Moduli of Twisted Sheaves and Generalized Azumaya Algebras

by

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Abstract

We construct and describe compactified moduli stacks of Azumaya algebras on a smooth projective morphism $X \to S$. These stacks are the algebro-geometric version of the (suitably compactified) stacks of principal PGL_n-bundles and they also have strong connections to arithmetic. A geometric approach to the problem leads one to study stacks of (semistable) twisted sheaves. We show that these stacks are very similar to the stacks of semistable sheaves. This gives a way of understanding the structure of the stack of principal PGL_n-bundles and its coarse moduli space in terms of fairly well-understood spaces. In particular, when $X \to S$ is a smooth projective curve or surface over an algebraically closed field, our method yields concrete theorems about the structure of these stacks (at least as certain natural invariants are allowed to increase without bound). On the arithmetic side, we use the geometry and rationality properties of these moduli spaces to study a classical question about the Brauer group of a function field K, known as the "period-index problem": for which classes α in $\mathrm{Br}(K)$ of order n does there exist a division algebra D of rank n^2 with $[D] = \alpha$? We give an answer to this question when K is the function field of a curve or surface over an algebraically closed, finite, or local field and when α is an unramified Brauer class of order prime to the characteristic of K. In the general case, we relate the unramified period-index problem to rationality questions on Galois twists of moduli spaces of semistable sheaves.

Thesis Supervisor: A. J. de Jong
Title: Professor of Mathematics

To my grandfather

Joseph J. Katz

mentor, inspiration, and friend

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Contents

Chapter 1

Introduction

This thesis was originally motivated by the problem of constructing compactified moduli of Azumaya algebras on projective algebraic surfaces. Despite its abstruse description, this problem brings together several strands of algebraic geometry, ranging from the purely geometric techniques of complex algebraic geometry to the Frobenius-laden methods of arithmetic geometers. Given a scheme X, an *Azumaya algebra of degree* n on X is simply an étale form of the matrix algebra $\mathrm{M}_n(\mathscr{O}_X)$. By the Skolem-Noether theorem, the set of isomorphism classes of Azumaya algebras of degree n is in bijection with the set of PGL_n-torsors, i.e., principal PGL_n-bundles. Viewing these objects as algebras yields a connection to arithmetic, while viewing them as PGL_n-bundles gives an algebraic version of the study of holomorphic bundles, an active area of complex geometry. We will describe in this introduction how these two viewpoints interact and how we will use this interaction to relate the structure theory of moduli spaces of bundles to the arithmetic of the Brauer group of function fields.

The set of Azumaya algebras on X admits an equivalence relation such that the quotient is naturally a group, called the *Brauer group* of X. This generalizes to schemes a classical construction first codified by Brauer in the 1930s, but ultimately tracing its way back through Noether's "crossed product algebras" to Hamilton's (temporarily) controversial quaternions and Frobenius's study of central division algebras over $\mathbf{R}$. The Brauer group of a number field plays an essential role in class field theory. In the 1960s, Grothendieck, building on earlier work of Auslander, Buchsbaum, and Azumaya, extended the Brauer group to an invariant of schemes (in fact, of ringed topoi) [34, 35, 36]. It was immediately observed that the Brauer group acts as an *algebraic* repository for certain *transcendental* information. For example, one can see that if X is a smooth, simply connected projective surface over the complex numbers, then $\mathrm{Br}(X) = (\mathbf{Q}/\mathbf{Z})^{b_2-\rho}$, where $b_2 = \dim \mathrm{H}^2(X, \mathbf{Q})$ is the second Betti number and $\rho = \mathrm{rk}\,\mathrm{NS}(X)$ is the dimension of the image of $\mathrm{Pic}(X) \otimes \mathbf{Q}$ in $\mathrm{H}^2(X, \mathbf{Q})$. Thus, the corank of $\mathrm{Br}(X)$ measures how many cohomology classes are non-algebraic. (One can also prove similar statements using ℓ-adic étale cohomology, as long as one works with elements of the Brauer group of order prime to the characteristic of the base field.) The fact that the Brauer group encodes non-algebraic cohomology classes is not special to surfaces; it is a general phenomenon.

Artin and Tate discovered that, over a finite field, the Brauer group of a sufficiently nice fibered surface $\pi : X \to C$ is isomorphic to the Tate-Shafarevich group of the Jacobian of the generic fiber of π. In particular, if $\pi : E \to \mathbf{P}^1$ is an elliptic surface (with a section), then $\mathrm{Br}(E) \cong Ш(k(\mathbf{P}^1), E_\eta)$ [36, 72]. This connects the Tate-Shafarevich conjecture for elliptic curves over equicharacteristic global fields with the Brauer group. Artin conjectured that any scheme proper over Spec $\mathbf{Z}$ has finite Brauer group, a conjecture which has proven to be extremely hard. It has been verified in many special cases, but there is as yet no unified approach which is expected to work universally (with enough effort).

One might hope that studying Azumaya algebras and their moduli will shed light on the Brauer group. For example, given a Brauer class α over a field K, it is natural to ask how large a field extension $L \supset K$ one must make to trivialize α. Similarly, given a projective variety X and a Brauer class α on X, one can ask for the minimal degree of a generically finite cover splitting α. When X is a surface, this number is the same as the minimal degree of an Azumaya algebra in the class α. This, in turn, is a question about the existence of points in a moduli space of Azumaya algebras of a given degree on X.

Of course, it is unlikely that such moduli spaces are compact. Thus, one should seek from the start a compactification of the moduli problem. This is in general a delicate task; one would like to add as few extra objects as possible. The most naïve approach to compactifying moduli of Azumaya algebras is to simply allow the space to parametrize flat limits of Azumaya algebras. Unfortunately, it is difficult to characterize such algebras. The model for this difficulty is given by considering $E = \mathrm{End}(M)$, where M is the first syzygy module of the maximal ideal of $A := k[\![x, y, z]\!]$. (In fact, this situation models what happens étale locally in general in degenerate fibers of a flat limit of Azumaya algebras on a smooth surface.) In an ideal world, one would hope that the natural map $E/xE \hookrightarrow \mathrm{End}(M/xM)$ is an isomorphism; unfortunately, it is easy to see that the cokernel is non-trivial. De Jong observed, on the other hand, that the dream comes true in the derived category: $\mathbf{R}\mathrm{End}(M) \overset{\mathbf{L}}{\otimes} A/xA \cong \mathbf{R}\mathrm{End}_{A/xA}(M/xM)$. He therefore suggested that one might somehow glue these derived algebras together to arrive at a nice compactification. Since gluing is notoriously difficult in the derived category, it takes a great deal of work to translate this intuition into mathematics.

The underlying philosophy, that a "generalized Azumaya algebra" is given locally by a module, with gluing only taking place up to maps which are trivial on endomorphism algebras, immediately brings one into contact with *twisted sheaves*. In order to describe these objects, let us consider a different way of describing Azumaya algebras. The Skolem-Noether theorem tells us that the automorphism sheaf of $\mathrm{M}_n(\mathscr{O}_X)$ is $\mathrm{PGL}_{n,X}$ (in the fpqc topology), so we see that the set of Azumaya algebras of degree n is in bijection with the set $\mathrm{H}^1(X, \mathrm{PGL}_n)$ of étale PGL_n-torsors. Consider the central extension $1 \to \mathbf{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$. Giraud's non-abelian cohomology [30] assigns to this exact sequence an exact sequence in cohomology, of which the relevant segment is

$$\mathrm{H}^1(X, \mathbf{G}_m) \to \mathrm{H}^1(X, \mathrm{GL}_n) \to \mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mathbf{G}_m).$$

Given a class $T \in \mathrm{H}^1(X, \mathrm{PGL}_n)$, the coboundary in $\mathrm{H}^2(X, \mathbf{G}_m)$ is interpretable in Giraud's theory as follows: thinking of T as a PGL_n-torsor, one can produce a $\mathbf{G}_m$-gerbe (in the étale topology) of lifts of T to a GL_n-torsor. This is a certain algebraic stack $\mathscr{X} \to X$, and one sees from its definition that it universally solves the problem of reducing the structure group of T to GL_n, in the sense that there is a GL_n-torsor $\mathscr{V}^*$ on $\mathscr{X}$ and an isomorphism of the PGL_n-torsor associated to $\mathscr{V}^*$ with T. The left-hand side of the cohomology sequence says that $\mathscr{V}^*$ is unique up to tensoring with a $\mathbf{G}_m$-torsor pulled back from X. The algebraic stack $\mathscr{X}$ has inertia stack $\mathbf{G}_m$, so the sheaf $\mathscr{V}^*$ comes equipped with a $\mathbf{G}_m$-action. In fact, one can see that this action agrees with the natural action given by the inclusion $\mathbf{G}_m \hookrightarrow \mathrm{GL}_n$. Replacing $\mathscr{V}^*$ with the associated locally free $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{V}$ of rank n, we see that the action of $\mathbf{G}_m$ on $\mathscr{V}^*$ becomes the action on $\mathscr{V}$ by scalars. A locally free sheaf on $\mathscr{X}$ whose inertial $\mathbf{G}_m$-action equals the scalar action is called a *twisted sheaf*. (There is a more elementary definition of these objects using Čech cohomology which we will discuss in section 2.1.3.) The Azumaya algebra associated to T is then precisely the algebra $\mathscr{E}nd(\mathscr{V})$.

The observation that an Azumaya algebra is the endomorphism sheaf of a locally free sheaf on a certain stack over X immediately suggests a globalization of de Jong's idea: a generalized Azumaya algebra should have the form $\mathbf{R}\mathscr{E}nd(\mathscr{F})$, where $\mathscr{F}$ is now a torsion free twisted sheaf. This also yields a surjective 1-morphism from the stack $\mathbf{Tw}_{\mathscr{X}}$ of torsion free twisted sheaves to the stack $\mathbf{GAz}_X$ of generalized Azumaya algebras on X sending $\mathscr{F}$ to $\mathbf{R}\mathscr{E}nd(\mathscr{F})$. One then expects to deduce information about $\mathbf{GAz}$ from corresponding information about $\mathbf{Tw}$.

This connects the study of $\mathbf{GAz}$ with another approach to compactifying the moduli of Azumaya algebras due to Artin. He made the observation that given two Azumaya algebras $\mathscr{A}$ and $\mathscr{B}$ of the same rank and Brauer class, there is an invertible right $\mathscr{A}$-module $\mathscr{L}$ such that $\mathscr{B} = \mathscr{E}nd_{\mathscr{A}}(\mathscr{L})$; furthermore, $\mathscr{L}$ is unique up to tensoring with an invertible sheaf L on X. This similarly yields a covering of the stack of Azumaya algebras by the stack of $\mathscr{A}$-modules. The compactification of the stack of Azumaya algebras should come from the natural compactification of the stack of invertible right $\mathscr{A}$-modules using torsion free $\mathscr{A}$-modules of rank 1, by taking the quotient by the action of $\mathrm{Pic}(X)$ in an appropriate sense.

It is well-known that $\mathscr{A}$-modules are the same thing as twisted sheaves [6]. Thus, the above discussion of $\mathbf{Tw} \to \mathbf{GAz}$ is displaying precisely the compactification suggested by Artin. In other words, Artin and de Jong's ideas are essentially the same. What have we gained by rephrasing everything in terms of sheaves on gerbes? The point is that a gerbe $\mathscr{X} \to X$ is a mildly stacky version of X itself. Thus, one expects from this point of view that sheaves on $\mathscr{X}$ should behave similarly to sheaves on X. Furthermore, by working with sheaves of $\mathscr{O}_{\mathscr{X}}$-modules, one can immediately apply classical tools from the theory of sheaves: Chern classes, the Riemann-Roch theorem, elementary transformations, the Harder-Narasimhan filtration, the deformation theory of Illusie, formation of reflexive hulls, Quot schemes, the Bogomolov inequality, and so on [40]. While it is true that many of these constructions have non-commutative analogues, certain technical difficulties disappear when working with $\mathscr{O}_{\mathscr{X}}$-modules. We also have a large supply of theorems – about semistable sheaves on surfaces – that we can now try to translate to the category of (semistable) *twisted* sheaves. This turns out to be entirely successful, so much so that at the time of this writing we have not yet gotten close to exhausting the statements amenable to such a translation.

While it is true that the geometry of the space of twisted sheaves is closely related to that of the space of untwisted sheaves, it is useful to recall that these two spaces capture different *arithmetic* information. This gives us a chance to apply the classical geometric methods used to study vector bundles to arithmetic problems about the Brauer group of a variety. In particular, in this thesis we will apply these techniques to a class of problems known as "period-index problems" – determining how large a function field extension one must make to trivialize a Brauer class.

Let us give a brief outline of the contents of this thesis: in chapter 2 we discuss the theory of twisted sheaves on topoi. These objects were first discovered by Giraud [30] and recently given an extensive treatment (when the base topos is a projective variety) in the thesis of Căldăraru [6]. We present a more general approach than Căldăraru which is less general than Giraud (and thus easier to apply), with an eye toward studying twisted sheaves on stacks. We also discuss fibered Morita theory and the Brauer group of a ringed topos. We warn the reader that section 2.1 is quite abstract and is likely to be unpleasant; it exists primarily as a foundational section, and we encourage the reader to skip it or skim it briefly for definitions.

In section 2.2, we study quasi-coherent twisted sheaves on schemes and we apply the abstract theory to re-prove several classical results on the Brauer group [28, 35, 36, 39] as well as some basic lemmas on the derived category of twisted sheaves paralleling the work of Neeman [14, 56]. We also discuss deformations and obstructions, in preparation for the discussion of moduli.

In chapter 3 we develop the theory of generalized Azumaya algebras, including the derived Skolem-Noether theorem. We then compare the stack of twisted sheaves with the stack of generalized Azumaya algebras, realizing rigorously the intuition described above.

In chapter 4, we use Artin's representability theorem [9] to prove that the stacks of twisted sheaves and generalized Azumaya algebras are algebraic stacks (in the sense of Artin). We also lay the foundation for the study of semistable twisted sheaves and generalized Azumaya algebras, and we show that semistability and stability define open substacks. Along the way, we study Quot spaces for twisted sheaves and applications of Geometric Invariant Theory (GIT) when working with twisted sheaves on surfaces. Using these results, we describe a scheme corepresenting the stack of semistable generalized Azumaya algebras on a surface.

In chapter 5, we take up the study of moduli of twisted sheaves on curves and surfaces. We prove a comparison theorem between the stack of semistable twisted sheaves and a Galois twist of the stack of semistable "untwisted" sheaves on a curve. We use this to realize the Artin-Tate isomorphism $\mathrm{Br}(X) \cong \text{Ш}(k(C), \mathrm{Pic}_{X/C})$ (described above) via twisted Picard spaces. Then we study moduli of twisted sheaves on surfaces. Building on recent work of Langer [50, 51] in the untwisted context, we prove a version of the Bogomolov inequality for twisted sheaves and some restriction theorems for semistable twisted sheaves. We then study the asymptotic properties of the moduli stacks of twisted sheaves, culminating in a proof that these stacks are eventually (for high enough second Chern class) irreducible generically smooth local complete intersections.

Finally, in chapter 6, we apply the structure theory of chapter 5 to prove period-index results for curves over function fields and local fields, and for surfaces over finite fields and local fields, by reducing these questions to rationality questions on the relevant moduli spaces.

Notation

Following standard conventions, we use $=$ for canonical isomorphisms.

Every locally free sheaf is assumed to have finite rank everywhere.

As should be universal by now, "algebraic stack" will mean "algebraic stack in the sense of Artin." Deligne-Mumford stacks will be called "DM stacks." All algebraic stacks are quasi-separated, as is any base scheme appearing in this thesis.

In order to prevent psychological problems, when given a topos X, we will write U in place of $X_{/U}$ to stand for the restriction of X to the object $U \in X$. For the sake of intuition, we will also interchangeably refer to "sheaves on X" and "objects in X" depending upon the context.

Following Huybrechts and Lehn [40], we use the notation "hom" for "$\dim \mathrm{Hom}$" and "ext" for "$\dim \mathrm{Ext}$." In general, we have tried to keep notations in common with their book when treating the twisted analogues of classical theorems so beautifully discussed there.

There is one pedantic grammatical convention we adopt which we hope will spread: a number with mathematical meaning is always written as a numeral, occasionally in contradiction to accepted "rules" of grammar (e.g., "rank 1", "characteristic 0"). The underlying philosophy is that numbers, when used in a mathematical context, are not "the same" as numbers when used in ordinary language. As such, they should be distinguished from their real-life counterparts.

Chapter 2

Twisted sheaves

2.1 Preliminaries: twisted sheaves on ringed topoi

In this section, we lay the foundations for the theory of twisted sheaves on algebraic spaces and stacks. The reader will note that much of the first three sections is written in the language of ringed topoi. We encourage those uncomfortable with this notion to substitute "ringed site" or even "ringed space" for "ringed topos"; the exposition will remain more or less the same after this substitution (but the reader should note that sites larger than the Zariski site of a scheme are essential for the theory to actually be interesting). One reason to write in this degree of generality is to make the theory apply to algebraic stacks, where one can only really understand the theory of sheaves from the topos-theoretic point of view.

In order to link Giraud's ideas with subsequent developments in the theory of algebraic stacks [52], we review foundations on the sites associated to a stack, sheaves on those sites, and classifying topoi associated to gerbes on (ringed) topoi. We only consider stacks in groupoids in this thesis; the task of extending the results to stacks in arbitrary (small) categories is left to the reader. (It will primarily consist of adding the word "Cartesian" in a few places.)

2.1.1 Sheaves and gerbes on stacks

Let X be a topos and $F : \mathscr{S} \to X$ a stack on X. The topology on X naturally induces a topology on $\mathscr{S}$.

Definition 2.1.1.1. The *site of* $\mathscr{S}$, denoted $\mathscr{S}^s$, has as underlying category

> Objects: morphisms $f : S \to \mathscr{S}$ of fibered categories over X, where S ranges over all sheaves on (=objects of) X
>
> Morphisms: a morphism from $f : S \to \mathscr{S}$ to $g : S' \to \mathscr{S}$ is a pair (φ, ψ) where $\varphi : S \to S'$ is a morphism in X and $\psi : f \xrightarrow{\sim} g \circ \varphi$ is a 2-isomorphism.

A covering is given by a morphism (φ, ψ) with φ a covering.

Remark 2.1.1.2. The site of $\mathscr{S}$ is also naturally a stack on X, and there is a 1-isomorphism of stacks $\mathscr{S}^s \to \mathscr{S}$. The stack $\mathscr{S}^s$ has the extra property that its natural pullback functors commute exactly, rather than up to coherent homotopies, i.e., $\mathscr{S}^s$ is *split* as a fibered category. ♦

Definition 2.1.1.3. The *classifying topos* of $\mathscr{S}$, denoted $\widetilde{\mathscr{S}}$, is the topos of sheaves on the site of $\mathscr{S}$.

There is a morphism of topoi $\pi : \widetilde{\mathscr{S}} \to X$: given a sheaf $\mathscr{F}$ on X, one gets a sheaf $\pi^*\mathscr{F}$ on $\widetilde{\mathscr{S}}$ by assigning to $f : S \to \mathscr{S}$ the object $\mathscr{F}(S)$. The obvious exactness properties show that this is the pullback of a morphism of topoi. In particular, when X is ringed, say by $\mathscr{O}$, $\widetilde{\mathscr{S}}$ is naturally ringed by $\pi^*\mathscr{O}$.

Remark 2.1.1.4. It is not difficult to check that the topos we have called the classifying topos agrees with Giraud's definition: $\widetilde{\mathscr{S}} = \mathrm{Cart}(\mathscr{S}, \mathrm{Fl(X)})$ [30, §5.1]. We leave this to the interested reader. ♦

Lemma 2.1.1.5. *Given a map $p : \mathscr{T} \to \mathscr{S}$ of X-stacks, there is an induced morphism $\widetilde{p} : \widetilde{\mathscr{T}} \to \widetilde{\mathscr{S}}$ of topoi.*

Proof. Given a sheaf F on $\widetilde{\mathscr{S}}$ and an object $f : T \to \mathscr{T}$ in the site of $\mathscr{T}$, set $\widetilde{p}^*F(f) = F(p \circ f)$. It is immediate that this is a sheaf on $\mathscr{T}$. The requisite exactness properties of this functor are readily checked, yielding a morphism of topoi as required. □

2.1.1.6. Given a stack $\mathscr{S} \to X$, there is an associated stack $\mathscr{I}(\mathscr{S}) \to \mathscr{S}$ called the *inertia stack*.

Lemma 2.1.1.7. *There is a natural equivalence of categories between sheaves on $\mathscr{S}^s$ and stacks $\mathscr{T}$ on X with morphisms $\mathscr{T} \to \mathscr{S}$ whose fibers are essentially discrete categories.*

The last condition of the lemma is equivalent to the requirement that for any object $S \to \mathscr{S}$, the fiber product $\mathscr{T} \times_{\mathscr{S}} S$ is the stack associated to a sheaf on S.

Proof. Laumon and Moret-Bailly [52, §14]. (Note that they call a sheaf on $\widetilde{\mathscr{S}}$ a "construction locale.") □

Definition 2.1.1.8. The assignment $(f : S \to \mathscr{S}) \mapsto \mathscr{A}ut(f)$ is a sheaf. The corresponding stack is denoted $\mathscr{I}(\mathscr{S}) \to \mathscr{S}$ and called the *inertia stack* of $\mathscr{S}$.

When X is the topos of sheaves on the big étale topology on affine schemes over a fixed base B and $\mathscr{S}$ is an algebraic stack, then one easily sees that $\mathscr{I}(\mathscr{S})$ is also an algebraic stack and the morphism $\mathscr{I}(\mathscr{S}) \to \mathscr{S}$ is representable, quasi-compact, and separated. In fact, it is easy to see that $\mathscr{I}(\mathscr{S}) = \mathscr{S} \times_{\mathscr{S}\times\mathscr{S}} \mathscr{S}$, the fiber self-square of the diagonal of $\mathscr{S}$.

Lemma 2.1.1.9. *Given a 1-morphism $f : \mathscr{S} \to \mathscr{S}'$ of stacks, there is an induced map $\mathscr{I}(\mathscr{S}) \to f^*\mathscr{I}(\mathscr{S})$ in $\widetilde{\mathscr{S}}$.*

Proof. As noted in 2.1.1.5, the sections of f^*F over $g : S \to \mathscr{S}$ are the sections of F over $f \circ g$. On the other hand, given an automorphism η of g, one gets an automorphism $f(\eta)$ of $f \circ g$ by functoriality. Combining these statements results in the lemma. □

Lemma 2.1.1.10. *Let F be a sheaf on $\widetilde{\mathscr{S}}$. There is a natural right group action $\mu : F \times \mathscr{I}(\mathscr{S}) \to F$.*

Proof. Let $f : S \to \mathscr{S}$ be an object of $\mathscr{S}^s$. A section of $\mathscr{I}(\mathscr{S})$ over S is an automorphism ψ of f. Given a section $s \in F(f)$, the action of ψ on s is given by pulling back along the natural map $\psi^* : F(f) \xrightarrow{\sim} F(f)$ (induced by the contravariant nature of sheaves). Since automorphisms of f are deemed by those in charge to act on the left, we are forced to have them act on the right on sections of F. (This will cause a headache in a moment.) □

Example 2.1.1.11. If $F = \mathscr{I}(\mathscr{S})$, then it is an easy exercise to see that the natural action is given by conjugation. Thus, if for example every automorphism group is abelian then the inertia stack acts trivially on itself. ◊

There is actually a more general version of 2.1.1.10 which holds for stacks. We will not have much use for it in the sequel, so we will not give a full proof.

Lemma 2.1.1.12. *Let $\mathscr{T} \to \mathscr{S}$ be a 1-morphism of stacks. There is an action of $\mathscr{I}(\mathscr{S})$ on $\mathscr{T}$ by 1-automorphisms which is strongly homotopy-commutative.*

Proof. We may replace $\mathscr{S}$ and $\mathscr{T}$ by $\mathscr{S}^s$ and $\mathscr{T}^s$ and thus assume that both stacks are split over X and the morphism $\mathscr{T} \to \mathscr{S}$ arises from a split stack on $\mathscr{S}^s$. We may mimic the construction of the action in 2.1.1.10 to get the desired result. Replacing $\mathscr{S}$ and $\mathscr{T}$ as described gives enough rigidity to the pullback functors that the action is homotopy-commutative. □

Thus, for example, one can see that a G-gerbe on $\mathscr{S}$ (see 2.1.1.13 below) with G abelian gives rise to a G-torsor on the stack $\mathscr{I}(\mathscr{S})$. This local system arises in the study of quantum cohomology [73].

2.1.1.13. We will be concerned throughout this thesis with gerbes.

Definition 2.1.1.14. The stack $\mathscr{S}$ is a *gerbe* on X if

1. For any $U \in X$ there exists a covering $U' \to U$ such that $\mathscr{S}_{U'} \neq \emptyset$.

2. For any $U \in X$ and any $s, s' \in \mathscr{S}_U$, there exists a covering $U' \to U$ such that $s|_U$ is isomorphic to $s'|_{U'}$.

In looser language, $\mathscr{S}$ has local sections everywhere and any two sections are locally isomorphic. There is a "moduli-theoretic" interpretation of this definition.

Definition 2.1.1.15. The *sheaf associated to $\mathscr{S}$*, denoted $\mathrm{Sh}(\mathscr{S})$, is the sheafification of the presheaf whose sections over $U \in X$ are isomorphism classes of objects in the fiber category $\mathscr{S}_U$.

Lemma 2.1.1.16. *The stack $\mathscr{S}$ is a gerbe on X if and only if the natural map $\mathrm{Sh}(\mathscr{S}) \to e_X$ is an isomorphism in X.*

Here e_X denotes the final object of the topos X. This will often be written as X by abuse of notation.

Proof. Suppose $\mathscr{S}$ is a gerbe. By functoriality of the natural map, it is enough to demonstrate the claim when $\mathscr{S}$ has a global section σ over X. But then every local section is locally isomorphic to σ, hence $\mathrm{Sh}(\mathscr{S})$ is a singleton and the natural map is an isomorphism.

