.$$

The Q_λ may also be defined as the specialization at $t = -1$ of one of the two equivalent defining formulae for Hall-Littlewood polynomials; see [3, III (2.1) (2.2)]. Let S_r act on $X = \{x_1, \ldots, x_r\}$ by permuting the variables, so that, when $\ell \leqslant r$, the Young subgroup $S_1^\ell \times S_{r-l}$ fixes each of $x_1, \ldots, x_\ell$. Let λ be a strict partition of length ℓ. If $\ell \leqslant r$, then

$$Q_\lambda(x_1, \ldots, x_r) = 2^\ell \sum_{[w] \in S_r / S_1^\ell \times S_{r-\ell}} w\left\{ x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell} \prod_{i=1}^{\ell} \prod_{j=i+1}^{r} \frac{x_i + x_j}{x_i - x_j} \right\}.$$

If λ has length greater than r, then $Q_\lambda(x_1, \ldots, x_r) = 0$. The Q_λ symmetric functions are obtained by taking the limit as the number of variables becomes infinite (for a mathematically precise definition of this limit see [3]).

Schur [8] defined these Q-functions in order to study the negative representations of symmetric groups. The fundamental connection is given by the following theorem. Let $m_i(\lambda) = \#\{j : \lambda_j = i\}$, the number of parts of λ equal to i. Define $z_\lambda = 1^{m_1(\lambda)}m_1(\lambda)!2^{m_2(\lambda)}m_2(\lambda)!\cdots$ The p_λ are the *power sum symmetric functions* defined by

$$\begin{aligned} p_n &= \sum_i x_i^n,\ n \geqslant 1 \quad (\text{with } p_0 = 1) \\ p_\lambda &= p_{\lambda_1}p_{\lambda_2}\cdots \end{aligned}$$

Theorem 1.5.1 (Schur, 1911).

$$Q_\lambda = \sum_{\pi \in \mathcal{P}^0(n)} 2^{[\ell(\lambda)+\ell(\pi)+\epsilon(\lambda)]/2}\langle\lambda\rangle(\pi)\frac{p_\pi}{z_\pi}.$$

Again consider the example with $n = 6$ and $\lambda = (51)$. We have

$$\begin{aligned} Q_{51} &= 2^{[2+6+0]/2}\langle 51\rangle(1^6)\frac{p_{1^6}}{z_{1^6}} + 2^{[2+4+0]/2}\langle 51\rangle(1^3 3)\frac{p_{1^3 3}}{z_{1^3 3}} + \\ &\quad 2^{[2+2+0]/2}\langle 51\rangle(15)\frac{p_{15}}{z_{15}} + 2^{[2+2+0]/2}\langle 51\rangle(3^2)\frac{p_{3^2}}{z_{3^2}} \\ &= 2^4 16\frac{p_{1^6}}{6!} + 2^3 2\frac{p_{1^3 3}}{3!3} - 2^2 1\frac{p_{15}}{5} - 2^2 2\frac{p_{3^2}}{18} \\ &= \frac{16}{45}p_{1^6} + \frac{8}{9}p_{1^3 3} - \frac{4}{5}p_{15} - \frac{4}{9}p_{3^2} \end{aligned}$$

Define $\deg(p_i) = 1$, so $\deg(p_\nu) = \ell(\nu)$. Then Theorem 1.3.1 gives us the following corollary.

Corollary 1.5.2. *The terms of lowest degree in Q_λ have degree at least* $\operatorname{srank}(\lambda)$. □

In our example $\operatorname{srank}(51) = 2$ and the p_ν satisfy $\ell(\nu) \geqslant 2$. Equivalently we can examine a specialization of the principal specialization of Q_λ, i.e.

$$ps_t^1(Q_\lambda) = Q_\lambda(\underbrace{1,1,\ldots,1}_{t\ 1's}) = Q_\lambda(1^t).$$

Since $p_\nu(1^t) = t^{\ell(\nu)}$, we can rephrase the above result.

Corollary 1.5.3. *$Q_\lambda(1^t)$ is divisible by $t^{\mathrm{srank}(\lambda)}$.* □

The following conjecture has been computationally verified (using John Stembridge's SF Package for Maple [14]) for all partitions $\lambda \vdash n$ for $1 \leqslant n \leqslant 12$.

Conjecture 1.5.4. *The terms of lowest degree in Q_λ have degree at exactly* $\mathrm{srank}(\lambda)$.

1.6 Number of strips in a skew shifted tableau

In this section we discuss minimal bar tableaux of skew shifted shapes. First we introduce shifted strip tableaux, then we define skew bar tableaux. Finally we give some partial results on the rank of skew shifted tableaux.

In what follows we only consider λ and μ having distinct parts. The j*th diagonal* of a skew diagram $D'_{\lambda/\mu}$ is the collection of squares $(1, j), (2, j+1), (3, j+2), \ldots$ in $D'_{\lambda/\mu}$. The first diagonal (which may be empty) is called the main diagonal.

A skew diagram $D'_{\lambda/\mu}$ is said to be a *strip* if it is rookwise connected and each diagonal has at most one square. The *height* h of a strip is the number of rows it occupies. For example, the squares labelled by 2 in the tableau on the right of Figure 1-5 form a strip of height 3. A *double strip* is a rookwise connected skew diagram formed by the union of two strips which both start on the main diagonal. Note that a double strip can be cut into two nonempty connected pieces-one piece (call it α) consisting of the diagonals of length two, the other piece (β) consisting of the strip formed by the diagonals of length one. The *depth* of a double strip is defined to be $|\alpha|/2 + h(\beta)$. For example, the squares labelled by 2 in the tableau on the left of Figure 1-5 form a double strip of depth 2.

A (skew shifted) *strip tableau* of shape λ/μ and content $\gamma \in \mathcal{P}^0(n)$ is defined to be a nested sequence of shifted diagrams

$$D'_\mu = D'_{\lambda^0} \subseteq D'_{\lambda^1} \subseteq \cdots \subseteq D'_{\lambda^l} = D'_\lambda$$

with $|\lambda^i| - |\lambda^{i-1}| = \gamma_i (1 \leqslant i \leqslant l)$ such that each intermediate diagram $D'_{\lambda^i/\lambda^{i-1}}$ is either a strip or a double strip. We illustrate two strip tableaux in Figure 1-5. Define the weight of a strip of height h to be $(-1)^{h-1}$, and define the weight of a double strip of depth d to be $2(-1)^{d-1}$. The *weight* of a strip tableau S, denoted $\mathrm{wt}(S)$, is the product of the weights of the component strips and double strips. For example the weight of the tableau on the left in Figure 1-5 is $(-1)^{1-1}2(-1)^{2-1} = -2$. The weight of the tableau on the right in Figure 1-5 is $2(-1)^{2-1}(-1)^{3-1} = -2$.

1	1	1
	2	2
		2

1	1	2
	1	2
		2

Figure 1-5: Two strip tableaux of the shape 321

In order to define the skew Q-functions $Q_{\lambda/\mu}$ first define an inner product on the algebra of symmetric functions by

$$[p_\mu, p_\nu] = z_\mu 2^{-\ell(\mu)} \delta_{\mu\nu}.$$

Define integers $f^\lambda_{\mu\nu}$ by

$$f^\lambda_{\mu\nu} = [Q_\lambda, 2^{-\ell(\mu)-\ell(\nu)} Q_\mu Q_\nu].$$

Now we can define

$$Q_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} Q_\nu.$$

This theory parallels that of the Schur functions s_λ and the Littlewood Richardson coefficents $c^\lambda_{\mu\nu}$. Both the $Q_{\lambda/\mu}$ and the $f^\lambda_{\mu\nu}$ can be defined combinatorially, but this involves another family of tableaux which we do not consider here. See [12] for more details.

Now we have enough background to introduce the skew shifted version of the Murnaghan-Nakayama rule, first proved by Morris [4]:

$$Q_{\lambda/\mu} = \sum_S 2^{\ell(\gamma)} wt(S) \frac{p_\gamma}{z_\gamma},$$

summed over all strip tableaux S of shape λ/μ and content γ. To illustrate this rule, we compute the coefficient of p_{3^2} in Q_{321} (of course $Q_{\lambda/\emptyset} = Q_\lambda$). Figure 1-5 in fact shows *all* strip tableaux of shape 321 with content 3^2. We have already computed that the weight of each tableaux is -2. Thus the coefficient of p_{3^2} in Q_{321} is $2\left[2^2(-2)\frac{1}{3^2 2!}\right] = -\frac{8}{9}$.

