Wave-current interaction in water of finite depth

by

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Abstract

In this thesis, the nonlinear interaction of waves and current in water of finite depth
is studied. Wind is not included.

In the first part, a 2D theory for the wave effect on a turbulent current over
rough or smooth bottom is presented. The logarithmic profile of the basic current
is modified by the waves due to an effective mean shear stress on the free surface.
Surface distortion of the eddy viscosity is shown to be important for the change of
the mean velocity profile by waves. Both wave-following and wave-opposing current
are studied. Comparisons are made with some existing laboratory experiments.

In the second part, an instability theory is presented for the initiation of Langmuir
cells due to waves interacting with a turbulent current maintained by tides or by an
external pressure gradient. With an infinitesimal span-wise disturbance, the free
surface experiences a new mean stress, which generates new vorticity to be diffused
downward, and induces further growth. Various contributions to the unstable growth
of Langmuir circulation are analyzed by examining the mechanical energy budget.
Evidences will be shown that the surface stress contributes significantly to instability.
Both wave-following current and wave-opposing current are studied. For the wave-
following current, two types of Langmuir cells can grow in time; while for the wave-
opposing current, only one can grow. Effects of current strength, wave conditions,
and water depth on the growth of Langmuir circulation will be studied by numerical
examples. Remarks on existing laboratory experiments are made

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Part I

Effects of surface waves on a turbulent current over a smooth or rough seabed
Chapter 1

Introduction

Fine sediments on the bottom of a shallow lake or sea can be resuspended by waves and transported by the current. Since these particles can be carriers of contaminants and nutrients, their distribution is crucial to the health of the water body. Quantitative understanding of the mutual influence between waves, current and wind is therefore of basic importance to the prediction of biological and/or chemical processes in shallow water. Omitting the direct effects of wind, Grant and Madsen (1979, 1986)[22, 23] have studied the effects of turbulence by a simple eddy-viscosity model, with attention focused on the region close to the seabed where sediment transport is often the most important. Based on experiments by Bakker and van Doorn (1978)[2] and Mathisen and Madsen (1996a, 1996b)[48, 47] and by others, they find that a current followed by waves experiences a reduction of speed near the bed, hence an increase of the apparent roughness. The record of Bakker and van Doorn (1978)[2] also shows a notable reduction of current velocity near the water surface. Later Kemp and Simons (1982, 1983)[32, 33], and Klopman (1994, 1997)[34, 35] reported full-depth profiles by similar experiments and showed that the near-surface velocity of a current is increased (reduced) if opposed (followed) by waves; consistent reduction was observed near the bed. A summary of some existing experiments regarding the bottom mean velocity is presented in Table (1.1), where $\Delta_k$ is the height of the roughness elements and $\Lambda_k$ the distance between the roughness elements (see Figure(1-1) ). There is no consistent trend about the change of the near bottom mean velocity among the existing experiments.
Table 1.1: Summary of the near bottom mean velocity in some experiments

<table>
<thead>
<tr>
<th>Authors</th>
<th>Roughness</th>
<th>$A_k (mm)$</th>
<th>$\Delta_k (mm)$</th>
<th>velocity with waves</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bakker &amp; Doorn (1978)</td>
<td>rectangular</td>
<td>15</td>
<td>2</td>
<td>reduced</td>
</tr>
<tr>
<td>Kemp &amp; Simons (1982)</td>
<td>smooth bed</td>
<td>0</td>
<td>0</td>
<td>increased</td>
</tr>
<tr>
<td>Kemp &amp; Simons (1982)</td>
<td>triangular</td>
<td>18</td>
<td>5</td>
<td>increased</td>
</tr>
<tr>
<td>Kemp &amp; Simons (1983)</td>
<td>triangular</td>
<td>18</td>
<td>5</td>
<td>increased</td>
</tr>
<tr>
<td>Mathisen &amp; Madsen (1996)</td>
<td>triangular</td>
<td>100</td>
<td>15</td>
<td>reduced</td>
</tr>
<tr>
<td>Mathisen &amp; Madsen (1996)</td>
<td>triangular</td>
<td>200</td>
<td>15</td>
<td>reduced</td>
</tr>
<tr>
<td>Klopman (1994)</td>
<td>sand</td>
<td>2</td>
<td>2</td>
<td>increased</td>
</tr>
</tbody>
</table>

In experiments by Nepf et al (1995) [57] on Langmuir circulation in a channel of finite width, mechanically generated breaking waves along a steady current also retards the current near the free surface along the center plane of the flume.

Since very fine sediments in lakes or shallow seas can be readily resuspended far above the bottom, the mean velocity distribution in the entire depth is of importance to their transport. Theoretical models for the prediction of wave-current interaction should therefore also include the region near the free surface. An eddy-viscosity model for this problem has been attempted by Nielsen and You (1996) [59]. Several ad hoc assumptions were added and the agreement with measured data is not very satisfactory. Dingemans et al (1996) [11] attributed the wave-induced change of the Eulerian mean velocity to Langmuir circulation induced by the lateral boundaries of the wave tank. The results of their three-dimensional computations based on a $k - \varepsilon$ model agree reasonably well with the measurement of Klopman only for waves following, but not opposing, the current. Allowing the current to be as strong as the phase velocity of waves, Groeneveld and Klopman (1998) [25] treated the two-dimensional problem by combining the method of Generalized Lagrangian Mean (GLM) and a numerical turbulence model. By an empirical estimate of the eddy viscosity and numerical computations, they found good agreement between the computed and measured profiles of the longitudinal velocity for both co-flowing and counter-flowing currents of Klopman (1994) [34]. Their method was further extended by Groeneveld and Battjes (2003) [24] to study the side-wall effects in the three-dimensional motion in a long flume of finite width. Transverse circulation due to the presence of the vertical sidewalls was calculated but was found to have only weak effects on the longitudinal velocity profiles.
Since simple eddy viscosity models facilitate analytical examination of the physical phenomenon, they have been used in early studies of wave generation by wind. Townsend (1972)\cite{71} proposed an eddy-viscosity model in which the mixing length is measured from the moving water surface. Thus the eddy viscosity near the sea surface is modified by waves and depends on time. In consequence the mean wind velocity near the water surface is significantly affected. Similar ideas about the wave-modified mixing length have been used by Janssen (1985), Jacobs (1987)\cite{28} and van Duin and Janssen(1992)\cite{73}. Essentially equivalently Miles (1993)\cite{53} assumed that the eddy viscosity is conserved along the streamlines. The common feature of these models is that the mixing length in the air flow is measured from the moving air-water interface, which restricts the size of local turbulent eddies. As a result the eddy viscosity is now the sum of a steady part and a transient part; the latter is associated with free-surface oscillations. While these models are still crude and heuristic representations of the complex physics of turbulence, they can help mimic the averaged motion without elaborate numerical computations.

Based on existing laboratory data on waves in finite depth, smooth and rough beds must be distinguished. Over a smooth bed, a steady current of the assumed magnitude is usually turbulent as the Reynolds threshold \(\langle \bar{u}^*_c \rangle h/\nu > 2100\) is easily superseded (for example, by a typical case with \(\langle \bar{u}^*_c \rangle > 21\text{cm/s},\) and \(h = 10\text{cm}\)). However a pure wave with comparable orbital velocity becomes turbulent in the bed boundary layer only if \(Re_w \equiv \omega a_b^2/\nu > N \times 10^4\) where \(\omega\) is the wave frequency and \(a_b\) the orbital amplitude just above the smooth bed. The coefficient \(N\) varies from 1.26 (Jonsson(1996)\cite{29}) to about 30 (Kamphuis (1975)\cite{31}). Laboratory waves often fall below this threshold (e.g., \(a_b = 5\text{ cm}\) and \(\omega = 2\pi \text{s}^{-1}\) gives \(Re_w = 1.57 \times 10^4\)). In this case, turbulence in the bed wave boundary layer is dominated by the pre-existing current. For a rough bed, it is known that the ratio \(a_b/k_N\), where \(k_N\) is the Nikuradse sand grain size, is another important factor. When \(a_b/k_N\) decreases, the threshold \(Re_w\) decreases. Only above a certain transitional range of \(Re_w\) is the bottom flow fully turbulent (see, Kamphuis (1975)\cite{31} or Sleath(1984) [65]). By defining \(\delta = O(\kappa u_f/\omega)\) as the thickness of the wave boundary layer (Kajiura (1968)\cite{30}), and \(u_f = \omega a_b \sqrt{f_w/2}\), it follows that \(\delta/k_N = O(\kappa \sqrt{f_w/2})(a_b/k_N)\). Only when \(\delta/k_N(\propto a_b/k_N) \gg 1\), are the roughness elements deeply buried inside the fully turbulent zone which can be
described as the boundary layer. The relations among the $k_N$, $z_B$, $\delta$ and the height of the roughness element $\Delta_k$ are illustrated in Figure (1-1). For the cases which are

Figure 1-1: Length scales relevant to the bottom wave boundary layer. $k_N = 30z_B$ is the equivalent Nikuradse sand grain size, which can be either larger or smaller than $\Delta_k$. $z_B$ usually is computed by fitting the measured velocity with the logarithmic profile. The thickness of the bottom wave boundary layer $\delta$ is computed by the bottom wave boundary layer theory and $\delta$ can be either larger or smaller than $k_N$.

relevant to most field-scale applications, a theoretical model for the wave -current boundary layer has been developed by Grant & Madsen (1979, 1986) by introducing an enhanced eddy viscosity. Their scheme has been found useful for quantitative modeling of coastal dynamics (Tang and Grimshaw(1999) [69]).

We shall give a boundary layer theory incorporating eddy viscosity models which account for the moving free surface and different seabed conditions. In particular the eddy viscosity $\nu_e$ diminishes to zero at both the moving free surface and the seabed and is the largest at mid core. For a smooth seabed, $\nu_e$ diminishes continuously from the core. For a rough bed, a different eddy viscosity linear in height with a larger friction velocity is assumed in the bed wave boundary layer (to be abbreviated by BWBL), and is discontinuous at the upper edge of the layer. After normalization and order estimates, the governing equation for the wave-perurbed current velocity in the core is derived by including the effects of wave-induced Reynolds stress, and energy dissipation. While the first order oscillatory motion gives the dissipation rate in BWBL, the second-order mean motion gives rise to wave Reynolds stress affecting the
core current. For both a smooth bed and a rough bed where the roughness elements are deeply immersed inside the wave boundary layer, a second-order analysis also provides the lower boundary value for the core current. Near the free surface, we first show that the wave boundary layer is unimportant. A second-order analysis gives the mean shear stress at the mean sea-level. With this as the upper boundary condition the core current is solved analytically. Determination of the friction velocities will be discussed. Quantitative predictions of the current profiles affected by waves in the same or opposite direction will be compared with experimental data at one station. For a smooth bed the agreement is excellent. For a rough bed, existing laboratory data are all for roughness either of moderate $a_b/k_N$, or formed by well separated strips. For these beds the boundary layer model is tenuous at best. Nevertheless we shall show that good agreement is still found for the damping rate and the velocity profiles can also be well predicted if empirical fitting at the lower boundary is made. Physical mechanisms responsible for current reduction or increase near the surface will be explained. For future experiments, we also present predictions of the spatial variation of current profiles in the direction of flow.
Chapter 2

Formulation

In the laboratory, the combined wave-current motion is usually realized by an open-channel flow with waves super-imposed on it. Thus, the combined wave-current motion can be viewed as the sum of the unperturbed motion (open-channel flow) and the perturbation due to waves. Wave motion will generate a bottom wave boundary layer (BWBL) and this BWBL will generate turbulence as well if the wave intensity is high enough. As a result, the eddy viscosity inside and outside the bottom wave boundary layer will be different. To study the wave effects on the turbulent current, the first mathematical task is to find a set of equations and boundary conditions for the perturbed motion.

2.1 Governing equations and boundary conditions in dimensional form

We shall use asterisks to distinguish dimensional from dimensionless variables, and from other parameters. Let waves propagate in the positive $x^*$ direction, and the mean current either follows or opposes waves in the positive or negative $x^*$— direction respectively.

Let $\{u^*_i\} \equiv \{u^*, w^*\}$ be the velocity components in $x^*_i = \{x^*, z^*\}$ directions, and $p^*$ the dynamic pressure, where the total pressure is $p^* - \rho g z^*$. The two-dimensional
motion is governed by the Reynolds equations

\[
\frac{\partial u^*_i}{\partial x^*_j} = 0
\]  
\[\rho \frac{\partial u^*_i}{\partial t^*} + \rho \frac{\partial u^*_i u^*_j}{\partial x^*_j} = -\frac{\partial p^*}{\partial x^*_i} + \frac{\partial \tau^*_{ij}}{\partial x^*_j}
\]  

(2.1) (2.2)

where \( \rho \) is the water density. The Reynolds stress is modeled by

\[
\tau^*_{ij} = \rho \nu_c \left( \frac{\partial u^*_i}{\partial x^*_j} + \frac{\partial u^*_j}{\partial x^*_i} \right)
\]  

(2.3)

On the free surface \( z^* = \eta^*(x^*, t^*) \), the kinematic condition requires

\[
\frac{\partial \eta^*}{\partial t^*} + u^* \frac{\partial \eta^*}{\partial x^*} - w^* = 0, \quad z^* = \eta^*
\]  

(2.4)

Assuming no wind, the shear and normal stresses must vanish on the moving free surface,

\[
- \left[ -P^* + \tau^*_{xx} \right] \frac{\partial \eta^*}{\partial x^*} + \tau^*_{zz} = 0, \quad z^* = \eta^*; \quad z^* = \eta^*
\]  

(2.5) (2.6)

where \( P^* = p^* - \rho g \eta^* \) is the total pressure. We define the seabed \( z^* = -h + z_B \) to be the depth where no slippage occurs,

\[
u^* = w^* = 0, \quad z^* = -h + z_B.
\]  

(2.7)

where \( z_B \) is the hydraulic roughness whose determination will be discussed later.

\subsection{2.2 Eddy viscosity}

The water motion is characterized by a three-layer structure: the surface wave boundary layer, the bottom wave boundary layer, and the core region in between. Turbulence is generated by the strong shear region near the bottom and diffused into the region far away from the bottom. For the pre-existing steady motion, the turbulence
generated from the boundary has enough time to diffuse throughout the whole water depth to reach a fully-developed state at which the turbulent eddies are all over the water. The wave motion, on the other hand, has a time scale of the wave period. The velocity induced by waves will change sign within one wave period. As a result, the turbulence generated by wave shear cannot be diffused into the region far away from the boundary, but limited in a thin layer adjacent to the bottom. Within the bottom wave boundary layer, these two kinds of the turbulence co-exist (see Grant and Madsen (1986) [23]).

Near the water surface which is assumed to be free of wind, the shear stresses related to waves and current at the moving surface are zero. As a result, no turbulence is generated from this boundary layer. Turbulence here is just the result of the diffusion by the pre-existing current. This means that there is no difference between the turbulence adjacent to the water surface and that in the core region.

In this thesis, a simple eddy viscosity model will be employed to represent the Reynolds stresses. This eddy viscosity model is a modification of the well-known quadratic model for the open-channel flow, and air flow over the sea.

For an open channel flow without waves, it is well-known that the total viscosity $\nu^*_e$ can be approximated by

$$\nu^*_e = \nu - \kappa u_f (z^* - \bar{\eta}_0) \left( 1 + \frac{z^*}{h^*} \right), \quad -h + z_B < z^* < \bar{\eta}_0 \tag{2.8}$$

where $\nu$ is the molecular viscosity, $u_f$ the friction velocity, $h$ the water depth and $\bar{\eta}_0$ the surface displacement of the open-channel flow. $z_B$ is the bottom roughness. For rough turbulent flow, $\kappa u_f z_B \gg \nu$ can be easily satisfied so that the molecular viscosity can be ignored.

The eddy size is restricted by the water surface as well as the rigid bottom. The water surface is assumed smooth, thus at the water surface, $\nu^*_e = \nu$, i.e., just next to the water surface there is a laminar layer with a thickness of $O(\nu/u_f)$. For the open-channel flow, the eddy viscosity (2.8) will render a velocity profile logarithmic in the distance from the bottom.

When waves are superimposed on the mean current, the diffusion length is restricted by the water surface, thus should be measured from the moving surface,
instead of the still water surface. This idea was first advanced by Townsend (1972) [71] and Jacobs (1987) [28], and further exploited by Janssen (1989) and van Duin and Janssen (1992) [73], Miles (1993) [53], Belcher and Hunt (1993) [3] in studying the air flow over water waves. We extend this idea to the water side and modify the eddy viscosity (2.8) to account for the moving surface.

Borrowing the ideas of Townsend (1972) [71] and Jacobs (1987) [28], in their studies of wind near the wavy sea surface, we adopt the following eddy viscosity model for the core region outside the BWBL

\[ \nu_c = -\kappa u_{fc} (z^* - \eta^*) \left( 1 + \frac{z^*}{h} \right), \quad -h + \delta_B < z^* < \eta^* \]  

(2.9)

where \( \eta^* \) is the water surface displacement, \( \kappa = 0.4 \) is the Kármán constant, \( h \) the water depth, \( u_{fc} \) the friction velocity and \( \delta_B \) is the thickness of BWBL to be determined.

The model (2.9) has the following desired properties:

- It is approximately linear in the distance from the bottom when \( z^* \to -h + \delta_B \)
- It is approximately linear in the distance from the moving surface when \( z^* \to \eta^* \).
- It has negligible wave effects near the bottom.
- It reduces to the eddy viscosity model for the open channel flow (2.8) when \( \eta^* \to \eta_0^* \).

Near the seabed, we distinguish three cases:

Case A: Laminar wave boundary layer over a smooth bed. \( Re_w < N \times 10^4 \) and \( a_b/k_N \to \infty \). Turbulence is dominated by the current, and the eddy viscosity is the near-bottom approximation of (2.9),

\[ \nu_c = \nu_b = \kappa u_{fc} (h + z^*), \quad -h + z_B < z^* = -h + \delta, \quad \text{smooth bed} \]  

(2.10)

Thus the eddy viscosity is continuous everywhere.

Case B: Rough seabed with \( (\delta/k_N, a_b/k_N) \gg 1 \). Wave induced turbulence can be described by a boundary layer theory. A larger eddy viscosity accounting for
contributions by both waves and currents is assumed,

\[ \nu_e \equiv \nu_b = \kappa u_{fb}(h + z^*), \quad -h + z_B < z^* = -h + \delta, \quad \text{rough bed} \quad (2.11) \]

where \( u_{fb} \neq u_{fc} \) is the friction velocity combining the effects of both current and waves. This friction velocity \( u_{fb} \) will be determined by the procedure of Grant & Madsen. Clearly \( \nu_e \) is discontinuous at the edge of BWBL.

Case C: Rough seabed with moderate (\( \delta/k_N \) or \( a_b/k_N \)). Wave-induced turbulence is two- or three-dimensional and cannot be described accurately by boundary-layer approximation. The threshold for \( a_b/k_N \) is around 10 to 115, according to Kamphuis (1975)[31], Sleath(1984) [65], Mathisen and Madsen (1996a, 1996b) [48, 47]. Beds roughened by well separated strips fall into this class.

### 2.3 Basic assumptions

We shall only consider waves of gentle slope so that

\[ \epsilon \equiv ka = \ll 1 \quad (2.12) \]

where \( a \) and \( k \) are the characteristic wave amplitude and wave number, respectively. In addition, the basic current is assumed to be comparable to the wave orbital velocity,

\[ \bar{u}^* \sim \tilde{u}^* = O(\epsilon C), \quad C = \omega/k = \text{phase velocity.} \quad (2.13) \]

where \( \bar{u}^* \) and \( \tilde{u}^* \) are the mean and fluctuating velocity of the water motion, respectively.

An a priori order estimate can be made of the friction velocities \( u_{fc} \) of the core current and \( u_{fb} \) of the bed boundary layer. For \( u_{fc} \) the knowledge of open channel flows \( \bar{u}_0^* \) given by

\[ \bar{u}_0^* = \pm \frac{u_{fc}}{\kappa} \ln \left( \frac{z^* + h}{z_B} \right) \quad (2.14) \]
is relevant. The plus sign is for the positive current (from left-to right) and minus sign for the negative current (from right to left). The friction velocity $u_{fc}$ is related to the friction factor $f_c$, commonly defined by

$$\rho(u_{fc})^2 = \rho \frac{f_c}{2} (\bar{u}_0^*)^2,$$

(2.15)

where $(\bar{u}_0^*)$ stands for the depth average of the basic current. It follows that

$$u_{fc} = \sqrt{\frac{f_c}{2} (\bar{u}_0^*)},$$

(2.16)

The common empirical value of the friction factor is about $f_c \sim 0.01$. In view of the assumption that $O((\bar{u}^*)) = O(\bar{u}^*) = O(\epsilon C)$, we shall regard $u_{fc}/\bar{u}^* = O(\epsilon)$, or equivalently,

$$u_{fc} = O(\epsilon^2 C).$$

(2.17)

Equation (2.14) is consistent with (2.17) if

$$\ln \frac{h}{z_B} = O \left( \frac{1}{\epsilon} \right)$$

(2.18)

which we shall assume. Note that the corresponding shear rate of the basic current is

$$\frac{h}{C} \frac{\partial \bar{u}_0^*}{\partial z^*} = \pm \frac{u_{fc}}{\kappa C} \frac{h}{z^* + h},$$

(2.19)

which is $O(\epsilon^2)$ when $z^* + h = O(h)$ but of order $O(1)$ when $(z^* + h)/h = O(\epsilon^2)$, i.e., near or inside the bed boundary layer. This difference is associated with the logarithmic variations near the bottom.

Though different in numerical values, the friction velocity in the bottom wave boundary layer must be of the same order of magnitude as those in the core current. This is because waves and current are of comparable strength here. Hence we estimate

$$\frac{u_{fb}}{C} \sim \frac{u_{fc}}{C} = O(\epsilon^2)$$

(2.20)

which can be checked after applying the procedure of Grant and Madsen (1979, 1986)[22, 23].
2.4 Normalization by core scales

We shall first normalize all equations by the scales appropriate for the core. In the bottom wave boundary layer, modification will later be made so that the vertical coordinate will be renormalized by the boundary layer thickness.

Let us denote the characteristic wavenumber by $k = 2\pi/\text{wavelength}$, the angular frequency by $\omega = 2\pi/T$, and the phase speed by $C = \omega/k$ according to the linearized theory. The normalized outer variables (without asterisks) are defined as follows

$$t = \omega t^*, \quad (x, z, \eta) = k(x^*, z^*, \eta^*), \quad u_i = u_i^*/C, \quad (\tau_{ij}, p) = (\tau_{ij}^*, p^*)/\rho C^2. \quad (2.1)$$

Conservation of mass requires

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (2.2)$$

In the momentum conservation laws, convective inertia terms are written in two equivalent forms,

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (wv)}{\partial z} \equiv \frac{\partial u}{\partial t} + \frac{\partial E}{\partial x} + w\Omega = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \quad (2.3)$$

$$\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} + \frac{\partial (ww)}{\partial z} \equiv \frac{\partial w}{\partial t} + \frac{\partial E}{\partial z} - u\Omega = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} \quad (2.4)$$

where

$$E = \frac{1}{2} (u^2 + w^2) \quad (2.5)$$

is the kinetic energy, and

$$\Omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \quad (2.6)$$

is the vorticity component in the $y-$ direction. The dynamic pressure $p$ is related to the total pressure $P$ by $p = P + (gk/\omega^2)z$. Using the eddy viscosity modeled by (2.9) in the core and (2.11) in the bed boundary layer, the dimensionless Reynolds stresses
are related to the strain rates by,

\[
\tau_{xx} = 2\alpha^2 S \frac{\partial u}{\partial x}, \quad \tau_{zz} = 2\alpha^2 S \frac{\partial w}{\partial z}, \quad \tau_{xz} = \alpha^2 S \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \quad (2.7)
\]

where \( S \) is the shape factor of the eddy viscosity,

\[
S = \begin{cases} 
S_c = -(z - \eta) \left( 1 + \frac{\kappa}{k_h} \right), & \text{core} \\
S_b = 1 + \frac{\kappa}{k_h}, & \text{BWBL}
\end{cases} \quad (2.8)
\]

and \( \alpha \) is the dimensionless friction velocity,

\[
\alpha = \alpha_c = \frac{\kappa u_f}{C^2}, \quad \text{core} \quad (2.9)
\]

\[
\alpha = \alpha_b = \begin{cases} 
\frac{\kappa u_f}{C^2}, & \text{BWBL, Case A} \\
\frac{\kappa u_f}{C^2}, & \text{BWBL, Cases B}
\end{cases} \quad (2.10)
\]

In accordance with (2.20), we have \( O(\alpha_c) = O(\alpha_b) = O(1) \), We further rewrite \( S_c \) in (2.8) as

\[
S_c = \bar{S}_c + \hat{S}_c \eta \quad (2.11)
\]

so that \( \bar{S}_c \) is the time mean of \( S_c \) and \( \hat{S}_c \eta \) is the surface distortion of the eddy viscosity, where

\[
\bar{S}_c = -z \left( 1 + \frac{z}{k_h} \right), \quad \hat{S}_c = 1 + \frac{z}{k_h} \quad (2.12)
\]

Note from (2.7) that the Reynolds stresses are of order \( O(\epsilon^3) \) in the core region.

For later use we give the vorticity equation by eliminating the pressure \( p \) from the momentum equations (2.3) and (2.4)

\[
\frac{\partial \Omega}{\partial t} + u \frac{\partial \Omega}{\partial x} + w \frac{\partial \Omega}{\partial z} = \Pi \quad (2.13)
\]
where

$$\Pi \equiv \frac{\partial}{\partial z} \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \tau_{zz}}{\partial x} + \frac{\partial \tau_{xx}}{\partial z} \right)$$  \hspace{1cm} (2.14)

We now introduce averages with respect to a wave period, and refer to all period averages\(^1\) as mean quantities from here on. Let us separate the mean current and the wave motion so that

$$(u, w, p, \eta, \Omega) = (\bar{u}, \bar{w}, \bar{p}, \bar{\eta}, \bar{\Omega}) + (\tilde{u}, \tilde{w}, \tilde{p}, \tilde{\eta}, \tilde{\Omega})$$

$$\left( \tau_{xx}, \tau_{xz}, \tau_{zz} \right) = (\bar{\tau}_{xx}, \bar{\tau}_{xz}, \bar{\tau}_{zz}) + (\tilde{\tau}_{xx}, \tilde{\tau}_{xz}, \tilde{\tau}_{zz})$$ \hspace{1cm} (2.15)

where quantities with bars represent the time mean and those with tildes the wave motion. By taking the time-average of the governing equations (2.2),(2.3) and (2.4), we obtain the governing equations of the current

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0$$ \hspace{1cm} (2.16)

$$\frac{\partial (\bar{u}u)}{\partial x} + \frac{\partial (\bar{w}w)}{\partial z} + \frac{\partial E_w}{\partial x} + \bar{w}\bar{\Omega} = - \frac{\partial \bar{p}}{\partial x} + \frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{zz}}{\partial z}$$ \hspace{1cm} (2.17)

$$\frac{\partial (\bar{u}w)}{\partial x} + \frac{\partial (\bar{w}u)}{\partial z} + \frac{\partial E_w}{\partial z} - \bar{u} \bar{\Omega} = - \frac{\partial \bar{p}}{\partial z} + \frac{\partial \bar{\tau}_{zz}}{\partial x} + \frac{\partial \bar{\tau}_{xx}}{\partial z}$$ \hspace{1cm} (2.18)

where \(\bar{E}_w\) is the wave kinetic energy

$$\bar{E}_w = \frac{1}{2} \left( \bar{u}^2 + \bar{w}^2 \right) = O(\epsilon^2)$$  \hspace{1cm} (2.19)

The wave-averaged Reynolds stresses are obtained by taking the time-average of (2.7)\(^1\)

\(^1\)These are period averages of stochastic averages.
, which are, in the core,

\[
\tau_{xx} = 2\alpha_c \varepsilon S_c \frac{\partial \bar{u}}{\partial x} + 2\alpha_c \varepsilon \bar{S}_c \frac{\partial \bar{u}}{\partial x} \quad (2.20)
\]

\[
\tau_{zz} = 2\alpha_c \varepsilon S_c \frac{\partial \bar{w}}{\partial z} + 2\alpha_c \varepsilon \bar{S}_c \frac{\partial \bar{w}}{\partial z} \quad (2.21)
\]

\[
\tau_{xz} = \alpha_c \varepsilon S_c \left( \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) + \alpha_c \varepsilon \bar{S}_c \hat{\eta} \left( \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) \quad (2.22)
\]

In the boundary layer, the wave-averaged Reynolds stresses are obtained simply by replacing \( \alpha_c \) with \( \alpha_b \), \( \bar{S}_c \) with \( S_b \) and setting \( \bar{S}_c \) to zero.

### 2.5 Basic current

In addition to the well known result (2.14) we summarize the salient relations of the basic current for later convenience. Let subscript \( _0 \) signify the basic current

\[
\bar{u} = \varepsilon \bar{u}_0, \quad \bar{w}_0 = 0, \quad \bar{p} = \varepsilon \bar{p}_0 \quad (2.1)
\]

Omitting the wave-perturbed parts in (2.17 ), we obtain the equation governing \( \bar{u}_0 \)

\[
\frac{\partial \bar{p}_0}{\partial x} = \varepsilon^2 \alpha_0 \frac{\partial}{\partial z} \left( \bar{S}_c \frac{\partial \bar{u}_0}{\partial z} \right), \quad \text{where} \quad \alpha_0 = \kappa u_{fc} C \varepsilon^2. \quad (2.2)
\]

to which the solution is (2.14). In (2.2), \( \bar{p}_0 \) is constant in depth from the vertical momentum equation. The pressure gradient of the basic current can be identified with the normalized friction velocity by

\[
- \frac{\partial \bar{p}_0}{\partial x} = \pm \varepsilon \frac{\alpha_0^2}{\kappa^2} \quad (2.3)
\]

For later use we note that the mean shear stress of the basic current has the following form

\[
(\tau_{xz})_0 = \pm \varepsilon \frac{\alpha_0^2}{\kappa^2} \left( \frac{-z}{kh} \right) \quad (2.4)
\]

and is of \( O(\varepsilon^4) \).
2.6 Length scale of wave attenuation

Due to dissipation, waves will attenuate in $x$. Since other mean quantities may in turn be affected, we must first ascertain the scale of attenuation by considering the mechanical energy in waves.

Let the leading order surface displacement of the surface wave be

$$\tilde{\eta} = \epsilon \left( \frac{A}{2} e^{i\theta} + \text{c.c.} \right) + O(\epsilon^2)$$

(2.5)

where $\epsilon A = O(\epsilon)$ is the dimensionless wave amplitude and $\theta = x - t$ the wave phase. Since by assumption the eddy viscosity is of $O(\epsilon^2)$ and the rotational core current of $O(\epsilon)$, the leading order wave field is irrotational,

$$\tilde{u} = \epsilon \left( \frac{A}{2} \frac{\cosh(kh + z)}{\sinh(kh)} e^{i\theta} + \text{c.c.} \right) + O(\epsilon^2)$$

(2.6)

$$\tilde{w} = \epsilon \left( -i \frac{A}{2} \frac{\sinh(kh + z)}{\sinh(kh)} e^{i\theta} + \text{c.c.} \right) + O(\epsilon^2)$$

(2.7)

$$\tilde{p} = \epsilon \left( \frac{A}{2} \frac{\cosh(kh + z)}{\sinh(kh)} e^{i\theta} + \text{c.c.} \right) + O(\epsilon^2)$$

(2.8)

Using standard arguments we can derive from the conservation laws the equation of mechanical energy in waves

$$\frac{\epsilon^2}{2} \int_{-kh + k\delta}^{kh + k\delta} \alpha_b S_b \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right)^2 dz + \frac{\epsilon^2}{2} \int_{-kh + k\delta}^{0} \alpha_c \tilde{S}_c \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right)^2 dz$$

$$= -\frac{\partial}{\partial x} \int_{-kh + k\delta}^{0} \overline{\tilde{u} p} dz$$

(2.9)

where $k\delta = O(\epsilon^2)$ is the dimensionless thickness of BWBL. For reference a derivation is sketched in Appendix A. Note that there are two depth integrals corresponding to the BWBL and the core. In the core $\tilde{S}_c = O(1)$, we can ignore the BWBL and get

$$\overline{\tilde{u} p} \propto \epsilon^2 A A^*$$

(2.10)

and

$$\int_{-kh}^{0} \tilde{S}_c \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right)^2 dz \propto \epsilon^2 A A^*$$

where $A^*$ is the complex conjugate of $A$. Because $S_b = O(k\delta)$, balance of the transient acceleration and the oscillatory viscous stress implies that $k\delta = O(\epsilon^2)$. Thus the
integral across the bed boundary layer is

\[
\int_{-kh+kz_B}^{h_b} S_b \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right)^2 \, dz \propto \epsilon^2 AA^* \tag{2.11}
\]

Two inferences may be made. First, dissipation in the two regions are comparably important; weak shear in the core is compensated by the large eddy viscosity there. Second, the length scale of horizontal length scale of attenuation is \(O(\epsilon^{-2})\) times that of the wavelength. This should affect all mean quantities associated with wave disturbances. Therefore, we introduce the slow coordinate \(x_2 = \epsilon^2 x\). The \(x\) derivative of every wave-periodic mean \(\bar{f}\) is

\[
\frac{\partial \bar{f}}{\partial x} = \epsilon^2 \frac{\partial \bar{f}}{\partial x_2} = O(\epsilon^2 \bar{f}) \tag{2.12}
\]

Also, (2.9) should yield formally

\[
\frac{\partial AA^*}{\partial x_2} = -\beta AA^* \tag{2.13}
\]

where \(\beta > 0\) is the wave energy dissipation rate, which will be calculated explicitly later.
Chapter 3

Mean and oscillatory motions in the core region

3.1 Order estimate of wave-induced core current

Extending (2.1), we express the total mean motion as the sum of the basic and wave-induced parts

\[ \bar{u} = \varepsilon u_0 + \mu_1(\varepsilon)\bar{u}_c, \quad \bar{w} = \mu_2(\varepsilon)\bar{w}_c, \quad \bar{p} = \varepsilon p_0 + \mu_1(\varepsilon)\bar{p}_c \]  

(3.1)

where \( \bar{u}_c, \bar{w}_c \) and \( \bar{p}_c \) are the velocity and pressure of the wave-induced current in the core, all of them of \( O(1) \). The gage functions \( \mu_i(\varepsilon), i = 1, 2 \) are to be determined. In an infinitely long flume or unbounded sea, waves are damped out at \( x \sim \infty \). The total steady-state discharge at any \( x, z \) must be equal to that of the pre-existing basic current,

\[ \int_{-kh+kz_B}^{0} u dz = \int_{-kh+kz_B}^{0} (\bar{u} + \bar{u}) dz = \varepsilon \int_{-kh+kz_B}^{0} \bar{u}_0 dz \]  

(3.2)

Taylor expansion of the middle integral of (3.2) around \( z = 0 \) and substitution of (3.1) into the resulting equation yields

\[ \mu_1(\varepsilon) \int_{-kh+k\delta}^{0} \bar{u}_c dz + \overline{[\bar{u}]}_0 \bar{\eta} = O(\varepsilon^3) \]  

(3.3)
where \( \bar{u} \) denotes the orbital velocity at \( z = 0 \). The contribution of the bottom wave boundary layer has been ignored due to the small thickness of \( O(\epsilon^2) \).

It is evident that \( \mu_1(\epsilon) = \epsilon^2 \). After using the linear wave solutions for \( \tilde{\eta} \) and \( \bar{u} \), the discharge condition (3.3) becomes

\[
\int_{-kh+k\delta}^{0} \bar{u}' dz = -\frac{AA^*}{2} \coth(kh). \tag{3.4}
\]

As waves propagate in the positive \( x \) direction, this term diminishes with \( x \) due to damping. Therefore the total flux of the Eulerian current

\[
\int_{-kh+k\delta}^{0} \bar{u} \, dz = \int_{-kh+k\delta}^{0} \left( \epsilon \tilde{u}_0 + \epsilon^2 \bar{u}'_1 \right) \, dz + O(\epsilon^3)
\]

which can be integrated from the measured current profile, must increase (decrease) its flux in the downstream direction for a wave-following (wave-opposing) current.

As a further inference, continuity requires

\[
\epsilon^2 \frac{\partial \bar{u}'}{\partial x} + \mu_2(\epsilon) \frac{\partial \bar{u}'}{\partial z} = 0 \tag{3.5}
\]

Since \( \partial \bar{u}_2'/\partial x = \epsilon^2 \partial \bar{u}_2'/\partial x_2 \) due to wave damping, we conclude that \( \mu_2 = \epsilon^4 \) so that (3.1) can be replaced by

\[
\bar{u} = \epsilon \tilde{u}_0 + \epsilon^2 \bar{u}_1', \quad \bar{w} = \epsilon^4 \bar{w}_1', \quad \bar{p} = \epsilon \bar{p}_0 + \epsilon^2 \bar{p}_1'. \tag{3.6}
\]

The presence of waves also modifies the friction velocity \( \alpha_{fc} \) from that of the pure current, hence the value of \( \alpha_c \), which in general differs from \( \alpha_0 \).

### 3.2 Momentum balance of mean motion

In the core region, the total Reynolds stresses (2.7) which appear on the right-hand sides of (2.17)-(2.18) can be approximated up to \( O(\epsilon^4) \) by using the linear wave solutions (2.6), (2.7) and (2.8), and the basic current (2.4),

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\[
\tau_{xx} = 2\alpha_c \epsilon^2 \dot{S}_c \frac{\partial \tilde{u}}{\partial x} + O(\epsilon^5)O(\epsilon^5), \quad \tau_{zz} = 2\alpha_c \epsilon^2 \dot{S}_c \frac{\partial \tilde{w}}{\partial z} + O(\epsilon^5)O(\epsilon^5)
\] (3.7)

\[
\tau_{xz} = \alpha_c \epsilon^2 \dot{\bar{S}}_c \left( \epsilon \frac{\partial \tilde{u}_0}{\partial z} + \dot{\epsilon} \frac{\partial \tilde{u}_c}{\partial z} \right) + \alpha_c \epsilon^2 \dot{\bar{S}}_c \left( \frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial x} \right) + O(\epsilon^5)
\] (3.8)

\[
= \epsilon^4 \left( \pm \frac{\alpha_0 \alpha_c}{\kappa^2} \frac{-z}{kh} \right) + \alpha_c \dot{\bar{S}}_c \frac{\partial \tilde{u}_c}{\partial z} + \alpha_c \dot{\bar{S}}_c \frac{\partial \tilde{w}}{\partial x} + \alpha_c \alpha^* \dot{\bar{S}}_c \frac{\sinh(kh + z)}{\sinh(kh)} + O(\epsilon^5)
\] (3.9)

The period-averaged Reynolds stress \(\tau_{xz}\) can be written as the sum of the basic and wave-induced parts

\[
\tau_{xz} = \pm \epsilon^4 \frac{\alpha_0^2}{\kappa^2} \frac{-z}{kh} + \epsilon^4 (\bar{\tau}_{xz})_c
\] (3.10)

with

\[
(\bar{\tau}_{xz})_c = \pm \frac{\alpha_0 (\alpha_c - \alpha_0)}{\kappa^2} \frac{-z}{kh} + \alpha_c \dot{\bar{S}}_c \frac{\partial \tilde{u}_c}{\partial z} + \alpha_c \alpha^* \dot{\bar{S}}_c \frac{\sinh(kh + z)}{\sinh(kh)}
\] (3.11)

Therefore, the right-hand side of (2.17) is of \(O(\epsilon^4)\) and the right-hand side of (2.18) is of \(O(\epsilon^2)\). The momentum equation (2.17) can be approximated up to \(O(\epsilon^4)\), by

\[
\epsilon^2 \frac{\partial}{\partial x} \left( \tilde{E}_w + \epsilon^2 \tilde{p}_c \right) + \frac{\omega \bar{\zeta}}{\partial z} = \epsilon^4 \frac{\partial (\bar{\tau}_{xz})_c}{\partial z} + O(\epsilon^5)
\] (3.12)

which governs the current shear in the core. Terms of the basic current disappear by cancellation.

To further simplify (3.12), we first note from the vertical momentum equation (2.18) that

\[
\frac{\partial}{\partial z} \left( \tilde{E}_w + \epsilon^2 \tilde{p}_c \right) = O(\epsilon^3)
\] (3.13)
Thus (3.12) can be written as

$$
\epsilon^2 \frac{\partial}{\partial x_2} \left[ \tilde{E}_w + \epsilon^2 \tilde{\rho}' \right] + \tilde{w} \tilde{\Omega} = \epsilon^4 \frac{\partial (\tilde{\rho}')}{\partial z} + O(\epsilon^5)
$$

(3.14)

To complete the equation governing the mean shear stress, we shall calculate in the next subsection the vortex force term $\tilde{w} \tilde{\Omega}$ up to $O(\epsilon^4)$.

### 3.3 Vortex force $\tilde{w} \tilde{\Omega}$

To get the mean vortex force we need to know the oscillatory part of vorticity $\tilde{\Omega}$ up to $O(\epsilon^3)$. The wave vorticity is governed by the oscillatory part of (2.13),

$$
\frac{\partial \tilde{\Omega}}{\partial t} + \tilde{u} \frac{\partial \tilde{\Omega}}{\partial x} + \tilde{w} \frac{\partial \tilde{\Omega}}{\partial z} + \tilde{u} \frac{\partial \tilde{\Omega}}{\partial x} + \tilde{w} \frac{\partial \tilde{\Omega}}{\partial x} + \tilde{w} \frac{\partial \tilde{\Omega}}{\partial x} = \tilde{\Pi}
$$

(3.15)

$\tilde{\Pi}$ is the oscillatory part of (2.14) which can be quickly worked out. From the definition (2.7), the oscillatory part of the Reynolds stresses are of $O(\epsilon^3)$,

$$
\tilde{\tau}_{xx} = 2\alpha_e \epsilon^2 \tilde{S}_c \frac{\partial \tilde{u}}{\partial x} + O(\epsilon^4), \quad \tilde{\tau}_{zz} = 2\alpha_e \epsilon^2 \tilde{S}_c \frac{\partial \tilde{w}}{\partial z} + O(\epsilon^4)
$$

(3.16)

$$
\tilde{\tau}_{xz} = \alpha_e \epsilon^2 \tilde{S}_c \left( \frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial x} \right) + O(\epsilon^4)
$$

(3.17)

Using these results and the linear wave solutions in (2.14), we get

$$
\tilde{\Pi} = 2\epsilon^2 \alpha_e \frac{\partial^2 \tilde{S}_c}{\partial z^2} \frac{\partial \tilde{u}}{\partial z} + O(\epsilon^4)
$$

(3.18)

For calculating $\tilde{w} \tilde{\Omega}$ at $O(\epsilon^4)$, we shall only need the first harmonic component of $\tilde{\Omega}$ at $O(\epsilon^3)$. Hence the last two terms on the left of (3.15) are of no concern. Using the fact that $\tilde{w} = \tilde{w}' = O(\epsilon^4)$, we get the governing equation for $\tilde{\Omega}$ correct to $O(\epsilon^3)$

$$
\frac{\partial \tilde{\Omega}}{\partial t} + \tilde{u} \frac{\partial \tilde{\Omega}}{\partial x} + \tilde{w} \frac{\partial \tilde{\Omega}}{\partial z} = 2\epsilon^2 \alpha_e \frac{\partial^2 \tilde{S}_c}{\partial z^2} \frac{\partial \tilde{u}}{\partial z} + SHT + O(\epsilon^4)
$$

(3.19)
where $SHT$ represents the second harmonic terms. After integrating (3.19) with respect to $t$, we obtain

$$\bar{\tilde{\Omega}} = -\bar{u} \int \frac{\partial \tilde{\Omega}}{\partial x} dt - \frac{\partial \tilde{\Omega}}{\partial z} \int \bar{\tilde{w}} dt + 2\epsilon^2 \alpha_c \frac{\partial^2 \tilde{S}_c}{\partial z^2} \int \frac{\partial \tilde{u}}{\partial z} dt + SHT + O(\epsilon^4) \quad (3.20)$$

Among all terms on the right, the second term dominates so that

$$\tilde{\Omega} = -\frac{\partial \tilde{\Omega}}{\partial z} \int \bar{\tilde{w}} dt + O(\epsilon^3) = O(\epsilon^2) \quad (3.21)$$

Thus the vorticity fluctuation is due to the convection of the mean vorticity by vertical oscillations. However, this term is out of phase with $\tilde{w}$, hence does not contribute to the vortex force. Note that the first term on the right of (3.20), which is of $O(\epsilon^3)$, can be estimated by (3.21).

$$\bar{u} \int \frac{\partial \tilde{\Omega}}{\partial x} dt = -\bar{u} \frac{\partial \tilde{\Omega}}{\partial z} \int \left( \int^t \frac{\partial \tilde{w}}{\partial x} dt' \right) dt + O(\epsilon^4) \quad (3.22)$$

which is also out of phase with $\tilde{w}$, hence does not affect the vortex force. Thus only the third term in (3.20) matters.

Finally substituting (3.20) in $\bar{\tilde{w}} \bar{\tilde{\Omega}}$, we find

$$\tilde{\bar{\tilde{\Omega}}} = 2\epsilon^2 \alpha_c \frac{\partial^2 \tilde{S}_c}{\partial z^2} \bar{\tilde{w}} \int \frac{\partial \tilde{u}}{\partial z} dt + O(\epsilon^4)$$

$$= -\alpha_c \epsilon^4 AA' \frac{\partial^2 \tilde{S}_c}{\partial z^2} \frac{\sinh^2(kh + z)}{\sinh^2(kh)} + O(\epsilon^5) \quad (3.23)$$

It is interesting that the vortex force is related to the curvature of the eddy viscosity.

### 3.4 Mean shear stress of wave-perturbed current

With the vortex force calculated, let us return to the governing equation for the wave-perturbed mean motion in the core, (3.14)

$$\frac{\partial}{\partial x_2} \left[ p'_c + \epsilon^{-2} \tilde{E}_w \right]_0 - \alpha_c AA' \frac{\partial^2 \tilde{S}_c}{\partial z^2} \frac{\sinh^2(kh + z)}{\sinh^2(kh)} = \frac{\partial (\tau'_c)}{\partial z} \quad (3.24)$$
Integrating once with respect to \(z\) from \(z = 0\), we obtain
\[
\frac{\partial}{\partial z} \left[ \frac{\partial^2 \tilde{S}_c}{\partial z^2} \left( \sinh(2(kh + z)) - \sinh(2kh) - 2z \right) \frac{1}{4 \sinh^2(kh)} \right] + \left( \tau'_{zz} \right)_c - \left[ (\tilde{\tau}'_{zz})_c \right]_0 = 0
\] (3.25)

where \(\frac{\partial^2 \tilde{S}_c}{\partial z^2} = -\frac{2}{kh} = \text{constant}\) has been used.

The wave-induced pressure gradient can be expressed in terms of the wave-induced mean shear \([ (\tilde{\tau}'_{zz})_c ]_+\), at the top edge of BWBL. By setting \(z = -kh + k\delta\) in (3.25), we have
\[
\frac{\partial}{\partial z} \left[ \frac{\partial^2 \tilde{S}_c}{\partial z^2} \left( \frac{-\sinh(2kh) + 2kh}{4 \sinh^2(kh)} \right) \right] + \left[ (\tilde{\tau}'_{zz})_c \right]_+ - \left[ (\tilde{\tau}'_{zz})_c \right]_0 = 0
\] (3.26)

This result can be used in (3.25) to eliminate the pressure gradient. On the other hand, we have from (3.11),
\[
\frac{z}{kh} \left[ (\tilde{\tau}'_{zz})_c \right]_+ + (\tilde{\tau}'_{zz})_c = \frac{z}{kh} \alpha_c \left[ \tilde{S}_c \frac{\partial \tilde{u}'}{\partial z} \right]_+ \left[ \alpha_c \tilde{S}_c \frac{\partial \tilde{u}'}{\partial z} + \alpha_c AA^+ \tilde{S}_c \frac{\sinh(kh + z)}{\sinh(kh)} \right] + \alpha_c AA^+ \tilde{S}_c \frac{\sinh(kh + z)}{\sinh(kh)}
\]

It then follows that (3.25) can be written as
\[
\alpha_c AA^+ \frac{\partial^2 \tilde{S}_c}{\partial z^2} \frac{(kh + z) \sinh(2kh) - kh \sinh(2(kh + z))}{4kh \sinh^2(kh)} + \left( 1 + \frac{z}{kh} \right) \left[ (\tilde{\tau}'_{zz})_c \right]_0 = 0
\] (3.27)

Next, we recall the definition of \(u_{fc}\) and its dimensionless equivalent \(\alpha_c^2\)
\[
\alpha_c^2 = \pm \kappa^2 \varepsilon^{-4} [\tilde{\tau}_{zz}]_+
\] (3.28)

Making use of (3.9) in (3.28), we have
\[
\alpha_c = \pm \kappa^2 \epsilon^{-1} \left[ \tilde{S}_c \frac{\partial \tilde{u}_0}{\partial z} \right]_+ \pm \kappa^2 \left[ \tilde{S}_c \frac{\partial \tilde{u}'}{\partial z} \right]_+ + O(\epsilon) = \alpha_0 \pm \kappa^2 \left[ \tilde{S}_c \frac{\partial \tilde{u}'}{\partial z} \right]_+ \] (3.29)

where (2.4) has been used in the last step. We obtain by definition, the expected
result

$$\alpha_c - \alpha_0 = \pm \kappa^2 \left[ \tilde{S}_c \frac{\partial \tilde{u}'_c}{\partial z} \right]_+$$ (3.30)

Finally, the equation for the wave-perturbed shear stress in the core is obtained,

$$\alpha_c \tilde{S}_c \frac{\partial \tilde{u}'_c}{\partial z} = \alpha_c AA^* \frac{\partial^2 \tilde{S}_c}{\partial z^2} \frac{(kh + z) \sinh(2kh) - kh \sinh(2(kh + z))}{4kh \sinh^2(kh)}$$

$$+ \left(1 + \frac{z}{kh}\right) [\tilde{\tau}'_{zz}]_0 \pm \frac{\alpha_c (\alpha_c - \alpha_0)}{\kappa^2} \left(\frac{-z}{kh}\right) - \alpha_c AA^* \tilde{S}_c \frac{\sinh(kh + z)}{\sinh(kh)}$$ (3.31)

This key result will be integrated later to get the wave-perturbed current \(\tilde{u}'_c\). Clearly two boundary values are needed: the velocity \(\tilde{u}'_c\) at the upper edge of the bed boundary layer and the mean shear stress \([\tilde{\tau}'_{zz}]_0\) on the mean free surface. For the former we need to analyze the wave-perturbed mean motion inside the bed boundary layer.
Chapter 4

Bottom boundary layer

A unified treatment can be given for all Cases A and B. Recalling that \( k\delta = O(\epsilon^2) \), an inner depth variable \( Z \) will be defined by

\[
Z = \frac{z + kh}{\alpha_b\epsilon^2}
\]  

(4.1)

For Case A one needs only to set \( \alpha_b = \alpha_c \).