Suppose conversely that $\mathrm{Sh}(\mathscr{S}) \to e_X$ is an isomorphism. In particular,

$$\mathrm{Sh}(\mathscr{S})(U) = \{\emptyset\}$$

for any $U \in X$. By the definition of $\mathrm{Sh}(\mathscr{S})$ and of sheafification, this says precisely that conditions 1 and 2 of 2.1.1.14 is satisfied. □

In moduli-theoretic terms, 2.1.1.16 roughly says that a stack is a gerbe over its moduli space if and only if the moduli space represents the sheaf associated to the moduli problem. This is not the same as the requirement that the moduli space be fine; the stack of stable sheaves with fixed determinant is a gerbe over its moduli space, but there is a well-known Brauer obstruction to the existence of a universal object (from the non-abelian cohomological point of view, the obstruction is precisely that gerbe!). In the Deligne-Mumford case, 2.1.1.16 may be interpreted as saying that the stack is a gerbe over the moduli space iff the completion of the strict localization of any point of the moduli space is identified via the projection with the universal deformation space of a lift of that geometric point into the stack. Finally, this is simply the statement that a universal object exist étale-locally on the moduli space.

Lemma 2.1.1.17. *If $\mathscr{S} \to X$ is a gerbe and F is a sheaf on $\mathscr{S}$ such that the inertia action $F \times \mathscr{I}(\mathscr{S}) \to F$ is trivial, then F is naturally the pullback of a unique sheaf on X up to isomorphism.*

Proof. We claim that $\pi^*\pi_*F \to F$ is an isomorphism. To verify this, it suffices to work locally on X, so we may assume that $\mathscr{S}$ has a section. One can then check using the hypothesis on the action that the pullback of F along this section equals the pushforward of F along the structure morphism. The result follows. □

Remark 2.1.1.18. This holds more generally when X is the coarse moduli space of a Deligne-Mumford stack $\mathscr{S}$ (with the action of inertia being studied in the big étale topology), but the proof is slightly more difficult: it follows without too much difficulty from the étale local structure of the stack as a finite group quotient stack [52, §6], [43]. ♦

Lemma 2.1.1.19. *If $\pi : \mathscr{S} \to X$ is a gerbe and $\mathscr{I}(\mathscr{S})$ is an abelian sheaf on $\mathscr{S}$, then there is an abelian sheaf A on X and an isomorphism $\widetilde{\pi}^*A \cong \mathscr{I}(\mathscr{S})$ as objects of $\widetilde{\mathscr{S}}$.*

Proof. This is an application of 2.1.1.17. □

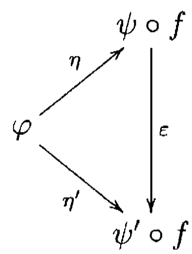
2.1.1.20. Given a stack $\mathscr{S}$, we can now try to think about stacks on the topos $\widetilde{\mathscr{S}}$. It turns out that this does not add any further complexity to the situation. We first recall a basic fact.

Lemma 2.1.1.21. *There is a natural 1-equivalence between the 2-category of stacks on the site of $\mathscr{S}$ and the 2-category of stacks on $\widetilde{\mathscr{S}}$.*

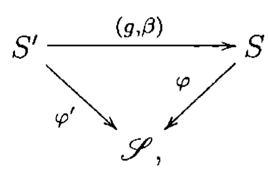
Proof. A full proof may be found in [30, II.1.3.3]. The functor taking a stack on $\widetilde{S}$ and giving a stack on the site of $\mathscr{S}$ is just the restriction functor. In the other direction (assuming we work in a fixed universe), a stack on the site of $\mathscr{S}$ gives a groupoid object in $\widetilde{\mathscr{S}}$ in the usual way, thus giving rise to a stack on $\widetilde{\mathscr{S}}$. □

Given a stack $\mathscr{T} \to X$ and a 1-morphism $f : \mathscr{T} \to \mathscr{S}$, we can construct a stack $\mathscr{T}_{\mathscr{S}} \to \mathscr{S}^s$ which is "the stack of sections of f."

Definition 2.1.1.22. The *stack of sections of f* has as objects over $\varphi : S \to \mathscr{S}$ pairs (ψ, η), where $\psi : S \to \mathscr{T}$ is an object of $\mathscr{T}^s$ and $\eta : \varphi \to \psi \circ f$ is an isomorphism. An isomorphism $(\psi, \eta) \xrightarrow{\sim} (\psi', \eta')$ over φ is a 2-isomorphism $\varepsilon : \psi \xrightarrow{\sim} \psi'$ such that the diagram



commutes. Given a diagram



where $\beta : \varphi' \xrightarrow{\sim} \varphi \circ g$ is a 2-isomorphism, the pullback of (ψ, η) along (g, β) is defined to be $(\psi \circ g, \alpha)$, where

$$\alpha : \varphi' \xrightarrow{\sim} \varphi \circ g \xrightarrow{\sim} (f \circ \psi) \circ g \xrightarrow{\sim} f \circ (\psi \circ g)$$

is the composition of β with the translation of η by g and the natural associativity isomorphism.

It is immediate that $\mathscr{T}_{\mathscr{S}}$ is a stack on $\mathscr{S}^s$. The reader familiar with the construction of the 1-fiber product given in [52, 2.2.2] will recognize the fibered category underlying $\mathscr{T}_{\mathscr{S}}$ as the natural 1-fiber product $\mathscr{T} \times_{\mathscr{S}} \mathscr{S}^s$. We will use this fact to simplify proofs below.

Proposition 2.1.1.23. *The functor* $(p : \mathscr{T} \to \mathscr{S}) \mapsto (\mathscr{T}_{\mathscr{S}} \to \mathscr{S}^s)$ *gives a 1-equivalence of 2-categories between* X*-stacks* $\mathscr{T}$ *over* $\mathscr{S}$ *and stacks on* $\mathscr{S}^s$.

Lemma 2.1.1.24. *If* $h : C \to C'$ *and* $g : C' \to C''$ *are categories fibered in groupoids, then* $f : C \to C''$ *is naturally a category fibered in groupoids.*

Proof. Up to equivalence, we may replace C' by a split fibered category over C''; one can easily check that pulling back C along any equivalence yields an equivalent category. We may then replace C by a split fibered category over C'. Once this has been done, it is clear how to define the pullback functors for $f : C \to C''$: for $c \in C$ and $c'' \to f(c)$, the pullbacks for $g : C' \to C''$ yield $c' \to g(c)$ Cartesian over $c'' \to f(c)$. Now the pullbacks for $h : C \to C'$ yields $\tilde{c} \to c$ Cartesian over $c' \to g(c)$. It now follows that $\tilde{c}$ is a pullback relative to f. (Note that the proof given here has nothing to do with categories fibered in groupoids, but is in fact completely general to fibered categories. The point is that if both fibered categories are fibered in groupoids, then the composition is also fibered in groupoids.) □

Proof of 2.1.1.23. An inverse to the given functor is the forgetful functor (which yields an X-stack by 2.1.1.24). The rest follows from the fact that $\mathscr{S}^s \to \mathscr{S}$ is a 1-isomorphism of stacks and that $\mathscr{T}_{\mathscr{S}} \cong \mathscr{T} \times_{\mathscr{S}} \mathscr{S}^s$. □

Remark 2.1.1.25. It is easy to see that the equivalence of 2.1.1.23 respects 1-fiber products. In other words, given two stacks $\mathscr{T}, \mathscr{T}' \to \mathscr{S}$, the stack on $\mathscr{T}^s$ associated to $\mathscr{T} \times_{\mathscr{S}} \mathscr{T}' \to \mathscr{T}$ is the pullback of the stack associated to $\mathscr{T}' \to \mathscr{S}$ on $\mathscr{S}^s$. Thus, for example, given a gerbe $G \to \mathscr{S}^s$, the morphism of stacks $\mathscr{G} \to \mathscr{S}$ representing G has the property that for any $S \to \mathscr{S}$, the pullback $\mathscr{G} \times_{\mathscr{S}} S \to S$ is a gerbe (equivalent to the restriction of G to S in the site $\mathscr{S}^s$). ♦

2.1.2 Twisted sheaves

Let $(X, \mathscr{O})$ be a ringed topos, A a sheaf of commutative groups on X, and $\chi : A \to \mathbf{G}_m$ a character.

Definition 2.1.2.1. An *A-gerbe* on X is a gerbe $\mathscr{S} \to X$ along with an isomorphism $A_{\mathscr{S}} \xrightarrow{\sim} \mathscr{I}(\mathscr{S})$ in $\widetilde{\mathscr{S}}$.

When A is non-commutative, this definition is not correct. One must instead choose any isomorphism as in the definition in the category of liens on $\mathscr{S}$ rather than the category of sheaves. The basic reason for this may be seen by thinking about the gerbe BG for a non-commutative group G. In general, the automorphism group of a left G-torsor is an inner form of G, not G itself. (The stack of liens on a topos is the universal stack receiving a 1-morphism from the stack of groups on that topos such that two inner forms naturally map to isomorphic objects.) This is of course described in great detail in [30].

Given a cohomology class $\alpha \in \mathrm{H}^2(X, A)$, there is a corresponding equivalence class of A-gerbes on X. We will fix such a class α and an A-gerbe $\mathscr{X} \to X$ in what follows.

The goal of this section is to single out a subcategory of sheaves on $\mathscr{X}$ which will play a fundamental role in what follows.

Given an $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{F}$, the resulting action $m : \mathbf{G}_m \times \mathscr{F} \to \mathscr{F}$ yields an associated right action $m' : \mathscr{F} \times \mathbf{G}_m \to \mathscr{F}$ with $m'(s, \varphi) = m(\varphi^{-1}, s)$. This will always be called the *associated right action.*

Definition 2.1.2.2. A *d-fold χ-twisted sheaf* on $\mathscr{X}$ is an $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{F}$ such that the natural action $\mu : \mathscr{F} \times A \to \mathscr{F}$ given by the A-gerbe structure makes the diagram

$$\begin{array}{ccc} \mathscr{F} \times A & \longrightarrow & \mathscr{F} \\ {\scriptstyle \chi^d}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathscr{F} \times \mathbf{G}_m & \xrightarrow{m'} & \mathscr{F} \end{array}$$

commute, where $\chi^d(s) = \chi(s)^d$. A 1-fold twisted sheaf will be called simply a twisted sheaf.

We will see below that if $\mathscr{X}^{(d)}$ is a gerbe representing $d \cdot \alpha \in \mathrm{H}^2(X, A)$, then twisted sheaves on $\mathscr{X}^{(d)}$ are equivalent to d-fold twisted sheaves on $\mathscr{X}$.

An attentive reader may have noticed that the morphism χ yields a "change of structure group" for the gerbe $\mathscr{X}$:

Lemma 2.1.2.3. *Let $f : A \to B$ be a morphism of abelian sheaves and $\alpha \in \mathrm{H}^2(X, A)$ a cohomology class with direct image $f_*(\alpha) = \beta \in \mathrm{H}^2(X, B)$. Given gerbes $\mathscr{X}_\alpha$ and $\mathscr{X}_\beta$ representing α and β, there is a 1-morphism $F : \mathscr{X}_\alpha \to \mathscr{X}_\beta$ over X such that for any section $\sigma : S \to \mathscr{X}_\alpha$, the induced morphism $A_S = \mathrm{Aut}(\sigma) \to \mathrm{Aut}(F(\sigma)) = B_S$ is f_S.*

Proof. The existence of the morphism of stacks $\mathscr{X}_\alpha \to \mathscr{X}_\beta$ is part of Giraud's theory of non-abelian cohomology. (Using 2.1.2.5 and 2.1.2.6 below, we may see this explicitly (non-canonically): let $U_\bullet$ be a hypercovering [5, Exposé V.7] which splits the gerbe α (hence also β). Change of structure group via f then gives rise to a functor on the stack of twisted torsors which satisfies the conditions of the lemma.) □

Under this identification, the stack of χ-twisted sheaves is identified with the stack of ι-twisted sheaves, where $\iota : \operatorname{im} \chi \hookrightarrow \mathbf{G}_m$ is the natural inclusion of the image. Thus, we have gained very little but canonicity by our formalism. However, one might in the future try something similar when $\mathscr{S}$ is not a gerbe and the inertia stack $\mathscr{I}(\mathscr{S})$ is *not* constant, in which case this setup is the correct one. Such a study is related to moduli of ramified Azumaya algebras and the ramified period-index problem. These issues will be explored in later work.

In preparation for the next section, we use the yoga of twisted sheaves to give an alternative description of any A-gerbe in terms of certain sheaves of A-modules on the associated site. Let $\mathscr{S}$ be an A-gerbe in what follows. We think of the inertia stack as a sheaf of groups on $\widetilde{\mathscr{S}}$.

Definition 2.1.2.4. A right A-torsor T on $\widetilde{\mathscr{S}}$ is *twisted* if the action of the inertia sheaf $T \times A \to T$ equals the A-torsor action. The fibered category that associates to

$S \in X$ the category of twisted right A-torsors on $\widetilde{\mathscr{S} \times_X S} = \widetilde{\mathscr{S}}_{/S}$ is a stack $\mathscr{T} \to X$, the *stack of twisted right A-torsors*.

The fact that $\mathscr{T}$ is a stack on X follows from the fact that any morphism of topoi is a stack, along with the fact that being twisted is a local condition on the base of a torsor.

Proposition 2.1.2.5. *There is a natural 1-isomorphism $\mathscr{S} \to \mathscr{T}$.*

Proof. We assume $\mathscr{S}$ is split as a fibered category for the sake of simplicity. Given a section $f : S \to \mathscr{S}$ we construct a section T of $\mathscr{T}$ over $S \in X$, i.e. a twisted right A-torsor on $\mathscr{S} \times_X S$. Given any $\pi : S' \to S$ and a lift $p : S' \to \mathscr{S}$, let $T((\pi, p))$ be the set of isomorphisms $f \circ \pi \xrightarrow{\sim} p$. Since $\mathscr{S}$ is a stack, one sees that this defines a sheaf on $\mathscr{S} \times_X S$. Furthermore, the inertia action on p makes T into a right A-torsor, hence into a twisted right A-torsor!

This construction is clearly local on X, so to show that it defines a 1-isomorphism, it suffices to do so under the assumption that $\mathscr{S}$ has a global section $\sigma : X \to \mathscr{S}$. In this case, given $T : S \to \mathscr{T}$, we can construct an element of $\mathrm{B}A$ (an ordinary right A-torsor on S) by taking $\mathrm{Isom}(P, T)$, where P is the twisted right A-torsor on X whose sections over $g : S' \to \mathscr{S}$ are $\mathrm{Isom}(\sigma, g)$. This defines a 1-isomorphism $\mathscr{T} \to \mathrm{B}A$. Composing the two 1-morphisms yields the natural map $\mathscr{S} \to \mathrm{B}A$ which is known to be a 1-isomorphism. □

There is an alternative way to describe the stack $\mathscr{T}$ which will be of use for us in comparing our definition of twisted sheaves with the earlier definition of Căldăraru. Let $x : U_0 \to \mathscr{S}$ be a section and $U_1 \to U_0 \times_X U_0$ a covering over which the two pullbacks of x are isomorphic. Using the theory of hypercoverings [5, Exposé V], we can find a simplicial object $U_\bullet$ with an augmentation to X containing U_0 and (an appropriately refined) U_1. Choosing an isomorphism $\psi : x_0 \xrightarrow{\sim} x_1$, one finds that $\psi_{02}^{-1}\psi_{01}\psi_{12} = a \in A$ gives a 2-cocycle in A. In fact, as we will show below, one can reconstruct the gerbe $\mathscr{S}$ (up to equivalence) from this datum.

Lemma 2.1.2.6. *A twisted right A-torsor on $S \in X$ is given by a right A-torsor $\mathscr{L}$ on U_{0S} and an isomorphism $\varphi : \mathscr{L}_1 \xrightarrow{\sim} \mathscr{L}_0$ on U_{1S} such that the coboundary $\varphi_{02}^{-1}\varphi_{01}\varphi_{12} = a \in \mathrm{Aut}(\mathscr{L}_2)$.*

Proof. Given a twisted right A-torsor, pulling back along x_S gives $\mathscr{L}$. The isomorphism of $x_0 \to x_1$ gives rise to φ and the twisted condition shows that the coboundary acts as a. The other direction will be proven in 2.1.3.9 below. □

Remark 2.1.2.7. The results of the next section will show that *any* hypercovering $U_\bullet$ with a 2-cocycle a representing the cohomology class α has the property that there is a section $x : U_0 \to \mathscr{X}$ and an isomorphism $\varphi : x_0 \xrightarrow{\sim} x_1$ whose coboundary is a. Thus, the seemingly more general equivalence of $\mathscr{T}$ with the pairs $(\mathscr{L}, \varphi)$ defined on an *arbitrary* hypercovering $U_\bullet$ with 2-cocycle a such that $[a] = \alpha$ will be demonstrated. ♦

2.1.3 Comparison with the formulation of Căldăraru

We explain in this section how our definition of twisted sheaves squares with that used by Căldăraru in [6]. The reader will note that this formulation seems more "user-friendly." We hope to make clear below, especially in our discussion of deformations and obstructions, why the more abstract approach is useful.

Throughout, we retain the notation of the previous section: $(X, \mathscr{O})$ is a ringed topos and $\chi : A \to \mathbf{G}_m$ is a character of a sheaf of commutative groups. Let $\alpha \in \mathrm{H}^2(X, A)$ be a fixed cohomology class. By a theorem of Verdier [5, Exposé V.7], there is a hypercovering $U_\bullet \to X$ and a scalar $a \in \Gamma(U_2, A)$ whose coboundary on U_3 is trivial and which represents α in cohomology. We fix such a representative in this section. We also fix a choice of A-gerbe $\mathscr{X}$ representing the cohomology class α.

Definition 2.1.3.1 (Căldăraru). A χ-twisted sheaf on X is a pair (F, g), where F is an $\mathscr{O}_{U_0}$-module and $g : F_1 \xrightarrow{\sim} F_0$ is a gluing datum on U_1 such that $\delta g \in \mathrm{Aut}(F_0)$ equals the cocycle $\chi(a)$.

We will (temporarily) call such an object a Căldăraru-χ-twisted sheaf.

Example 2.1.3.2. Suppose $A = \mathbf{G}_m$, $\chi = \mathrm{id}$, and X is a complex analytic space. We may take the hypercovering $U_\bullet$ to be the Čech hypercovering generated by an open covering of X, i.e., we may replace $U_\bullet$ by an open covering $\{U_i\}$ of X. Then a χ-twisted sheaf on X is given by

1. a sheaf of modules $\mathscr{F}_i$ on each U_i

2. for each i and j an isomorphism of modules $g_{ij} : \mathscr{F}_j|_{U_{ij}} \xrightarrow{\sim} \mathscr{F}_i|_{U_{ij}}$

subject to the requirement that on U_{ijk}, $g_{ik}^{-1} g_{ij} g_{jk} : \mathscr{F}_k|_{U_{ijk}} \xrightarrow{\sim} \mathscr{F}_k|_{U_{ijk}}$ is equal to multiplication by the scalar $a \in \boldsymbol{\mu}_n(U_{ijk})$ giving the 2-cocycle. ◇

Our first goal in this section is to show that the stack of Căldăraru-χ-twisted right A-torsors is actually an A-gerbe with class α. This is not as trivial as it seems.

Definition 2.1.3.3. The stack of *explicit twisted right A-torsors*, denoted $\mathscr{X}(U_\bullet, a)$, has as objects over $S \in X$ the groupoid of pairs $(\mathscr{L}, \varphi)$, where $\mathscr{L}$ is a right A-torsor on $U_0 \times S$ and $\varphi : \mathscr{L}_0 \xrightarrow{\sim} \mathscr{L}_1$ is an isomorphism of the pullbacks to $U_1 \times S$ such that $\delta\varphi = a|U_2 \times S$. Isomorphisms are isomorphisms of the $\mathscr{L}$ which commute with the gluing data φ.

Proposition 2.1.3.4. *$\mathscr{X}(U_\bullet, a)$ is an A-gerbe representing* $[a]$.

We begin the proof by proving a series of results about explicit Čech 2-cohomology on hypercoverings. First, a trivial lemma which will be useful below and which is not easily extractable from the literature. It is a simple case of something more subtle which does not clearly work as stated in higher indices without extra hypotheses (such as working in the small étale site of a Deligne-Mumford stack).

Lemma 2.1.3.5. *Let X be a topos and $U_\bullet$ a hypercovering of X. Given a covering $\widetilde{U}_1 \to U_1$, there is a hypercovering $U'_\bullet$ and a morphism $U'_\bullet \to U_\bullet$ such that the map $U'_0 \to U_0$ is an isomorphism and $U'_1 \to U_1$ has the form $\widetilde{U}_1 \coprod U_0 \to U_1$, with the U_0 summand mapping to U_1 via the degeneracy map.*

Proof. Let $U''_1 = \widetilde{U_1} \coprod U_0$, $U''_0 = U_0$. Then U''_1 and U''_0 may be made into a 1-truncated simplicial object $U''_\bullet[1]$ with a map to $\operatorname{sk}_1 U_\bullet$ (which sends the U_0 summand of U''_1 to U_1 via the degeneracy map $U_0 \to U_1$). Thus, we get a map $\operatorname{cosk}_1 U''_\bullet[1] \to \operatorname{cosk}_1 \operatorname{sk}_1 U_\bullet$. The pullback $\operatorname{cosk}_1 U''_\bullet[1] \times_{\operatorname{cosk}_1 \operatorname{sk}_1 U_\bullet} U_\bullet$ gives a hypercover mapping to $U_\bullet$ whose term in degree 1 is $(\widetilde{U}_1 \coprod U_0) \times_{U_1} U_1 = \widetilde{U}_1 \coprod U_0$. □

Proposition 2.1.3.6. *There is a section $x : U_0 \to \mathscr{X}$ whose pullbacks $x_0, x_1 : U_1 \to \mathscr{X}(U_\bullet, a)$ are isomorphic. Fixing an isomorphism $\varphi : x_0 \to x_1$, one has $\delta\varphi = a$.*

Proof. The content of this proposition lies in the fact that we are not changing the hypercovering $U_\bullet$. We will make a descent-theoretic argument.

Changing the base of $U_\bullet$ to U_0, we find that the augmentation has a section. Using this section, we can make a morphism $U_{0,\mathrm{const}} \to U_\bullet \times_X U_0$ of simplicial objects augmented over U_0, where $U_{0,\mathrm{const}}$ denotes the constant object, i.e., every object of the simplicial object is U_0 and all morphisms are the identity. Since we are now augmented over U_0, the object $U_{0,\mathrm{const}}$ is now a *hypercovering*. Since hypercoverings compute cohomology (in the limit) and the Čech cohomology of $U_{0,\mathrm{const}}$ is clearly trivial, we see that $[a]|_{U_0} = 0 \in \mathrm{H}^2(U_0, A)$. Thus, as $[\mathscr{X}] = [a] \in \mathrm{H}^2(X, A)$, we find that there is a morphism $x : U_0 \to \mathscr{X}$. It is not *a priori* the case that the pullbacks x_0 and x_1 are isomorphic on U_1, but there will be a covering $\widetilde{U}_1 \to U_1$ such that $x_0|_{\widetilde{U}_1} \cong x_1|_{\widetilde{U}_1}$. Furthermore, x_0 and x_1 are isomorphic when restricted to the degeneracy map $U_0 \to U_1$. By 2.1.3.5, there is a hypercovering $U'_\bullet$ and a morphism $h : U'_\bullet \to U_\bullet$ such that $U'_0 = U_0$ and $U'_1 \cong \widetilde{U}_1 \coprod U_0$; thus, $x_0|_{U'_1} \cong x_1|_{U'_1}$. Consider the diagram

$$\begin{array}{ccccccc}
U''_2 & \Rrightarrow & U''_1 & \rightrightarrows & U''_0 & = & U_0 \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
U'_2 & \Rrightarrow & U'_1 & \rightrightarrows & U'_0 & = & U_0 \\
{\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow & \swarrow{\scriptstyle \mathrm{id}} & \\
U_2 & \Rrightarrow & U_1 & \rightrightarrows & U_0, & &
\end{array}$$

where $U'' := U' \times_U U'$.

Choosing an isomorphism $x_0 \xrightarrow{\sim} x_1$ on U'_1 and taking the coboundary yields a 2-cocycle $a' \in A(U'_2)$ such that $[a'] = [a] \in \mathrm{H}^2(X, A)$. There is a spectral sequence

$$E_2^{p,q} = \check{\mathrm{H}}^p(U'_\bullet, \mathscr{H}^q(A)) \Rightarrow \mathrm{H}^{p+q}(X, A)$$

from which we deduce that the kernel of the natural map $\check{\mathrm{H}}^2(U'_\bullet, A) \to \mathrm{H}^2(X, A)$ is the image of $\check{\mathrm{H}}^0(U'_\bullet, \mathscr{H}^1(A))$. This is just the set of A-torsors $\mathscr{L}$ on U'_0 such that $\mathscr{L}_0 \cong \mathscr{L}_1$ on U'_1; choosing such an isomorphism yields the 2-cocycle which is cohomologically trivial. We know that h^*a and a' map to the same thing in $H^2(X, A)$. Thus, they differ by the cocycle coming from some A-torsor on U_0'. Since $U_0' = U_0$, we may adjust the section x by $\mathscr{L}$ (using the fact that we are dealing with A-gerbes). Thus, we may assume that $h^*a = a'$. In other words, we may assume that there is a section σ of $\mathrm{Isom}(x_0, x_1)$ over U_1' whose coboundary is equal to a as an element of $\mathrm{Isom}(x_0, x_0)(U_2')$. We wish to show that σ will descend to U_1. On the hypercovering $U_\bullet''$, we may consider the two pullbacks σ_0 and σ_1 of σ to U_1''. Since $\mathrm{Isom}(x_0, x_1)$ is a sheaf, it is enough to show that $\sigma_0 = \sigma_1$. But we know that $\delta\sigma_0 = \delta\sigma_1 = a|_{U_2''}$. Thus, $\sigma_1^{-1} \circ \sigma_0$ is a section $b'' \in A(U_1'')$ such that $\delta(b'') = 0$. We conclude that there is an A-torsor M on X with a trivialization $\alpha : A_{U_0''} \xrightarrow{\sim} M_{U_0''}$ whose coboundary on U_1'' is b. As $U_0'' = U_0$, we see that b is the pullback of a section $b \in A(U_1)$. Now pulling back the relation $\sigma_0 = \sigma_1 \circ b$ along the diagonal $U_1' \to U_1''$, we find $\sigma = \sigma \circ b$, whence $b|_{U_1'} = \mathrm{id}_A$ and we see that $\sigma_0 = \sigma_1$. □

Lemma 2.1.3.7. *Let X be a topos, $U_\bullet$ a hypercover such that the augmentation has a section, a a 2-cocycle for A. Then a is a coboundary.*

Proof. By the argument above, $[a] = 0 \in \mathrm{H}^2(X, A)$, so there is an A-torsor L on U_0 and an isomorphism $\varphi : L_0 \xrightarrow{\sim} L_1$ whose coboundary is a. We wish to show that L_0 and L_1 are trivial. In that case, choosing trivializations, φ gets identified with a scalar b whose coboundary is a. Suppose first that the pullback of L along the section to the augmentation is trivial. Then, as above, there is a refinement $U_\bullet' \to U_\bullet$ such that the pullbacks of L_0 and L_1 are trivial on U_1'. Fix trivializations of both over U_1'. In this case, φ yields a section $b \in A(U_1')$ such that $\delta(b) = a$. We wish to show that b descends to U_1. As before, on U_1'', $\delta(b_0) = a = \delta(b_1)$. Thus, $\delta(b_1 - b_0) = 0$. Writing c for $b_1 - b_0$, we see that c gives a descent datum for an A-torsor on X relative to the covering $U_\bullet''$. Thus, there is some M on X such that $M|_{U_0''}$ is trivial, and choosing the right trivialization $A|_{U_0''} \xrightarrow{\sim} M|_{U_0''}$ we see that the descent datum on M is identified with c. As $U_0'' = U_0$, we see that c comes by pullback from U_1. Taking the coboundary d of c on U_1''', we see from the fact that c is a coboundary (being $b_1 - b_0$) that $d = 0$. On the other hand, as $c \in A(U_1)$, we see that $d = c$. Thus $c = 0$ and b descends, making a a coboundary. □

Question 2.1.3.8. How much of this remains true for higher cohomology classes? In other words, what do we need to know about the hypercover to conclude that a trivial cocycle is a coboundary? The proof given here uses various special facts about the index 2, among them the fact that for the constant simplicial object, the 2-coboundary map is not zero! (All odd degree maps are zero; all even degree maps are the identity.) One thing which is still clear is that when the augmentation has a section, the Čech cohomology along the hypercover contributes nothing to the sheaf cohomology. Thus, any cohomology class defined on $U_\bullet$ will split on U_0.