In order to apply our knowledge of bar tableaux, we need to generalise the definition to apply to the skew case. Define a *skew bar tableau* of shape λ/μ to be an assignment of nonnegative integers to the squares of $S(\lambda)$ such that:

(a) the entries are weakly increasing across rows,

(b) each positive integer i appears an odd number of times,

(c) a positive integer i can appear in at most two rows, and if it does, it must begin both rows (equivalent to the bar being of type 3),

(d) the partition remaining if we remove all squares labelled by integers larger than some i then reorder has distinct parts,

(e) the partition remaining if we remove all squares labelled by positive integers and reorder is μ.

We illustrate for example a skew bar tableau of the shape 86541/821, and the various partitions remaining if we remove all squares labelled by integers larger than some i:

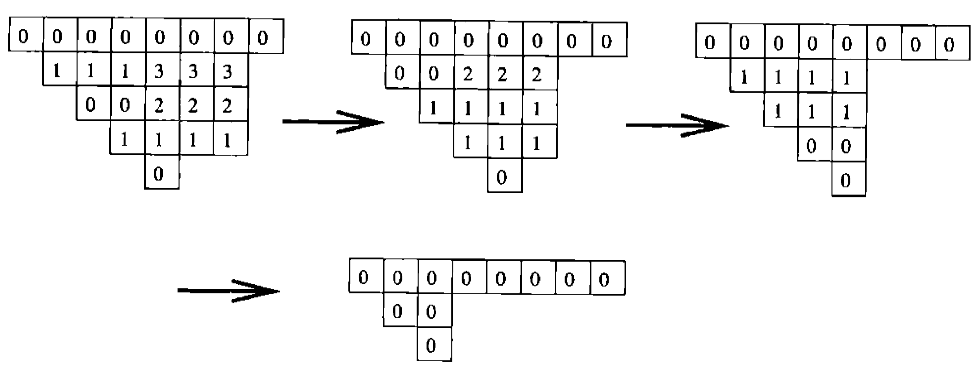


Figure 1-6: Checking legality of the skew bar tableau of the shape 86541

We give a bijection between skew bar tableaux and skew shifted strip tableaux.

Begin with a skew bar tableau. For every bar of type 3, mark the labels of the longer sub bar with an apostrophe. In the sequel, we use the ordering on labels $1' < 1 < 2' < 2 \cdots$. We describe an algorithm ϕ.

let i be the biggest label in the bar tableau.
repeat
  let $(m, n+1)$ be the rightmost square labelled by i.
  repeat
    if there is a square $(m+1, n+1)$ and it is labelled smaller than i **then**
      switch the initial $n-m+1$ labels in row m with the initial $n-m+1$ labels in row $m+1$.
      $m := m+1$
    else
      $n := n-1$
    end if
  until the label in square (m, n) is not i
  let i be the next smallest label
until $i = 0$
remove all squares labelled by 0

This algorithm terminates because after every step of the inner loop there is one fewer square labelled by i to the left of square (m, n). It is also clear that ϕ preserves the content of the tableau. We will refer to the part of the algorithm where we are examining boxes with label i as *step* i. For example we illustrate ϕ applied to the skew bar tableau from Figure 1-6.

After step i of ϕ, denote the tableau remaining if we remove all squares with labels $< i$ by ST_i. Denote the tableau remaining if we remove all boxes with labels $\geqslant i$ by BT_i. Denote the tableau remaining when we remove all squares with labels $\geqslant i$ from the *original* bar tableau and reorder by RT_i. We illustrate with our running example:

Lemma 1.6.1. *For every label j, $BT_j = RT_j$.*

Proof. Assume by way of induction that $BT_{i+1} = RT_{i+1}$. It suffices to show that removing the bar labelled i from BT_{i+1} and reordering is the same operation as applying step i of ϕ to BT_{i+1} and then removing the squares labelled i. Say that the squares labelled by i are in row p before the ith step begins, and say that there are q

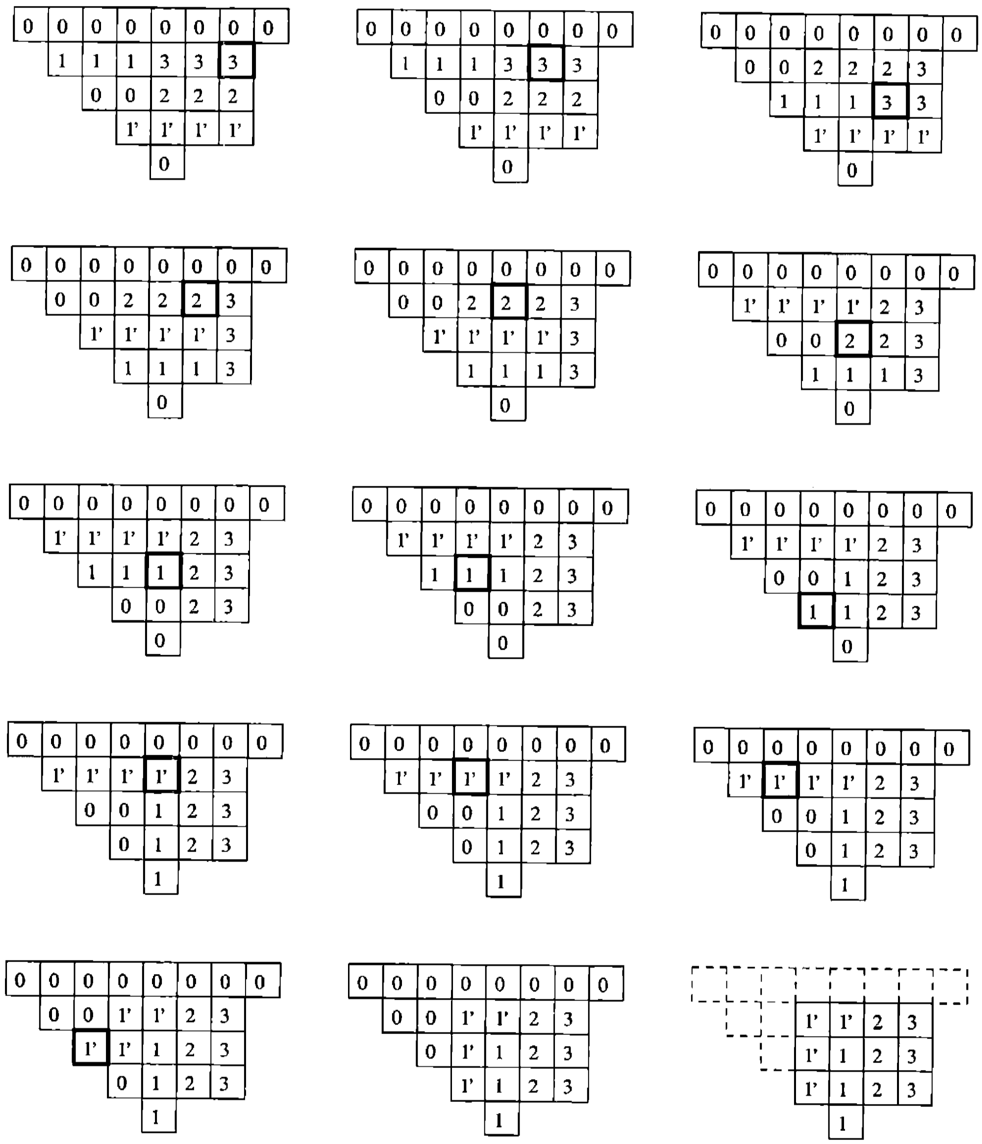
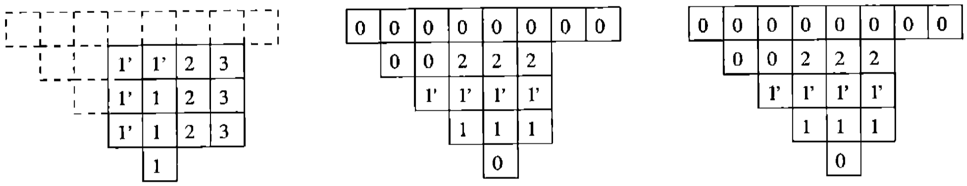


Figure 1-7: Applying ϕ to a bar tableau

squares not labelled by i in row p. Let r be maximal such that $\lambda_r > q$. We show by induction that for $p < j \leqslant r, \phi$ moves the labels of row j up one row and labels the remaining $\lambda_j - \lambda_{j-1}$ squares with i. We show that ϕ leaves the first q labels of row p in row r, and that ϕ does not affect any other rows.