4.1 Order of magnitude estimate

Let us first estimate the order of magnitudes of the vertical velocities, both mean and oscillatory, in the bed boundary layer. The horizontal velocities and dynamic pressures are more obvious. Let us express the mean and oscillatory flow fields as follows.

\[
\breve{u} = \epsilon \breve{u}_0 + \epsilon^2 \breve{u}_b', \quad \breve{w} = \sigma_1(\epsilon) \breve{w}_b' + \cdots, \quad \breve{p} = \epsilon \breve{p}_0 + \epsilon^2 \breve{p}_b'
\]  

(4.2)

\[
\tilde{u} = \epsilon \tilde{u}_b, \quad \tilde{w} = \sigma_2(\epsilon) \tilde{w}_b, \quad \tilde{p} = \epsilon \tilde{p}_b
\]  

(4.3)
where the gage functions $\sigma_1(\epsilon)$ and $\sigma_2(\epsilon)$ are to be determined. From the time averaged continuity equation,

$$\epsilon^4 \frac{\partial \tilde{u}_b'}{\partial x_2} + \frac{\sigma_1(\epsilon)}{\alpha_b^2} \frac{\partial \tilde{w}_b'}{\partial Z} = 0 \quad (4.4)$$

Clearly $\sigma_1 = O(\epsilon^6)$ so that we can write $\tilde{w} = \epsilon^6 \tilde{w}_b' = O(\epsilon^6)$.

To find the order of the oscillatory motion inside the boundary layer, we invoke the continuity equation,

$$\epsilon \frac{\partial \tilde{u}_b}{\partial x} + \epsilon^3 \frac{\partial \tilde{u}_b}{\partial x_2} + \frac{\sigma_2(\epsilon)}{\alpha_b^2} \frac{\partial \tilde{w}_b}{\partial Z} = 0 \quad (4.5)$$

hence $\sigma_2 = O(\epsilon^3)$. It follows from (4.5) that

$$\tilde{w} = \epsilon^3 \tilde{w}_b' = -\epsilon^3 \alpha_b \int_{Z_b}^{Z} \frac{\partial \tilde{u}_b}{\partial x} dZ + O(\epsilon^4), \quad \text{with} \quad Z_B = \frac{kz_B}{\alpha_b \epsilon^2} \quad (4.6)$$

With these estimates we can now simplify the mean horizontal momentum equation in order to show just what is needed from the oscillatory motion. Explicit calculations of the needed oscillatory motion will then be worked out, and finally the horizontal component of wave-perturbed current velocity will be found.

### 4.2 Approximate momentum balance of the mean motion

Substituting (4.2) and (4.3) in (2.7), we obtain the total Reynolds stress inside the boundary layer

$$\tau_{xx} = 2\epsilon^8 \alpha_b^2 Z \frac{\partial \tilde{u}_b}{\partial x_2}, \quad \tau_{zz} = 2\epsilon^8 \alpha_b Z \frac{\partial \tilde{w}_b}{\partial Z}, \quad \tau_{xz} = \pm \epsilon^4 \alpha_b^2 + \epsilon^4 (\tau'_{xz})_b \quad (4.7)$$

where the first term in $\tau_{xz}$ is the shear stress of the basic current, and the second term is the wave-induced mean shear $(\tau'_{xz})_b$, given by

$$(\tau'_{xz})_b = \alpha_b Z \frac{\partial \tilde{u}_b}{\partial Z} + \epsilon^{-2}(\alpha_b - \alpha_0) S_b \frac{\partial \tilde{u}_0}{\partial Z} = \alpha_b Z \frac{\partial \tilde{u}_b}{\partial Z} \pm (\alpha_b - \alpha_0) \frac{\alpha_0}{\kappa^2} \quad (4.8)$$
The last term above has been simplified by using (2.4). After substituting (4.2) and (4.3) and (4.7) in the horizontal mean momentum equation (2.17), we obtain,

$$\frac{\partial \bar{u}_b \bar{w}_b}{\partial Z} = \frac{\partial \bar{\tau}_{zz}}{\partial Z} + O(\epsilon)$$  \hspace{1cm} (4.9)

Integrating this with respect to $Z$ from the upper edge of the boundary layer,

$$Z = Z_b \equiv \frac{k \delta}{\alpha_b \epsilon^2}$$  \hspace{1cm} (4.10)

we get

$$\frac{\bar{u}_b \bar{w}_b}{\bar{u}_b \bar{w}_b} - [(\bar{u}_b \bar{w}_b) +] = (\bar{\tau}_{zz})_b - [\bar{\tau}_{zz}]_c + (\bar{\tau}_{zz})_b - [\bar{\tau}_{zz}]_c + O(\epsilon)$$  \hspace{1cm} (4.11)

The last equality follows by the continuity of stress at the top edge of BWBL, signified by the subscript $\cdot \cdot \cdot +$. Physically (4.9) or (4.11) states that the total mean shear stress is not constant across the bed boundary layer$^1$. Second-order mean shear is forced by the wave-induced Reynolds stress. This result is similar to that of Eulerian streaming in a oscillatory laminar boundary layer of Stokes.

By using (3.11), (4.8) and then (3.30), the right-hand side of (4.11) can be written as

$$(\bar{\tau}_{zz})_b - [\bar{\tau}_{zz}]_c = \alpha_b Z \frac{\partial \bar{u}_b}{\partial Z} \pm \frac{\alpha_0 \alpha_b - \alpha_c^2}{\kappa_0^2} + O(\epsilon)$$  \hspace{1cm} (4.12)

It then follows from (4.8), (4.11) and (4.12) that

$$Z \frac{\partial \bar{u}_b}{\partial Z} = \frac{\bar{u}_b \bar{w}_b}{\alpha_b} - \left( \pm \frac{\alpha_0 \alpha_b - \alpha_c^2}{\alpha_b \kappa_0^2} + \frac{[\bar{u}_b \bar{w}_b]_+}{\alpha_b} \right) + O(\epsilon)$$  \hspace{1cm} (4.13)

which can be integrated from $Z = Z_b$, to give the wave -perturbed current in the bed boundary layer

$$\bar{u}_b(Z) = \int_{Z_b}^{Z} \frac{\bar{u}_b \bar{w}_b}{\alpha_b Z} dZ - \left( \pm \frac{\alpha_0 \alpha_b - \alpha_c^2}{\alpha_b \kappa_0^2} + \frac{[\bar{u}_b \bar{w}_b]_+}{\alpha_b} \right) \ln \left( \frac{Z}{Z_b} \right) + O(\epsilon)$$  \hspace{1cm} (4.14)

$^1$In the theory by Grant & Madsen, this additional shear stress was not accounted for, therefore the current profile remains logarithmic in height even with waves.
Its value at upper edge of the BWBL \((Z = Z_6)\) gives the lower boundary condition for \(\tilde{u}_c\)

\[
[\tilde{u}_c]_+ = \int_{Z_B}^{Z_6} \frac{\overline{u_b} \bar{w}_b}{\alpha_b Z} dZ - \left( \frac{\alpha_c \alpha_b - \alpha_c^2}{\alpha_c \alpha_b} + \frac{[\bar{u}_b \bar{w}_b]_+}{\alpha_b} \right) \ln \left( \frac{Z_6}{Z_B} \right) + O(\epsilon) \tag{4.15}
\]

which holds for both Cases A and B. In case A, one only needs to set \(\alpha_c = \alpha_b\). In (4.15) the wave-Reynolds stress is to be determined in the next section after solving the oscillatory motion inside BWBL. Note that \([\tilde{u}_c]_+\) depends implicitly on \(\alpha_c\) and \(\alpha_b\) which are yet to be found.

### 4.3 Oscillatory motion: \(\tilde{u}_b\)

Making use of (4.6), the wave-induced Reynolds stress in (4.13) can be expressed as

\[
\overline{u_b} \bar{w}_b = -\alpha_b \tilde{u}_b \int_{Z_B}^{Z} \frac{\partial \tilde{u}_b}{\partial Z} dZ + O(\epsilon) \tag{4.16}
\]

The oscillatory flow \(\tilde{u}_b\) is needed only to the leading order in order to integrate (4.14) for \(\tilde{u}_c\). After using (4.3) and (4.6), and the oscillatory parts of the Reynolds stress (2.7), the boundary value problem for the oscillatory motion is that of the Stokes problem with a depth-linear eddy viscosity. The solution has been given by [30]

\[
\tilde{u}_b = \frac{A}{2 \sinh(kh)} (1 - K(Z)) e^{i \theta} + c.c., \quad \text{where} \quad K(Z) = \frac{K_0(2 \sqrt{Z} e^{-i \pi/4})}{K_0(2 \sqrt{Z_B} e^{-i \pi/4})} \tag{4.17}
\]

with \(K_0\) being the Kelvin function of the zeroth order. We now follow [22] and define the outer edge of the bottom wave boundary layer \(Z_6\) by the condition \([\tilde{u}_b/[\tilde{u}_b]_+] = 0.95^2\) or from (4.17)

\[
|K(Z_6)| = 0.05 \tag{4.18}
\]

\(^2\text{We have checked for Case A, that this numerical value is immaterial as long as it is close to 1.}\)
Now the wave-induced Reynolds stress (4.16) can be calculated from (4.17)

$$\overline{\overline{w}} \overline{w}(Z) = -\frac{\alpha_b A^*}{4 \sinh^2(kh)} \left[ (1 - K^*) \int_{Z_B}^Z i(1 - K) dZ + \text{c.c.} \right] + O(\epsilon)$$

(4.19)

At the upper edge of the boundary layer, $K \to 0$, thus,

$$\overline{[\overline{u}_b \overline{w}_b]}_+ = -\frac{\alpha_b A^*}{2 \sinh^2(kh)} \int_{Z_B}^{Z_b} \text{Im}(K) dZ + O(\epsilon)$$

(4.20)

### 4.4 Wave damping rate

After finding the oscillatory flow field (4.17) inside the bed boundary layer, we can now calculate the rate of wave damping explicitly. In view of (4.3) and (4.6), the first integral on the left-hand side of the mechanical energy equation (2.9) representing dissipation inside the bed boundary layer can be approximated as follows

$$\int_{Z_B}^{Z_b} Z \left( \frac{\partial \tilde{u}_b}{\partial Z} \right)^2 dZ = \frac{\alpha_b A^*}{2 \sinh^2(kh)} \int_{Z_B}^{Z_b} Z \frac{\partial K(Z)}{\partial Z} \left( \frac{\partial K(Z)}{\partial Z} \right)^* dZ$$

(4.21)

The integral on the right can be evaluated numerically. The second term in (2.9) which represents energy dissipation in the core, can be easily calculated from the linear wave solution

$$\epsilon^2 \frac{1}{2} \int_{-kh+k\delta}^{0} \alpha_c \tilde{5}_c \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right)^2 dz = \epsilon^4 \alpha_c AA^* \frac{2kh \cosh^2(kh) - \sinh(2kh)}{2kh \sinh^2(kh)}$$

(4.22)

where (2.8) has been used for $\tilde{5}_c$ and terms smaller by the factor of $k\delta = O(\epsilon^2)$ have been ignored. Similarly the rate of pressure working on the right-hand side of (2.9) can also be evaluated

$$-\frac{\partial}{\partial x} \int_{-kh+k\delta}^{0} \overline{\overline{w}} \overline{p} dZ \approx -\left( \frac{\sinh(2kh) + 2kh}{8 \sinh^2(kh)} \right) \frac{\partial AA^*}{\partial x_2}$$

(4.23)

Summarizing (2.9) and (4.21)-(4.23), we have

$$\frac{\partial AA^*}{\partial x_2} = -\beta AA^*$$

(4.24)
where

\[
\begin{align*}
\beta &= \alpha_c \beta_c + \alpha_b \beta_b \\
\beta_c &= 4 \frac{2kh \cosh^2(kh) - \sinh(2kh)}{kh (\sinh(2kh) + 2kh)} \\
\beta_b &= \frac{2AA*}{\sinh(2kh) + 2kh} \int_{Z_B}^{\infty} Z \frac{\partial K(Z)}{\partial Z} \left( \frac{\partial K(Z)}{\partial Z} \right) \, dZ
\end{align*}
\]

with \( \beta_c \) and \( \beta_b \) mark the contributions from the core and the bed boundary layer respectively.

Since the damping rate affects the evolution over distances much longer than both the wave length and the roughness size, local inaccuracies due to boundary layer approximation may not be crucial. Hence the results here may even apply for a rough bed with large and/or well separated roughness elements (Case C). This will be checked later.
Chapter 5

Free surface

Near the surface, a wave boundary layer also exists in principle. Because of the stress condition on the free surface, this boundary layer is weak. We shall first show that, to the desired accuracy, the boundary layer corrections can be ignored in calculating the current. Next, the approximate surface boundary condition for the mean current will be derived at the mean water level.

5.1 Unimportance of surface wave boundary layer

Dynamically the tangential and normal stresses must vanish on the moving wind-free surface, and the dimensionless boundary conditions (cf. (2.4), (2.6), (2.6)) are

\[-[P + \tau_{xx}] \frac{\partial \eta}{\partial x} + \tau_{zz} = 0, \quad z = \eta\]  

\[-[P + \tau_{zz}] - \tau_{zz} \frac{\partial \eta}{\partial x} = 0, \quad z = \eta\]  

where $P$ is the total pressure (static and dynamic). In addition the kinematic surface boundary condition requires

\[\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} - w = 0, \quad z = \eta\]  

(5.3)
Let us denote the boundary layer corrections of the flow field by \((U', W', P')\), i.e.,

\[ u = u_c + U', \quad w = w_c + W', \quad p = p_c + P' \quad (5.4) \]

where \((u_c, w_c, p_c)\) are the values of \((u, w, p)\) evaluated at the outer edge of the surface boundary layer, where all correction terms should vanish.

First, the oscillatory shear stress must vanish on the free surface, hence,

\[ \frac{\partial u_c}{\partial z} = O \left( \frac{\partial U'}{\partial z} \right) \quad (5.5) \]

In view of the smallness of the eddy viscosity, \(O(\epsilon^2 k\delta)\), where the dimensionless boundary layer thickness near the free surface is also of the order \(k\delta = O(\epsilon^2)\), we have, within surface wave boundary layer

\[ \tilde{U}' = O(\epsilon^2 u_c) = O(\epsilon^3) \quad (5.6) \]

Now the mean current. The boundary-layer correction to the perturbed core current is the result of nonlinear interaction between the irrotational waves and the boundary layer corrections, hence

\[ \tilde{U}' = O(\epsilon \tilde{U}') = O(\epsilon^4) \quad (5.7) \]

The continuity equations for the surface layer corrections to the oscillatory and the mean motions are, respectively,

\[ \frac{\partial \tilde{U}'}{\partial x} + \frac{\partial \tilde{W}'}{\partial z} = 0, \quad \frac{\partial \tilde{U}'}{\partial z} + \frac{\partial \tilde{W}'}{\partial x} = 0 \quad (5.8) \]

From these we find

\[ \tilde{W}' = O(\epsilon^2 \tilde{U}') = O(\epsilon^5), \quad \tilde{W}' \leq O(\epsilon^2 \tilde{U}') \leq O(\epsilon^6) \quad (5.9) \]

From the vertical momentum equation for the surface layer corrections, the fol-
following order of magnitude estimate is obtained

\[ \tilde{W}' \sim \frac{\partial \tilde{P}'}{\partial z} \quad (5.10) \]

Since \( \Delta z = O(\epsilon^2) \), it follows that

\[ \tilde{P}' = O(\epsilon^2 \tilde{W}') = O(\epsilon^7) \quad (5.11) \]

We further estimate that \( \tilde{P}' = O(\epsilon \tilde{P}') = O(\epsilon^8) \) because the perturbed mean boundary layer correction is due to the nonlinear interaction between the irrotational waves and the perturbed wave boundary layer corrections.

In view of the smallness of the correction terms, only the outer solution suffices near the free surface up to an accuracy to be determined soon.

## 5.2 Perturbed mean Reynolds stress on the still water surface

To derive the boundary condition for the perturbed mean Reynolds stress, we integrate the governing equations, (2.2), (2.3) and (2.4), from the moving surface down to the still water surface at \( z = 0 \), invoke the surface boundary conditions and then approximate the results to the desired accuracy. This approach has been employed before by Liu and Davis (1977) [44] in their study of wave-induced current, and is essentially the same as that in the theory of radiation stresses arising in the depth-integrated and time-averaged momentum balance of a wave field (see, e.g., Phillips (1977) [61], Mei (1989) [50]).

By integrating the horizontal momentum equation (2.3) from \( z = 0 \) to the instantaneous sea level, applying the boundary conditions, and taking the time averages, we obtain the following exact result,

\[ \left[ \tilde{r}_{xz} \right]_0 - [\tilde{u} \tilde{w}]_0 = -\frac{\partial}{\partial x} \int_0^\eta (uu + P - \tau_{xz}) \, dz \quad (5.12) \]

On \( z = 0 \), the basic current \( \tilde{u}_0 \) has zero shear stress. It follows from (3.9) that the
leading term is the result of wave disturbance, i.e., $[\tilde{r}_{xx}]_0 = \epsilon^4 [\tilde{r}'_{xx}]_0$. Up to $O(\epsilon^4)$, not only the wave boundary layer corrections can be ignored, but the right-hand side of (5.12) can be approximated so that

$$\epsilon^4 [\tilde{r}'_{xx}]_0 = [\bar{u}\bar{w}]_0 - \frac{\partial}{\partial x} \int_0^\ell Pdz + O(\epsilon^5)$$

(5.13)

On the right-hand side the first term represents the averaged wave-induced momentum flux (wave-induced Reynolds stress) while the second is the averaged net pressure force in the horizontal direction; their sum makes up the mean shear stress at the mean sea level. We now calculate these two terms.

From the definition of the dynamic pressure, $p = P + \frac{\rho a^2}{c^2} z$, we get

$$\frac{\partial}{\partial x} \int_0^\ell Pdz = \epsilon^2 \frac{\partial}{\partial x_2} \left( [\bar{p}] \partial_{\eta} \right) - \epsilon^2 \frac{\partial}{\partial x_2} \left( \frac{\rho^2 g k}{2 \omega^2} \right) + O(\epsilon^5)$$

(5.14)

Making use of the linear wave solutions on the right-hand side, we get

$$\frac{\partial}{\partial x} \int_0^\ell Pdz = \epsilon^4 \frac{\partial \coth(kh) \partial AA^*}{\partial x_2} + O(\epsilon^5)$$

(5.15)

Clearly this variation owes its existence to wave attenuation.

Next we compute the wave-induced Reynolds stress on the mean surface, $[\bar{u}\bar{w}]_0$. For this we turn to the core and recall the following identity

$$\frac{\partial u^2}{\partial x} + \frac{\partial u w}{\partial z} = \frac{\partial E}{\partial x} + w \Omega$$

(5.16)

Since $u = \epsilon \bar{u}_0 + \epsilon^2 \bar{u}'$, $\partial \bar{f}/\partial x_2 = O(\epsilon^2 f)$, and $\bar{w} = O(\epsilon^3)$, we have

$$\frac{\partial \bar{u} \bar{w}}{\partial z} = \frac{\partial \bar{u} \bar{w}}{\partial z} + O(\epsilon^5) = \epsilon^2 \frac{1}{2} \frac{\partial}{\partial x_2} \left( \bar{w}'^2 - \bar{u}'^2 \right) + \bar{w} \bar{\Omega} + O(\epsilon^5)$$

(5.17)

Making use of (3.23) for $\bar{w} \bar{\Omega}$ and the linear wave solutions in (5.17), we get

$$\frac{\partial \bar{u} \bar{w}}{\partial z} = -\alpha \epsilon^4 AA^* \frac{\partial^2 S_c}{\partial z^2} \frac{\sinh^2(kh + z)}{\sinh^2(kh)} - \frac{\epsilon^4}{4 \sinh^4(kh)} \frac{\partial AA^*}{\partial x_2} + O(\epsilon^5)$$

This equation can be integrated from the upper edge of the bottom boundary layer.
to give

$$\tilde{u}\tilde{w}(z) = [\tilde{u}\tilde{w}]_+ - \alpha_c \epsilon^4 AA^* \frac{\partial^2 \mathcal{S}_c \sinh(2(kh + z)) - 2(kh + z)}{4 \sinh^2(kh)}$$

$$\frac{\epsilon^4(kh + z)}{4 \sinh^2(kh) \partial x_2} \frac{\partial AA^*}{\partial x_2} + O(\epsilon^5), \quad kh + z > k\delta. \quad (5.18)$$

Because of (4.3), the first term on the right can be matched to the boundary layer solution given by (4.20),

$$[\tilde{u}\tilde{w}]_+ = \epsilon^4 [\tilde{u}_b\tilde{w}_b]_+ = -\frac{\alpha_b AA^*}{2 \sinh^2(kh)} \int_{Z_B}^{Z_b} \text{Im}(K) dZ \quad (5.19)$$

where $Z_\delta = \delta/(\alpha_b \epsilon^2)$. Substituting this result in (5.18), we obtain

$$\tilde{u}\tilde{w} = \epsilon^4 \frac{\alpha_b AA^*}{2 \sinh^2(kh)} \int_{Z_B}^{Z_b} \text{Im}(K) dZ - \frac{\epsilon^4(kh + z)}{4 \sinh^2(kh)} \frac{\partial AA^*}{\partial x_2}$$

$$- \alpha_c \epsilon^4 AA^* \frac{\partial^2 \mathcal{S}_c \sinh(2(kh + z)) - 2(kh + z)}{4 \sinh^2(kh)} + O(\epsilon^5), \quad kh + z > \delta. \quad (5.20)$$

for all $z$ in the core. In particular its value at $z = 0$ is

$$[\tilde{u}\tilde{w}]_0 = -\frac{\alpha_b AA^*}{2 \sinh^2(kh)} \int_{Z_B}^{Z_b} \text{Im}(K) dZ - \frac{\epsilon^4 kh}{4 \sinh^2(kh)} \frac{\partial AA^*}{\partial x_2}$$

$$- \alpha_c \epsilon^4 AA^* \frac{\partial^2 \mathcal{S}_c \sinh(2kh) - 2kh}{4 \sinh^2(kh)} + O(\epsilon^5) \quad (5.21)$$

The shear stress at the mean sea level then follows by combining (5.15) and (5.21) with (5.13)

$$[\tilde{r}_{zz}]_0 = -\frac{\alpha_b AA^*}{2 \sinh^2(kh)} \int_{Z_B}^{Z_b} \text{Im}(K) dZ - \left( \frac{\coth(kh)}{4} + \frac{kh}{4 \sinh^2(kh)} \right) \frac{\partial AA^*}{\partial x_2}$$

$$- \alpha_c AA^* \frac{\partial^2 \mathcal{S}_c \sinh(2kh) - 2kh}{4 \sinh^2(kh)} \quad (5.22)$$

This upper boundary condition for the core current (3.31) is a key result. Physically, despite the absence of wind, the mean shear stress is not zero on $z = 0$, due to the combined effects of the wave-induced Reynolds stress at the bottom, wave damping and the curvature of the eddy viscosity. There is a similar result in the theory for
deep-water waves, where molecular viscosity gives rise to a finite shear stress on the mean surface due to wave damping ([61]).
Chapter 6

Final solution for the core

6.1 Formulas for mean turbulent stress and current velocity

With the surface boundary condition (5.22), the perturbed turbulent stress follows from (3.31), which can be rearranged as

\[
\tilde{S}_c \frac{\partial \tilde{u}_c'}{\partial z} = -AA^* \tilde{S}_c \frac{\sinh(\kappa h + z)}{\sinh(\kappa h)} \pm \left( \frac{\alpha_c - \alpha_0}{\kappa^2} \frac{-z}{\kappa h} \right) \tau_\delta
\]
\[
+ \beta AA^* (z + \kappa h) (2\kappa h + \sinh(2\kappa h)) \frac{\alpha_c}{8\kappa h \sinh^2(\kappa h)} \tau_a
\]
\[
- \frac{\alpha_b AA^* (\kappa h + z)}{\alpha_c 2\kappa h \sinh^2(\kappa h)} \int_{Z_B}^{Z_s} \text{Im}(K) dZ \tau_b
\]
\[
- AA^* \frac{\partial^2 \tilde{S}_c \sinh(2(\kappa h + z)) - 2(\kappa h + z)}{4 \sinh^2(\kappa h)} \tau_{\tilde{S}}
\]

(6.1)

where (4.24) for the wave damping rate has been used.

As labeled above, the perturbed turbulent stress \( \tilde{S}_c (\partial \tilde{u}_c' / \partial z) \) is the sum of the following parts:

- \( \tau_\delta \): Surface distortion of the eddy viscosity, which is always negative and
greatest on $z = 0$.

- $\tau_{\alpha c}$: Wave-induced change of friction velocity, where the upper (lower) sign corresponds to wave-following (-opposing) current.

- $\tau_\beta$: Wave damping, which is always positive.

- $\tau_B$: Wave-induced Reynolds stress from the bottom boundary layer, which is always negative.

- $\tau_g$: Curvature of the eddy viscosity, which is originated from the vortex force (cf. (3.23)) and is always positive.

A negative stress should lead to reduction (increase) of surface current velocity if waves and current are in the same (opposite) direction. The opposite of course holds for a positive stress. The magnitude of each term will be examined later to find the net consequence. Upon integration from the outer edge of the wave boundary layer, we obtain the wave-perturbed current velocity $\bar{u}_c'$:

$$
\bar{u}_c'(z) = [\bar{u}_c]' + AA^* \int_{-kh+\delta}^{z} \frac{\sinh(kh + z')}{-z'} \sinh(kh) dz' \pm \frac{\alpha_c - \alpha_0}{\kappa^2} \ln \left( \frac{kh + z}{\delta} \right) - \frac{\beta AA^*}{\alpha_c} \left( \frac{2kh + \sinh(2kh)}{8 \sinh^2(kh)} \right) \ln \left( \frac{-z}{kh - \delta} \right) + \left( \frac{\alpha_0}{\alpha_c} \frac{AA^*}{2 \sinh^2(kh)} \int_{Z_B}^{Z_B} \operatorname{Im}(K(Z)) dZ \right) \ln \left( \frac{-z}{kh - \delta} \right) - \frac{AA^*}{\partial z^2} \int_{-kh+\delta}^{z} \left( \frac{\sinh(2(kh + z') - 2(kh + z'))}{4 S_c \sinh^2(kh)} \right) dz' \tag{6.2}
$$

where $[\bar{u}_c]'$ given by (4.15). The final current velocity in the core is $\bar{u} = \epsilon \bar{u}_0 + \epsilon^2 \bar{u}_c'$ by summing up (2.14) and (6.2). Note that $\bar{u}_c'$ depends on two parameters $\alpha_c$ and $\alpha_0$, which remain to be determined. For a smooth bed (Case A), we simply set $\alpha_c = \alpha_0$.

### 6.2 Friction velocities

For a smooth bed (Case A), the friction velocity $u_{fc}$ or its dimensionless equivalent $\alpha_c$ is determined by requiring the constancy of discharge (3.3).
For a rough bed (Cases B), \( \alpha_b \neq \alpha_c \); we must further invoke the scheme of Grant and Madsen. To facilitate comparison we begin with the dimensional form of the friction velocity inside the bottom boundary layer. The total friction velocity \( u_{fb} \) is defined by the magnitude of the total bottom shear stress

\[
\rho u_{fb}^2 = \max \left( \tau_{xz}^* \right)_B
\]

(6.3)

where \( \tau_{xz}^* \) represents a quantity evaluated at the bottom \( z^* = -h + z_B \) in physical coordinates, or \( z = -kh + kz_B \) in the dimensionless coordinates.

By requiring that waves and current assume the same eddy viscosity \( \nu_b \), (2.11), the Reynolds stress \( \tau_{xz}^* \) can be written as

\[
\tau_{xz}^* = \rho \kappa u_{fb} \left[ (h + z^*) \frac{\partial \tilde{u}^*}{\partial z^*} + (h + z^*) \frac{\partial \tilde{u}^*}{\partial z^*} \right]_B
\]

(6.4)

where terms smaller than the retained by a factor of \( k \delta \) have been ignored. In view of the definition of \( u_{fb} \), (6.3), it follows from (6.4) that

\[
\tau_{xz}^* = \kappa \max \left[ (h + z^*) \frac{\partial \tilde{u}^*}{\partial z^*} + (h + z^*) \frac{\partial \tilde{u}^*}{\partial z^*} \right]_B
\]

(6.5)

After normalizing \( u_{fb} \) by \( C_r^2 \), and substituting (3.1) and (4.3) in (6.5), we obtain

\[
\alpha_b = \kappa^2 \max \left[ \kappa + Z \left( \frac{\partial \tilde{u}_0}{\partial Z} + Z \left( \frac{\partial \tilde{u}_b}{\partial Z} \right) + \frac{Z \partial \tilde{u}_b}{\partial Z} \right) \right]_B + O(\epsilon)
\]

(6.6)

In accordance with (2.4), the first term in brackets above is just \( \pm \alpha_0 / \kappa^2 \). From the governing equation of the mean current inside the bed boundary layer, (4.13), the second term in the brackets of (6.6) is

\[
\left[ Z \frac{\partial \tilde{u}_b'}{\partial Z} \right]_B = - \left( \pm \frac{\alpha_0 \alpha_b - \alpha_c^2}{\alpha_b \kappa^2} + \frac{[\tilde{u}_b \tilde{w}_b]_+}{\alpha_b} \right) + O(\epsilon)
\]

(6.7)

where \( [\tilde{u}_b \tilde{w}_b]_+ \) is given by (4.20 ). Use has been made of the fact that the wave-induced Reynolds stress vanishes on the seabed. Lastly, the wave solution inside the
The bed boundary layer, (4.17), gives the third term

\[
\left[ -\frac{A}{2 \sinh(kh)} \left( Z \frac{\partial K}{\partial Z} e^{i\theta} + c.c. \right) \right]_B
\] (6.8)

For small \( Z_B \) as in many field cases, we can approximate the Kelvin function ([1]) to get

\[
\left[ Z \frac{\partial K}{\partial Z} \right]_B = \frac{1}{2 \ln \left( \frac{1 + \gamma}{\gamma} \right) + \ln(Z_B) + 2\gamma} + O(Z_B)
\] (6.9)

where \( \gamma = 0.57722 \). Therefore, for sufficiently small \( Z_B \), (6.8) is practically of order \( O(\epsilon) \) and all terms on the right of (6.6) are of order unity.

For any complex number \( a = |a|e^{i\Theta} \), where \( \Theta \) is the phase of \( a \), the following is an identity,

\[
\max [ae^{i\Theta} + c.c.] = \max [|a|e^{i(\Theta+\Theta)} + c.c.] = 2|a|
\] (6.10)

With this, (6.6) can be reduced to

\[
\alpha_b = \kappa^2 \left[ \pm \frac{\alpha_0}{\kappa^2} - \left( \pm \frac{\alpha_0 \alpha_b - \alpha_c^2}{\alpha_b \kappa^2} + \frac{\overline{\omega_b \omega_b}}{\alpha_b} \right) \right] + \frac{\kappa^2}{\epsilon} \frac{A}{\sinh(kh)} \left( Z \frac{\partial K}{\partial Z} \right)_B
\] (6.11)

This implicit relation between \( \alpha_b \) and \( \alpha_c \) reduces to that of Grant & Madsen only if we disregard the wave-induced Reynolds stress \( \overline{\omega_b \omega_b} \), which is important here.

### 6.3 Numerical procedure

In water of given depth \( h \), we suppose that the basic current and the wave parameters are known in advance.

For a steady current, the bottom roughness \( z_B \) is usually determined by fitting the logarithmic profile with the measured profile of the pure current. Based on experiments [48, 47] have shown that the same value can be taken if waves are also
The numerical procedure of predicting the friction velocity $u_{fb}$ (hence $\alpha_b$) is iterative:

1. Calculate from the basic current $f_c$ according to (2.16) for $z_B$ given by fitting empirical data, and then compute the friction velocity $u_{fc} = \langle \bar{u}_0 \rangle \sqrt{f_c/2}$, which gives the dimensionless friction velocity $\alpha_0$.

2. Starting from $\alpha_c = \alpha_0$, we compute $\alpha_b$ from (6.11) by iteration. The value of $Z_\delta$ defined by (4.18) is found during the iteration.

3. Solve for $\bar{u}_c'$ according to (6.2).

4. Check if the discharge condition (3.4) is satisfied within the allowed error. If not, Step (b) is repeated with a new trial $\alpha_c$. The iteration procedure is continued until the discharge condition is satisfied. The solution of $\bar{u}_c'$ is then found.

5. Compute the total mean velocity in the core from $\bar{u} = \epsilon \bar{u}_0 + \epsilon^2 \bar{u}_c'$.

In the next sections, we shall compare the predicted current profile at one station with available experiments, and predict further the velocity profiles at different stations in the direction of wave propagation.
Chapter 7

Comparison with experiments

7.1 Past experiments

Laboratory studies in long flumes on wave-following currents have been reported by Bakker and van Doorn (1978) [2], Brevik and Aas (1980) [5], Myrhaug (1987) [55], and Mathisen and Madsen (1996a, 1996b) [48, 47], only for the region near a rough bottom. Myrhaug (1987) [55] also reported results for the region near the smooth bottom. Measurements of the current velocity profile for the entire depth are more limited. We are aware of only two experiments by Kemp and Simons (1982, 1983) [32, 33] and by Klopman (1994, 1997) [34, 35] for rough beds. These experiments are classified in Table 7.1, according to the empirical criteria of Kamphuis (1975) [31]. It is evident that only Kemp and Simons (1982) [32] with smooth bed falls clearly in Case A. Further discussions will be limited to the works of Kemp & Simons and Klopman.

Kemp and Simons have reported the current profiles for wave-following currents

<table>
<thead>
<tr>
<th>Authors</th>
<th>Roughness</th>
<th>( \Lambda_k )</th>
<th>( \Delta_k )</th>
<th>( Re_w(10^4) )</th>
<th>( \alpha_b/k_N )</th>
<th>Bed Conditions</th>
</tr>
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<tr>
<td>Bakker &amp; Doorn (1978)</td>
<td>rectangular</td>
<td>15</td>
<td>2</td>
<td>1.9 \sim 2.8</td>
<td>3.35 \sim 4.11</td>
<td>C</td>
</tr>
<tr>
<td>Kemp &amp; Simons (1982)</td>
<td>smooth bed</td>
<td>0</td>
<td>0</td>
<td>0.07 \sim 0.36</td>
<td>\infty</td>
<td>A</td>
</tr>
<tr>
<td>Kemp &amp; Simons (1982)</td>
<td>triangular</td>
<td>18</td>
<td>5</td>
<td>0.07 \sim 0.36</td>
<td>0.94 \sim 1.76</td>
<td>C</td>
</tr>
<tr>
<td>Kemp &amp; Simons (1983)</td>
<td>triangular</td>
<td>18</td>
<td>5</td>
<td>0.07 \sim 0.36</td>
<td>0.94 \sim 1.76</td>
<td>C</td>
</tr>
<tr>
<td>Mathisen &amp; Madsen (1996)</td>
<td>triangular</td>
<td>100</td>
<td>15</td>
<td>0.5 \sim 1.6</td>
<td>0.2 \sim 0.44</td>
<td>C</td>
</tr>
<tr>
<td>Mathisen &amp; Madsen (1996)</td>
<td>triangular</td>
<td>200</td>
<td>15</td>
<td>0.5 \sim 1.8</td>
<td>1.14 \sim 1.74</td>
<td>C</td>
</tr>
<tr>
<td>Klopman (1994)</td>
<td>sand</td>
<td>2</td>
<td>2</td>
<td>1.1</td>
<td>4.25</td>
<td>C</td>
</tr>
</tbody>
</table>

Table 7.1: Wave and bed conditions of available experiments: \( \Delta_k \) (mm) = roughness height and \( \Lambda_k \) (mm) = distance between the roughness elements.
over smooth and rough beds (1982), and wave-opposing currents (1983) over a rough bed only, in a small wave tank of 14.5m length, 0.457m width and 0.69m height. Roughness was created with 5 mm triangular strips separated at 18 mm intervals along the channel. The still water depth was 0.2m. Along the centerline the depth-averaged mean current velocity was 18.5 cm/s (1982) and -11 cm/s (1983). The corresponding steady discharge rates were $Q = 0.037 m^2/s$ and $-0.022 m^2/s$ respectively. Waves with period $T = 1.006s$ were added to the turbulent current. The wave amplitudes $a$ ranged from 11.35 to 23.3 mm for the wave-following current of 1982, and 13.95 to 29.55 mm for the wave-opposing current of 1983. All full-depth profiles of currents were measured on top of a strip at a station 8.07 m away from the wave maker. As shown in Ridler and Sleath (2000) for pure waves of similarly roughened bed, current profiles measured at different stations between two strips can be vastly different. Wave-affected current profiles may likely vary with the horizontal station. Such data is unfortunately unavailable for the full depth.

Klopman (1994) performed similar experiments in a larger wave flume 45 m long and 1m wide. The still water depth was 0.5m. The bed was roughened by coarse sand of 2 mm mean diameter. The test section was located at 22.5 m away from the wave maker. The discharge was $Q = 0.08 m^2/s$. In the absence of waves the mean velocity profile of the turbulent current was essentially logarithmic. A monochromatic wave train of period $T = 1.44s$ and amplitude $a = 0.06m$ was then superposed on the current which either followed or opposed the current. No tests with waves of other amplitude or frequency were reported. Later Klopman (1997) repeated the experiments under identical conditions, and measured the transverse velocity in addition with a view to examining Langmuir circulation due to the finite tank width. The mean velocity profile along the centerline was measured at the same station but at only five different depths. The current velocities at all five measuring depths were however greater than the 1994 data by 0.0128 m/s. We shall only use the more comprehensive measurements of Klopman (1994) for discussion. It is worth pointing out however, that corresponding to the ratio $a_b/k_N$ listed in Table 7.1, the ratio $\delta/k_N \sim 1.4$ is not large. Thus the sand grains are not deeply immersed in the boundary layer.

Being the only full-depth observations available, the measurements by Kemp and
7.2 Wave-following current over a smooth bed by Kemp & Simons

In Table 7.2 the experimental parameters are summarized. Also included is the reflection coefficient $R$, which is $O(\epsilon)$ in all four cases. It is known (Mei(1989) [50]) that reflection of this intensity does not affect the induced streaming except at the order $O(\epsilon^2 R)$ which is negligible here. We also display the flux discrepancy $\Delta_{fux}$ which is a measure of the difference between the measured discharge and the theoretical current discharge, and is implied by (3.4) with $AA^* = 1$.

$$\Delta_{fux} = 2 \int_{-k\delta}^{0} \frac{\tilde{u}_c' dz}{\coth(kh)} + 1$$  \hspace{1cm} (7.1)

When the measured value of $\tilde{u}_c'$ is used in the preceding formula, departure from zero suggests difficulty in the velocity measurements. We also compare the values of $[\tilde{u}]_+$ obtained from both theory and experiments to show that (4.15) can indeed provide good lower boundary value for the core current over a smooth bed.

Table 7.2: Parameters of Kemp and Simons (1982) for wave-following current over a smooth bed. Wavelength $L = 1.23m$, $k = 5.14m^{-1}$, $C = 1.23m/s$ and $kh = 1.03$. The four tests WCA1 to WCA5 are ordered by the wave steepness $\epsilon$. WCA2 was not reported.

Simons (1982, 1983) [32, 33] and Klopman(1994) [34] are compared with our theory in the next section, despite the differences in seabed conditions.
Figure 7-1: Various contributions to the mean stress for wave-opposing current over a smooth bed (Kemp and Simons(1982)[32]), Run WCA5: Solid line: total mean shear stress due to waves. The contributing factors are: $\tau_S$: surface distortion of eddy viscosity; $\tau_{oc}$: change of friction velocity; $\tau_B$: wave damping; $\tau_B$: wave-induced Reynolds stress from BWBL; $\tau_{Sc}$: Curvature of eddy viscosity due to vortex force.

To help understand the physics, let us first examine a typical stress distribution for the perturbed current, as defined in (6.1). Figure 7-1 shows the wave-induced mean stress for Run WCA5. Both wave damping $\tau_B$ and curvature of the eddy viscosity $\tau_{Sc}$ are positive and increase from zero at the bed to their maximum at the water surface. Negative stresses, which tend to retard the current, are contributed by $\tau_{oc}$, $\tau_B$, and $\tau_S$. Due to the surface distortion of eddy viscosity, $\tau_S$ contributes the most to slow down the current near the water surface, while $\tau_{oc}$ (also negative) in this case contributes little.

In Figure 7-2 the velocity profiles of all reported runs by Kemp & Simons are compared with the predictions. The agreement between theory and experiments is quite good. In particular, the reduction of the current velocity near the free surface is well predicted. This suggests that not only the shear stress but also the edge velocity are well described by (6.1) and (4.15) respectively. For Run WCA1 the relative error in the total discharge $\Delta_{flux}$ is the largest, the agreement is poorer, due likely to the smaller wave steepness and the smaller effects on the current.

No wave-opposing current profiles over a smooth bed have been reported by Kemp & Simons.
Figure 7-2: Comparisons with WCA1, WCA3, WCA4 and WCA5 of Kemp & Simons (1982) for wave-following current over a smooth bed. Solid line: predicted current with waves; dashed line: predicted pure current; circles: measured current with waves. crosses: measured pure current.

7.3 Wave-following current over a rough bed by Kemp & Simons

From Kemp and Simons (1982) [32] a summary of experimental parameters is shown in Table 7.3. The roughness height $z_B$ is computed by fitting the measured velocity of pure current with the logarithmic profile, with the tank bottom (the base of the strips) defined as $z = -kh$. Note that for the weaker waves WCR1, WCR3 and WCR4, the flux discrepancy is very large, suggesting possible difficulty in measuring small velocity variations and boundary layer effects near the side walls. Comparisons are shown for Runs WCR1, WCR3, WCR4 and WCR5. It also turns out that the lower boundary value $[\bar{u'}_+^2]$ given by (4.15) differs markedly from the data, see Table 7.3. Only by using the measured velocity $\bar{u}_r$ at a reference height $z_r$, shown in the same table, a fair agreement between predicted and measured velocities is achieved for WCR4 and WCR5, as shown in Figure 7-3. The need for this empirical fitting is not surprising since the profiles were all measured at the station where a strip is located, and are most likely different from the profiles at other stations between the
<table>
<thead>
<tr>
<th>Parameter</th>
<th>WCR1</th>
<th>WCR3</th>
<th>WCR4</th>
<th>WCR5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$</td>
<td>0.06</td>
<td>0.08</td>
<td>0.10</td>
<td>0.13</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>2.12</td>
<td>1.08</td>
<td>0.59</td>
<td>0.42</td>
</tr>
<tr>
<td>$R$</td>
<td>0.088</td>
<td>0.078</td>
<td>0.074</td>
<td>0.072</td>
</tr>
<tr>
<td>$\Delta_{flux}$</td>
<td>8.102</td>
<td>4.268</td>
<td>1.312</td>
<td>0.205</td>
</tr>
<tr>
<td>$kz_r$</td>
<td>0.13</td>
<td>0.15</td>
<td>0.13</td>
<td>0.15</td>
</tr>
<tr>
<td>$\bar{u}_r$</td>
<td>2.13</td>
<td>1.7</td>
<td>1.21</td>
<td>0.92</td>
</tr>
<tr>
<td>$[\bar{u}]_+$</td>
<td>1.70</td>
<td>1.33</td>
<td>1.03</td>
<td>0.77</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>0.47</td>
<td>0.14</td>
<td>0.13</td>
<td>0.19</td>
</tr>
<tr>
<td>$Z_\delta$</td>
<td>7.48</td>
<td>7.1</td>
<td>6.5</td>
<td>6.13</td>
</tr>
<tr>
<td>$k\delta \times 10$</td>
<td>0.9</td>
<td>0.9</td>
<td>0.86</td>
<td>1.12</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>1.50</td>
<td>0.46</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>$\alpha_b$</td>
<td>3.42</td>
<td>1.98</td>
<td>1.32</td>
<td>1.08</td>
</tr>
</tbody>
</table>

Table 7.3: Parameters of Kemp and Simons (1982) for wave-following current over a rough bed. Wavelength $L = 1.23m$, $k = 5.14m^{-1}$, $C = 1.23m/s$, and $kh = 1.03$. Hydraulic roughness $z_B = 0.208cm$ ($kz_B = 0.0105$). The four tests WCR1 to WCR5 are ordered by the wave steepness $\epsilon$. WCR2 was not reported.

strips. With similar fitting, the agreement is of course worse for Run WCR4 than for Run WCR5 in view of the very large discharge error $\Delta_{flux}$ for the former. Due to the relative large flux error in WCR1 and WCR3, the agreement between the predictions and the measurements are poor for WCR1 and WCR3. We also remark that when the error in the measured flux is large the predicted profile will be greatly affected by the location of $z_r$.

To help understand the physics, we plot in Figure 7-4 various wave-induced stress components as defined in (6.1), for case WCR5. Note that $\tau_\beta$ and $\tau_{S'}$ are again both positive and the greatest near the free surface. All these tend to increase the current speed near the free surface. $\tau_B$ is negative but is almost canceled by $\tau_\beta$. Due to waves the dimensionless core friction velocity $\alpha_0$ is smaller that $\alpha_0$ of the pure current $(\alpha_c - \alpha_0 = 0.29 - 0.42 = -0.13)$. Thus $\tau_{ac}$ attains the negative maximum at the bottom. Most important, the shear stress $\tau_\delta$ due to the distortion of eddy viscosity is negative and large and is the dominating factor for reducing the current velocity near the free surface. The combined effect is much stronger than the case over a smooth bed.

Although the current profiles were measured only at a single station, records are available for the wave height along the channel, for Run WCR5 only. In terms of the
Figure 7-3: Profiles of wave-following current over a rough bed by Kemp and Simons (1982), Runs WCR1, WCR3, WCR4 and WCR5. Solid line: theoretical profile with waves; dashed line: theoretical profile of pure current; circles: measured profile with waves. Cross shows the height ($kz_r$) where empirical fitting with ($\bar{u}_r$) is made.

Figure 7-4: Predicted contributions to the mean stress for Run WCR5 of Kemp & Simons (1982). Solid line: total mean shear stress due to waves. Various contributing parts are: $\tau_S$: surface distortion of eddy viscosity; $\tau_{\alpha_e}$: change of friction velocity; $\tau_d$: wave damping; $\tau_B$: wave-induced Reynolds stress from the bottom wave boundary layer; $\tau_{h_{\theta^v}}$: curvature of the eddy viscosity due to vortex force.
normalized slow coordinate $x_2$ the length of the tank is only $\Delta x_2 = 0.69$. Hence we use the damping coefficient $\beta$ computed for the starting station $x_2 = 0$ for calculating the wave attenuation. The predicted and measured wave heights are shown in Figure 7-5. The reasonable agreement gives partial confirmation of the boundary layer model which predicts comparable importance of dissipation in the bottom boundary layer and in the core.

### 7.4 Wave-opposing current over a rough bed by Kemp & Simons

A summary of dimensionless parameters for all the wave-opposing current tests, both given and predicted, are listed in Table 7.4.

Note that the discharge discrepancy is relatively small in all cases. Using input parameters computed from data as listed in Table 2, numerical results are obtained for the various components of the shear stresses and the current velocity profile. The predicted parameters $\alpha_0$ and $\alpha_0'$ are also recorded in Table 7.4 above. Also, in these runs the predicted $\langle u \rangle_+$ nearly equals the measured value at the same station with a strip. This coincidence can only be accidental in view of the separation of strips.

Let us first examine, for the representative Run WDR5, the stress distribution for the perturbed current, as defined in (6.1). Again both wave damping $\tau_0$ and curvature of the eddy viscosity $\tau_0^r$ are positive, but they now tend to slow down the current.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>WDR1</th>
<th>WDR2</th>
<th>WDR3</th>
<th>WDR4</th>
<th>WDR5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$</td>
<td>0.07</td>
<td>0.09</td>
<td>0.10</td>
<td>0.13</td>
<td>0.15</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>0.84</td>
<td>0.55</td>
<td>0.43</td>
<td>0.26</td>
<td>0.19</td>
</tr>
<tr>
<td>$R$</td>
<td>0.025</td>
<td>0.03</td>
<td>0.023</td>
<td>0.026</td>
<td>0.02</td>
</tr>
</tbody>
</table>

From data:

| $\Delta_{flux}$ | 1.610 | 0.520 | -0.309 | -0.222 | 0.104 |
| $k_z r$ | 0.13 | 0.18 | 0.15 | 0.18 | 0.15 |
| $\bar{u}_r$ | -0.59 | -0.67 | -0.55 | -0.45 | -0.32 |
| $[\bar{u}_c]_+$ | -0.500 | -0.471 | -0.469 | -0.357 | -0.285 |

Calculated:

| $k \delta \times 10$ | 6.8 | 6.62 | 6.45 | 6.00 | 5.62 |
| $\alpha_c$ | 1.32 | 0.86 | 0.64 | 0.47 | 0.36 |
| $\alpha_b$ | 2.75 | 2.02 | 1.67 | 1.22 | 0.97 |

Table 7.4: Parameters of Kemp and Simons (1983) for wave-opposing current over rough bottom. Wavelength $L = 1.23m$, $k = 5.14m^{-1}$, $C = 1.23m/s$, and $kh = 1.03$. Hydraulic roughness $z_B = 0.203cm$ ($kz_B = 0.0104$). The five tests WDR1 to WDR5 are ordered by the wave steepness $\epsilon$.

Figure 7-6: Various contributions to the mean stress for wave-opposing current over rough bottom of Kemp and Simons (1983) [33], Case WDR5. Solid line: the total mean shear stress due to waves. The contributing factors are: $\tau_S$: surface distortion on the eddy viscosity; $\tau_{\alpha_c}$: change of the friction velocity in the core region; $\tau_\beta$: wave damping; $\tau_B$: wave-induced Reynolds stress from the bottom wave boundary layer; $\tau_S^\nu$: curvature of the eddy viscosity.
Figure 7-7: Current profile of WDR5 by Kemp and Simons (1983) for wave-opposing current over a rough bed. Solid line: theoretical profile; dashed line: theoretical profile of pure current; circles: measured profile. Cross shows the height ($kz_s$) where empirical fitting with ($\bar{u}_r$) is made.

They both attain their maximum at the water surface and vanish at the bottom.