Proof of 2.1.3.4. It is clear that the natural map from $A \to \mathrm{Aut}(\mathscr{L}, \varphi)$ is an isomorphism, so to see that $\mathscr{X}(U_\bullet, a)$ is an A-gerbe, it remains to check that it has local sections everywhere and that any two sections are locally isomorphic. By 2.1.3.7, a is a coboundary after changing the base to U_0, say $a = \delta(b)$. Thus, taking the trivial

A-torsor and composing the descent datum with b yields a section (A, b) of $\mathscr{X}(U_\bullet, a)$ over U_0. In fact, one easily sees that sending a pair $(\mathscr{L}, \varphi)$ to $(\mathscr{L}, \varphi \circ b^{-1})$ yields an equivalence of the fiber category $\mathscr{X}(U_\bullet, a)_{U_0}$ with the category of right A-torsors on U_0 (via the category of torsors on $U_0 \times U_0$ with descent data for $U_\bullet \times U_0$). As any A-torsor is locally isomorphic to A, we see that $\mathscr{X}(U_\bullet, a)$ is an A-gerbe. To see that $[\mathscr{X}(U_\bullet, a)] = [a]$, note that in general one can compute the class of a gerbe by choosing a hypercovering $V_\bullet$ such that

(1) There is a section $x : V_0 \to \mathscr{X}$ with $x_0 \cong x_1$ on V_1.

(2) Choosing $\varphi : x_0 \xrightarrow{\sim} x_1$, the coboundary $\delta\varphi$ will be a 2-cocycle for A which represents the cohomology class of $\mathscr{X}$ in the Čech cohomology of $V_\bullet$.

But this is precisely what 2.1.3.6 does for the hypercover $U_\bullet$ with the 2-cocycle a. □

Arguing as in 2.1.2.6, one sees that there is a 1-morphism of stacks $\varepsilon : \mathscr{T} \to \mathscr{X}(U_\bullet, a)$.

Corollary 2.1.3.9. *ε is a 1-isomorphism.*

Proof. Both are A-gerbes in the same cohomology class, so any 1-morphism between them which is A-linear (i.e., such that the natural map of 2.1.1.9 is an isomorphism) is a 1-isomorphism. (Indeed, the map is essentially surjective because any two sections are locally isomorphic and it is fully faithful on fiber categories by the assumption on A-linearity.) But it is clear that the natural map ε is A-linear by construction. □

Let $\mathscr{F}$ be a χ-twisted sheaf. Pulling back by the section $x : U_0 \to \mathscr{X}$ of 2.1.3.6 we get an $\mathscr{O}_{U_0}$-module F. The isomorphism $\varphi : x_0 \xrightarrow{\sim} x_1$ gives rise to an isomorphism $\psi : F_1 \xrightarrow{\sim} F_0$ (where the first factor, to be completely rigorous, is the pullback of F_1 along the identity map). The coboundary is multiplication by a (via the natural right action) by the requirement on the character of the inertia action built into the definition of χ-twisted sheaf. In other words, we have defined a morphism from the stack of χ-twisted sheaves to the stack of Căldăraru-χ-twisted sheaves. We will refer to this as the natural map.

Lemma 2.1.3.10. *If $\mathscr{X}$ is trivial, the choice of a section $X \to \mathscr{X}$ identifies the stack of χ-twisted sheaves with the stack of sheaves on X.*

Proof. The section induces an isomorphism $\mathrm{B}A \xrightarrow{\sim} \mathscr{X}$, which induces an isomorphism of the stack of χ-twisted sheaves on $\mathscr{X}$ with the stack of χ-twisted sheaves on $\mathrm{B}A$. (We are implicitly identifying the inertia stacks using 2.1.1.9.) Given a sheaf on $\mathrm{B}A$, pullback along the section gives a sheaf on X. Given a sheaf F on X, associating to every A-torsor L on $U \to X$ the sheaf $F \otimes \chi(L)$ gives a χ-twisted sheaf on $\mathrm{B}A$. Here $\chi(L)$ means the right $\mathbf{G}_m$-torsor associated to L via the character χ. It is easy to check that these define an equivalence of categories. □

Proposition 2.1.3.11. *The natural map induces an equivalence between the stack of χ-twisted sheaves and the stack of Căldăraru-χ-twisted sheaves.*

Proof. By the naturality of the map, it suffices to work locally, so we may assume that a is a coboundary by 2.1.3.7, say $a = \delta(b)$. Changing the gluing datum by b as in 2.1.3.4, we see by 2.1.3.10 that the natural map is isomorphic to the map sending a sheaf to its pullback to $U_\bullet$ along with its descent datum. This is an equivalence by the definition of a topos! $\square$

On the other hand, it is obvious that the Căldăraru-χ-twisted sheaves are equivalent to twisted sheaves for the cocycle in $\mathbf{G}_m$ induced by χ. Let $\mathscr{X}_\alpha \to \mathscr{X}_{\chi(\alpha)}$ be as in 2.1.2.3, where $\chi(\alpha) \in \mathrm{H}^2(X, \mathbf{G}_m)$.

Definition 2.1.3.12. If $\chi : A \to \mathbf{G}_m$ is the natural inclusion of a subsheaf (e.g., $\boldsymbol{\mu}_n$ or $\mathbf{G}_m$), a χ-twisted sheaf on $\mathscr{X}$ will be called an *$\mathscr{X}$-twisted sheaf*.

Corollary 2.1.3.13. *The map $\mathscr{X}_\alpha \to \mathscr{X}_{\chi(\alpha)}$ induces by pullback a 1-isomorphism of the stack of $\mathscr{X}_{\chi(\alpha)}$-twisted sheaves with the stack of χ-twisted sheaves.*

Proof. This is immediate once one chooses a hypercovering splitting α: the Căldăraru forms are then identical, so we are done by 2.1.3.11. $\square$

2.1.4 Fibered Morita theory

Our goal in this section is to provide a geometric version of Morita theory: we answer the question "When are the fibered categories of modules for two rings in a topos fibered-equivalent and how can the equivalences be described (up to 2-isomorphism)?" This provides the correct framework for understanding the Brauer group of a ringed topos, as developed in the next section. Throughout this section, $(X, \mathscr{O})$ is a locally ringed topos with enough points.

Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves of (unital associative not-necessarily commutative) $\mathscr{O}$-algebras.

Definition 2.1.4.1. The assignment of any object $U \in X$ to the category of sheaves of $\mathscr{A}|_U$-modules on U defines a fibered $\mathscr{O}$-linear category $\mathscr{M}\mathrm{od}_{\mathscr{A}} \to X$. Two $\mathscr{O}$-algebras $\mathscr{A}$ and $\mathscr{B}$ are *fibered Morita equivalent* if there is an $\mathscr{O}$-linear fibered equivalence $\mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{A}} \to \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{B}}$.

Here, the superscript "fp" denotes the sub-fibered category of (everywhere locally) finitely presented modules. Our ad hoc proofs unfortunately seem to require this condition (in the absence of any other conditions on X or the structure of the modules being studied). Of course, any fibered Morita equivalence induces an equivalence on the fibered category of all modules (as this is just the ind-category of finitely presented modules). The condition of $\mathscr{O}$-linearity on a functor F means that given any $M, N \in \mathscr{M}\mathrm{od}_{\mathscr{A},U}$, the natural map $\mathscr{H}om_{\mathscr{A}}(M, N) \to \mathscr{H}om_{\mathscr{B}}(F(M), F(N))$ is $\mathscr{O}$-linear.

The following proposition will not be used in the sequel, but it is reassuring to know that it is true. Given a point P of X, we define the *stalk* of $\mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{A}}$ to be the category $\mathrm{Mod}^{\mathrm{fp}}_{\mathscr{A}_P}$ of finitely presented $\mathscr{A}_P$-modules.

Lemma 2.1.4.2. *Given an $\mathscr{O}$-linear functor of fibered categories $\mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{A}} \to \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{B}}$, there is an induced functor on stalks.*

Proof. Any $M \in \mathrm{Mod}^{\mathrm{fp}}_{\mathscr{A}_P}$ can be described by a matrix with entries in $\mathscr{A}_P$. By the usual properties of points in a topos (see Sga 4.1.6) there is a neighborhood of P over which this matrix extends, i.e., a section of $\mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{A}}$ over some object $V \in X$ whose stalk at a point lifting P is isomorphic to M. Taking the image by F and taking the stalk (which is the "germ" in the fibered category) yields an object $F_P(M) \in \mathrm{Mod}^{\mathrm{fp}}_{\mathscr{B}_P}$. More generally, from $\mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{A}}$, one gets an ind-category indexed by the neighborhoods of P, and one sees that this ind-category admits a functor to the constant ind-category on $\mathrm{Mod}^{\mathrm{fp}}_{\mathscr{A}_P}$ which is an equivalence. The functor on stalks in question is then the functor induced by the ind-functor associated to F. ◻

We will call a set of points of X *dense* if the corresponding functors $X \to \mathbf{Set}$ form a conservative family. See [4] for more information about this concept. The reader unfamiliar with these terms is encouraged to think of the set of points of a scheme: the behavior of a diagram of sheaves in the localizations determines its global behavior. In other words, given a functor $h : I \to X$ from a small (resp. finite) indexing category, a natural transformation from h to a constant diagram (resp. from a constant diagram to h) is a colimit (resp. limit) in X if and only if it is in all the stalks of the conservative family. For a down-to-earth example: a map of sheaves $F \to G$ is a surjection (resp. injection) if and only if $F_P \to G_P$ is a surjection (resp. injection) for all P in the family. *Caveat emptor*: a set of a points of a scheme may be dense in the topological sense without being dense in our sense (e.g., the generic point).

Proposition 2.1.4.3. *An $\mathscr{O}$-linear functor of fibered categories $F : \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{A}} \to \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{B}}$ is an equivalence if and only if it is an equivalence in the stalk at a dense set of points of X.*

Proof. Given $M, N \in \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{A},U}$, F induces a map of sheaves of $\mathscr{O}$-modules

$$\mathscr{H}om_{\mathscr{A}}(M, N) \to \mathscr{H}om_{\mathscr{B}}(F(M), F(N)).$$

Furthermore, as M, N, $F(M)$, and $F(N)$ are finitely presented, this map localizes to give the map on stalks

$$\mathrm{Hom}_{\mathscr{A}_P}(M_P, N_P) \to \mathrm{Hom}_{\mathscr{B}_P}(F(M)_P, F(N)_P) = \mathrm{Hom}_{\mathscr{B}_P}(F_P(M_P), F_P(N_P)).$$

If F is an equivalence, then we see that the map of sheaves is an isomorphism, whence the localized map is an isomorphism. Conversely, if the localizations are isomorphisms at a dense set of points P, the original map is an isomorphism. Thus, F is fully faithful (on fiber categories) if and only if F_P is fully faithful for a dense set of points P. (We have implicitly used the fact that the restriction of a topos with enough points to an object still has enough points.) It remains to check essential surjectivity in fiber categories. Arguing as in 2.1.4.2 we see that if F is essentially surjective on fiber categories then F_P is essentially surjective for all P. Conversely, as the fibered categories of modules are stacks and we have just checked that F is fully faithful if it is at a dense set of points, it is enough to check that any section $N \in \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{B},U}$ is *locally* in the essential image of F. But this is clear, again by an argument similar to that in 2.1.4.2. $\square$

As in ordinary Morita theory, any fibered Morita equivalence is given by a Morita context in the category of $\mathscr{O}$-modules.

Proposition 2.1.4.4. *Any fibered Morita equivalence $F : \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{A}} \to \mathscr{M}\mathrm{od}^{\mathrm{fp}}_{\mathscr{B}}$ is isomorphic to a functor of the form $\mathscr{H}\!om_{\mathscr{A}}(\mathscr{P}, \,\cdot\,)$, where $\mathscr{P}$ is a locally faithfully projective $\mathscr{A}$-module of finite presentation. In this case, $\mathscr{B} \cong \mathscr{E}\!nd_{\mathscr{A}}(\mathscr{P})$.*

As X is locally ringed with enough points, the concept of "locally faithfully projective" is easily understood: $\mathscr{P}_P$ is a progenerator for $\mathrm{Mod}_{\mathscr{A}_P}$ for a dense set of points P.

Proof. One way to prove this would be to lift and glue local progenerators using classical Morita theory [65]. We give a more global proof (which only reduces to the local case once the global construction has been made).

First, let $\mathscr{P}$ be a locally faithfully projective $\mathscr{A}$-module of finite presentation. Let $\mathscr{B} := \mathscr{E}\!nd_{\mathscr{A}}(\mathscr{P})$. Then $\mathscr{P}$ is naturally an $(\mathscr{A}, \mathscr{B})$-bimodule, and so $\mathscr{P}^* := \mathscr{H}\!om_{\mathscr{A}}(\mathscr{P}, \mathscr{A})$ is naturally a $(\mathscr{B}, \mathscr{A})$-bimodule. Furthermore, as $\mathscr{P}$ is finitely presented,

$$\mathscr{H}\!om_{\mathscr{A}}(\mathscr{P}, M) = \mathscr{P}^* \underset{\mathscr{A}}{\otimes} M$$

(and this isomorphism respects the left $\mathscr{B}$-module structures). Thus, there is a natural isomorphism $\mathscr{P}^* \otimes_{\mathscr{A}} \mathscr{P} \cong \mathscr{B}$. We claim that there is also a natural isomorphism $\mathscr{P} \otimes_{\mathscr{B}} \mathscr{P}^* \cong \mathscr{A}$ of $(\mathscr{A}, \mathscr{A})$-bimodules. There is certainly a natural $(\mathscr{A}, \mathscr{A})$-bimodule map given by "contraction": $p \otimes \varphi \mapsto \varphi(p) \in \mathscr{A}$ on sections. This is functorial, and thus we may take stalks and reduce to the classical Morita theory, where the map is known to be an isomorphism. One can now check that $\mathscr{H}\!om_{\mathscr{B}}(\mathscr{P}^*, \,\cdot\,)$ defines a quasi-inverse functor.

Conversely, suppose given an equivalence F as in the statement of the lemma, and suppose $\mathscr{P}$ has the property that $F(\mathscr{P}) \cong \mathscr{B}$. Then, since F is fully faithful, $\mathscr{P}$ inherits an $(\mathscr{A}, \mathscr{B})$-bimodule structure from a choice of isomorphism $F(\mathscr{P}) \cong \mathscr{B}$. We claim that $F \cong \mathscr{H}\!om_{\mathscr{A}}(\mathscr{P}, \,\cdot\,)$ via the following natural map: as F is a fibered equivalence, given any two $\mathscr{A}$-modules M and N, F_U induces an isomorphism $\mathrm{Hom}_{\mathscr{A}_U}(M_U, N_U) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}_U}(F(M)_U, F(N)_U)$. This means that F induces an isomorphism $\mathscr{H}\!om_{\mathscr{A}}(M, N) \xrightarrow{\sim} \mathscr{H}\!om_{\mathscr{B}}(F(M), F(N))$. Thus, using the chosen isomorphism $F(\mathscr{P}) \xrightarrow{\sim} \mathscr{B}$, there is an induced isomorphism

$$\mathscr{H}\!om_{\mathscr{A}}(\mathscr{P}, M) \xrightarrow{\sim} \mathscr{H}\!om_{\mathscr{B}}(F(\mathscr{P}), F(M)) \xrightarrow{\sim} \mathscr{H}\!om_{\mathscr{B}}(\mathscr{B}, F(M)) \xrightarrow{\sim} F(M).$$

It remains to show that $\mathscr{P}$ is locally faithfully projective. It suffices to check this on stalks, as $\mathscr{P}$ is already finitely presented by hypothesis. But any surjection of *finitely presented* modules on a stalk is a localization of a (local) surjection of finitely presented sheaves, so we may apply the fact that F is exact (being an equivalence) to conclude that $\mathscr{P}$ is locally projective. It is a local generator by a similar argument. $\square$

Remark 2.1.4.5. I am not sure if the finite presentation hypothesis is necessary. (In the topos of sets, of course, it is not.) ♦

The global version of Morita theory immediately gives us the following corollary. A proof using more abstract methods (which works in the absence of sufficiently many points globally) is indicated in 2.1.4.2 above.

Corollary 2.1.4.6. *If $\mathscr{A}$ and $\mathscr{B}$ are fibered Morita equivalent then $\mathscr{A}_P$ is Morita equivalent to $\mathscr{B}_P$ (in the sense of finitely presented modules) for any point P.*

Given $\mathscr{A}$ and $\mathscr{B}$ as above, we can consider the fibered category of all fibered Morita equivalences in the following sense: let $\mathscr{M}(\mathscr{A},\mathscr{B})$ be the category of triples $(U,\mathscr{P},\varphi)$ where $U \in X$, $\mathscr{P}$ is a locally faithfully projective $\mathscr{A}|_U$-module, and $\varphi : \mathscr{E}nd_{\mathscr{A}|_U}(\mathscr{P}) \xrightarrow{\sim} \mathscr{B}|_U$ is an isomorphism of $\mathscr{O}$-algebras. Isomorphisms $(U,\mathscr{P},\varphi) \xrightarrow{\sim} (U,\mathscr{P}',\varphi')$ in the fiber categories are $\mathscr{A}$-linear isomorphisms $\psi : \mathscr{P}' \xrightarrow{\sim} \mathscr{P}$ such that $\varphi \circ \psi^* = \varphi'$, where $\psi^* : \mathscr{E}nd_{\mathscr{A}}(\mathscr{P}') \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{A}}(\mathscr{P})$ is the induced isomorphism of $\mathscr{O}$-algebras. We will investigate this stack after a few non-commutative preliminaries.

Recall that a ring A is *local* if A/J is a simple ring, where J is the Jacobson radical of A (i.e., A has a unique isomorphism class of simple left modules). We will say that A is *strongly local* if A/J is a simple left-Artinian ring. This has the effect of making the category of left A/J modules semisimple [24]. (Note that when A is commutative, A/J will always be a field, so every local commutative ring is strongly local.)

Definition 2.1.4.7. An $\mathscr{O}$-algebra is *locally (strongly) local* if $\mathscr{A}_P$ is a (strongly) local ring for a dense set of points P of X.

Remark 2.1.4.8. This definition is made only for the case when X has enough points. I have not thought about what the correct condition should be when X is more general. (By a theorem of Deligne, any coherent topos has enough points, so, in particular, "any topos arising in algebraic geometry" will have enough points.) I am not even sure if it is true that a locally local sheaf of algebras by our definition has local stalks at every point. ♦

Lemma 2.1.4.9. *Let A be a strongly local ring and P and Q finitely presented projective A-modules. Any isomorphism $\mathrm{End}_A(P) \xrightarrow{\sim} \mathrm{End}_A(Q)$ of rings is induced by an isomorphism $P \xrightarrow{\sim} Q$. The automorphisms of P inducing the trivial automorphism of $\mathrm{End}_A(P)$ are precisely the elements of $Z(A^\times)$.*

Proof. Let J be the Jacobson radical of A (which is the maximal left ideal and is in fact a two-sided ideal.) Let $\psi : \mathrm{End}_A(P) \xrightarrow{\sim} \mathrm{End}_A(Q)$ be an isomorphism. Since P is projective, the map $\mathrm{End}_A(P) \to \mathrm{End}_{A/J}(P/JP)$ is surjective, hence (as A is local) is identified with the quotient of $\mathrm{End}_A(P)$ by its Jacobson radical. Furthermore, the number of idempotents of $\mathrm{End}_{A/J}(P/JP)$ is 2^n, where n is the number of simple summands of P/JP. Similar results hold for Q, so we deduce that that P/JP and Q/JQ

have the same number of simple summands and hence that there is an isomorphism $P \to Q$ (using projectivity).

It remains to show that any automorphism of $\mathrm{End}_A(P)$ is induced by an automorphism of P. Since A/J is a simple left-Artinian ring, we see by reducing to the case where A/J is a division ring (using Morita theory again) that any automorphism of $\mathrm{End}_{A/J}(P/JP)$ is induced by an automorphism of P/JP. Thus, since P is projective, it suffices to show that any automorphism of $\mathrm{End}_A(P)$ which reduces to the identity modulo J is induced by an automorphism of P.

Write $B = \mathrm{End}_A(P)$ and let $\varphi : B \xrightarrow{\sim} B$ be an automorphism. This results in two right B-module structures on P: the natural structure and the natural structure twisted by φ. Denote the twisted structure by P'. By Morita theory, P with the right B-structure projective right B-module [65]. (Indeed, one shows that P^* is a projective left B-module and that $N \mapsto \mathrm{Hom}_B(P^*, N)$ gives the inverse Morita equivalence to $M \mapsto \mathrm{Hom}_A(P, M)$. Thus, $P = (P^*)^*$ is the dual of a finitely presented projective left B-module.) On the other hand, P and P' are isomorphic modulo J, so there is a B-linear map $\psi : P \to P'$ lifting such an isomorphism. Since twisting by φ is an exact functor, we see that P' is a finitely generated projective B-module, hence finitely presented. We conclude by Nakayama's Lemma that ψ is an isomorphism. This is easily seen to imply that φ is inner, as required.

The statement about the automorphisms $\varphi : P \xrightarrow{\sim} P$ inducing the trivial automorphism follows from the fact that $Z(A^\times)$ is the automorphism group of the identity functor on Mod_A. □

In order to retain uniform notation, we will say that two rings A and B are "fibered Morita equivalent" if they are Morita equivalent in such a way that finitely presented modules are preserved.

Lemma 2.1.4.10. *$\mathscr{M}(\mathscr{A}, \mathscr{B})$ is a stack. It has a section in a neighborhood of P iff $\mathscr{A}_P$ and $\mathscr{B}_P$ are fibered Morita equivalent. If $\mathscr{A}_P$ is strongly local then any two sections of $\mathscr{M}(\mathscr{A}, \mathscr{B})$ are locally isomorphic around P. When $\mathscr{A}$ is locally strongly local and $\mathscr{A}_P$ is fibered Morita equivalent to $\mathscr{B}_P$ at a dense set of points P, $\mathscr{M}(\mathscr{A}, \mathscr{B})$ is a gerbe whose band is the center of the sheaf $\mathscr{A}^\times$ of units of $\mathscr{A}$.*

Proof. That $\mathscr{M}$ is a stack is immediate. The second statement is a consequence of the definition of $\mathscr{M}$ and 2.1.4.4. The third statement is 2.1.4.9. □

Remark 2.1.4.11. Suppose A is strongly local and let M be the simple left module. A *projective cover* of M is a finitely generated projective left A-module P such that $P/JP \cong M$. If a projective cover P exists then it is a progenerator for left A-modules, and in fact every finitely generated progenerator is a direct sum of copies of P. We conclude in this case that the fibered Morita equivalence class of A consists of the matrix algebras $\mathrm{M}_n(\mathrm{End}_A(P))$ for all $n \geq 0$ (with $\mathrm{M}_0(\mathrm{End}_A(P)) := A$). If A is a finite algebra over a Noetherian local ring R (which we will assume to be excellent starting with the next sentence), then there is a projective cover for M over the completion of R [10, 1.1.5]. As the functor of Morita equivalence is locally of finite presentation, we see by Popescu's theorem [17, 59, 60] that the algebras which are fibered Morita equivalent to A are all étale locally isomorphic to matrix algebras over a fixed ring. This structure is a mild generalization of the concept of an Azumaya algebra; see 2.1.5.

Corollary 2.1.4.12. *If $\mathscr{A}$ is fibered Morita equivalent to $\mathscr{C}$, then $\mathscr{M}(\mathscr{A},\mathscr{B})$ is fibered equivalent to $\mathscr{M}(\mathscr{C},\mathscr{B})$. Similarly for Morita equivalences in $\mathscr{B}$.*

Proof. Left to the reader. □

As expected, the stack $\mathscr{M}(\mathscr{A},\mathscr{B})$ carries the "universal Morita equivalence of $\mathscr{A}$ and $\mathscr{B}$".

Lemma 2.1.4.13. *There is a finitely presented locally faithfully projective $\mathscr{A}$-module $\mathscr{F}$ on $\mathscr{M}(\mathscr{A},\mathscr{B})$ and an isomorphism $\varphi : \mathscr{E}nd_{\mathscr{A}}(\mathscr{F}) \xrightarrow{\sim} \mathscr{B}$ in $\mathscr{M}(\mathscr{A},\mathscr{B})$. The pullback of the pair $\mathscr{F},\varphi$ along any section $s : U \to \mathscr{M}(\mathscr{A},\mathscr{B})$ is isomorphic to the equivalence on U associated to s.*

Proof. Sending a triple $(U,\mathscr{P},\varphi)$ to $\mathscr{P}$ defines the sheaf $\mathscr{F}$. The rest is a tautology. Note that when $\mathscr{M}$ is a $Z(\mathscr{A}^\times)$-gerbe, the sheaf $\mathscr{P}$ is twisted (in the obvious sense generalizing the definitions above). □

Now suppose that both conditions of 2.1.4.10 are satisfied, so that $\mathscr{M}(\mathscr{A},\mathscr{B})$ is a $Z(\mathscr{A}^\times)$-gerbe. Note that the existence of local sections comes from the fibered Morita structures of $\mathscr{A}$ and $\mathscr{B}$, while the description of the band only involves $\mathscr{A}$. Thus, the conditions are invariant under Morita equivalences in $\mathscr{B}$. Let $\mathscr{C}$ be another $\mathscr{O}$-algebra which is Morita equivalent to $\mathscr{B}$ on a dense set of stalks.