It is clear that the ith step of ϕ does not affect any rows higher than row p. Let j be such that $p < j \leqslant r$. Say we have just reset $m := m + 1 = j$ in ϕ. Assume inductively that row j now contains the first q labels of row p and ends with $\lambda_j - q$ squares labelled i. Also assume inductively that every row k such that $p \leqslant k < j$



$ST_{1'}$ BT_3 RT_3

contains the first λ_{k+1} labels from row $k+1$ and ends with $\lambda_k - \lambda_{k+1}$ squares labelled i.

First suppose that $j < r$, i.e. that $\lambda_{j+1} > q$. ϕ is now examining square $(m,n) = (j, j+\lambda_j - 1)$. ϕ does nothing until it examines square $(m,n) = (j, j+\lambda_{j+1} - 1)$. Then it will switch the labels as required.

Now suppose $j = r$. If $\lambda_{j+1} = q$ then BT_{i+1} would have two equal parts when the squares labelled by i were removed, which is impossible. So $\lambda_{j+1} < q$. ϕ is now examining square $(m,n) = (j, j+\lambda_j - 1)$. ϕ does nothing until it examines square $(m,n) = (j, j+q-1)$ because before then for each (m,n) there is no square $(m+1, n+1)$. But square $(j, j+q-1)$ is not labelled by i, so ϕ will stop, leaving the first q labels of row p in row r as required. □

Since by definition RT_j is a skew bar tableau, it follows that BT_j is a skew bar tableau.

Theorem 1.6.2. *Let T be a skew bar tableau. Then $\phi(T)$ is a skew shifted strip tableau.*

Proof. We prove by induction that ST_j is a skew shifted strip tableau. First we show that after step i the squares labelled by i form a border strip. The squares labelled by i initially form a horizontal bar which is a border strip. During ϕ when we are examining the square (m,n), the square $(m+1,n)$ is also labelled by i. So when we switch the initial $n-m+1$ labels in row m with the initial $n-m+1$ labels in row $m+1$, the strip remains rookwise connected. Clearly no 2×2 squares are introduced by this operation. So inductively after step i the squares labelled by i form a border strip.

After step i the union of this new border strip with ST_{i+1} is just λ/π, where π is the shape occupied by BT_i. Hence the shape ST_i occupies is a legal skew shifted shape, and ST_i is a legal skew shifted strip tableau. □

Given an ST_i and a BT_i which occur during ϕ, it is trivial to invert step i of ϕ and recover uniquely ST_{i+1} and BT_{i+1}. So ϕ is an injection. We describe the exact algorithm to invert ϕ below.

```
create μ_j squares at the start of row j and label them 0
let i be the smallest label in the skew shifted strip tableaux.
repeat
  let (m, n) be the bottom leftmost square labelled by i.
  repeat
    if there is a square (m, n + 1) and it is labelled i then
      n := n + 1
    else if  the label in square (m − 1, n) is equal to i then
      switch the initial n − m + 1 labels in row m with the initial n − m + 1 labels
      in row m − 1.
      m := m − 1
    end if
  until Neither condition was satisfied
  let i be the next biggest label
until i is bigger than the biggest label present in the tableau
```

To show that ϕ is surjective, we must show that given an arbitrary skew shifted strip tableau ST of shape λ/π and an arbitrary skew bar tableau BT of shape π, we can move the squares containing the smallest label of the strip tableau to the bar tableau, and get a legal bar tableau. The only property that needs to be checked is that removing all parts bigger than some j from the bar tableau leaves distinct parts. However if $j < i$ the parts remaining are the same as they were before in BT and so must be distinct.

So suppose $j \geqslant i$, that the lowest square labelled i in ST is in row r, and that the top rightmost square labelled i is in position (m, n). Thus row m of π has length at most $n - m$. The label in square $(m-1, n)$ is smaller than i, and so row $m-1$ of π has length at least $n-m+2$. The bottom leftmost square labelled by i is in position $(r, \pi_r + r)$ so there are $q = r - m + n - \pi_r - r + 1 = n - m - \pi_r + 1$ squares labelled by i.

Remove all parts bigger than j from the updated bar tableau. The remaining parts are the rows of π (without row r) which are distinct because they were before, and also a new row $\pi_r + q = n - m + 1$. But row m and necessarily any lower rows of π have length at most $n - m$, and row $m - 1$ and necessarily any higher rows of π have length at least $n - m + 2$, so this new row is distinct from all the others as required. So we have shown that ϕ is a bijection.

Lemma 1.6.3. *Given a skew bar tableau of λ, let o_r (and e_r) be the number of odd (even respectively) rows of λ which do not have any square labelled 0. Let o_s (and e_s) be the number of rows of λ which do have a square labelled 0 but end with an odd (even respectively) number of squares not labelled 0. Then the minimal number of (nonzero) bars in the tableau is*

$$o_s + 2e_s + \max(o_r, e_r + (e_r + o_r \bmod 2))).$$

Proof. Rows which are counted by none of o_s, e_s, o_r or e_r are entirely labelled with zeroes and so have no nonzero bars. The rows counted by o_s and e_s contain only bars of type 1 because the first square is labelled zero. If a row counted by o_s contained more than one (nonzero) label, we could relabel with some new large integer and save a strip. So these rows contribute exactly o_s bars to the total minimum. Rows counted by e_s cost at least 2 nonzero labels (since each label occurs an odd number of times), and if they contained more we could relabel the first square with some new large label, and the remainder with some other new large integer, saving a bar. So these rows contribute $2e_s$ to the total minimum.

Finally, rows counted by o_r and e_r can be thought of as a non skewed shape, independent of the other rows. From Theorem 1.3.1 we know that we need at least $\max(o_r, e_r + (e_r + o_r \bmod 2)))$ bars to fill these rows. So the total minimum is $o_s + 2e_s + \max(o_r, e_r + (e_r + o_r \bmod 2)))$ as required. □

Thus we can count the minimal number of bars required in a skew bar tableau when the squares labelled 0 are fixed. However varying the locations of the zeros will vary the minimal number of bars required in the remainder; finding the overall minimum (or rank) is an open problem.

Chapter 2

Straight Shapes

After recalling some necessary background, we analyse the bottom Schur functions and derive an expression for them as a minor of the Jacobi-Trudi matrix. Finally we show that this minor is itself the Jacobi-Trudi matrix for some skew shape, and that this skew shape has useful properties. Then we apply these results to give a basis for the space spanned by the bottom Schur functions. This gives us the dimensions of the spaces spanned by the bottom Schur functions, which turns out to be a well known classical sequence.

2.1 Definitions

In this section we define the bottom Schur functions and give some other related definitions. First we express the Schur functions s_λ in terms of the power sum symmetric functions p_ν.

Let λ be a partition of n with Frobenius rank k. Recall that k is the length of the main diagonal of the diagram of λ, or equivalently, the largest integer i for which $\lambda_i \geqslant i$. As before let $m_i(\lambda) = \#\{j : \lambda_j = i\}$, the number of parts of λ equal to i. Define $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$. A *border strip* (or *rim hook* or *ribbon*) is a connected skew shape with no 2×2 square. An example is 75443/4332 whose diagram is illustrated in Figure 2-1. Define the *height* $\mathrm{ht}(B)$ of a border strip B to be one less than its number of rows.

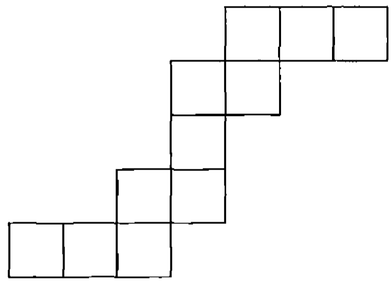


Figure 2-1: The border strip 75443/4332

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n. Define a *border strip tableau* of shape λ and type α to be an assignment of positive integers to the squares of λ such that:

(a) every row and column is weakly increasing,

(b) the integer i appears α_i times, and

(c) the set of squares occupied by i forms a border strip.