Negative stresses, which tend to speed up the current, are contributed by $\tau_{ac}$, $\tau_B$ and $\tau_S$. Again, $\tau_B$ due to the wave-induced Reynolds stress at the outer edge of the bottom wave boundary layer, nearly cancels the opposite effect of $\tau_B$. The negative stress due to the surface distortion of eddy viscosity $\tau_S$ contributes the most to speed up the current near the water surface. Now since $\alpha_c - \alpha_0 = 0.36 - 0.19 = 0.17$, $\tau_{ac}$ attains a very large negative maximum at the bottom, contributing to the flattening of the velocity profile near the bed, in comparison with the pure current. The net shear stress shown in solid line is negative throughout the entire depth but more so that the wave-following case of WCR5. Now the negative shear stress near the free surface also causes the surface current to be stronger than that of a pure current. Figure 7-7 compares the predicted current profile $\bar{u}_0 + \epsilon \bar{u}_r$, with the measured data by Kemp & Simons for case (WDR5). The velocity profile of a pure current (logarithmic, dotted) is also included for reference. The overall agreement between the theory and experiments is very good. Reduction near the bottom and increase near the surface of the mean velocity are correctly predicted. Note in Table 2, that there is an increase in friction velocity due to waves, i.e., $\alpha_b > \alpha_c$, which is responsible for the reduction of the mean current velocity near the bed, consistent with most past observations, contributing to an increase of the apparent bottom roughness (Grant and Madsen(1986) [23]).
Figure 7-8: Current profiles for wave-opposing current over a rough bed. Solid line: predicted profile with waves; dashed line: predicted profile of pure current; circles: measured profile with waves. Cross shows the height \( (k_z) \) where empirical fitting with \( (\ddot{u}_r) \) is made.

Additional comparisons for all other Runs WDR1-WDR4 are presented in Figures 7-8, with satisfactory agreement. The stress distributions resemble those shown in Figure 7-6 and are omitted.

For Run WDR4, the attenuation of wave height \( H = 2a \) with distance has been reported by Kemp and Simons (1983)[33]. In terms of the normalized slow variable \( x_2 \) the length of the tank is \( \Delta x_2 = 0.93 \). Using the damping coefficient \( \beta \) at \( x_2 = 0 \) (the current-measuring station at mid tank) we compare in Figure (7-9) the computed \( H \) and those measured against the distance from the wave maker. The agreement between prediction and data is good, confirming again the comparable importance of dissipation within the boundary layer and in the core above.

For WDR4, the eddy viscosity near the bottom was reported by Kemp and Simons (1983). The scatter of data was so large that no meaningful comparison with theory can be made.
7.5 Wave-following/opposing currents over a rough bed by Klopman

In Klopman (1994) [34], the wave number is \( k = 2.34m^{-1} \) according to the linear wave theory. Bottom is roughened by sand of a diameter 2mm. Other experimental and calculated parameters are summarized in Table 7.5. From Table 7.5 we find \( \delta/k_N \sim 1.4 \) for both wave-following and opposing currents. Therefore, wave-induced turbulence is likely three dimensional. Nevertheless after fitting the velocity at one height, \( z_r \), the predicted mean velocity profile compares with Klopman’s data very well for both wave-following and wave-opposing currents, as shown in Figure 7-10.

7.6 Longitudinal variation of current

In all the existing experiments cited here, current measurements for the full depth were made only at a single station. In view of the empirical assumptions needed in the theory, it would be desirable to have new measurements at many stations in order to provide firmer checks. For this purpose we have extended our calculations to include the change of the current profiles along the channel, provided that the wave and current conditions are specified at one station. Predictions are shown only for a smooth bed.

Recalling the definition of \( x_2 = k^3a^2x^* \), we start from station \( x_2 = 0 \) where
Table 7.5: Parameters of Klopman (1994) for wave-following and wave-opposing currents over a rough bed. Dimensionless hydraulic roughness $k z_B = 9.2 \times 10^{-4}$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>wave-following current</th>
<th>wave-opposing current</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$</td>
<td>0.14</td>
<td>0.14</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>0.119</td>
<td>0.119</td>
</tr>
<tr>
<td>$R$</td>
<td>&lt; 0.02</td>
<td>&lt; 0.02</td>
</tr>
<tr>
<td>From data</td>
<td>$\Delta_{flux}$</td>
<td>0.021</td>
</tr>
<tr>
<td></td>
<td>$k z_r$</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>$\bar{u}_r$</td>
<td>0.56</td>
</tr>
<tr>
<td></td>
<td>$[\bar{u}<em>c]</em>+$</td>
<td>0.4</td>
</tr>
<tr>
<td>Calculated</td>
<td>$k \delta \times 10^2$</td>
<td>3.74</td>
</tr>
<tr>
<td></td>
<td>$\alpha_c$</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>$\alpha_b$</td>
<td>0.47</td>
</tr>
</tbody>
</table>

Figure 7-10: Comparisons with Klopman (1994) for wave-following and wave-opposing currents. In this figure the basic current is from left to right. A : waves are from left to right. B: waves are from right to left. Solid line : theoretical profile with waves; dashed line : theoretical profile of pure current; circles : experimental data for currents with waves. Crosses show the heights $(k z_r)$ where empirical fitting with $(\bar{u}_r)$ are made.
$A(0) = 1$ and advance to larger $x_2 > 0$ in the direction of wave propagation. For a smooth bed the discharge of the pure current, or the depth-averaged velocity $\bar{u}_0$, can be specified so that $\alpha_0$ can be easily computed. Afterward, the numerical algorithm to compute $\bar{u}(x_2)$ is straightforward. Let the wave and current conditions be known at $x_2 = X_{k-1}$. To predict the current velocity at $x_2 = X_k = X_{k-1} + \Delta x_2$, we first compute the wave amplitude $A(X_k) = A(X_{k-1}) - \frac{1}{2} \beta(X_{k-1}) \Delta x_2$, where $\beta(X_{k-1})$ is the energy dissipation rate at $x_2 = X_{k-1}$. Then $\alpha_c(X_k)$ and $\alpha_b(X_k)$ are calculated by the procedures described in (§6.3).

Two examples are presented in Figure 7-11 for a smooth bed. One is for the wave-following current and the other for the wave-opposing current. The wave and current conditions at the start ($x_2 = 0$) are the same as those in Run WCA5 of Kemp and Simons (1982)[32]. In both cases, as the distance from $x_2 = 0$ increases, the current profile approaches the limit of a pure current without waves. Reduction or increase of the current velocity near the bottom becomes smaller and smaller. When the waves are eventually damped out, the current profile becomes logarithmic in depth. Note that the total Eulerian flux decreases (increases) with distance when waves and current have the opposite (same) direction, as expected by mass conservation (3.4).

Similar predictions have been made for a rough bed and with very large $a_b/k_N$, for possible comparisons with future experiments in a large flume or in the field. Results resemble those in Figure 7-11 and are not presented.

Figure 7-11: Changes along the tank. The damping scale $x_2$ is defined by $x = \varepsilon^2 x_2$ and the spatial grid step is $\Delta x_2 = 2$. 
Chapter 8

Concluding remarks of part I

In this part, a boundary-layer theory for predicting the wave effects on a turbulent current over a smooth or rough bed has been presented. The current is assumed to be as strong as the orbital velocity of waves and renders the flow turbulent throughout the depth. For a smooth bed, a model of continuous eddy viscosity parabolic in depth is used. For a rough bed, a discontinuous eddy viscosity is used instead to account for the combined effects of waves and current in the bed wave boundary layer. At the leading order turbulence in both the bed boundary layer and the core result in wave attenuation over a long distance of many wave lengths. At the second order waves modify the mean flow through wave-induced Reynolds stresses due to oscillations in all regions of the flow. In particular the wave-induced mean shear stress is not zero on the still water surface despite the absence of wind, and is not constant in depth above the bottom boundary layer. Of special importance is that this mean shear stress on the free surface is opposite in direction to the waves. As a consequence, a wave-following current must experience a reduction, while a wave-opposing current experiences an increase, in speed. This phenomenon is due largely to the distortion of eddy viscosity at the free surface.

Comparison with existing experiments for a smooth bed is successful. For a rough bed, the roughness elements in existing experiments are either relatively large or well separated, making a boundary layer analysis difficult. Nevertheless the present theory still gives a good prediction of current for the entire depth, if empirical matching is made at one depth just above the bed boundary layer (as in Grant & Madsen). For
beds roughened by well separated strips, this theory appears adequate for predicting quantities that vary slowly in the horizontal direction, such as the attenuation rate, and may likely give a good prediction of the horizontally averaged velocities profiles, without empirical fitting. To check this speculation additional measurements either for deeply submerged roughness, or of current profiles at several closely spaced stations between strips and over a long fetch, would be worthwhile.

Extension of this study to the three-dimensional problem of Langmuir circulation in shallow water is of value to the transport of fine sediments in lakes and coastal waters, and will be discussed in next part of the thesis.
Part II

Langmuir circulation in water of finite depth
Chapter 9

Introduction

In a clear day, windrows (streaks) of foam or floating materials often can be observed on the surface of lakes or oceans. These windrows (streaks) are oriented approximately parallel to the wind. Langmuir (1938) [37] was the first one to make a general field investigation of the nature of windrows (streaks) on lakes or oceans.

Figure (9-1) shows windrows (streaks) on Rodeo Lagoon in Marin County, California, 1994. The surface signatures shown in the photo appears as windrows of bubbles, oriented approximately along the wind direction. The average depth of the Lagoon is 1.4m and the width shown in the photo is about 200m.

From the studies on Lake George, New York, Langmuir (1938) [37] found that the streaks were associated with the vertical motions and oriented nearly parallel to the wind. The field measurements of Langmuir (1938) [37] and others (Myer (1971)[54], Weller (1985)[75], Weller and Price (1988)[76], etc.) confirmed the presence of counter-rotating vortices whose axes were aligned approximately parallel to the wind. Surface floating tracers formed lines when swept slowly into the surface convergent regions between the adjacent vortices. The motion associated with these vortices are now called Langmuir circulation. The formation of the windrows, as described by Langmuir is shown in Figure (9-2). The streaks or windrows were the surface convergences beneath which water move downward (down-welling), while between streaks water move upward (up-welling). Figure (9-3) shows the three components of the velocity field measured by Weller et al (1985)[75]. As shown in Figure (9-3), Weller et al (1985)[75] and Weller and Price (1988)[76] found that the maximum vertical
velocity was about 20 cm/s, much larger than previously believed (see, e.g., Pollard (1977)[62]) and the maximum vertical velocity appeared well below the surface.

Earlier observations focused on the velocity scales and spacing of Langmuir circulation at the steady state or the quasi-steady state. A fairly complete list of the field observations conducted before 1983 has been given by Leibovich (1983)[39]. Recent field observations focused on the evolution of the Langmuir circulation in space and time (Weller et al (1985) [75], Smith et al (1987) [67], Smith (1992) [66], Gramham and Hall (1997) [21], etc).

From the field observations mentioned above, the following conclusions can be drawn on the typical scales of Langmuir circulation:

- The typical spacing of Langmuir cells is comparable to the wave length.
- The typical penetration depth of the cells is also comparable to the wave length, thus the aspect ratio of the cell is about unity.
- The horizontal and vertical velocities associated with the Langmuir circulation

Figure 9-1: Windrows or streaks on Rodeo Lagoon in Marin County, California. From Seriz (1996) [68]
Figure 9-2: Formation of windrows or streaks on the surface of the deep ocean. D is the penetration depth of the Langmuir cells and L the spacing between the cells. From Leibovich (1983)[39]

are comparable to the Stokes drift. Numerically, the value of the longitudinal velocity is larger than that of the transverse velocities.

- Small cells exist between large cells (Langmuir (1938)[37], Weller and Price(1988)[76], Gramham and Hall (1996)[21], etc). Thus there is a spectrum of the cell spacing. The large and very small spacings have relatively small frequency of appearance, as shown by Figure (9-4).

- The time scale over which the Langmuir circulation evolves is about several hundreds wave periods (Smith(1992)[66]).

- The streaks or windrows can extend a distance 100 times the spacing before breakdown. Thus the Langmuir cells are relatively stable and localized (Gramham and Hall (1997)[21]).

Langmuir (1938) [37] was the first one who believed that such relatively large-
scale, organized, three dimensional vortex motion should play a significant role in the vertical mixing. As vertical mixing is important for the transport of heat, oxygen, nutrients and fine suspended sediment particles, Langmuir circulation would be important for the life in the water body. Also because the surface temperature depends partially on the exchange of heat between air and water, which in turn depends on the vertical mixing, Langmuir circulation can be potentially important for the climate temperature change (Smith (1992)).

Weller and Price (1988) [76] found that Langmuir circulation can rapidly erase the thermoclines and is indeed an important vertical mixing mechanism in the upper surface layer. Li, Zahariiev and Garrett (1995) [42] discussed the deepening of the surface mixed layer by Langmuir circulation. Gnanadesikan (1996) [19] predicted by numerical examples that Langmuir circulation dominates the vertical transport of momentum and density. Li and Garrett (1997) [43] modeled the surface mixed layer deepening due to Langmuir circulation. Not many investigations have considered the Langmuir circulations when dealing with the vertical mixing in the general circulation model, and no attempt has been made to relate the Langmuir circulation to the transport of the suspended sediment in lakes or the near shore regions.

There are not many experimental studies of Langmuir circulation. Faller and Caponi (1977) [14] was the first one to investigate the Langmuir circulation in a
wind-wave tank. They studied the correlation of the cell spacing and the wind speed and compared with the field data available at the time. Nepf, et al (1991) [58] studied Langmuir circulation for mechanically-generated waves riding on an open-channel flow over smooth bottom. Streaks formed by floating tracers were found for both wave-following and wave-opposing current. Waves in these experiments were very short, $kh = O(9)$, where $k$ is the wavenumber and $h$ the water depth. Nepf, et al (1995) [57] also investigated the impact of wave-breaking on the generation of Langmuir circulation. Klopman (1997)[35], measured the secondary current due to Langmuir circulation for the mechanically-generated waves riding on the open-channel flow over a rough bottom. One pair of Langmuir cells were found for both wave-following current and wave-opposing current. Recently, Melville, Shear and Veron (1998) [51] investigated the generation of Langmuir circulation due to wind-generated waves and current. They focused on the initial stage of a time scale of about half minute. At this initial stage, waves are short, not fully developed, and the current is very similar to that of a laminar flow. In all the laboratory studies of Langmuir circulation, the sidewall effect cannot be ignored. Unfortunately, the sidewall effects have not been
given enough attention in the past.

For the theoretical study, Leibovich (1983) [39] reviewed the early ideas concerning the generation mechanism of Langmuir circulation. Craik and Leibovich (1976) [10] found that a pair of prescribed wave trains can produce Langmuir cells, and now this mechanism is called CL-I theory. Craik (1977) [9] found that the instability of the laterally uniform wave-current system can also lead to Langmuir circulation through an instability mechanism. Now the instability mechanism of Craik and Leibovich (Craik and Leibovich (1976) [10], Craik (1977) [9], Leibovich (1977) [38], and Leibovich (1983) [39], etc) is often referred to as the CL-II theory and widely accepted. In CL-II theory the "vortex force", which is basically a mean wave Reynolds stress, is responsible to the formation of Langmuir circulation through an instability mechanism. In particular it is assumed that the mean shear stress associated with Langmuir circulation is zero on the mean water surface. One necessary condition of the instability predicted by CL-II theory (Craik (1977) [9]) is that somewhere in the water column.

\[ M(z) = \frac{\partial U_s}{\partial z} \frac{\partial U_0}{\partial z} > 0 \]

where \( U_s \) is the Stokers drift and \( U_0 \) is the velocity of mean current without Langmuir circulation. Thus, no Langmuir circulation can exist in a wave-opposing current by the CL-II mechanism.

Based on the equations laid out by Craik and Leibovich (1976) [10], Leibovich (1983) [39], etc, the linear and nonlinear instability of the stratified/non-stratified deep ocean to the Langmuir circulation and the effect of the Langmuir circulation on the surface mixed layer have been studied by many authors (Leibovich (1977) [38], Leibovich and Radhakrishmnan (1977), [41], Leibovich and Paolucci (1981) [40], Cox and Leibovich (1994) [8], Skyllingstad and Denbo (1995) [64], Li and Garrett (1997) [43], etc.). In all these previous theoretical or numerical studies, the theory was applied to deep water with the bottom of the Langmuir cells being the bottom of the surface mixed layer, where the shear stress is assumed to vanish. The eddy viscosity in all these studied was assumed to be constant in depth and time. The shortcoming of the current theoretical studies has been noticed by a number of authors (see e.g., Smith
In this thesis, a depth-varying eddy viscosity is adopted. We shall re-derive the linearized equations for Langmuir circulation. In particular, the dynamic surface boundary condition will be found to be significantly different from that in CL-II theory. This new boundary condition leads to a new generation mechanism. It is found that Langmuir cells can be formed in both wave-following and wave-opposing current. The effects of wave slope, current strength, relative water depth and the cell spacing on the growth rate of Langmuir cells will be studied by examples. We shall also examine the energy budget of Langmuir circulation to understand the physics of instability.
Chapter 10

Equations and boundary conditions for total motion

In this chapter, we present the dimensionless equations and boundary conditions for the total motion. We first lay out the normalization scheme in section (10.1), and then present the normalized equations governing the total motion in section (10.2). Dimensionless surface boundary conditions for the total motion at the moving surface and the mean surface are derived in section (10.3).

10.1 Normalization

The normalization scheme is the same as in the part one except that we have three velocity components and three coordinates in a 3D problem. For clarity, we lay out here the normalization scheme used in the part II.

We denote the dimensional velocity vector by \( \mathbf{q}^* = (q_1^*, q_2^*, q_3^*) = (u^*, v^*, w^*) \), the dimensional time \( t^* \), dimensional coordinates by \( (x_1^*, x_2^*, x_3^*) = (x^*, y^*, z^*) \), dimensional Reynolds stress tensor by \( \tau_{ij}^* \), dimensional dynamic pressure by \( p^* \), and finally the dimensional surface displacement by \( \eta^* \).

We normalize the length by the inverse of the wave number \( k \) and time by the inverse of the wave angular frequency \( \omega \). The velocity is then normalized by the wave phase velocity \( C = \omega/k \), and the pressure and Reynolds stresses by \( \rho C^2 \). We denote
all the dimensionless variables by removing the superscripts *, therefore, we have

\[ \tilde{q} \equiv (q_1, q_2, q_3) \equiv \frac{q^*}{C}, \quad \text{or} \quad (u, v, w) \equiv \frac{(u^*, v^*, w^*)}{C} \]

\[ (x_1, x_2, x_3) \equiv k(x_1^*, x_2^*, x_3^*), \quad \text{or} \quad (x, y, z) \equiv k(x^*, y^*, z^*) \]

\[ t = \omega t^*, \quad \eta = k\eta^*, \quad \rho = \frac{p^*}{\rho C^2}, \quad \tau_{ij} = \frac{\tau_{ij}^*}{\rho C^2} \]

Before proceeding to the dimensionless equations, let us first discuss the normalization of eddy viscosity. As in part one, the surface waves will modify the eddy viscosity of the open-channel flow. The dimensional eddy viscosity outside the bottom wave boundary layer is assumed to be

\[ \nu_e = \nu + \kappa u_f (-z^* + \eta^*) \left( 1 + \frac{z^*}{h} \right) \quad (10.1) \]

where \( \nu \) is the laminar kinetic viscosity and \( u_f \) the friction velocity, \( \eta^* \) the surface displacement, \( h \) the water depth and \( \kappa = 0.4 \). The inclusion of \( \nu \) is to remove the singularity at the mean surface \( z = 0 \) in the eigenvalue problem derived later in Chapter (12). Normalizing the eddy viscosity (10.1) by \( \omega/k^2 \), the dimensionless eddy viscosity \( \nu_e \) is thus

\[ \nu_e = \frac{k^2}{\omega} + \kappa \frac{u_f}{C} (-z + \eta) \left( 1 + \frac{z}{k} \right) \]

which can be re-written as

\[ \nu_e = \kappa \frac{u_f}{C} \left[ z_\nu + (-z + \eta) \left( 1 + \frac{z}{k} \right) \right] \quad (10.2) \]

where

\[ \kappa \frac{u_f}{C} = O(\epsilon^2) \quad (10.3) \]

give numerical values to support the assumption on the strength of turbulent current, (2.20). \( z_\nu \) comes from the dimensionless laminar viscosity, and is defined by

\[ z_\nu \equiv \frac{k \nu}{\kappa u_f} \quad (10.4) \]
which is a small number not scaled by wave slope $\epsilon$. Later we shall see that $z_\nu$ is much smaller $O(\epsilon^2)$ numerically.

The dimensionless eddy viscosity (10.2) can be written as

$$\nu_e = \alpha \epsilon^2 S(z)$$

(10.5)

where the function $S(z)$ represents the shape of the eddy viscosity and $\alpha$ is the dimensionless friction velocity defined by

$$\alpha = \kappa \frac{u_f}{\epsilon^2}$$

(10.6)

which is of $O(1)$ by assumption on the current strength, (2.20), in part I.

For a smooth bottom, we study the case where the bottom wave boundary layer of pure waves are laminar, i.e., there is no turbulence generated by waves from the bottom. In other words, turbulence is dominated by the current. Therefore, the eddy viscosity is continuous at the outer edge of the BWBL and $\alpha$ is the same inside and outside the BWBL.

The shape function $S$ inside and outside the BWBL can be written as

$$S = \begin{cases} 
S_b = kh + z & k \delta \leq kh + z \leq k \delta \\
S_c = z_\nu - (z - \eta) \left(1 + \frac{z}{kh}\right) & k \delta < kh + z \leq kh 
\end{cases}$$

(10.7)

where $k \delta$ is the dimensionless bottom wave boundary layer thickness to be determined. $z_\nu$ is defined by (10.4) and $kz_B$ is the dimensionless equivalent bottom roughness for a smooth bottom, which is a small number not scaled by the wave slope $\epsilon$. The value of $kz_B$ is normally much smaller that $k \delta = O(\epsilon^2)$, which is the thickness of BWBL. The dimensional equivalent bottom roughness $z_B$ is obtained by fitting the measured velocity profile if the such a velocity profile is available. Madsen and Wood (2002) [46]) adopted the following empirical relation

$$z_B = \frac{\nu}{9u_f}$$

(10.8)

We remark that in $S_b$ the laminar kinematic viscosity $\nu$ is not needed, as is accounted for by the equivalent bottom roughness $z_B$, which is related to the viscous sublayer
near the bottom. For simplicity, we shall adopt the following notation

\[ \delta_b = k\delta, \quad z_b = kZ_B \quad (10.9) \]

and from here on all equations and boundary conditions will be presented in the dimensionless form.

### 10.2 Governing equations of total motion

We denote the gradient operator by \( \nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3) = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \).

The laws of conservation of mass and momentum are given by

\[ \nabla \cdot \vec{q} = 0 \quad (10.10a) \]
\[ \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} + \nabla p = \nabla \cdot \mathbf{T} \quad (10.10b) \]

where \( \mathbf{T} \) is the dimensionless Reynolds stress tensor, defined by

\[ \mathbf{T} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \quad (10.11) \]

In terms of the dimensionless eddy viscosity \( \nu_e \), the Reynolds stress components can be written as

\[ \tau_{xx} = 2\epsilon^2\alpha S \frac{\partial u}{\partial x}, \quad \tau_{yy} = 2\epsilon^2\alpha S \frac{\partial v}{\partial y}, \quad \tau_{zz} = 2\epsilon^2\alpha S \frac{\partial w}{\partial z} \quad (10.12a) \]
\[ \tau_{xy} = \epsilon^2\alpha S \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \tau_{xz} = \epsilon^2\alpha S \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \quad (10.12b) \]
\[ \tau_{yz} = \epsilon^2\alpha S \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \quad (10.12c) \]

The leading order waves are assumed to be irrotational, thus the use of the vorticity facilitates the evaluation of wave-wave interaction. The transport of the vorticity \( \vec{\Omega} \) is governed by the following equation, which is obtained by taking the curl of
where the vorticity $\vec{\Omega}$ is defined by

$$\vec{\Omega} = (\xi, \theta, \zeta) = \nabla \times \vec{q}$$

Making use of the following equality for an incompressible fluid,

$$\vec{q} \cdot \nabla \vec{q} = \nabla E - \vec{q} \times \vec{\Omega}, \quad E = \frac{1}{2} \vec{q} \cdot \vec{q}$$

where $E$ is the dimensionless kinetic energy per unit volume, we can rewrite the momentum equation (10.10b) as

$$\frac{\partial \vec{q}}{\partial t} + \nabla p = -\nabla E + \vec{q} \times \vec{\Omega} + \nabla \cdot \mathbf{T}$$

In the term $\vec{q} \times \vec{\Omega}$, the part related to the wave motion is called "vortex force" by Craik and Leibovich (1976)[10] and Leibovich (1983)[39]. Alternatively the three components of the momentum equation (10.16) can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = F_x$$
$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = F_y$$
$$\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = F_z$$

where the forcing terms $F_x, F_y, F_z$ are defined by

$$F_x = -\frac{\partial E}{\partial x} + v\zeta - w\vartheta + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$
$$F_y = -\frac{\partial E}{\partial y} + w\xi - u\zeta + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$
$$F_z = -\frac{\partial E}{\partial z} + u\vartheta - v\xi + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

Bottom boundary conditions are simply the no-flux ($w = 0$) and no-slip ($u = v = 0$)
conditions on the bottom $z = -kh + z_b$. Next, we discuss the surface boundary conditions.

10.3 Surface boundary conditions

In this section, we first present the surface boundary conditions at the moving surface $z = \eta$, and then derive a set of equivalent boundary conditions at $z = 0$ by integrating the conservation of mass and momentum from $z = \eta$ to $z = 0$. The dimensional boundary conditions on a free surface have been given in Mei (1989) [50].

10.3.1 Exact surface boundary conditions on free surface

On the moving surface, the surface kinematic boundary condition is

$$\frac{\partial \eta}{\partial t} + q_j \frac{\partial \eta}{\partial x_j} - w = 0, \quad j = 1, 2, \quad z = \eta$$ (10.19)

where the index notation is used for simplicity, i.e., $(x_1 \equiv x, x_2 \equiv y)$ and $(q_1 \equiv u, q_2 \equiv v)$. In this section, the repeated index $j$ indicates the summation over $j = 1, 2$. The dynamic surface boundary conditions in the $i$-th directions are

$$[-(p - \coth(kh)\eta)\delta_{ij} + \tau_{ij}]n_j + [-(p - \coth(kh)\eta)\delta_{i3} + \tau_{i3}]n_3 = 0$$ (10.20)

where $i = 1, 2, 3$ and $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. The repeated index $j$ indicates the summation over $j = 1, 2$. $p$ is the dynamic pressure and $n_i$ the $x-, y-, z-$ components of the unit vector normal to the water surface $z = \eta$, given by

$$(n_1, n_2, n_3) = \left(\frac{\partial \eta}{\partial x_1}, \frac{\partial \eta}{\partial x_2}, 1\right) / |\nabla F|$$

$$|\nabla F| = \sqrt{1 + \left(\frac{\partial \eta}{\partial x_1}\right)^2 + \left(\frac{\partial \eta}{\partial x_2}\right)^2}$$ (10.21)
Therefore, surface boundary condition (10.20) can be written as

$$\tau_{i3} = [- (p - \coth(kh)\eta)\delta_{ij} + \tau_{ij}] \frac{\partial \eta}{\partial x_j} + (p - \coth(kh)\eta)\delta_{i3}, \; i = 1, 2, 3 \quad (10.22)$$

On the moving surface the shear stress in $i$-direction is determined by terms on the right-hand side of (10.22). The boundary condition (10.22) is nonlinear because $\eta$ is part of the solution. The unimportance of the surface boundary layer corrections is similar to the 2D problem, and is not repeated here.

### 10.3.2 Surface boundary conditions at $z = 0$

The surface boundary conditions given above are specified on the moving surface. The familiar method to obtain the boundary condition at the mean level is to expand the free surface condition in a Taylor series about the mean level. But this method may not be appropriate for waves riding on a current varying with depth. As commented by Peregrine (1976) [60]: "If this approach is adopted, the mean flow must also be expanded in a Taylor series. While this may be sensible for a flow chosen for its mathematical convenience, such as a linear profile with constant vorticity, it is quite inappropriate if actual velocity measurements are used, since even second derivatives may be quite uncertain". A number of transformations of the equations of motion and boundary conditions enable this problem to be avoided, for example, Lagrangian coordinates (see e.g., Unluata and Mei (1970) [72]), wave-following coordinates (see, e.g., Longuet-Higgins (1953)[45]), etc.

In the following, we present another way to deal with the surface boundary conditions for waves on currents. The idea is shown in Figure (10-1). By studying the conservation of mass and momentum in the control volume shown in Figure (10-1), we can derive a set of the boundary conditions applied at $z = 0$. Similar approach has been used by Liu and Davis (1977)[44] and Phillips (1977) [61] for wave-induced mass transport in a laminar flow.

In the following, the surface conditions at $z = 0$ will be derived by integrating the continuity and momentum equations from the moving surface $z = \eta$ to $z = 0$. We shall adopt the notations used in Mei (1989)[50] for the derivation of the radiation stress, where the integrations were carried out from the bottom to the moving surface.
Kinematic surface boundary condition at \( z = 0 \)  

The continuity equation for an incompressible fluid can be written as

\[
\frac{\partial q_j}{\partial x_j} + \frac{\partial w}{\partial z} = 0 \tag{10.23}
\]

where repeated index \( j \) indicates the summation over \( j = 1, 2 \).

Integrating the continuity equation (10.23) from \( z = 0 \) to \( z = \eta \), we can obtain

\[
[w]_0 - [w]_\eta + \int_0^\eta \frac{\partial q_j}{\partial x_j} dz = 0 \tag{10.24}
\]

Making use of Leibniz’s identity

\[
\int_0^\eta \frac{\partial f}{\partial x} dz = \frac{\partial}{\partial x} \int_0^\eta f dz - [f]_\eta \frac{\partial \eta}{\partial x} \tag{10.25}
\]

we get from (10.24)

\[
[w]_0 - [w]_\eta - \frac{\partial}{\partial x} \int_0^\eta q_j dz + [q_j]_\eta \frac{\partial \eta}{\partial x_j} = 0 \tag{10.26}
\]

After applying the surface boundary condition (10.19) in (10.26), we get

\[
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_j} \int_0^\eta q_j dz - [w]_0 = 0, \quad j = 1, 2 \tag{10.27}
\]
where the second term on the left is the instantaneous flux in the $j$-th direction between $z = 0$ and $z = \eta$. Equation (10.27) states the conservation of mass in a column of height $\eta$ and unit cross-section shown in Figure (10-1). Equation (10.27) is exact, and regarded as the surface kinematic boundary condition on $z = 0$.

**Dynamic surface boundary conditions at $z = 0$** In order to integrate the momentum equation (10.10b) from $z = 0$ to $z = \eta$, we rewrite them as

$$
\frac{\partial q_i}{\partial t} + \frac{\partial q_j q_i}{\partial x_j} + \frac{\partial w q_i}{\partial z} + \frac{\partial p}{\partial x_i} = \frac{\partial r_{ij}}{\partial x_j} + \frac{\partial r_{33}}{\partial z}, \quad i = 1, 2
$$

(10.28)

$$
\frac{\partial w}{\partial t} + \frac{\partial q_j w}{\partial x_j} + \frac{\partial w w}{\partial z} + \frac{\partial p}{\partial x_j} = \frac{\partial r_{3j}}{\partial x_j} + \frac{\partial r_{33}}{\partial z}
$$

(10.29)

so that the vertical derivative terms in (10.28) and (10.29) can be integrated explicitly. Here and from here on the repeated index $j$ indicates the summation over $j = 1, 2$.

In the following, we first derive the normal stress condition from (10.29) and then the tangential stress conditions in $x$- and $y$- directions from (10.28).

**Normal stress condition** Integrating the vertical momentum equation (10.29) from $z = 0$ to $z = \eta$, we have

$$
\int_0^\eta \frac{\partial w}{\partial t} dz + \int_0^\eta \frac{\partial g_j w}{\partial x_j} dt + [ww]_\eta - [ww]_0 + [q_j w]_\eta - [q_j w]_0
$$

$$
= \int_0^\eta \frac{\partial r_{3j}}{\partial x_j} dz + [r_{33}]_\eta - [r_{33}]_0
$$

(10.30)

which, after using Leibniz’s identity (10.25), can be rewritten as

$$
\frac{\partial}{\partial t} \int_0^\eta w dz - [w]_\eta \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_j} \int_0^\eta q_j w dt - [q_j w]_\eta \frac{\partial \eta}{\partial x_j} + [ww]_\eta - [ww]_0
$$

$$
+ [p]_\eta - [p]_0 = \frac{\partial}{\partial x_j} \int_0^\eta \tau_{3j} dz - [\tau_{3j}]_\eta \frac{\partial \eta}{\partial x_j} + [\tau_{33}]_\eta - [\tau_{33}]_0
$$

(10.31)
In view of the surface kinematic boundary condition, (10.19), equation (10.31) can be written as

$$\begin{align*}
\frac{\partial}{\partial t} \int_0^\eta w dt + \frac{\partial}{\partial x_j} \int_0^\eta q_j w dt - [w w]_0 \\
+ [p]_\eta - [p]_0 = \frac{\partial}{\partial x_j} \int_0^\eta \tau_{3j} dz - [\tau_{3j}]_\eta \frac{\partial \eta}{\partial x_j} + [\tau_{33}]_\eta - [\tau_{33}]_0
\end{align*}$$

(10.32)

From (10.22), the normal stress condition on the moving surface \( z = \eta \) can be written as

$$[	au_{33}]_\eta = [	au_{3j}]_\eta \frac{\partial \eta}{\partial x_j} + [p]_\eta - \eta \coth(kh)$$

(10.33)

Substitution of (10.33) in (10.32) gives

$$\begin{align*}
\eta \coth(kh) - [p]_0 &= \frac{\partial}{\partial x_j} \int_0^\eta \tau_{3j} dz - [\tau_{33}]_0 \\
- \frac{\partial}{\partial t} \int_0^\eta w dt - \frac{\partial}{\partial x_j} \int_0^\eta q_j w dt + [w w]_0, \quad j = 1, 2
\end{align*}$$

(10.34)

which can be rearranged as

$$\begin{align*}
[	au_{33}]_0 &= [p]_0 - \eta \coth(kh) + \frac{\partial}{\partial x_j} \int_0^\eta \tau_{3j} dz \\
- \frac{\partial}{\partial t} \int_0^\eta w dt - \frac{\partial}{\partial x_j} \int_0^\eta q_j w dt + [w w]_0, \quad j = 1, 2
\end{align*}$$

(10.35)

Equation (10.35) is exact and serves as the normal stress condition at \( z = 0 \). Equation (10.35) states the balance of vertical forces in vertical direction of the control volume shown in Figure (10-1). Next, we derive the tangential stress conditions in \( x \)- and \( y \)-directions at \( z = 0 \).

**Tangential stress conditions** Integrating the momentum equation (10.28) from \( z = 0 \) to \( z = \eta \), we obtain

$$\begin{align*}
\int_0^\eta \frac{\partial q_i}{\partial t} dz + \int_0^\eta \frac{\partial q_j q_i}{\partial x_j} dz + [w q_i]_0 - [w q_i]_\eta + \int_0^\eta \frac{\partial p}{\partial x_i} dz \\
= \int_0^\eta \frac{\partial \tau_{ij}}{\partial x_j} dz + [\tau_{ii}]_0 - [\tau_{ii}]_\eta, \quad i = 1, 2
\end{align*}$$

(10.36)
which, after using the Leibniz’s identity (10.25), can be rewritten as

\[
\frac{\partial}{\partial t} \int_0^n q_i dz - [q_i]\eta \frac{\partial \eta}{\partial x_j} + \frac{\partial}{\partial x_j} \int_0^n q_j q_i dz \quad \text{and} \quad \frac{\partial q_i}{\partial t} = \frac{\partial}{\partial x_j} \int_0^n \tau_{ij} dz - [\tau_{ij}]_{\eta} \frac{\partial \eta}{\partial x_j} + [w_{qi}]_z - [w_{qi}]_\eta
\]

where \(i, j = 1, 2\). Making use of the kinematic surface boundary condition, (10.19), to replace the term \(\partial \eta/\partial t\) in (10.37), we obtain

\[
\frac{\partial}{\partial t} \int_0^n q_i dz - [q_i]\eta \frac{\partial \eta}{\partial x_j} + [w_{qi}]_\eta - [w_{qi}]_0 + \frac{\partial}{\partial x_j} \int_0^n q_j q_i dz - [q_j q_i]_{\eta} \frac{\partial \eta}{\partial x_j} + [w_{qi}]_\eta - [w_{qi}]_0
\]

(10.38)

which can be simplified to

\[
\frac{\partial}{\partial t} \int_0^n q_i dz + \frac{\partial}{\partial x_j} \int_0^n q_j q_i dz - [w_{qi}]_0 + \frac{\partial}{\partial x_i} \int_0^n q_i dz - [q_i]_{\eta} \frac{\partial \eta}{\partial x_i} + [\tau_{ij}]_{\eta} \frac{\partial \eta}{\partial x_j} + [\tau_{ij}]_{\eta} - [\tau_{ij}]_0
\]

(10.39)

Making use of the surface dynamic boundary condition (10.22) to replace the shear stress \([\tau_{ij}]_\eta\) in (10.39), we get

\[
\frac{\partial}{\partial t} \int_0^n q_i dz + \frac{\partial}{\partial x_j} \int_0^n q_j q_i dz - [w_{qi}]_0 + \frac{\partial}{\partial x_i} \int_0^n q_i dz - [q_i]_{\eta} \frac{\partial \eta}{\partial x_i} = \frac{\partial}{\partial x_j} \int_0^n \tau_{ij} dz - [\tau_{ij}]_{\eta} \frac{\partial \eta}{\partial x_j} + [\tau_{ij}]_{\eta} - [\tau_{ij}]_0
\]

(10.39)

(2)

\[
\frac{\partial}{\partial x_j} \int_0^n \tau_{ij} dz - [\tau_{ij}]_{\eta} \frac{\partial \eta}{\partial x_j} + \left( \frac{[\tau_{ij}]_{\eta} \frac{\partial \eta}{\partial x_j} - [q_i]_{\eta} \frac{\partial \eta}{\partial x_i} + \frac{\coth(kh)}{2} \frac{\partial \eta^2}{\partial x_i}}{2} \right) - [\tau_{ij}]_0
\]

(10.39)
After cancellations of under-braced terms, we finally have

\[ [\tau_{3}]_0 = -\frac{\partial}{\partial t} \int_0^\eta q_i dz - \frac{\partial}{\partial x_j} \int_0^\eta (q_j q_i) dz + [w_0]_0 \]

\[- \frac{\partial}{\partial x_i} \int_0^\eta p dz + \frac{\partial}{\partial x_j} \int_0^\eta \tau_{ij} dz + \frac{\coth(kh)}{2} \frac{\partial \eta^2}{\partial x_i}, \ i = 1, 2 \]  \hspace{1cm} (10.40)

which is the exact tangential stress condition at \( z = 0 \). Equation (10.40) states the balance of horizontal forces in vertical direction shown in Figure (10-1).

The surface boundary conditions on the mean surface given so far are exact and independent of models used for the viscous (Reynolds) stresses. Thus, the results apply to laminar flow as well as the turbulent flow.

### 10.3.3 Mean surface boundary conditions at \( z = 0 \)

The boundary condition (10.40) has some important implications on the mean shear stress at \( z = 0 \). After taking the time average of (10.40) over a wave period, we get

\[ [\bar{\tau}_{3}]_0 = -\frac{\partial}{\partial t} \int_0^\eta q_i dz - \frac{\partial}{\partial x_j} \int_0^\eta (q_j q_i) dz + [w_0]_0 \]

\[- \frac{\partial}{\partial x_i} \int_0^\eta p dz + \frac{\partial}{\partial x_j} \int_0^\eta \bar{\tau}_{ij} dz + \frac{\coth(kh)}{2} \frac{\partial \eta^2}{\partial x_i}, \ i = 1, 2 \]  \hspace{1cm} (10.41)

which is the tangential stress conditions for the mean current in water. For a water surface free of wind, the mean shear stress at \( z = 0 \) may not be zero due to several factors. The terms on the right-hand side of (10.41) have the following physical meanings:

1. Term \( \frac{\partial}{\partial t} \int_0^\eta q_i dz \) indicates that wave damping over a long time scale will affect the mean shear stress at \( z = 0 \).

2. Term \( \frac{\partial}{\partial x} \int_0^\eta (u q_i) dz, \frac{\partial}{\partial x} \int_0^\eta p dz, \frac{\coth(kh)}{2} \frac{\partial \eta^2}{\partial x} \), and \( \frac{\partial}{\partial x} \int_0^\eta \tau_{ij} dz \) represent the contribution of wave damping in space.

3. Term \( \frac{\partial}{\partial y} \int_0^\eta (v q_i) dz, \frac{\partial}{\partial y} \int_0^\eta p dz, \frac{\coth(kh)}{2} \frac{\partial \eta^2}{\partial y} \), and \( \frac{\partial}{\partial x} \int_0^\eta \tau_{ij} dz \) represent the contribution of the short-crested waves.

4. Term \( [w_0]_0 \) is the wave-Reynolds stress evaluated at \( z = 0 \). It is known that
wave-induced Reynolds stresses of the irrotational wave field are zero. It can be shown that the non-zero \( [\bar{wq_i}]_0 \) is the results of eddy viscosity, wave damping in space and rotational current.

We remark that terms like \( \frac{\partial}{\partial y} \int_0^d (vq_i) dz \) may not be zero for short-crested waves or a pair of wave strains. In the theory of CL-1, a pair of wave strains was assumed. However, they imposed without proof that \( [\bar{\tau}_{i3}]_0 = 0 \) which is not correct based on the present analysis.

Even for waves that are long-crested to the leading order, the instability of Langmuir circulation will make the waves at higher order short-crested, as will be shown in the following chapters. Therefore, terms like \( \frac{\partial}{\partial y} \int_0^d (vq_i) dz \) will have important role in the mean boundary conditions for Langmuir circulation.

In summary, an important conclusion is that the right-hand side of (10.41) is not zero. For the simplest case of a constant eddy viscosity (pseudo-laminar), it may be recalled that Phillips (1977) [61] and Liu and Davis (1977) [44] have shown that wave attenuation in time will result in a positive shear stress near the surface.\(^1\)

---

\(^1\)Longuet-Higgins (1953) [45] assumed that the surface is free of stress and the wave dissipation can be ignored. Phillips (1977) [61] pointed out that waves studied must be attenuated in space if the surface is free of any stress. If waves are not damped, a pressure force is needed on the surface to overcome wave dissipation.
Chapter 11

Basic state

In this chapter, we define the basic flow to be a flow which is uniform laterally. Some of the key results obtained in the part I are needed and cited here.

11.1 Basic state for Langmuir circulation

We take the basic state to be the superposition of the long-crested wave field, the open-channel flow and the second order mean current induced by the wave-current interaction.

We denote the surface displacement of the basic state by $H(x, z, t)$, the pressure by $P(x, z, t)$, and the three velocity components of the basic state by $(U, V, W)$. 

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Denoting the wave phase by $\theta = x - t$, the basic state can be expressed by

$$
U = \epsilon U_0^{(1)} + \epsilon^2 U_0^{(2)} + \epsilon \left( U_1^{(1)} e^{i\theta} + \text{c.c.} \right) \\
+ \epsilon^2 \left( U_1^{(2)} e^{i\theta} + \text{c.c.} \right) + \epsilon^2 \left( U_2^{(2)} e^{2i\theta} + \text{c.c.} \right) + O(\epsilon^3) 
$$

(11.1a)

$$
V = 0 
$$

(11.1b)

$$
W = \epsilon \left( W_1^{(1)} e^{i\theta} + \text{c.c.} \right) \\
+ \epsilon^2 \left( W_1^{(2)} e^{i\theta} + \text{c.c.} \right) + \epsilon^2 \left( W_2^{(2)} e^{2i\theta} + \text{c.c.} \right) + O(\epsilon^3) 
$$

(11.1c)

$$
P = \epsilon P_0^{(1)} + \epsilon^2 P_0^{(2)} + \epsilon \left( P_1^{(1)} e^{i\theta} + \text{c.c.} \right) \\
+ \epsilon^2 \left( P_1^{(2)} e^{i\theta} + \text{c.c.} \right) + \epsilon^2 \left( P_2^{(2)} e^{2i\theta} + \text{c.c.} \right) + O(\epsilon^3) 
$$

(11.1d)

$$
H = \epsilon H_0^{(1)} + \epsilon^2 H_0^{(2)} + \epsilon \left( H_1^{(1)} e^{i\theta} + \text{c.c.} \right) \\
+ \epsilon^2 \left( H_1^{(2)} e^{i\theta} + \text{c.c.} \right) + \epsilon^2 \left( H_2^{(2)} e^{2i\theta} + \text{c.c.} \right) + O(\epsilon^3) 
$$

(11.1e)

where $f_j(z, x_2)$ represents the amplitude of the $j$-th harmonic component of $f$ and $f_j^{(k)}(z, x_2)$ is the $k$-th order approximation of $f_j(z, x_2)$. The $x_2$ dependency of $f_j^{(k)}(z, x_2)$ comes from the anticipated wave dissipation in space $^1$. We emphasis here that basic solutions are different inside and outside the BWBL.

Next, we first recall the wave solution and then the mean current solution in the basic state (11.1). Outside the BWBL, the first order wave solutions for the irrotational linear waves are given by

$$
U_1^{(1)} = \frac{A \cosh(kh + z)}{2 \sinh(kh)}, \quad P_1 = \frac{A \cosh(kh + z)}{2 \sinh(kh)} 
$$

(11.2a)

$$
W_1^{(1)} = -i \frac{A \sinh(kh + z)}{2 \sinh(kh)}, \quad H_1 = \frac{A}{2} 
$$

(11.2b)

where $i = \sqrt{-1}$ and $A = A(x_2)$ due to wave dissipation by turbulent eddy viscosity.

Inside the BWBL, the oscillatory longitudinal velocity (4.17) is recalled

$$
U_1^{(1)} = \frac{A}{2 \sinh(kh)} \left( 1 - K(Z) \right) e^{i\theta} + \text{c.c.}, \quad \text{and} \quad K(Z) = \frac{K_0(2\sqrt{Z}e^{-i\pi/4})}{K_0(2\sqrt{Z}e^{-i\pi/4})} 
$$

(11.3)

$^1$It has been shown in Part one that the wave energy dissipation due to the turbulent eddy viscosity $O(\epsilon^2)$ happens over a dimensionless length $O(1/\epsilon^2)$.
where \( K_0(Z) \) is the Kelvin function of the zero-th order, and \( Z, Z_B \) and \( Z_{\delta} \) are defined by

\[
Z = \frac{kh + z}{\alpha \epsilon^2}, \quad Z_B = \frac{z_b}{\alpha \epsilon^2}, \quad Z_{\delta} = \frac{\delta_b}{\alpha \epsilon^2}
\]  
(11.4)

with \( \delta_b \) being the thickness of the BWBL defined in (10.9). Recalling the oscillatory vertical velocity inside BWBL, (4.6), we have

\[
W^{(1)}_1 = W^{(2)}_1 = 0, \quad W^{(3)}_1 = -\alpha \int_{Z_B}^{Z} \frac{\partial U^{(1)}_1}{\partial x} dZ
\]  
(11.5)

At the second order, the first harmonic solutions \( (U^{(2)}_1, W^{(2)}_1, P^{(2)}_1, H^{(2)}_1) \) are the results of the first order irrotational wave motion and the first order rotational turbulent current \( U^{(1)}_1 \). As a result, these components are rotational. The second harmonic wave motion at the second order, on the other hand, are the results of the nonlinear interaction of the first order irrotational waves. Thus, \( (U^{(2)}_2, W^{(2)}_2, P^{(2)}_2, H^{(2)}_2) \) are irrotational.

For the leading order current, \( H^{(1)}_0 \) and \( P^{(1)}_0 \) are the driving force for the open-channel flow. The velocity profile of the leading order current is logarithmic,

\[
U^{(1)}_0 = \frac{\alpha_0 \epsilon}{\kappa} \ln \left( \frac{kh + z}{z_b} \right)
\]  
(11.6)

where \( \alpha_0 = O(1) \) is the dimensionless friction velocity, \( \epsilon \) the wave slope, and \( \kappa = 0.4 \) the Karman constant. For a turbulent flow, the no-slip condition is specified at an apparent bottom characterized by the bottom roughness \( z_b \). It will be assumed in this thesis that \( U^{(1)}_0(0) = O(1) \).

The second order mean current \( U^{(2)}_0 \) is induced by the interaction of open-channel flow and the long-crested waves. The solution of \( U^{(2)}_0 \) itself is the 2D theory of the effect of surface waves on the turbulent current. The governing equations and boundary conditions of \( U^{(2)}_0 \) found in the 2D theory early, are summarized here for
\[ \alpha_c = \alpha_b = \alpha. \] In the core region,

\[
\begin{align*}
\tilde{S}_c \frac{\partial U_0^{(2)}}{\partial z} &= -AA^* \frac{\tilde{S}_c \sinh(kh + z)}{\sinh(kh)} \pm (\alpha - \alpha_0) \left( \frac{-z}{kh} \right) \\
&\quad + \frac{\beta AA^* (z + kh)(2kh + \sinh(2kh))}{\alpha} 8kh \sinh^2(kh) \\
&\quad + \frac{AA^* (kh + z)}{2kh \sinh^2(kh)} \int_{Z_b}^{Z} \text{Im}(K) dZ \\
&\quad - \frac{AA^* \frac{\partial^2 \tilde{S}_c \sinh(2(kh + z)) - 2(kh + z)}{\partial z^2}}{4 \sinh^2(kh)} \\
\end{align*}
\]

where \( U_0^{(2)} \) is \( \bar{u}' \) in the 2D theory. In (11.7), \( \beta \) is the dissipation rate of wave energy, \( \alpha \) the dimensionless friction velocity. The function \( \tilde{S}_c = 1 + z/kh \) is the surface distortion of eddy viscosity in the core region. Inside the bottom wave boundary layer, \( U_0^{(2)} \) is solved as in the 2D theory of Part I (cf. equation (4.14)).

\[
U_0^{(2)}(Z) = \int_{Z_B}^{Z} \frac{\bar{u}_b \bar{w}_b}{\alpha Z} dZ - \left( \pm \frac{\alpha_0 - \alpha}{\kappa^2} + \frac{\bar{u}_b \bar{w}_b}{\alpha} \right) \ln \left( \frac{Z}{Z_B} \right)
\]

where

\[
\frac{\bar{u}_b \bar{w}_b}{\kappa^2} = -\frac{\alpha AA^*}{4 \sinh^2(kh)} \left[ (1 - K^*) \int_{Z_B}^{Z} i(1 - K) dZ + c.c. \right]
\]

The conclusion of the 2D theory is that wave damping, bottom wave boundary layer and the surface distortion of the eddy viscosity all contribute to the solution of \( U_0^{(2)} \).

### 11.2 Shear rate of basic current

In this section, we discuss the open-channel flow, the concept of asymptotic and numeric order of small parameters, and the treatment of the logarithmic profile as coefficient in the formulation of the problem.

As shown in section (11.1), when the velocity is normalized by the wave phase speed and the length by the inverse of wavenumber, the dimensionless velocity of the
open-channel flow is given by

\[ U_0^{(1)} = \frac{\alpha_0 \epsilon}{\kappa} \ln \left( \frac{kh + z}{z_b} \right) \]  

(11.10)

where \( \alpha_0 = O(1) \) is the dimensionless friction velocity.