Lemma 2.1.4.14. *The $Z(\mathscr{A}^\times)$-gerbes $\mathscr{M}(\mathscr{A},\mathscr{B})$ and $\mathscr{M}(\mathscr{A},\mathscr{C})$ are equivalent if and only if $\mathscr{B}$ and $\mathscr{C}$ are fibered Morita equivalent.*

Proof. The "if" direction follows from 2.1.4.12. Suppose that $F : \mathscr{M}(\mathscr{A},\mathscr{B}) \xrightarrow{\sim} \mathscr{M}(\mathscr{A},\mathscr{C})$ is an equivalence of $Z(\mathscr{A}^\times)$-gerbes. Since $\mathscr{A}$ is the pullback of an algebra on X to $\mathscr{M}(\mathscr{A},\,\cdot\,)$, we see that $F^*\mathscr{A} = \mathscr{A}$. Thus, the universal equivalence $\mathscr{E}nd_{\mathscr{A}}(\mathscr{P}) \xrightarrow{\sim} \mathscr{C}$ on $\mathscr{M}(\mathscr{A},\mathscr{C})$ pulls back to give such an equivalence on $\mathscr{M}(\mathscr{A},\mathscr{B})$, and $F^*\mathscr{P}$ remains twisted as above by the condition that F be a map of $Z(\mathscr{A}^\times)$-gerbes. Let $\mathscr{F}$ be the twisted finitely presented locally faithfully projective $\mathscr{A}$-module giving the fibered Morita equivalence of $\mathscr{A}$ with $\mathscr{B}$. We see that $\mathscr{G} := (F^*\mathscr{P})^* \otimes \mathscr{F}$ is an untwisted finitely presented locally faithfully projective left $\mathscr{C}$-module which defines a fibered Morita equivalence of $\mathscr{C}$ and $\mathscr{B}$ on $\mathscr{M}(\mathscr{A},\mathscr{B})$. As $\mathscr{G}$ has trivial stabilizer action, it is the pullback of such a module on X by 2.1.1.17, and this defines a fibered Morita equivalence of $\mathscr{B}$ and $\mathscr{C}$ on X. □

Remark 2.1.4.15. When $\mathscr{A} = \mathscr{O}$, we see that classifying Azumaya algebras on X in a fixed fibered Morita class $[\mathscr{B}]$ (see the following section) is essentially the same as classifying *locally free twisted sheaves* on the $\mathbf{G}_m$-gerbe $\mathscr{M}(\mathscr{O},\mathscr{B})$. Thus, by absorbing the fibered Morita structure into the stack $\mathscr{M}$, we have reduced the problem of studying Azumaya algebras in a fixed Brauer class to studying moduli of ordinary sheaves (with a given stabilizer action) on $\mathscr{M}$. In other words, we can replace the non-commutative aspect of this particular problem by a commutative stacky structure. Having done this, all of the tools of commutative algebraic geometry may be brought to bear on the situation, which thus becomes quite tractable. In the case of Azumaya algebras, these ideas have applications outside of the world of non-commutative algebra, as one is then classifying PGL_n-bundles, one of the principal goals of this thesis.

2.1.5 The Brauer group of a ringed topos

Let $(X, \mathscr{O})$ be a ringed topos as above.

2.1.5.1. Recall that an *Azumaya algebra* on X is a sheaf $\mathscr{A}$ of $\mathscr{O}$-algebras which is locally isomorphic to $\mathrm{M}_n(\mathscr{O})$, i.e., it is a form of a matrix algebra.

Definition 2.1.5.2. With the above notation, the number n is the *degree* of $\mathscr{A}$.

Lemma 2.1.5.3. *If X is locally ringed, then the sheaf of automorphisms of $\mathrm{M}_n(\mathscr{O})$ is naturally isomorphic to $\mathrm{PGL}_{n,X}$.*

Proof. When X has enough points, this follows from 2.1.4.9. The general case may be found in [30, §V.4]. □

Thus, by the usual theory of descent in a topos, there is a bijection

$$\{\text{Azumaya algebras of degree } n \text{ on } X\}/\text{isom} \leftrightarrow \mathrm{H}^1(X, \mathrm{PGL}_n).$$

Given any Azumaya algebra $\mathscr{A}$ of degree n, we may apply the boundary map to its class in $\mathrm{H}^1(X, \mathrm{PGL}_n)$ to yield a class in $\mathrm{H}^2(X, \mathbf{G}_m)$. On the other hand, by 2.1.4.10, $\mathscr{M}(\mathscr{O}, \mathscr{A})$ is a $\mathbf{G}_m$-gerbe. We will denote this stack by $\mathscr{X}(\mathscr{A})$ and call it the *gerbe of trivializations of $\mathscr{A}$*.

Remark 2.1.5.4. One should be very careful in keeping track of the directions of morphisms in $\mathscr{X}(\mathscr{A})$. We have made a map $(U, F, \varphi) \to (U', F', \varphi')$ (using the notation for sections of the stack from the previous section) consist of a morphism $\alpha : U \to U'$ and an isomorphism $\alpha^* F' \xrightarrow{\sim} F$ commuting with the trivializations. A moment of thought shows that this is the most natural way to proceed, as α^* is a left adjoint. Thus, in the fiber category over U, the natural isomorphisms $(U, F, \varphi) \xrightarrow{\sim} (U, F', \varphi')$ are isomorphisms $F' \xrightarrow{\sim} F$ commuting with the trivializations. ♦

Lemma 2.1.5.5. *The 1-isomorphism class of $\mathscr{X}(\mathscr{A})$ equals the boundary of the torsor corresponding to $\mathscr{A}$.*

Proof. A complete proof is in [30, §V.4]. The point is that if T is the PGL_n-torsor associated to $\mathscr{A}$, $\mathscr{X}(\mathscr{A})$ is identified with the $\mathbf{G}_m$-gerbe of lifts of T to a GL_n-torsor, which is the coboundary of $[T]$. (The reader is encouraged to carefully pay attention to the natural directions of arrows as in the remark above.) □

Proposition 2.1.5.6. *Two Azumaya algebras $\mathscr{A}$ and $\mathscr{B}$ have the same cohomology class in $\mathrm{H}^2(X, \mathbf{G}_m)$ if and only if they are fibered Morita equivalent.*

Proof. This is an application of 2.1.4.14. □

Definition 2.1.5.7. The *Brauer group* of X, denoted $\mathrm{Br}(X)$, is the set of isomorphism classes of Azumaya algebras on X modulo fibered Morita equivalence. The *cohomological Brauer group* of X, denoted $\mathrm{Br}'(X)$, is $\mathrm{H}^2(X, \mathbf{G}_m)_{\mathrm{tors}}$.

Given two Azumaya algebras $\mathscr{A}$ and $\mathscr{B}$ on X, one sees that $\mathscr{A} \otimes \mathscr{B}$ is an Azumaya algebra. In fact, it is not hard to see that $[\mathscr{A} \otimes \mathscr{B}] = [\mathscr{A}] + [\mathscr{B}] \in \mathrm{H}^2(X, \mathbf{G}_m)$ [30, Lemme V.4.3]. Furthermore, $[\mathscr{A}] + [\mathscr{A}^\circ] = 0$ (as $\mathscr{A} \otimes \mathscr{A}^\circ \cong \mathscr{E}nd_{\mathscr{O}}(\mathscr{A})$). Thus, $\mathrm{Br}(X)$ is a group as claimed above, and sending $\mathscr{A}$ to $\mathscr{X}(\mathscr{A})$ gives an injection of groups into $\mathrm{H}^2(X, \mathbf{G}_m)$; in fact, the image lies in $\mathrm{Br}'(X)$. The question of when $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$ is an isomorphism is a difficult question. We present some of the known results using our techniques below in section 2.2.3.

The reader already familiar with the Brauer group will be happy to see the next lemma.

Lemma 2.1.5.8. *Two Azumaya algebras $\mathscr{A}$ and $\mathscr{B}$ are fibered Morita equivalent iff there are locally free sheaves $\mathscr{V}$ and $\mathscr{W}$ on X and an isomorphism of $\mathscr{O}$-algebras $\mathscr{A} \otimes \mathscr{E}nd(\mathscr{V}) \xrightarrow{\sim} \mathscr{B} \otimes \mathscr{E}nd(\mathscr{W})$.*

Proof. We know that $\mathscr{A} \sim \mathscr{B}$ if and only if $\mathscr{A} \otimes \mathscr{B}^\circ \sim \mathscr{O}$. It is easy to see that it is enough to show that if $\mathscr{A} \sim \mathscr{O}$ then $\mathscr{A} \cong \mathscr{E}nd_{\mathscr{O}}(\mathscr{V})$ for some locally free $\mathscr{O}$-module $\mathscr{V}$. Indeed, in this case the stack $\mathscr{X}(\mathscr{A})$, being equivalent to $\mathscr{X}(\mathscr{O}) \cong \mathrm{B}\mathbf{G}_m$, has a global section. By the definition of the stack of trivializations, this completes the proof. □

There are a lot of interesting basic things to say about the Brauer group, most of which we postpone until we have discussed the theory of twisted sheaves on a scheme.

2.2 The case of a scheme

When X is the flat or étale topos of a scheme, we can use the abstract machinery developed above to re-prove several classical results about the Brauer group and its relation to the cohomological Brauer group. Among the statements (re)proven below are Gabber's celebrated result on the relationship between the Brauer group and the cohomological Brauer group of an affine scheme [28, 39], and Grothendieck's basic results on this question for Noetherian schemes of low dimension [35]. We also sketch a proof of a fact which we have not seen in the literature: the derived category of quasi-coherent twisted sheaves on a separated quasi-compact scheme is compactly generated by perfect complexes (of rank 0, unfortunately). This may be seen as laying the groundwork for algebraic twisted K-theory, to be developed fully along the lines of Thomason [74] in future work. Finally, we will specialize to the case of $\boldsymbol{\mu}_n$-gerbes on X and develop the deformation theory necessary to analyze the moduli of twisted sheaves in section 4

2.2.1 Quasi-coherent twisted sheaves

Let X be a scheme, A a group scheme which is faithfully flat and locally of finite presentation over X, $\alpha \in \mathrm{H}^2(X_{\mathrm{fppf}}, A)$ a flat cohomology class, and $\chi : A \to \mathbf{G}_m$ an algebraic character. Fix a gerbe $\mathscr{X}$ representing α in the big fppf topology on X. When A is smooth, a theorem of Grothendieck [35, Appendix] says that the restriction of $\mathscr{X}$ to the (big or small) étale topos of X is an A-gerbe (and this defines an isomorphism $\mathrm{H}^2(X_{\mathrm{fppf}}, A) \xrightarrow{\sim} \mathrm{H}^2(X_{\text{ét}}, A)$). (In fact, Grothendieck's theorem holds for the cohomology in all degrees.)

Lemma 2.2.1.1. *The gerbe $\mathscr{X}$ is an algebraic stack locally of finite presentation over X. If X is quasi-separated and A is finitely presented then $\mathscr{X}$ is finitely presented. The scheme X is (locally) Noetherian if and only if $\mathscr{X}$ is (locally) Noetherian.*

Proof. We apply [52, 10.1]: it is enough to check that $\mathscr{X}$ is an fppf stack, has representable separated quasi-compact diagonal, and has a representable fppf cover by an algebraic space. The first condition is a consequence of the definition of a gerbe. The second and third conditions follow immediately once we note that $\mathscr{X}$ has a cover by an fppf X-scheme and that the sheaf of isomorphisms of any two sections of $\mathscr{X}$ is an fppf A-torsor (hence is representable by an algebraic space by [52, 10.4.1]). There is a cover of $\mathscr{X}$ which is of finite presentation over X because any locally finitely presented morphism from an affine to a quasi-separated scheme is quasi-compact (hence finitely presented). The last statement on the diagonal is left to the reader. □

Remark 2.2.1.2. In our study of twisted sheaves on surfaces (when we actually want to say something!), we will take $A = \boldsymbol{\mu}_n$ with n prime to the characteristics of X. In this case, the reader will immediately verify that any A-gerbe is in fact a DM stack.

As usual, when X is Noetherian we may define quasi-coherent and coherent twisted sheaves. To be careful, one should first check that this is independent of the topology chosen.

Lemma 2.2.1.3. *The obvious morphism $F : \widetilde{\mathscr{X}}_{\mathrm{fppf}} \to \widetilde{\mathscr{X}}_{\text{lis-ét}}$ induces by pullback an equivalence of the stacks of quasi-coherent sheaves. In fact, the map $\mathscr{F} \to F_*F^*\mathscr{F}$ is an isomorphism (for any sheaf on $\mathscr{X}_{\text{lis-ét}}$).*

Proof. When $\mathscr{X}$ is a scheme, both are naturally the same as Zariski quasi-coherent sheaves. We will use this fact repeatedly. For a proof, one could consult [1, Exposé VIII]. A sheaf $\mathscr{M}$ on $\mathscr{X}_{\text{lis-ét}}$ is quasi-coherent if and only if there is a smooth cover $f : S \to \mathscr{X}$ with S a scheme such that $f^*\mathscr{M}$ is quasi-coherent (in the *lisse-étale site* of S) [52, §12]. Similarly, a sheaf on $\mathscr{X}_{\mathrm{fppf}}$ is quasi-coherent if and only if there is an fppf cover $S \to \mathscr{X}$ such that the pullback is quasi-coherent on the fppf site of S. A consideration of this fact combined with classical descent theory shows that it is enough to prove the lemma when $\mathscr{X}$ is a scheme. (For a host of arguments of this flavor, see [52, §13].) The last statement is a simple computation. □

The notion of quasi-coherence is also independent of the group chosen.

Lemma 2.2.1.4. *Under the equivalence of 2.1.3.13, quasi-coherent sheaves are taken to quasi-coherent sheaves.*

Proof. This follows from the fact that a sheaf is quasi-coherent if and only if it pulls back to a quasi-coherent lisse-étale sheaf on some smooth cover and the fact that any smooth cover of $\mathscr{X}_\alpha$ maps to a smooth cover of $\mathscr{X}_{\chi(\alpha)}$ under the map $\mathscr{X}_\alpha \to \mathscr{X}_{\chi(\alpha)}$. □

It is natural to wonder if a quasi-coherent twisted sheaf is the colimit of its coherent twisted subsheaves. With the proper foundation, this is now a triviality, as any subsheaf of a twisted sheaf is twisted and any quasi-coherent sheaf on a Noetherian algebraic stack is the colimit of its coherent subsheaves. We have shown the following.

Proposition 2.2.1.5. *Suppose X is Noetherian and A is group scheme faithfully flat of finite presentation over X. A quasi-coherent χ-twisted sheaf is the colimit of its coherent χ-twisted subsheaves.*

In fact, when A is special, we can split up the category of quasi-coherent sheaves into pieces indexed by characters. Suppose D is a diagonalizable affine group scheme (i.e., the Cartier dual of D is a constant finitely generated abelian group). Write C for the dual group of D, which is the group of homomorphisms $D \to \mathbf{G}_m$. Let $\mathscr{X}$ be a D-gerbe and $\mathscr{F}$ a quasi-coherent sheaf on $\mathscr{X}$. Given $\chi \in C$, there is a χ-eigensheaf $\mathscr{F}_\chi \subset \mathscr{F}$.

Proposition 2.2.1.6. *Suppose $\mathscr{F}$ is a quasi-coherent sheaf on $\mathscr{X}$. The natural maps induce an isomorphism*

$$\bigoplus_{\chi \in C(X)} \mathscr{F}_\chi \xrightarrow{\sim} \mathscr{F}.$$

The eigensheaves $\mathscr{F}_\chi$ are quasi-coherent.

Proof. By functoriality, it is enough to check this when $\mathscr{X}$ has a section, hence when $\mathscr{X} = \mathrm{B}D$. Applying descent theory to the covering $X \to \mathrm{B}D$ given by the trivial torsor, we see that a quasi-coherent sheaf in the fppf topology on $\mathrm{B}D$ is naturally identified with a quasi-coherent comodule for the group algebra of D on X. By the elementary theory of algebraic representations for diagonalizable group schemes [78], we are done. □

Remark 2.2.1.7. Note that since $\mathscr{F}_\chi$ may now be defined as the image of a certain morphism $\mathscr{F} \to \mathscr{F}$, we see *a posteriori* that it is quasi-coherent. Since the summands are quasi-coherent, we see that if we work in the lisse-étale topology set on $\mathscr{X}$, we get a similar decomposition. It is not entirely clear if the eigensheaves can be defined as equalizers in the lisse-étale topology when D has non-smooth factors. ♦

Let $Y \to X$ be a quasi-compact morphism of schemes and $\mathscr{X}$ a D-gerbe on X. Define $\mathscr{Y} := Y \times_X \mathscr{X}$; this is naturally a D-gerbe on Y. Denote the morphism $\mathscr{Y} \to \mathscr{X}$ by π.

Lemma 2.2.1.8. *π respects the decomposition of 2.2.1.6. In other words, if $\mathscr{F}$ is a quasi-coherent sheaf on $\mathscr{Y}$, then the natural map $\pi_*(\mathscr{F}_\chi) \to \pi_*\mathscr{F}$ identifies $\pi_*(\mathscr{F}_\chi)$ with $(\pi_*\mathscr{F})_\chi$.*

Proof. Since π is quasi-compact, π_* sends quasi-coherent sheaves to quasi-coherent sheaves. It is easy to see (because of the easy explicit description of the direct sum of modules) that π_* commutes with the formation of direct sums (but not colimits in general!). Thus, $\pi_*\mathscr{F} \cong \bigoplus \pi_*\mathscr{F}_\chi$. It is therefore enough to show that the action of D on $\pi_*\mathscr{F}_\chi$ is via χ. To see this, consider the action on sections. Since π is D-linear, we see that for any $U \to \mathscr{X}$, the diagram

$$\begin{array}{ccc}
\pi_*\mathscr{F}_\chi(U) \times D(U) & \longrightarrow & \pi_*\mathscr{F}_\chi(U) \\
\downarrow & & \\
\mathscr{F}_\chi(U \times_{\mathscr{X}} \mathscr{Y}) \times D(U) & & \downarrow \\
\downarrow & & \\
\mathscr{F}_\chi(U \times_{\mathscr{X}} \mathscr{Y}) \times D(U \times_{\mathscr{X}} \mathscr{Y}) & \longrightarrow & \mathscr{F}_\chi(U \times_{\mathscr{X}} \mathscr{Y})
\end{array}$$

commutes. Thus, if D is acting by χ on $\mathscr{F}_\chi$, it follows that it acts by χ on $\pi_*\mathscr{F}_\chi$. (One can also see this last statement by working in Căldăraru form, but that is less natural than the approach taken here.) □

Remark 2.2.1.9. When $A = \boldsymbol{\mu}_n$ with n invertible on X, one can show without too much trouble that in fact *all* $\mathscr{O}_{\mathscr{X}}$-modules decompose as a sum of eigensheaves. The point is that an action of $\boldsymbol{\mu}_n$ is étale locally (say over U) the same *on sections* as an action of the ring $\mathscr{O}(U)[x]/(x^n - 1)$ on the $\mathscr{O}(U)$-module $\mathscr{F}(U)$. When U is fine enough, $x^n - 1$ factors as $\prod(x - \zeta^i)$, which will split the module $\mathscr{F}(U)$ into factors which are seen to be precisely the sections of the eigensheaves. This trick does not work when n has zeros on X, unfortunately. ♦

Question 2.2.1.10. The natural question is "What happens when n is not invertible but $\mathscr{F}$ is required to be Cartesian?" One could ask this question on the lisse-étale topos or the fppf topos. (In either case, one must be careful about the definition of "Cartesian." Laumon and Moret-Bailly only require a Cartesian module on the lisse-étale site to have the property that $f^*M_V \cong M_U$ for $f : U \to V$ over $\mathscr{X}$ when f is *smooth*. In the case of quasi-coherent sheaves, this actually implies that condition for all f, making this distinction disappear.) It is quite confusing to think about this, and I do not recommend doing it for very long. (But if you figure it out, please tell me!)

2.2.2 Elementary applications

Here we investigate the applications of the theory we have developed so far to the study of the Brauer group. While seemingly vacuous, the theory of twisted sheaves yields many of the basic results on the Brauer group without requiring recourse to étale cohomology. For a sketch of a result on the structure of the derived category, see 2.2.4.

Fix a Noetherian scheme X and a $\mathbf{G}_m$-gerbe $\mathscr{X}$. Note that the condition that $n[\mathscr{X}] = 0 \in \mathrm{H}^2(X, \mathbf{G}_m)$ is equivalent to the existence of an n-fold twisted invertible sheaf on $\mathscr{X}$.

Lemma 2.2.2.1. *Let $\eta \subset X$ be the scheme of generic points. Any coherent $\mathscr{X}|_\eta$-twisted sheaf extends to a coherent $\mathscr{X}$-twisted sheaf.*

Proof. That such a sheaf extends to a quasi-coherent sheaf follows immediately from the expression in Căldăraru form 2.1.3.11. Applying 2.2.1.5 will complete the proof (as the generic fiber is coherent, the colimit will fill the generic fiber at some finite stage, yielding a coherent extension.) □

Lemma 2.2.2.2. *There exists a non-zero coherent twisted sheaf on X.*

Proof. Over the reduced structure on the generic scheme of X, we have a $\mathbf{G}_m$-gerbe over the spectrum of a finite product of fields. Thus, if we can produce a non-zero coherent twisted sheaf when X is the spectrum of a field, we can push it forward to get a such an object on the generic scheme of X and then apply 2.2.2.1. When X is $\operatorname{Spec} K$, any étale covering is finite over X. Thus, there is a finite free morphism $Y \to X$ such that the gerbe $\mathscr{Y} := Y \times_X \mathscr{X}$ has a section. Once there is a section, there is a natural equivalence between sheaves and twisted sheaves. Thus, there is a non-zero (in fact, locally free) twisted sheaf on $\mathscr{Y}$. Pushing forward along the morphism $\mathscr{Y} \to \mathscr{X}$ yields a non-zero coherent (in fact, locally free) $\mathscr{X}$-twisted sheaf by 2.2.1.8. (For a generalization of this argument, see 2.2.3.5 below.) A reader who is uncomfortable with this argument is invited to make it in Căldăraru form. □

Proposition 2.2.2.3. *If X is regular and quasi-compact then $\mathrm{H}^2(X, \mathbf{G}_m)$ is torsion.*

Proof. We may assume X is connected (and therefore irreducible). By 2.2.2.2, there exists a coherent $\mathscr{X}$-twisted sheaf F of rank $n > 0$. As X is regular and $\mathbf{G}_m$ is smooth and connected, the algebraic stack $\mathscr{X}$ is also regular and irreducible. Thus, F is perfect as an object of $\mathbf{D}(\mathrm{QCoh}(\mathscr{X}))$ with globally bounded amplitude. Applying the Mumford-Knudsen determinant to F is easily seen to yield an n-fold twisted invertible sheaf on $\mathscr{X}$. Thus, $n[\mathscr{X}] = 0 \in \mathrm{H}^2(X, \mathbf{G}_m)$. As $\mathscr{X}$ was an arbitrary $\mathbf{G}_m$-gerbe, we are done. □

Remark 2.2.2.4. It is natural to conjecture that the order of $[\mathscr{X}]$ in $\mathrm{Br}(X)$ is equal to the minimal positive rank of a coherent $\mathscr{X}$-twisted sheaf on X. The problem of deciding whether or not this is true is a special case of what is known as the *period-index problem*. We will study this problem below in chapter 6.

Proposition 2.2.2.5. *If X is regular and integral with generic scheme η, then the restriction map*

$$\mathrm{H}^2(X, \mathbf{G}_m) \to \mathrm{H}^2(\eta, \mathbf{G}_m)$$

is an injection.

Proof. Let $\mathscr{X} \times_X \eta$ represent the trivial cohomology class. This means that there is an invertible $\mathscr{X}_\eta$-twisted sheaf L_η. By 2.2.1.5, L_η has a coherent extension L on all of $\mathscr{X}$. In fact, there is a reflexive such extension. On the other hand, this extension has rank 1 by construction. But X, and therefore $\mathscr{X}$, is regular. As any reflexive module *of rank 1* over a regular local ring of arbitrary dimension is free [15, §VII.4], we conclude that L is an invertible twisted sheaf, whence $[\mathscr{X}] = 0 \in \mathrm{H}^2(X, \mathbf{G}_m)$. □

Proposition 2.2.2.6. *If X is local of dimension at most 1 then there exists a locally free $\mathscr{X}$-twisted sheaf of positive rank.*

Proof. This will be subsumed by 2.2.3.3 below; thus, we only give a sketch here.

First suppose X has dimension 0, so that X is the spectrum of an Artinian semilocal ring R. There is certainly a locally free twisted sheaf of positive rank over the residue field of R, and its deformations are unobstructed, yielding a locally free twisted sheaf on all of $\operatorname{Spec} R$. In fact, one can check that there is a unique locally free twisted sheaf of a given rank.

Applying 2.2.3.5 and 2.2.3.6 below, we may replace X by a finite semi-local extension $Y \to X$ such that $\mathscr{Y} := \mathscr{X} \times_X Y$ is trivial over every localization of Y. Choosing trivializations at each closed point of Y, it is immediate that there exists an integer N such that for any closed point $y \in Y$, there is a locally free twisted sheaf on $\operatorname{Spec} \mathscr{O}_{y,Y}$ of rank N. By the previous paragraph, these are all isomorphic on the generic scheme of Y (as y varies). It follows (since Y is of dimension 1 at each closed point) that we can glue the local twisted sheaves to produce a locally free $\mathscr{Y}$-twisted sheaf. □

Corollary 2.2.2.7. *If X is a local 1-dimensional Noetherian scheme,* $\mathrm{H}^2(X, \mathbf{G}_m)$ *is torsion and equals* $\mathrm{Br}(X)$.