Equivalently, one may think of a border-strip tableau as a sequence $\emptyset = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \lambda^r \subseteq \lambda$ of partitions such that each skew shape λ^i/λ^{i+1} is a border-strip of size α_i. For instance, Figure 2-2 shows a border strip tableau of 53321 of type $(7, 0, 3, 1, 3)$. It

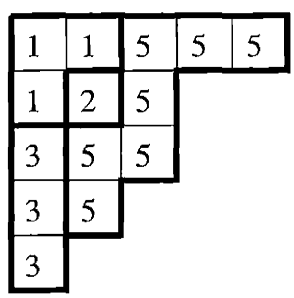


Figure 2-2: A border strip tableau of 53321 of type $(7, 0, 3, 1, 3)$

is easy to see (in this nonskew case) that the smallest number of strips in a border-strip tableau is $\mathrm{rank}(\lambda)$. Define the *height* $\mathrm{ht}(T)$ of a border-strip tableau T to be

$$\mathrm{ht}(T) = \mathrm{ht}(B_1) + \mathrm{ht}(B_2) + \cdots + \mathrm{ht}(B_k)$$

where $B_1, \ldots, B_k$ are the (nonempty) border strips appearing in T. In the example we have $\mathrm{ht}(T) = 1 + 0 + 2 + 3 = 6$. Now we can define

$$\chi^\lambda(\nu) = \sum_T (-1)^{\mathrm{ht}(T)},$$

summed over all border-strip tableaux of shape λ and type ν. Since there are at least $\mathrm{rank}(\lambda)$ strips in every tableau, we have that $\chi^\lambda(\nu) = 0$ if $\ell(\nu) < \mathrm{rank}(\lambda)$.

Finally we can express the Schur function s_λ in terms of power sums p_ν

$$s_\lambda = \sum_\nu \chi^\lambda(\nu) \frac{p_\nu}{z_\nu}.$$

As we saw in Chapter 1 the coefficients $\chi^\lambda(\nu)$ for $\lambda, \nu \vdash n$ have a fundamental algebraic interpretation: They are the values of the irreducible (ordinary) characters of the symmetric group S_n. More precisely, the irreducible characters χ^λ are indexed in a natural way by partitions $\lambda \vdash n$, and $\chi^\lambda(\nu)$ is the value of χ^λ at an element $\omega \in S_n$ of cycle type ν.

Define $\deg(p_i) = 1$, so $\deg(p_\nu) = \ell(\nu)$. The bottom Schur function $\hat{s}_\lambda$ is defined to be the lowest degree part of s_λ, so

$$\hat{s}_\lambda = \sum_{\nu : \ell(\nu) = \mathrm{rank}(\lambda)} \chi^\lambda(\nu) \frac{p_\nu}{z_\nu}.$$

Also write $\tilde{p}_i = \frac{p_i}{i}$. For instance,

$$s_{321} = \frac{1}{45} p_1^6 - \frac{1}{9} p_3 p_1^3 + \frac{1}{5} p_1 p_5 - \frac{1}{9} p_3^2.$$

Hence

$$\begin{aligned} \hat{s}_{321} &= \frac{1}{5} p_1 p_5 - \frac{1}{9} p_3^2 \\ &= \tilde{p}_1 \tilde{p}_5 - \tilde{p}_3^2. \end{aligned}$$

Let e be an edge of the lower envelope of λ, i.e. no square of λ has e as its upper or left-hand edge. We will define a certain subset S_e of squares of λ, called a *snake.* If e is horizontal and (i,j) is the square of λ having e as its lower edge, define

$$\begin{aligned} S_E &= (\lambda) \cap \{(i,j),(i-1,j),(i-1,j-1), \\ &\quad (i-2,j-1),(i-2,j-2),\ldots\}. \end{aligned} \tag{2.1.1}$$

If e is vertical and (i,j) is the square of λ having e as its right-hand edge, define

$$\begin{aligned} S_E &= (\lambda) \cap \{(i,j),(i,j-1),(i-1,j-1), \\ &\quad (i-1,j-2),(i-2,j-2),\ldots\}. \end{aligned} \tag{2.1.2}$$

In Figure 2-3 the nonempty snakes of the shape 533322 are shown with dashed paths through their squares, with a single bullet in the two snakes with just one square. The *length* $\ell(S)$ of a snake S is one fewer than its number of squares; a snake of length $i-1$ (so with i squares) is call an *i-snake.* Call a snake of even length a *right snake* if it has form 2.1.1 and a *left snake* if it has form 2.1.2. It is clear that the snakes are linearly ordered from lower left to upper right. In this linear ordering, replace a left snake with the symbol L, a right snake with R, and a snake of odd length with O. The resulting sequence (which does *not* determine λ) is called the *snake sequence* of λ, denoted $\mathrm{SS}(\lambda)$. For instance, from Figure 2-3 we see that

$$\mathrm{SS}(533322) = LLOOLORROOR.$$

Lemma 2.1.1. *The L's in the snake sequence correspond exactly to horizontal edges of the lower envelope of λ which are below the line $x+y=0$. The R's correspond exactly to vertical edges of the lower envelope of λ which are above the line $x+y=0$. All other edges of the lower envelope of λ are labelled by O's.*

Clearly we could have defined the snake sequence this way, however the definitions above apply to skew shapes also. This Lemma only holds when λ is a straight shape.

Proof. Let e be an edge of the lower envelope of λ below the line $x+y=0$. Let (i,j)

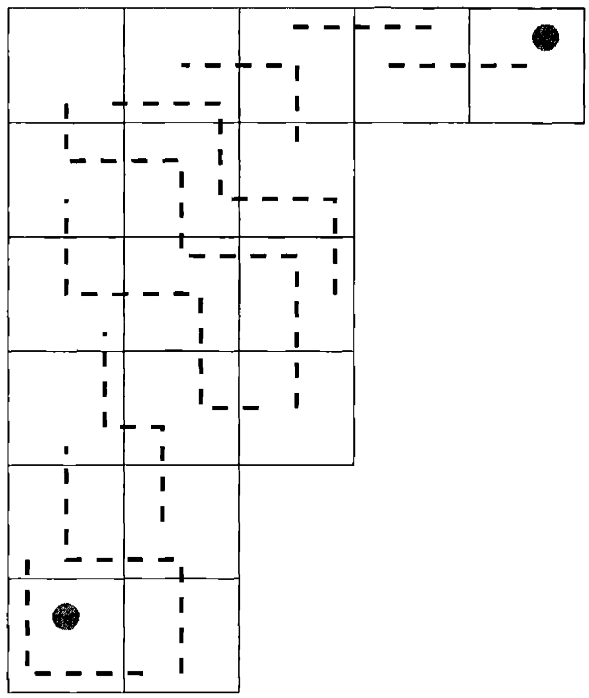


Figure 2-3: Snakes for the shape 533322

be the square of λ having e as its lower edge. The last square in the snake is some square in the first column of λ. So if e is horizontal the last square is $(i-j+1,1)$, the snake has an odd number of squares, so has even length, and is labelled by L. If e is vertical the last square is $(i-j,1)$, the snake has an even number of squares, so has odd length, and is labelled by R. The case when e is above $x+y=0$ is proved similarly. □

Corollary 2.1.2. *In the snake sequence of λ, the L's occur strictly to the left of the R's.* □

The number of horizontal edges of the lower envelope of λ which are below the line $x+y=0$ equals the length of the main diagonal of the diagram of λ, which is the rank of λ. Similarly the number of vertical edges of the lower envelope of λ which are above the line $x+y=0$ also equals the rank of λ. Henceforth we fix $k=\text{rank}(\lambda)$.

Let $\text{SS}(\lambda)=q_1q_2\cdots q_m$, and define an *interval set* of λ to be a collection $\mathcal{I}$ of k ordered pairs,

$$\mathcal{I}=\{(u_1,v_1),\ldots,(u_k,v_k)\},$$

satisfying the following conditions:

(a) the u_i's and v_i's are all distinct integers,

(b) $1 \leqslant u_i < v_i \leqslant m$,

(c) $q_{u_i} = L$ and $q_{v_i} = R$.

Figure 2-4 illustrates the interval set $\{(1,11),(2,7),(5,8)\}$ of the shape 533322.

Figure 2-4: An interval set of the shape 533322

Define the *crossing number* $c(\mathcal{I})$ of an interval set $\mathcal{I} = \{(u_1,v_1),\ldots,(u_k,v_k)\}$ to be the number of *crossings* of $\mathcal{I}$, i.e. the number of pairs (i,j) for which $u_i < u_j < v_i < v_j$.

Let $\boldsymbol{T}$ be a border strip tableau of shape λ. Recall that

$$\mathrm{ht}(\boldsymbol{T}) = \sum_B \mathrm{ht}(B),$$

where B ranges over all border strips in $\boldsymbol{T}$ and $\mathrm{ht}(B)$ is one less than the number of rows of B. Define $z(\lambda)$ to be the height $\mathrm{ht}(\boldsymbol{T})$ of a "greedy border strip tableau" $\boldsymbol{T}$ of shape λ obtained by starting with λ and successively removing the largest possible border strip. (Although $\boldsymbol{T}$ may not be unique, the set of border strips appearing in $\boldsymbol{T}$ is unique, so $\mathrm{ht}(\boldsymbol{T})$ is well-defined.)