There are two small parameters in (11.10), \( z_b \) and \( \epsilon \). The apparent bottom roughness \( z_b \) is normally much smaller than \( \epsilon \). Strictly speaking, asymptotic ordering of \( z_b \) in terms of \( \epsilon \), say, \( z_b(\epsilon) \), is not possible due to the independence of \( z_b \) on \( \epsilon \). Numerically ordering \( z_b \) in terms of \( \epsilon \) is possible for a given problem with specific \( z_b \) and \( \epsilon \).

If \( U_0^{(1)}(0) = O(1) \) and \( kh = O(1) \) are assumed, then velocity profile (11.10) implies

\[ \epsilon \ln \left( \frac{1}{z_b} \right) = O(1) \] 

(11.11)

or,

\[ z_b = \mathcal{K} \epsilon^{-1/\epsilon}, \quad \mathcal{K} = O(1) \] 

(11.12)

There is a special feature of non-uniformity in the shear rate of (11.10). The logarithmic profile makes the shear rate of \( U_0^{(1)} \)

\[ \frac{\partial U_0^{(1)}}{\partial z} = \frac{\alpha_0 \epsilon}{\kappa} \left( \frac{1}{kh + z} \right) \] 

(11.13)

non-uniform in \( z \). Assuming \( \alpha_0 = O(1), U_0^{(1)} = O(1) \) and \( kh = O(1) \), the shear rate (11.13) is of \( O(\epsilon) \) in the core region where \( kh + z = O(1) \). At the bottom \( z + kh = z_b \), the shear rate of \( U_0 \) is

\[ \frac{\partial U_0^{(1)}}{\partial z} = O \left( \frac{\epsilon}{z_b} \right) = O(\epsilon \epsilon^{1/\epsilon}) \] 

(11.14)

according to (11.12). For wave slope \( \epsilon = 0.1 \), the numeric order \( \epsilon \epsilon^{1/\epsilon} \sim 2202 \gg O(1) \). Therefore, it is impossible to treat (11.13) uniformly in different regions of \( z \), even numerically.
We notice that even though (11.13) is non-uniform in \( z \), the product

\[
(kh + z) \frac{\partial U_0^{(1)}}{\partial z} = \epsilon \frac{\alpha_0}{\kappa} = O(\epsilon)
\]

is of \( O(\epsilon) \) in the entire depth of water. Equation (11.15) is the result of the logarithmic feature of \( U_0^{(1)} \) and suggests that in performing the perturbation analysis, the velocity \( U_0^{(1)} \) and the shear rate \( \partial U_0^{(1)}/\partial z \) should be treated separately.

For a function \( f(z) \) which satisfies the boundary condition \( f = 0 \) at \( kh + z = z_b \), it can be approximated by

\[
f(z) = (kh + z - z_b) \left[ \frac{\partial f}{\partial z} \right]_{kh+z=z_b} + ...
\]

near the bottom \( kh + z = z_b \). Thus, we have from (11.13) and (11.16) that

\[
f \frac{\partial U_0}{\partial z} = \epsilon \frac{\alpha_0}{\kappa} \left( \frac{kh + z - z_b}{kh + z} \right) \left[ \frac{\partial f}{\partial z} \right]_{kh+z=z_b} + ...
\]

Right at the bottom, we have

\[
\left[ f \frac{\partial U_0^{(1)}}{\partial z} \right]_{kh+z=z_b} = 0
\]

and near the bottom \( kh + z = O(z_b) \),

\[
f \frac{\partial U_0^{(1)}}{\partial z} = O \left( z_b \frac{\epsilon}{z_b} \right) = O(\epsilon)
\]

If \( f = O(1) \) away from the bottom, then

\[
f \frac{\partial U_0^{(1)}}{\partial z} = O(\epsilon)
\]

is valid in the entire depth of water.

Based on the analysis given above, we make the following conclusions:

- \( U_0^{(1)} = O(1) \) for all \( z \);
- If the shear rate of \( U_0^{(1)} \) appear in the problem as coefficient of \( f(z) = O(1) \)
which is zero at \( kh + z = z_b \), then \( f \frac{\partial U_0^{(1)}}{\partial z} = O(\epsilon) \) for all \( z \).
Chapter 12

Linearized equations and boundary conditions for Langmuir circulation

In this chapter, we first introduce the spanwise perturbation to the basic flow in section (12.1). Next we solve for secondary waves in section (12.2). In section (12.3) we derive the linearized equations governing Langmuir circulation; and in sections (12.4) and (12.5), we derive the linearized surface and bottom boundary conditions for Langmuir circulation.

12.1 Spanwise Perturbations

It is assumed that primary (long-crested) waves of $O(\epsilon)$ ride on the unidirectional current of $O(\epsilon)$. Langmuir circulation, which has spanwise variations, comes from the instability of the initial system to the spanwise current perturbation. We denote the strength of this spanwise perturbation by $O(\epsilon \delta)$ where $\delta$ is the measure of the relative strength of the spanwise perturbation to the initial current. We take $\delta$ to be an infinitesimally small parameter, independent of $\epsilon$, so that the interactions between the perturbations, $O(\delta^2 \epsilon^2)$, can be ignored in the linear analysis.

Through the interaction between Langmuir circulation of $O(\epsilon \delta)$ and primary waves of $O(\epsilon)$, the so-called "secondary waves" of $O(\delta \epsilon^2)$ will be generated. The secondary waves are short-crested due to the lateral variations of Langmuir circulation. The interaction of the primary and secondary waves will give rise to a mean Reynolds stress
of $O(\epsilon^3\delta)$, which has a transverse variation. According to the CL-II theory, this wave-induced Reynolds stress is responsible for the generation of Langmuir circulation.

Let us introduce a slow time $t_\ell = \epsilon^\ell t$ as the growth time scale for Langmuir circulation. From here on waves will be assumed to be steady in time, i.e., the wave energy is constant in time, but attenuates in space over length scale $x_2 = \epsilon^2 x$, as shown in the problem for $U_0^{(2)}$ in the part I. Therefore, we have two time scales and two spatial scales to describe the fast wave motion and the slow attenuation of wave energy. The introduction to perturbation methods and multiple-scale methods is referred to Nayfeh (1981)[56] and the discussion of the perturbation methods in fluid mechanics is referred to Van Dyke (1964) [13].

By the chain rule, the $x$- and $t$- derivatives can be expanded as

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon^\ell \frac{\partial}{\partial t_\ell}$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon^2 \frac{\partial}{\partial x_2}$$

The total horizontal velocity $u$ can be written as

$$u = U + \epsilon\delta u_0(z, y, x_2, t_\ell) + \epsilon^2\delta \left\{u_1(z, y, x_2, t_\ell)e^{i\theta} + \text{c.c.} \right\} + O(\epsilon^3, \epsilon^3\delta, \epsilon^4\delta^2) \quad (12.2)$$

where $U$ is the longitudinal velocity of the basic flow given in (11.1a) , $\epsilon\delta u_0$ the longitudinal velocity of Langmuir circulation and $\epsilon^2\delta u_1 e^{i\theta} + \text{c.c.}$ the longitudinal velocity of secondary waves. In equation (12.2), c.c. stands for the complex conjugate. Secondary waves are of $O(\epsilon^2\delta)$ because they come from the interaction between Langmuir circulation of $O(\epsilon\delta)$ and primary (long-crested) waves of $O(\epsilon)$. The higher harmonic components of the perturbed waves are smaller than the first harmonic component by a factor of $O(\epsilon)$, hence of $O(\epsilon^3\delta)$ and not included in (12.2). They will also be absent in the expressions of the other two velocity components, pressure and the surface displacement from here on. Similarly the velocity, $v$ and $w$, can be written as

$$v = \epsilon\delta v_0(z, y, x_2, t_\ell) + \epsilon^2\delta \left\{v_1(z, y, x_2, t_\ell)e^{i\theta} + \text{c.c.} \right\} + O(\epsilon^3, \epsilon^3\delta, \epsilon^4\delta^2) \quad (12.3a)$$

$$w = W + \epsilon\delta w_0(z, y, x_2, t_\ell) + \epsilon^2\delta \left\{w_1(z, y, x_2, t_\ell)e^{i\theta} + \text{c.c.} \right\} + O(\epsilon^3, \epsilon^3\delta, \epsilon^4\delta^2) \quad (12.3b)$$
where \( W \) is the vertical velocity of the basic flow given in (11.1c), while \((\epsilon \delta v_0, \epsilon \delta u_0)\) are the lateral and vertical mean velocities of Langmuir circulation, respectively. The order of magnitudes of \((u_0, v_0)\), relative to \( u_0 = O(1) \), are to be determined. In (12.3a) and (12.3b), \( \epsilon^2 \delta v_1 \) and \( \epsilon^2 \delta w_1 \) are the lateral and vertical velocity of secondary waves, respectively.

Finally, dynamic pressure \( p \) and surface displacement \( \eta \) can also be written as

\[
p = P + \epsilon^2 \delta p_0(z, y, x_2, t) + \epsilon^2 \delta \{p_1(z, y, x_2, t_\ell)e^{i\theta} + c.c.\} + O(\epsilon^3, \epsilon^3 \delta, \epsilon^2 \delta^2) \quad (12.4a)
\]
\[
\eta = H + \epsilon^2 \delta \eta_0(y, x_2, t_\ell) + \epsilon^2 \delta \{\eta_1(y, x_2, t_\ell)e^{i\theta} + c.c.\} + O(\epsilon^3, \epsilon^3 \delta, \epsilon^2 \delta^2) \quad (12.4b)
\]

where \( P \) and \( H \) are the dynamic pressure and surface displacement of the basic flow given in (11.1d) and (11.1e). The mean dynamic pressure and the mean surface displacement due to Langmuir circulation are the results of the interaction between the basic current of \( O(\epsilon) \) and Langmuir circulation of \( O(\epsilon \delta) \), thus we have \( \epsilon^2 \delta p_0 \) and \( \epsilon^2 \delta \eta_0 \) represent the steady spanwise perturbations to the mean pressure and mean surface displacement, respectively. Finally, \( \epsilon^2 \delta p_1 \) and \( \epsilon^2 \delta \eta_1 \) are the dynamic pressure and surface displacement associated with secondary waves, in accordance with (12.2) and (12.3).

### 12.2 Secondary waves

As we shall see that it is the vortex force that drives Langmuir circulation. Recall (10.16) that the vortex force is

\[
\overrightarrow{q} \times \Omega = \underbrace{\text{Primary waves}}_{O(\epsilon)} \times \underbrace{\text{Secondary waves}}_{O(\epsilon \delta)} = O(\epsilon^3 \delta)
\]

Thus even though Langmuir circulation itself is of \( O(\epsilon \delta) \), the governing equations need to be obtained at \( O(\epsilon^3 \delta) \). In order to compute the vortex force, the solutions of secondary waves need to be solved first.

After substituting (12.2) –(12.4b) in the continuity equation (10.10a) and momentum equations (10.10b), and collecting the coefficients of \( \epsilon^2 \delta e^{i\theta} \), the following
linearized equations are then obtained for secondary waves

\begin{align}
&iu_1 + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = O(\epsilon, \delta) \quad (12.5a) \\
&-iu_1 + ip_1 = X_1 + O(\epsilon, \delta) \quad (12.5b) \\
&-iw_1 + \frac{\partial p_1}{\partial y} = Y_1 + O(\epsilon, \delta) \quad (12.5c) \\
&-iw_1 + \frac{\partial p_1}{\partial z} = Z_1 + O(\epsilon, \delta) \quad (12.5d)
\end{align}

where \(X_1, Y_1, Z_1\) represent the wave-current interaction and are given by

\begin{align}
X_1 &= -iu_0U_1^{(1)} - w_0 \frac{\partial U_1^{(1)}}{\partial z} - \frac{W_1^{(1)}}{\partial z} \quad (12.6a) \\
Y_1 &= -W_1^{(1)} \frac{\partial v_0}{\partial z} \quad (12.6b) \\
Z_1 &= -iu_0W_1^{(1)} - W_1 \frac{\partial w_0}{\partial z} - \frac{\partial W_1^{(1)}}{\partial z} \quad (12.6c)
\end{align}

In expression (12.5), we have kept terms of the leading order in \(\epsilon\) and \(\delta\). Terms proportional to \(\delta^n, n > 1\), represent the nonlinear interaction of Langmuir circulation, and ignored in the linear analysis.

From (12.5) and (12.6), we see that the basic current, \(U_0(z)\), does not affect secondary waves directly, but through Langmuir circulation \((u_0, v_0, w_0)\), which is affected by \(U_0\).

The three vorticity components of the secondary waves, \((\xi_1, \vartheta_1, \zeta_1)\), are defined by

\begin{align}
\xi_1 &= \frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial z} \quad (12.7a) \\
\vartheta_1 &= \frac{\partial u_1}{\partial z} - iw_1 \quad (12.7b) \\
\zeta_1 &= iu_1 - \frac{\partial u_1}{\partial y} \quad (12.7c)
\end{align}

and the governing equations of \((\xi_1, \vartheta_1, \zeta_1)\) can be obtained from (12.5) by cross-
Making use of the linear wave solutions (11.2) in (12.6) and substituting the results in (12.8), we get the following expressions for vorticity \((\xi_1, \vartheta_1, \zeta_1)\),

\[
\begin{align*}
\xi_1 &= \frac{A \sinh(kh + z)}{2 \sinh(kh)} \left( \frac{\partial^2 v_0}{\partial z^2} - \frac{\partial^2 w_0}{\partial z \partial y} - i \frac{\partial u_0}{\partial y} \right) + \frac{A \cosh(kh + z)}{2 \sinh(kh)} \left( \frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial y} \right) \\
\vartheta_1 &= -\frac{A \sinh(kh + z)}{2 \sinh(kh)} \frac{\partial^2 u_0}{\partial z^2} \\
\zeta_1 &= -\frac{A \sinh(kh + z)}{2 \sinh(kh)} \left( i \frac{\partial v_0}{\partial z} - \frac{\partial^2 u_0}{\partial z \partial y} - i \frac{\partial w_0}{\partial y} \right) - \frac{A \cosh(kh + z)}{2 \sinh(kh)} \frac{\partial u_0}{\partial y} 
\end{align*}
\]  

(12.9a)  

(12.9b)  

(12.9c)

Note that, in (12.9), the vorticity of the secondary waves is linear in \((u_0, v_0, w_0)\). This vorticity field is the result of the interaction between the first order waves and Langmuir circulation.

Similarly, substituting (12.2) -(12.4b) in the kinematic surface boundary condition (10.27) and the normal stress condition, (10.35) on \(z = 0\), and then collecting the coefficients of \(\varepsilon^2 \delta e^{i\theta}\), we get

\[
\begin{align*}
-i \eta_1 - w_1 &= O(\varepsilon, \delta) \\
p_1 - \coth(kh) \eta_1 &= O(\varepsilon, \delta)
\end{align*}
\]  

(12.10)  

(12.11)

which are the surface boundary conditions for the secondary waves.

With the solutions of the secondary waves formally obtained in this section, it is now convenient to derive the linearized equations and boundary conditions for the steady spanwise perturbation (Langmuir circulation).
12.3 Linearized equations governing the initial Langmuir circulation

12.3.1 Continuity equation

Substituting (12.2)-(12.4b) in the continuity equation (10.10a), collecting the zero-th harmonic terms up to \(O(\epsilon \delta w_0)\), we have the following linearized continuity equation for Langmuir circulation

\[
\frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0 
\]  
(12.12)

which implies that

\[
O(w_0) = O(v_0) 
\]  
(12.13)

The continuity equation (12.12) also implies that a stream function \(\psi\) can be defined by

\[
v_0 = -\frac{\partial \psi}{\partial z}, \quad w_0 = \frac{\partial \psi}{\partial y} 
\]  
(12.14)

It then follows that the mean longitudinal vorticity \(\xi_0\) can be written as

\[
\xi_0 = \frac{\partial w_0}{\partial y} - \frac{\partial v_0}{\partial z} = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial y^2} 
\]  
(12.15)

The relative order of magnitude \(u_0\) and \(w_0\) is undetermined.

12.3.2 Equation of mean longitudinal momentum

Langmuir circualtion is of \(O(\epsilon \delta)\), but the governing equations need to be obtained at \(O(\epsilon^3 \delta)\), as indicated in the section (12.2). Substituting (12.2)-(12.4b) in equations (10.10b), collecting the zero-th harmonic terms up to \(O(\epsilon^{l+1} \delta u_0)\), \(O(\epsilon^3 \delta u_0)\), and \(O(\epsilon^3 \delta w_0)\) in \(x\)- direction, so that the leading order inertia term and viscous terms are retained in the resulting equation. We then get the following linearized longitudinal
momentum equation for spanwise current perturbations.

\[ \epsilon^{t+1} \frac{\partial u_0}{\partial t} + \epsilon^2 \delta w_0 \left( \frac{\partial U_0^{(1)}}{\partial z} + \epsilon \frac{\partial U_0^{(2)}}{\partial z} \right) + \epsilon^3 \delta \left( W_1^{(1)} \right)^* \frac{\partial u_1}{\partial z} + \epsilon^3 \delta W_1^{(1)} \frac{\partial u_1^*}{\partial z} \]

interaction of primary & secondary waves

\[ + \epsilon^3 \delta w_1 \frac{\partial (U_1^{(1)})^*}{\partial z} + \epsilon^3 \delta w_1^* \frac{\partial U_1^{(1)}}{\partial z} \]

interaction of primary & secondary waves

\[ = \epsilon^3 \delta \alpha \left[ \tilde{\Sigma}_c \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial}{\partial z} \left( \tilde{\Sigma}_c \frac{\partial u_0}{\partial z} \right) \right] + O(\epsilon^4 \delta, \epsilon^3 \delta^2) \quad (12.16) \]

where the superscript * represents the complex conjugate and the solutions of basic flow (current and waves) given in section (11.1) and \( U_0^{(1)} \) given by (11.10) are needed.

In the core region where \( kh + z = O(1) \), the shear rate of \( U_0^{(1)} \), given in (11.13), is of \( O(\epsilon) \), thus the shear rate of \( U_0^{(1)} \) is the same as that of \( U_0^{(2)} \) which is in general not logarithmic. Near the bottom where \( kh + z \ll O(1) \), the vertical velocity \( w_0 \) can be written as

\[ w_0 = \left[ w_0 \right]_{kh+z} + \frac{\partial w_0}{\partial z} (kh + z) + \frac{1}{2} \frac{\partial^2 w_0}{\partial z^2} (kh + z)^2 + O(kh + z)^3 \quad (12.17) \]

Upon using the continuity equation (12.12) and no-flux and the no-slip conditions at the bottom, we have

\[ w_0 = -\frac{\partial v_0}{\partial y} (kh + z) + \frac{1}{2} \frac{\partial^2 w_0}{\partial z^2} (kh + z)^2 + O(kh + z)^3 \]

\[ = \frac{1}{2} \frac{\partial^2 w_0}{\partial z^2} (kh + z)^2 + O(kh + z)^3 = O(kh + z)^2 \quad (12.18) \]

Thus equations (11.13) and (12.18) lead to

\[ \epsilon^2 \delta w_0 \left( \frac{\partial U_0^{(1)}}{\partial z} + \epsilon \frac{\partial U_0^{(2)}}{\partial z} \right) = \epsilon^2 \left( \frac{\alpha_0}{\kappa} \frac{w_0}{kh + z} + w_0 \frac{\partial U_0^{(2)}}{\partial z} \right) \ll O(\epsilon^3 \delta) \quad (12.19) \]

near the bottom where \( kh + z \ll O(1) \). Because \( w_0 = 0 \) at the bottom, according to
section (11.2), we have

\[
\frac{\alpha_0}{\kappa} \frac{w_0}{kh + z} = O \left( w_0 \frac{\partial U^{(2)}_0}{\partial z} \right)
\]  \quad (12.20)

in the entire depth of water.

In order to allow the unstable growth of \((u_0, v_0, w_0)\), we must chose \(\ell = 2\) and \(w_0 = O(u_0)\) in equation (12.16). As a result, equation (12.16) can be written as

\[
\frac{\partial u_0}{\partial t} + \frac{\alpha_0}{\kappa} \frac{w_0}{kh + z} + w_0 \frac{\partial U^{(2)}_0}{\partial z} + \left( W^{(1)}_1 \right)^* \frac{\partial u_1}{\partial z} + W^{(1)}_1 \frac{\partial u^*_1}{\partial z} + u_0 + w_0 \frac{\partial U^{(1)}_1}{\partial z} + w_1 \frac{\partial U^{(1)}_1}{\partial z}
\]

interaction of primary & secondary waves

\[
= \alpha \left[ \mathcal{S}_c \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial}{\partial z} \left( \mathcal{S}_c \frac{\partial u_0}{\partial z} \right) \right] + O(\epsilon, \delta)
\]  \quad (12.21)

In view of (12.9), the interaction of primary and secondary waves are linear in \((u_0, v_0, w_0)\), consequently, equation (12.21) are linear in \((u_0, v_0, w_0)\).

The interaction between the primary and secondary waves in (12.21) can be simplified from the definition of the vorticity \(\vartheta_1\). From (12.7b), we obtain

\[
\frac{\partial u_1}{\partial z} = \vartheta_1 + iw_1
\]  \quad (12.22)

In view of (12.9), we have

\[
\frac{\partial u_1}{\partial z} = -\frac{A\sinh(kh + z)}{2 \sinh(kh)} \frac{\partial^2 u_0}{\partial z^2} + iw_1
\]  \quad (12.23)

It then follows from (12.23) and the linear wave solutions (11.2) that the interaction
between the primary and the secondary waves in (12.21) vanishes

\[
\begin{align*}
(W_1^{(1)})^* \frac{\partial u_1}{\partial z} + \epsilon^2 W_1^{(1)} \frac{\partial u_1^*}{\partial z} + w_1 \frac{\partial (U_1^{(1)})^*}{\partial z} + w_1^* \frac{\partial U_1^{(1)}}{\partial z} \\
= (W_1^{(1)})^* \left( -\frac{A \sinh(kh + z)}{2} \sinh(kh) \frac{\partial^2 u_0}{\partial z^2} + iw_1 \right) + W_1^{(1)} \left( -\frac{A^* \sinh(kh + z)}{2} \sinh(kh) \frac{\partial^2 u_0}{\partial z^2} - iw_1^* \right) \\
+ w_1 ( -iw_1^{(1)} )^* + w_1^* ( -iw_1^{(1)} ) = 0
\end{align*}
\]

where the fact that \( W_1^{(1)}/A \) is imaginary has been used (cf. (11.2b)). The cancellation of those underlined terms is remarkable. Thus, equation (12.21) can be simplified to

\[
\frac{\partial u_0}{\partial t_2} + \left( \frac{\alpha_0}{\kappa} \frac{w_0}{kh + z} + w_0 \frac{\partial U_0^{(2)}}{\partial z} \right) \text{d}t_2 + \text{O}(e^3 \delta v_0) = \alpha \left[ \tilde{S}_c \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial}{\partial z} \left( \tilde{S}_c \frac{\partial u_0}{\partial z} \right) \right] + \text{O}(e, \delta)
\]

Note that the shear rate of \( U_0^{(1)} \) and \( U_0^{(2)} \) appear at the same order due to the logarithmic feature of \( U_0^{(1)} \) (see equation (12.19)).

In summary, the longitudinal velocity of Langmuir circulation is forced by the interaction between the vertical velocity \( w_0 \) and the shearing rate in the basic current and second order current correction. In the core region, the shearing rate in the second order correction is at the same order of magnitude as that in the basic current because of (11.13).

### 12.3.3 Equation of mean lateral momentum

Again, the equation governing Langmuir circulation is obtained at \( O(e^3 \delta) \). Substituting (12.2) - (12.4b) in the lateral momentum equation in (10.17b), collecting the zero-th harmonic terms up to \( O(e^{\ell+1} \delta v_0) \) and \( O(e^3 \delta v_0) \), taking \( \ell = 2 \), we have

\[
\begin{align*}
\epsilon^2 \frac{\partial v_0}{\partial t_2} - \epsilon^2 (W_1^* \xi_1 + \text{c.c.}) + \epsilon^2 (U_1^* \zeta_1 + \text{c.c.}) + \epsilon^2 \frac{\partial \Pi_0}{\partial y} \\
= \alpha \epsilon^2 \left[ \frac{\partial}{\partial y} \left( 2 \tilde{S}_c \frac{\partial v_0}{\partial y} \right) + \frac{\partial}{\partial z} \left( \tilde{S}_c \left( \frac{\partial v_0}{\partial y} + \frac{\partial v_0}{\partial z} \right) \right) \right]
\end{align*}
\]

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where $\Pi_0$ is the pressure head defined by

$$\Pi_0 = \epsilon^{-1}(p_0 + E_0) \quad (12.26)$$

with $E_0$ given by

$$E_0 = \frac{1}{2} \left[U_0 \omega_0 + \epsilon (U_1^* u_1 + \text{c.c.}) + \epsilon (W_1^* w_1 + \text{c.c.}) \right] \quad (12.27)$$

The first order approximation to $E_0$ balances the first order approximation to $p_0$. We shall see that $\Pi_0$ can be eliminated from the final vorticity equations and the expression of $\Pi_0$ is immaterial.

Recalling (12.9a) and (12.9c), we have

$$W_1^* \xi_1 + \text{c.c.} = \frac{AA^* \sinh^2(kh + z)}{2 \sinh^2(kh)} \frac{\partial u_0}{\partial y} \quad (12.28)$$

$$U_1^* \xi_1 + \text{c.c.} = \frac{AA^* \sinh(2kh + z)}{4 \sinh^2(kh)} \frac{\partial^2 u_0}{\partial z \partial y} - \frac{AA^* \cosh^2(kh + z)}{2 \sinh^2(kh)} \frac{\partial u_0}{\partial y} \quad (12.29)$$

Collecting terms, we have

$$\epsilon^2 \frac{\partial v_0}{\partial t_2} + \epsilon^2 \left( \frac{AA^* \sinh(2kh + z)}{4 \sinh^2(kh)} \frac{\partial^2 u_0}{\partial z \partial y} - \frac{AA^* \cosh(2kh + z)}{2 \sinh^2(kh)} \frac{\partial u_0}{\partial y} \right) + \epsilon^2 \frac{\partial \Pi_0}{\partial y}$$

$$= \alpha \epsilon^2 \left[ \frac{\partial}{\partial y} \left( 2 \bar{S}_c \frac{\partial v_0}{\partial y} \right) + \frac{\partial}{\partial z} \left( \bar{S}_c \left( \frac{\partial u_0}{\partial y} + \frac{\partial \xi_0}{\partial z} \right) \right) \right] \quad (12.30)$$

Let us define

$$G_0 = \Pi_0 + \left( \frac{AA^* \sinh(2kh + z)}{4 \sinh^2(kh)} \frac{\partial u_0}{\partial z} - \frac{AA^* \cosh(2kh + z)}{2 \sinh^2(kh)} u_0 \right) \quad (12.31)$$

then equation (12.30) can be written as

$$\frac{\partial v_0}{\partial t_2} + \frac{\partial G_0}{\partial y} = \alpha \left[ \frac{\partial}{\partial y} \left( 2 \bar{S}_c \frac{\partial v_0}{\partial y} \right) + \frac{\partial}{\partial z} \left( \bar{S}_c \left( \frac{\partial u_0}{\partial y} + \frac{\partial \xi_0}{\partial z} \right) \right) \right] \quad (12.32)$$

Now $\Pi_0$ is hidden in $G_0$, which will be eliminated from the final vorticity equation.
12.3.4 Equation of mean vertical momentum

Substituting (12.2) - (12.4b) in the longitudinal momentum equation in (10.17c), taking \( \ell = 2 \) and collecting the zero-th harmonic terms up to \( O(\epsilon^3 \delta w_0) \), we have

\[
\epsilon^2 \frac{\partial w_0}{\partial t} - \epsilon^2 \left( U_1 \varphi^* + c.c. \right) + \epsilon^2 \frac{\partial \Pi_0}{\partial z} \nonumber \\
= \alpha \epsilon^2 \left[ \frac{\partial}{\partial y} \left( \tilde{S}_e \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left( 2 \tilde{S}_e \frac{\partial w_0}{\partial z} \right) \right] \tag{12.33}
\]

Making use of (12.9b) and the linear wave solutions, we have

\[
U_1 \varphi^* + c.c. = -\frac{AA^* \sinh(2(kh + z))}{4 \sinh^2(kh)} \frac{\partial^2 u_0}{\partial z^2} \tag{12.34}
\]

Collecting terms, we get

\[
\epsilon^2 \frac{\partial w_0}{\partial t} + \epsilon^2 \left( \frac{AA^* \sinh(2(kh + z))}{4 \sinh^2(kh)} \frac{\partial^2 u_0}{\partial z^2} \right) + \epsilon^2 \frac{\partial \Pi_0}{\partial z} \nonumber \\
= \alpha \epsilon^2 \left[ \frac{\partial}{\partial y} \left( \tilde{S}_e \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left( 2 \tilde{S}_e \frac{\partial w_0}{\partial z} \right) \right] \tag{12.35}
\]

In view of the definition of \( G_0 \), (12.31), equation (12.35) can be written as

\[
\frac{\partial w_0}{\partial t^2} + \frac{AA^* u_0 \sinh(2(kh + z))}{\sinh^2(kh)} \frac{\partial G_0}{\partial z} = \alpha \left[ \frac{\partial}{\partial y} \left( \tilde{S}_e \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left( 2 \tilde{S}_e \frac{\partial w_0}{\partial z} \right) \right] \tag{12.36}
\]

Again, \( G_0 \) will be eliminated from the final vorticity equation. The second term on the left of (12.36) is the vortex force of Craik and Leibovich (1977) and Craik (1977), which drives the Langmuir circulation.

12.3.5 Equation of mean longitudinal vorticity

It is convenient to identify the driving force of Langmuir circulation from the longitudinal vorticity equation. By cross-differentiation of (12.32) and (12.36), we shall obtain the vorticity equation for the longitudinal vorticity \( \xi_0 \).
First, we note that
\[
\begin{align*}
\alpha \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial y} \left( \ddot{S} \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial y} \right) \right) + \frac{\partial}{\partial z} \left( 2\ddot{S} \frac{\partial w_0}{\partial z} \right) \right] \\
-\alpha \frac{\partial}{\partial z} \left[ \frac{\partial}{\partial y} \left( 2\ddot{S} \frac{\partial v_0}{\partial y} \right) + \frac{\partial}{\partial z} \left( \ddot{S} \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial y} \right) \right) \right] \\
= \ddot{S} \left( \frac{\partial^3 w_0}{\partial z^2 \partial y} - \frac{\partial^3 v_0}{\partial y^2 \partial z} + \frac{\partial^3 w_0}{\partial y^3} - \frac{\partial^3 v_0}{\partial z^3} \right) \\
-2 \left( \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial z^2} \right) \frac{\partial \ddot{S}}{\partial z} - \frac{\partial^2 \ddot{S}}{\partial z^2} \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial y} \right) \\
= \alpha \ddot{S}_c \left( \frac{\partial^2 \xi_0}{\partial z^2} + \frac{\partial^2 \xi_0}{\partial y^2} \right) \\
-2 \left( \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial z^2} \right) \frac{\partial \ddot{S}}{\partial z} - \frac{\partial^2 \ddot{S}}{\partial z^2} \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial y} \right) + O(\epsilon, \delta)
\end{align*}
\]  
(12.37)

Therefore, after taking the cross-differentiation of (12.32) and (12.36), the longitudinal vorticity \( \xi_0 \) is governed by
\[
\frac{\partial \xi_0}{\partial t} + AA^* \frac{\partial u_0}{\partial y} \frac{\sinh(2(kh + z))}{\sinh^2(kh)} = \alpha \ddot{S}_c \left( \frac{\partial^2 \xi_0}{\partial z^2} + \frac{\partial^2 \xi_0}{\partial y^2} \right)
\]
- vortex-force
\[
-2 \left( \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial z^2} \right) \frac{\partial \ddot{S}}{\partial z} - \frac{\partial^2 \ddot{S}}{\partial z^2} \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial y} \right) + O(\epsilon, \delta)
\]  
(12.38)

Because there is no \( x_2 \) derivative in (12.24) and (12.38), Langmuir circulation is local in \( x_2 \). Note that \( G_0 \), hence \( \Pi_0 \), disappears here. The only forcing term left is the vorticity force found by Craik and Leibovich (Craik and Leibovich (1976)[10], Leibovich (1983) [39], etc.).

For primary waves, the Stokes drift is
\[
e^2 \bar{U}_s = e^2 \frac{AA^* \cosh(2(kh + z))}{2} \frac{1}{\sinh^2(kh)}
\]  
(12.39)

thus the vortex force in (12.38) can be written as
\[
AA^* \frac{\partial u_0}{\partial y} \frac{\sinh(2(kh + z))}{\sinh^2(kh)} = \frac{\partial u_0}{\partial y} \frac{\partial \bar{U}_s}{\partial z}
\]  
(12.40)
12.3.6 Equation of mean vertical vorticity

For later analysis of physics, we derive the vertical vorticity equation. Because the Langmuir circulation is local in \( x_2 \), the mean vertical vorticity, \( \zeta_0 \), is simply

\[ \zeta_0 = -\frac{\partial u_0}{\partial y} \]  

(12.41)

After taking the \( y \)-derivative of (12.24) and making use of (12.41), we have

\[
\frac{\partial \zeta_0}{\partial t_2} - \frac{\partial w_0}{\partial y} \left( \frac{\alpha_0}{\kappa} \frac{1}{kh + z} + \frac{\partial U_0^{(2)}}{\partial z} \right) = \alpha \left[ \tilde{S}_e \frac{\partial^2 \zeta_0}{\partial y^2} + \frac{\partial}{\partial z} \left( \tilde{S}_e \frac{\partial \zeta_0}{\partial z} \right) \right] + O(\epsilon, \delta) \]  

(12.42)

Thus the mean vertical vorticity is forced by the interaction of \( \partial w_0/\partial y \) and the shearing rate in the basic current.

In summary, the governing equations (12.24) and (12.38) modifies those of Craik and Leibovich by adding the effect of varying eddy viscosity. In the next section, a major change of the surface boundary condition will be introduced.

12.4 Linearized free-surface boundary conditions for Langmuir circulation

First we show that \( [w_0]_0 = 0 \) from the kinematic surface boundary condition and then we derive the shear stress conditions for Langmuir circulation.

In approximating the surface boundary conditions, we shall make use of the following approximation

\[
\int_0^{\eta} f dz = [f \eta]_0 + O(f \eta^2). \]  

(12.43)

for \( \eta = O(\epsilon) \ll 1 \). We shall see that terms of \( O(f \eta^2) \) will not appear in the approximate surface boundary conditions to the accuracy same as the governing equations (12.24) and (12.38).
Kinematic surface boundary condition  After taking the time average of (10.27) over a wave period and noting the following relations

\[ \frac{\partial \bar{f}}{\partial t} = \varepsilon \frac{\partial \bar{f}}{\partial t_2}, \quad \frac{\partial \bar{f}}{\partial x} = \varepsilon \frac{\partial \bar{f}}{\partial x_2} \]  

we have

\[ \varepsilon^2 \frac{\partial \bar{\eta}}{\partial t_2} + \varepsilon \frac{\partial}{\partial x_2} \int_0^n u dz + \frac{\partial}{\partial y} \int_0^n udz - [\bar{w}]_0 = 0 \]  

(12.45)

Langmuir circulation is of \( O(\varepsilon\delta) \), so the surface boundary condition for \( w_0 \) is obtained at \( O(\varepsilon\delta) \) from (12.45). Substituting (12.2)-(12.4b) in (12.45), and collecting terms of \( O(\varepsilon\delta) \) for the zero-th harmonic components, we get

\[ [w_0]_0 = O(\varepsilon, \delta) \]  

(12.46)

which is the kinematic surface boundary condition for \( w_0 \) at \( z = 0 \). Recall from (12.3b) that the mean vertical velocity is \( \varepsilon\delta[w_0]_0 \) at the surface.

The dynamic condition of mean longitudinal stress  After taking the time average of (10.40) over a wave period for \( i = 1 \), and making use of (12.44), we get

\[ [\tau_{xz}]_0 = -\varepsilon^2 \frac{\partial}{\partial t_2} \int_0^n u dz - \varepsilon^2 \frac{\partial}{\partial x_2} \int_0^n u^2 dz - \frac{\partial}{\partial y} \int_0^n (uv) dz + [\bar{w} u]_0 
- \varepsilon^2 \frac{\partial}{\partial x_2} \int_0^n p dz + \varepsilon^2 \frac{\partial}{\partial x_2} \int_0^n \tau_{11} dz + \frac{\partial}{\partial y} \int_0^n \tau_{12} dz + \varepsilon^2 \coth(kh) \frac{\partial \eta_0^2}{\partial x_2} \]  

(12.47)

The dimensionless eddy viscosity (10.2) is of \( O(\varepsilon^2) \), and Langmuir circulation \( O(\varepsilon\delta) \), secondary waves \( (\varepsilon^3\delta) \), as shown in the previous section. Therefore, from (10.12) the mean Reynolds stresses induced by Langmuir circulation on the left-hand side of (12.47) is \( O(\varepsilon^3\delta) \) and the oscillatory parts of Reynolds stress induced by Langmuir circulation (associated with the secondary waves) is of \( O(\varepsilon^4\delta) \). We then have the following estimates:

\[ \varepsilon^2 \frac{\partial}{\partial t_2} \int_0^n u dz = O(\varepsilon^5\delta), \quad -\varepsilon^2 \frac{\partial}{\partial x_2} \int_0^n u^2 dz = O(\varepsilon^5\delta) \]
Substituting (12.2)-(12.4b) in equation (12.47), and collecting terms of $O(\epsilon^2\delta)$ and $O(\epsilon^3\delta)$ in (12.47) for the zero-th harmonic components, we have the following results:

At $O(\epsilon^2\delta)$,

$$U^{(1)}_0 w_0 = 0, \quad z = 0$$

(12.48)

which is satisfied automatically because of (12.46).

At $O(\epsilon^3\delta)$,

$$\alpha S_c \frac{\partial u_0}{\partial z} = -\left(U_1^{(1)} \left(H_{-1}^{(1)}\right)^* + \left(U_1^{(1)}\right)^* H_1^{(1)}\right) \frac{\partial v_0}{\partial y} + u_1 \left(W_1^{(1)}\right)^* + (u_1)^* W_1^{(1)}$$

$$+ U_1^{(1)} (w_1)^* + (U_1^{(1)})^* w_1$$

(12.49)

where (12.46) has been used.

Substituting the linear wave solutions (11.2) in (12.49), we can simplify the first two terms in (12.49) to

$$- \left(U_1^{(1)} \left(H_{-1}^{(1)}\right)^* + \left(U_1^{(1)}\right)^* H_1^{(1)}\right) \frac{\partial v_0}{\partial y} = - \coth(kh) \frac{AA^*}{2} \frac{\partial v_0}{\partial y}$$

(12.50)

and the last four terms to

$$u_1 \left(W_1^{(1)}\right)^* + (u_1)^* W_1^{(1)} + U_1^{(1)} (w_1)^* + (U_1^{(1)})^* w_1$$

$$= A^* \left\{ iu_1 + \coth(kh) w_1 \right\}_0 + c.c.$$  

(12.51)

From the momentum equation (12.5b), and the surface boundary conditions (12.10)
and (12.11), we have for the secondary waves

\textbf{Momentum equation:} \quad i[u_1]_0 = i[p_1]_0 - [X_1]_0 \quad (12.52a)

\textbf{Kinematic S.B.C.:} \quad -i\eta_1 - [w_1]_0 = 0 \quad (12.52b)

\textbf{Normal stress S.B.C.:} \quad [p_1]_0 - \coth(kh)\eta_1 = 0 \quad (12.52c)

Hence, (12.51) may be written as

\begin{align*}
[iu_1 + \coth(kh)w_1]_0 &= i[p_1]_0 - [X_1]_0 + \coth(kh)[w_1]_0 \\
&= i[p_1]_0 - [X_1]_0 - i\coth(kh)\eta_1 = -[X_1]_0 \\
\end{align*}

so that

\begin{equation}
\frac{A^*}{2} [iu_1 + \coth(kh)w_1]_0 + \text{c.c.} = -\left( \frac{A^*}{2} [X_1]_0 + \frac{A}{2} [(X_1)^*]_0 \right) \quad (12.54)
\end{equation}

In view of the expression (12.6) of \(X_1\), we have

\begin{align*}
\frac{A^*}{2} [X_1]_0 + \frac{A}{2} [(X_1)^*]_0 \\
&= \frac{A^*}{2} \left\{ -iu_0U_1^{(i)} - w_0 \frac{\partial U_1^{(i)}}{\partial z} - W_1^{(i)} \frac{\partial u_0}{\partial z} \right\} + \text{c.c.} \quad (12.55)
\end{align*}

which, after using the linear wave solutions, reduces to

\begin{equation}
\frac{A^*}{2} [X_1]_0 + \frac{A}{2} [(X_1)^*]_0 = \frac{AA^*}{4} \left\{ -iu_0 \coth(kh) + i \frac{\partial u_0}{\partial z} \right\} + \text{c.c.} \quad (12.56)
\end{equation}

where \([w_0]_0 = 0\) has been used in \([X_1]_0\). The right-hand side of (12.56) vanishes because both \(AA^*\) and \(u_0\) are real.

Summarizing the results obtained from (12.49) to (12.56), the surface boundary condition (12.49) can be simplified to

\begin{equation}
\alpha \bar{S}_c \frac{\partial u_0}{\partial z} = -\coth(kh) \frac{AA^*}{2} \frac{\partial u_0}{\partial y} \quad (12.57)
\end{equation}

at \(z = 0\). In previous studies (see, Craik and Leibovich (1977)[9] ,Leibovich (1983)[39]) , it was assumed in an ad hoc manner that \(\partial u_0/\partial z = 0\) at the mean surface where
wind is absent. This is an important difference between the present theory and that
of Craik and Leibovich.

The mean lateral shear stress condition  After taking the time average of
(10.40) over a wave period for \( i = 2 \), and making use of (12.44), we get

\[
[\overline{\tau}_{yz}]_0 = -\varepsilon^2 \frac{\partial}{\partial t_2} \int_0^n v dz - \varepsilon^2 \frac{\partial}{\partial x_2} \int_0^n u v dz - \frac{\partial}{\partial y} \int_0^n (v^2) dz + [\overline{w v}]_0
\]

\[
- \frac{\partial}{\partial y} \int_0^n p dz + \varepsilon^2 \frac{\partial}{\partial x_2} \int_0^n \tau_{12} dz + \frac{\partial}{\partial y} \int_0^n \tau_{22} dz + \frac{\coth(kh)}{2} \frac{\partial \eta^2}{\partial y} \tag{12.58}
\]

The dimensionless eddy viscosity (10.2) is \( O(\varepsilon^2) \), and Langmuir circulation
\( O(\varepsilon \delta) \), secondary waves \( O(\varepsilon^2 \delta) \), as shown in the previous section. Again, from (10.12),
the mean Reynolds stresses induced by Langmuir circulation is \( O(\varepsilon^3 \delta) \). We also have
the following order-of-magnitude estimates:

\[
\varepsilon^2 \frac{\partial}{\partial t_2} \int_0^n v dz = O(\varepsilon^5 \delta), \quad -\varepsilon^2 \frac{\partial}{\partial x_2} \int_0^n u v dz = O(\varepsilon^6 \delta)
\]

\[
- \frac{\partial}{\partial y} \int_0^n (v^2) dz = O(\varepsilon^5 \delta^2), \quad [\overline{w v}]_0 = O(\varepsilon^3 \delta)
\]

\[
\varepsilon^2 \frac{\partial}{\partial x_2} \int_0^n \tau_{12} dz = O(\varepsilon^5 \delta), \quad \frac{\partial}{\partial y} \int_0^n \tau_{22} dz = O(\varepsilon^4 \delta)
\]

\[
\frac{\partial}{\partial y} \int_0^n p dz = O(\varepsilon^3 \delta), \quad \frac{\coth(kh)}{2} \frac{\partial \eta^2}{\partial y} = O(\varepsilon^3 \delta)
\]

Substituting (12.2)-(12.4b) in the equations (12.58), and collecting terms of \( O(\varepsilon^2 \delta) \)
and \( O(\varepsilon^3 \delta) \) in (12.58) for the zero-th harmonic components, respectively, we find that
up to \( O(\varepsilon^2 \delta) \), the surface boundary condition (12.58) is satisfied automatically. At
\( O(\varepsilon^3 \delta) \),

\[
\alpha_S \frac{\partial v_0}{\partial z} = \left[ W_1^{(1)}(v_1)^* + \left( W_1^{(1)} \right)^* v_1 \right]_0
\]

\[
- \frac{\partial}{\partial y} \left[ \left( P_1^{(1)} \right)^* \eta + P_1^{(1)}(\eta_1)^* + p_1 \left( H_1^{(1)} \right)^* + (p_1)^* H_1^{(1)} \right]_0
\]

\[
+ \coth(kh) \frac{\partial}{\partial y} \left[ \left( H_1^{(1)} \right)^* \eta + H_1^{(1)}(\eta_1)^* \right]_0 + O(\varepsilon, \delta) \tag{12.59}
\]

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Making use of the linear wave solutions (11.2), the first two terms can be simplified to

\[
\left[ W_1^{(1)}(v_1) + (W_1^{(1)})^* v_1 \right]_0 = \frac{i A^*}{2} v_1 + c.c. \quad (12.60)
\]

The remaining terms can be reduced to

\[
\frac{\partial}{\partial y} \left[ \left( P_1^{(1)} \right)^* \eta_1 + P_1^{(1)}(\eta_1)^* \right]_0 + coth(kh) \frac{\partial}{\partial y} \left[ \left( H_1^{(1)} \right)^* \eta_1 + H_1^{(1)}(\eta_1)^* \right]_0 = -\frac{1}{2} \coth(kh) \frac{\partial}{\partial y} (A^* \eta_1 + c.c.) \quad (12.61)
\]

It then follows that the right-hand side of (12.59) can be simplified to

\[
A^* \left( i[v_1]_0 - \coth(kh) \frac{\partial \eta_1}{\partial y} \right) + c.c. \quad (12.62)
\]

In view of the last condition in (12.52), we have

\[
A^* \left( i[v_1]_0 - \coth(kh) \frac{\partial \eta_1}{\partial y} \right) = A^* \left( i[v_1]_0 - \frac{\partial [\rho_1]_0}{\partial y} \right) \quad (12.63)
\]

which, after using the lateral momentum equation (12.5c) and the expression for \( Y_1 \), (12.6b), becomes

\[
A^* \left( i[v_1]_0 - \frac{\partial [\rho_1]_0}{\partial y} \right) = -i AA^* \frac{AA^*}{2} \left[ \frac{\partial v_0}{\partial z} \right]_0 \quad (12.64)
\]

Therefore,

\[
A^* \left( i[v_1]_0 - \frac{\partial [\rho_1]_0}{\partial y} \right) + c.c. = 0 \quad (12.65)
\]

because both \( AA^* \) and \( v_0 \) are real.

Summarizing the results from (12.59) to (12.65), the boundary condition (12.59) is simplified to

\[
\frac{\partial v_0}{\partial z} = 0, \quad z = 0 \quad (12.66)
\]
Summary of surface boundary conditions  The surface boundary conditions for Langmuir circulation are given by

\[ \alpha \bar{S}_c \frac{\partial \theta_0}{\partial z} = - \coth(kh) \frac{A_A^*}{2} \frac{\partial \theta_0}{\partial y} + O(\epsilon, \delta) \]  
\[ \frac{\partial \nu_0}{\partial z} = O(\epsilon, \delta) \]  
\[ w_0 = O(\epsilon, \delta) \]

at \( z = 0 \),

The surface boundary condition (12.67a) differs from that used in Craik and Leibovich (1976), Craik (1977) and Leibovich (1983). In their theory, the right-hand side of (12.67a) is assumed to be zero. We have however shown that the longitudinal shear stress for Langmuir circulation is coupled with the transverse motion. The effect of this coupled surface boundary condition on the instability of Langmuir circulation is one of the key tasks in the present theory.

12.5 Bottom boundary conditions for perturbed current

In this section, we derive the equations governing the perturbed current associated with Langmuir circulation inside the BWBL in order to provide the bottom boundary conditions for Langmuir circulation in the core region.

Inside the BWBL, the vertical coordinate should be the inner coordinate \( Z \), defined by

\[ Z = \frac{z + kh}{\alpha \epsilon^2} \]

where \( \alpha \) is the dimensionless friction velocity and \( \epsilon \) the wave slope. By the chain rule, we have

\[ \frac{\partial}{\partial z} \rightarrow \frac{1}{\alpha \epsilon^2} \frac{\partial}{\partial Z} \]

In terms of the inner coordinates, the governing equations (10.10a) and (10.10b) can
be written as

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\alpha^2 Z} \frac{\partial w}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{1}{\alpha^2 Z} \frac{\partial \tau_{zz}}{\partial z} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial p}{\partial y} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{1}{\alpha^2 Z} \frac{\partial \tau_{yz}}{\partial z} \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{1}{\alpha^2 Z} \frac{\partial \tau_{zz}}{\partial z} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{1}{\alpha^2 Z} \frac{\partial \tau_{zz}}{\partial z}
\end{align*}
\]

(12.69a) (12.69b) (12.69c) (12.69d)

where the Reynolds stresses are given by

\[
\begin{align*}
\tau_{xx} &= 2\epsilon^2 \alpha S_b \frac{\partial u}{\partial x}, \quad \tau_{yy} = 2\epsilon^2 \alpha S_b \frac{\partial v}{\partial y}, \quad \tau_{zz} = 2\epsilon^2 \alpha S_b \left( \frac{1}{\alpha^2} \frac{\partial w}{\partial z} \right) \\
\tau_{xy} &= \epsilon^2 \alpha S_b \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \tau_{xz} = \epsilon^2 \alpha S_b \left( \frac{1}{\alpha^2} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\
\tau_{xy} &= \epsilon^2 \alpha S_b \left( \frac{\partial w}{\partial y} + \frac{1}{\alpha^2} \frac{\partial v}{\partial z} \right)
\end{align*}
\]

(12.70a) (12.70b)

The bottom boundary conditions are those of no-slip and no-flux at the bottom

\[
u = v = w = 0, \quad Z = Z_B
\]

(12.71)

where \(Z_B\) is the bottom roughness in the inner coordinates

\[
Z_B = \frac{z_b}{\alpha^2}
\]

The upper boundary conditions are the continuity of the velocity and the shear stress at the outer edge of the BWBL \(Z_\delta\), defined by

\[
Z_\delta = \frac{\delta_b}{\alpha^2}
\]

(12.72)

where \(\delta_b\) is the dimensionless thickness of the BWBL.
12.5.1 Spanwise perturbations

As in the core region, the longitudinal velocity inside the BWBL can be written as

\[ u = U(Z) + \epsilon \delta u_0(Z, y, x_2, t_2) + \epsilon^2 \delta \{ u_1(Z, y, x_2, t_2)e^{i\theta} + \text{c.c.} \} + O(\epsilon^3, \epsilon^3 \delta, \epsilon \delta^2) \]  

(12.73)

where \( U(Z) \) is the longitudinal velocity of the basic flow inside BWBL, given in (11.1a) and (11.3), \( \epsilon \delta u_0 \) the longitudinal velocity of Langmuir circulation, and \( \epsilon^2 \delta u_1 e^{i\theta} + \text{c.c.} \) the longitudinal velocity of secondary waves. In equation (12.73), c.c. stands for the complex conjugate. Secondary waves are of \( O(\epsilon^2 \delta) \) because they come from the interaction between Langmuir circulation of \( O(\epsilon \delta) \) and primary (long-crested) waves of \( O(\epsilon) \). The higher harmonic components of the perturbed waves are smaller than the first harmonic component by a factor of \( O(\epsilon) \), hence of \( O(\epsilon^3 \delta) \) and not included in (12.73). They will also be absent in the expressions of the other two velocity components, pressure and the surface displacement as well.