Proof. As usual, once one has a locally free $\mathscr{X}$-twisted sheaf of positive rank n, one sees by taking the determinant that $n[\mathscr{X}] = 0 \in \mathrm{H}^2(X, \mathbf{G}_m)$. Applying 2.2.2.6, we are done. □

Proposition 2.2.2.8. *If X is integral and Noetherian there is a coherent twisted sheaf which is locally free at every point of codimension 1. If X is regular, there is a coherent twisted sheaf which is locally free at every point of codimension 2.*

Proof. There is certainly some open $V \subset X$ over which there is a locally free twisted sheaf F. Suppose $p \in X \setminus V$ has codimension 1 in X. Let $i : \operatorname{Spec} \mathscr{O}_{p,X} \hookrightarrow X$ and $j : V \hookrightarrow X$. By 2.2.2.6, there exists a locally free twisted sheaf G on $\operatorname{Spec} \mathscr{O}_{p,X}$ such that $G_\eta = F_\eta$ (identifying the generic schemes of $\operatorname{Spec} \mathscr{O}_{p,X}$ and V with η using i and j, respectively). Consider $Q := i_*G \cap j_*F \subset F_\eta$. This is a quasi-coherent twisted sheaf on X which equals F when restricted to V and G when pulled back by i. Applying 2.2.1.5 and the fact that localization commutes with colimits, we see that there is a coherent subsheaf $P \subset Q$ such that $P|_V \cong F$ and $P|_{\operatorname{Spec} \mathscr{O}_{p,X}} \cong G$. The locus where P is locally free is thus an open set containing $V \cup \{p\}$. By Noetherian induction, the first statement is proven. (More concretely, there can only be finitely many codimension 1 points p not in V, as they must be generic points of irreducible components of $X \setminus V$.) The second statement follows from the fact that any reflexive module over a regular local ring of dimension at most 2 is free. Thus, the reflexive hull of any coherent twisted sheaf will be locally free in codimension 2. □

Corollary 2.2.2.9. *If X is regular of dimension at most 2 everywhere then the inclusion* $\mathrm{Br}(X) \hookrightarrow \mathrm{H}^2(X, \mathbf{G}_m)$ *is an isomorphism.*

Proof. This follows from 2.2.2.8 and the fact that $[\mathscr{X}] \in \mathrm{Br}(X)$ if and only if there is a locally free $\mathscr{X}$-twisted sheaf of positive rank. □

2.2.2.10. Let $\mathscr{A}$ be an Azumaya algebra on X of degree n. In general, there is an exact sequence in the flat topology

$$1 \to \boldsymbol{\mu}_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1$$

which maps to the sequence

$$1 \to \mathbf{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

(When n is invertible on X, everything also holds as described in the étale topology.) Thus, the coboundary $\mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mathbf{G}_m)$ giving the Brauer class of an Azumaya algebra of degree n factors through the natural map $\mathrm{H}^2(X, \boldsymbol{\mu}_n) \to \mathrm{H}^2(X, \mathbf{G}_m)$. We will call the image of $[\mathscr{A}]$ in $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ the *class of* $\mathscr{A}$ and write $\mathrm{cl}(\mathscr{A}) \in \mathrm{H}^2(X, \boldsymbol{\mu}_n)$. There is an explicit construction of a $\boldsymbol{\mu}_n$-gerbe with this class.

Definition 2.2.2.11. The *gerbe of trivialized trivializations* of $\mathscr{A}$ has as sections over $V \to X$ triples $(\mathscr{V}, \varphi, \psi)$, where $\mathscr{V}$ is locally free of rank n on $\mathscr{V}$, $\varphi : \mathscr{E}nd(\mathscr{V}) \xrightarrow{\sim} \mathscr{A}$ is an isomorphism, and $\psi : \mathscr{O} \xrightarrow{\sim} \det \mathscr{V}$ is a trivialization. The isomorphisms of pairs $(\mathscr{V}, \varphi, \psi) \xrightarrow{\sim} (\mathscr{V}', \varphi', \psi')$ are isomorphisms $\mathscr{V}' \xrightarrow{\sim} \mathscr{V}$ which are compatible with the given maps, just as in 2.1.5.

The reader can easily check that this is a $\boldsymbol{\mu}_n$-gerbe representing the class of $\mathscr{A}$.

Let $\mathscr{X}$ be a fixed $\boldsymbol{\mu}_n$-gerbe. Note that the determinant of any $\mathscr{X}$-twisted sheaf of rank n naturally lies in $\mathrm{Pic}(X)$.

Proposition 2.2.2.12. *If $\mathscr{V}$ is a locally free $\mathscr{X}$-twisted sheaf of rank n, then*

$$\mathrm{cl}(\mathscr{E}nd(\mathscr{V})) - [\mathscr{X}] = \delta[\det \mathscr{V}],$$

where δ is the natural inclusion $\mathrm{Pic}\, X / n \,\mathrm{Pic}\, X \hookrightarrow \mathrm{H}^2(X, \boldsymbol{\mu}_n)$.

Proof. We work in Căldăraru form, i.e., we work with explicit cocycles on coverings. It is easy to see that the $\boldsymbol{\mu}_n$-gerbe corresponding to an Azumaya algebra $\mathscr{A}$ has as its sections over U triples $(\mathscr{W}, \varphi, \tau)$ where $\mathscr{W}$ is a locally free sheaf, $\varphi : \mathscr{E}nd(\mathscr{W}) \xrightarrow{\sim} \mathscr{A}$ is an isomorphism of $\mathscr{O}$-algebras, and $\tau : \det \mathscr{W} \xrightarrow{\sim} \mathscr{O}$ is a trivialization of the determinant. We may assume without loss of generality that $\mathscr{X}$ is associated to a cocycle α for $\boldsymbol{\mu}_n$ on a hypercovering $U_\bullet$ of X such that $\mathscr{V}$ is a free sheaf on U_0 with a global isomorphism of $\mathscr{V}_0$ and $\mathscr{V}_1$ on U_1. Since $\mathscr{V}$ is free on U_0, it has a trivial determinant. Fixing an isomorphism $\tau : \det \mathscr{V} \xrightarrow{\sim} \mathscr{O}_{U_0}$, we see that the gluing datum $\varphi : \mathscr{V}_0 \xrightarrow{\sim} \mathscr{V}_1$ yields via τ a gluing datum $\gamma : \mathscr{O} \xrightarrow{\sim} \mathscr{O}$ on the trivial line bundle. By the multiplicative property of the determinant and the fact that α is in $\boldsymbol{\mu}_n$, γ is actually a descent datum, and the resulting invertible sheaf L on X is just (the pushforward of) the determinant of $\mathscr{V}$. To construct the cocycle corresponding to the $\boldsymbol{\mu}_n$-gerbe of $\mathscr{A}$, we need to alter φ so that $\gamma = 1$. In other words, we need to multiply φ by a chosen nth root of γ. Writing this out shows exactly that this changes the cohomology class by adding the coboundary of L. □

Corollary 2.2.2.13. *Let $\mathscr{X}$ be a $\boldsymbol{\mu}_n$-gerbe. An Azumaya algebra $\mathscr{A}$ of degree n has class $[\mathscr{X}]$ if and only if there is a locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$ of rank n and trivial determinant such that $\mathscr{A} \cong \mathscr{E}nd(\mathscr{V})$.*

2.2.3 Gabber's theorems

It is well-known that the Brauer group and cohomological Brauer group of an affine scheme coincide. This result is originally in Gabber's thesis [28]. We wish to show how twisted sheaves may be used to give an especially streamlined proof. Our argument is a simplification of the argument of Hoobler [39] which is itself a simplification of Gabber's proof. The point here is that using twisted sheaves allows one to "think in modules" from the beginning, making the final recourse to K-theory completely natural and eliminating certain invertible annoyances which appear when one is always working with endomorphism algebras. Our approach notably also avoids the comparison of the "Mayer-Vietoris sequence" in non-Abelian flat cohomology with that in ordinary Abelian flat cohomology, by absorbing all of the cohomology (Abelian and otherwise) into the underlying gerbe $\mathscr{X}$. The outline of our proof comes from Hoobler's paper [*ibid.*].

Let R be a commutative unital ring. Following Hoobler, let $M(R)$ denote the set of faithfully projective R-modules modulo the equivalence relation $P \sim Q$ if there are positive integers n and m such that $P^{\oplus n} \cong Q^{\oplus m}$. The tensor power operation induces a $\mathbf{Z}$-action on $M(R)$.

Proposition 2.2.3.1. *$M(R)$ is a $\mathbf{Q}$-vector space.*

Proof. The proof of the proposition is non-trivial. See [39] for a (slightly cryptic) reference. □

Corollary 2.2.3.2. *Given a faithfully projective R-module P and a positive integer n, there exist positive integers m and m' and a faithfully projective R-module $\overline{P}$ such that $\left(P \otimes \overline{P}^{\otimes n}\right)^{\oplus m} \cong R^{\oplus m'}$.*

Theorem 2.2.3.3. *Let X be an affine scheme and $\mathscr{X}$ an fppf $\boldsymbol{\mu}_n$-gerbe on X. There exists an $\mathscr{X}$-twisted locally free sheaf of constant finite non-zero rank.*

Corollary 2.2.3.4. *If X is an affine scheme then the natural injection $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$ is an isomorphism.*

Proof. Any torsion class $\alpha \in \mathrm{H}^2(X, \mathbf{G}_m)$ (taken in the flat topology or étale topology) has a lift to a flat cohomology class in $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$. The theorem gives a twisted vector bundle on a gerbe in any such class. Taking its endomorphism algebra yields an Azumaya algebra with class α. □

To prove 2.2.3.3 we first need a few lemmas.

Lemma 2.2.3.5. *If $Y \to X$ is a finite locally free covering and $\alpha \in \mathrm{H}^2(X, \mathbf{G}_m)$, then there is a nowhere zero locally free twisted sheaf on Y if and only if there is such a twisted sheaf on X. If X is quasi-compact, the same holds for locally free twisted sheaves of finite constant non-zero rank.*

Proof. Fixing a gerbe $\mathscr{X}$ representing α, we see that $\mathscr{Y} := \mathscr{X} \times_X Y$ represents the pullback of α to Y. Furthermore, $\pi : \mathscr{Y} \to \mathscr{X}$ is a finite locally free morphism. Thus, given any locally free twisted sheaf F on $\mathscr{Y}$, $\pi_* F$ will be a locally free twisted sheaf on $\mathscr{X}$. The second statement is left to the reader. □

Lemma 2.2.3.6. *Given a local ring A and a local-étale A-algebra B, there exists a finite free A-algebra C such that for all maximal ideals $\mathfrak{m} \subset C$, there is a map $B \to C_{\mathfrak{m}}$.*

Proof. The proof uses the local structure of étale morphisms: B is a localization of $A[x]/f(x)$ at a prime not containing $f'(x)$. The algebra C is then the "universal splitting algebra" of f. See [28, 39] for details of the lemma and [63] for the local structure of étale maps. □

Combining 2.2.3.6 and 2.2.3.5, we see that to prove 2.2.3.3 it is enough to prove the following.

Proposition 2.2.3.7. *Let $X = \operatorname{Spec} A$ be affine, $\mathscr{X}$ a $\boldsymbol{\mu}_n$-gerbe such that there is everywhere Zariski-locally on X a locally free $\mathscr{X}$-twisted sheaf. Then there is a global $\mathscr{X}$-twisted sheaf.*

Proof. By Quillen induction, this reduces to the following: we have $X = U \cup V$ with U, V, and $U \cap V$ all affine and locally free twisted sheaves on each of U, V, and $U \cap V$. (The words "Quillen induction" are just meant to intimidate the reader. The point is that it is enough to show that

$$J = \{f \in A \mid \text{there is a locally free twisted sheaf on } \operatorname{Spec} A_f\}$$

is an ideal. The Zariski-local existence hypothesis shows that J cannot be contained in any maximal ideal, hence if it is an ideal $J = A$. The situation above arises from the only non-trivial part of checking that J is an ideal, namely that it is closed under addition.)

Let $\mathscr{P}$ be a locally free twisted sheaf on U and $\mathscr{Q}$ a locally free twisted sheaf on V. Since $\mathscr{X}$ is a $\boldsymbol{\mu}_n$-gerbe, we see that $\mathscr{P}^{\otimes n}$ is naturally identified with a locally free untwisted sheaf, and similarly for $\mathscr{Q}$. By 2.2.3.2, we see that there are non-zero locally free sheaves $\overline{P}$ on U, $\overline{Q}$ on V and non-zero finite free modules F_0, F_1 on U, G_0, G_1 on V such that

$$\mathscr{P}^{\otimes n} \otimes \overline{P}^{\otimes n} \otimes F_0 \cong F_1$$

and

$$\mathscr{Q}^{\otimes n} \otimes \overline{Q}^{\otimes n} \otimes G_0 \cong G_1.$$

Thus, replacing $\mathscr{P}$ by $\mathscr{P} \otimes \overline{P} \otimes F_0$ and $\mathscr{Q}$ by $\mathscr{Q} \otimes \overline{Q} \otimes G_0$, we may assume that $\mathscr{P}^{\otimes n}$ and $\mathscr{Q}^{\otimes n}$ are free modules on U and V respectively of the same rank. Now consider the situation on $U \cap V$. Note that $\mathscr{P} \otimes (\mathscr{Q}^\vee \otimes \mathscr{Q}) \cong \mathscr{Q} \otimes (\mathscr{Q}^\vee \otimes \mathscr{P})$. Letting $P = \mathscr{Q}^\vee \otimes \mathscr{Q}$ and $Q = \mathscr{P} \otimes \mathscr{Q}^\vee$, we have an isomorphism of locally free twisted sheaves $\mathscr{P} \otimes P \cong \mathscr{Q} \otimes Q$. Furthermore, taking nth tensor powers, we see that $P^{\otimes n} \sim Q^{\otimes n}$ (in the equivalence relation of 2.2.3.1). Thus, by 2.2.3.1, $P \sim Q$, so there exist positive integers N and M such that $P^{\oplus N} \cong Q^{\oplus M}$. Furthermore, there exists $\overline{Q}$ such that $Q \otimes \overline{Q} \cong F_0$, a non-zero free module. As $P \sim Q$, there are non-zero free modules F_1 and F_2 with $P \otimes \overline{Q} \otimes F_1 \cong F_2$. Letting $\widetilde{Q} = \overline{Q} \otimes F_1$, we have $Q \otimes \widetilde{Q} \cong F_3$ and $P \otimes \widetilde{Q} \cong F_4$ for some non-zero free modules F_3 and F_4 of rank r. Finally,

$$\mathscr{P}^{\oplus r} \cong \mathscr{P} \otimes P \otimes \widetilde{Q} \cong \mathscr{Q} \otimes Q \otimes \widetilde{Q} \cong \mathscr{Q}^{\oplus r}$$

on $U \cap V$. Thus, the locally free twisted sheaves $\mathscr{P}^{\oplus r}$ on U and $\mathscr{Q}^{\oplus r}$ on V glue to give an everywhere non-zero locally free twisted sheaf on X. □

Corollary 2.2.3.8. *If X admits an ample invertible sheaf, then the natural injection* $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$ *is an isomorphism.*

Proof. See [19]. The idea is to start with a supply of twisted sheaves which are locally free at selected points, and by making kernels of general morphisms between them to increase the locus where the twisted sheaf is locally free. The ample invertible sheaf enables one to make a Bertini argument when studying a general map between twisted sheaves (tensored with powers of the ample invertible sheaf). □

2.2.4 Compact generation of the derived category

In this section, we indicate how to prove a result about the K-theory of twisted sheaves. It is an immediate corollary of 2.2.3.3 and the methods of Neeman [14, 56]. In fact we prove a slightly more general result about the K-theory of stacks with certain "moduli-like" spaces and a large supply (locally) of vector bundles. We also sketch a proof that the natural map $\mathbf{D}(\mathrm{QCoh}(\mathscr{X})) \to \mathbf{D}_{\mathrm{QCoh}}(\mathscr{X})$ is an equivalence for certain algebraic stacks $\mathscr{X}$. In particular, this holds for gerbes over schemes, so in our setting we see that the derived category $\mathbf{D}_{\mathrm{QCoh}}(\mathscr{X})$ breaks up as a coproduct of categories according to twisting class. As none of the results of this section are strictly necessary for the rest of this thesis, we only sketch the proofs. Details will appear along with a development of algebraic twisted K-theory in future work. The reader who is uninterested in this material or unwilling to read something with no details should look at 2.2.4.7 and skip the rest of the section.

Fix a quasi-compact quasi-separated base scheme S, and let $\mathscr{X}$ be an Artin stack over S. Let X be an algebraic space over S.

Definition 2.2.4.1. A map $\pi : \mathscr{X} \to X$ is called a *cohomological moduli space* if π_* preserves quasi-coherence and $\mathbf{R}^i\pi_* F = 0$ for all quasi-coherent sheaves F on $\mathscr{X}$ and all $i > 0$. If X is a scheme, it will be called a cohomological moduli scheme.

The property of being a cohomological moduli space is clearly preserved by flat base change.

Remark 2.2.4.2. It is not hard to show that the moduli space of a tame DM stack is a cohomological moduli space. If $\mathscr{X}$ is corepresented by an algebraic space X, then any cohomological moduli space X' for $\mathscr{X}$ admits a map $X \to X'$. If $\mathscr{X}$ is DM, it is corepresented by its moduli space $\pi : \mathscr{X} \to X$. Furthermore, for any quasi-coherent

F on X, the natural map $F \to \pi_*\pi^*F$ is an isomorphism. When $\mathscr{X}$ is Noetherian with separated coarse moduli space, we conclude by Serre's theorem that in this case there is an equivalence between cohomological moduli spaces for $\mathscr{X}$ and affine morphisms $X \to X'$. Thus, for example, if a DM stack admits a (quasi-projective) cohomological moduli *scheme*, then its coarse moduli space is also a (quasi-projective) scheme. ♦

The following definition is an adaptation of a definition made by Olsson and Starr [58].

Definition 2.2.4.3. A locally free sheaf F on $\mathscr{X}$ is a *generating sheaf relative to* π (or a π*-generating sheaf*) if the natural map

$$\pi^*\pi_*\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(F, Q) \otimes F \to Q$$

is a surjection for every quasi-coherent sheaf Q on $\mathscr{X}$.

Definition 2.2.4.4. A cohomological moduli scheme $\pi : \mathscr{X} \to X$ is *locally generating* if there is an affine cover $\{U_i\}$ of X such that $\mathscr{X} \times_X U_i$ admits a π-generating locally free sheaf for all i.

Remark 2.2.4.5. The question of when a cohomological moduli scheme is locally generating is a subtle question. Olsson and Starr [58] have shown that any separated tame *quotient* DM stack over a quasi-compact algebraic space is globally generating over its moduli space. Combining this with results of de Jong [19] and Vistoli-Kresch [47], it follows that any smooth separated tame DM stack over a field is globally generating over its moduli space. It follows from the remarks in 2.2.4.2 that a separated DM stack admits a locally generating cohomological moduli scheme if and only if its coarse moduli space is a scheme and is locally generating. ♦

Proposition 2.2.4.6. *If* $\mathscr{X}$ *is an algebraic stack admitting a quasi-compact separated locally generating cohomological moduli scheme then the natural map* $\mathbf{D}(\mathrm{QCoh}(\mathscr{X})) \to \mathbf{D}_{\mathrm{QCoh}}(\mathscr{X})$ *is an equivalence.*

Remark 2.2.4.7. It is important to note that the methods of Hartshorne [37] may be applied to show that for a locally Noetherian algebraic stack $\mathscr{X}$, the functor $\mathbf{D}^+(\mathrm{QCoh}(\mathscr{X})) \to \mathbf{D}^+_{\mathrm{QCoh}}(\mathscr{X})$ is an equivalence. As usual, one finds enough quasi-coherent injectives which remain injective in $\mathrm{Mod}_{\mathscr{O}_{\mathscr{X}}}$. Since we will usually have boundedness conditions on our complexes, this will suffice. (Sketch of proof: for a smooth map $f : U \to \mathscr{X}$ from a locally Noetherian scheme, f_* of any quasi-coherent injective is a quasi-coherent injective. It will be injective in the category of all modules if it was on U. Thus, the result is reduced to the result on a locally Noetherian scheme.) So, modulo the work needed to classify injective modules on a Noetherian scheme, it is very easy to show the equivalence in the bounded case with enough Noetherian hypotheses, independent of the existence of any kind of cohomological moduli space. ♦

Lemma 2.2.4.8. *If* $\mathscr{X}$ *is a quasi-compact quasi-separated algebraic stack then the category* $\mathrm{QCoh}(\mathscr{X})$ *has all limits.*

We have added the adjective "quasi-separated" to emphasize that it is needed for the truth of the lemma. The reader will recall that all algebraic stacks considered in this thesis are quasi-separated (as in [52]).

Proof. This is a simple result of the theory of the "coherator" [2, 74]: let $f : X \to \mathscr{X}$ be a smooth covering by an affine scheme and $Y \to X \times_{\mathscr{X}} X$ a covering by an affine; let $g : Y \to \mathscr{X}$ denote the map. Given any sheaf F on X, let $Q_X F$ be the quasi-coherent sheaf associated to $\Gamma(X, F)$, and similarly for Y. Any sheaf $\mathscr{F}$ on $\mathscr{X}$ gives rise to a pair of maps $h_i : f_* Q_X \mathscr{F}_X \to g_* Q_Y \mathscr{F}_Y$, $i = 1, 2$. As f and g are quasi-compact, f_* and g_* preserve quasi-coherence. Thus, the equalizer of h_1 and h_2 will be a quasi-coherent sheaf which we will denote by $Q_{\mathscr{X}} \mathscr{F}$. This is easily seen to provide a right adjoint to the inclusion $\mathrm{QCoh}(\mathscr{X}) \to \mathrm{Mod}_{\mathscr{O}_{\mathscr{X}}}$. It is then immediate that limits exist in $\mathrm{QCoh}(\mathscr{X})$. For example, to form the product $\prod_i \mathscr{F}_i$ in $\mathrm{QCoh}(\mathscr{X})$, one simply takes $Q_{\mathscr{X}} \prod_i^{\mathscr{O}-\mathrm{mod}} \mathscr{F}_i$. □

Lemma 2.2.4.9. *If $\mathscr{X}$ is an algebraic stack admitting an affine generating cohomological moduli space then* $\mathrm{QCoh}(\mathscr{X})$ *satisfies AB4* (products of exact sequences are exact).*

Proof. Let $F_i \to G_i$ be a set of surjections of quasi-coherent sheaves on $\mathscr{X}$. We know that $\prod_i F_i$ and $\prod_i G_i$ exist in $\mathrm{QCoh}(\mathscr{X})$ (but their images in the category of all sheaves is not the product). We wish to show that the natural map $\prod_i F_i \to \prod_i G_i$ is a surjection. Let

$$0 \to K \to \prod_i F_i \to \prod_i G_i \to C \to 0$$

be exact. Let $\mathscr{E}$ be a π-generating sheaf. Applying the functor $\mathscr{H}om(\mathscr{E}, \cdot\,)$ preserves exactness. Since π is acyclic for quasi-coherent sheaves and X is affine, we see that there is an exact sequence

$$0 \to \mathrm{Hom}(\mathscr{E}, K) \to \mathrm{Hom}(\mathscr{E}, \prod_i \mathscr{F}_i) \to \mathrm{Hom}(\mathscr{E}, \prod_i \mathscr{G}_i) \to \mathrm{Hom}(\mathscr{E}, C) \to 0.$$

But these products are *in the category* $\mathrm{QCoh}(\mathscr{X})$. We conclude from the fact that the category of Abelian groups satisfies AB4* that $\mathrm{Hom}(\mathscr{E}, C) = 0$. Since $\mathscr{E}$ is a generator for QCoh, we see that $C = 0$ and therefore $\mathrm{QCoh}(\mathscr{X})$ satisfies AB4*. □

As a consequence, products of exact sequences in $\mathrm{QCoh}(\mathscr{X})$ are exact, and cohomology sheaves of products are just the products of the cohomology sheaves (all products being taken in QCoh).

Proof of 2.2.4.6. We only sketch the proof, the lemmas above providing some of the details which are necessary to fill in the skeleton provided by the methods of Bökstedt and Neeman [14]. The reader interested in filling everything in (and reading a beautiful paper) is advised to consult the source. Let $E(\mathscr{X})$ be the full subcategory of $K(\mathscr{X})$ of acyclic complexes. It is easy to see that $E(\mathscr{X})$ is a localizing subcategory (in the sense of Bousfield). The proof proceeds in several steps.

1. Any complex $Z \in K(\mathscr{X})$ admits a Bousfield localization $Z \to Z_\ell$. To prove this, note that classical methods using injective resolutions allow one to produce such a map $Z^{\geq n} \to Z_\ell^{\geq n}$ for any truncation. One would then like to say that since Z is the homotopy limit of the $Z^{\geq n}$, we can construct Z_ℓ by forming the homotopy limit of the $Z^{\geq n}$. Unfortunately, the category $K(\mathscr{X})$ does not satisfy AB4*, so this argument needs careful justification. Using the fact that the cohomology sheaves are quasi-coherent (hence have vanishing cohomology on an affine), one can show that everything works. (Recent work [7] has provided a more general proof than Neeman and Bökstedt's proof, but the method of Neeman and Bökstedt is used elsewhere in the proof of the present result.)

2. When the cohomological moduli space X is affine, one can now show using the locally free generator and 2.2.4.9 that the map $\mathbf{D}(\mathrm{QCoh}(\mathscr{X})) \to \mathbf{D}(\mathscr{X})$ is fully faithful. In other words, given Y and Z, one wants to show that $\mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}(\mathscr{X}))}(Y, Z) = \mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(Y, Z)$. The key is to use the expression of Z as the homotopy limit of its truncations in $K(\mathrm{QCoh}(\mathscr{X}))$ and as the homotopy limit of the Bousfield localizations of its truncations in $K(\mathscr{X})$ to reduce the problem to the case where Z is bounded below. Using easier homotopy colimit arguments (easier because the map $\mathbf{D}(\mathrm{QCoh}(\mathscr{X})) \to \mathbf{D}(\mathscr{X})$ preserves homotopy colimits), one reduces to the case where Y is bounded above. Now one can use the locally free generator $\mathscr{E}$ to replace Y by a complex of sheaves of the form $\mathscr{E}^I$, a direct sum indexed by the set I. Using the fact that such a complex is the colimit of its finite subcomplexes, one is reduced to the case $Y = \mathscr{E}$. Now by standard techniques, one reduces to the case $Z = \Sigma^n M$ is a single module placed in degree $-n$. Thus, one is reduced to showing that $\mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}(\mathscr{X}))}(\mathscr{E}, \Sigma^n M) = \mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\mathscr{E}, \Sigma^n M)$. Both sides vanish for $n < 0$ and are equal to $\mathrm{Hom}(\mathscr{E}, M)$ for $n = 0$. When $n > 0$, the left side vanishes because $\mathrm{Hom}(\mathscr{E}, \,\cdot\,)$ is exact on QCoh, while the right side vanishes because it is the same as $\mathrm{H}^n(\mathscr{X}, \mathscr{H}om(\mathscr{E}, M))$, and one can apply the Leray spectral sequence for π and the vanishing of quasi-coherent cohomology on an affine.