The connection between bottom Schur functions and interval sets was given by Stanley [11, Theorem 5.2]:

$$\hat{s}_\nu = (-1)^{z(\nu)} \sum_{\mathcal{I}=\{(u_1,v_1),\ldots,(u_k,v_k)\}} (-1)^{c(\mathcal{I})} \prod_{i=1}^{k} \tilde{p}_{v_i-u_i},$$

where $\mathcal{I}$ ranges over all interval sets of ν.

For example the shape 321 has snake sequence $LOLROR$. There are two interval sets, $\{(1,4),(3,6)\}$ with crossing number 1, and $\{(1,6),(3,4)\}$ with crossing number

0. So as we saw before

$$\hat{s}_{321} = \tilde{p}_1\tilde{p}_5 - \tilde{p}_3^2.$$

2.2 Bottom Schur Functions of straight shapes

In this section we first give a lemma about the reverse lexicographical order on snake sequences which will be used later. Then we analyse the bottom Schur functions and derive an expression for them as a minor of the Jacobi-Trudi matrix. Finally we show that this minor is itself the Jacobi-Trudi matrix for some skew shape, and that this skew shape has useful properties.

Lemma 2.2.1. *The lexicographic order on shapes ν whose length $\ell(\nu)$ equals their rank k is equal to the reverse lexicographical order on their snake sequences.*

Proof. Since $\ell(\nu) = k$, the snake sequence begins with k L's. If the length of the ith row of ν is $k + j$, then there are j O's to the left of the $(k - i + 1)$st R. □

Define the *complete homogeneous symmetric functions* h_λ for $\lambda \in \mathcal{P}$ by the formula

$$\begin{aligned} h_n &= \sum_{i_1 \leqslant \cdots i_n} x_{i_1} \cdots x_{i_n}, \quad n \geqslant 1 \quad (\text{with } h_0 = 1) \\ h_\lambda &= h_{\lambda_1} h_{\lambda_2} \cdots \quad \text{if } \lambda = (\lambda_1, \lambda_2, \ldots). \end{aligned}$$

We will use the fact that

$$h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$$

The h_λ give us a determinantal expression for the s_λ, the Jacobi-Trudi identity:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1}^n,$$

where we define $h_i = 0$ for $i < 0$. For example

$$s_{554421} = \det \begin{bmatrix} h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} \\ h_4 & h_5 & h_6 & h_7 & h_8 & h_9 \\ h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0 & 0 & 1 & h_1 & h_2 & h_3 \\ 0 & 0 & 0 & 0 & 1 & h_1 \end{bmatrix}.$$

Since $h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$, the term of lowest degree (in p) in the expansion of a given h_n in terms of the p_j is just $\frac{p_n}{n} = \tilde{p}_n$. For a product $h_{n_1} h_{n_2} \cdots h_{n_j}$ the term of lowest degree in the expansion in terms of the p_j is just $\tilde{p}_{n_1} \tilde{p}_{n_2} \cdots \tilde{p}_{n_j}$. So we have that $\hat{s}_\lambda =$ terms of lowest order in $\det(\tilde{p}_{\lambda_i - i + j})_{i,j=1}^n$ (since the p_λ are algebraically independent, and since $\det(h_{\lambda_i - i + j}) = s_\lambda \neq 0$, this determinant will not vanish). For example

$$\hat{s}_{554421} = \text{ terms of lowest order in } \det \begin{bmatrix} \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_7 & \tilde{p}_8 & \tilde{p}_9 & \tilde{p}_{10} \\ \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_7 & \tilde{p}_8 & \tilde{p}_9 \\ \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_7 \\ \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_6 \\ 0 & 0 & 1 & \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 \\ 0 & 0 & 0 & 0 & 1 & \tilde{p}_1 \end{bmatrix}.$$

Since $p_0 = 1$, the terms of lowest order are those which contain the most number of 1's.

Row i of the matrix will have a 1 in position (i, j) if $\lambda_i - i + j = 0$, i.e. if $\lambda_i < i$ (this shows that the number of rows of JT_λ which do not contain a 1 is another definition of $\text{rank}(\lambda)$).

Let JT_p^* be the matrix obtained from the original Jacobi-Trudi matrix by removing every row and column which contains a 1 and replacing the h_i with $\tilde{p}_i$. We show below that this matrix is not singular and so we have

$$\hat{s}_\lambda = \det\ JT_p^*.$$

For example

$$\hat{s}_{554421} = \det \begin{bmatrix} \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_8 & \tilde{p}_{10} \\ \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_7 & \tilde{p}_9 \\ \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_5 & \tilde{p}_7 \\ \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_4 & \tilde{p}_6 \end{bmatrix}.$$

Any minor of the Jacobi-Trudi matrix for a shape λ is the Jacobi-Trudi matrix for some skew shape μ/σ. For let JT^* be some minor of size m of some Jacobi-Trudi matrix JT. If the entry in position (i,j) is h_x put $jt^*_{i,j} = x$. Now we can set

$$\sigma_i = jt^*_{1,m} - jt^*_{1,i} - m + i,$$

and

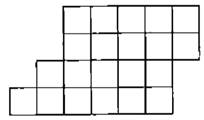
$$\mu_i = jt^*_{i,i} + \sigma_i.$$

Again note that since the p_λ are algebraically independent and $\det JT^* = s_{\mu/\sigma} \neq 0$, we have $\det JT_p^* \neq 0$.

In our running example, we have $\sigma_1 = 10 - 5 - 4 + 1 = 2, \sigma_2 = 10 - 6 - 4 + 2 = 2, \sigma_3 = 10 - 8 - 4 + 3 = 1$ and $\sigma_4 = 10 - 10 - 4 + 4 = 0$. Hence $\sigma = (2210)$. Also $\mu_1 = 5 + 2, \mu_2 = 5 + 2, \mu_3 = 5 + 1$ and $\mu_4 = 6 + 0$. Thus $\mu = (7766)$. Therefore we have that $\hat{s}_{554421}$ equals the determinant of the Jacobi-Trudi matrix of (7766/2210) with the h's replaced by $\tilde{p}$'s.

Lemma 2.2.2. *If the skew shape μ/σ has the Jacobi-Trudi matrix JT^* obtained by removing all rows and columns with a 1 from a Jacobi-Trudi matrix JT of a shape λ with rank k, then μ/σ contains a square of size k.*

The rank of 554421 is 4, and the diagram of (7766/2210) does indeed contain a square of size 4:



Proof. We show that $\mu_k \geqslant k+\sigma_1$. JT^* is a matrix of size k, so from the observations above $\mu_k = jt^*_{k,k}$. The rows of JT^* are the first k rows of JT, so $jt^*_{k,k} = jt_{k,i}$ for some i. Any columns of JT past the $\ell(\lambda)$th will have a 1 in them and be removed. The $\ell(\lambda)$th column will not have a 1 and so must be the last column of JT^*. So we have

$$\mu_k = jt^*_{k,k} = jt_{k,\ell(\lambda)} = \lambda_k - k + \ell(\lambda).$$

Also $\sigma_1 = jt^*_{1,k} - jt^*_{1,1} - k + 1$. The first column of JT does not contain a 1, so it must also be the first column of JT^*. We know the kth column of JT^* is the $\ell(\lambda)$th column of JT. So

$$\begin{aligned} \sigma_1 &= jt^*_{1,k} - jt^*_{1,1} - k + 1 \\ &= jt_{(1,\ell(\lambda))} - jt_{(1,1)} - k + 1 \\ &= \lambda_1 - 1 + \ell(\lambda) - \lambda_1 - k + 1 \\ &= \ell(\lambda) - k. \end{aligned}$$

It remains to show that

$$\lambda_k - k + \ell(\lambda) \geqslant \ell(\lambda) - k + k.$$

This is true because λ has rank k. □

2.3 The space spanned by the bottom Schur functions

In this section we use the previous results to give a basis for the space spanned by the bottom Schur functions. First we recall some classical tableaux theory.

If $\lambda \vdash n$, a *standard Young tableau* (SYT) is a labelling of the squares of λ with the numbers $1, 2, \ldots, n$, each number appearing once, so that every row and column is increasing. A *semistandard Young tableau* (SSYT) is a labelling of the squares of λ with positive integers that is weakly increasing in every row and strictly increasing in every column. We say that T has *type* $\alpha = (\alpha_1, \alpha_2, \ldots)$ if T has α_i parts equal to i.