Similarly the other two components of the velocity, \( v \) and \( w \), can be written as

\[ v = \epsilon \delta v_0(Z, y, x_2, t_2) + \epsilon^2 \delta \{ v_1(Z, y, x_2, t_2)e^{i\theta} + \text{c.c.} \} + O(\epsilon^3, \epsilon^3 \delta, \epsilon \delta^2) \]  

(12.74a)

\[ w = W(Z) + \epsilon \delta w_0(Z, y, x_2, t_2) + \epsilon^2 \delta \{ w_1(Z, y, x_2, t_2)e^{i\theta} + \text{c.c.} \} + O(\epsilon^3, \epsilon^3 \delta, \epsilon \delta^2) \]  

(12.74b)

where \( W(Z) = O(\epsilon^3) \) is the vertical velocity of the basic flow given in (11.1c) and (11.5), while \( (\epsilon \delta v_0, \epsilon \delta w_0) \) are the lateral and vertical mean velocities of Langmuir circulation, respectively.

Finally, dynamic pressure \( p \) can also be written as

\[ p = P + \epsilon^2 \delta p_0(Z, y, x_2, t_2) + \epsilon^2 \delta \{ p_1(Z, y, x_2, t_2)e^{i\theta} + \text{c.c.} \} + O(\epsilon^3, \epsilon^3 \delta, \epsilon \delta^2) \]  

(12.75)

where \( P \) is the dynamic pressure of the basic flow given in (11.1d). The mean dynamic pressure due to Langmuir circulation is the results of the interaction between the basic current of \( O(\epsilon) \) and Langmuir circulation of \( O(\epsilon \delta) \), thus we have \( \epsilon^2 \delta p_0 \) represents the steady spanwise perturbations to the mean pressure. Finally, \( \epsilon^2 \delta p_1 \) is the dynamic
pressure associated with secondary waves, in accordance with (12.73) and (12.74).

12.5.2 Mean motion associated Langmuir circulation inside BWBL

As in the core region, before deriving the governing equations for \((u_0, v_0, w_0, p_0)\), we first need to solve the secondary waves in terms of \((u_0, v_0, w_0, p_0)\).

From (12.69a), continuity for the secondary waves requires

\[
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{1}{\alpha^2} \frac{\partial w_1}{\partial Z} = 0 \tag{12.76}
\]

which, after integrating from the bottom \(Z = Z_B\) to any \(Z\), gives

\[
\frac{\partial w_1}{\partial Z} = -\alpha^2 \int_{Z_B}^{Z} \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) d\zeta \tag{12.77}
\]

Now we estimate the terms linear in \(\delta\) in the momentum equations. After taking the time average (12.69), we have

\[
e^2 \frac{\partial \bar{u}}{\partial t} + u \frac{\partial \bar{u}}{\partial x} + v \frac{\partial \bar{u}}{\partial y} + \frac{w}{\alpha^2} \frac{\partial \bar{u}}{\partial Z} + e^2 \frac{\partial \bar{p}}{\partial x} = e^2 \frac{\partial \bar{r}_{xx}}{\partial x} + \frac{\partial \bar{r}_{xy}}{\partial y} + \frac{1}{\alpha^2} \frac{\partial \bar{r}_{xz}}{\partial Z} \tag{12.78a}
\]

\[
e^2 \frac{\partial \bar{v}}{\partial t} + u \frac{\partial \bar{v}}{\partial x} + v \frac{\partial \bar{v}}{\partial y} + \frac{w}{\alpha^2} \frac{\partial \bar{v}}{\partial Z} + e^2 \frac{\partial \bar{p}}{\partial y} = e^2 \frac{\partial \bar{r}_{xy}}{\partial x} + \frac{\partial \bar{r}_{yy}}{\partial y} + \frac{1}{\alpha^2} \frac{\partial \bar{r}_{yz}}{\partial Z} \tag{12.78b}
\]

\[
e^2 \frac{\partial \bar{w}}{\partial t} + u \frac{\partial \bar{w}}{\partial x} + v \frac{\partial \bar{w}}{\partial y} + \frac{w}{\alpha^2} \frac{\partial \bar{w}}{\partial Z} + \frac{1}{\alpha^2} \frac{\partial \bar{p}}{\partial Z} = e^2 \frac{\partial \bar{r}_{xz}}{\partial x} + \frac{\partial \bar{r}_{yy}}{\partial y} + \frac{1}{\alpha^2} \frac{\partial \bar{r}_{zz}}{\partial Z} \tag{12.78c}
\]

where the continuity equation (12.69a) has been used.

Let \(L_\delta(f)\) be the sorting operator which takes out from \(f\) only terms linear in \(\delta\). Recalling (12.73)- (12.75), (12.89) and (12.77), we can estimate the mean Reynolds
stresses in (12.78a)-(12.78c) as follows

\[ L_\delta (\tilde{\tau}_{xx}) = 2\epsilon^6 \alpha^2 Z L_\delta \left( \frac{\partial \tilde{u}}{\partial x_2} \right) = O(\epsilon^7 \delta) \quad (12.79) \]

\[ L_\delta (\tilde{\tau}_{yy}) = 2\epsilon^6 \alpha^2 Z L_\delta \left( \frac{\partial \tilde{v}}{\partial y} \right) = O(\epsilon^5 \delta) \quad (12.80) \]

\[ L_\delta (\tilde{\tau}_{zz}) = 2\epsilon^2 \alpha Z L_\delta \left( \frac{\partial \tilde{w}}{\partial Z} \right) = O(\epsilon^3 \delta) \quad (12.81) \]

\[ L_\delta (\tilde{\tau}_{xy}) = \epsilon^4 \alpha^2 Z L_\delta \left( \frac{\partial \tilde{u}}{\partial y} \right) = O(\epsilon^5 \delta) \quad (12.82) \]

\[ L_\delta (\tilde{\tau}_{xz}) = \epsilon^4 \alpha^2 Z L_\delta \left( \frac{\partial \tilde{w}}{\partial x} + \epsilon^2 \frac{\partial \tilde{u}}{\partial x_2} \right) = O(\epsilon^3 \delta) \quad (12.83) \]

Recalling (12.73)- (12.75), (12.89) and (12.77), terms in the momentum equation (12.78a) can be estimated as follows

\[ L_\delta \left( \frac{\partial \tilde{u}}{\partial t_2} \right) = O(\epsilon \delta), \quad L_\delta \left( u \frac{\partial \tilde{u}}{\partial x} \right) = O(\epsilon^3 \delta), \quad L_\delta \left( \tilde{u} \frac{\partial \tilde{u}}{\partial y} \right) = O(\epsilon^3 \delta) \]

\[ L_\delta \left( \frac{\partial \tilde{u}}{\partial x} \right) = O(\epsilon^3 \delta), \quad L_\delta \left( \tilde{u} \frac{\partial \tilde{u}}{\partial x} \right) = O(\epsilon^3 \delta), \quad L_\delta \left( \frac{\partial \tilde{u}}{\partial x_2} \right) = O(\epsilon^7 \delta) \]

\[ L_\delta \left( \frac{\partial \tilde{\tau}_{xy}}{\partial y} \right) = O(\epsilon^5 \delta), \quad L_\delta \left( \frac{\partial \tilde{\tau}_{xz}}{\partial Z} \right) = O(\epsilon^3 \delta) \]

It then follows that

the left-hand side of (12.78a) \( = O(\epsilon^3 \delta) \)

the right-hand side of (12.78a) \( = O(\epsilon \delta) \)

Thus, the longitudinal momentum equation for the Langmuir circulation inside the BWBL is, to the leading order,

\[ \frac{\partial}{\partial Z} \left( Z \frac{\partial \tilde{u}_0}{\partial Z} \right) = O(\epsilon^2, \delta) \quad (12.84) \]

Similarly, the lateral momentum equation for the Langmuir circulation inside the
BWBL is

\[
\frac{\partial}{\partial Z} \left( Z \frac{\partial v_0}{\partial Z} \right) = O(\epsilon^2, \delta) \tag{12.85}
\]

and the vertical momentum equation for the Langmuir circulation inside the BWBL is

\[
\frac{\partial p_0}{\partial Z} = O(\epsilon^2, \delta) \tag{12.86}
\]

The bottom conditions for (12.84) and (12.85) are

\[
u_0 = v_0 = 0, \text{ at } Z = Z_B (or z = -kh + z_b) \tag{12.87}
\]

Inside the BWBL, \(w_0\) can be computed from the mean continuity equation. From (12.69a), continuity for the mean motion requires

\[
\epsilon^2 \frac{\partial w_0}{\partial x_2} + \frac{\partial v_0}{\partial y} + \frac{1}{\alpha \epsilon^2} \frac{\partial w_0}{\partial Z} = 0 \tag{12.88}
\]

where (12.1) has been used. After integrating from the bottom and making use of the no-flux condition on the bottom, we have

\[
w_0 = -\alpha \epsilon^2 \int_{z_B}^{z} \frac{\partial v_0}{\partial y} dZ' + O(\epsilon^3) \tag{12.89}
\]

thus \(w_0\) is computed by (12.89). The boundary conditions at the top of the BWBL is the continuity of the velocity and the shear stress. So in principle, the problems for the Langmuir circulation inside and outside the BWBL are coupled.

We remark that rewriting the core governing equations (12.24), (12.32), and (12.36) in terms of the inner coordinates, we can also obtain (12.89), (12.84)-(12.86), if \(U_0\) is taken as the solution inside BWBL. In other words, we can use (12.24) and (12.38) as the unified equations governing the Langmuir circulation inside and outside the BWBL, and the bottom conditions for (12.24) and (12.38) are simply

\[
u_0 = v_0 = w_0 = 0, \text{ at } z = -kh + z_b \tag{12.90}
\]
with the understanding that

\[
\frac{\partial U_0}{\partial z} = \frac{\alpha_0}{\kappa} \frac{1}{kh + z} + \frac{\partial \bar{U}_0^{(2)}}{\partial z}
\]

in (12.24) is computed differently inside and outside the BWBL.

The numerical scheme of solving \( u_0, v_0 \) and \( w_0 \) will be presented in chapter (15). Before that we discuss the energy budget of Langmuir circulation.
Chapter 13

Initial Energy Budget of Langmuir Circulation

13.1 Equation of mechanical energy

Now we examine the energy exchange between Langmuir circulation, irrotational waves and the basic current. In order to examine the role of the viscous terms, we shall use the strain rate to rewrite the viscous terms in the momentum equations, (12.24), (12.32) and (12.36).

The components of the strain rate of Langmuir circulation are defined by

\[ e_{xz} = e_{zx} = \frac{1}{2} \frac{\partial u_0}{\partial z}, \quad e_{xy} = e_{yx} = \frac{1}{2} \frac{\partial u_0}{\partial y} \]  \hspace{1cm} (13.1)

\[ e_{yy} = \frac{\partial v_0}{\partial y}, \quad e_{zz} = \frac{\partial w_0}{\partial z} \]  \hspace{1cm} (13.2)

\[ e_{yz} = e_{zy} = \frac{1}{2} \left( \frac{\partial v_0}{\partial y} + \frac{\partial v_0}{\partial z} \right) \]  \hspace{1cm} (13.3)

In terms of the strain rate, the momentum equations (12.24), (12.32) and (12.36), can be written as follows:
• Longitudinal momentum equation:

\[
\frac{\partial u_0}{\partial t} + \left( \frac{\alpha_0}{\kappa \, kh + z} + w_0 \frac{\partial U_0^{(2)}}{\partial z} \right) = 2\alpha \left[ \frac{\partial}{\partial y} (\tilde{S}_c e_{xy}) + \frac{\partial}{\partial z} (\tilde{S}_c e_{xz}) \right]
\]  

(13.4)

• Transverse momentum equation:

\[
\frac{\partial v_0}{\partial t} + \frac{\partial G_0}{\partial y} = 2\alpha \left[ \frac{\partial}{\partial y} (\tilde{S}_c e_{xy}) + \frac{\partial}{\partial z} (\tilde{S}_c e_{yz}) \right]
\]

(13.5)

where \( G_0 \) is defined by (12.31).

• Vertical momentum equation:

\[
\frac{\partial w_0}{\partial t} + AA^* u_0 \frac{\sinh(2(kh + z))}{\sinh^2(kh)} + \frac{\partial G_0}{\partial z} = 2\alpha \left[ \frac{\partial}{\partial y} (\tilde{S}_c e_{yz}) + \frac{\partial}{\partial z} (\tilde{S}_c e_{zz}) \right]
\]

(13.6)

The boundary conditions are

• On the mean surface \( z = 0 \)

\[
\frac{\partial v_0}{\partial z} = 0 \quad \text{(13.7a)}
\]

\[
w_0 = 0 \quad \text{(13.7b)}
\]

\[
\alpha \tilde{S}_c \frac{\partial u_0}{\partial z} = -\frac{AA^*}{2} \coth(kh) \frac{\partial v_0}{\partial y} \quad \text{(13.7c)}
\]

• On the bottom, \( z = z_+ = -kh + z_b \)

\[
u_0 = v_0 = w_0 = 0 \quad \text{(13.8)}
\]

The dimensionless energy of Langmuir circulation per unit volume and per unit mass is defined by

\[
E = \frac{1}{2} \left( u_0^2 + v_0^2 + w_0^2 \right)
\]

(13.9)
Multiplying $u_0$, $v_0$, $w_0$ on both sides of equations (13.4)-(13.6), respectively, we get

\[
u_0 \left( \frac{\partial u_0}{\partial t_2} + \left( \frac{\alpha_0 w_0}{\kappa (k h + z)} + w_0 \frac{\partial v_0^{(2)}}{\partial z} \right) \right) = 2\nu_0 \alpha \left[ \frac{\partial}{\partial y} (\tilde{S}_c e_{xy}) + \frac{\partial}{\partial z} (\tilde{S}_c e_{zz}) \right] \quad (13.10)
\]

\[
v_0 \left( \frac{\partial v_0}{\partial t_2} + \frac{\partial G_0}{\partial y} \right) = 2\alpha v_0 \left[ \frac{\partial}{\partial y} (\tilde{S}_c e_{yy}) + \frac{\partial}{\partial z} (\tilde{S}_c e_{yz}) \right] \quad (13.11)
\]

\[
w_0 \left( \frac{\partial w_0}{\partial t_2} + u_0 \frac{\partial U_z}{\partial z} + \frac{\partial G_0}{\partial z} \right) = 2\alpha w_0 \left[ \frac{\partial}{\partial y} (\tilde{S}_c e_{yz}) + \frac{\partial}{\partial z} (\tilde{S}_c e_{zz}) \right] \quad (13.12)
\]

where

\[
\frac{\partial U_z}{\partial z} = \frac{A A^* \sinh(2(kh + z))}{\sinh^2(kh)} \quad (13.13)
\]

is the gradient of the Stokes drift of the irrotational wave field.

The sum of (13.10)-(13.12) is the mechanical energy equation of Langmuir circulation. Next, we work out the algebra term by term.

1. The sum of the time-derivative terms on left-hand side of equations (13.10)-(13.12)

\[
u_0 \frac{\partial u_0}{\partial t_2} + v_0 \frac{\partial v_0}{\partial t_2} + w_0 \frac{\partial w_0}{\partial t_2} = \frac{1}{2} \frac{\partial}{\partial t_2} \left( u_0^2 + v_0^2 + w_0^2 \right) = \frac{\partial E}{\partial t_2} \quad (13.14)
\]

is the time rate of change of the mechanical energy of Langmuir circulation per unit volume and per unit mass.

2. The sum of those terms associated with $G_0$ is

\[
v_0 \frac{\partial G_0}{\partial y} + w_0 \frac{\partial G_0}{\partial z} = \frac{\partial G_0 v_0}{\partial y} + \frac{\partial G_0 w_0}{\partial z} \quad (13.15)
\]

where the continuity equation for Langmuir circulation, (12.12), has been used.
3. The sum of the right-hand side of (13.10)-(13.12) is

\[
2 \alpha u_0 \left[ \frac{\partial}{\partial y} (S_c e_{xy}) + \frac{\partial}{\partial z} (S_c e_{zz}) \right] + 2 \alpha v_0 \left[ \frac{\partial}{\partial y} (S_c e_{yy}) + \frac{\partial}{\partial z} (S_c e_{yz}) \right]
\]

\[
+ 2 \alpha w_0 \left[ \frac{\partial}{\partial y} (S_c e_{yz}) + \frac{\partial}{\partial z} (S_c e_{zz}) \right]
\]

(13.16)

which can be rewritten as

\[
\left\{ 2 \alpha \left[ \frac{\partial}{\partial y} (S_c e_{xy} u_0) + \frac{\partial}{\partial z} (S_c e_{xx} u_0) \right] + 2 \alpha \left[ \frac{\partial}{\partial y} (S_c e_{yy} v_0) + \frac{\partial}{\partial z} (S_c e_{yz} v_0) \right]
\right\}
\]

\[
+ 2 \alpha \left[ \frac{\partial}{\partial y} (S_c e_{yz} w_0) + \frac{\partial}{\partial z} (S_c e_{zz} w_0) \right]
\}

\[
- \left\{ 2 \alpha \left[ \frac{\partial u_0}{\partial y} (S_c e_{xy}) + \frac{\partial u_0}{\partial z} (S_c e_{xx}) \right] + 2 \alpha \left[ \frac{\partial v_0}{\partial y} (S_c e_{yy}) + \frac{\partial v_0}{\partial z} (S_c e_{yz}) \right]
\right\}
\]

\[
+ 2 \alpha \left[ \frac{\partial w_0}{\partial y} (S_c e_{yz}) + \frac{\partial w_0}{\partial z} (S_c e_{zz}) \right]
\}

\[
= 2 \alpha \frac{\partial}{\partial y} \left[ S_c (e_{xy} u_0 + e_{yy} v_0 + e_{yz} w_0) \right] + 2 \alpha \frac{\partial}{\partial z} \left[ S_c (e_{xx} u_0 + e_{yy} v_0 + e_{zz} w_0) \right]
\]

\[
\text{total work by Reynolds stress of Langmuir circulation}
\]

\[
-2 \alpha \tilde{S}_c \left( 2e_{xy}^2 + 2e_{xx}^2 + 2e_{yy}^2 + 2e_{yz}^2 + 2e_{zz}^2 \right)
\]

(13.17)

viscous dissipation

where the definitions of the strain rate \( e_{ij} \) have been used in obtaining the last equation.

By summing up (13.10)-(13.12), the mechanical energy equation of Langmuir circulation is obtained

\[
\frac{\partial E}{\partial t_2} = -u_0 w_0 \frac{\partial U_s}{\partial z} - \left( \frac{\alpha_0 w_0 u_0}{\kappa kh + z} + w_0 u_0 \frac{\partial U^{(2)}_0}{\partial z} \right)
\]

\[
-2 \alpha \tilde{S}_c \left( 2e_{xy}^2 + 2e_{xx}^2 + 2e_{yy}^2 + 2e_{yz}^2 + 2e_{zz}^2 \right)
\]

\[
+ \frac{\partial}{\partial y} \left( 2 \alpha \tilde{S}_c (e_{xy} u_0 + e_{yy} v_0 + e_{yz} w_0) - G_0 v_0 \right)
\]

\[
+ \frac{\partial}{\partial z} \left( 2 \alpha \tilde{S}_c (e_{xx} u_0 + e_{yy} v_0 + e_{zz} w_0) - G_0 w_0 \right)
\]

(13.18)

Let us compute the total energy of Langmuir circulation contained in a box which extends laterally over a transverse wavelength, horizontally over a unit length, and
vertically from bottom to the mean surface. This can be done in two steps:

1. Integrating (13.18) with respect to $y$ over one lateral wavelength and making use of periodicity, we get

\[
\frac{\partial}{\partial t_2} \langle E \rangle = -\langle w_0 u_0 \rangle \frac{\partial U_y}{\partial z} \left( \frac{\alpha_0 \langle w_0 u_0 \rangle}{\kappa kh + z} + \langle w_0 u_0 \rangle \frac{\partial U_y^{(2)}}{\partial z} \right) - 2\alpha \bar{S}_c \langle 2e_{xy}^2 + 2e_{xz}^2 + e_{yy}^2 + 2e_{yz}^2 + e_{zz}^2 \rangle + \frac{\partial}{\partial z} \left( 2\alpha \bar{S}_c \langle e_{zz} u_0 + e_{yz} v_0 + e_{zz} w_0 \rangle - \langle G_0 w_0 \rangle \right)
\]

(13.19)

where $\langle . \rangle$ represents the integration over one lateral wavelength in $y$.

2. Integrating (13.19) with respect to $z$ from the bottom to the mean surface, and making use of the bottom and surface boundary conditions, (13.7) and (13.8), we get

\[
\frac{\partial}{\partial t_2} \int_{z_+}^{0} \langle E \rangle \, dz = -\int_{z_+}^{0} \langle w_0 u_0 \rangle \frac{\partial U_y}{\partial z} \, dz - \int_{z_+}^{0} \frac{\alpha_0 \langle w_0 u_0 \rangle}{\kappa kh + z} \, dz - \int_{z_+}^{0} \bar{S}_c \langle 2e_{xy}^2 + 2e_{xz}^2 + e_{yy}^2 + 2e_{yz}^2 + e_{zz}^2 \rangle \, dz
\]

\[
+ \int_{z_+}^{0} \left( 2\alpha \bar{S}_c \langle e_{zz} u_0 \rangle \right) \text{work done by surface stress}
\]

(13.20)

where $z_+$ is denoted to the vertical coordinate at bottom, $z_+ = -kh + z_b$. Note that in (13.20) $G_0$ disappears.

Each term on the right-hand side of (13.20) has the physical meaning similar to the energy budget for a turbulent shear flow (see, e.g., pp436, Kundu( 1990) [36]):

- Viscous dissipation function

\[
\varepsilon = 2\alpha \int_{z_+}^{0} \bar{S}_c \langle 2e_{xy}^2 + 2e_{xz}^2 + e_{yy}^2 + 2e_{yz}^2 + e_{zz}^2 \rangle \, dz
\]

(13.21)

which is always positive.
In analogy to the turbulent shear flow (Tennekes and Lumley (1972) [70]; Kundu (1990) [36]), the energy production by the Reynolds stress of Langmuir circulation against the mean shear is

\[ - \int_{z^+}^{0} \langle u_0 w_0 \rangle \frac{\partial U^z}{\partial z} dz - \int_{z^+}^{0} \frac{\alpha_0}{\kappa} \langle w_0 u_0 \rangle dz - \int_{z^+}^{0} \langle w_0 u_0 \rangle \frac{\partial U_0^{(2)}}{\partial z} dz \]  

(13.22)

There are three kinds of shear productions of Langmuir circulation in (13.22):

1. The energy production by the Reynolds stress of Langmuir circulation against the shear of Stokes drift

\[ P_{st} = - \int_{z^+}^{0} \langle u_0 w_0 \rangle \frac{\partial U^z}{\partial z} dz, \]  

(13.23)

which represents the energy exchange between the Langmuir circulation and irrotational wave motion through the nonlinear wave-wave interaction. Because \( w_0 = 0 \) at the bottom and surface and \( u_0 = 0 \) at the bottom, the most important contribution comes from Stokes drift in the core region.

2. The energy production by the Reynolds stress of Langmuir circulation against the shear of the basic current \( U_0^{(1)} \)

\[ P_{u_1} = - \int_{z^+}^{0} \frac{\alpha_0}{\kappa} \langle w_0 u_0 \rangle \frac{\partial U^z}{\partial z} dz. \]  

(13.24)

Because \( w_0 = 0 \) at two boundaries, \( \bar{u}_0 = 0 \) at the bottom, and \( kh + z = z_b \) is small on the bottom, the most important contribution comes from in the core region and possibly the bottom. Again, the logarithmic feature of \( U_0^{(1)} \) makes the shear rate of \( U_0^{(1)} \) comparable to \( U_0^{(2)} \) as shown in (12.19).

3. The energy production by the Reynolds stress of Langmuir circulation against the shear of the wave-induced current \( U_0^{(2)} \)

\[ P_{u_2} = - \int_{z^+}^{0} \langle w_0 u_0 \rangle \frac{\partial U_0^{(2)}}{\partial z} dz. \]  

(13.25)

Because \( U_0^{(2)} \) is induced by the nonlinear interaction between the irrotational wave and the rotational wave motion, thus this part of the shear...
production also represents the energy exchange between the Langmuir circulation and the rotational wave motion through the nonlinear wave-wave interaction in the bulk of water. Because \( w_0 = 0 \) at the two boundaries and the shear rate of \( U_0^{(2)} \) is large both near the surface and bottom, the entire region from the bottom to the mean surface are important for this part of shear production.

- Work done by the surface stress:

\[
W_{sf} = 2\alpha \left[ \bar{S}_z \langle e_{zz} u_0 \rangle \right]_0
\]  

(13.26)

If \( e_{xx} = 0 \) on the surface, then there would be no force on the surface and the work done by surface force is zero. This is the case of CL-I and CL-II theory for Langmuir circulation. In the present theory, the surface boundary condition reads

\[
\alpha \bar{S}_z \frac{\partial u_0}{\partial z} = -AA^* \frac{\coth(kh)}{2} \frac{\partial v_0}{\partial y}, \quad e_{xz} = \frac{1}{2} \frac{\partial u_0}{\partial z}
\]  

(13.27)

hence the work done by the surface force, (13.26), is

\[
W_{sf} = -\left[ \langle u_0 \frac{\partial v_0}{\partial y} \rangle \right]_0 AA^* \frac{\coth(kh)}{2}
\]  

(13.28)

Because the mean surface shear stress is the result of the nonlinear interaction between the primary and secondary waves, this part of work represents the energy exchange between the Langmuir circulation and the rotational wave motion through the mean shear stress on the water surface.

In summary the mechanical energy equation of Langmuir circulation is

\[
\frac{\partial}{\partial t_2} \int_{x,z} \langle E \rangle \, dz = \mathcal{P}_{st} + \mathcal{P}_{u_1} + \mathcal{P}_{u_2} + W_{sf} - \varepsilon
\]  

(13.29)

The production and dissipation of Langmuir circulation on the right-hand side of (13.29) can be computed once \((u_0, v_0, w_0)\) are obtained by solving the eigenvalue problem formulated in the chapter (12). In the following we shall assume the normal
mode and solve for the growth rate and eigenfunctions of \((u_0, v_0, w_0)\). The corresponding eigenvalue is related to the time rate of change of the mechanical energy of Langmuir circulation (13.29), which will be used to check the numerical results and to help with physical understanding of generation of Langmuir circulation.

### 13.2 Growth rate of the normal mode

We examine the stability of the normal mode of the form

\[
\begin{align*}
  u_0 &= e^{\sigma t_2} \hat{u}_0 e^{iK_y} + \text{c.c.}, \quad v_0 = e^{\sigma t_2} \hat{v}_0 e^{iK_y} + \text{c.c.} \tag{13.30} \\
  w_0 &= e^{\sigma t_2} \hat{w}_0 e^{iK_y} + \text{c.c.}, \quad G_0 = e^{\sigma t_2} \hat{G}_0 e^{iK_y} + \text{c.c.} \tag{13.31}
\end{align*}
\]

where \(K\) is the dimensionless transverse wavenumber and assumed real. \(\sigma\) is the eigenvalue to be found. In general, \(\sigma\) is complex and can be written as \(\sigma = \sigma_e + i\sigma_i\) with both \(\sigma_e\) and \(\sigma_i\) being real.

Substituting (13.30) and (13.31) in the linearized equation of mechanical energy for Langmuir circulation, (13.20), we get

- **mechanical energy**

  \[
  \langle E \rangle = e^{2\sigma_e t_2} \left( |\hat{u}_0|^2 + |\hat{v}_0|^2 + |\hat{w}_0|^2 \right) \tag{13.32}
  \]

- **strain rate**

  \[
  \begin{align*}
  e_{xy} &= iK \frac{1}{2} \hat{u}_0 e^{\sigma t_2} e^{iK_y} + \text{c.c.}, \quad e_{zz} = \frac{1}{2} \frac{\partial \hat{u}_0}{\partial z} e^{\sigma t_2} e^{iK_y} + \text{c.c.} \\
  e_{yy} &= iK \hat{v}_0 e^{\sigma t_2} e^{iK_y} + \text{c.c.}, \quad e_{zz} = \frac{\partial \hat{v}_0}{\partial z} e^{\sigma t_2} e^{iK_y} + \text{c.c.} \\
  e_{yz} &= \frac{1}{2} \left( iK \hat{w}_0 + \frac{\partial \hat{v}_0}{\partial z} \right) e^{\sigma t_2} e^{iK_y} + \text{c.c.}
  \end{align*}
  \]

Thus we have

\[
\begin{align*}
  \langle e_{xy}^2 \rangle &= \frac{1}{2} K^2 |\hat{u}_0|^2 e^{2\sigma_e t_2}, \quad \langle e_{zz}^2 \rangle = \frac{1}{2} \left| \frac{\partial \hat{u}_0}{\partial z} \right|^2 e^{2\sigma_e t_2}, \quad \langle e_{zz}^2 \rangle = 2 \left| \frac{\partial \hat{v}_0}{\partial z} \right|^2 e^{2\sigma_e t_2} \\
  \langle e_{yz}^2 \rangle &= \frac{1}{2} \left| iK \hat{w}_0 + \frac{\partial \hat{v}_0}{\partial z} \right|^2 e^{2\sigma_e t_2}, \quad \langle e_{yy}^2 \rangle = 2K^2 |\hat{v}_0|^2 e^{2\sigma_e t_2}
\end{align*}
\]
viscous dissipation function

\[ \varepsilon = \alpha \int_{z_+}^{0} \mathcal{S}_c \left( 4e_{xy}^2 + 4e_{xz}^2 + 2e_{yy}^2 + 4e_{yz}^2 + 2e_{zz}^2 \right) dz \]

\[ = 2\alpha e^{2\pi t_2} \int_{z_+}^{0} \mathcal{S}_c \left( K^2 |\ddot{u}_0|^2 + \left| \frac{\partial u_0}{\partial z} \right|^2 + 2K^2 |\ddot{v}_0|^2 + 2 \left| \frac{\partial \ddot{v}_0}{\partial z} \right|^2 + \left| \frac{\partial \ddot{v}_0}{\partial z} + iK \ddot{v}_0 \right|^2 \right) dz \]

energy production by Stokes drift

\[ \mathcal{P}_{st} = -e^{2\pi t_2} \int_{z_+}^{0} (\ddot{u}_0 \dddot{\omega}_0 + \text{c.c.}) \frac{\partial U_0}{\partial z} dz \] (13.33)

where \( \dddot{\omega}_0 \) is the complex conjugate of \( \dddot{\omega}_0 \).

energy production by basic current \( U_0^{(1)} \)

\[ \mathcal{P}_{u_1} = -e^{2\pi t_2} \int_{z_+}^{0} \alpha_0 \left( \dddot{u}_0 \dddot{\omega}_0 + \text{c.c.} \right) \frac{\partial U_0^{(1)}}{\partial z} dz \] (13.34)

energy production by wave-induced current \( U_0^{(2)} \)

\[ \mathcal{P}_{u_2} = -e^{2\pi t_2} \int_{z_+}^{0} (\dddot{u}_0 \dddot{\omega}_0 + \text{c.c.}) \frac{\partial U_0^{(2)}}{\partial z} dz \] (13.35)

work done by the surface stress:

\[ \mathcal{W}_{sf} = -e^{2\pi t_2} K [i\dot{v}_0 \dddot{\omega}_0 + \text{c.c.}]_0 AA^* \frac{\text{coth}(kh)}{2} \] (13.36)

It then follows that the linearized equation of mechanical energy of Langmuir circu-
lation (13.29) can be written as

\[
2\sigma_r \int_{z_+}^0 \left( |\hat{u}_0|^2 + |\hat{v}_0|^2 + |\hat{w}_0|^2 \right) dz = -\int_{z_+}^0 (\hat{u}_0\hat{w}_0^* + c.c.) \frac{\partial U_s}{\partial z} dz
\]

\[
-\int_{z_+}^0 \frac{\alpha_0}{\kappa} \frac{(\hat{u}_0\hat{w}_0^* + c.c.)}{kh + z} dz - \int_{z_+}^0 (\hat{u}_0\hat{w}_0^* + c.c.) \frac{\partial U_s^{(2)}}{\partial z} dz
\]

\[
-K [i\hat{v}_0\hat{w}_0^* + c.c.]_0 AA^* \frac{\coth(kh)}{2}
\]

\[
-2\alpha \int_{z_+}^0 \tilde{S}_c \left( K^2 |\hat{u}_0|^2 + \left| \frac{\partial \hat{u}_0}{\partial z} \right|^2 + 2K^2 |\hat{v}_0|^2 + 2 \left| \frac{\partial \hat{v}_0}{\partial z} \right|^2 + \left| \frac{\partial \hat{v}_0}{\partial z} + iK\hat{w}_0 \right|^2 \right) dz
\]

which can be written in terms of the growth rates

\[
\sigma_r = \sigma_{st} + \sigma_{u_1} + \sigma_{u_2} + \sigma_{sw} + \sigma_\varepsilon
\]  

(13.37)

where

\[
\sigma_{st} = -\frac{\int_{z_+}^0 (\hat{u}_0\hat{w}_0^* + c.c.) \frac{\partial U_s}{\partial z} dz}{2 \int_{z_+}^0 (|\hat{u}_0|^2 + |\hat{v}_0|^2 + |\hat{w}_0|^2) dz}
\]  

(13.38a)

\[
\sigma_{u_1} = -\frac{\int_{z_+}^0 \frac{\alpha_0}{\kappa} \frac{(\hat{u}_0\hat{w}_0^* + c.c.)}{kh + z} dz}{2 \int_{z_+}^0 (|\hat{u}_0|^2 + |\hat{v}_0|^2 + |\hat{w}_0|^2) dz}
\]  

(13.38b)

\[
\sigma_{u_2} = -\frac{\int_{z_+}^0 (\hat{u}_0\hat{w}_0^* + c.c.) \frac{\partial U_s^{(2)}}{\partial z} dz}{2 (|\hat{u}_0|^2 + |\hat{v}_0|^2 + |\hat{w}_0|^2)}
\]  

(13.38c)

\[
\sigma_{sw} = -\frac{K [i\hat{v}_0\hat{w}_0^* + c.c.]_0 AA^* \frac{\coth(kh)}{2}}{2 \int_{z_+}^0 (|\hat{u}_0|^2 + |\hat{v}_0|^2 + |\hat{w}_0|^2) dz}
\]  

(13.38d)

\[
\sigma_\varepsilon = -\frac{\alpha \int_{z_+}^0 \tilde{S}_c \left( K^2 |\hat{u}_0|^2 + \left| \frac{\partial \hat{u}_0}{\partial z} \right|^2 + 2K^2 |\hat{v}_0|^2 + 2 \left| \frac{\partial \hat{v}_0}{\partial z} \right|^2 + \left| \frac{\partial \hat{v}_0}{\partial z} + iK\hat{w}_0 \right|^2 \right) dz}{\int_{z_+}^0 (|\hat{u}_0|^2 + |\hat{v}_0|^2 + |\hat{w}_0|^2) dz}
\]  

(13.38e)

Physically

- \( \sigma_{st} \) is the rate of the mechanical energy production by shear in Stokes drift, which is also the rate of the energy exchange between the Langmuir circulation and wave motion in the bulk body of water.

- \( \sigma_{u_1} \) is the rate of the mechanical energy production by shear in the unperturbed current.
\* \( \sigma_{u2} \) is the rate of the mechanical energy production by shear in the wave-induced current correction.

\* \( \sigma_{sw} \) is the rate of the mechanical energy production by the work done by the surface stress, which is also the rate of the energy exchange between the Langmuir circulation and wave motion through the surface boundary of water.

\* \( \sigma_c \) is the rate of mechanical energy dissipation by eddy viscosity.

### 13.3 Contribution of BWBL to the growth rate

Inside the BWBL, the basic current is logarithmic and has a large shear, but we shall show that the contribution of the BWBL to the growth rate is small in comparison with that in the core region.

Inside the BWBL, the governing equations of Langmuir circulation are (12.89), (12.84) and (12.85). Integrating (12.84) and (12.85) with respect to \( Z \) from the outer edge of the BWBL, we have

\[
Z \frac{\partial u_0}{\partial Z} = \left[ Z \frac{\partial u_0}{\partial Z} \right]_+ \tag{13.39}
\]

\[
Z \frac{\partial v_0}{\partial Z} = \left[ Z \frac{\partial v_0}{\partial Z} \right]_+ \tag{13.40}
\]

where \([.]+\) means evaluating at the outer edge of the BWBL, \( Z = Z_\delta \), defined by (12.72). The right-hand sides of (13.39) and (13.40) are computed by the continuity of the shear stress at the outer edge of the BWBL.

Integrating (13.39) and (13.40) once more with respect to \( Z \) from the bottom and making use of no-flux and no-slip condition at the bottom \( Z = Z_B \), we get

\[
u_0 = \left[ Z \frac{\partial u_0}{\partial Z} \right]_+ \ln \left( \frac{Z}{Z_B} \right) \tag{13.41}
\]

\[
u_0 = \left[ Z \frac{\partial u_0}{\partial Z} \right]_+ \ln \left( \frac{Z}{Z_B} \right) \tag{13.42}
\]

Because \( u_0 \) and \( v_0 \) are of \( O(1) \) at the outer edge of the BWBL, thus small \( Z_B \) will always accompany with small \( [Z \frac{\partial u_0}{\partial Z}]_+ \) and \( [Z \frac{\partial v_0}{\partial Z}]_+ \).
The vertical velocity inside BWBL is given by (12.89),

\[ w_0 = -\alpha e^2 \int_{Z_B}^{Z} \frac{\partial \hat{v}_0}{\partial y} dZ' + O(\epsilon^3) \]  
\[ (13.43) \]

It then follows from (13.41) and (13.43) that inside the BWBL,

\[ \hat{u}_0 \hat{w}_0^* + c.c. = \alpha K e^2 \left\{ i \left[ Z \frac{\partial \hat{u}_0}{\partial Z} \right]_+ \ln \left( \frac{Z}{Z_B} \right) \int_{Z_B}^{Z} \hat{v}_0^* dZ' + c.c. \right\} \]
\[ (13.44) \]

From (13.42) we get

\[ \int_{Z_B}^{Z} \hat{v}_0^* dZ' = \left[ Z \frac{\partial \hat{v}_0^*}{\partial Z} \right]_+ \left( Z \ln \left( \frac{Z}{Z_B} \right) - Z + Z_B \right) \]

thus,

\[ \hat{u}_0 \hat{w}_0^* + c.c. = \alpha K e^2 F_B(Z, Z_B) \left\{ i \left[ Z \frac{\partial \hat{u}_0}{\partial Z} \right]_+ \left[ Z \frac{\partial \hat{v}_0^*}{\partial Z} \right]_+ + c.c. \right\} \]
\[ (13.45) \]

where \( F_B(Z, Z_B) \) is defined by

\[ F_B(Z, Z_B) = \ln \left( \frac{Z}{Z_B} \right) \left( Z \ln \left( \frac{Z}{Z_B} \right) - Z + Z_B \right) \]
\[ (13.46) \]

Now we examine the the contribution of the BWBL to \( \sigma_{u_1} \). Inside the BWBL, we have

\[ \frac{\alpha_0}{\kappa} \frac{(\hat{u}_0 \hat{w}_0^* + c.c.)}{h + z} = \alpha e^2 \frac{\alpha_0}{\kappa} \frac{(\hat{u}_0 \hat{w}_0^* + c.c.)}{Z} \]
\[ = \frac{\alpha_0}{\kappa} \frac{e^4}{\kappa} \left\{ i \left[ Z \frac{\partial \hat{u}_0}{\partial Z} \right]_+ \left[ Z \frac{\partial \hat{v}_0^*}{\partial Z} \right]_+ + c.c. \right\} \frac{F_B(Z, Z_B)}{Z} \]
\[ (13.47) \]

where \( \alpha_0 = O(1), \alpha = O(1) \), and

\[ \int_{Z_B}^{Z} (|\hat{u}_0|^2 + |\hat{v}_0|^2 + |\hat{w}_0|^2) dz = O(1) \]
\[ (13.48) \]

at most because small \( Z_B \) will always accompany with small \( \left[ Z \frac{\partial \hat{u}_0}{\partial Z} \right]_+ \) and \( \left[ Z \frac{\partial \hat{v}_0^*}{\partial Z} \right]_+ \).

Function \( F_B(Z, Z_B)/Z \) is plotted in Figure (13-1) for several \( Z_B \). It is clear that
function $O(F_B(Z, Z_B)/Z)$ increases with decreasing $Z_B$.

We take $Z_B = 0.0001, \epsilon = 0.1$ for example. From (13.47) and (13.48), and $\alpha_0 = O(1), \alpha = O(1)$, we have

\[
\frac{\alpha_0 (\bar{u}_0 \bar{w}_0^* + c.c.)}{\kappa kh + z} = O(0.01)
\]

(13.49)

inside the BWBL. If the thickness of the BWBL (12.72) is

\[
Z_\delta = \frac{\delta_b}{\alpha \epsilon^2} = O(5)
\]

(13.50)

we have

\[
\int_{-kh + \delta_b}^{kh + \delta_b} \frac{\alpha_0 (\bar{u}_0 \bar{w}_0^* + c.c.)}{\kappa kh + z} \, dz = \int_{Z_B}^{Z_B} \frac{\alpha_0 (\bar{u}_0 \bar{w}_0^* + c.c.)}{\kappa Z} \, dZ = O(0.05)
\]

(13.51)

Thus the contribution of the BWBL to $\sigma_{u_1}$ would be $O(0.05)$ at most. Thus, even though, the basics shear is larger inside the BWBL and $u_0, \nu_0$ are logarithmic inside BWBWL, the contribution of the BWBL to the growth rate of $\sigma_{u_1}$ is small in comparison with that in the core region. Similar conclusions can be draw for $\sigma_{u_2}$ and $\sigma_{st}$, that is, the bottom wave boundary layer is not important for the growth rate of Langmuir circulation.
The contribution of the BWBL to the growth rate is small does not mean that the bottom is not important to the growth rate of Langmuir circulation. The viscous boundary conditions at the bottom still contribute to the growth rate by affecting the shapes of the eigenfunctions.
Chapter 14

Generation mechanism of Langmuir circulation

In this chapter, we shall discuss the generation of Langmuir circulation by two different mechanisms: the feedback mechanism of Craik and Leibovich and the surface-stress-forcing mechanism. Since past experiments were performed in a wave flume of finite width, we shall also remark on the sidewall effect in order to interpret the laboratory observations on Langmuir circulation.

From the analysis of the energy production of Langmuir circulation, it has been shown that eddy viscosity will always dissipate energy, and cause stability. To study the instability mechanism, it is helpful to examine first the inviscid limit of the original problem.

14.1 Review of inviscid problem

The inviscid limits of the governing equations, which have been studied by Criak(1977) [9] for deep water, are:

- Conservation of mass:

\[
\frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0
\]  

(14.1)
• The longitudinal momentum equation (cf. (12.24)):

\[
\frac{\partial u_0}{\partial t_2} + w_0 \frac{\partial U_0}{\partial z} = 0
\]  

(14.2)

where \( U_0 \) is the total Eulerian mean current, including the unperturbed basic current and the wave-induced current correction.

• The transverse momentum equation (cf. (12.32)):

\[
\frac{\partial v_0}{\partial t_2} + \frac{\partial G_0}{\partial y} = 0
\]  

(14.3)

• The vertical momentum equation (cf. (12.36)):

\[
\frac{\partial w_0}{\partial t_2} + u_0 \frac{\partial U_s}{\partial z} + \frac{\partial G_0}{\partial z} = 0
\]  

(14.4)

Boundary conditions in the inviscid limit are the no-flux condition on the mean surface (\( z=0 \)) and the bottom (\( z = -kh + z_b \))

\[
w_0 = 0
\]  

(14.5)

Eliminating \( G_0 \) from (14.3) and (14.4), we get the following longitudinal vorticity equation

\[
\frac{\partial \xi_0}{\partial t_2} + \frac{\partial u_0}{\partial y} \frac{\partial U_s}{\partial z} = 0
\]  

(14.6)

where the longitudinal vorticity \( \xi_0 \) is defined by

\[
\xi_0 = \frac{\partial w_0}{\partial y} - \frac{\partial v_0}{\partial z}
\]  

(14.7)

After taking the \( t_2 \)-derivative of (14.6), we get

\[
\frac{\partial^2 \xi_0}{\partial t_2^2} + \frac{\partial^2 u_0}{\partial y \partial t_2} \frac{\partial U_s}{\partial z} = 0
\]  

(14.8)
Eliminating $u_0$ from (14.8) and (14.2), we have

$$\frac{\partial^2 \xi_0}{\partial t^2} = \frac{\partial w_0}{\partial y} M(z) \quad (14.9)$$

where

$$M(z) = \frac{\partial U_0 \partial U_s}{\partial z \partial z} \quad (14.10)$$

which is a function independent of the spanwise perturbations.

### 14.2 The feedback mechanism of Craik and Leibovich

Based on the inviscid equations, we shall first review the instability criterion and the feedback mechanism for the generation of Langmuir circulation, which is first discovered by Craik and Leibovich (Craik and Leibovich (1976) [10], Criak (1977) [9], and Leibovich (1983) [39], etc.)

#### 14.2.1 Instability criterion for inviscid limit

For the normal mode given in (13.30) and (13.31), we have from (14.9)

$$\sigma^2 \dot{\xi}_0 = iK \dot{w}_0 M(z), \text{ where } \dot{\xi}_0 = iK \dot{w}_0 - \frac{\partial \dot{v}_0}{\partial z} \quad (14.11)$$

It is convenient to introduce the transverse stream function defined by

$$\frac{\partial \psi}{\partial z} = -\dot{v}_0, \quad iK \psi = \dot{w}_0 \quad (14.12)$$

which satisfy the conservation of mass (14.1) automatically.
The governing equation (14.9) can be reduced to a single equation for \( \psi \)

\[
\sigma^2(D^2\psi - K^2\psi) + K^2M(z)\psi = 0
\]

or

\[
D^2\psi + K^2\left(\frac{M(z)}{\sigma^2} - 1\right)\psi = 0
\]  \hspace{1cm} (14.13)

where \( D = \partial/\partial z \). In terms of the stream function, the inviscid boundary condition \( (14.5) \), is

\[
\psi = 0 \text{ at } z = 0, -kh + z_b
\]  \hspace{1cm} (14.14)

This limit has been studied by Craik (1977) [9] and Leibovich (1977) [38] for deep water.

Because there is no dissipation, the perturbation will either grow or oscillate, but not decay. Therefore, Craik (1977) and Leibovich (1977) have found the following:

1. The principle of the exchange of stability holds.

2. A necessary condition for instability is

\[
M(z) = \frac{\partial U_0}{\partial z} \frac{\partial U_s}{\partial z} > 0
\]  \hspace{1cm} (14.15)

in some interval of \( z \).

3. If the perturbation is unstable then the basic flow is unstable to all lateral wavenumbers.

4. The growth rate increase with increasing lateral wavenumber \( K \).

Now we revisit this simple inviscid problem for the finite depth of water. Note that equation (14.13) constitutes the standard Sturm-Liouville problem. From the well-known theory, all the eigenfunctions are orthogonal to each other. Because \( M(z) \) and \( K \) are real, thus if \( \psi \) is an eigenfunction for \( \sigma^2 \), then \( \psi^* \) must be another eigenfunction for \( (\sigma^2)^* \) (see, e.g., Craik (1977) [9]). It then follows from the orthogonality of the eigenfunctions that \( (\sigma^2)^* = \sigma^2 \), i.e., \( \sigma^2 \) is real. In other words, \( \sigma \) is either real (the basic state is unstable) or \( \sigma \) is pure imaginary (the perturbation is oscillatory).
Now let’s examine the conditions for instability. Multiplying \( \psi^* \) on both side of (14.13), integrating from the bottom to the surface and then applying the integration by part once, we have

\[
\int_{z_+}^{0} |D\psi|^2 dz = K^2 \int_{z_+}^{0} \left( \frac{M(z)}{\sigma^2} - 1 \right) |\psi|^2 dz
\]  

(14.16)

where \( z_+ = -kh + z_b \). The left-hand side of this equation is positive. In order to have nontrivial solutions, it is necessary that

\[
\frac{M(z)}{\sigma^2} > 1, \quad \Rightarrow \quad \sigma^2 < M(z)
\]

somewhere in the water column.

For \( M(z) < 0 \) everywhere in \( z = [-kh + z_b, 0] \), we must have \( \sigma^2 < 0 \) in order to have nontrivial solution, i.e., \( M(z) < 0 \) means oscillatory stability. It then follows that the necessary condition for the instability is \( M(z) > 0 \) somewhere in the water column, the same conclusions as in deep water (Craik(1977) [9]; Leibovich(1977) [38]).

Alternatively, we can write (14.16) as

\[
\sigma^2 \left( \int_{z_+}^{0} |D\psi|^2 dz + K^2 \int_{z_+}^{0} |\psi|^2 dz \right) = K^2 \int_{z_+}^{0} M(z)|\psi|^2 dz
\]

(14.18)

Because terms in the parenthesis on the left-hand side of this equation and \( K^2 \) are positive, \( M(z) < 0 \) everywhere in the water column must lead to \( \sigma^2 < 0 \).

If the basic state is unstable, then from (14.18) the growth rate \( \sigma^2 \) can be computed by

\[
\sigma^2 = \frac{K^2 \int_{z_+}^{0} M(z)|\psi|^2 dz}{\left( \int_{z_+}^{0} |D\psi|^2 dz + K^2 \int_{z_+}^{0} |\psi|^2 dz \right)}
\]

(14.19)

Because of the boundary conditions (\( \psi = 0 \)) at the bottom and the mean surface, it is interesting to note that the most important contribution of \( M(z) \) comes from the core region where \( \psi \) is not small.
From (14.19), we get the following limits

\[ \sigma^2 \to \frac{\int_{z_0}^{0} M(z)|\psi|^2 dz}{\int_{z_0}^{z_*} |\psi|^2 dz}, \text{ for } K^2 \to \infty \]  
(14.20)

\[ \sigma^2 \to 0, \text{ for } K^2 \to 0 \]  
(14.21)

The growth rate of \( \sigma^2 \) with \( K^2 \) is illustrated in Figure (14-1).