3. To finish the proof when the cohomological moduli space is affine, it remains to show that every object Z of $\mathbf{D}_{\mathrm{QCoh}}(\mathscr{X})$ is in the essential image of $\mathbf{D}(\mathrm{QCoh}(\mathscr{X}))$. If Z is bounded, then this follows by induction on the number of cohomology sheaves from the case where Z is a quasi-coherent sheaf. When Z is bounded below, the expression of Z as the homotopy colimit of $Z^{\leq n}$ and the compatibility of the map $\mathbf{D}(\mathrm{QCoh}(\mathscr{X})) \to \mathbf{D}(\mathscr{X})$ reduces this to the bounded case. The general case now follows from the expression of a Bousfield local Z as the homotopy limit of $Z^{\geq -n}$, along with 2.2.4.9.

4. The general case now follows by induction on the number of affines in a cover of X. Given one affine $j : U \to X$ in the covering, with complement $i : X \backslash U \to X$, one uses the local cohomology triangle $i_* i^! Z \to Z \to j_* j^* Z$ to reduce to the case of U and $X \setminus U$, which has a cover by one fewer affines.

□

Lemma 2.2.4.10. *Let $\pi : \mathscr{X} \to X$ be a generating affine cohomological moduli space. The category $\mathbf{D}_{\mathrm{QCoh}}(\mathscr{X})$ is compactly generated by perfect complexes. Any compact object is perfect.*

Proof. See [56]. It is easy to see that the suspensions of a generator give a family of compact generators which are perfect. One shows that any perfect complex is compact (as the problem may be made local, this follows immediately from the case of a scheme). Using Neeman's version of the Thomason localization theorem, one sees that in fact any compact object is perfect. The proof is purely formal. □

Lemma 2.2.4.11. *Let $\pi : \mathscr{X} \to X$ be a generating affine cohomological moduli scheme, $U \subset X$ an open subscheme. The category $\mathbf{D}_{\mathscr{X}\setminus\mathscr{U},\mathrm{QCoh}}(\mathscr{X})$ is compactly generated by the suspensions of a single perfect complex, where $\mathscr{U} = \mathscr{X} \times_X U$.*

Proof. The category in question is the subcategory of complexes supported on $\mathscr{X}\setminus\mathscr{U}$. This is just as in [14, 6.1], using the generator in place of the structure sheaf. □

Lemma 2.2.4.12 (Neeman). *Suppose $\mathscr{X} \to X$ is a generating cohomological moduli scheme, with X quasi-compact and separated. Let $U \subset X$ be a quasi-compact open subscheme. Suppose $x \in \mathbf{D}(\mathrm{QCoh}(\mathscr{X}))$ is arbitrary, $u \in \mathbf{D}(\mathrm{QCoh}(\mathscr{U}))$ is perfect, and we are given a map $u \to x$ in $\mathbf{D}(\mathrm{QCoh}(\mathscr{U}))$. There exists a perfect complex $u' \in \mathbf{D}(\mathrm{QCoh}(\mathscr{U}))$ such that*

(i) there is a perfect complex $\tilde{u} \in \mathbf{D}(\mathrm{QCoh}(\mathscr{X}))$ such that $\tilde{u}|_{\mathscr{U}} \cong u \oplus u'$, and

(ii) the map $u \oplus u' \to x|_{\mathscr{U}}$ lifts to a map $\tilde{u} \to x$.

Proof. The proof is exactly as in [56, 2.6] (with slight modifications to account for the replacement of U with $\mathscr{U}$, etc.). □

Proposition 2.2.4.13. *Let $\mathscr{X}$ be an algebraic stack admitting a quasi-compact separated locally generating cohomological moduli scheme. Then $\mathbf{D}(\mathrm{QCoh}(\mathscr{X}))$ is compactly generated by perfect complexes.*

Proof. One applies 2.2.4.12 as in [56, 2.5]. The fact that $\mathbf{D}(\mathrm{QCoh}(\mathscr{X}))$ is compactly generated is crucial for applying Neeman's form of the Thomason localization theorem. □

Corollary 2.2.4.14. *Let X be a quasi-compact separated scheme and $\mathscr{X}$ a $\mathbf{G}_m$-gerbe on X. The derived category of quasi-coherent twisted sheaves is compactly generated by perfect complexes.*

Proof. This follows from 2.2.4.13 combined with the decomposition of $\mathrm{QCoh}(\mathscr{X}) = \bigoplus \mathrm{QCoh}^i(\mathscr{X})$ and the fact that a summand of a perfect complex is perfect. □

2.2.5 Deformations and obstructions

Since twisted sheaves are modules in a topos, we can try to apply the deformation theory of Illusie to study deformations and obstructions of twisted sheaves. The condition that a deformation preserve the character of the inertial action and that an obstruction take this into account makes the situation slightly more complicated than in Illusie's bare theory. We present two approaches to the deformation theory of twisted sheaves. The first, which we call the "naïve approach," originates with Grothendieck (and thus we should point out to the reader that by "naïve" we mean "devoid of extensive machinery" and not "silly"!). The second is simply to make enough hypotheses that Illusie's theory works in our context. In particular, restricting attention to quasi-coherent twisted sheaves will turn out to be sufficient for Illusie's machine to work for twisted sheaves. (One of the reasons we have included the first approach at all is to actually write down some of the proofs using a slightly more "homotopical" approach to the derived category, which we feel illuminates things.)

2.2.5.1. *The naïve approach.* We first indicate an approach to deformation and obstruction theory for modules, due to Grothendieck, which does not use Illusie's machinery. As we will be primarily interested in quasi-coherent twisted sheaves, where Illusie's theory will apply as we will see, we only sketch this approach. In this section, we work in the abelian category of twisted sheaves. Thus, all Ext groups are computed in this category and not in the larger category of all sheaves on $\mathscr{X}$. When we specialize to quasi-coherent twisted sheaves, this will no longer matter, as both Ext spaces are naturally isomorphic.

Lemma 2.2.5.2. *The category of $\mathscr{X}$-twisted sheaves contains enough injectives and enough flat objects.*

Proof. Let $U \in X$ be an object over which $\mathscr{X}$ splits and let $\mathscr{U} = \mathscr{X} \times_X U$, with natural map $f : \mathscr{U} \to \mathscr{X}$. Then $\mathscr{U} \cong \mathbf{B}\mathbf{G}_{m,U}$ as U-stacks. By 2.1.3.10, there are enough twisted injectives and flat objects on $\mathscr{U}$. Taking f_* of injectives and $f_!$ of flat objects yields the desired result. The details are left to the reader (or see [6]). □

The reader will note that the proof of 2.2.5.2 actually shows that any bounded above complex C of twisted sheaves admits a resolution $F \to C$ where each term of F is a sum of sheaves of the form $\pi_! L$, with $\pi : \mathscr{X} \times_X U \to \mathscr{X}$ and L a twisted invertible sheaf on $\mathscr{X} \times_X U$. This explicit description will be used below.

We will use the "cher à Cartan" isomorphism to produce a naïve deformation and obstruction theory for twisted sheaves (without making use of the whole topos of sheaves on the gerbe). This is a slightly sticky issue, so we give a brief exposition combining ideas of Neeman-Bökstedt and Spaltenstein. For the most part, we only sketch the results without giving full proofs (as the full-blown theory of Illusie will apply in our case).

Remark 2.2.5.3. The reader should note that this whole rigamarole is necessary only owing to the fact that it is not at all clear than an injective twisted sheaf is injective in the category of all modules on $\mathscr{X}$. This may be true, but I have no idea how to prove it (when the twisted sheaf is not quasi-coherent on a scheme!). ♦

We recall some definitions due to Spaltenstein [71]. Let $\mathbf{A}$ be a monoidal abelian category satisfying AB4 (filtered colimits exist and are exact). For example, $\mathbf{A}$ could be the category of abelian sheaves on a site. Given complexes A and B of objects of $\mathbf{A}$, write $\underline{\mathrm{Hom}}(A, B)$ for the usual complex of homomorphisms.

Definition 2.2.5.4. A complex $C \in K(\mathbf{A})$ is *K-injective* if the complex $\underline{\mathrm{Hom}}(D, C)$ is acyclic for every acyclic object D. (In other words, C is Bousfield local for the localizing subcategory of acyclic complexes.) It is *K-flat* if $D \otimes C$ is acyclic for every acyclic complex D. It is *weakly K-injective* if $\underline{\mathrm{Hom}}(D, C)$ is acyclic for every acyclic K-flat complex D (in other words, if it is Bousfield local for the localizing subcategory of K-flat acyclic complexes).

Lemma 2.2.5.5. *If C is a complex of $\mathscr{X}$-twisted sheaves then C admits a K-flat resolution $F \to C$ and a K-injective resolution $C \to I$.*

Proof. The first statement follows from standard techniques in the theory of homotopy colimits [14], combined with 2.2.5.2. The second statement has been proven recently by Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio [7] for any Grothendieck abelian category. When C is bounded below, the second statement is a standard result (again using 2.2.5.2) [37] dating back to Cartan and Eilenberg. □

Lemma 2.2.5.6. *Let $B \to B_0$ be a morphism of rings in a topos X. A K-injective complex C of twisted B_0-modules is weakly K-injective as a complex of twisted B-modules.*

Proof. If S is a K-flat acyclic complex of B-modules, then

$$\underline{\mathrm{Hom}}_B(S, C) = \underline{\mathrm{Hom}}_{B_0}(S \otimes B_0, C).$$

As S is K-flat and acyclic, one can show (using the existence of K-flat resolutions of B_0) that $S \otimes B_0$ is acyclic. As C is K-injective over B_0, we conclude that $\underline{\mathrm{Hom}}(S, C)$ is acyclic, whence C is weakly K-injective. □

Lemma 2.2.5.7. *Let A and B be complexes of twisted sheaves. If $F \to A$ is a K-flat resolution and $B \to I$ is a weakly K-injective resolution, then $\underline{\mathrm{Hom}}(F, I)$ is quasi-isomorphic to $\mathbf{R}\mathrm{Hom}(A, B)$ (defined on the category of twisted sheaves).*

Proof. It suffices to show that if $I \to J$ is a K-injective resolution then $\mathrm{Hom}(F, I) \to \mathrm{Hom}(F, J)$ is a quasi-isomorphism. As any K-injective complex is weakly K-injective, we easily see that the homotopy cofiber M (mapping cone) of $I \to J$ is weakly K-injective and acyclic. By the theory of homotopy colimits, we may replace F by a complex whose terms are all sums of sheaves of the form $\pi_! L$, L an invertible twisted sheaf on some $\mathscr{X} \times_X U$. It suffices to show that for one such term the complex $\underline{\mathrm{Hom}}(\pi_! L, M)$ is acyclic. Since $\pi_!$ preserves K-flatness and tensoring by L preserves weak K-injectivity, we are reduced to showing that given a weakly K-injective complex M sheaves on a topos and any object U of the topos, the complex of sections $M(U)$ is acyclic. The clever argument may be found in [71, 5.16]. (Note that while Spaltenstein's argument is written on topological spaces, nothing about the result quoted uses this degree of specificity.) □

Proposition 2.2.5.8. *Let $B \to B_0$ be a morphism of rings in X and $\mathscr{X} \to X$ a $\mathbf{G}_m$-gerbe. Given a complex of $\mathscr{X}$-twisted B-modules M and a complex of $\mathscr{X}$-twisted B_0-modules J, there is a natural isomorphism in the derived category*

$$\mathbf{R}\mathrm{Hom}_B(M, J) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{B_0}(M \overset{\mathbf{L}}{\underset{B}{\otimes}} B_0, J).$$

Proof. Note that $M \overset{\mathbf{L}}{\otimes}_B B_0$, while *a priori* a B-module, has a natural structure of B_0-module. Thus, the statement makes sense! To prove it, we just apply 2.2.5.7: let $F \to M$ be a K-flat resolution of B-modules and $J \to I$ a K-injective resolution of B_0-modules. Then the complex $\mathrm{Hom}(F, I)$ computes $\mathbf{R}\mathrm{Hom}_B(M, J)$, but clearly $\mathrm{Hom}_B(F, I) = \mathrm{Hom}_{B_0}(F \otimes_B B_0, I)$, and the latter represents $\mathbf{R}\mathrm{Hom}_{B_0}(F \overset{\mathbf{L}}{\otimes} B_0, I)$, as I is K-injective in Mod_{B_0}. □

This is the derived adjointness of $\overset{\mathbf{L}}{\otimes}_B B_0$ and (derived) restriction of scalars to B. (Note that this also applies when the gerbe $\mathscr{X}$ is trivial, so in particular in any topos.) We will use this below in 2.2.5.17 to deduce localization and constructibility properties for the deformation and obstruction theory of twisted sheaves.

Corollary 2.2.5.9 (cher à Cartan). *Let $B \to B_0$ be a surjection of rings in X and $\mathscr{X} \to X$ a $\mathbf{G}_m$-gerbe. Given two $\mathscr{X}$-twisted B_0-modules M_0 and J, there is a natural isomorphism*

$$\mathbf{R}\mathrm{Hom}_B(M_0, J) \to \mathbf{R}\mathrm{Hom}_{B_0}(M_0 \overset{\mathbf{L}}{\underset{B}{\otimes}} B_0, J)$$

in the derived category of $\mathscr{X}$-twisted sheaves.

Let $B \to B_0$ is a square-zero extension of rings in X with kernel I. Suppose M_0 and J are twisted B_0-modules. We wish to know when there exists an I-flat extension of M_0 by J. Given any such extension, there is a naturally resulting morphism $I \otimes_{B_0} M_0 \to J$, which is an isomorphism if and only if the extension is I-flat. Fix a morphism $u : I \otimes_{B_0} M_0 \to J$. As in [42], we have the following proposition.

Proposition 2.2.5.10. *There is an exact sequence*

$$0 \to \mathrm{Ext}^1_{B_0}(M_0, J) \to \mathrm{Ext}^1_B(M_0, J) \to \mathrm{Hom}_{B_0}(I \underset{B_0}{\otimes} M_0, J) \xrightarrow{\partial} \mathrm{Ext}^2_{B_0}(M_0, J)$$

with the property that there exists an extension with associated morphism u if and only if $\partial(u) = 0$. The space of all such extensions is a torsor under $\mathrm{Ext}^1_{B_0}(M_0, J)$.

Proof. The exact is the sequence of low degree terms arising from 2.2.5.9 and the composition of functors spectral sequence. That the maps agree with the interpretation given is checked carefully in [42, p. 252ff]. Note that Illusie's proof works in the derived category of twisted sheaves; it is not necessary to work in the category of all modules in the topos. (The Ext groups are different, but the functorialities are the same.)

□

2.2.5.11. We now have enough information to describe an obstruction theory for the problem of twisted sheaves on a scheme. In this section, $f : X \to S$ will be a proper morphism from a scheme X to a Noetherian excellent scheme S and $\mathscr{X} \to X$ will be a fixed $\mathbf{G}_m$-gerbe. We will develop the deformation-theoretic tools necessary to apply Artin's Existence Theorem. Let A_0 be a reduced Noetherian ring. We recall some terminology from Artin's paper [9].

Definition 2.2.5.12. A *deformation situation* is a commutative diagram of Noetherian rings $A' \to A \to A_0$ such that

1. $A \to A_0$ and $A' \to A$ are infinitesimal extensions (i.e., they have nilpotent kernels)

2. $\ker(A' \to A) = M$ is a finite A_0-module.

In the classical study of versal deformations, one often takes A_0 to be a field and A, A' to be local Artinian rings with residue field k.

Let F be a stack on S.

Definition 2.2.5.13. An *obstruction theory* for F consists of two parts.

(i) For each infinitesimal extension $A \to A_0$ and element $a \in F(A)$, a functor $\mathrm{Ob}_a : \mathrm{Mod}_{A_0}^{\mathrm{finite}} \to \mathrm{Mod}_{A_0}^{\mathrm{finite}}$

(ii) For each deformation situation and $a \in F(A)$, there is an element $o_a(A') \in \mathrm{Ob}_a(M)$ which vanishes if and only if there is an element $F(A')$ whose reduction to A is isomorphic to a.

These data are subject to two further constraints:

F: Given a diagram

$$\begin{array}{ccc} B & & \\ & \searrow & \\ \downarrow f & & A_0 \\ & \nearrow & \\ A & & \end{array}$$

one has $\mathrm{Ob}_a = \mathrm{Ob}_{f(a)}$ as functors $\mathrm{Mod}_{A_0}^{\mathrm{finite}} \to \mathrm{Mod}_{A_0}^{\mathrm{finite}}$.

L: For any diagram of deformation situations

$$\begin{array}{ccccc} A' & \longrightarrow & A & & \\ & & & \searrow & \\ \downarrow g & & \downarrow & & A_0 \\ & & & \nearrow & \\ B' & \longrightarrow & B & & \end{array}$$

giving rise to an A_0-linear map of kernels $M_A \to M_B$, we get for any $a \in F(A)$ an A_0-linear map $\mathrm{Ob}_a(M_A) \to \mathrm{Ob}_a(M_B)$ taking $o_a(A')$ to $o_a(B')$.

We will call **F** *functoriality* and **L** *linearity* of the obstruction theory.

Let F be the groupoid which assigns to any Noetherian affine scheme $\operatorname{Spec} A \to S$ the groupoid of A-flat families of coherent $\mathscr{X}$-twisted sheaves $\mathscr{F}$ on $X \otimes_S A$.

Lemma 2.2.5.14. *If $\mathscr{F}$ and $\mathscr{G}$ are A-flat coherent $\mathscr{X} \otimes A$-twisted sheaves then $\mathrm{Ext}^i(\mathscr{F}, \mathscr{G})$ is a finite A-module.*

Proof. This follows from the local to global spectral sequence for Ext and the finiteness of coherent cohomology for a proper morphism. (Coherence of the sheaf Exts is a local computation in the lisse-étale topology of $\mathscr{X}$, hence follows from the corresponding fact for locally Noetherian schemes.) □

Proposition 2.2.5.15. *The following give an obstruction theory for F.*

1. *Given an infinitesimal extension $A \to A_0$ and $\mathscr{F} \in F(A)$,*

$$\mathrm{Ob}_{\mathscr{F}}(M) = \mathrm{Ext}^2_{X \otimes A_0}(\mathscr{F}_0, M \underset{A_0}{\otimes} \mathscr{F}_0) = \mathrm{Ext}^2_{X \otimes A}(\mathscr{F}, M \underset{A}{\otimes} \mathscr{F}).$$

2. *Given a deformation situation $A' \to A \to A_0$ with kernel M, $o_{\mathscr{F}}(A') = \partial(\mathrm{id} : M \otimes_A \mathscr{F} \to M \otimes_A \mathscr{F})$ in 2.2.5.10*

The equality $\mathrm{Ob}_{\mathscr{F}}(M) = \mathrm{Ext}^2_{X \otimes A_0}(\mathscr{F}_0, M \otimes_{A_0} \mathscr{F}_0) = \mathrm{Ext}^2_{X \otimes A}(\mathscr{F}, M \otimes_A \mathscr{F})$ follows from the fact that $\mathscr{F}$ is A-flat. This is a simple consequence of the cher à Cartan isomorphism and the fact that $\mathscr{F} \overset{\mathbf{L}}{\otimes}_A A_0 = \mathscr{F}_0$ (by flatness).

Proof. Using 2.2.5.10, it suffices to check **F** and **L** and prove that $\mathrm{Ext}^2(\mathscr{F}_0, M \otimes_{A_0} \mathscr{F}_0)$ is a finite A_0-module. The finiteness is 2.2.5.14. **F** follows from the description of the obstruction group in terms of A_0 and **L** follows from the naturality of 2.2.5.10. □

2.2.5.16. We recall some of Illusie's results: Let X be a topos, and $0 \to I \to \mathscr{O} \to \overline{\mathscr{O}}$ a square-zero extension of rings in X. Let $\mathscr{A}$ be a commutative $\mathscr{O}$-algebra in X and $\overline{F}$ an $\overline{\mathscr{A}}$-module. Then there is a class

$$\mathfrak{o}(\overline{F}) \in \mathrm{Ext}^2_{\overline{\mathscr{A}}}(\overline{F}, I \underset{\overline{\mathscr{O}}}{\otimes} \overline{F})$$

which vanishes if and only if there exists an I-flat $\mathscr{A}$-module F with $\overline{\mathscr{A}} \otimes_{\mathscr{A}} F \cong \overline{F}$. When this is the case, the set of such F is principal homogeneous under the group $\mathrm{Ext}^1_{\overline{\mathscr{A}}}(\overline{F}, I \otimes \overline{F})$. Furthermore, given such an extension F the set of automorphisms of F which reduce to the identity on $\overline{F}$ is principal homogeneous under $\mathrm{Hom}_{\overline{\mathscr{A}}}(\overline{F}, I \otimes \overline{F})$. Note that if $\overline{F}$ is $\overline{\mathscr{O}}$-flat, then F is $\mathscr{O}$-flat if and only if it is I-flat by the local criterion of flatness.

In fact, these groups are packaged into a short exact sequence exactly as in 2.2.5.10. In the case of twisted sheaves, this exact sequence (in the topos) receives a functorial morphism from the exact sequence of 2.2.5.10. When M_0 and J are quasi-coherent, this morphism is an isomorphism. In other words, *in the quasi-coherent case*, Illusie's formalism (with the attendant functorialities and filigree) carries over precisely, without needing to first restrict attention to a subcategory of modules in the topos (where things like the Atiyah class may not work as expected). We will return to the details of Illusie's construction in future work when we study the virtual fundamental classes of the stacks of generalized Azumaya algebras and twisted sheaves.

We can use this formalism to prove several "localization and constructibility" results about the deformation theory of coherent twisted sheaves. These are the "conditions (4.1)" of Artin's famous [9]. Let A_0 be a reduced Noetherian ring, $A_0 \to B_0$ a flat ring extension, $X \to \operatorname{Spec} A_0$ a proper morphism, $\mathscr{X} \to X$ a $\mathbf{G}_m$-gerbe, $\mathscr{F}$ an A_0-flat family of coherent $\mathscr{X}$-twisted sheaves, and M an A_0-module.

Proposition 2.2.5.17. *For any $i \geq 0$ the following hold.*

1. $\operatorname{Ext}^i_X(\mathscr{F}, M \otimes \mathscr{F}) \otimes_{A_0} B_0 \cong \operatorname{Ext}^i_{X_{B_0}}(\mathscr{F}_{B_0}, M_{B_0} \otimes \mathscr{F}_{B_0})$.

2. *If $\mathfrak{m} \subset A_0$ is a maximal ideal then*

$$\operatorname{Ext}^i_X(\mathscr{F}, M \otimes \mathscr{F}) \otimes \widehat{A_0} \cong \varprojlim \operatorname{Ext}^i_X(\mathscr{F}, M/\mathfrak{m}^n M \otimes \mathscr{F}),$$

the completion being taken with respect to $\mathfrak{m}$.

3. *There is a dense open set of points (of finite type) $p \in \operatorname{Spec} A_0$ such that*

$$\operatorname{Ext}^i(\mathscr{F}, M \otimes \mathscr{F}) \otimes \kappa(p) \cong \operatorname{Ext}^i_{X_{\kappa(p)}}(\mathscr{F}_{\kappa(p)}, M_{\kappa(p)} \otimes \mathscr{F}_{\kappa(p)}).$$

Proof. The proof of 1 is immediate. To prove 2, we work in Căldăraru form. This makes it clear that one can easily understand formal twisted sheaves on the formal completion of a scheme along a closed subscheme. We wish to prove that if $X \to \operatorname{Spec} A$ is a proper scheme over a complete Noetherian local ring and $\mathscr{F}$ and $\mathscr{G}$ are coherent twisted sheaves on X then

$$\operatorname{Ext}^i_X(\mathscr{F}, \mathscr{G}) = \varprojlim \operatorname{Ext}^i_{X \otimes A/\mathfrak{m}^n}(\mathscr{F} \otimes A/\mathfrak{m}^n, \mathscr{G} \underset{A}{\otimes} A/\mathfrak{m}^n).$$

This works just as in [33, 4.5]: one shows that the completion of the sheaf $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ along the closed fiber is naturally isomorphic to the sheaf $\mathscr{E}xt^i(\widehat{\mathscr{F}}, \widehat{\mathscr{G}})$ of extensions over the formal scheme. The rest comes by taking the local-to-global Ext spectral sequence and using the finiteness of coherent cohomology to make an Artin-Rees argument. The interested reader should consult [33] for further details.

The proof of 3 is slightly subtle. The discussion here similar to that of [12] (which is written in the analytic untwisted context). Suppose first that A_0 is excellent (in general, this case suffices for applications to Artin's theorem). Among other things, this means that the regular locus of A_0 is open. We may assume (since A_0 is reduced and we are just looking for points in a dense open set) that A_0 is a regular Noetherian domain. In fact, we may assume that A_0 is a finite type $\mathbf{Z}$-algebra. Thus, the dimension of A_0 at any point is bounded by a fixed number.

Given $X \to \operatorname{Spec} A_0$ proper and flat, $\mathscr{F}$ and $\mathscr{G}$ flat coherent twisted sheaves on X, define a functor $T : M \mapsto \operatorname{Ext}^i_X(\mathscr{F}, M \otimes \mathscr{G})$. We can produce a complex $C^\bullet$ such that $C^\bullet \otimes M$ computes $\mathbf{R}\operatorname{Hom}_X(\mathscr{F}, M \otimes \mathscr{G})$ for all M (and hence for all base changes by the cher à Cartan isomorphism) as follows: we may resolve $\mathscr{F}$ by a complex $L^\bullet$ whose terms have the form $\bigoplus \pi_! \mathscr{L}$, where $\pi : U \to X$ splits $\mathscr{X}$ with U affine and $\mathscr{L}$ an invertible twisted sheaf on U. The complex $C^\bullet(\mathscr{G}) = \operatorname{Hom}(L^\bullet, \mathscr{G})$ is the desired complex. To show this, it is easy to see that it is enough to show that $\mathscr{G} \mapsto C^\bullet(\mathscr{G})$ is an exact functor from quasi-coherent twisted sheaves to complexes and that $\mathrm{H}^0(C^\bullet(\mathscr{G})) = \operatorname{Hom}(\mathscr{F}, \mathscr{G})$. The second statement is immediate, while the first follows from the fact that $\pi_!$ is a left adjoint to π^* and that affines have no quasi-coherent cohomology. There is a spectral sequence

$$E_2^{p,q} = \operatorname{Tor}_{-q}(M, \mathrm{H}^p(C^\bullet(\mathscr{G}))) \Rightarrow \mathrm{H}^{p-q}(C^\bullet(\mathscr{G} \otimes M)).$$

Whenever M has homological dimension d, we see that if $\mathrm{H}^j(C^\bullet)$ is A_0-flat for all $j < i+d$ then $\mathrm{H}^i(C^\bullet) \otimes M \cong \mathrm{H}^i(C^\bullet \otimes M)$, as desired. As A_0 is a finite type $\mathbf{Z}$-algebra, there is a universal upper bound for the homological dimension of any residue field, so we see that if we pass to a sufficiently small open $U \subset \operatorname{Spec} A_0$, we will know that for any point $p \in U$, the map $T(A) \otimes \kappa(p) \to T(\kappa(p))$ will be an isomorphism.