1	3	5	6
2	7	8	9
4			

SYT

1	1	3	3
2	4	4	4
5			

SSYT

Now we define an operation on standard Young tableaux called a *jeu de taquin slide*. This was invented by M. P. Schützenberger. Given a skew shape λ/μ, consider the squares b that can be added to λ/μ, so that b shares at least one edge with λ/μ, and $\{b\} \cup \lambda/\mu$ is a valid skew shape. Suppose that b_0 shares a lower or right edge with λ/μ (the other situation is completely analogous). There is at least one square b_1 in λ/μ that is adjacent to b_0; if there are two such squares, then let b_1 be the one with a smaller entry. Move the entry occupying b_1 into b_0. Then repeat this procedure, starting at b_1. The resulting tableau will be a standard Young tableau. Analogously if b_0 shares an upper or left edge, the operation is the same except we let b_1 be the square with the bigger entry from two possibilities. For example we illustrate both situations; the tableau on the right results from playing jeu de taquin beginning at the square marked by a bullet on the tableau on the left (and vice versa).

●	1	5	6
2	3	8	9
4	7		

1	3	5	6
2	7	8	9
4	●		

Figure 2-5: Jeu de taquin slides

Two tableaux T and T' are called *jeu de taquin equivalent* if one can be obtained from another by a sequence of jeu de taquin slides.

The *reading word* of a (semi)standard Young tableau is the sequence of entries of T obtained by concatenating the rows of T bottom to top. For example, the tableau on the left in Figure 2-5 has the reading word 472389156. The *reverse reading word* of a tableau is simply the reading word read backwards; so the tableau on the left in Figure 2-5 has the reverse reading word 651983274.

A *lattice permutation* is a sequence $a_1a_2\cdots a_n$ such that in any initial factor $a_1a_2\cdots a_j$, the number of i's is at least as great as the number of $i+1$'s (for all i). For example 123112213 is a lattice permutation.

The *Littlewood-Richardson coefficients* $c^{\lambda}_{\mu\nu}$ are the coefficients in the expansion of a skew Schur function in the basis of Schur functions:

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

The Littlewood-Richardson rule is a combinatorial description of the coefficients $c^{\lambda}_{\mu\nu}$. We will use two different versions of the rule.

Theorem 2.3.1. *Fix an SYT P of shape ν. The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of SYT of shape λ/μ that are jeu de taquin equivalent to P.*

P=

1	2	3
4	5	6
7	8	

For example, let $\lambda = (5,3,3,1), \mu = (3,1)$, and $\nu = (3,3,2)$. Consider the tableau P of shape ν. There are exactly two SYTs T of shape λ/μ such that $jdt(T) = P$, namely,

			2	3
	1	6		
4	5	8		
7				

and

			2	3
	5	6		
1	7	8		
4				

Theorem 2.3.2. *The Littlewood-Richardson coefficient* $c^{\lambda}_{\mu\nu}$ *is equal to the number of semistandard Young tableaux of shape* λ/μ *and type* ν *whose reverse reading word is a lattice permutation.*

For example, with $\lambda = (5,3,3,1), \mu = (3,1)$, and $\nu = (3,3,2)$ as above, there are exactly two SSYTs T of shape λ/μ and type ν whose reverse reading word is a lattice permutation:

			1	1
	1	2		
2	2	3		
3				

			1	1
	2	2		
1	3	3		
2				

Now we have enough machinery to state and prove this chapter's main theorem.

Theorem 2.3.3. *Fix* n *and* k. *The set* $\{\hat{s}_\nu : \nu \vdash n,\ \mathrm{rank}(\nu) = k \text{ and } \ell(\nu) = k\}$ *is a basis for the space* $\mathrm{span}_{\mathbb{Q}}\{\hat{s}_\lambda : \lambda \vdash n \text{ and } \mathrm{rank}(\lambda) = k\}$.

For example if $n = 12$ and $k = 3$, we have that $\{\hat{s}_{633}, \hat{s}_{543}, \hat{s}_{444}\}$ is a basis for $\mathrm{span}_{\mathbb{Q}}\{\hat{s}_{633}, \hat{s}_{543}, \hat{s}_{5331}, \hat{s}_{444}, \hat{s}_{4431}, \hat{s}_{4332}, \hat{s}_{43311}, \hat{s}_{3333}, \hat{s}_{33321}, \hat{s}_{333111}\}$.

Proof. First we prove that the $\hat{s}_\nu$ are linearly independent. We show that given any such ν, there is some term in the expansion of $\hat{s}_\nu$ which does not occur in the expansion of any $\hat{s}_{\nu'}$ for ν' lexicographically less than ν.

From Stanley [11, Theorem 5.2] we have that

$$\hat{s}_\nu = (-1)^{z(\nu)} \sum_{\mathcal{I} = \{(u_1,v_1),\ldots,(u_k,v_k)\}} (-1)^{c(\mathcal{I})} \prod_{i=1}^{k} \tilde{p}_{v_i - u_i},$$

where $\mathcal{I}$ ranges over all interval sets of ν. Let $t = p_{j_1 \geqslant \cdots \geqslant j_k}$ be the term corresponding to the non crossing interval set $\mathcal{I}$ of the snake sequence of ν. We claim that t does not occur in the expansion of any $\hat{s}_{\nu'}$ for ν' lexicographically less than ν. Assume by way of contradiction that it does occur for some such ν' with corresponding interval set $\mathcal{I}'$.

Assume inductively that the first $i-1$ L's are matched with the last $i-1$ R's without crossings in $\mathcal{I}'$. Let r_j (and r'_j respectively) be the position of the jth R in the snake sequence of $\nu(\nu'$ respectively). By lemma 2.2.1 $r_j \geqslant r'_j$. But the length of the interval matching the ith R from the right in $\mathcal{I}$ is $r_{k-i+1} - i$. So for there to be an interval of this length in $\mathcal{I}'$ we must match the ith R from the right with the ith L; this interval has no crossing. Proceeding by induction we see that $\mathcal{I}'$ is also non crossing, and so must equal $\mathcal{I}$. This shows that the snake sequences corresponding to ν and ν' are equal, and so $\nu = \nu'$, a contradiction.

Now we prove that the $\hat{s}_\nu$ span the space of all $\hat{s}_\lambda$. We have shown that $\hat{s}_\lambda = \hat{s}_{\mu/\sigma}$. Expand $s_{\mu/\sigma}$ in terms of (straight) Schur functions using the Littlewood Richardson rule

$$s_{\mu/\sigma} = \sum_\nu c^\mu_{\sigma\nu} s_\nu \ .$$

We need to show that $c^\mu_{\sigma\nu} = 0$ unless ν is of rank k and length k.

Fix an SYT P of shape ν. The Littlewood-Richardson coefficient $c^\mu_{\sigma\nu}$ is equal to the number of SYT of shape μ/σ that are jeu de taquin equivalent to P. Playing jeu de taquin on a straight-shape tableau of shape ν can only increase the length of the shape. Hence if $c^\mu_{\sigma\nu} \neq 0, \ell(\nu) \leqslant k$.

The Littlewood-Richardson coefficient $c^\mu_{\sigma\nu}$ is also equal to the number of semistandard Young tableaux of shape μ/σ and type ν whose reverse reading word is a lattice permutation. But we know that μ/σ contains a square of size k (by Lemma 2.2.2). Therefore the bottom row of this square must be filled with k 'k's, and so $\nu_k \geqslant k$, i.e. $\mathrm{rank}(\nu) \geqslant k$. Since $\ell(\nu) \leqslant k$, we must have $\mathrm{rank}(\nu) = k$.

Taking terms of lowest degree on both sides of

$$s_{\mu/\sigma} = \sum_\nu c^\mu_{\sigma\nu} s_\nu \ ,$$

we have that

$$\hat{s}_\lambda = \hat{s}_{\mu/\sigma} = \sum_\nu c^\mu_{\sigma\nu} \hat{s}_\nu \ ,$$

where the sum is over ν of length k and rank k as required. □

2.4 Dimension of the space spanned by the bottom Schur functions

In this section we derive the dimension of the space spanned by the bottom Schur functions, and show that the sequence of dimensions is a well known classical sequence.

Let $p_{\leqslant k}(n)$ be the number of partitions of n with length at most k, and define $p_{\leqslant k}(0) = 1$.

Corollary 2.4.1. *The dimension of the space of bottom Schur functions* $\mathrm{span}_{\mathbb{Q}}\{\hat{s}_\lambda : \lambda \vdash n\}$ *is*

$$\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} p_{\leqslant k}(n-k^2).$$

For example, the first 27 terms in this sequence are

$$1, 1, 1, 2, 2, 3, 3, 4, 5, 6, 7, 9, 10, 12, 14, 17, 19, 23, 26, 31, 35, 41, 46, 54, 61, 70, 79.$$

There is a nice bijection between the above partitions and the set of partitions $\{\lambda \vdash n : \lambda_i - \lambda_{i+1} \geqslant 2\}$. For, given a k and a partition $\lambda^* \vdash n - k^2$ with fewer than k parts, we can set $\lambda_i = \lambda_i^* + 2k - 2i + 1$. This gives a partition of n with k rows with $\lambda_i - \lambda_{i+1} \geqslant 2$ as required. This is clearly a bijection.