**14.2.2 Role of viscosity**

Note that the criterion (14.15) is independent of the lateral wavenumber. It was known from the Rayleigh-Benard instability problem that when the viscosity is taken into account, the growth rate depends on \( K \). There will be a lower cut-off wavenumber \( K_{c,r1} \) and an upper cut-off wavenumber \( K_{c,r2} \) so that the basic state is unstable only when \( K_{c,r1} < K < K_{c,r2} \) (see, e.g., Drazin and Reid(1981) [12]).

From the energy equation (13.37), we have shown that eddy viscosity is a stabilizing force. The inclusion of eddy viscosity will reduce the growth rate found in the inviscid case. The viscous stabilizing force (13.38e) will increase with the increasing \( K \). As a result, the growth rate of Langmuir circulation is expected to be negative for very large \( K \). For very small \( K \to 0 \), the growth rate found in the inviscid limit
Figure 14-2: $\sigma^2$ as a function of $K^2$ (with viscosity)

is (14.21) and the viscous stabilizing force (13.38e) reduces to

$$\sigma_c \rightarrow -\alpha \int_{z_+}^{z_-} \int_0^1 \left( \left| \frac{\partial \tilde{u}_0}{\partial z} \right|^2 + 2 \left| \frac{\partial \tilde{u}_0}{\partial y} \right|^2 + \left| \frac{\partial \tilde{u}_0}{\partial z} \right|^2 \right) dz < 0, \quad \text{as } K \rightarrow 0$$

Therefore, $\sigma^2$ will be negative for very small $K$ as well. The change of $\sigma^2$ with $K$ in the viscous case is illustrated in Figure (14-2)

### 14.2.3 Feedback mechanism of CL-II

In this section, we shall explain the feedback mechanism of CL-II under the assumption $M(z) > 0$ for some $z$ in the water column.

Suppose that initially there is a wave-like perturbation $u_0(y, z)$ as sketched in Figure (14-4). In the part $\pi/K > y > 0$, the perturbation in $u_0$ will generate a vertical vorticity $\zeta_0 = -\partial u_0/\partial y > 0$. The longitudinal vorticity equation (14.6) can be written as

$$\frac{\partial \zeta_0}{\partial t_2} = \zeta_0 \frac{\partial U_s}{\partial z} \quad (14.22)$$

which says that the perturbed vertical vorticity can be bent to the longitudinal di-
rection by the shear in Stokes drift, thus generates the longitudinal vorticity

$$\xi_0 = \frac{\partial w_0}{\partial y} - \frac{\partial v_0}{\partial z} > 0$$

in the part $\pi/K > y > 0$.

Taking the $y$- derivative of the longitudinal momentum equation, (14.2), we obtain the equation governing the vertical vorticity $\zeta_0$

$$\frac{\partial \zeta_0}{\partial t_2} = \frac{\partial w_0}{\partial y} \frac{\partial U_0}{\partial z}$$

(14.23)

Thus the transverse motion $w_0$ will generate a vertical vorticity by extracting energy from the basic current $U_0$.

Within the Langmuir cells, the sign of $\partial w_0/\partial y$ can be derived from the sign of the longitudinal vorticity $\xi_0$. The relation between $\zeta_0$ and $\xi_0$ is shown in Figure (14-3),

![Figure 14-3: Relation between $\xi_0$ and $w_0$ within the Langmuir cells.](image)

from which we have

$$\xi_0 > 0 \Rightarrow \partial w_0/\partial y > 0, \quad \text{for } K/\pi > y > 0, z < 0 \quad (14.24a)$$

$$\xi_0 < 0 \Rightarrow \partial w_0/\partial y < 0, \quad \text{for } -K/\pi < y < 0, z < 0 \quad (14.24b)$$

Hence, positive longitudinal vorticity $\xi_0$ will generate positive vertical vorticity $\zeta_0$.
for $\partial U_0/\partial z > 0$, according to (14.23). As a result, both the longitudinal vorticity and vertical vorticity can be enhanced in this manner. This is the "feedback mechanism" of CL-II theory (see, e.g., Leibovich (1983) [39], Nepf (1991) [58], etc.).

The feedback mechanism is illustrated in Figure (14-4). Because $w_0$ is zero at both the surface and the bottom, the feedback mechanism is important only in the core region.

The feedback mechanism is related to the energy production rate $\sigma_{u_1}$ and $\sigma_{u_2}$ in equation (13.37). Because $w_0$ is zero near the bottom and surface, thus we emphasis that the feedback mechanism mainly happens in the core region where $w_0$ is relatively large.

The longitudinal vorticity is the result of the vertical vorticity bent to the longitudinal direction by Stokes drift. This mechanism contributes directly to $\sigma_{St}$ in equation (13.37).
14.3 The surface stress mechanism

From the energy equation, we showed that the surface stress will also produce energy for Langmuir circulation. In this section, we illustrate this new mechanism in terms of the vorticity generation.

The generation of longitudinal vorticity is still governed by the equation (14.22), which can be written as

\[
\frac{\partial \xi_0}{\partial t} = \zeta_0 \frac{\partial U_s}{\partial z} \tag{14.25}
\]

Now we examine the generation of the vertical vorticity, \( \zeta_0 \). Consider the part \( \pi/K > y > 0 \). The initial perturbation of the vertical vorticity \( \zeta_0 = -\partial u_0/\partial y \), is bent by the Stokes drift to give a longitudinal vorticity \( \xi_0 \), as shown in Figure (14-5). For the initial perturbation shown in Figure (14-5), we have

\[
\zeta_0 = -\frac{\partial u_0}{\partial y} > 0 \Rightarrow \xi_0 > 0, \quad \text{for } \pi/K > y > 0 \tag{14.26a}
\]
Near the surface, \( w_0 \sim 0 \), hence \( \partial w_0 / \partial y \sim 0 \), and the feedback mechanism of CL-II in (14.23) is not effective. In this region, the resupply of the vertical vorticity, \( \zeta_0 = -\partial u_0 / \partial y \), is achieved by the surface stress. Recall the surface boundary condition for \( u_0 \), (12.67a),

\[
\alpha_s \frac{\partial u_0}{\partial z} = -AA^* \frac{\coth(kh)}{2} \frac{\partial v_0}{\partial y}, \quad z = 0
\]

which can be written as

\[
\alpha_s \frac{\partial u_0}{\partial z} = AA^* \frac{\coth(kh)}{2} \frac{\partial w_0}{\partial z}
\] (14.27)

after using the continuity equation for Langmuir circulation, (12.12). Taking \( y \)-derivative of (14.27), we obtain

\[
\alpha_s \frac{\partial}{\partial z} \left( \frac{\partial u_0}{\partial y} \right) = AA^* \frac{\coth(kh)}{2} \frac{\partial}{\partial z} \left( \frac{\partial w_0}{\partial y} \right), \quad z = 0
\] (14.28)

The coefficient on the right-hand side of (14.28) is related to the Stokes drift by

\[
AA^* \frac{\coth(kh)}{2} = \frac{1}{4} \left[ \frac{\partial U_s}{\partial z} \right]_0
\] (14.29)

Hence the surface condition (14.28) can be written as

\[
-\alpha_s \frac{\partial \zeta_0}{\partial z} = \alpha_s \frac{\partial}{\partial z} \left( \frac{\partial u_0}{\partial y} \right) = \frac{1}{4} \left[ \frac{\partial U_s}{\partial z} \right]_0 \frac{\partial}{\partial z} \left( \frac{\partial w_0}{\partial y} \right), \quad z = 0
\] (14.30)

Once the longitudinal vorticity is generated by the Stokes drift, \( \partial w_0 / \partial y \) generates new \( \zeta_0 = -\partial u_0 / \partial z \) from the surface. Thus, the energy source of \( \zeta_0 = -\partial u_0 / \partial y \) is still the wave motion. Next, we examine the sign of the vertical vorticity generated by the surface stress, according to (14.30).

The surface-stress-driven mechanism is illustrated in Figure (14-5). In the part \( K/\pi > y > 0 \) the longitudinal vorticity \( \xi_0 > 0 \), which, in view of Figure (14-3), leads to that \( w_0 \) increases from a negative value at \( y = 0 \) to a positive value at \( y = K/\pi \). The surface boundary condition of \( w_0 \) leads to \( \partial w_0 / \partial y = 0 \) on the surface \( z = 0 \), thus
from (14.24) we have

\[
\frac{\partial}{\partial z} \left( \frac{\partial w_0}{\partial y} \right) < 0, \quad \text{for } K/\pi > y > 0, z < 0
\]

(14.31a)

\[
\frac{\partial}{\partial z} \left( \frac{\partial w_0}{\partial y} \right) > 0, \quad \text{for } -K/\pi < y < 0, z < 0
\]

(14.31b)

The gradient of the Stokes drift is always positive and \(\alpha \tilde{S}_c > 0\), it then follows from

(14.30) that

\[
\frac{\partial}{\partial z} \left( \frac{\partial u_0}{\partial y} \right) < 0, \quad \text{for } K/\pi > y > 0, z = 0
\]

(14.32a)

\[
\frac{\partial}{\partial z} \left( \frac{\partial u_0}{\partial y} \right) > 0, \quad \text{for } -K/\pi < y < 0, z = 0
\]

(14.32b)

which means that

\[
\frac{\partial u_0}{\partial y} < \left[ \frac{\partial u_0}{\partial y} \right]_0, \quad \text{or } \frac{\partial \zeta_0}{\partial z} > 0, \quad \text{for } K/\pi > y > 0
\]

(14.33)

\[
\frac{\partial u_0}{\partial y} \leq \left[ \frac{\partial u_0}{\partial y} \right]_0, \quad \text{or } \frac{\partial \zeta_0}{\partial z} < 0, \quad \text{for } -K/\pi < y < 0
\]

(14.34)

or in terms of the vertical vorticity \(\zeta_0\),

\[
\zeta_0 \leq |\zeta_0|_0, \quad \text{for } K/\pi > y > 0
\]

(14.35)

\[
\zeta_0 \geq |\zeta_0|_0, \quad \text{for } -K/\pi < y < 0
\]

(14.36)

The vertical vorticity generated at the mean surface will diffuse downward according to the vorticity equation (12.42). Therefore, the magnitude of this vertical vorticity must have its maximum on the mean surface where it is generated,

\[
|\zeta_0|_0 \geq |\zeta_0|
\]

(14.37)

i.e.,

\[
|\zeta_0|_0 \geq \zeta_0, \quad \text{if } \zeta_0 > 0
\]

(14.38)

\[
|\zeta_0|_0 < \zeta_0, \quad \text{if } \zeta_0 < 0
\]

(14.39)
Thus, from (14.35), (14.36), and (14.38), we conclude that

\[
\zeta_0 = - \left[ \frac{\partial u_0}{\partial y} \right]_0 > 0, \quad \text{for } K/\pi > y > 0, z = 0
\]

\[
\zeta_0 = - \left[ \frac{\partial u_0}{\partial y} \right]_0 < 0, \quad \text{for } -K/\pi < y < 0, z = 0
\]

which are of the same signs as that by the initial perturbation of \( u_0 \) (cf. (14.26a)). Therefore, the surface stress, (12.67a), serves as a mechanism to re-generate new vertical vorticity through Stokes drift on the surface.

The surface stress mechanism corresponds directly to \( \sigma_{sw} \) in the energy equation (13.37)

\[
\sigma_{sw} = -\frac{K [i\tilde{v}_0 \tilde{u}_0^* + c.c.]_0 AA^* \coth(kh)}{2 \int_{z+}^0 (|\tilde{u}_0|^2 + |\tilde{v}_0|^2 + |	ilde{v}_0|^2) \, dz}
\]

(14.41)

The surface-stress mechanism we just illustrated implies that \( \sigma_{sw} \) should be positive. The positive \( \sigma_{sw} \) can also be proved in the following way.

The surface stress is given by (14.27). For the normal mode given in (13.30) and (13.31), we have

\[
i\alpha \tilde{S} \frac{\partial \tilde{u}_0}{\partial z} = KA^* \coth(kh) \frac{\coth(kh)}{2} \tilde{v}_0, \quad z = 0
\]

(14.42)

where \( \alpha, \tilde{S}, K \) and \( AA^* \) are real and positive. Because the eigenfunction is a homogeneous solution, it can be multiplied by a constant, we can choose to make \( \tilde{u}_0 \) real and positive near the surface, so that \( \tilde{v}_0 \) is pure imaginary on \( z = 0 \) according to (14.42). Near the surface, the longitudinal velocity \( \tilde{u}_0 \) is driven by this surface shear stress, thus for positive \( \tilde{u}_0 \), the vertical gradient of \( \tilde{u}_0 \) is always positive, which leads to \( \tilde{v}_0 \) imaginary and positive according to (14.42). As a result, \([-i\tilde{v}_0 \tilde{u}_0^* + c.c.]_0 \) is real and positive and thus \( \sigma_{sw} > 0 \) according to (14.41).

The regeneration of the vertical vorticity by the surface mechanism is controlled by equation (14.28), which implies the surface stress mechanism can become stronger for shallower water (small \( kh \)) due to the factor \( \coth(kh) \).

Note that in this mechanism, the basic current \( U_0 \) has no direct role. Thus, the surface-stress driven Langmuir circulation can occur in a wave-following current, a
wave-opposing current, or pure wave field without current.

### 14.4 Circulation generated by sidewalls

In all the laboratory experiments, the effect of the side-wall can not be ignored. The typical longitudinal velocity profile across a tank is illustrated in Figure(14-6). Note that with sidewalls, $u_0(y, z)$ is due not to the instability, but to the sidewall boundary layers. Even if there is no wave, $\partial u_0(y, z)/\partial y \neq 0$ near the wall. In the following, we shall show $u_0(y, z)$ due to the sidewalls will also generate a circulation when waves are present.

There are two effects that sidewalls have on the generation of circulation. One has to do with the mode due to feedback mechanism of CL-II and another does not, and we shall call this circulation the "sidewall-driven circulation". We begin with existence of the "sidewall-driven circulation" and then discuss its pattern for both the wave-following and wave-opposing currents.

![Figure 14-6: Top view of the typical variation of $u_0$ with $y$ due to the sidewalls: $U_0(z)$ is the $y$-averaged longitudinal velocity and $u_0(y, z)$ is the difference between the total longitudinal velocity and $U_0(z)$.](image-url)
Again, let us begin with the longitudinal vorticity equation (14.6),

\[ \frac{\partial \xi_0}{\partial t_2} = -\frac{\partial u_0}{\partial y} \frac{\partial U_s}{\partial z}, \tag{14.43} \]

where \( U_s \) is the dimensionless Stokes drift. Because \( \partial U_s/\partial z \) is always positive, the growing longitudinal vorticity \( \xi_0 \) always has a sign the same as the vertical vorticity \( \zeta_0 = -\partial u_0/\partial y \).

When waves are superimposed on the current, the vertical vorticity \( \zeta_0 = -\partial u_0/\partial y \) due to the sidewall boundary layers will be bent to the longitudinal direction by the Stokes drift. For small time \( t_2 \), \( \xi_0 \) is small, and it will not affect \( u_0 \), hence \( \zeta_0 \). Thus \( \xi_0 \) can be estimated by

\[ \xi_0 \sim \zeta_0 \frac{\partial U_s}{\partial z} t_2 \tag{14.44} \]

when \( t_2 \ll O(1) \). When \( t_2 = O(1) \), \( \xi_0 \) will become comparable to \( \zeta_0 = -\partial u_0/\partial y \) due to the sidewalls (i.e., the initial \( \partial u_0/\partial y \)), and the feedback mechanism of CL-II will also become important if the circulation can enhance the vertical vorticity.

Now we illustrate the sidewall-generated circulations for both the wave-following and wave-opposing currents. Let waves be in the positive \( x \)-direction, i.e., \( \partial U_s/\partial z \) is always positive. It then follows from (14.43) that

\[ \frac{\partial u_0}{\partial y} > 0 \Rightarrow \xi_0 < 0, \text{ for all } y \tag{14.45} \]
\[ \frac{\partial u_0}{\partial y} < 0 \Rightarrow \xi_0 > 0, \text{ for all } y \tag{14.46} \]

Because the \( \partial u_0/\partial y \) has opposite sign near the two sidewalls, the longitudinal vorticity \( \xi_0 \) generated by the sidewall boundary layer has opposite signs as well.

The longitudinal vorticity \( \xi_0 \) generated by the two sidewall boundary layers will diffuse toward the center of the tank, and eventually, these two vortices with opposite directions will meet at the center of the tank, and a pair of cells form. The spacing of the sidewall-driven circulation is the width of the tank \( W_{tank} \) and the aspect ratio of this type of the circulation is \( W_{tank}/(2h) \), where \( h \) is the water depth.

For a wave-following current, the longitudinal vorticity according to (14.43) is
illustrated in Figure (14-7). The transverse gradient of the longitudinal velocity \( \frac{\partial u_0}{\partial y} < 0 \) for \( y > 0 \); while \( \frac{\partial u_0}{\partial y} > 0 \) for \( y < 0 \). The gradient of the Stokes drift is always positive, thus the longitudinal vorticity, \( \xi_0 \), is positive for \( y > 0 \) and negative for \( y < 0 \), as shown in Figure (14-7). The sign of \( v_0 \) and \( w_0 \) can be derived from Figure (14-3).

![Figure 14-7: Sidewall-driven Langmuir circulation for a wave-following current](image)

Initially, the transverse gradient of the longitudinal velocity is not the result of the instability, and the feedback mechanism of CL-II is not effective in the entire depth of water. For a wave-following current, the floating materials on the surface will eventually collect in the center line of the tank, as shown in Figure (14-7). We point out that the surface signature just described can be altered if the feedback mechanism of CL-II also leads to instability (i.e., the vertical vorticity can by enhanced by the generated circulation).

For a wave-opposing current, the transverse gradient of the longitudinal velocity, Stoke drift, and the generated longitudinal vorticity are depicted in Figure (14-8). Because the direction of the generated longitudinal vorticity here has signs opposite
to that of a wave-following current, floating materials on the surface will collect themselves slowly along two lines next to wall if the feedback mechanism of CL-II does not lead to instability.

Figure 14-8: Sidewall-driven Langmuir circulation for a wave-opposing current

In summary, for both wave-following and wave-opposing currents, circulation can be generated by the sidewall boundary layers.

14.5 Summary

Circulation cells can be driven by the feedback mechanism of CL-II, the surface-stress mechanism and the sidewall boundary layers. The differences among them are:

1. In classical CL-II theory, the vertical vorticity is provided by the feedback mechanism. The instability depends on the sign of $M(z)$, defined by (14.10).

2. The surface-stress-driven circulation is independent of the $U_0$. It obtains its energy from waves through the surface mean stress.
3. In the sidewall-driven circulation, the vertical vorticity is provided mainly by the sidewall boundary layer.

4. Stokes drift is crucial to all three mechanisms. It is the Stokes drift which bends the vertical vorticity into the longitudinal direction. The total energy of the circulation include $u_0$, and $(v_0, w_0)$. Stoked drift only provide energy for $(v_0, w_0)$. Therefore, that Stokes drift is important does not necessarily mean that $\sigma_{St}$ will be equally important in all three mechanisms. Small $(v_0, w_0)$ normally go with small $\sigma_{St}$.

In the open sea, both the CL-II and surface stress mechanisms are present. In the laboratory experiments, sidewall, CL-II mechanism, and surface stress mechanism are all present, but the relative importance of these three mechanisms depends on the tank width and the problem under consideration. The distinction of these three mechanisms is important for the interpretation of the observed Langmuir circulation in the laboratory experiments.
Chapter 15

Numerical Scheme

In this chapter we present the numerical scheme used to solve the eigenvalue problem formulated in the chapter (12). To reduce the order of the differential operators, we shall work with equations (12.12), (12.24), (12.32) and (12.36). The numerical scheme is checked with an artificial boundary value problem which can be solved exactly.

15.1 Normal mode

For the normal modes given in (13.30) and (13.31), the continuity equation (12.12) and the momentum equations (12.24), (12.32) and (12.36) can be written as follows:

- continuity equation:
  \[
  iK\hat{\nu}_0 + \frac{\partial \hat{\nu}_0}{\partial z} = 0
  \]  (15.1)

- longitudinal momentum equation:
  \[
  \sigma\hat{\nu}_0 + \hat{\nu}_0 \frac{\partial U_0}{\partial z} = \alpha \left[-\delta e K^2 \hat{\nu}_0 + \frac{\partial}{\partial z} \left( \delta e \frac{\partial \hat{\nu}_0}{\partial z} \right) \right]
  \]  (15.2)

where

\[
\frac{\partial U_0}{\partial z} = \frac{\alpha_0}{\kappa \kappa h + z} + \frac{\partial U_0^{(2)}}{\partial z}
\]  (15.3)
\textbf{lateral momentum equation:}

\[ \sigma \dot{v}_0 + iK \hat{G}_0 = \alpha \left[ -2K^2 \hat{S}_c \dot{v}_0 + \frac{\partial}{\partial z} \left( \hat{S}_c \left( -iK \dot{v}_0 + \frac{\partial \hat{v}_0}{\partial z} \right) \right) \right] \]  \hspace{1cm} (15.4)

where \( \hat{G}_0 \) is the amplitude of \( \hat{G}_0 \).

\textbf{vertical momentum equation:}

\[ \sigma \dot{v}_0 + AA^* \dot{u}_0 \frac{\sinh(2(kh + z))}{\sinh^2(kh)} + \frac{\partial \hat{G}_0}{\partial z} = \alpha \left[ -iK \left( \hat{S}_c \left( -iK \dot{v}_0 + \frac{\partial \hat{v}_0}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left( 2\hat{S}_c \frac{\partial \hat{v}_0}{\partial z} \right) \right] \]  \hspace{1cm} (15.5)

\( \hat{G}_0 \) will be eliminated after the discretization. We work with (15.4) and (15.5), instead of the vorticity equation with \( \hat{G}_0 \) being eliminated, so that a small condition number of the differentiation matrix can be achieved (see e.g., Canuto, et al (1988) [6], Boyd(2002) [4]).

For the normal mode, the boundary conditions (13.7) and (13.8) become

- On the mean surface \( z = 0 \)
  \[ \frac{\partial \hat{v}_0}{\partial z} = 0 \]  \hspace{1cm} (15.6a)
  \[ \dot{v}_0 = 0 \]  \hspace{1cm} (15.6b)
  \[ \alpha \hat{S}_c \frac{\partial \hat{u}_0}{\partial z} = -iK \frac{\partial AA^*}{2} \coth(kh) \dot{v}_0 \]  \hspace{1cm} (15.6c)

- On the bottom, \( z = z_+ \equiv -kh + z_b \)
  \[ \hat{u}_0 = \dot{v}_0 = \dot{w}_0 = 0 \]  \hspace{1cm} (15.7)

\textbf{15.2 Rescaling}

Note that the coefficients in equations (15.1)-(15.5) and the stress condition (15.6) are complex. To make the coefficients real and the gradient of \( \dot{u}_0 \) at the surface about unity, we need to rescale \( \hat{u}_0 \) and \( \hat{v}_0 \) before numerically solving the eigenvalue problem.
For this purpose, we define the following new unknowns

$$
\bar{u}_0 = C_s \hat{u}_0, \quad \bar{v}_0 = iK \hat{v}_0, \quad \bar{w}_0 = \hat{w}_0, \quad \bar{G}_0 = \hat{G}_0
$$

(15.8)

where

$$
C_s = \alpha [\bar{S}_c]_0
$$

(15.9)

It then follows that the boundary conditions (15.6) and (15.7) can be written as

- On the surface, \( z = 0 \):

$$
\bar{w}_0 = 0, \quad \frac{\partial \bar{w}_0}{\partial z} = 0, \quad \frac{\partial \bar{u}_0}{\partial z} = -AA^* \frac{\coth(kh)}{2} \bar{v}_0
$$

(15.10)

- On the bottom, \( z = z_+ \):

$$
\bar{w}_0 = 0, \quad \bar{v}_0 = 0, \quad \bar{u}_0 = 0
$$

(15.11)

After rewriting equations (15.1)-(15.5) in terms of the new unknowns, we have the following equations:

- Continuity equation:

$$
\bar{v}_0 + \frac{\partial \bar{w}_0}{\partial z} = 0
$$

(15.12)

- Longitudinal momentum equation:

$$
\sigma \bar{u}_0 + C_s \left( \frac{\alpha_0}{\kappa} \frac{\bar{w}_0}{kh + z} + \bar{w}_0 \frac{\partial U_0^{(2)}}{\partial z} \right) = \alpha \left[ -\bar{S}_c K^2 \bar{u}_0 + \frac{\partial}{\partial z} \left( \bar{S}_c \frac{\partial \bar{u}_0}{\partial z} \right) \right]
$$

(15.13)

- Lateral momentum equation:

$$
\sigma \bar{v}_0 - K^2 \bar{G}_0 = \alpha \left[ -2K^2 \bar{S}_c \bar{v}_0 + \frac{\partial}{\partial z} \left( \bar{S}_c \left( -K^2 \bar{w}_0 + \frac{\partial \bar{v}_0}{\partial z} \right) \right) \right]
$$

$$
= \alpha \left[ -K^2 \bar{S}_c \bar{v}_0 + \frac{\partial}{\partial z} \left( \bar{S}_c \frac{\partial \bar{v}_0}{\partial z} \right) - K^2 \bar{w}_0 \frac{\partial \bar{S}_c}{\partial z} \right]
$$

(15.14)
where the continuity equation (15.12) has been used.

- Vertical momentum equation:

\[
\begin{align*}
\sigma C_s \ddot{w}_0 + \ddot{u}_0 \frac{\partial U_s}{\partial z} + C_s \frac{\partial \ddot{G}_0}{\partial z} &= \alpha C_s \left[ \left( \ddot{S}_c \left( -K^2 \ddot{w}_0 + \frac{\partial \ddot{v}_0}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left( 2 \ddot{S}_c \frac{\partial \ddot{v}_0}{\partial z} \right) \right] \\
&= \alpha C_s \left[ \left( \ddot{S}_c \left( -K^2 \ddot{w}_0 + \frac{\partial^2 \ddot{w}_0}{\partial z^2} \right) \right) + 2 \ddot{S}_c \frac{\partial \ddot{w}_0}{\partial z} \right] \\
&= \alpha C_s \left[ \left( -\ddot{S}_c K^2 \ddot{w}_0 + \frac{\partial \ddot{S}_c}{\partial z} \frac{\partial \ddot{w}_0}{\partial z} + \ddot{S}_c \frac{\partial^2 \ddot{w}_0}{\partial z^2} \right) - \ddot{v}_0 \frac{\partial \ddot{S}_c}{\partial z} \right] 
\end{align*}
\]

(15.15)

where the continuity equation (15.12) has been used to derive the last result.

After rescaling, all coefficients in the continuity and momentum equations, as well as the boundary conditions, are real. The gradient of \( \ddot{u}_0 \) near the surface is about unity and easy to deal with numerically.

15.3 Discretization Scheme

In the following, we first discretize the continuity and the momentum equations and then eliminate \( \ddot{G}_0 \) and \( \ddot{v}_0 \). The resulting numerical matrix equation is different from that obtained by first eliminating \( \ddot{G}_0 \) and then making the discretization (see, e.g., Canuto, et al (1988) [6]). For the later, we would have to work with a fourth order ODE, and the large numerical error and spurious mode may occur. The pseudo-spectral method (see, e.g., Gottlieb and Orszag (1977) [20], Fornberg (1996) [16], Fornberg (1998) [15], Boyd (2002) [4]) has been used to discretize the vertical derivative. The Matlab code described by Weideman and Reddy (2000) [74] has been adopted by this thesis to compute the vertical derivatives. The treatment of the boundary conditions can be found in Fornberg (1998) [15] and Boyd (2002) [4].

We first write the continuity and momentum equations in the following matrix-
vector form:

\[
\begin{bmatrix}
0, 0, 0, 0 \\
1, 0, 0, 0 \\
0, 1, 0, 0 \\
0, 0, C_s, 0
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_0 \\
\tilde{v}_0 \\
\tilde{w}_0 \\
\tilde{g}_0
\end{bmatrix}
= \begin{bmatrix}
0, 1, A_{13}, 0 \\
A_{21}, A_{22}, A_{23}, 0 \\
0, A_{32}, A_{33}, K^2 \\
A_{41}, A_{42}, A_{43}, B_G
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_0 \\
\tilde{v}_0 \\
\tilde{w}_0 \\
\tilde{g}_0
\end{bmatrix}
\]

(15.16)

where

\[
B_G = -C_s \frac{\partial}{\partial z}, \quad A_{13} = \frac{\partial}{\partial z}
\]

(15.17a)

\[
A_{21} = \alpha \left(-K^2 \tilde{s}_c + \tilde{s}_c \frac{\partial^2}{\partial z^2} + \frac{\partial \tilde{s}_c}{\partial z} \right), \quad A_{22} = 0, \quad A_{23} = -C_s \frac{\partial \tilde{u}_0}{\partial z}
\]

(15.17b)

\[
A_{32} = \alpha \left(-K^2 \tilde{s}_c + \tilde{s}_c \frac{\partial^2}{\partial z^2} + \frac{\partial \tilde{s}_c}{\partial z} \right), \quad A_{33} = -\alpha K^2 \frac{\partial \tilde{s}_c}{\partial z}
\]

(15.17c)

\[
A_{41} = -\frac{\partial U_s}{\partial z}, \quad A_{42} = -2\alpha C_s \frac{\partial \tilde{s}_c}{\partial z}, \quad A_{43} = \alpha C_s \left(-K^2 \tilde{s}_c + \tilde{s}_c \frac{\partial^2}{\partial z^2} \right)
\]

(15.17d)

Before the discretization, the domain \(z = [-kh + z_b, 0]\) needs to be mapped into \(\tilde{z} = [-1, 1]\), according to the following mapping

\[
z = -kh + z_b + \frac{kh - z_b}{2} (\tilde{z} + 1)
\]

(15.18)

The differential operators after mapping from \(z\) to \(\tilde{z}\) are given in the Appendix (C).

After mapping the differential operators, we apply the Chebyshev Gauss-Lobatto grid (see e.g., Canuto, et al. (1988) [6], Fornberg (1996) [16]) on the domain \(\tilde{z} = [-1, 1]\),

\[
\tilde{z}_k = -\cos \left(\frac{k - 1}{N - 1} \pi \right), \quad k = 1..N
\]

(15.19)

where \(N\) is the number of the grid points. The Chebyshev Gauss-Lobatto grid (15.19) has its nodes clustered near the two boundaries, which will increase the numerical resolution near the two boundaries.

On the grid (15.19), the \(m\)-th order derivative at \(\tilde{z}_k, k = 1..N\) can be written in
the following form

\[
\frac{\partial^m f}{\partial z^m} \bigg|_{z_k} = \sum_{j=1}^{N} d_{jk}^{(m)} f(z_j)
\]

(15.20)

where \(d_{jk}^{(m)}\) are the weights (elements/entries) of the differentiation matrix (see Fornberg (1996)[16]) for the given grid.

After discretizing differential equation (15.16) with boundary conditions (15.10) and (15.11), we get the differentiation matrices, \(A_{ij}\), and \(B_G\). Because there is no boundary conditions for \(\bar{G}_0\), the number of the grid points for \(B_G\) is two less than that for velocities which all have two boundary conditions. Because the surface boundary condition for \(\hat{u}_0\) is coupled with \(\hat{v}_0\), the differentiation matrix of \(A_{22}\) will become non-zero. In other words, differentiation matrix of \(A_{22}\) will be determined during discretizing the differential operator \(A_{21}\) (see section (15.4)).

For simplicity, from here on \(A_{ij}\) and \(B_G\) will be used to represent the differentiation matrix of the corresponding differential operators, and we are working with the numerical version of (15.16). First, we eliminate \(\bar{G}_0\), which can be written as the following numerical matrix-vector form:

\[
\bar{G}_0 = (\sigma \bar{v}_0 - A_{32} \bar{v}_0 - A_{33} \bar{u}_0) K^{-2}
\]

(15.21)

After eliminating \(\bar{G}_0\) from (15.16), we get

\[
\sigma \begin{bmatrix}
0, 0, 0 \\
I, 0, 0 \\
0, -B_G K^{-2}, C_s
\end{bmatrix}
\begin{bmatrix}
\bar{u}_0 \\
\bar{v}_0 \\
\bar{\omega}_0
\end{bmatrix}
= \begin{bmatrix}
0, I, A_{13} \\
A_{21}, A_{22}, A_{23} \\
A_{41}, A_{42} - B_G A_{32} K^{-2}, A_{43} - B_G A_{33} K^{-2}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_0 \\
\bar{v}_0 \\
\bar{\omega}_0
\end{bmatrix}
\]  

(15.22)

where \(A_{ij}\) and \(B_G\) are the differentiation matrices with order less than or equal to two, and \(I\) is the unit matrix.

Now we can further eliminate \(\bar{v}_0\). The continuity equation of Langmuir circulation
(the first row in (15.22)), gives

\[ \tilde{v}_0 = -A_{13} \tilde{w}_0 \]  \hspace{1cm} (15.23)

so that (15.22) can be written as

\[
\sigma \begin{bmatrix}
1, 0 \\
0, B_G A_{13} K^{-2} + C_s
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_0 \\
\tilde{w}_0
\end{bmatrix} = \begin{bmatrix}
A_{21}, A_{23} - A_{22} A_{13} \\
A_{41}, M
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_0 \\
\tilde{w}_0
\end{bmatrix}
\]  \hspace{1cm} (15.24)

where

\[ M = -(A_{42} - B_G A_{32} K^{-2}) A_{13} + A_{43} - B_G A_{33} K^{-2} \]  \hspace{1cm} (15.25)

The general eigenvalue problem (15.24) can be solved by standard routine provided by Matlab (see, e.g., Using MATLAB (2002) [49]).

**Condition numbers** The reciprocal of the condition number of matrix \( M \), denoted by \( \text{rcond}(M) \), determines how well the matrix \( M \) is conditioned and how accurate the computed eigenvalues will be. If the reciprocal of the condition number is close to one, then the matrix is well conditioned; If the reciprocal of the condition number is close to zero, then the matrix is badly conditioned and a large numerical error is expected. For the pseudo-spectral method on the Chebyshev grid, the reciprocal of the condition number of the differentiation matrix is of \( O(N^{-2p}) \), where \( N \) is the number of the grid points and \( p \) the order of the derivative (see, e.g., Fornberg (1996)[16]). For the second order derivative on \( N = 600 \), \( \text{rcond}(M) = O(10^{-12}) \). Because \( A_{ij} \), \( M \) in (15.24) all are combination of the differentiation matrices of second order differential operators, the condition number of these differential operators are of \( O(N^{-2p}) \). That is why we choose to make discretization before the elimination of unknown variables.
15.4 Coupled surface boundary condition

In this section, we discuss how to discretize $\frac{\partial^m u}{\partial \bar{z}^m}$ with the following coupled boundary condition

$$\frac{\partial u}{\partial \bar{z}} = C_{\beta} v, \quad \text{at } \bar{z} = 1 \tag{15.26}$$
$$u = 0, \quad \bar{z} = -1 \tag{15.27}$$

with the collocation method so that the differentiation matrix for the differential operator (see, e.g., Fornberg (1996) [16]) can be obtained.

The boundary conditions of $v$ are assumed to be

$$\frac{\partial v}{\partial \bar{z}} = 0, \quad \bar{z} = 1 \tag{15.28}$$
$$v = 0, \quad \bar{z} = -1 \tag{15.29}$$

The discretized surface boundary condition (15.26) at $\bar{z} = 1$ can be written as

$$\tilde{d}_{N,2} u_2 + \cdots + \tilde{d}_{N,N-1} u_{N-1} + \tilde{d}_{N,N} u_N = C_{\beta} v_N \tag{15.30}$$

where $\tilde{d}_{N,i}, i = 2..N$ are the pseudo-spectral weights of the first derivative of $u$ at $\bar{z} = 1$. It then follows that

$$u_N = \tilde{d}_{N,N}^{-1} C_{\beta} v_N - \tilde{d}_{N,N}^{-1} \left( \tilde{d}_{N,2} u_2 + \cdots + \tilde{d}_{N,N-1} u_{N-1} \right) \tag{15.31}$$

The $m$-th order derivative of $u$ at $\bar{z}_k, k = 1..N$ can be written as

$$\begin{bmatrix}
\frac{\partial^m u}{\partial \bar{z}^m} \\
\frac{\partial^m u}{\partial \bar{z}^m} \\ 
\vdots \\
\frac{\partial^m u}{\partial \bar{z}^m} \\
\frac{\partial^m u}{\partial \bar{z}^m}
\end{bmatrix}_{\bar{z}_1} =
\begin{bmatrix}
\tilde{d}_{1,2}^{(m)} \\
\tilde{d}_{2,2}^{(m)} \\
\vdots \\
\tilde{d}_{N-1,2}^{(m)} \\
\tilde{d}_{N,2}^{(m)}
\end{bmatrix} \begin{bmatrix}
\tilde{d}_{1,1}^{(m)} \\
\tilde{d}_{1,1}^{(m)} \\
\vdots \\
\tilde{d}_{N-1,1}^{(m)} \\
\tilde{d}_{N,1}^{(m)}
\end{bmatrix} 
\begin{bmatrix}
\tilde{d}_{1,N}^{(m)} \\
\tilde{d}_{2,N}^{(m)} \\
\vdots \\
\tilde{d}_{N-1,N}^{(m)} \\
\tilde{d}_{N,N}^{(m)}
\end{bmatrix} \begin{bmatrix}
u_2 \\
\vdots \\
u_{N-1}
\end{bmatrix} +
\begin{bmatrix}
\tilde{d}_{1,1}^{(m)} \\
\tilde{d}_{2,1}^{(m)} \\
\vdots \\
\tilde{d}_{N-1,1}^{(m)} \\
\tilde{d}_{N,N}^{(m)}
\end{bmatrix} \begin{bmatrix}
u_{N-1} \\
\vdots \\
u_N
\end{bmatrix} \tag{15.32}$$
After using (15.31), the $m$-th derivative of $u$, (15.32), becomes

\[
\begin{bmatrix}
\frac{\partial^m u}{\partial x^m} & \frac{\partial^m u}{\partial y^m} & \cdots & \frac{\partial^m u}{\partial z^m}
\end{bmatrix}
\begin{bmatrix}
\tilde{d}_{1,2}^{(m)} & \cdots & \tilde{d}_{1,N-1}^{(m)} \\
\tilde{d}_{2,2}^{(m)} & \cdots & \tilde{d}_{2,N-1}^{(m)} \\
\vdots & & \vdots \\
\tilde{d}_{N-1,2}^{(m)} & \cdots & \tilde{d}_{N-1,N-1}^{(m)}
\end{bmatrix}
\begin{bmatrix}
u_2 \\
\vdots \\
\nu_{N-1}
\end{bmatrix}
\]

Now we express $v_N$ in terms of the interior points of $v$. After discretization, the surface boundary condition for $v$, (15.28), gives

\[
\tilde{d}_{N,2}^{(m)} v_2 + \cdots + \tilde{d}_{N,N-1}^{(m)} v_{N-1} + \tilde{d}_{N,N}^{(m)} v_N = 0
\]  

where $\tilde{d}_{N,i}$, $i = 2..N$ are the pseudo-spectral weights of the first derivative of $v$ at $\tilde{z} = 1$. Alternatively, (15.34) can be written as

\[
v_N = -\tilde{\ell}_{N,N}^{-1} \left( \tilde{\ell}_{N,2} v_2 + \cdots + \tilde{\ell}_{N,N-1} v_{N-1} \right)
\]

\[
= -\tilde{\ell}_{N,N}^{-1} \left[ \tilde{\ell}_{N,2}, \cdots, \tilde{\ell}_{N,N-1} \right] \begin{bmatrix}
u_2 \\
\vdots \\
\nu_{N-1}
\end{bmatrix}
\]

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Finally, we have

\[
\begin{bmatrix}
\frac{\partial^m u}{\partial z^m} & \frac{\partial^n u}{\partial z^n} & \frac{\partial^m u}{\partial z^m} & \frac{\partial^n u}{\partial z^n} & \cdots & \frac{\partial^m u}{\partial z^m} \\
\frac{\partial^m u}{\partial z^m} & \frac{\partial^n u}{\partial z^n} & \frac{\partial^m u}{\partial z^m} & \frac{\partial^n u}{\partial z^n} & \cdots & \frac{\partial^m u}{\partial z^m} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^m u}{\partial z^m} & \frac{\partial^n u}{\partial z^n} & \frac{\partial^m u}{\partial z^m} & \frac{\partial^n u}{\partial z^n} & \cdots & \frac{\partial^m u}{\partial z^m}
\end{bmatrix}
\begin{bmatrix}
d_1^{(m)} \\
d_2^{(m)} \\
\vdots \\
d_{N-1}^{(m)} \\
d_N^{(m)}
\end{bmatrix}
= 
\begin{bmatrix}
d_1^{(m)} \\
d_2^{(m)} \\
\vdots \\
d_{N-1}^{(m)} \\
d_N^{(m)}
\end{bmatrix}
\begin{bmatrix}
\tilde{d}_{1,2} \\
\tilde{d}_{2,2} \\
\vdots \\
\tilde{d}_{N-1,2} \\
\tilde{d}_{N,2}
\end{bmatrix}
\begin{bmatrix}
u_2 \\
\vdots \\
u_{N-1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
d_1^{(m)} \\
d_2^{(m)} \\
\vdots \\
d_{N-1}^{(m)} \\
d_N^{(m)}
\end{bmatrix}
\begin{bmatrix}
\tilde{d}_{N,2} \\
\tilde{d}_{N,N} \\
\vdots \\
\tilde{d}_{N,N-1}
\end{bmatrix}
= 
\begin{bmatrix}
d_1^{(m)} \\
d_2^{(m)} \\
\vdots \\
d_{N-1}^{(m)} \\
d_N^{(m)}
\end{bmatrix}
\begin{bmatrix}
\tilde{d}_{N,2} \\
\tilde{d}_{N,N} \\
\vdots \\
\tilde{d}_{N,N-1}
\end{bmatrix}
\begin{bmatrix}
u_2 \\
\vdots \\
u_{N-1}
\end{bmatrix}
\]

\[
-C_0
\begin{bmatrix}
\tilde{d}_{1,2} \\
\tilde{d}_{2,2} \\
\vdots \\
\tilde{d}_{N-1,2} \\
\tilde{d}_{N,2}
\end{bmatrix}
\begin{bmatrix}
\tilde{d}_{N,2} \\
\tilde{d}_{N,N} \\
\vdots \\
\tilde{d}_{N,N-1}
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{d}_{1,2} \\
\tilde{d}_{2,2} \\
\vdots \\
\tilde{d}_{N-1,2} \\
\tilde{d}_{N,2}
\end{bmatrix}
\begin{bmatrix}
\tilde{d}_{N,2} \\
\tilde{d}_{N,N} \\
\vdots \\
\tilde{d}_{N,N-1}
\end{bmatrix}
\begin{bmatrix}
u_2 \\
\vdots \\
u_{N-1}
\end{bmatrix}
\]

Basically, when \( u \) is discretized with the boundary condition coupled with \( v \), it will give rise to a companion differentiation matrix for \( v \).

The correctness of the treatment of the coupled boundary condition presented in this section will be verified by an example in the next section.

### 15.5 A test example

To test the differentiation matrix obtained under the given boundary conditions, we create the following artificial problem by setting

\[
S = 1, \quad \frac{\partial U_0}{\partial z} = 1, \quad \frac{\partial U_2}{\partial z} = 1, \quad \alpha = 1
\]  

(15.37)
in (15.16) and add an non-homogeneous term to (15.16), so that we have the following boundary value problem.

\[
\ddot{v}_0 + \frac{\partial \ddot{w}_0}{\partial \bar{z}} = 0 \\
\sigma \ddot{u}_0 + C_s \ddot{w}_0 = \frac{\partial^2 \ddot{u}_0}{\partial \bar{z}^2} - K^2 \ddot{u}_0 + 1 \\
\sigma \ddot{v}_0 - K^2 \ddot{G}_0 = \frac{\partial^2 \ddot{v}_0}{\partial \bar{z}^2} - K^2 \ddot{v}_0 \\
\sigma C_s \ddot{w}_0 + \ddot{u}_0 + C_s \frac{\partial \ddot{G}_0}{\partial \bar{z}} = C_s \left( \frac{\partial^2 \ddot{w}_0}{\partial \bar{z}^2} - K^2 \ddot{w}_0 \right)
\]

Boundary conditions are (15.10) and (15.11). The set of equations (15.38) can also be written as

\[
\sigma \begin{bmatrix}
0, 0, 0, 0 \\
1, 0, 0, 0 \\
0, 1, 0, 0 \\
0, 0, C_s, 0
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_0 \\
\ddot{v}_0 \\
\ddot{w}_0 \\
\ddot{G}_0
\end{bmatrix}
= \begin{bmatrix}
0, 1, A_{13}, 0 \\
A_{21}, A_{22}, A_{23}, 0 \\
0, A_{32}, A_{33}, K^2 \\
A_{41}, A_{42}, A_{43}, B_G
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_0 \\
\ddot{v}_0 \\
\ddot{w}_0 \\
\ddot{G}_0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\]

where

\[
B_G = -C_s \frac{\partial}{\partial \bar{z}}, \quad A_{13} = \frac{\partial}{\partial \bar{z}} \\
A_{21} = -K^2 + \frac{\partial^2}{\partial \bar{z}^2}, A_{22} = 0, \quad A_{23} = -C_s \\
A_{32} = -K^2 + \frac{\partial^2}{\partial \bar{z}^2}, A_{33} = 0 \\
A_{41} = -1, A_{42} = -2C_s, \quad A_{43} = C_s \left( -K^2 + \frac{\partial^2}{\partial \bar{z}^2} \right)
\]

After discretization of equation (15.39), we can eliminate \( \ddot{G}_0 \). From the third row in (15.39), we have

\[
\ddot{G}_0 = K^{-2} (\sigma \ddot{w}_0 - A_{32} \ddot{v}_0 - A_{33} \ddot{u}_0)
\]

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After eliminating $\bar{G}_0$ from fourth row in (15.39), we have

$$
\sigma \begin{bmatrix}
0,0,0 \\
1,0,0 \\
0, -B_G K^{-2}, C_s
\end{bmatrix} \begin{bmatrix}
\bar{u}_0 \\
\bar{v}_0 \\
\bar{w}_0
\end{bmatrix}
= \begin{bmatrix}
0,1,A_{13} \\
A_{21}, A_{22}, A_{23} \\
A_{41}, A_{42} - B_G A_{32} K^{-2}, A_{43} - B_G A_{33} K^{-2}
\end{bmatrix} \begin{bmatrix}
\bar{u}_0 \\
\bar{v}_0 \\
\bar{w}_0
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
$$

Now we can further eliminate $\bar{v}_0$. The continuity equation of Langmuir circulation gives

$$
\bar{v}_0 = -A_{13} \bar{w}_0
$$

so that (15.22) can be written as

$$
\sigma \begin{bmatrix}
I,0 \\
0,B_G A_{13} K^{-2} + C_s
\end{bmatrix} \begin{bmatrix}
\bar{u}_0 \\
\bar{w}_0
\end{bmatrix} = \begin{bmatrix}
A_{21}, A_{23} - A_{22} A_{13} \\
A_{41}, M
\end{bmatrix} \begin{bmatrix}
\bar{u}_0 \\
\bar{w}_0
\end{bmatrix} + \begin{bmatrix}
1 \\
0
\end{bmatrix}
$$

where

$$
M = -(A_{42} - B_G A_{32} K^{-2}) A_{13} + A_{43} - B_G A_{33} K^{-2}
$$

Equation (15.43) can be solved numerically by standard Matlab routine,

The difference between (15.43) and (15.24) is that (15.43) is an inhomogeneous problem.

We take $\sigma = 1$, $K = 2, C_s = 10^{-4}$ and $kh = 1$ for example. The exact solutions of (15.38) can be obtained analytically in principle. But solving a sixth order ordinary differential equation analytically still need some effort. Fortunately, Maple can provide us an analytical solution. The following solution is obtained by Maple 8 ( see,
e.g., Gander and Hrebicek (1997) [17]).

\[
\begin{align*}
\tilde{u}_0 &= 0.2083333334 - 0.0338294424 e^{1.732050808 z} - 0.0150789444 e^{-1.732050808 z} \\
&+ (0.0083455423 e^{-2.361869413 z} - 0.0198543063 e^{2.361869413 z}) \sin(0.280048433 z) \\
&- (0.0087891096 e^{-2.361869413 z} + 0.0390129974 e^{2.361869413 z}) \cos(0.280048433 z) \\
\tilde{v}_0 &= -1171.886271 e^{1.732050808 z} + 522.3499575 e^{-1.732050808 z} \\
&- (219.2813176 e^{-2.361869413 z} + 1112.660666 e^{2.361869413 z}) \sin(0.280048433 z) \\
&- (343.6715719 e^{-2.361869413 z} - 964.3274710 e^{2.361869413 z}) \cos(0.280048433 z) \\
\tilde{w}_0 &= -416.6666667 + 676.5888537 e^{1.732050808 z} + 301.5788885 e^{-1.732050808 z} \\
&- (74.54127912 e^{-2.361869413 z} - 416.8219106 e^{2.361869413 z}) \sin(0.280048433 z) \\
&- (154.3466960 e^{-2.361869413 z} + 457.7127709 e^{2.361869413 z}) \cos(0.280048433 z)
\end{align*}
\]

which will be regarded as the exact solution. The Maple solutions and computed solutions of \( \tilde{u}_0, \tilde{v}_0, \tilde{w}_0 \) are plotted in Figure (15-1). The absolute error is, by definition, the absolute value of the difference between the numerical solution and the exact solution. This examples shows that the differentiation matrices of the operators (except those associated with \( \partial \tilde{S}_c/\partial z \)) and the treatment of the boundary conditions are correct.

## 15.6 Solution procedure

### 15.6.1 Typical parameters

For linear waves, the dispersion relationship is

\[
\omega^2 = \frac{g}{h} k h \tanh(kh)
\]

(15.45)

where \( g = 9.8 m/s^2 \) is the gravitational acceleration. For \( kh = 1 \), we get wave angular frequency \( \omega = 3.9 s^{-1} \) for \( h = 0.5 m \) and \( \omega = 1.2 s^{-1} \) for \( h = 5 m \).

Take the following typical wave amplitude: \( a = 0.05 m \) for \( h = 0.5 m \) and \( a = 0.3 m \) for \( h = 5 m \), we have the following typical wave orbital velocity: \( \omega a = 0.19 m/s \) for \( h = 0.5 m \) and \( \omega a = 0.37 m/s \) for \( h = 5 m \).
Figure 15-1: Comparison between the Maple and computed solutions for the test problem: solid lines are the Maple solutions and the dotes are the numerical results. N = 50
The wave phase speed $C = \omega/k$ is

$$C = \sqrt{gh \frac{\tanh(kh)}{kh}}$$  \hspace{1cm} (15.46)

For $kh = 1$, we get $C = 1.9 m/s$ for $h = 0.5 m$ and $C = 6.1 m/s$ for $h = 5 m$.