Now suppose $f : \operatorname{Spec} A \to \operatorname{Spec} A_0$ is a dominant map from an integral Noetherian affine scheme to our finite type $\mathbf{Z}$-scheme. We claim that $f^{-1}U \subset \operatorname{Spec} A$ has the same base change property. Indeed, it is clear that the map $T(A) \otimes \kappa(p) \to T(\kappa(p))$ is still surjective for any $p \in f^{-1}U$. But (using step 2) once the natural map is surjective at p it is in fact an isomorphism there. This standard argument may be found in [38, III.12.10], for example. $\square$

Chapter 3

Generalized Azumaya algebras

In this section we lay the foundation for studying compactified moduli of Azumaya algebras on a projective variety. Our approach originates in an observation of de Jong. An Azumaya algebra is a sheaf of algebras which is étale locally the endomorphism algebra of a locally free sheaf. De Jong suggested that the objects in the compactification should still be characterized by being local "endomorphism algebras," but clearly if there are to be limits, one must allow endomorphism algebras of torsion free sheaves which are not locally free. However, taking (locally) $\mathscr{E}nd(\mathscr{F})$ for a torsion free $\mathscr{F}$ is not good enough, as when $\mathscr{F}$ is not locally free the formation of $\mathscr{E}nd(\mathscr{F})$ is not compatible with base change. However, de Jong observes, the formation of $\mathbf{R}\mathscr{E}nd(\mathscr{F})$ *is* compatible with base change. Thus, one should somehow glue "derived endomorphism algebras" to make generalized Azumaya algebras. We carry out this procedure in this section. It turns out that the resulting objects are closely related to twisted sheaves, as we will describe.

3.1 The stack GAz

The key to gluing local derived endomorphism algebras lies in a derived analogue of the Skolem-Noether theorem. We take this up in 3.1.1. We then present the construction of the stack of generalized Azumaya algebras on a scheme X and a refinement when X is a smooth surface.

3.1.1 Derived Skolem-Noether

In what follows, we work primarily in the derived category of modules over a local commutative ring $(\mathscr{O}, \mathfrak{m}, k)$. For the sake of a smoother exposition, we assume that $\mathscr{O}$ is Noetherian, but this is unnecessary. On occasion, we will work in the category of chain complexes. However, we will use the word "complex" ambiguously, and we leave it to the reader to determine from the context whether we mean an object of $\mathbf{D}(\mathscr{O})$ or an object of $\mathrm{K}(\mathscr{O})$. Similarly, "isomorphism" will be consistently used in place of "quasi-isomorphism" and we will always assume that isomorphisms preserve whatever additional structures of objects are implicit. Given a scheme X, the symbol $\mathbf{D}(X)$

will denote a derived category of sheaves of $\mathscr{O}_X$-modules, with various conditions (boundedness, perfection, quasi-coherence of cohomology) clear from context. In the end, it will suffice to work in the category denoted $\mathbf{D}_{\mathrm{fTd}}(X)$ by Hartshorne in [37], so the hypotheses on $\mathbf{D}$ will not be a focus of attention.

Definition 3.1.1.1. Given a scheme X, an object $A \in \mathbf{D}(X)$ will be called a *weak $\mathscr{O}$-algebra* if there are maps $\mu : A \overset{\mathbf{L}}{\otimes} A \to A$ and $i : \mathscr{O} \to A$ in $\mathbf{D}(X)$ which satisfy the usual axioms for an associative unital algebra, the diagrams being required to commute in the derived category.

In other words, a weak algebra is an algebra object of the derived category. Note that the derived tensor product makes $\mathbf{D}(X)$ into a symmetric monoidal additive category (as the universal property of derived functors ensures that all different associations of an iterated tensor product are naturally isomorphic). Thus, it makes sense to speak of "associative" algebra structures.

Given an additive symmetric monoidal category, one can define most of the usual objects and maps of the theory of algebras: (unital) modules, bimodules, linear maps, derivations, inner derivations, maps of algebras, etc. We leave it to the reader to write down precise definitions of these terms, giving two examples: Given a map of weak algebras $A \to B$, an *$\mathscr{O}$-linear derivation* from A to B is a map $\delta : A \to B$ in $\mathbf{D}(X)$ such that $\delta \circ \mu_A = \mu_B \circ (\mathrm{id} \overset{\mathbf{L}}{\otimes} \delta + \delta \overset{\mathbf{L}}{\otimes} \mathrm{id})$ in $\mathbf{D}(X)$. A derivation from A to A is *inner* if there is an $\alpha : \mathscr{O} \to A$ such that $\delta = \mu \circ (\alpha \overset{\mathbf{L}}{\otimes} \mathrm{id}) - \mu \circ (\mathrm{id} \overset{\mathbf{L}}{\otimes} \alpha)$.

Given a ring map $\mathscr{O} \to \mathscr{O}'$, the derived functor $(\cdot) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}} \mathscr{O}' : \mathbf{D}(\mathscr{O}) \to \mathbf{D}(\mathscr{O}')$ respects the monoidal structure. There results a natural base change operation for weak algebras and modules. (This operation will be consistently written as a change of base on the right to avoid sign errors.)

Similarly, given a weak algebra A and a left A-module P, the functor $P \overset{\mathbf{L}}{\otimes} (\cdot)$ takes objects of $\mathbf{D}(\mathscr{O})$ to A-modules. This follows from the natural associativity of the derived tensor product.

The first non-trivial example of a weak algebra is given by

$$\mathbf{R}\mathrm{End}(K) := \mathbf{R}\mathrm{Hom}(K, K)$$

for a perfect complex K. Replacing K by a projective resolution, one easily deduces the weak algebra structure from the usual composition of functions: if we write K as a finite complex of free modules (which we will also call K), then $\mathbf{R}\mathrm{End}(K)$ has as nth module $\prod_p \mathrm{Hom}(K^p, K^{p+n})$, with differential $\partial^n(\alpha_p)_q = (-1)^{n+1}\alpha_{q+1}d + d\alpha_q$. Since K is perfect, the nth module of $\mathbf{R}\mathrm{End}(K) \overset{\mathbf{L}}{\otimes} \mathbf{R}\mathrm{End}(K)$ is equal to

$$\prod_{a+b=n} \prod_{s,t} \mathrm{Hom}(K^s, K^{s+a}) \otimes \mathrm{Hom}(K^t, K^{t+b})$$

and the multiplication projects to the factors with $s + a = t$ and then composes functions as usual. Setting $K^\vee = \mathbf{R}\mathrm{Hom}(K, \mathscr{O})$ (the derived dual of K), we have the following basic lemma.

Lemma 3.1.1.2. *Let K be a perfect complex.*

(*i*) *There is a natural isomorphism* $K \overset{\mathbf{L}}{\otimes} K^\vee \xrightarrow{\sim} \mathbf{R}\mathrm{End}(K)$.

(*ii*) *There is a natural left action of* $\mathbf{R}\mathrm{End}(K)$ *on* K.

Tensoring the action

$$\mathbf{R}\mathrm{End}(K) \overset{\mathbf{L}}{\otimes} K \to K$$

with $K^\vee$ *on the right and using* (*i*) *yields the multiplication of* $\mathbf{R}\mathrm{End}(K)$.

It is essential that the action be written on the left (when using the standard sign convention for forming the total complex of a double complex [42, I.1.2.1], [54, Appendix]) and that $K^\vee$ be written on the right for the signs to work out. We will not usually point out these sign sensitivities, but the reader should remain vigilant.

An algebra of the form $\mathbf{R}\mathrm{End}(K)$ will be called a *derived endomorphism algebra*. Our goal is to re-prove the classical results about matrix algebras for derived endomorphism algebras of perfect complexes.

Notation 3.1.1.3. The symbols P and Q will always be taken to mean perfect complexes with a chosen realization as a bounded complex of finite free modules. Thus, maps $P \to Q$ in the derived category will always come from maps of the "underlying complexes" (taken to mean the chosen realizations). Similarly, $\mathbf{R}\mathrm{End}(P)$ will have as chosen representative the complex constructed from the underlying complex of P as above: $\mathbf{R}\mathrm{End}(P)^n = \prod_t \mathrm{Hom}(P^t, P^{t+n})$ with differential $\partial(\alpha_t)_s = (-1)^{n+1}\alpha_{s+1}d + d\alpha_s$.

These conventions facilitate making certain basic arguments without speaking of replacing the object by a projective resolution, etc., but it is ultimately only important for this book-keeping reason; the reader may ignore it without fear (until it is explicitly invoked!).

Definition 3.1.1.4. Given $M \in \mathbf{D}(\mathscr{O})$, the *annihilator of* M is the kernel $\mathrm{Ann}(M)$ of the natural map from $\mathscr{O}$ to $\mathrm{End}_{\mathbf{D}(\mathscr{O})}(M)$. The quotient $\mathscr{O}/\mathrm{Ann}(M)$ will be denoted by $\mathscr{O}_M$.

Given an isomorphism $\psi : P \to Q(n)$, there is an isomorphism $\psi^* : \mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$ given by functorial conjugation by ψ followed by the natural identification of $\mathbf{R}\mathrm{End}(Q(n))$ with $\mathbf{R}\mathrm{End}(Q)$. We will call this the *induced map*. The map ψ^* may also be described as follows: under the natural identification of $\mathbf{R}\mathrm{End}(P)$ with $P \overset{\mathbf{L}}{\otimes} P^\vee$, ψ^* is identified with $\psi \overset{\mathbf{L}}{\otimes} (\psi^\vee)^{-1}$.

Theorem 3.1.1.5. *Let P and Q be non-zero perfect complexes of $\mathscr{O}$-modules. If* $\mathbf{R}\mathrm{End}(P) \cong \mathbf{R}\mathrm{End}(Q)$ *as weak algebras, then there exists a unique n such that the map*

$$\mathrm{Isom}(P, Q(n)) \to \mathrm{Isom}(\mathbf{R}\mathrm{End}(P), \mathbf{R}\mathrm{End}(Q))$$

is surjective with each fiber a torsor under $\mathscr{O}_P^\times$. If $P = Q$, then $n = 0$ and the kernel is naturally a split torsor.

Corollary 3.1.1.6. *The sequence*

$$0 \to \mathscr{O}_P \to \mathrm{End}(P) \to \mathrm{Der}(\mathbf{R}\mathrm{End}(P)) \to 0$$

is exact. More generally, if P and $Q(n)$ are isomorphic, then the map

$$\mathrm{Hom}(P, Q(n)) \to \mathrm{Der}(\mathbf{R}\mathrm{End}(P), \mathbf{R}\mathrm{End}(Q))$$

is surjective with each fiber naturally a torsor under $\mathscr{O}_P$.

Proof. Apply Theorem 3.1.1.5 to $P[\varepsilon]$ over $\mathscr{O}[\varepsilon]$ and look at automorphisms of the weak algebra $\mathbf{R}\mathrm{End}_{\mathscr{O}[\varepsilon]}(P[\varepsilon])$ reducing to the identity modulo ε. □

The proof of Theorem 3.1.1.5 will make use of the completion of $\mathscr{O}$ to lift the classical theorems on matrix algebras from the closed fiber by "infinitesimal induction."

Proposition 3.1.1.7. *If $\mathscr{O}$ is a field k then Theorem 3.1.1.5 holds.*

Proof. The bounded derived category of k is naturally identified with the category of $\mathbf{Z}$-graded finite k-modules by sending a complex to the direct sum of its cohomology spaces. Given perfect complexes P and Q, the algebra $\mathbf{R}\mathrm{End}(P)$ (resp. $\mathbf{R}\mathrm{End}(Q)$) is then identified with a matrix algebra, carrying the induced grading from the grading of the vector space P (resp. Q), and an isomorphism from $\mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$ is identified with an isomorphism of matrix algebras which respects the gradings. By the allowance of a shift, we may restrict our attention to graded spaces whose minimal non-zero graded piece is in degree 0; any reference in what follows to graded vector spaces will implicitly assume this hypothesis. (The reader should note that the algebras involved will still carry non-trivial graded pieces with negative degrees.) Let A be a graded matrix algebra of rank n^2 and V and W two graded n-dimensional vector spaces with non-trivial graded A-actions. By the Skolem-Noether theorem, there is an A-linear isomorphism $\alpha : V \to W$. We claim that α is graded. To prove this, it suffices to show that given a non-zero vector $v \in V_0$, $\alpha(v)$ is in W_0 (because V and W are simple A-modules). Write $\alpha(v) = \sum w_i$. Since V is a simple A- module, $A_n \cdot v = V_n$; a similar statement holds for W (given a choice of non-zero weight 0 vector, which exists by the hypothesis on the gradings). Thus, the highest non-trivial grading N of A will equal the highest non- trivial grading of both V and W. Furthermore, given any i such that $w_i \neq 0$, the fact that $A_{-i} \cdot w_i = W_0$ means that $A_{-i} \neq 0$. Given $i > 0$ such that $w_i \neq 0$, we have for all $\tau \in A_{-i}$ that

$$0 = \alpha(0) = \alpha(\tau(v)) = \tau(\alpha(v)) = \tau\Big(\sum w_j\Big) = \tau(w_i) + \text{higher terms.}$$

Thus, $\tau(w_i) = 0$, which implies that $W_0 = 0$. This contradicts the assertion that W_0 is the minimal non-trivial graded piece. So $w_i = 0$ for all $i > 0$ and therefore $w \in W_0$. Translating this back into the derived language, we have proven that given an isomorphism $\varphi : \mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$, there is an isomorphism $P \to Q$ in $\mathbf{D}(k)$ which induces φ by functoriality. In fact, we have shown the rest of the proposition as well, because α is the unique choice for such an isomorphism up to scalars by the classical theory of matrix algebras. □

Lemma 3.1.1.8. *Theorem 3.1.1.5 is true for $\mathscr{O}$ if it is true for $\widehat{\mathscr{O}}$.*

Proof. We proceed by reducing the problem to a question of linear algebra and then using the faithful flatness of completion.

Suppose given P and Q and an isomorphism $\varphi : \mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$; this defines an action of $A := \mathbf{R}\mathrm{End}(P)$ on Q. We claim that finding $u : P \to Q$ such that $\varphi = u^*$ is equivalent to finding an A-linear isomorphism from P to Q. Indeed, suppose $u : P \to Q$ is A-linear, so that the diagram

$$\begin{array}{ccc} \mathbf{R}\mathrm{End}(P) \overset{\mathbf{L}}{\otimes} P & \xrightarrow{\varphi \overset{\mathbf{L}}{\otimes} u} & \mathbf{R}\mathrm{End}(Q) \overset{\mathbf{L}}{\otimes} Q \\ \downarrow & & \downarrow \\ P & \xrightarrow{u} & Q \end{array}$$

commutes, where the vertical arrows are the actions. Tensoring the left side with $P^\vee$ and the right side with $Q^\vee$, we see that the resulting diagram

$$\begin{array}{ccc} \mathbf{R}\mathrm{End}(P) \overset{\mathbf{L}}{\otimes} P \overset{\mathbf{L}}{\otimes} P^\vee & \xrightarrow{\varphi \overset{\mathbf{L}}{\otimes} u \overset{\mathbf{L}}{\otimes} (u^\vee)^{-1}} & \mathbf{R}\mathrm{End}(Q) \overset{\mathbf{L}}{\otimes} Q \overset{\mathbf{L}}{\otimes} Q^\vee \\ \downarrow & & \downarrow \\ P \overset{\mathbf{L}}{\otimes} P^\vee & \xrightarrow{u \overset{\mathbf{L}}{\otimes} (u^\vee)^{-1}} & Q \overset{\mathbf{L}}{\otimes} Q^\vee \end{array}$$

also commutes. Applying Lemma 3.1.1.2 and writing B for $\mathbf{R}\mathrm{End}(Q)$, we find that the diagram

$$\begin{array}{ccc} A \overset{\mathbf{L}}{\otimes} A & \xrightarrow{u^* \overset{\mathbf{L}}{\otimes} \varphi} & B \overset{\mathbf{L}}{\otimes} B \\ \downarrow & & \downarrow \\ A & \xrightarrow{u^*} & B \end{array}$$

commutes. Considering the units in the algebras, one readily concludes the proof of the claim. Note that to conclude that any such u as above is an isomorphism, it suffices for its reduction to the residue field to be an isomorphism (e.g. because the complexes are bounded above).

It is easy to see that $\mathrm{Hom}_{\mathbf{D}(\mathscr{O})}$ is compatible with flat base change when restricted to the category of perfect complexes: given a flat ring extension $\mathscr{O} \to \mathscr{A}$, there is a natural isomorphism $\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(M \overset{\mathbf{L}}{\otimes} \mathscr{A}, N \overset{\mathbf{L}}{\otimes} \mathscr{A}) \cong \mathrm{Hom}_{\mathbf{D}(\mathscr{O})}(M, N) \otimes \mathscr{A}$ for all perfect M and N in $\mathbf{D}(\mathscr{O})$. Furthermore, given a perfect weak algebra Ξ, the realization of the module of Ξ-linear maps as a kernel of maps of Hom-modules shows that the same statement is true for Hom_{Ξ}. Thus there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\Xi}(M,N) & \longrightarrow & \mathrm{Hom}_{\widehat{\Xi}}(\widehat{M},\widehat{N}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\Xi}(M,N)\otimes_{\mathscr{O}} k & = & \mathrm{Hom}_{\widehat{\Xi}}(\widehat{M},\widehat{N})\otimes_{\widehat{\mathscr{O}}} k \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}_{\Xi\overset{\mathbf{L}}{\otimes}k}(M\overset{\mathbf{L}}{\otimes}k, N\overset{\mathbf{L}}{\otimes}k). & \end{array}$$

This immediately applies to our situation to show that the map of Theorem 3.1.1.5 is surjective for $\mathscr{O}$ if it is for $\widehat{\mathscr{O}}$ (for a fixed n, which may be determined from the reduction to the residue field). Similarly, to verify that an isomorphism $\xi : P \xrightarrow{\sim} P$ in the kernel of the automorphism map is homotopic to a constant, it suffices to show that an element $\xi \in \mathrm{End}_{\mathbf{D}(\mathscr{O})}(P)$ is in $\mathscr{O}_P$ if and only if this is true after completing. But the module of maps homotopic to a constant is also clearly compatible with flat base change and completion is moreover *faithfully* flat (all modules involved are finite over $\mathscr{O}$ because the complexes involved are perfect), so ξ is in a submodule Z of $\mathrm{End}(P)$ if and only if its image in $\mathrm{End}(P)\otimes\widehat{\mathscr{O}}$ is contained in $Z\otimes\widehat{\mathscr{O}}$. □

From this point onward, *we assume that $\mathscr{O}$ is a complete local Noetherian ring.* Recall that a quotient of local rings $0 \to I \to \mathscr{O} \to \overline{\mathscr{O}} \to 0$ is *small* if I is generated by an element ε which is annihilated by the maximal ideal of $\mathscr{O}$ (so that, in particular, $\varepsilon^2 = 0$). We can choose a filtration $\mathscr{O} \supset \mathfrak{m} = I_0 \supset I_1 \supset I_2 \supset \cdots$ which is separated (i.e., so that $\cap_i I_i = 0$) and defines a topology equivalent to the $\mathfrak{m}$-adic topology such that for all $i \geq 0$, the quotient $0 \to I_i/I_{i+1} \to \mathscr{O}/I_{i+1} \to \mathscr{O}/I_i \to 0$ is a small extension, with I_i/I_{i+1} generated by ε_i. We fix such a filtration for remainder of this section, and we denote $\mathscr{O}/I_n$ by $\mathscr{O}_n$.

Lemma 3.1.1.9. *Let $0 \to I \to R \to \overline{R} \to 0$ be a surjection of rings. Let A be a weak R-algebra and P and Q two left A-modules. Let T denote the triangle in $\mathbf{D}(R)$ arising from the quotient map $R \to \overline{R}$ as above.*

(*i*) *The maps in $P\overset{\mathbf{L}}{\otimes}T$ are A-linear (with the natural A-module structures).*

(*ii*) *Any A-linear map $\psi : P \to Q\overset{\mathbf{L}}{\otimes}\overline{R}$ factors through an A-linear map $\overline{\psi} : P\overset{\mathbf{L}}{\otimes}\overline{R} \to Q\overset{\mathbf{L}}{\otimes}\overline{R}$ which is the derived restriction of scalars of an $A\overset{\mathbf{L}}{\otimes}\overline{R}$-linear map from $P\overset{\mathbf{L}}{\otimes}\overline{R}$ to $Q\overset{\mathbf{L}}{\otimes}\overline{R}$.*

(*iii*) *If $R \to \overline{R}$ is a small extension of local rings with residue field k, then the natural identification $P\overset{\mathbf{L}}{\otimes}I \xrightarrow{\sim} P_k$ induced by a choice of basis for I over k is A-linear.*

Proof. Note that basic results about homotopy colimits allow us to replace any object of $\mathbf{D}(R)$ by a complex of projectives, so there are no boundedness conditions on any of the complexes involved. Part (i) follows immediately from the fact that $P \overset{\mathbf{L}}{\otimes}(\cdot)$ is a functor from $\mathbf{D}(R)$ to A-modules. Part (ii) follows from writing P and A as complexes of projectives and representing the map $P \to Q \overset{\mathbf{L}}{\otimes} \overline{R}$ as a map on complexes. (Note that this factorization need not be unique as a map in $\mathbf{D}(R)$, but it is unique as the derived restriction of scalars from a map in $\mathbf{D}(\overline{R})$.) Part (iii) follows similarly from looking at explicit representatives of P and A. ☐

Lemma 3.1.1.10. *Suppose $f, g : P \to Q$ are two maps of perfect complexes in $K(\mathscr{O})$. Let $P_n = P \otimes \mathscr{O}_n$, $Q_n = Q \otimes \mathscr{O}_n$, $f_n = f \otimes \mathscr{O}_n$, $g_n = g \otimes \mathscr{O}_n$. Suppose there are homotopies*

$$h(n) \in \prod_t \operatorname{Hom}(P^t, Q^{t-1} \otimes I_n)$$

such that for all n,

$$f_n - g_n = d\Big(\sum_{s<n} \bar{h}(s)\Big) + \Big(\sum_{s<n} \bar{h}(s)\Big)d$$

as maps of complexes, where $\bar{h}$ denotes the induced map. Then f is homotopic to g.

Proof. The element $h = \sum_{s=0}^{\infty} h(s)$ converges and defines the homotopy. ☐

Lemma 3.1.1.11. *Let $0 \to I \to R \to \overline{R} \to 0$ be a small extension of local rings with residue field k. Let P and Q be perfect complexes of R-modules (with chosen realizations) and $\varphi : \mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$ an isomorphism of the derived endomorphism algebras, written as a map in that direction on the underlying complexes. If there exists an isomorphism of the underlying complexes $\overline{u} : \overline{P} \xrightarrow{\sim} \overline{Q}$ such that $\overline{\varphi} = \overline{u}^*$ as maps of complexes, then there is a lift u of $\overline{u}$ and a homotopy h between φ and u^* such that $h(\mathbf{R}\mathrm{End}(P)) \subset \mathbf{R}\mathrm{End}(Q) \otimes I$. In particular, $\varphi = u^*$ in $\mathbf{D}(R)$.*

Proof. Let $A = \mathbf{R}\mathrm{End}(P)$ and let A act on Q via φ. The identification of $\overline{\varphi}$ with $\overline{u}^*$ provides an $\overline{A}$-linear isomorphism $\overline{\gamma} : \overline{P} \to \overline{Q}$, and we wish to lift this to an A-linear isomorphism $P \to Q$. Consider the composition $P \to \overline{Q} \to Q \otimes I(1) \cong Q_k(1)$ in the derived category. By Lemma 3.1.1.9, this map is A-linear and factors through an A-linear map $\alpha : P_k \to Q_k(1)$ which comes by derived restriction of scalars from an A_k-linear map in $\mathbf{D}(k)$. By Proposition 3.1.1.7, we see that α is either zero or an isomorphism. But $P_k \cong Q_k \not\cong 0$, which implies that $\alpha = 0$. This means that there is an *R-linear* lift γ of $\overline{\gamma}$. Now $(\gamma^*)^{-1} \circ \varphi - \mathrm{id}$ is identified with a map $\mathbf{R}\mathrm{End}_k(P_k) \to \mathbf{R}\mathrm{End}_k(P_k)$ in $\mathbf{D}(k)$ which is a derivation of the algebra, hence is homotopic to the inner derivation induced by a map $\omega_k : P_k \to P_k$ in $\mathbf{D}(k)$. Writing ω for the composition

$$P \longrightarrow P_k \xrightarrow{\omega_k} P_k \xrightarrow{\cong} P \overset{\mathbf{L}}{\otimes} I \longrightarrow P,$$

we see that there is a homotopy between φ and $\gamma(1+\omega)^*$ with image in $\mathbf{R}\mathrm{End}(Q) \otimes I$, and that $\gamma(1+\omega)$ is a lift of $\overline{\gamma}$ as maps of complexes. ☐

Lemma 3.1.1.12. *Let $0 \to I \to R \to \overline{R} \to 0$ be a small extension of local rings with residue field k. Let P be a perfect complex of R-modules (with a chosen realization) and $\psi : P \to P$ an automorphism of the underlying complex such that $\overline{\psi} = \overline{\alpha}$ for some $\overline{\alpha} \in \overline{R}_{\overline{P}}$ as maps of the complex $\overline{P}$ and such that ψ^* is homotopic to the identity as a map of weak algebras. Then there is a unit α lifting $\overline{\alpha}$ and a homotopy h between ψ and α such that $h(\mathbf{R}\mathrm{End}(P)) \subset \mathbf{R}\mathrm{End}(P) \otimes I$.*

Proof. The proof is quite similar to the proof of Lemma 3.1.1.11, using the left half of the exact sequence of Corollary 3.1.1.6 rather than the right half. □

Proposition 3.1.1.13. *Theorem 3.1.1.5 holds for $\mathscr{O}$ (now assumed complete).*

Proof. Given an isomorphism $\varphi : \mathbf{R}\mathrm{End}(P) \xrightarrow{\sim} \mathbf{R}\mathrm{End}(Q)$, we may assume after adding zero complexes to P and Q, shifting Q, and applying a homotopy to φ, that there is an isomorphism $\psi_0 : P_0 \to Q_0$ such that $\varphi_0 = \psi_0^*$ as maps of complexes. We can now apply Lemma 3.1.1.11 to arrive at an isomorphism ψ_1 lifting ψ_0 and a homotopy $\overline{h}(0)$ with image in $\mathbf{R}\mathrm{End}_{\mathscr{O}_1}(Q_1) \otimes_{\mathscr{O}_1} I_0/I_1$ between φ_1 and ψ_1^*. Lift $\overline{h}(0)$ to a homotopy $h(0)$ with image in $\mathbf{R}\mathrm{End}(Q) \otimes I_0$. Then $(\varphi - (dh(0) + h(0)d))_1 = \psi_1^*$ as maps of complexes, and we may find a homotopy $h(1)$, etc. By Lemma 3.1.1.10, we see that there is an isomorphism $\psi : P \to Q$ such that $\varphi = \psi^*$ in $\mathbf{D}(\mathscr{O})$. A similar argument shows that the kernel is $\mathscr{O}_P^*$. □

3.1.2 The construction of GAz

In this section, we define a stack which we will use to compactify the stack of Azumaya algebras. While the definition is rather technical in general, in the case of a relative surface it assumes a simpler and more intuitive form. A more subtle version of this definition which includes an A_∞-structure is in some sense more natural, but at the expense of yielding a higher stack whose truncation will be what we describe below. Work in the general ∞-direction is currently in progress by Jacob Lurie, Bertrand Toen, Gabriele Vezzosi, and others.