This classical sequence also gives the number of partitions of n into parts $5k+1$ or $5k-1$; equivalently these numbers are the coefficients in the expansion of the Rogers-Ramanujan identity

$$1 + \sum_{n\geqslant 1} \frac{t^{n^2}}{(1-t)(1-t^2)\cdots(1-t^n)} = \prod_{n\geqslant 1} \frac{1}{(1-t^{5n-1})(1-t^{5n-4})}$$

2.5 2 bottom Schur functions

We have shown that a basis for the space spanned by the bottom Schurs consists of the $\hat{s}_\lambda$ where $\ell(\lambda) \leqslant \mathrm{rank}(\lambda)$. Thus is it natural to consider those λ for which $\ell(\lambda) \leqslant \mathrm{rank}(\lambda) + 1$. The number of such partitions of n coincides with the dimension of the space of double bottom Schurs for $n < 15$ (using John Stembridge's SF Package for Maple [14]). In particular, this sequence is $1, 2, 2, 3, 4, 6, 7, 9, 11, 14, 17, 22, 26, \ldots$. This suggests:

Conjecture 2.5.1 (Stanley). *A basis for the space spanned by the bottom 2 degree terms of all Schur functions consists of all 2 bottom Schurs $\hat{\hat{s}}_\lambda$, where λ is a partition of n satisfying $\ell(\lambda) \leqslant rank(\lambda) + 1$.*

The clear generalisation is to look at the k bottom Schurs. However in the $k = 3$ case, the dimensions of the spaces spanned by the 3 bottom Schurs are $1, 2, 3, 4, 6, 9, 11, 15, 19, 24, 30, \ldots$. The numbers of partitions λ of n satisfying $\ell(\lambda) \leqslant \mathrm{rank}(\lambda) + 2$ are $1, 2, 3, 4, 5, 8, 10, 14, 17, 22, 27, \ldots$. Unfortunately these are two different sequences.

Chapter 3

A Symmetric Function Identity

Having introduced some definitions related to interval sets, we prove several lemmas about the properties of labelled interval sets. We use these results to give for each shape λ a relation between the power sum symmetric functions and the monomial symmetric functions. The functions which occur are those indexed by the type μ of any minimal bar tableau of λ.

3.1 Definitions

In this section we introduce some definitions related to interval sets.

Fix a shape λ of rank k. Given an interval set $\mathcal{I} = \{(u_1, v_1), \ldots, (u_k, v_k)\}$, of λ and a labelling of the intervals $(\alpha_1, \ldots, \alpha_k)$ such that $\alpha_i \in \mathbb{P}$, define

$$x^{\mathcal{I}} = \prod_{i=1}^{k} x_{\alpha_i}^{v_i - u_i}.$$

Recall that $c(\mathcal{I})$ is the number of crossings of the interval set $\mathcal{I}$. For instance Figure 3-1 shows a labelled interval set of the shape 533322 with the snake sequence *LLOOLORROOR*. For this interval set $c(\mathcal{I}) = 1$ and for this labelling $x^{\mathcal{I}} = x_4^{10} x_2^5 x_4^3 = x_2^5 x_4^{13}$.

Figure 3-1: A labelled interval set of the shape 533322

3.2 Labelled Interval Sets

In this section we prove several lemmas about the properties of labelled interval sets.

Example 3.2.1. For the shape $\lambda = (4, 4, 4)$ with snake sequence $LLLORRR$, Figure 3-2 shows some labelled interval sets. In the top left we have $(-1)^{c(\mathcal{I})}x^{\mathcal{I}} = (-1)^3 x_a^4 x_a^4 x_b^4 = -x_a^8 x_b^4$. In the top right we have $(-1)^{c(\mathcal{I})}x^{\mathcal{I}} = (-1)^2 x_a^5 x_a^3 x_b^4 = x_a^8 x_b^4$. In fact in every row the term $(-1)^{c(\mathcal{I})}x^{\mathcal{I}}$ in the left column is exactly the negative of the corresponding term $(-1)^{c(\mathcal{I})}x^{\mathcal{I}}$ in the right column.

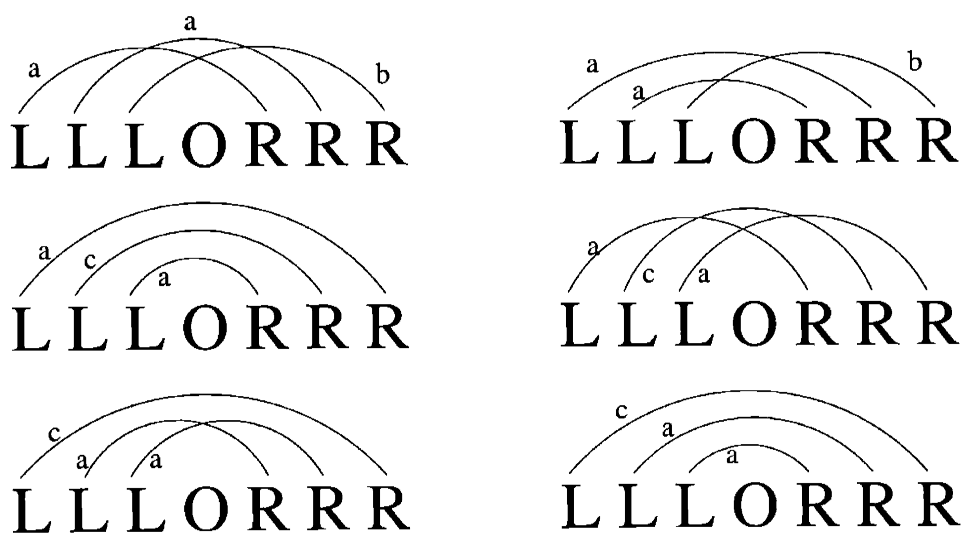


Figure 3-2: Some labelled interval sets of the shape 444

Lemma 3.2.1. *Fix a shape λ. Then $\sum(-1)^{c(\mathcal{I})}x^{\mathcal{I}} = 0$, where the sum is over all labelled interval sets of λ such that the labels are not distinct.*

Proof. We give a sign reversing involution on these labelled interval sets. Examine a specific labelled interval set $\mathcal{I}$. Since we are dealing with straight (non-skew) shapes, we know by Corollary 2.1.2 that the snake sequence has all the 'L's before any of the

'R's, or $u_k < v_1$. So any two intervals i and $j(> i$ say) either intersect ($u_i < u_j < v_i < v_j$) or are nested ($u_i < u_j < v_j < v_i$).

Let a be some label which is repeated. The intervals in $\mathcal{I}$ are ordered by where they start, so identifying the first two intervals i and $j(> i)$ labelled by a is well defined.

Simply change the interval (u_i, v_i) to (u_i, v_j) and (u_j, v_j) to (u_j, v_i), while preserving the label a. Where the intervals start remains unchanged, so these intervals remain the first two intervals labelled by a. So this operation is an involution. Note that if the intervals initially nested, they now intersect, and if they initially intersected, they now nest. In other words the number of crossings has changed by one, and so this involution is sign reversing. The rows of Figure 3-2 are some examples of this involution (if $a \neq c$).

Thus given any labelled interval set with repeated labels, there is a unique labelled interval set with one more (or fewer) crossings, and so the sum of all such terms $(-1)^{c(\mathcal{I})} x^{\mathcal{I}}$ is zero. $\square$

Fix a shape λ. Let $\boldsymbol{T}$ be a border strip tableau of shape λ. Recall that

$$\operatorname{ht}(\boldsymbol{T}) = \sum_B \operatorname{ht}(B),$$

where B ranges over all border strips in $\boldsymbol{T}$ and $\operatorname{ht}(B)$ is one less than the number of rows of B. Define $z(\lambda)$ to be the height $\operatorname{ht}(\boldsymbol{T})$ of a "greedy border strip tableau" $\boldsymbol{T}$ of shape λ obtained by starting with λ and successively removing the largest possible border strip. (Although $\boldsymbol{T}$ may not be unique, the set of border strips appearing in $\boldsymbol{T}$ is unique, so $\operatorname{ht}(\boldsymbol{T})$ is well-defined.)