The typical wavenumber, $k = \omega/C$, is $k = O(2 m^{-1})$ for $h = 0.5 m$ and $k = O(0.2 m^{-1})$ for $h = 5 m$. It then follows that the typical wave slopes are $\epsilon = O(0.1)$ for $h = 0.5 m$ and $\epsilon = O(0.06)$ for $h = 5 m$.

For both the field ($h = O(5 m)$) and the laboratory ($h = O(0.5 m)$) conditions, we take the typical depth-averaged current velocity $\langle \bar{u} \rangle = O(0.3 m/s)$, which is comparable to the wave orbital velocity, the friction velocity is

$$u_f = \sqrt{\frac{f_c}{2}} \langle \bar{u} \rangle = O\left(\sqrt{\frac{0.01}{2}} \times 0.3\right) = O(0.02) m/s$$  \hspace{1cm} (15.47)

where the typical friction factor $f_c \sim 0.01$ was used. Thus, we have $u_f/C = O(0.003)$ for $h = 5 m$ and $u_f/C = 0.01$ for $h = 0.5 m$.

### 15.6.2 Bottom roughness

The dimensional bottom roughness $z_B$ is an empirical parameter. If there are measured mean velocity profile available, $z_B$ should be computed by fitting the measured velocity with the logarithmic profile. If there is no such data available, $z_B$ can be determined by some empirical formula (see, e.g., Tennekes and Lumley(1972) [70], Madsen (2002) [46], etc.). Madsen (2002) [46] suggested the following formula to determine the dimensional bottom roughness for smooth bottom

$$z_B = \frac{\nu}{N u_f}, \hspace{1cm} N = 9$$  \hspace{1cm} (15.48)

where $u_f$ is the friction velocity and $\nu$ the laminar kinetic viscosity and $N = 9$ is an empirical value. It then follows that the dimensionless bottom roughness $z_b$ is

$$z_b = k z_B = \frac{k \nu}{N u_f}$$  \hspace{1cm} (15.49)

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Thus, in terms of the dimensionless parameters $kh, u_f/C$, (15.49) can be written as

$$ z_b = \frac{k\nu/C}{N u_f/C} = \frac{1}{N} \left( \frac{(kh)^{3/2}}{\sqrt{\tanh(kh)}} \right) \left( \frac{1}{u_f/C} \right) \left( \frac{\nu}{\sqrt{gh^3}} \right) $$  \hspace{1cm} (15.50)

### 15.6.3 Maximum wave slope

According to Miche (1951) [52] (see also, e.g., Mei (1989) [50]), at the threshold of breaking the fluid velocity at the wave crest is equal to the phase speed, and the maximum wave slope $\epsilon_{\text{max}}$ is determined by

$$ \epsilon_{\text{max}} = 0.147 \tanh(kh) $$  \hspace{1cm} (15.51)

Miche’s criterion gives $\epsilon_{\text{max}} = 0.33$ for $kh = 1$, and $\epsilon_{\text{max}} = 0.27$ for $kh = 0.7$. In practice, waves may break even when $\epsilon < \epsilon_{\text{max}}$. When $\epsilon$ is close to $\epsilon_{\text{max}}$, nonlinearity will become important. Thus, we take the upper limit of $\epsilon$ as 0.2. It is expected that result obtained for $\epsilon$ close to the upper limit $\epsilon = 0.2$ will likely be affected by nonlinearity, even though waves are not broken.

### 15.6.4 Solution procedure

The eigenvalue problem (15.24) is solved in the following way:

1. For given $kh, u_f/C, \epsilon, z_b$, solve for $\bar{U}_0$ inside and outside the BWBL according to the 2D theory in Part I.

2. With $\bar{U}_0$ known, solve for the eigenvalue $\sigma(K)$ for the given transverse wavenumber $K$ and the given parameters $(\epsilon, kh, z_b, u_f/C)$.

The ranges of the dimensionless parameters in the examples studied in the next two chapters are $kh = 0.7 \sim 1.3$, $\epsilon = 0.03 \sim 0.2$, and $u_f/C = 0.003 \sim 0.01$.

We shall take $z_b = 2 \times 10^{-6}$ in the following examples. For the given parameter ranges of $u_f/C, \epsilon$ and $kh$, we can compute the corresponding water depth $h$ from (15.50).

- For field condition, say, $u_f/C \sim 0.003, kh \sim 1.3$, we have from (15.50) that $h \sim 4.5m$. 

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• For laboratory condition, say, \(kh = 0.7, uf/C \sim 0.01\), we have from (15.50) that \(h \sim 1.2m\).

We point out that the relation between the water depth \(h\) and the dimensionless bottom roughness \(z_b\), (15.50), depends on the empirical value of \(N\) and the friction factor \(f_c\). Thus we regard that \(z_b = 2 \times 10^{-6}\) covers the water depth \(h = O(1m)\) to \(h = O(5m)\).

### 15.7 Numerical accuracy

In this section, we determine the minimum number of grid points needed to obtain accurate eigenvalues. We take

\[
kh = 1, uf/C = 0.01, \epsilon = 0.15, z_b = 2 \times 10^{-6}, K = 3 \quad (15.52)
\]
as an example. We examine \(N = 100, 200, 300, 400, 500,\) and \(600\).

The first four eigenvalues (EVs) obtained by directly solving the eigenvalue problem are listed in Table (15.1) for different \(N\).

<table>
<thead>
<tr>
<th>Index of EVs</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N=100)</td>
<td>3.3238</td>
<td>0.3732</td>
<td>-1.0286 - 0.2857i</td>
<td>-1.0286 + 0.2857i</td>
</tr>
<tr>
<td>(N=200)</td>
<td>2.4420</td>
<td>0.3613</td>
<td>-1.0289 - 0.2830i</td>
<td>-1.0289 + 0.2830i</td>
</tr>
<tr>
<td>(N=300)</td>
<td>2.2883</td>
<td>0.3578</td>
<td>-1.0291 - 0.2823i</td>
<td>-1.0291 + 0.2823i</td>
</tr>
<tr>
<td>(N=400)</td>
<td>2.2520</td>
<td>0.3568</td>
<td>-1.0291 - 0.2821i</td>
<td>-1.0291 + 0.2821i</td>
</tr>
<tr>
<td>(N=500)</td>
<td>2.2427</td>
<td>0.3566</td>
<td>-1.0291 - 0.2821i</td>
<td>-1.0291 + 0.2821i</td>
</tr>
<tr>
<td>(N=600)</td>
<td>2.2403</td>
<td>0.3565</td>
<td>-1.0291 - 0.2821i</td>
<td>-1.0291 + 0.2821i</td>
</tr>
</tbody>
</table>

Table 15.1: Sensitivity of eigenvalues to the number of grid points

The first 10 eigenvalues obtained by \(N = 100, 200, 300, 400, 500,\) and \(N = 600\) are plotted in Figure (15-2). For \(N = 400\), the first 10 eigenvalues were computed with acceptable accuracy.

Now we study the numerical accuracy in computing the energy production of Langmuir circulation. \(\sigma_{st}, \sigma_{u_1}, \sigma_{u_2}, \sigma_{sw}, \sigma_{\epsilon}\) and \(\sigma_r\), defined in (13.37), are listed in Table (15.2) and (15.3) for the first and second eigen-mode, respectively. We shall
use $\sigma_r(EVP)$ represent the eigenvalues in Table (15.1), which is computed by directly solving the eigenvalue problem.

<table>
<thead>
<tr>
<th>N</th>
<th>$\sigma_{St}$</th>
<th>$\sigma_{u_1}$</th>
<th>$\sigma_{u_2}$</th>
<th>$\sigma_{SW}$</th>
<th>$\sigma_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.0049</td>
<td>0.0027</td>
<td>-0.0040</td>
<td>7.3796</td>
<td>-14.2188</td>
</tr>
<tr>
<td>200</td>
<td>0.0097</td>
<td>0.0060</td>
<td>-0.0067</td>
<td>11.4132</td>
<td>-11.7791</td>
</tr>
<tr>
<td>300</td>
<td>0.0112</td>
<td>0.0071</td>
<td>-0.0074</td>
<td>12.8049</td>
<td>-11.4067</td>
</tr>
<tr>
<td>400</td>
<td>0.0115</td>
<td>0.0074</td>
<td>-0.0076</td>
<td>13.3390</td>
<td>-11.3981</td>
</tr>
<tr>
<td>500</td>
<td>0.0116</td>
<td>0.0075</td>
<td>-0.0076</td>
<td>13.5789</td>
<td>-11.4593</td>
</tr>
<tr>
<td>600</td>
<td>0.0116</td>
<td>0.0075</td>
<td>-0.0076</td>
<td>13.7050</td>
<td>-11.5220</td>
</tr>
</tbody>
</table>

Table 15.2: Numerical accuracy in the computation of the energy production for the first eigen-mode

<table>
<thead>
<tr>
<th>N</th>
<th>$\sigma_{St}$</th>
<th>$\sigma_{u_1}$</th>
<th>$\sigma_{u_2}$</th>
<th>$\sigma_{SW}$</th>
<th>$\sigma_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.0650</td>
<td>0.5553</td>
<td>-0.0634</td>
<td>0.5050</td>
<td>-1.3337</td>
</tr>
<tr>
<td>200</td>
<td>0.0624</td>
<td>0.5386</td>
<td>-0.0620</td>
<td>1.0248</td>
<td>-1.4324</td>
</tr>
<tr>
<td>300</td>
<td>0.0617</td>
<td>0.5340</td>
<td>-0.0617</td>
<td>1.2143</td>
<td>-1.4650</td>
</tr>
<tr>
<td>400</td>
<td>0.0615</td>
<td>0.5328</td>
<td>-0.0616</td>
<td>1.2801</td>
<td>-1.4808</td>
</tr>
<tr>
<td>500</td>
<td>0.0614</td>
<td>0.5324</td>
<td>-0.0615</td>
<td>1.3061</td>
<td>-1.4906</td>
</tr>
<tr>
<td>600</td>
<td>0.0614</td>
<td>0.5322</td>
<td>-0.0615</td>
<td>1.3183</td>
<td>-1.4973</td>
</tr>
</tbody>
</table>

Table 15.3: Numerical accuracy in the computation of the energy production for the second eigen-mode
With a grid of $N = 600$, the error between $\sigma_r$ and $\sigma_r(EVP)$ is of $O(10^{-2})$ for the first eigen-mode and $O(10^{-3})$ for the second eigen-mode. This is due to the relative numerical error that may occur in computing $\sigma_{sw}$. Recall that $\sigma_{sw}$ in (13.37) is computed by the surface values of $u_0$ and $v_0$ and the value of $[v_0]_0$ is computed by the local derivative of $w_0$. The differentiation matrix has the largest weight near the surface where the grid has the small size (see, e.g., Fornberg (1996)[16]). As a result, even though $u_0$ and $w_0$ are computed with higher accuracy at the grid point, the computed value of $v_0$ will not be as accurate as $u_0$ and $w_0$. For this reason, the eigenvalue $\sigma_r(EVP)$ is more accurate than $\sigma_r$. For the purpose of investigating the relative importance of the energy productions, an error of $O(10^{-2})$ is acceptable. In theory, small error in $\sigma_r$, when computed by the energy equation, can be achieved by using large $N$. But a large $N$ will result in a large condition number for the differentiation matrix, and the numerical error due the machine precision will become the dominant one.

If a relative error of $O(0.001)$ in the real part of the eigenvalue, $\sigma_r$, is acceptable, then $N = 400$ is the minimum number of grid points to construct the curves of neutral stability and fast growing mode. To illustrate the contributions to the growth rate, the number of grid points should take $N = 600$ at least.
Chapter 16

Langmuir circulation in a wave-following current

In this chapter, we discuss by examples the unstable modes for a wave-following current. For the parameters chosen to be typical of laboratory and field scales, we shall find that there are at most two unstable modes for a wave-following current. The first unstable mode is driven mainly by the work done by the surface stress and the second mode is the combined result of the surface stress and the CL-II mechanisms. The shapes and the cell patterns of the first and the second unstable modes will be discussed as well.

16.1 Unstable eigen-modes

In this section, we show that instability is possible for a wave-following current. We take $kh = 1.0$, $u_f/C = 0.003$, $\epsilon = 0.06$, $z_0 = 2 \times 10^{-6}$ for example. The velocity profile of the basic current is plotted in Figure (16-1), and the shear rates of the unperturbed and perturbed current are plotted in Figure (16-2).

Note that the shear rate of the perturbed current is negative with its maximums near the bottom and the surface. The slight reduction of the mean current near the surface by waves can be noticed. This reduction of the mean velocity will become more significant for a large wave slope. To show this, we take $u_f/C = 0.003$, $kh = 0.7$, $\epsilon = 0.12$, and $z_0 = 2 \times 10^{-6}$, and the velocity profile of the basic current is shown
Figure 16-1: The unperturbed current (dashed-line) and the total basic current (solid line) for wave-following current for $kh = 1, \frac{u_f}{C} = 0.003, \epsilon = 0.06, z_b = 2 \times 10^{-6}$

Figure 16-2: Shear rate of the unperturbed current (dashed-line) and the perturbed current (solid line) for wave-following current for $kh = 1, \frac{u_f}{C} = 0.003, \epsilon = 0.06, z_b = 2 \times 10^{-6}$

in Figure (16-3), and the shear rates of the unperturbed and perturbed current in Figure (16-4).

Note in Figure (16-4) that for a large wave slope the shear rate of the perturbed current is more negative in the entire depth of water, indicating the strong effect of surface waves on the mean current as shown in Figure (16-3).

Now we study the stability/instability of the basic current which includes both the unperturbed current and the perturbed current. We first take $K = 3.5$ and examine the growth rate of this prescribed spanwise perturbation.
Figure 16-3: The unperturbed current (dashed-line) and the total basic current (solid line) for wave-following current for \( kh = 0.7, u_f/C = 0.003, \epsilon = 0.12, z_b = 2 \times 10^{-6} \)

Figure 16-4: Shear rate of the unperturbed current (dashed-line) and the perturbed current (solid line) for wave-following current for \( kh = 0.7, u_f/C = 0.003, \epsilon = 0.12, z_b = 2 \times 10^{-6} \)

### 16.1.1 Eigenvalue spectrum

The real and imaginary parts of the first 10 eigenvalues obtained by directly solving the eigenvalue problem (EVP), (15.24), for spanwise perturbation with \( K = 3.5 \), are plotted in Figure (16-5). The first eigen-mode has a growth rate \( \sigma_r = 2.1679 \) and the second eigen-mode \( \sigma_r = 0.0774 \), indicating the instability. Thus under the conditions studied here, the basic state is unstable to both the first and the second eigen-modes.

In Figure (16-5), the eigenvalue spectrum is symmetric about \( \sigma_i = 0 \). This is because all the coefficients of the eigenvalue problem are real, which guarantees that if \( \sigma \) is one eigenvalue, then its complex conjugate \( \sigma^* \) is another eigenvalue.
Figure 16-5: The first 10 eigenvalues for wave-following current: $kh = 1, u_f/C = 0.003, \epsilon = 0.06, z_b = 2 \times 10^{-6}$ and $K = 3.5$. Numbers next to the symbols indicate the index of the eigen-modes.

**Principle of exchange of stability**

The principle of exchange of stability holds, if every perturbation which is not damped out does not oscillate with respect to time (see, e.g., Georgescu(1985) [18]). Mathematically, the principle of exchange of stability means

$$\text{if } \sigma_r(K; p_1, p_2, ...) \geq 0, \text{ then } \sigma_i(K; p_1, p_2, ...) = 0$$

(16.1)

where $p_r$ are the parameters determining the basic flow.

If the principle of exchange of stability holds, then the marginal stability is characterized by

$$\sigma_r = \sigma_i = 0$$

(16.2)

There are several analytical ways to prove the principle of exchange of stability for certain problems:

1. Definite integral method. This method has been extensively used to problems of
constant coefficients (see, e.g., Chandrasekhar (1961) [7], or Georgescu (1985) [18]).

2. Variational principle. (Chandrasekhar (1961) [7]).

3. Definite operator method. This method is more robust and can deal with certain problems of variable coefficients (Herron (1996, 2000) [26, 27]).


For the present problem, we shall show the principle of exchange of stability numerically. We can see from Figure (16-5) that \( \sigma_1 \) and \( \sigma_2 \) have no imaginary part and all other \( \sigma_i \)'s have imaginary parts, indicating that the principle of exchange of stability holds. In all the examples we have computed in this thesis, the numerical results show that the principle of exchange of stability holds. Also, in all the examples presented here, there are at most two eigen-modes which can grow with time.

16.1.2 Energy budget

Now we examine the energy budget discussed in section (13.2) so as to give a better understanding on the generation of Langmuir circulation.

The computed values of \( \sigma_{St}, \sigma_{u_1}, \sigma_{u_2}, \sigma_{sw}, \) and \( \sigma_\varepsilon \), defined in equation (13.37), are given in Table (16.1) for the first and the second eigen-modes. The difference between

<table>
<thead>
<tr>
<th></th>
<th>( \sigma_{St} )</th>
<th>( \sigma_{u_1} )</th>
<th>( \sigma_{u_2} )</th>
<th>( \sigma_{sw} )</th>
<th>( \sigma_\varepsilon )</th>
<th>( \sigma_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>0.0334</td>
<td>0.0556</td>
<td>-0.0200</td>
<td>14.1311</td>
<td>-12.3334</td>
<td>1.8668</td>
</tr>
<tr>
<td>2nd</td>
<td>0.0409</td>
<td>0.7658</td>
<td>-0.0413</td>
<td>1.5839</td>
<td>-2.2996</td>
<td>0.0497</td>
</tr>
</tbody>
</table>

Table 16.1: Initial energy budget for wave-opposing current with \( kh = 1.0, u_f/C = 0.003, \varepsilon = 0.06, K = 3.5 \)

values of \( \sigma_r \) in Table (16.1) and those obtained by directly solving the EVP is due to the numerical error. Note that \( \sigma_{u_2} \) is negative, indicating that it is a stabilizing force for both the first and the second eigen-modes. \( \sigma_{u_1} \) is positive, so that it is a destabilizing force for a wave-following current.

For the first eigen-mode, the contributions to the growth rate are dominated by the work done by the surface stress \( \sigma_{sw} \) and the viscous dissipation \( \sigma_\varepsilon \). Other contributions are smaller than these two terms by an order of magnitude. Thus, under
the conditions studied here, the first eigen-mode is driven mainly by the surface stress mechanism discussed in section (14.3). Under certain conditions, the CL-II mechanism can be important for the first eigen-mode, which we shall see later in this chapter.

For the second eigen-mode, the energy production due to the Reynolds stress of Langmuir circulation against the basic shear, $\sigma_{uw}$, is comparable to $\sigma_{w}$ and $\sigma_{\epsilon}$. Other contributions are much smaller. Thus the second eigen-mode is driven by both the surface stress mechanism and the CL-II mechanism discussed in section (14.2.3).

The small value of $\sigma_{Sl}$ for both the first and the second eigen-modes indicates that Langmuir circulation does not gain much energy in bending the vertical vorticity into the longitudinal direction. For the second eigen-mode, $\sigma_{u1}$ becomes comparable to $\sigma_{sw}$, indicating that energy obtained by Langmuir circulation due to bending the lateral vorticity into vertical direction is comparable to the work done by the surface stress.

### 16.1.3 Eigenfunctions and Langmuir cells for the first eigen-mode

Eigenfunctions $\hat{u}_0$, $\hat{v}_0$ and $\hat{w}_0$ of the first eigen-mode are plotted in Figure (16-6). It

![Figure 16-6](image-url)

Figure 16-6: The modal shape for the first eigen-mode: solid lines—real part, dashed-lines—imaginary part. $kh = 1$, $u_f/C = 0.003$, $\epsilon = 0.06$, $z_0 = 2 \times 10^{-6}$ and $K = 3.5$.  

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can be seen that \( \hat{u}_0 \) has only one zero, which is located at the bottom. \( \hat{v}_0 \) has two zeros: one is located at the bottom, and another at the location where \( \partial \hat{w}_0 / \partial z = 0 \). \( \hat{w}_0 \) has two zeros, which are located at the surface and the bottom, respectively. These eigenfunctions are the simplest for the given boundary conditions.

Note that the numerical value of \( \hat{v}_0 \) is smaller than \( \hat{u}_0 \) by a factor of about 10 and \( \hat{w}_0 \) is smaller than \( \hat{v}_0 \) by a factor of about 10. From the eigenfunction of \( \hat{u}_0 \), we can see that the vertical vorticity \( \zeta_0 = -\partial u_0 / \partial y \) has its maximum near the surface and then decreases rapidly with increasing distance from the mean surface. The longitudinal vorticity \( \xi_0 = \partial w_0 / \partial y - \partial v_0 / \partial z \) is strong only near the surface and weak near the bottom. Because this eigen-mode is driven by the surface stress mechanism, these features are the results of the vorticity diffusion from the mean surface.

We remark that the relative smallness of \( \hat{u}_0 \) and \( \hat{v}_0 \) depends on the values of \( \alpha \), \( K \), and \( \epsilon \), i.e., the diffusion processes. We shall have examples showing this in the next section when we discuss the effects of \( K \) on Langmuir circulation.

The pattern of the computed Langmuir cells is shown in Figure (16-7). The values of \( u_0 \), \( v_0 \), \( w_0 \) are computed according to (lc.normal.mode.1) and (lc.normal.mode.2). The origin of \( y \) is chosen such that \( v_0 = 0 \) at \( y = Ky/\pi \). In the top plot of Figure (16-7), the velocities \( u_0 \) and \( v_0 \) are not right on the mean surface \( z = 0 \), but one grid point below the mean surface.

From Figure (16-7), the down-welling zone is signified by the negative vertical velocity \( w_0 \), and the up-welling zone by the positive vertical velocity. Near the free surface, fluid moves toward the maximum value of \( u_0 \), which is the surface convergence. The down-welling zone is beneath the surface convergence. The centers of the Langmuir cells are more close to the mean surface than to the bottom.

16.1.4 Eigenfunctions and Langmuir cells for the second eigen-mode

The growth rate of the second eigen-mode in this case is small. We include the results here for comparison with later results.

The eigenfunctions \( \hat{u}_0 \), \( \hat{v}_0 \) and \( \hat{w}_0 \) of the second eigen-mode are plotted in Figure (16-8), which is quite different from the first eigen-mode. Now \( \hat{u}_0 \) has two zeros, one
Figure 16-7: The pattern of Langmuir cells due to the first eigen-mode: top figure—velocity $u_0$ (solid line) and $v_0$ (dashed-line) on the surface; bottom figure— the contour of the stream function $\psi$. Conditions same as in Figure (16-6). Arrows indicate the directions of the velocity vectors at the corresponding locations.

at the bottom and another one next to the surface. $\tilde{v}_0$ has three zeros: one is located at the bottom, and the second one next to the surface and the third one in the core region. $\tilde{w}_0$ has three zeros, two next to the surface and one at the bottom. Because the value of $\tilde{v}_0$ and $\tilde{w}_0$ are small near the surface, the enlarged views of $\tilde{v}_0$ and $\tilde{w}_0$ near surface region are provided in Figure (16-8).

Note that the numerical values of $\tilde{v}_0$ and $\tilde{w}_0$ are comparable in the core region. The vertical vorticity $\zeta_0 = -\partial u_0 / \partial y$ still has its maximum near the surface, but does not decrease as quickly as for the first eigen-mode. The longitudinal vorticity $\xi_0 = \partial w_0 / \partial y - \partial v_0 / \partial z$ is large in the core region but small near the surface and bottom. Near the surface $\tilde{w}_0$ is about $O(10^{-3})$, the CL-II mechanism is very weak here. The vertical vorticity near the surface is due to the surface stress and the vorticity diffusion from the core region where the CL-II mechanism is strong.
Figure 16-8: Eigenfunctions for the second eigen-mode: solid lines—real part, dashed-lines—imaginary part. \( kh = 1, u_f/C = 0.003, \varepsilon = 0.06, z_b = 2 \times 10^{-6} \) and \( K = 3.5 \).

The pattern of the computed cells for the second eigen-mode is shown in Figure (16-9). There are two cells: the smaller one close to the surface and the larger one in the core. Near the free surface, fluid moves toward the maximum value of \( u_0 \), which is the surface convergence. But in the core region it is the up-welling zone that is beneath the surface convergence. The center of the larger cells are more close to the bottom surface than to the mean surface.

At the bottom of the small cells \((w_0 = 0)\), the shear of the longitudinal velocity, \( \partial u_0/\partial z \), is not zero, as seen in Figure (16-8). This suggests that there will be momentum exchange (thus, energy exchange) between the small cells and the large cells below. Note that the small cells are weak, and above the wave trough in this example.

When both the first and the second eigen-modes can grow with time, the interesting differences between the first and the second eigen-mode are:

1. *The location of the down-welling zone in the core region.* For the first eigen-mode, the down-welling zone is beneath the surface convergence, but for the second eigen-mode, the down-welling zone in the core region is between the surface convergences.

2. *The center of the Langmuir cell.* For the first eigen-mode, the centers of the cells
are closer to the mean surface, while for the second eigen-mode, the centers are closer to the bottom. Thus the effect of the bottom on the Langmuir circulation is more direct for the second eigen-mode than for the first.

16.2 Effect of $K$ on growth rate and fastest-growing Langmuir cell

Among all the possible Langmuir cells with different $K$, we want to find which one amplifies most rapidly. Presumably, this fastest growing Langmuir cell is the one most likely to be observed. Typically, the values of basic state parameters such as current strength, wave slope are fixed and the transverse wavenumber $K$ is varied until the maximum value of the real part of the characteristic exponent $\sigma$ is found.
This gives the transverse wavenumber, $K$, of the expected Langmuir cell as well as its growth rate.

To illustrate the determination of the fastest-growing Langmuir cells, we take the wave and current conditions as follows

$$kh = 1, \quad \epsilon = 0.03, \quad u_f/C = 0.003, \quad z_b = 2 \times 10^{-6} \quad (16.3)$$

and vary $K$ from 0.2 to 9. We choose these parameters to demonstrate the existence of the fastest-growing cells of the first eigen-mode. The growth rates of the first and the second eigen-modes are plotted in Figure (16-10). It can be seen that the basic state is stable to the second eigen-mode for all $K$, but unstable to the first eigen-mode with transverse wavenumber $0.5 < K < 4.25$. The growth rate of the first eigen-mode first increases with increasing $K$. At $K = 2.6$, the growth rate reaches the maximum value $\sigma_r = 0.87$. Further increasing $K$ will reduce the growth rate. Under the wave and current conditions given here, the fastest-growing Langmuir cell is due to the first eigen-mode with $K = 2.6$. 

![Figure 16-10: Growth rate $\sigma_r$ against $K$ for the first (squares) and the second (circles) eigen-modes: $kh = 1, u_f/C = 0.003, \epsilon = 0.03, z_b = 2 \times 10^{-6}$.](image-url)
16.2.1 Eigenfunctions and pattern of Langmuir cells due to the first eigen-mode

The eigenfunctions of the first eigen-mode for \( K = 1.0, 2.6 \) and 4.2 are plotted in Figure (16-11). In all cases, \( \hat{v}_0 \) has only one zero at the bottom and \( \hat{w}_0 \) has two zeros located at the surface and the bottom. \( \hat{v}_0 \) has two zeros, one at the bottom and another at \( \partial \hat{w}_0 / \partial z = 0 \), where \( \hat{w}_0 \) takes its maximum value. Note that the location where \( \hat{w}_0 \) takes maximum moves toward the mean surface when the transverse wavenumber \( K \) is increased. Also note that, among the three \( K \)'s, the magnitude of \( \hat{w}_0 \) is the largest for \( K = 2.6 \).

The cell pattern of the first eigen-mode with \( K = 1 \) is plotted in Figure (16-12). In comparison with (16-7) where the center of the cells are closer to the surface, the center of the cells in this case is approximately in the mid-depth, suggesting that both the CL-II and surface stress mechanisms are important in producing these cells.

16.2.2 Energy budget

Now we examine the energy budget in order to understand the pattern of the Langmuir cells discussed in the previous section.

Different contributions to the energy production of the first eigen-mode for \( K = 1.0, 2.6, 4.2 \) are listed in Table (16.2). The value of \( \sigma_{sw} \) increases with increasing

<table>
<thead>
<tr>
<th>( K )</th>
<th>( \sigma_{St} )</th>
<th>( \sigma_{u_1} )</th>
<th>( \sigma_{u_2} )</th>
<th>( \sigma_{sw} )</th>
<th>( \sigma_{r} )</th>
<th>( \sigma_{r} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0241</td>
<td>0.5020</td>
<td>-0.0096</td>
<td>0.2052</td>
<td>-0.4592</td>
<td>0.2625</td>
</tr>
<tr>
<td>2.6</td>
<td>0.0826</td>
<td>1.7323</td>
<td>-0.0353</td>
<td>1.9538</td>
<td>-2.8937</td>
<td>0.8398</td>
</tr>
<tr>
<td>4.2</td>
<td>0.1019</td>
<td>1.6855</td>
<td>-0.0451</td>
<td>5.6532</td>
<td>-7.4663</td>
<td>-0.0708</td>
</tr>
</tbody>
</table>

Table 16.2: Energy budget for the first eigen-mode

\( K \), which is easy to understand from the expression of \( \sigma_{sw} \) in (13.38d). For all \( K \) discussed here, \( \sigma_{u_1} \) and \( \sigma_{sw} \) are comparable under the given wave/current conditions, suggesting that both the CL-II and the surface stress mechanisms contribute to the growth of the first eigen-mode. This is why the center of the Langmuir cells in Figure (16-12) is approximately at the mid-depth. This is quite different from the case discussed in Table (16.1) where \( \sigma_{u_1} \) is much smaller than \( \sigma_{sw} \) and the circulation is
Figure 16-11: Eigenfunctions of the first eigen-mode. From the top to bottom, $K = 1.0, 2.6, 4.2$, respectively. solid line(real part) and dashed-line (imaginary part): $kh = 1, u_f/C = 0.003, \epsilon = 0.03, z_b = 2 \times 10^{-6}$. 

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Figure 16-12: The pattern of Langmuir cells due to the first eigen-mode: top figure—velocity $u_0$ (solid line) and $v_0$ (dashed line) right on the surface; bottom figure—the contour of the stream function $\psi$. Conditions same as (a) in Figure (16-11). Arrows indicate the directions of the velocity vectors at the corresponding locations.

Driven mainly by the surface stress. Thus, as far as the first eigen-mode is concerned, the surface stress mechanism is always important, but the CL-II mechanism can be important for the first eigen-mode under certain conditions.

Now we analyze the conditions under which CL-II mechanism can be important for the first eigen-mode. The current strengths in Table (16.2) and Table(16.1) are the same, $u_f/C = 0.003$, but the wave slope here is only one half of that in Table (16.1). As a result, the dimensionless friction velocity $\alpha$, which also measured the diffusivity of the vorticity, is two times larger than that in Table (16.1). Hence under the condition given here, the vorticity generated from the surface can be easily diffused into the core and enhance the CL-II mechanism in the core region. In other words, the strong current and weak waves will help the vorticity diffusion. That is why the values of $(\hat{u}_0, \hat{v}_0$ and $\hat{w}_0)$ in Figure (16-11) are larger than that in Figure (16-6) and both the CL-II and surface stress mechanisms are important in Figure (16-11). We
conclude that a large value of the dimensionless friction velocity $\alpha$ will make the CL-II mechanism important for the first eigen-mode.

Note that $\sigma_{ST}$ are comparable to $\sigma_r$ for $K = 4.2$, indicating that in this case Langmuir cells can gain some non-negligible energy in bending the vertical vorticity into the longitudinal direction.

Even though the CL-II mechanism can be important for the first eigen-mode, there is a fundamental difference between the first eigen-mode and the second eigen-mode (beside the difference of the eigenfunctions). The relative importance of the CL-II instability mechanism in the first eigen-mode crucially depends on the diffusion of the vorticity generated by the surface stress; while in the second eigen-mode, CL-II instability mechanism is affected by, but not dependent on, the vorticity generated by the surface stress.

16.3 Effect of wave slope $c$ on Langmuir circulation

16.3.1 Growth rate $\sigma_r$

To study the effect of the wave slope on the Langmuir circulation, we begin with the following example. The input parameters of this example are:

$$kh = 1, u_f/C = 0.003, \epsilon = [0.02, 0.2], z_b = 2 \times 10^{-6}, K = 3.5 \quad (16.4)$$

The growth rates of the first and the second eigen-mode are plotted in Figure (16-13) for various wave slopes. From Figure (16-13), we conclude that the basic state will be unstable to the first eigen-mode with $K = 3.5$ if $\epsilon > 0.025$; while the basic state will be unstable to the second eigen-mode if $\epsilon > 0.06$.

The growth rate of the first eigen-mode increases with increasing wave slope without reaching the maximum within $0 < \epsilon < 0.2$. After $\epsilon > 0.14$, the growth rate of the first eigen-mode only change slightly with $\epsilon$. The growth rate of the second eigen-mode first increases with increasing $\epsilon$, and reaches the maximum value at $\epsilon = 0.1$. Further increasing $\epsilon$ will reduce the growth rate of the second eigen-mode. The suppression of the growth rate by large wave slope will be discussed shortly from the viewpoint of the perturbed current.
Figure 16-13: Growth rate $\sigma_t$ against wave slope $\epsilon$ for the first (squares) and the second (circles) eigen-modes: $kh = 1, u_f/C = 0.003, K = 3.5$ and $z_0 = 2 \times 10^{-6}$.

16.3.2 The first eigen-mode

Now we examine the effect of the wave slope on the first eigen-mode in some details.

Eigenfunctions and Langmuir cells

The eigenfunctions, $\hat{\omega}_0, \hat{v}_0$ and $\hat{\psi}_0$, of the first eigen-mode for $\epsilon = 0.04, 0.1, 0.18$ are plotted in Figure (16-14), from which we can see that the values of $\hat{v}_0$ and $\hat{\psi}_0$ decrease with increasing $\epsilon$. Under the same current strength, large wave slope corresponds to a small dimensionless friction velocity $\alpha$, thus the vorticity diffusion from the surface becomes harder. As a result, the vorticity for large wave slope will be more confined to the surface than that for small wave slope.

The pattern of the Langmuir cells for $\epsilon = 0.1$ is shown in Figure (16-15), which shows that the center of Langmuir cell is close to the mean surface and the downwelling zone is beneath the surface convergence.

Energy budget

For the first eigen-mode, the computed values of $\sigma_{St}, \sigma_{u_1}, \sigma_{u_2}, \sigma_{sw}$, and $\sigma_{e}$, defined in (13.37), are given in Table (16.3) for $\epsilon = 0.04, 0.1, 0.18$. The vorticity is confined in a narrow region near the surface for large wave slope, as shown in Figure(16-14), hence
Figure 16-14: Eigenfunctions of the first eigen-mode. solid line (real part) and dashed-line (imaginary part): $kh = 1, u_f/C = 0.003, K = 3.5, z_0 = 2 \times 10^{-6}$. 
the CL-II mechanism becomes not effective in the core region. Therefore, the value of $\sigma_{u_1}$ decreases with increasing wave slope. For small wave slope, $\sigma_{u_1}$ is of the same order of magnitude as $\sigma_{sw}$, thus the CL-II mechanism is as important as the surface stress mechanism due to the relative large value of $\alpha$ for small wave slope. Other contributions (except the dissipation) to the growth rate of the first eigen-mode are negligible.

![Graph showing $u_0$ and $v_0$](image)

Figure 16-15: The pattern of Langmuir cells due to the first eigen-mode: top figure—velocity $u_0$ (solid line) and $v_0$ (dashed-line) right on the surface; bottom figure—the contour of the stream function $\psi$. Conditions same as (b) in Figure (16-14). Arrows indicate the directions of the velocity vectors at the corresponding locations.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\sigma_{St}$</th>
<th>$\sigma_{u_1}$</th>
<th>$\sigma_{u_2}$</th>
<th>$\sigma_{sw}$</th>
<th>$\sigma_{e}$</th>
<th>$\sigma_{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.0905</td>
<td>0.7063</td>
<td>-0.0402</td>
<td>8.6417</td>
<td>-8.1682</td>
<td>1.2301</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0048</td>
<td>0.0017</td>
<td>-0.0067</td>
<td>15.3061</td>
<td>-12.8484</td>
<td>2.4574</td>
</tr>
<tr>
<td>0.18</td>
<td>0.0002</td>
<td>0.0000</td>
<td>-0.0015</td>
<td>12.9250</td>
<td>-10.0919</td>
<td>2.8318</td>
</tr>
</tbody>
</table>

Table 16.3: Effect of wave slope on the energy budget of the first eigen-mode
16.3.3 The second eigen-mode

For the second eigen-mode, the growth rate for $\epsilon \leq 0.056$ is negative, so we only examine the cases where $\epsilon = 0.1, 0.18$.

Eigenfunctions and Langmuir cells

The eigenfunctions $\hat{u}_0, \hat{v}_0$ and $\hat{w}_0$ of the second eigen-mode for $\epsilon = 0.1, 0.18$ are plotted in Figure (16-16). The shapes of these eigenfunctions are generally similar to those in Figure (16-8). The value $\hat{v}_0$ on the surface tends to zero as we increase the wave slope. For large wave slope, the number of zeros in the eigenfunction $\hat{w}_0$ reduces from

![Graphs of eigenfunctions for $\epsilon = 0.1$ and $\epsilon = 0.18$]
three to two, one on the surface and another on the bottom. As indicated by the enlarged view of the eigenfunction near the surface, the zero next to the surface that exists for small wave slope becomes unresolvable for large wave slope even under the grid with $N = 600$, which is already very fine near the two boundaries. Also note that for large wave slope, $\hat{u}_0, \hat{v}_0$ and $\hat{w}_0$ are comparable numerically, while for small wave slope, the values of $\hat{v}_0$ and $\hat{w}_0$ are normally smaller than that of $\hat{u}_0$. Thus we conclude that under large wave slope the surface stress mechanism is weak while the CL-II mechanism is strong for the second eigen-mode.

The pattern of the cells for $\epsilon = 0.18$ is given in Figure (16-17), which shows that the center of the cell is close to the bottom, indicating that the surface stress mechanism is not effective for this cell. The up-welling zone is beneath the surface convergences, while the down-welling zone is located between the surface convergences.

Figure 16-17: The pattern of Langmuir cells due to the second eigen-mode: top figure—velocity $u_0$ (solid line) and $v_0$ (dashed-line) right on the surface; bottom figure— the contour of the stream function $\psi$. Conditions same as (b) in Figure (16-16). Arrows indicate the directions of the velocity vectors at the corresponding locations.
Energy budget

For the first eigen-mode, the computed values of $\sigma_{St}$, $\sigma_{u_1}$, $\sigma_{u_2}$, $\sigma_{sw}$, and $\sigma_\epsilon$, defined in (13.37), are given in Table (16.4) for $\epsilon = 0.1, 0.18$. First of all, $\sigma_{sw}$ decreases with increasing wave slope, indicating that the surface stress mechanism is not effective for the second eigen-mode when wave slope is large. This can also be seen from the eigenfunctions in Figure (16-16). For $\epsilon = 0.18$, the value of $\hat{u}_0$ on the surface is practically zero, resulting in a very small value of $\sigma_{sw}$. Next, $\sigma_{u_2}$ become more negative for larger wave slope. This is because a larger wave slope generates a larger reduction of the mean velocity near the surface. The second order current correction is the stabilizing force and it is $\sigma_{u_2}$ that makes the growth rate decrease for large wave slope, as shown in Figure (16-13).

Also note that $\sigma_{St}$ increases with increasing wave slope. In view of the eigenfunction shown in Figure (16-16), $\hat{u}_0$ and $\hat{w}_0$ are of the same order of magnitude, and the Reynolds stress of circulation is larger for large wave slope. Consequently the energy production due to the Reynolds stress of circulation against the Stokes shear is also larger for larger wave slope.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\sigma_{St}$</th>
<th>$\sigma_{u_1}$</th>
<th>$\sigma_{u_2}$</th>
<th>$\sigma_{sw}$</th>
<th>$\sigma_\epsilon$</th>
<th>$\sigma_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.0961</td>
<td>0.5327</td>
<td>-0.0743</td>
<td>0.8083</td>
<td>-0.9772</td>
<td>0.3856</td>
</tr>
<tr>
<td>0.18</td>
<td>0.1424</td>
<td>0.2860</td>
<td>-0.1109</td>
<td>0.0123</td>
<td>-0.0666</td>
<td>0.2633</td>
</tr>
</tbody>
</table>

Table 16.4: Effect of wave slope on the energy budget of the second eigen-mode

16.4 Growth rate $\sigma_r(K, \epsilon)$

In the $(K, \epsilon)$ parameter space, we can construct the relationship between $\epsilon$ and $K$ for the neutral stability and the fastest-growing Langmuir cells.

The $K - \epsilon$ relationship/curve for the neutral stability curve is the border between stability and instability, and is defined by $\sigma_r(K, \epsilon) = 0$ for given $kh$, $z_b$ and $u_f/c$. If the basic state is unstable, then among all the unstable modes corresponding different transverse wavenumber $K$, the unstable mode with the largest growth rate $\sigma_r$ will be the one which is most likely to be observed. Langmuir cell with the largest $\sigma_r$ for all $K$ is the fastest-growing cell.
For each fixed \( \epsilon \), \( \sigma_r(K; \epsilon) \) is a function of \( K \). The maximum value of \( \sigma_r \) is obtained at

\[
\frac{\partial \sigma_r}{\partial K}\bigg|_\epsilon = 0 \tag{16.5}
\]

where \( \big|_\epsilon \) means fixing \( \epsilon \). Equation (16.5) gives \( K = K_{\text{max}} \) at which \( \sigma_r \) has its maximum. For any \( \sigma_r(K, \epsilon) = \text{cons.} \), \( d\sigma_r = 0 \) along the contour of \( \sigma_r \), thus we have

\[
4\sigma_r = \frac{\partial \sigma_r}{\partial \epsilon} \bigg|_K \, d\epsilon + \frac{\partial \sigma_r}{\partial K} \bigg|_\epsilon \, dK = 0 \tag{16.6}
\]

which gives

\[
\frac{d\epsilon}{dK} = -\frac{\frac{\partial \sigma_r}{\partial K}\bigg|_\epsilon}{\frac{\partial \sigma_r}{\partial \epsilon} \bigg|_K} \tag{16.7}
\]

In view of (16.5), \( K_{\text{max}} \) can also be obtained from the condition (16.8)

\[
\frac{d\epsilon}{dK} = 0 \tag{16.8}
\]

on a given contour of \( \sigma_r \).\(^1\)

16.4.1 Growth rate \( \sigma_r(\epsilon, K) \) of the first eigen-mode

The growth rates \( \sigma_r(\epsilon, K) \) of the first eigen-mode for \( kh = 1.0, \frac{u_f}{C} = 0.003 \) and \( z_b = 2 \times 10^{-6} \) are plotted in Figure (16.18). As discussed in section (16.3), the growth rate of the first eigen-mode increases with increasing \( K \) for given \( \epsilon \). In other words, small Langmuir cells reach the nonlinear stage early.

The \( \epsilon - K \) relationship/curve of the neutral stability is represented by the curve \( \sigma_r = 0 \). For each \( \epsilon < 0.04 \), there are two values of \( K \) for which Langmuir cells do not grow or decay. Between these two values of \( K \), Langmuir cells grow. For \( \epsilon > 0.04 \), there is only one value of \( K \) for which the Langmuir cells do not grow or decay. The critical wave slope, \( \epsilon_{cr} = 0.021 \), is found at \( d\epsilon/dK = 0 \) on the curve \( \sigma_r = 0 \). If wave slope is less than 0.021 then the basic state is stable for all \( K \) under the chosen conditions.

\(^1\)Thanks Professor Mei for the derivation of equations (16.5)-(16.8).
Figure 16-18: Growth rate of the first eigen-mode \( kh = 1.0, u_f/C = 0.003 \) and \( z_b = 2 \times 10^{-6} \) conditions.

16.4.2 Growth rate \( \sigma_r(\epsilon, K) \) of the second eigen-mode

The growth rates \( \sigma_r(\epsilon, K) \) of the second eigen-mode for \( kh = 1.0, u_f/C = 0.003 \) and \( z_b = 2 \times 10^{-6} \) are plotted in Figure (16-19). Again, the \( \epsilon - K \) relationship/curve of the neutral stability is represented by the curve \( \sigma_r = 0 \). The critical wave slope for the second eigen-mode is \( \epsilon_{cr} = 0.058 \). For each \( \epsilon > 0.058 \), there are two values of \( K \) at which Langmuir cells do not grow or decay. Between these two values of \( K \), Langmuir cells grow.

In Figure (16-19), there is a maximum growth rate at \( \epsilon = 0.105 \) and \( K = 3.8 \). For the fastest growing Langmuir cells \( (de/dK = 0 \) on the contour lines), the growth rate first increases with increasing wave slope and when the wave slope is larger than 0.105, the growth rate decreases with further increasing wave slope. As we have discussed in the previous section, large wave slope makes the reduction of the mean velocity near the surface larger, hence the stability force due to the perturbed current larger. As a result, the growth rate can be suppressed by large wave slope.
Figure 16-19: Growth rate of the second eigen-mode $kh = 1.0, u_f/C = 0.003$ and $z_b = 2 \times 10^{-6}$. The irregularity of the contour lines is due to the numerical error.

In summary, large waves make the basic flow more unstable, and small Langmuir cells grow faster than large cells.

16.5 Effect of $kh$ on growth rate $\sigma_r(\epsilon, K)$

To study the effect of the $kh$ on the growth rate of Langmuir circulation, we compute the growth rates for three water depths, $kh = 0.7, 1.0, 1.3$ with $u_f/C = 0.006, z_b = 2 \times 10^{-6}$ fixed.

16.5.1 Growth rate $\sigma_r(K, \epsilon)$ for the first eigen-mode

The growth rates $\sigma(K, \epsilon)$ of the first eigen-mode for $kh = 0.7, kh = 1.0$ and $kh = 1.3$ are plotted in Figure (16-20), which leads to the following conclusions:

- The effect of water depth $kh$ on the first eigen-mode is relative weak. This is largely due to the fact that this mode is derived mainly by the surface stress. The vorticity will be relatively weak when it is diffused from the surface to the bottom.
• For given \((\epsilon, K)\), the growth rate is decreasing with increasing water depth. This is mainly due to the factor coth\((kh)\) in the surface stress condition.

• The spacing of the fastest-growing Langmuir cells for small wave slope are larger in deep water than that in shallower water.

16.5.2 Growth rate \(\sigma_r(K, \epsilon)\) for the second eigen-mode

The growth rates \(\sigma(K, \epsilon)\) for the second eigen-mode are plotted in Figure (16-21) for \(kh = 0.7\), \(kh = 1.0\), and \(kh = 1.3\) with \(u_f/C = 0.006\) and \(z_b = 2 \times 10^{-6}\). From Figure (16-21), we find that the effects of water depth on the growth rate of the second eigen-mode are similar to that for the first eigen-mode, that is,

• The minimum wave slope needed to have instability is not sensitive to \(kh\).

• For given \((\epsilon, K)\), the growth rate is decreasing with increasing water depth \(kh\).

• The cell spacing of the fastest-growing Langmuir cells are larger in deep water than in shallower water.

• The spacing distribution of the unstable Langmuir circulation is wider in shallower water than in deep water.

In the CL-II theory for deep water (see Leibovich (1983) [39]), the eigenvalue problem depends on a single parameter, the so-called "Langmuir number" \(La\), defined by

\[
La = \sqrt{\frac{\nu_r^2 k^2}{\omega a^2 u_f^2}}
\]  

(16.9)

where \(\nu_r\) is a constant eddy viscosity and \(a\) the wave amplitude. Parameter \(La\) is a measure of the relative importance of current to waves.

For a variable eddy viscosity, let us measure the relative importance of current to waves by

\[
La_{cr} = \sqrt{\frac{u_f/C}{\epsilon_{cr}^2}}
\]  

(16.10)
where $\epsilon_{cr}$ is the minimum wave slope required to have instability for given current strength.

We remark that if we take $\nu_T = u_f/k$, then (16.9) can be also written as

$$La = \sqrt{\frac{u_f/C}{\epsilon^2}} \quad (16.11)$$

Thus $La_{cr}$ defined in (16.10) is closely related to the Langmuir number, $La$, defined by (16.9) for deep water and constant eddy viscosity. The values of $La_{cr}$ for the current strength given in Figure (16-23) are listed in Table (16.5). Note that $La$ decreases

<table>
<thead>
<tr>
<th>$kh$</th>
<th>0.7</th>
<th>1.0</th>
<th>1.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$La_{cr}$</td>
<td>0.993</td>
<td>0.956</td>
<td>0.945</td>
</tr>
</tbody>
</table>

Table 16.5: $La_{cr}$ for the second eigen-mode

only slightly with increasing water depth $kh$ and can be regard as independent of $kh$ for the given current strength.

To further examine how water depth affects the growth rate, we study the effect of the water depth on the energy production of Langmuir circulation. In Table (16.6), the energy productions of the second eigen-mode are listed for three water depths.

<table>
<thead>
<tr>
<th>$kh$</th>
<th>$\sigma_{St}$</th>
<th>$\sigma_{u_1}$</th>
<th>$\sigma_{u_2}$</th>
<th>$\sigma_{sw}$</th>
<th>$\sigma_{e}$</th>
<th>$\sigma_{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0.0614</td>
<td>0.7225</td>
<td>-0.1279</td>
<td>2.0204</td>
<td>-2.3356</td>
<td>0.3407</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0557</td>
<td>0.7010</td>
<td>-0.0560</td>
<td>1.3914</td>
<td>-1.8337</td>
<td>0.2584</td>
</tr>
<tr>
<td>1.3</td>
<td>0.0458</td>
<td>0.6616</td>
<td>-0.0255</td>
<td>0.6614</td>
<td>-1.2735</td>
<td>0.0699</td>
</tr>
</tbody>
</table>

Table 16.6: Energy productions of the second eigen-mode for $\epsilon = 0.10$, $z_0 = 2 \times 10^{-6}$, $u_f/C = 0.006$ and $K = 3.5$.

For the given conditions, the energy production by the Stokes drift is relatively small for all three water depths. The energy dissipation for the three water depths is compensated mainly by the energy production by $\sigma_{sw}$ and $\sigma_{u_1}$. For small $kh$, $\sigma_{u_2}$ is also important. Because $\sigma_{u_1}$ and $\sigma_{St}$ do not change much with $kh$, the change of $\sigma_r$ with $kh$ is mainly due to that change of $\sigma_{sw}$ which has a factor $\coth(kh)$ in its expression.
We have argued in section (13.3) that the contribution of the bottom wave boundary layer to the growth rate is about $O(0.05)$ at most (see section (13.3)). That conclusion has been supported here by the fact that $\sigma_{St}$, $\sigma_{u_1}$ does not change much with $kh$.

In summary, $kh$ affects the growth rate indirectly though the second order current correction and the surface stress. The most significant effect of $kh$ is that the water depth limit the size of Langmuir cells.

### 16.6 Effect of the current strength on the growth rate

Now, we study the effect of the current strength on the growth rate by varying $u_f/C$ from 0.003 to 0.006 and 0.01. Other parameters are fixed at $kh = 1.0$, $z_b = 2 \times 10^{-6}$.

#### 16.6.1 Growth rate $\sigma_r(K, \epsilon)$ for the first eigen-mode

The growth rates $\sigma(K, \epsilon)$ for the first eigen-mode are plotted in Figure (16-22) for $u_f/C = 0.003, u_f/C = 0.006$ and $u_f/C = 0.01$. From Figure (16-22), we have the following conclusions:

- For given $(\epsilon, K)$, the growth rate of the stronger current is smaller than that of the weaker current. This is partially due to the increased dissipation by strong current.

- The minimum wave slope required to have instability increases with increasing current strength, thus strong current need large waves in order to have instability.

- The spacing of the fastest-growing cells slightly increases with increasing current strength.

To examine how the current strength affect the energy production, we listed the values of $\sigma_{St}, \sigma_{u_1}, \sigma_{u_2}, \sigma_{sw}$, and $\sigma_\epsilon$ of the first eigen-mode in Table (16.7) for $u_f/C = 0.003, 0.006, 0.01$. Other parameters are fixed at $kh = 1.0$, $z_b = 2 \times 10^{-6}$, $\epsilon = 0.1$ and...
Table 16.7: Effect of current strength on the energy budget of the first eigen-mode

\[ \begin{array}{cccccc}
  u_f/C & \sigma_{St} & \sigma_{u1} & \sigma_{u2} & \sigma_{zw} & \sigma_c \\
  0.003 & 0.0048 & 0.0017 & -0.0067 & 15.3061 & -12.8484 & 2.4574 \\
  0.006 & 0.0185 & 0.0172 & -0.0119 & 14.9878 & -12.9138 & 2.0979 \\
  0.01 & 0.0437 & 0.1018 & -0.0193 & 13.4395 & -11.8133 & 1.7525 \\
\end{array} \]

The first eigen-modes listed in Table (16.7) are driven mainly by the surface stress. The balance of the energy production due to the work done by the surface stress and the dissipation controls the growth rate of Langmuir cells. Increasing current strength increases the (eddy viscosity) diffusivity which in turn increases the turbulent dissipation. For the first eigen-mode, the net effect is the reduced growth rate by the increased current strength (eddy viscosity).

16.6.2 Growth rate \( \sigma_r(K, \epsilon) \) for the second eigen-mode

The growth rates \( \sigma(K, \epsilon) \) for the second eigen-mode are plotted in Figure (16-23) for \( u_f/C = 0.003, u_f/C = 0.006 \) and \( u_f/C = 0.01 \).

As for the first eigen-mode, for given \( (\epsilon, K) \), the growth rate increases with decreasing current strength, and the minimum wave slope required to have instability increases with increasing current strength, thus strong current need large waves in order to have instability. The spacing of the fastest growing Langmuir cells is not sensitive to the change of the current strength.

The value of \( La_{cr} \), define in (16.10), are listed in Table (16.8)) for the current strength given in Figure (16-23), which shows that \( La_{cr} \) is almost a constant for the

\[ \begin{array}{ccc}
  u_f/C & 0.003 & 0.006 & 0.01 \\
  La_{cr} & 0.928 & 0.956 & 0.971 \\
\end{array} \]

Table 16.8: \( La_{cr} \) for the second eigen-mode

given dimensionless water depth \( kh \).

From Table (16.5) and Table (16.8), an averaged value of \( La_{cr} \), \( \langle La_{cr} \rangle \), is found by taking the averaged of all values of the \( La_{cr} \) in these two tables.

\[ \langle La_{cr} \rangle = 0.959 \quad (16.12) \]
Thus (16.12) can be viewed as an empirical criterion for instability for the problems studied here.

To examine how the current strength affect the energy budget, we listed the values of $\sigma_{St}, \sigma_{u1}, \sigma_{u2}, \sigma_{sw},$ and $\sigma_e$ of the second eigen-mode in Table (16.7) for $u_f/C = 0.003, 0.006, 0.01$. Other parameters are fixed at $kh = 1.0$, $z_b = 2 \times 10^{-6}$, $\epsilon = 0.1$ and $K = 3.5$. It is easy to understand that strong current will result in a large $\sigma_{sw}$

<table>
<thead>
<tr>
<th>$u_f/C$</th>
<th>$\sigma_{St}$</th>
<th>$\sigma_{u1}$</th>
<th>$\sigma_{u2}$</th>
<th>$\sigma_{sw}$</th>
<th>$\sigma_e$</th>
<th>$\sigma_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.003</td>
<td>0.0961</td>
<td>0.5327</td>
<td>-0.0743</td>
<td>0.8083</td>
<td>-0.9772</td>
<td>0.3856</td>
</tr>
<tr>
<td>0.006</td>
<td>0.0557</td>
<td>0.7010</td>
<td>-0.0560</td>
<td>1.3914</td>
<td>-1.8337</td>
<td>0.2584</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0350</td>
<td>0.8034</td>
<td>-0.0434</td>
<td>1.6443</td>
<td>-2.5460</td>
<td>-0.1066</td>
</tr>
</tbody>
</table>

Table 16.9: Effect of current strength on the energy budget of the second eigen-mode dissipation through the eddy viscosity, as indicated by the values of $\sigma_e$. The energy production due to the Stoke drift decreases slightly with increasing current strength, and its contribution to the total growth rate is small. The strong current produces large transverse vorticity $\partial u_0 / \partial z$ and results in a strong CL-II mechanism. Thus $\sigma_{u1}$ for a stronger current is larger than that for a weaker current. $\sigma_{u2}$ decreases with increasing current strength because the second order current correction in a stronger current is weaker than that in a weaker current under the same wave conditions.

To explain the change of $\sigma_{sw}$ with the current strength, we examine the eigenfunctions, which are plotted in Figure (16-24) for $u_f/C = 0.003, 0.006, 0.01$.

The significant effect of the current strength on the eigenfunction is that a strong current results in a large transverse velocity $\hat{v}_0$ near the surface. Strong current will always accompany the large eddy viscosity. For the same wave slope, a strong current means a large dimensionless diffusivity (eddy viscosity), $\alpha$. The larger $\alpha$ will help the vorticity diffusion and deepen the region influenced by the vorticity generated on the surface. Therefore, the destabilizing force of the surface stress is more influential for a stronger current. This is consistent with the effect of the wave slope, where small waves slope results in large $\alpha$ and deepen the region influenced by the vorticity generated on the surface, hence the transverse velocity $\hat{v}_0$ and $\sigma_{sw}$.  

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Figure 16-20: The growth rate of the first eigen-mode for $kh = 0.7$, $kh = 1.0$ and $kh = 1.3$ with $u_f/C = 0.006$ and $z_b = 2 \times 10^{-6}$. Numbers on the contour lines are the growth rates.
Figure 16-21: The growth rate of the second eigen-mode for $kh = 0.7$, $kh = 1.0$ and $kh = 1.3$ with $u_f/C = 0.006$ and $z_b = 2 \times 10^{-6}$. Numbers on the contour lines are the growth rates. The irregularity of the contour lines is due to the numerical error.
Figure 16-22: The growth rate of the first eigen-mode for $u_f/C = 0.003$, $u_f/C = 0.006$ and $u_f/C = 0.01$ with $kh = 1.0$ and $z_b = 2 \times 10^{-6}$. 

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Figure 16-23: The growth rate of the second eigen-mode for $u_f/C = 0.003$, $u_f/C = 0.006$ and $u_f/C = 0.01$ with $kh = 1.0$ and $z_b = 2 \times 10^{-6}$. 
Figure 16-24: Eigenfunctions of the first eigen-mode. solid line (real part) and dashed-line (imaginary part): \(kh = 1.0, \epsilon = 0.1, K = 3.5, z_b = 2 \times 10^{-6}\).
Chapter 17

Langmuir circulation in a wave-opposing current

In this chapter, we shall discuss the unstable mode in a wave-opposing current. We shall find that for the parameter range chosen there is only one unstable mode, which is driven by the surface stress. The shape of this unstable mode and effects of transverse wavenumber, wave slope, current strength and water depth on the growth rate of this unstable mode will also be discussed.

As we have argued before, the surface-stress driven circulation depends only weakly on the basic current. We have also seen in the previous chapter that the bottom is not important for the growth rate. Thus it can be expected that Langmuir circulation of the first eigen-mode in a wave-opposing current will be similar to that in a wave-following current.

17.1 Unstable eigen-mode

Let us first show that instability is possible for a wave-opposing current. We take $kh = 1.0, u_f/C = 0.003, \epsilon = 0.06, z_b = 2 \times 10^{-6}$ for example. The velocity profile of the basic current is plotted in Figure (17-1), and the shear rates of the unperturbed and perturbed currents are plotted in Figure (17-2).

For a wave-opposing current, the shear rate of the unperturbed current is negative. Note that the shear rate of the perturbed current is still negative with maximums near
Figure 17-1: The unperturbed current (dashed-line) and the total basic current (solid line) for wave-opposing current for $kh = 1, u_f/C = 0.003, \epsilon = 0.06, z_b = 2 \times 10^{-6}$

bottom and the surface. The slight increase of the mean current near the surface by waves can be noticed. This increase of the mean velocity will become more significant for large wave slope. To show this, we take $u_f/C = 0.003, kh = 0.7, \epsilon = 0.12$, and $z_b = 2 \times 10^{-6}$, and the velocity profile of the basic current is shown in Figure (17-3), and the shear rates of the unperturbed and perturbed currents in Figure (17-4).

Note that in Figure (17-4) that shear rate of the perturbed current is more negative in the entire depth of water for large wave slope, which again indicates the strong effect of waves on the mean current as shown in Figure (17-3).

Now we study the stability/instability of a wave-opposing current which includes both the unperturbed current and the perturbed current. As in a wave-following current, we first take $K = 3.5$ and examine the growth rate of the prescribed spanwise
Figure 17-3: The unperturbed current (dashed-line) and the total basic current (solid line) for wave-opposing current for $kh = 0.7, u_f/C = 0.003, \epsilon = 0.12, z_b = 2 \times 10^{-6}$

Figure 17-4: Shear rate of the unperturbed current (dashed-line) and the perturbed current (solid line) for wave-opposing current for $kh = 0.7, u_f/C = 0.003, \epsilon = 0.12, z_b = 2 \times 10^{-6}$

perturbation.

The real and imaginary parts of the first 9 eigenvalues are plotted in Figure (17-5). There is only one unstable eigen-mode with a growth rate $\sigma_r = 1.6722$. The principle of exchange of stability still holds.

The computed values of $\sigma_{St}, \sigma_{u1}, \sigma_{u2}, \sigma_{sw1},$ and $\sigma_\epsilon$ are given in Table (17.1). The dominant contribution to the growth rate comes from the work done by the surface stress, which is order of magnitude larger than other contributions. Even though $\sigma_{u1}$ is small, but it is negative for a wave-opposing current, indicating the unperturbed current is a stabilizing force, while for a wave-following current the unperturbed current is a destabilizing force. The CL-II mechanism is absent here. Therefore Langmuir cells generated by this unstable mode are the results of the vorticity diffusion from
Figure 17-5: Eigenvalue spectrum for wave-opposing current with \( kh = 1.0, u_f/C = 0.003, \epsilon = 0.06, K = 3.5 \)

\[
\begin{array}{cccccc}
\sigma_{St} & \sigma_{u1} & \sigma_{u2} & \sigma_{sw} & \sigma_{e} & \sigma_{r} \\
0.0311 & -0.0304 & -0.0186 & 13.4566 & -11.8086 & 1.6301 \\
\end{array}
\]

Table 17.1: Initial energy production for wave-opposing current with \( kh = 1.0, u_f/C = 0.003, \epsilon = 0.06, K = 3.5 \)

the mean surface.

The eigenfunctions of \( \hat{u}_0, \hat{v}_0 \) and \( \hat{w}_0 \) are plotted in Figure (17-6). Near the surface (except right on the mean surface), \( \hat{v}_0 \) is smaller than but comparable to \( \hat{u}_0 \), which has two zeros: one in the core region and another at the bottom, indicating that the vertical vorticity has two zeros at the same locations as well.

The pattern of Langmuir cells due to this unstable eigen-mode is shown in Figure (17-7). Water particles near the surface move first toward the surface convergences and then downward. The center of the cell is closer to the surface than to the bottom.

We conclude that for unstable Langmuir cells found in a wave-opposing current, the vertical vorticity, \( \zeta_0 = \partial u_0 / \partial y \), is generated by the surface stress, and bent into the longitudinal direction, and then diffuse downwards.
Figure 17-6: Eigenfunctions for wave-opposing current with $kh = 1.0, U_f/C = 0.003, \epsilon = 0.06, K = 3.5$. Solid lines – real parts of the eigenfunctions; dashed-lines–imaginary parts of the eigenfunctions.

17.2 Effect of $K$

To show the effect of the transverse wavenumber in a wave-opposing current, we take $kh = 1.0, U_f/C = 0.003, \epsilon = 0.06$, and examine the growth rates of the spanwise perturbations with $0.2 \leq K \leq 9$.

The first 9 eigen-values are plotted in Figure (17-8) for $K = 3.4$ and Figure (17-9) for $K = 9$, which show that there is only one unstable eigen-mode and the principle of exchange of stability holds.

The initial energy budget analysis shows that this unstable mode is forced again mainly by the surface stress, as indicated by Table (17.2), in which $\sigma_{u_1} < 0$, indicating

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\sigma_{St}$</th>
<th>$\sigma_{u_1}$</th>
<th>$\sigma_{u_2}$</th>
<th>$\sigma_{sw}$</th>
<th>$\sigma_S$</th>
<th>$\sigma_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.4</td>
<td>0.0313</td>
<td>-0.0303</td>
<td>-0.0184</td>
<td>12.7629</td>
<td>-11.3839</td>
<td>1.3616</td>
</tr>
<tr>
<td>9</td>
<td>0.0163</td>
<td>-0.0155</td>
<td>-0.0184</td>
<td>31.7933</td>
<td>-27.7370</td>
<td>4.0387</td>
</tr>
</tbody>
</table>

Table 17.2: Initial energy budget for wave-opposing current with $kh = 1.0, U_f/C = 0.003, \epsilon = 0.06$ and $K = 3.4, 9$
Figure 17-7: The pattern of Langmuir cells: Top figure—velocity $u_0$ (solid line) and $v_0$ (dashed-line) on the surface; bottom figure—the contour of the transverse stream function $\psi$. Conditions as in Figure (17-6). Arrows indicate the direction of the velocity vectors at the corresponding locations.

that the unperturbed current is a stabilizing force for a wave-opposing current.

For $K = 9$, the eigenfunctions, $\hat{u}_0$, $\hat{v}_0$, and $\hat{w}_0$, are plotted in Figure (17-10) and the pattern of the Langmuir cell is plotted in Figure (17-11). In comparison with the case of small $K$, the cell center for larger $K$ is closer to the mean surface, indicating the diffusion distance of vorticity is smaller for larger $K$. For $K = 9$, both $\hat{v}_0$ and $\hat{w}_0$ are of $O(10^{-3})$, thus the dominant velocity component is $\hat{u}_0$ that is confined in a narrow region near the surface. But the energy of this Langmuir cell is still carried by both the longitudinal velocity and the transverse velocity, as the latter is relatively large in the core region.

Only one unstable eigen-mode has been found numerically for $0.2 \leq K \leq 9$. The growth rate of this unstable eigen-mode is plotted in Figure (17-12), which shows that the Langmuir cell with smaller spacing (larger $K$) will grow faster.
Figure 17-8: Eigenvalue spectrum for $kh = 1.0, \frac{u_f}{C} = 0.003, \epsilon = 0.06, K = 3.4$

Figure 17-9: Eigenvalue spectrum for $kh = 1.0, \frac{u_f}{C} = 0.003, \epsilon = 0.06, K = 9$

For large $K$ or small spacing, the dominant velocity component is $\tilde{u}_0$, it is likely that for Langmuir cell of small spacing is difficult to be observed by surface tracers because of the small value of $\tilde{u}_0$ near the surface.
17.3 Effect of wave slope

Now, we show the effects of the wave slope on the Langmuir circulation. We take $kh = 1.0, u_f/C = 0.003, \epsilon = 0.06, K = 9$. Solid lines - real parts of the eigenfunctions; dashed-lines - imaginary parts of the eigenfunctions.

The first 9 eigen-values are plotted in Figure (17-13) for $\epsilon = 0.06$, and Figure (17-14) for $\epsilon = 0.2$, which show that there is only one unstable eigen-mode and the principle of exchange of stability holds. The unstable mode again is driven by the surface stress, as indicated by the energy budget analysis given in Table (17.3).

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\sigma_{St}$</th>
<th>$\sigma_{u_1}$</th>
<th>$\sigma_{u_2}$</th>
<th>$\sigma_{sw}$</th>
<th>$\sigma_{\epsilon}$</th>
<th>$\sigma_{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>0.0366</td>
<td>-0.0350</td>
<td>-0.0123</td>
<td>2.7369</td>
<td>-2.6354</td>
<td>0.0907</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0050</td>
<td>-0.0005</td>
<td>-0.0033</td>
<td>4.7738</td>
<td>-4.0247</td>
<td>0.7503</td>
</tr>
</tbody>
</table>

Table 17.3: Initial energy budget for wave-opposing current with $kh = 1.0, u_f/C = 0.003, K = 1$ and $\epsilon = 0.06, 0.2$

Eigenfunctions, $\hat{u}_0$, $\hat{v}_0$, and $\hat{w}_0$, are plotted in Figure (17-15) and the pattern of the Langmuir cell is plotted in Figure (17-16) for $\epsilon = 0.06$ and $\epsilon = 0.2$. It can be seen that the cell center for large wave slope is closer to the mean surface, indicating that the
Figure 17-11: The pattern of Langmuir cells: Top figure—velocity $u_0$ (solid line) and $v_0$ (dashed-line) on the surface; bottom figure—the contour of the transverse stream function $\psi$. Conditions as in Figure(17-10). Arrows indicate the direction of the velocity vectors at the corresponding locations.

diffusion distance is smaller. Note that the cell center for $\epsilon = 0.06$ is approximately in the mid-depth. Because of the absence of the CL-II theory, this feature is the result of the vorticity diffusion alone. For $\epsilon = 0.06$, $\bar{u}_0 = O(10^{-1})$ and $\bar{v}_0 = O(10^{-3})$, thus it is likely that the Langmuir circulation of large spacing can be observed in the field conditions.

Computations show that there is only one unstable eigen-mode for $0.02 \leq \epsilon \leq 0.2$. As plotted in Figure (17-17), the growth rate of the unstable eigen-mode increases with increasing wave slope. The growth rate $\sigma_r$ does not have a maximum value between $0.02 \leq \epsilon \leq 0.2$, while in a wave-following current there is a range of $\epsilon$ within which $\sigma_r$ has maximum.

Large wave slope has the same effects as large transverse wavenumber on the diffusion distance of vorticity, i.e., both large $\epsilon$ and large $K$ will reduce the diffusion
Figure 17-12: Effect of transverse wave number $K$ on the growth rate for $kh = 1.0, u_f/C = 0.003, \varepsilon = 0.06$

Figure 17-13: Eigenvalue spectrum for $kh = 1.0, u_f/C = 0.003, \varepsilon = 0.06, K = 1$

distance of the vorticity and move the center of the Langmuir cells toward the mean surface.
17.4 Growth rate $\sigma_r(K, \epsilon)$

Similar to the wave-following current, we can plot the growth rate $\sigma_r$ in the $(\epsilon, K)$ parameter space and construct the relationship between $\epsilon$ and $K$ for the neutral (marginal) stability.

For $kh = 1, u_f/C = 0.003$ and $z_b = 2 \times 10^{-6}$, the growth rate $\sigma_r(K, \epsilon)$ is plotted in Figure (17-18). It can be seen that the growth rate increases with increasing wave slope $\epsilon$ and the transverse wavenumber $K$. The fastest growing Langmuir cells have transverse wavenumber $K > 9$. This feature is different from that of wave-following current where the fastest growing cells may have a transverse wavenumber $K < 9$ for small wave slope.

As expected, the small cells will first be observed, and these cells will quickly reach the nonlinear state.

The $\epsilon - K$ relationship of the neutral stability is represented by $\sigma_r = 0$. In Figure (17-18), the critical wave slope is $\epsilon_{cr} = 0.056$, below which the basic state is stable for all $K$ under the chosen conditions.

In comparison with (b) in Figure (16-20) for the wave-following current, it can been seen that the critical wave slope for wave-opposing current is larger, thus a
wave-opposing current is relatively stable compared with a wave-following current.

We have computed the growth rate $\sigma_r(\epsilon, K)$ for the combinations of $kh = 0.7, 1.0, 1.3$ and $u_f/C = 0.003, 0.006, 0.01$. The results are given in Figure (17-19). We remark that, for a wave-opposing current, there is no instability based on the classic CL-II theory. The instability found here is the result of work done by the mean surface stress. The vorticity generated on the surface will diffuse downward and form the Langmuir cells. The slight differences of $\sigma_r(\epsilon, K)$ between a wave-following current and a wave-opposing current for the first eigen-mode is partially due to the small difference in the diffusion processes.
Figure 17-16: The pattern of Langmuir cells: Top figure–velocity $u_0$ (solid line) and $v_0$ (dashed-line) on the surface; bottom figure–the contour of the transverse stream function $\psi$. Conditions same as in Figure(17-15). Arrows indicate the direction of the velocity vectors at the corresponding locations.
Figure 17-17: Effect of wave slope $\epsilon$ on the growth rate for $kh = 1.0, u_f/C = 0.003, K = 1$

Figure 17-18: Growth rate of $kh = 1.0, u_f/C = 0.003, z_b = 2 \times 10^{-6}$
Figure 17-19: Growth rate $\sigma_r(\epsilon, K)$. Clockwise: (a) $u_f/C = 0.003, kh = 0.7$; (b) $u_f/C = 0.006, kh = 0.7$; (c) $u_f/C = 0.01, kh = 0.7$; (d) $u_f/C = 0.003, kh = 1.0$; (e) $u_f/C = 0.01, kh = 1.0$; (f) $u_f/C = 0.003, kh = 1.3$; (g) $u_f/C = 0.006, kh = 1.3$; (h) $u_f/C = 0.01, kh = 1.3$. $z_b = 2 \times 10^{-6}$ for all cases.
17.5 Effect of current strength

We show the effect of the current strength on the Langmuir circulation by examining the curve of the neutral stability for $u_f/C = 0.003, 0.006$ and $0.01$ under $kh = 1.0$, $z_b = 2 \times 10^{-6}$, which are plotted in Figure (17-20). The critical wave slope increases with increasing current strength $u_f/C$. Thus, stronger current need larger waves to make the basic flow unstable, which is the same as in a wave-following current.

For each current strength, when $K$ is larger than some critical value, the neutral stability curves becomes flat and independent of $K$. The critical $K$ is about 5 for the conditions considered here. This feature is also similar to that found in a wave-following current, where, under the same conditions, this critical $K$ is about 7.

17.6 Effect of water depth

Now we show the effect of the dimensionless depth $kh$ on the Langmuir circulation by an example. We take the current strength $u_f/C = 0.006$, and $kh = 0.7, 1.0, 1.3$. For all cases, $z_b = 2 \times 10^{-6}$.
The curves of the neutral stability for the conditions stated above are plotted in Figure (17-21). Again, the critical wave slope increases with increasing water depth $h$.

![Figure 17-21: Effect of water depth on Langmuir circulation on Langmuir circulation for wave-opposing current](image)

$k_h$, indicating that deep water is more stable than shallower water. Again, the curve of the neutral stability becomes flat when $K$ is larger than some critical value.

As we have shown that the Langmuir cells found in a wave-opposing current is the result of the vorticity diffusion from the surface and the velocity components have their maximums near the surface, thus the dissipation due to the bottom is negligible. The water depth $k_h$ affects the growth rate through the dissipation in the core and the eigenfunctions. The water depth also will affect the work done by the surface stress through the factor $\coth(kh)$ in surface shear stress condition.

### 17.7 Summary

In summary, Langmuir circulation can occur in a wave-opposing current by an instability mechanism, driven by the surface stress. A wave-opposing current is more
stable than a wave-following current. As in a wave-following current, small cells have relative large growth rate and will reach the nonlinear stage quickly. Deeper water is more stable than shallower water and large waves make the basic flow more unstable. Because the origin of circulation is the mean surface stress, the pattern and the growth rate of it will not be directly affected by the basic current and the bottom. In other words, wave-following or wave-opposing current will produce similar cells. The slight differences due to the current direction is mainly because the diffusion processes differ in details for these two types of currents.
Chapter 18

Concluding remarks of part II

In the second part of the thesis, a linear theory for Langmuir circulation in water of finite depth has been presented. As in the part I, the current is assumed to be as strong as the wave orbital velocity. Only the smooth bottom is studied here. The interaction between the primary waves and the secondary waves render a non-zero mean stress on the surface, which leads to the surface-stress generation mechanism of Langmuir circulation, in addition to the CL-II mechanism. It is found that Langmuir cells can be formed in both a waves-following and a wave-opposing current. For a wave-following current, both the first and the second eigen-mode can grow in time; while for a wave-opposing current, only the first eigen-modes can grow in time. It is found that the principle of exchange of stability holds for both wave-following and wave-opposing currents.

For both the wave-following and wave-opposing currents, the analysis of the energy budget of Langmuir circulation shows that the stabilizing forces are the second order current correction and the eddy viscosity, and the destabilizing forces are Stokes drift and the surface-stress. The unperturbed current is a destabilizing force for a wave-following current, but a stabilizing force for a wave-opposing current. For the second eigen-mode, the most significant contributions to the growth rate come from the unperturbed current, eddy viscosity and the surface-stress. The first eigen-mode is driven mainly by the surface stress for large waves or large transverse wavenumber. The second eigen-mode is driven mainly by the CL-II mechanism for large waves. For weak waves both the CL-II and surface-stress mechanisms can be important to the
large-spacing cells of the first eigen-mode. For strong waves, both the surface-stress and the CL-II mechanisms are important to cells of the second eigen-mode.

The center of Langmuir cells is close to the mean surface if the surface-stress mechanism dominates the CL-II mechanism and the center of cells is close to the bottom if the CL-II mechanism dominates the surface-stress mechanism. When both the CL-II and surface stress mechanisms are important, the cell center will be in the mid-depth.

Direct comparison with laboratory observations is difficult currently. All the past experiments were performed in a tank with the width less than 1 m. As we argued in the sidewall-generated circulation (see section (14.4)), the circulation due to the sidewalls can be significant in these laboratory experiments. The observed circulation in Nepf, et al (1991) [58] were greatly affected by the sidewall and the circulation patterns in their experiments were similar to that generated by the sidewalls (see section (14.4)). In another set of experiments of Nepf, et al (1995) [57], the cell patterns other than that of the sidewall-generated circulation were found for wave-breaking. In both Nepf, et al (1991)[58] and Nepf, et al (1995) [57], the dimensionless water depths $kh$ were much larger than unity. The observed circulations of Klopman (1997) [35] for both the wave-following and wave-opposing currents over rough bottom were also similar to that generated by the sidewalls (see section (14.4)). Other experiments (Faller and Caponi (1977)[14] and Melville, et al(1998)[51]) had wind blowing over the surface. Therefore, additional experiments for non-breaking wave in a water of finite depth and and in tank with large width are worthwhile to check the present theory.

The present theory can be extended to a rough bottom where the roughness element is deeply buried inside the BWBL, in which case the equations governing Langmuir circulation inside and outside the BWBL will be different. The solutions of Langmuir circulation inside the BWBL provide the bottom boundary conditions for Langmuir circulation in the core region.

There is also no difficulty in including wind in the model as long as a reasonable eddy viscosity model can be described. As an immediate next step, it is also desirable to study the nonlinear evolution of the Langmuir circulation. The order of estimates of Langmuir circulation at the nonlinear stage has been given in Appendix (B), and a
set of nonlinear equations based on those order of estimates can be derived to study
the nonlinear evolution of Langmuir cells. Finally, it is of practical interests to study
the impact of Langmuir circulation on the suspended sediment transport.
Appendix A

Wave energy equation

The governing equation for the wave energy in a 2D flow can be derived in a way similar to the kinetic energy equation of turbulent flows. For compactness, we denote

\[ q_1 \equiv u, \quad q_3 \equiv w, \quad x_1 \equiv x, \quad x_3 \equiv z \]  \hfill (A.1)

The dimensionless continuity and momentum equations of the total motion for an incompressible fluid are

\[ \frac{\partial q_i}{\partial x_i} = 0 \]  \hfill (A.2)
\[ \frac{\partial q_i}{\partial t} + q_j \frac{\partial q_i}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} \]  \hfill (A.3)

where \( i = 1, 3 \) and \( j = 1, 3 \) and

\[ \sigma_{ij} = -p \delta_{ij} + \tau_{ij} \]  \hfill (A.4)
\[ \tau_{ij} = \epsilon^2 \alpha S \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right) \]  \hfill (A.5)

where \( \alpha \) is the dimensionless friction velocity and \( S \) the shape function of the eddy viscosity. \( p \) is the dynamic pressure, related to the total pressure \( P \) by

\[ P = p - \frac{gk}{\omega^2} z \]  \hfill (A.6)
where $g$ is the gravity acceleration, $k$ the wavenumber, and $\omega$ the wave angular frequency (see e.g., section 10.2, Mei, 1989).

The dimensionless surface boundary conditions are:

- The no-flux condition at the surface $z = \eta$

$$\frac{\partial \eta}{\partial t} + q_1 \frac{\partial \eta}{\partial x} - q_3 = 0, \ z = \eta$$

(A.7)

where $\eta$ is the surface displacement.

- The zero normal and tangential stress condition at the surface $z = \eta$

$$-\left[ -(p - \frac{gk}{\eta^2}) + \tau_{11} \right] \frac{\partial \eta}{\partial x} + \tau_{13} = 0$$

(A.8)

$$-\left[ -(p - \frac{gk}{\eta^2}) + \tau_{33} \right] - \tau_{13} \frac{\partial \eta}{\partial x} = 0$$

(A.9)

At the bottom, $z = z_+ \equiv -kh + z_b$, where $z_b = k z_B$ with $z_B$ being the dimensional bottom roughness, we have the no-slip condition

$$q_1 = 0, \ z = z_+$$

(A.10)

and the no-flux condition

$$q_3 = 0, \ z = z_+$$

(A.11)

To derive the energy equation, we multiply $q_i$ on both side of (A.3) to obtain

$$\frac{\partial q_i^2/2}{\partial t} + q_i q_j \frac{\partial q_i}{\partial x_j} = q_i \frac{\partial \sigma_{ij}}{\partial x_j}$$

(A.12)

In view of the following two identities for an incompressible fluid

$$q_i q_j \frac{\partial q_i}{\partial x_j} = q_i \frac{\partial q_i^2/2}{\partial x_j} = \frac{\partial q_i q_j^2/2}{\partial x_j} - q_i^2 \frac{\partial q_j}{\partial x_j} = \frac{\partial q_i (q_i^2/2)}{\partial x_j}$$

(A.13)

$$q_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial q_j \sigma_{ij}}{\partial x_j} - \sigma_{ij} \frac{\partial q_i}{\partial x_j}$$

(A.14)
(A.12) can be written as
\[
\frac{\partial q_i^2/2}{\partial t} = \frac{\partial}{\partial x_j} \left( q_i \sigma_{ij} - q_j q_i^2 \right) - \sigma_{ij} \frac{\partial q_i}{\partial x_j}
\]
\[
= -\sigma_{ij} \frac{\partial q_i}{\partial x_j} + \frac{\partial}{\partial x} \left( q_i \sigma_{i1} - q_1 q_i^2 \right) + \frac{\partial}{\partial z} \left( q_i \sigma_{i3} - q_3 q_i^2 \right) \tag{A.15}
\]
Integrating this equation with respect to \( z \) from the bottom \( z = z_+ \) to the moving surface \( z = \eta \), we have
\[
\int_{z_+}^{\eta} \left( \frac{\partial q_i^2/2}{\partial t} \right) dz = -\int_{z_+}^{\eta} \sigma_{ij} \frac{\partial q_i}{\partial x} dz + \int_{z_+}^{\eta} \frac{\partial}{\partial x} \left( q_i \sigma_{i1} - q_1 q_i^2 \right) dz
\]
\[
+ \left[ q_i \sigma_{i3} - q_3 q_i^2 \right]_{z_+} - \left[ q_i \sigma_{i3} - q_3 q_i^2 \right]_{\eta} \tag{A.16}
\]
In view of the no-slip and no-flux conditions at the bottom \( z = z_+ \), (A.10) and (A.11), we have
\[
\int_{z_+}^{\eta} \left( \frac{\partial q_i^2/2}{\partial t} \right) dz = -\int_{z_+}^{\eta} \sigma_{ij} \frac{\partial q_i}{\partial x} dz + \int_{z_+}^{\eta} \frac{\partial}{\partial x} \left( q_i \sigma_{i1} - q_1 q_i^2 \right) dz
\]
\[
+ \left[ q_i \sigma_{i3} - q_3 q_i^2 \right]_{\eta} \tag{A.17}
\]
Making use of Leibnitz rule, we have
\[
\int_{z_+}^{\eta} \frac{\partial}{\partial x} \left( q_i \sigma_{i1} - q_1 q_i^2 \right) dz = \frac{\partial}{\partial x} \int_{z_+}^{\eta} \left( q_i \sigma_{i1} - q_1 q_i^2 \right) dz - \left[ q_i \sigma_{i1} - q_1 q_i^2 \right]_{z_+} \frac{\partial \eta}{\partial x}
\]
\[
\int_{z_+}^{\eta} \left( \frac{\partial q_i^2/2}{\partial t} \right) dz = \frac{\partial}{\partial t} \int_{z_+}^{\eta} \frac{q_i^2}{2} dz - \frac{\partial}{\partial t} \int_{z_+}^{\eta} \frac{q_i^2}{2} dz - \frac{\partial}{\partial t} \int_{z_+}^{\eta} \frac{q_i^2}{2} dz
\]
\[
hence (A.17) can be rewritten as
\[
\frac{\partial}{\partial t} \int_{z_+}^{\eta} \frac{q_i^2}{2} dz = -\int_{z_+}^{\eta} \sigma_{ij} \frac{\partial q_i}{\partial x} dz + \frac{\partial}{\partial x} \int_{z_+}^{\eta} \left( q_i \sigma_{i1} - q_1 q_i^2 \right) dz
\]
\[
+ \left[ q_i \sigma_{i3} - q_3 q_i^2 \right]_{\eta} - \left[ q_i \sigma_{i1} - q_1 q_i^2 \right]_{\eta} \frac{\partial \eta}{\partial x} + \left[ q_i \sigma_{i3} - q_3 q_i^2 \right]_{\eta} \frac{\partial \eta}{\partial t} \tag{A.18}
\]
By virtue of the surface kinematic boundary condition (A.7), the following equation
holds

\[
\left[ q_i \sigma_{i3} - q_3 \frac{q_i^2}{2} \right]_\eta - \left[ q_i \sigma_{i1} - q_1 \frac{q_i^2}{2} \right]_\eta \frac{\partial \eta}{\partial t} - \left[ \frac{q_i^2}{2} \right]_\eta \frac{\partial \eta}{\partial t} = \left[ \frac{q_i^2}{2} \right]_\eta \left( -q_3 + q_i \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} \right) + \left[ q_i \sigma_{i3} - \sigma_{i1} \frac{\partial \eta}{\partial x} \right]_\eta
\]

\[
= \left[ q_i \left( \sigma_{i3} - \sigma_{i1} \frac{\partial \eta}{\partial x} \right) \right]_\eta \tag{A.19}
\]

It then follows that (A.18) can be written as

\[
\frac{\partial}{\partial t} \int_{-h}^h q_i^2 \, dz = - \int_{-h}^h \left( \sigma_{ij} \frac{\partial q_i}{\partial x_j} \right) \, dz + \frac{\partial}{\partial x} \int_{-h}^h \left( q_i \sigma_{i1} - q_1 \frac{q_i^2}{2} \right) \, dz
\]

\[
+ \left[ q_i \left( \sigma_{i3} - \sigma_{i1} \frac{\partial \eta}{\partial x} \right) \right]_\eta \tag{A.20}
\]

Making use of the surface dynamic boundary conditions (A.8) and (A.9), and the kinematic boundary condition (A.7), we get

\[
\left[ q_i \left( \sigma_{i3} - \sigma_{i1} \frac{\partial \eta}{\partial x} \right) \right]_\eta = \left[ q_1 \left( \tau_{31} - (-p - \tau_{11}) \frac{\partial \eta}{\partial x} \right) \right]_\eta
\]

\[
+ \left[ q_3 \left( (-p + \tau_{33}) - \tau_{31} \frac{\partial \eta}{\partial x} \right) \right]_\eta
\]

\[
= \left[ q_1 \frac{gk}{\omega^2} \frac{\partial \eta}{\partial x} - q_3 \frac{gk}{\omega^2} \right]_\eta
\]

\[
= - \frac{gk}{2 \omega^2} \frac{\partial \eta}{\partial t} \tag{A.21}
\]

Thus (A.20) can be rewritten as

\[
\frac{\partial}{\partial t} \int_{z_+}^h \left( \frac{q_i^2}{2} + \frac{gk \eta^2}{2} \right) \, dz = - \int_{z_+}^h \left( \sigma_{ij} \frac{\partial q_i}{\partial x_j} \right) \, dz
\]

\[
+ \frac{\partial}{\partial x} \int_{z_+}^h \left( q_i \sigma_{i1} - q_1 \frac{q_i^2}{2} \right) \, dz \tag{A.22}
\]
In view of the definition of $\sigma_{ij}$ (A.4), and the continuity equation (A.2), we have

$$
\int_{z_+}^{n} \left( \sigma_{ij} \frac{\partial q_i}{\partial x_j} \right) dz = \int_{z_+}^{n} \left( -p \delta_{ij} + \tau_{ij} \right) \frac{\partial q_i}{\partial x_j} dz
$$

$$
= \int_{z_+}^{n} \left( \tau_{ij} \frac{\partial q_i}{\partial x_j} \right) dz = \epsilon^2 \frac{1}{2} \int_{z_+}^{n} \alpha S \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right)^2 dz. \quad (A.23)
$$

Thus the depth-integrated energy equation (A.22) can be written as

$$
\frac{\partial}{\partial t} \int_{z_+}^{n} \left( \frac{q_i^2}{2} + \frac{g k \eta^2}{2} \right) dz
$$

$$
= -\epsilon^2 \frac{1}{2} \int_{z_+}^{n} \alpha S \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right)^2 dz + \frac{\partial}{\partial x} \int_{z_+}^{n} \left( q_i \sigma_{ij} - \frac{q_i q_j^2}{2} \right) dz \quad (A.24)
$$

where the factor $(kh - z_+)^{-1}$ has been canceled on both sides of (A.24).

Taking the time average of (A.24), we have

$$
\frac{\partial}{\partial t} \int_{z_+}^{n} \left( \frac{q_i^2}{2} + \frac{g k \eta^2}{2} \right) dz
$$

$$
= -\epsilon^2 \frac{1}{2} \int_{z_+}^{n} \alpha S \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right)^2 dz
$$

$$
+ \frac{\partial}{\partial x} \int_{z_+}^{n} \left( q_i \sigma_{ij} - \frac{q_i q_j^2}{2} \right) dz \quad (A.25)
$$

The left-hand side of (A.25) is time rate of the change of the total energy (sum of the potential energy and the kinetic energy). When the total energy is constant in time (i.e., waves are not damped in time), we have

$$
\epsilon^2 \frac{1}{2} \int_{z_+}^{n} \alpha S \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right)^2 dz
$$

$$
= \frac{\partial}{\partial x} \int_{z_+}^{n} \left( -\delta_{ij} pq_i + \epsilon^2 \alpha q_i \left( \frac{\partial q_i}{\partial x_i} + \frac{\partial q_i}{\partial x_1} \right) - \frac{q_i q_j^2}{2} \right) dz \quad (A.26)
$$

where the definition of $\sigma_{ij}$ (A.4) and $\tau_{ij}$ (A.5) have been used in the last equation.

After using the continuity equation on the right-hand side of (A.26), we get

$$
\epsilon^2 \frac{1}{2} \int_{z_+}^{n} \alpha S \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right)^2 dz
$$

$$
= \frac{\partial}{\partial x} \int_{z_+}^{n} \left( -\delta_{ij} pq_i + \epsilon^2 \alpha S \left( \frac{\partial q_i q_i}{\partial x_i} + \frac{\partial q_i q_i^2}{2} \right) - \frac{q_i q_j^2}{2} \right) dz \quad (A.27)
$$

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The unperturbed current is assumed time-independent. The surface displacement of the mean flow is denoted by $\eta$, and velocity and pressure by $\tilde{q}_i, \tilde{p}$. As a special case of (A.27), the energy equation for the mean current reads

$$\varepsilon^2 \frac{1}{2} \int_{z_+}^{\eta+ \tilde{\eta}} \alpha S \left( \frac{\partial \tilde{q}_i + \tilde{q}_i}{\partial x_j} + \frac{\partial \tilde{q}_j + \tilde{q}_j}{\partial x_i} \right)^2 \, dz$$

$$= \frac{\partial}{\partial x} \int_{z_+}^{\eta+ \tilde{\eta}} \left( -\delta_{ij} \tilde{p} \tilde{q}_i + \varepsilon^2 \alpha S \left( \frac{\partial \tilde{q}_i}{\partial x_i} + \frac{\partial \tilde{q}_j}{\partial x_j} \right) - \tilde{q}_i \tilde{q}_i \right) \, dz$$ \hspace{1cm} (A.28)

When both waves and current are present, the total velocity and pressure can be written as $q_i = \tilde{q}_i + \tilde{q}_i, \eta = \tilde{\eta} + \tilde{\eta}$ and $p = \tilde{p} + \tilde{p}$, respectively. Now the energy equation (A.27) can be written as

$$\varepsilon^2 \frac{1}{2} \int_{z_+}^{\eta+ \tilde{\eta}} \alpha (S + \tilde{S}) \left( \frac{\partial \tilde{q}_i + \tilde{q}_i}{\partial x_j} + \frac{\partial \tilde{q}_j + \tilde{q}_j}{\partial x_i} \right)^2 \, dz$$

$$= -\frac{\partial}{\partial x} \int_{z_+}^{\eta+ \tilde{\eta}} (\delta_{ij} (\tilde{p} + \tilde{p}) (\tilde{q}_i + \tilde{q}_i)) \, dz$$

$$+ \varepsilon^2 \alpha \frac{\partial}{\partial x} \int_{z_+}^{\eta+ \tilde{\eta}} \left( \frac{\partial}{\partial x_i} \left( (\tilde{q}_i + \tilde{q}_i)(\tilde{q}_i + \tilde{q}_i) \right) + \frac{1}{2} \frac{\partial}{\partial x_1} (\tilde{q}_i + \tilde{q}_i)^2 \right) \, dz$$

$$- \frac{1}{2} \frac{\partial}{\partial x} \int_{z_+}^{\eta+ \tilde{\eta}} (\tilde{q}_i + \tilde{q}_i)(\tilde{q}_i + \tilde{q}_i)^2 \, dz$$ \hspace{1cm} (A.29)

Under the assumptions

$$\tilde{q}_i = O(\varepsilon), \tilde{q}_i = O(\varepsilon), \tilde{\eta} = O(\varepsilon), \tilde{\eta} = O(\varepsilon^2), \tilde{S} = O(\varepsilon), \frac{\partial f}{\partial x} = O(\varepsilon^2)$$ \hspace{1cm} (A.30)

and making use of the following identity

$$\int_{z_+}^{\eta+ \tilde{\eta}} f \, dz = \int_{z_+}^{0} f \, dz + \int_{0}^{\eta+ \tilde{\eta}} f \, dz,$$ \hspace{1cm} (A.31)

an approximate equation of the wave energy can be derived as follows.

Subtracting (A.28) from (A.27), we have the energy equation of the perturbed motion. Some of the order of magnitude estimates are listed below
\[
\begin{align*}
\alpha \left( S^+ S \right) & \left( \frac{\partial \bar{q}_i + \tilde{q}_i}{\partial x_i} + \frac{\partial \bar{q}_j + \tilde{q}_j}{\partial x_j} \right)^2 dz - \alpha \left( S^+ S \right) \left( \frac{\partial \bar{q}_i}{\partial x_i} + \frac{\partial \bar{q}_j}{\partial x_j} \right)^2 dz
\end{align*}
\]
\[
= \varepsilon^2 \frac{1}{2} \int_{z^+}^{\eta+\bar{\eta}} \alpha \left( S^+ S \right) \left( \frac{\partial \bar{q}_i}{\partial x_i} + \frac{\partial \bar{q}_j}{\partial x_j} \right)^2 dz + O(\varepsilon^5) \tag{A.32}
\]

\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial x} \int_{z^+}^{\eta+\bar{\eta}} (\tilde{q}_1 + \bar{q}_1)(\tilde{q}_1 + \bar{q}_1)^2 dz - \frac{\partial}{\partial x} \int_{z^+}^{\eta} \bar{q}_1 \frac{\partial^2}{\partial x_1^2} dz = O(\varepsilon^5)
\end{align*}
\tag{A.33}
\]

\[
\begin{align*}
\varepsilon^2 \frac{\partial}{\partial x} \int_{z^+}^{\eta+\bar{\eta}} S \left( \frac{\partial}{\partial x_1} ((\tilde{q}_1 + \bar{q}_1)(\tilde{q}_1 + \bar{q}_1)) + \frac{1}{2} \frac{\partial}{\partial x_1} (\tilde{q}_1 + \bar{q}_1)^2 \right) dz
\end{align*}
\]
\[
- \varepsilon^2 \frac{\partial}{\partial x} \int_{z^+}^{\eta} S \left( \frac{\partial \bar{q}_1}{\partial x_1} + \frac{\partial \bar{q}_1^2}{\partial x_1^2} \right) dz = O(\varepsilon^5)
\tag{A.34}
\]

\[
\begin{align*}
\frac{\partial}{\partial x} \int_{z^+}^{\eta+\bar{\eta}} (\bar{\delta}_1 + \bar{p})(\bar{q}_1 + \bar{q}_1) dz - \frac{\partial}{\partial x} \int_{z^+}^{\eta} \bar{\delta}_1 p \bar{q}_1 dz
\end{align*}
\]
\[
= \frac{\partial}{\partial x} \int_{-h}^{0} (\tilde{u} \bar{p}) dz + O(\varepsilon^5)
\tag{A.35}
\]

Thus to the first approximation, the energy equation of the wave motion is

\[
\begin{align*}
\varepsilon^2 \frac{1}{2} \int_{z^+}^{0} \alpha S \left( \frac{\partial \bar{q}_i}{\partial x_i} + \frac{\partial \bar{q}_j}{\partial x_j} \right)^2 dz = - \frac{\partial}{\partial x} \int_{-h}^{0} (\tilde{u} \bar{p}) dz + O(\varepsilon^5)
\end{align*}
\tag{A.36}
\]

which in turn verifies the assumption

\[
\frac{\partial \bar{f}}{\partial x} = O(\varepsilon^2 \bar{f})
\tag{A.37}
\]
Appendix B

Strength of Langmuir circulation at nonlinear stage

To estimate the strength of the Langmuir circulation at nonlinear stage, or the scale of \( \delta \), we notice that the nonlinear interaction of Langmuir circulation must be comparable to the driving force when Langmuir circulation become strong enough. Let examine the longitudinal momentum equation of Langmuir circulation. From equation (12.24), the driving force is

\[
\epsilon^3 \delta \left( \frac{w_0}{kh + z} \right)
\]  

(B.1)

where the factor \( \epsilon \delta \) which indicates the order of magnitude of equation (12.24) itself has been incorporated.

From (12.2), (12.3a), and (12.3b), it then follows that the nonlinear interaction of the Langmuir circulation in the longitudinal momentum equation is

\[
\epsilon^2 \delta^2 \left( v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} \right)
\]  

(B.2)

Term (B.1) must be comparable to (B.2) when the Langmuir circulation becomes large enough, thus we have

\[
\delta = O(\epsilon)
\]  

(B.3)

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which means that Langmuir circulation at the nonlinear stage is comparable to the Stokes drift, just like the case studied by Craik and Leibovich [10] for an $O(\epsilon^2)$ basic current whose shear rate is of the same order of the magnitude in the entire depth of water.
Appendix C

Mapping of differential operators

Here we map the differential operators $A_{ij}$ and $B_G$, defined by (15.17), from the coordinate $z$ to the new coordinate $\tilde{z}$ defined by (15.18).

By the chain rule, we have

$$\frac{\partial}{\partial z} = \left( \frac{2}{kh - z_b} \right) \frac{\partial}{\partial \tilde{z}} \equiv \beta_m \frac{\partial}{\partial \tilde{z}} \quad (C.1)$$

where

$$\beta_m = \frac{2}{kh - z_b} \quad (C.2)$$

Thus, the differential operators $A_{ij}$ and $B_G$ in the new coordinate are

$$B_G = -C_s \beta_m \frac{\partial}{\partial \tilde{z}}, \quad A_{13} = \beta_m \frac{\partial}{\partial \tilde{z}} \quad (C.3a)$$

$$A_{21} = \alpha_c \left( -K^2 \tilde{S}_c + \tilde{S}_c \beta_m \frac{\partial^2}{\partial \tilde{z}^2} + \beta_m \frac{\partial \tilde{S}_c}{\partial \tilde{z}} \frac{\partial}{\partial \tilde{z}} \right), \quad A_{23} = -C_s \frac{\partial U_0}{\partial \tilde{z}} \quad (C.3b)$$

$$A_{32} = \alpha_c \left( -K^2 \tilde{S}_c + \tilde{S}_c \beta_m \frac{\partial^2}{\partial \tilde{z}^2} + \beta_m \frac{\partial \tilde{S}_c}{\partial \tilde{z}} \frac{\partial}{\partial \tilde{z}} \right), \quad A_{33} = -\alpha_c K^2 \beta_m \frac{\partial \tilde{S}_c}{\partial \tilde{z}} \quad (C.3c)$$

$$A_{41} = -\frac{\partial U_s}{\partial \tilde{z}}, \quad A_{42} = -2\alpha_c C_s \frac{\partial \tilde{S}_c}{\partial \tilde{z}} \quad A_{43} = \alpha_c C_s \left( -K^2 \tilde{S}_c + \tilde{S}_c \beta_m \frac{\partial^2}{\partial \tilde{z}^2} \right) \quad (C.3d)$$

In (C.3), for given $\tilde{z}$, the following functions

$$\tilde{S}_c, \quad \frac{\partial \tilde{S}_c}{\partial \tilde{z}}, \quad \frac{\partial^2 \tilde{S}_c}{\partial \tilde{z}^2}, \quad \frac{\partial U_s}{\partial \tilde{z}}, \quad \frac{\partial U_0}{\partial \tilde{z}}$$
all are evaluated at $z$ computed by equation (15.18)

$$z = -kh + z_0 + \frac{kh - z_0}{2} (\bar{z} + 1).$$  \hspace{1cm} (C.4)
Bibliography


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