From Stanley [11, Theorem 5.2] we have that

$$\hat{s}_\lambda = (-1)^{z(\lambda)} \sum_{\mathcal{I}=\{(u_1,v_1),\ldots,(u_k,v_k)\}} (-1)^{c(\mathcal{I})} \prod_{i=1}^{k} \tilde{p}_{v_i - u_i},$$

where $\mathcal{I}$ ranges over all interval sets of λ. For another shape μ, define c_μ to be the coefficient of $\tilde{p}_\mu$ in the above sum , i.e.

$$c_\mu = (-1)^{z(\lambda)} \sum_{\mathcal{I}} (-1)^{c(\mathcal{I})},$$

where the sum is over all interval sets of type μ.

Lemma 3.2.2. $c_\mu p_\mu = (-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}}$, *where the sum is over all labelled interval sets of type* μ.

Example 3.2.2. Examine the shape $\lambda = (4,4,4)$ with snake sequence $LLLORRR$ as before. In particular, consider the interval set $\{(1,7),(2,5),(3,6)\}$ of type $(6,3,3)$ labelled by (a,b,c). This interval set is illustrated in Figure 3-3. If $a = b = c$ then $x^{\mathcal{I}} = x_a^{12}$ and the sum over all such labellings will give $x_1^{12} + x_2^{12} + \cdots = m_{(12)}$. If $a = b \neq c$ the sum over all such labellings will give $x_1^6 x_1^3 x_2^3 + x_1^6 x_1^3 x_3^3 + x_2^6 x_2^3 x_1^3 + \cdots = x_1^9 x_2^3 + x_1^9 x_3^3 + x_2^9 x_1^3 + \cdots = m_{93}$. Similarly if $a = c \neq b$ we will get m_{93}, and if $b = c \neq a$ we will get $x_1^6 x_2^3 x_2^3 + x_2^6 x_1^3 x_1^3 + \cdots = 2x_1^6 x_2^6 + \cdots = 2m_{66}$. Finally if the three labels are distinct, the sum will give $x_1^6 x_2^3 x_3^3 + x_1^6 x_3^3 x_2^3 + \cdots = 2m_{633}$. So the sum over all such labellings is $m_{(12)} + 2m_{66} + 2m_{93} + 2m_{633} = p_{633}$.

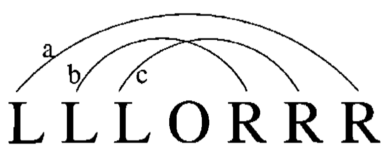


Figure 3-3: A labelled interval set of the shape 444

Proof. We need to show that for every interval set $\mathcal{I}$ of type μ, $\sum x^{\mathcal{I}} = p_\mu$, where the sum is over all labellings of $\mathcal{I}$. First note that the intervals can be ordered largest first and left to right among intervals of the same length. So the ith interval is well defined, and has length μ_i.

By definition $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{\ell(\mu)}} = (x_1^{\mu_1} + x_2^{\mu_1} + \cdots)(x_1^{\mu_2} + x_2^{\mu_2} + \cdots) \cdots (x_1^{\mu_{\ell(\mu)}} + x_2^{\mu_{\ell(\mu)}} + \cdots)$. But if we expand this product into monomials $x_{i_1}^{\mu_1} x_{i_2}^{\mu_2} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}$, each monomial corresponds uniquely to the labelling of $\mathcal{I}$ where the jth interval is labelled by i_j, and so occurs exactly once in $\sum x^{\mathcal{I}}$ as required. □

Given a partition μ, let $m_i(\mu) = \#\{j : \mu_j = i\}$, the number of parts of μ equal to i.

Lemma 3.2.3. *$c_\mu m_1(\mu)!m_2(\mu)!\cdots m_\mu = (-1)^{z(\lambda)}\sum(-1)^{c(\mathcal{I})}x^{\mathcal{I}}$, where the sum is over all interval sets of type μ with labellings with no label repeated.*

Note that we have already demonstrated this result in Example 3.2.2. Indeed for that interval set when the labels were all distinct we saw that $\sum x^{\mathcal{I}} = 2m_{633}$.

Proof. Fix an interval set $\mathcal{I}$. We need to show $m_1(\mu)!m_2(\mu)!\cdots m_\mu = \sum x^{\mathcal{I}}$ where the sum is over all labellings with no label repeated. As before we can order the intervals and say that the ith interval is labelled by α_i. Note that if $\mu_j = \mu_{j+1}$, the two labellings $(\alpha_1, \alpha_2, \ldots, \alpha_j, \alpha_{j+1}, \ldots)$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{j+1}, \alpha_j, \ldots)$ both produce the same term $x^{\mathcal{I}} = x_{\alpha_1}^{\mu_1}x_{\alpha_2}^{\mu_2}\cdots$. So we have $\sum_{(\alpha_1,\alpha_2,\ldots)} x^{\mathcal{I}} = \sum_{(\beta_1,\beta_2,\ldots)} m_1(\mu)!m_2(\mu)!\cdots x^{\mathcal{I}}$ where we impose the condition that if $\mu_j = \mu_{j+1}$, then $\beta_j < \beta_{j+1}$.

But by the definition of the m_μ, we have $m_\mu = \sum_{(\beta_1,\beta_2,\ldots)} x^{\mathcal{I}}$ as required. □

3.3 A symmetric function identity

In this section we use the preceding results to give for each shape λ a relation between the power sum symmetric functions and the monomial symmetric functions.

Theorem 3.3.1. *For each shape λ, write the bottom Schur function $\hat{s}_\lambda = \sum_\mu c_\mu \tilde{p}_\mu$. Then $\sum_\mu c_\mu p_\mu = \sum_\mu c_\mu m_1(\mu)!m_2(\mu)!\cdots m_\mu$.*

Example 3.3.1. For $\lambda = (4, 4, 4)$ we have

$$\hat{s}_\lambda = -\tilde{p}_{642} + \tilde{p}_{633} + \tilde{p}_{552} - 2\tilde{p}_{543} + \tilde{p}_{444}.$$

So our result states that

$$-p_{642} + p_{633} + p_{552} - 2p_{543} + p_{444} = -m_{642} + 2m_{633} + 2m_{552} - 2m_{543} + 6m_{444}.$$

Proof. From Lemma 3.2.2 we have $c_\mu p_\mu = (-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}}$, where the sum is over all labelled interval sets of type μ. But by Lemma 3.2.1 $(-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}} = 0$ if we sum over all labelled interval sets with a repeated label, and by Lemma 3.2.3 $(-1)^{z(\lambda)} \sum (-1)^{c(\mathcal{I})} x^{\mathcal{I}} = c_\mu m_1(\mu)! m_2(\mu)! \cdots m_\mu$ if we sum over all labelled interval sets with no label repeated. So $\sum_\mu c_\mu p_\mu = \sum_\mu c_\mu m_1(\mu)! m_2(\mu)! \cdots m_\mu$. □

Suppose that symmetric functions $f = \sum_\mu d_\mu \tilde{p}_\mu$ and $g = \sum_\mu e_\mu \tilde{p}_\mu$ are such that $\sum_\mu d_\mu p_\mu = \sum_\mu d_\mu m_1(\mu)! m_2(\mu)! \cdots m_\mu$ and $\sum_\mu e_\mu p_\mu = \sum_\mu e_\mu m_1(\mu)! m_2(\mu)! \cdots m_\mu$. Then clearly we have $\sum_\mu (d_\mu + e_\mu) p_\mu = \sum_\mu (d_\mu + e_\mu) m_1(\mu)! m_2(\mu)! \cdots m_\mu$. We have shown in Chapter 2 that the space of Bottom Schur functions is spanned by $\{\hat{s}_\nu : \nu \vdash n,\ \mathrm{rank}(\nu) = k \text{ and } \ell(\nu) = k\}$ and has dimension

$$d(n) = \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} p_{\leqslant k}(n - k^2).$$

Therefore Theorem 3.3.1 gives $d(n)$ "linearly independent" identities for each n. However, there are more such identities. For example,

$$p_{511} - 3p_{421} + p_{331} + p_{322} = 2m_{511} - 3m_{421} + 2m_{331} + 2m_{322}$$

is not generated by our results (when $n = 7$ the rank of λ is at most 2). Let $R_{\lambda\mu}$ be the transition matrix from the power sum symmetric functions to the monomial symmetric functions, ie

$$p_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu.$$

Let $R'_{\lambda\mu}$ be the same matrix except with $R'_{\lambda\lambda} = 0$. It is easy to see that the nullspace of $R'_{\lambda\mu}$ corresponds to all identities of this form. We have given a combinatorial interpretation for certain basis vectors of this nullspace; finding a combinatorial interpretation for the remainder is an open problem.

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