Let $(X, \mathscr{O})$ be a ringed topos.

Definition 3.1.2.1. A sheaf $\mathscr{F}$ on X is *totally supported* if the natural map $\mathscr{O} \to \mathscr{E}nd(\mathscr{F})$ is an injection.

Definition 3.1.2.2. Let X be a ringed topos. A *pre-generalized Azumaya algebra* on X is a perfect algebra object $\mathscr{A}$ of the derived category $\mathbf{D}(X)$ of $\mathscr{O}_X$-modules such that there exists an object $U \in X$ covering the final object and a totally supported perfect sheaf $\mathscr{F}$ on U with $\mathscr{A}|_U \cong \mathbf{R}\mathscr{E}nd_U(\mathscr{F})$ as weak algebras. An isomorphism of pre-generalized Azumaya algebras is an isomorphism in the category of weak algebras.

3.1.2.3. Consider the fibered category $\mathscr{P} \to Sch_{\text{ét}}$ of pre-generalized Azumaya algebras on the large étale topos over $\operatorname{Spec} \mathbf{Z}$. We will stackify this to yield the stack of generalized Azumaya algebras. This is slightly different from the construction given in [52, 3.2], as we do not assume below that the fibered category is a pre-stack.

Lemma 3.1.2.4. *Suppose T is a topos and $\mathscr{C} \to T$ is a category fibered in groupoids. There exists a stack $\mathscr{C}^s$, unique up to 1-isomorphism and a 1-morphism $\mathscr{C} \to \mathscr{C}^s$ which is universal among 1-morphisms to stacks (up to 2-isomorphism).*

Proof. The proof is the usual type of argument. A reader interested in seeing a generalization to stacks in categories larger than groupoids should consult [30]. First, we may assume that in fact $\mathscr{C} \to T$ admits a splitting (after replacing $\mathscr{C}$ by a 1-isomorphic fibered category). Define a new fibered category $\mathscr{C}^p$ as follows: the objects will be the same, but the morphisms between two objects a and b over $t \in T$ will be the global sections of the sheafification of the presheaf $\mathrm{Hom}_t(a,b) : (s \xrightarrow{\varphi} t) \mapsto \mathrm{Hom}_s(\varphi^*a, \varphi^*b)$ on t. Clearly $\mathscr{C}^p$ is a prestack (i.e., given any two sections a and b over t, the hom-presheaf just described is a sheaf) and the natural map $\mathscr{C} \to \mathscr{C}^p$ of fibered categories over T is universal up to 1-isomorphism for 1-morphisms of $\mathscr{C}$ into prestacks. Now we apply [52, 3.2] to construct $\mathscr{C}^s$ as the stackification of $\mathscr{C}^p$. □

Definition 3.1.2.5. The stack of *generalized Azumaya algebras* on schemes is defined to be the stack in groupoids $\mathscr{P}^s \to Sch_{\text{ét}}$ associated to the fibered category of pre-generalized Azumaya algebras.

Remark 3.1.2.6. Explicitly, given a scheme X, to give a generalized Azumaya algebra on X is to give an étale 3-hypercovering $Y'' \substack{\to\\\to\\\to} Y' \rightrightarrows Y \longrightarrow X$, a totally supported sheaf $\mathscr{F}$ on Y, and a gluing datum for $\mathbf{R}\mathscr{E}nd_Y(\mathscr{F})$ in the derived category $\mathbf{D}(Y')$ whose coboundary in $\mathbf{D}(Y'')$ is trivial. Two such objects $(Y_1, \mathscr{F}_1, \delta_1)$ and $(Y_2, \mathscr{F}_2, \delta_2)$ are isomorphic if and only if there is a common refinement Y_3 of the 3-hypercovers Y_1 and Y_2 and an isomorphism $\varphi : \mathscr{F}_1|_{Y_3} \xrightarrow{\sim} \mathscr{F}_2|_{Y_3}$ commuting with the restrictions of δ_1 and δ_2. Thus, a generalized Azumaya algebra is gotten by gluing "derived endomorphism algebras" together in the étale topology. When X is a quasi-projective smooth surface, or, more generally, a quasi-projective scheme smooth over an affine with fibers of equidimension 2, then the sections of $\mathscr{P}$ over X are the same as the sections of $\mathscr{P}^s$ over X; see 3.1.3.

Example 3.1.2.7. Let $\pi : \mathscr{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe and $\mathscr{F}$ a totally supported perfect $\mathscr{X}$-twisted sheaf. The complex $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}}(\mathscr{F}) \in \mathbf{D}(X)$ is a pre-generalized Azumaya algebra, hence has a naturally associated generalized Azumaya algebra. We will see below that the global sections of the stack $\mathscr{P}^s$ are precisely the weak algebras of this form.

Definition 3.1.2.8. Let $\mathscr{A}$ be a generalized Azumaya algebra on X. The *gerbe of trivializations* of $\mathscr{A}$, denoted $\mathscr{X}(\mathscr{A})$, has sections over $V \to X$ given by pairs $(\mathscr{F}, \varphi)$, where $\mathscr{F}$ is a totally supported sheaf on V and $\varphi : \mathbf{R}\mathscr{E}nd_V(\mathscr{F}) \xrightarrow{\sim} \mathscr{A}|_V$ is an isomorphism of generalized Azumaya algebras. The isomorphisms in the fiber categories are isomorphisms of the sheaves which respect the identifications with $\mathscr{A}$, as usual.

This is entirely analogous to the gerbe produced in 2.1.4. The reader should as usual observe the *caveat* of 2.1.5.4.

Lemma 3.1.2.9. *$\mathscr{X}(\mathscr{A})$ is a $\mathbf{G}_m$-gerbe.*

Proof. This follows immediately from the derived Skolem-Noether theorem and the fact that all of the sheaves $\mathscr{F}$ are totally supported. □

Let $\pi : \mathscr{X}(\mathscr{A}) \to X$ denote the natural projection.

Lemma 3.1.2.10. *There is an $\mathscr{X}(\mathscr{A})$-twisted sheaf $\mathscr{F}$ and an isomorphism of generalized Azumaya algebras $\varphi : \mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}(\mathscr{A})}(\mathscr{F}) \xrightarrow{\sim} \mathscr{A}$. The datum $(\mathscr{X}(\mathscr{A}), \mathscr{F}, \varphi)$ is functorial in $\mathscr{A}$.*

Proof. As usual, the construction of $\mathscr{X}(\mathscr{A})$ yields by first projection a twisted sheaf $\mathscr{F}$. Whenever $\mathscr{X}(\mathscr{A})$ has a section f over V, there is an isomorphism $\mathbf{R}\mathscr{E}nd_V(f^*\mathscr{F}) \to \mathscr{A}|_V$ by construction, and this is natural in V and f. This is easily seen to imply the remaining statements of the lemma. □

Let $\mathscr{D} \to Sch_{\text{ét}}$ denote the fibered category of derived categories which to any scheme X associates the derived category $\mathbf{D}(X)$ of étale $\mathscr{O}_X$-modules.

Proposition 3.1.2.11. *There is a faithful morphism of fibered categories $\mathscr{P}^s \to \mathscr{D}$ which identifies $\mathscr{P}^s$ with the subcategory $\mathscr{D}$ whose sections over X are weak algebras of the form $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}}(\mathscr{F})$, where $\pi : \mathscr{X} \to X$ is a $\mathbf{G}_m$-gerbe, and whose isomorphisms $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}}(\mathscr{F}) \xrightarrow{\sim} \mathbf{R}\pi'_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}'}(\mathscr{F}')$ are a pseudo-torsor under the group $\Gamma(\mathscr{X}, \mathrm{Aut}(\mathscr{F})/\mathbf{G}_{m,\mathscr{X}})$, with a section if and only if $\mathscr{X}$ is 1-isomorphic to $\mathscr{X}'$ and under such an identification, there is an isomorphism $\mathscr{F} \xrightarrow{\sim} \mathscr{L} \otimes \mathscr{F}'$, with $\mathscr{L}$ an invertible sheaf.*

Proof. The morphism $\mathscr{P}^s \to \mathscr{D}$ comes from the universality of the construction of the stackification (or from 3.1.2.10, which provides a functorial object of $\mathscr{D}$). That the functor is faithful is a consequence of the derived Skolem-Noether theorem and the definition of $\mathscr{P}^s$. The characterization of the morphisms again follows from the derived Skolem-Noether, which says that given an isomorphism $\mathscr{A} \xrightarrow{\sim} \mathscr{A}'$, there is an induced 1-isomorphism $\mathscr{X}(\mathscr{A}) \xrightarrow{\sim} \mathscr{X}(\mathscr{A}')$ in such a way that the twisted sheaf corresponding to $\mathscr{A}'$ pulls back to be isomorphic to an invertible sheaf $\mathscr{L}$ on X tensored with the twisted sheaf corresponding to $\mathscr{A}$. The automorphisms of $\mathbf{R}\mathscr{E}nd(\mathscr{F})$ are then seen to be identified with isomorphisms $\mathscr{F} \xrightarrow{\sim} \mathscr{M} \otimes \mathscr{F}$, where $\mathscr{M}$ is some invertible sheaf, modulo scalar multiplications. The sheaf of such things is precisely $\mathrm{Aut}(\mathscr{F})/\mathbf{G}_m$. □

Corollary 3.1.2.12. *A generalized Azumaya algebra $\mathscr{A}$ has a class in $\mathrm{H}^2(X, \mathbf{G}_m)$. When the rank of $\mathscr{A}$ is n^2, $\mathscr{A}$ has a class in $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ (in the fppf topology).*

Proof. The first statement is trivial. The second statement follows from the analogous construction of the $\boldsymbol{\mu}_n$-gerbe associated to an Azumaya algebra using "trivialized trivializations" and is left to the reader. □

Definition 3.1.2.13. When $\mathrm{rk}\,\mathscr{A} = n^2$, we will call the cohomology class in $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ the *class of* $\mathscr{A}$, and write $\mathrm{cl}(\mathscr{A})$.

One can check that all of the formal properties of the $\boldsymbol{\mu}_n$-gerbe of trivialized trivializations introduced in 2.2.2.10 carry over to the generalized setting. In particular, given a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$, a generalized Azumaya algebra $\mathscr{A}$ of degree n has class $[\mathscr{X}]$ if and only if it is isomorphic to $\mathbf{R}\mathscr{E}nd(\mathscr{F})$ for some $\mathscr{F}$ on $\mathscr{X}$ of rank n and trivial determinant.

Thus, at the end of the complex process of stackification, one is left simply with the derived endomorphism algebras of twisted sheaves, with special morphisms giving the isomorphisms.

3.1.2.14. Now we wish to use "Weil restriction" to define relative stacks of generalized Azumaya algebras.

Definition 3.1.2.15. Let $f : X \to S$ be a morphism. A *relative generalized Azumaya algebra* on X/S is a generalized Azumaya on X whose local sheaves are S-flat. This is equivalent to writing $\mathscr{A} \cong \mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}}(\mathscr{F})$ with $\mathscr{X} \to X$ a $\mathbf{G}_m$-gerbe and $\mathscr{F}$ an S-flat $\mathscr{X}$-twisted sheaf which is totally supported in every geometric fiber.

Before continuing, we prove a lemma which shows that the condition of total support on fibers implies total support on the total space. This will make later arguments smoother.

Lemma 3.1.2.16. *Suppose $f : Y \to Z$ is a flat morphism of locally Noetherian schemes and $\mathscr{F}$ is a Z-flat coherent sheaf on Y. If the restriction of $\mathscr{F}$ to every fiber of f is totally supported, then $\mathscr{F}$ is totally supported on Y.*

Proof. We may assume that $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$ are local schemes and that f is the map associated to a local homomorphism $\varphi : A \to B$. Write F for the stalk of $\mathscr{F}$ at the closed point of B. Choosing generators $x_1, \ldots, x_n$ for F, we find a surjection $B^n \twoheadrightarrow F$ which yields an injection $\operatorname{End}(F) \hookrightarrow F^n$. The composition of this injection with the natural inclusion of B sends $1 \in B$ to the n-tuple $(x_1, \ldots, x_n) \in F^n$. We will show that this map $\iota : B \to F^n$ is an injection. Note that ι respects base change in the sense that for any A-algebra C, $\iota \otimes_A C$ is the map corresponding to the composition $C \to \operatorname{End}_C(F \otimes_A C) \to (F \otimes_A C)^n$. As the right-hand map in that sequence is always an injection, we find that the left-hand map is an injection if and only if $\iota \otimes_A C$ is an injection.

We proceed by "infinitesimal induction" relative to A, i.e., write A with the $\mathfrak{m}_A$-adic topology as an inverse limit of small extensions $\{A_m\}$ with $A_0 = k(A)$, the residue field of A. We will show that $\varprojlim \iota_m : \widehat{B} \to \widehat{F}^n$ is an injection. Krull's theorem and the obvious compatibility then show that ι itself is an injection.

By hypothesis ι_0 is an injection. Suppose by induction that ι_m is an injection. Let ε generate the kernel of $A_{m+1} \to A_m$. By flatness, there are identifications $\varepsilon B_{m+1} \cong (\varepsilon) \otimes_{A_{m+1}} B_{m+1} \cong B_0$ and $\varepsilon F^n_{m+1} \cong (\varepsilon) \otimes F^n_{m+1} \cong F^n_0$, and under these identifications, $\varepsilon \cdot \iota_{m+1}$ is identified with ι_0. Now consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varepsilon B_{m+1} & \longrightarrow & B_{m+1} & \longrightarrow & B_m & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varepsilon F^n_{m+1} & \longrightarrow & F^n_{m+1} & \longrightarrow & F^n_m & \longrightarrow & 0. \end{array}$$

By the Snake Lemma and the inductive hypothesis, the kernel of the left-hand vertical map is identified with the kernel of the middle map (which is ι_{m+1}). But the left-hand map is identified with ι_0, hence is injective. □

As above, one may define the class of such a generalized Azumaya algebra. Let $\mathscr{X} \to X$ be a fixed $\boldsymbol{\mu}_n$-gerbe, with $n \in \mathscr{O}_S(S)^\times$.

Definition 3.1.2.17. Let $\mathbf{GAz}_{\mathscr{X}/S}(n)$ denote the stack of generalized Azumaya algebras on X/S of rank n^2 in every geometric fiber whose class agrees with $[\mathscr{X}]$ étale locally around every point on the base.

When $X \to S$ is proper and n is invertible on S, the last condition is equivalent to agreeing with $[\mathscr{X}]$ in every geometric fiber.

3.1.3 Stackification is unnecessary on a surface

Let $f : X \to S$ be a smooth projective relative surface. We will prove here that pre-generalized Azumaya algebras on X as in 3.1.2 form a stack on S.

Lemma 3.1.3.1. *Let $f : X \to S$ be a flat morphism of locally Noetherian schemes. Suppose $\mathscr{F}$ is an f-flat sheaf on X whose restrictions to the fibers of f are torsion free (i.e., have no embedded points). If $Z \subset X$ is a quasi-finite scheme over S, then there is no local section of $\mathscr{F}$ supported on Z. Furthermore, there are no local sections of $\mathscr{E}nd(\mathscr{F})$ having support in Z.*

Proof. To prove the theorem, we may replace S by its localization $\operatorname{Spec} A$ at a point and X by its localization $\operatorname{Spec} B$ at a closed point over the point of A. Therefore we may take f to be a local map $\varphi : A \to B$ of local Noetherian rings, and we may replace $\mathscr{F}$ by its stalk M at the closed point of B. In this case, Z will be the closed point of $\operatorname{Spec} B$. The method of proof is an infinitesimal induction formally similar to that of Lemma 3.1.2.16, and we leave the details to the reader. □

Given a pre-generalized Azumaya algebra $\mathscr{A}$ on X, 3.1.2.10 produces a $\mathbf{G}_m$-gerbe $\mathscr{X}$, an $\mathscr{X}$-twisted sheaf $\mathscr{F}$, and an isomorphism of *generalized* Azumaya algebras $\mathscr{B} := \mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{F}) \xrightarrow{\sim} \mathscr{A}$. We will show that in fact $\mathscr{B}$ and $\mathscr{A}$ are isomorphic as *pre-generalized Azumaya algebras*. We will temporarily call $\mathscr{B}$ the *associated twisted derived endomorphism algebra* (or TDEA for short).

Proposition 3.1.3.2. *Suppose $\pi : X \to S$ is a smooth quasi-projective surface over an affine scheme. Any pre-generalized Azumaya algebra is isomorphic to the associated TDEA in $\mathscr{P}$. Furthermore, the isomorphisms of two such weak algebras form a sheaf on S.*

Proof. Fix an ample sheaf $\mathscr{O}(1)$ on X. Let $\mathscr{A}$ be a generalized Azumaya algebra on X, and let $\mathscr{B}$ be the associated TDEA. As S is affine and π is quasi-projective, we may choose a cochain complex representing $\mathscr{A}$ of the form $F^\bullet$, where each F^i is a locally free sheaf of the form $\mathscr{O}(-n)^m$ and $F^i = 0$ for $i > 1$. We may assume in what follows that each n is arbitrarily large (with succeeding steps possibly depending upon previous ones). We may similarly choose a representative for $\mathscr{B}$ of the form $C^{-1} \to C^0 \to C^1$, with the C^i coherent sheaves on X such that C^0 and C^1 are locally free and $C^{-1} \to C^0$ is injective. We will identify C^{-1} with its image in C^0 below hopefully without confusion. (This implies that C^{-1} is also locally free because π is smooth of relative dimension 2, but this is not important to us. Similarly, this fact tells us that we could assume that the F^i vanish for $i < -1$, but again this is of no importance to us.)

By the definition of pre-generalized Azumaya algebra, we see that both of these complexes are concentrated in degree zero away from a locus Z which is quasi-finite over S. Lemma 3.1.3.1 further shows that the zeroth cohomology (which is the same for both sheaves by étale descent) has no sections with support contained in Z. By definition, we may find an étale hypercovering $Y_\bullet \to X$ with each Y_i an affine scheme such that there is an isomorphism $\varphi : \mathscr{A}_{Y_0} \to \mathscr{B}_{Y_0}$ respecting the derived descent data on $\mathscr{A}$ and $\mathscr{B}$ on Y_1. Since locally free sheaves are projective objects in the category of modules over an affine, we see that φ may be represented by a map $f : F_{Y_0} \to C_{Y_0}$ of complexes. Compatibility with the descent data means that the two maps of complexes $p_1^* f$ and $p_2^* f$ are homotopic on Y_1 (as maps from F_{Y_1} to C_{Y_1}), say by $h \in \prod \operatorname{Hom}(F^n_{Y_1}, C^{n-1}_{Y_1})$. This h has the property that its coboundary on Y_2 is a zero homotopy, i.e., $dh + hd = 0$. Let $\mathscr{H}$ be the sheaf of tuples of morphisms $F^n \to C^{n-1}$, and $\mathscr{Z} \subset \mathscr{H}$ the subsheaf of such tuples h such that $dh + hd = 0$. Consider the sequence

$$0 \to \mathscr{Z} \to \mathscr{H} \to \mathscr{C} \to 0$$

of coherent sheaves on X. The homotopy h gives rise to an element of $\mathrm{H}^1(X, \mathscr{C})$ which we wish to vanish. (Indeed, as all sheaves are coherent and the Y_i are affine, the Čech cohomology of $Y_\bullet$ computes the sheaf cohomology, so we can then split the cocycle *on the given hypercovering*, which will descend the map of complexes.) We claim that the entire cohomology group is zero. By the assumption on the negativity of the F^i, we may assume that $\mathrm{H}^1(\mathscr{C}) = \mathrm{H}^2(\mathscr{Z})$. Let us present $\mathscr{Z}$ in a different way. Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & \operatorname{Hom}(F^1, C^{-1}) & \longrightarrow & \mathscr{Q}' & \longrightarrow & 0 \\
 & & & \nearrow & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{Z} & \longrightarrow & \operatorname{Hom}(F^1, \ker d_C^0) & \longrightarrow & \mathscr{Q} & \longrightarrow & 0.
\end{array}$$

The left-hand map factors through the diagonal arrow and the vertical inclusion on $U = X \setminus Z$: on U, the first cohomology of F and C vanishes. Therefore, since the components are locally free, we may split the complexes $F^\bullet$ and $C^\bullet$ at the right end. This in fact shows that $\mathscr{Z}$ is *identified* with the subsheaf $\operatorname{Hom}(F^1, C^{-1})$ on U. We claim that this implies that the left-hand map always factors through the subsheaf $\operatorname{Hom}(F^1, C^{-1})$ of $\operatorname{Hom}(F^1, \ker d_C^0)$ and that the cokernel $\mathscr{Q}'$ is supported only on Z. Indeed, any morphism $\varphi : F^1 \to \ker d_C^0$ (over an étale open of X) maps to 0 in the zeroth cohomology of $C^\bullet$ away from the finite locus Z. By the fact that $\mathrm{H}^0(C^\bullet) = \mathscr{E}nd(\mathscr{F})$ and 3.1.3.1, we see that φ must lie in $\operatorname{Hom}(F^1, C^{-1})$ over *every* point. On the other hand, by the negativity assumption on F^1, $\mathrm{Hom}(F^1, C^{-1})$ has no cohomology. Thus, $\mathscr{Z}$ has no H^2.

The last statement of the proposition follows by an identical proof which we leave to the reader. □

3.2 The relation with twisted sheaves

By taking advantage of the derived Skolem-Noether theorem and the construction of the stack of generalized Azumaya algebras, we can see that the stack-theory of twisted sheaves is closely bound to that of generalized Azumaya algebras. *We assume throughout that n is invertible on the base S.*

3.2.1 A covering

Let $\mathscr{X} \to X \to S$ be a $\boldsymbol{\mu}_n$-gerbe on a proper flat morphism of finite presentation to a locally Noetherian scheme S. It is clear that $\mathscr{X}$-twisted sheaves form a stack on S, with flat families with torsion free fibers forming a substack. (We will show this substack is open in 4.1.2 below.)

Definition 3.2.1.1. The *stack of torsion free perfect $\mathscr{X}$-twisted sheaves of rank n*, denoted $\mathbf{Tw}_{\mathscr{X}/S}(n)$, is the stack whose sections over $T \to S$ are T-flat families of coherent $\mathscr{X}$-twisted sheaves $\mathscr{F}$ such that for every geometric point $t \to T$, the fiber $\mathscr{F}_t$ is a torsion free perfect $\mathscr{X}_t$-twisted sheaf of rank n.

We remind the reader that "perfect" means "perfect as an object of the derived category $\mathbf{D}(\mathscr{X})$." Using the construction of Mumford and Knudsen [45], there is a natural map $\mathbf{Tw}_{\mathscr{X}/S}(n) \to \mathrm{Pic}_{X/S}$ given by sending $\mathscr{F}$ to $\det \mathscr{F}$. The stack-theoretic fiber over $[\mathscr{O}_X]$, which we will denote $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O})$, is the stack of twisted sheaves with *trivialized* determinant, i.e., pairs $(\mathscr{F}, \psi)$ with $\mathscr{F}$ a torsion free perfect $\mathscr{X}$-twisted sheaf of rank n and ψ is a trivialization $\det \mathscr{F} \xrightarrow{\sim} \mathscr{O}$. (This follows from the construction of the natural 1-fiber product of stacks.)

There is a natural map $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}) \to \mathbf{GAz}_{\mathscr{X}/S}(n)$ given by sending an $\mathscr{X}$-twisted sheaf $\mathscr{F}$ to the generalized Azumaya algebra $\mathbf{R}\mathscr{E}nd(\mathscr{F})$.

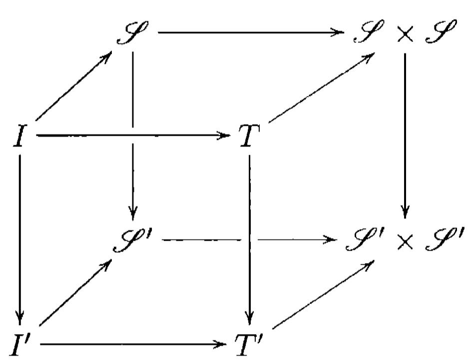
Proposition 3.2.1.2. *The morphism $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}) \to \mathbf{GAz}_{\mathscr{X}/S}(n)$ is an epimorphism of algebraic stacks. There exists a stack $\mathscr{M}$ with a factorization $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}) \to \mathscr{M} \to \mathbf{GAz}_{\mathscr{X}/S}(n)$ such that $\mathscr{M} \to \mathbf{GAz}$ is representable by $\mathrm{Pic}_{X/S}[n]$-torsors and $\mathbf{Tw} \to \mathscr{M}$ is representable by $\boldsymbol{\mu}_n$-gerbes.*

Proof. The epimorphism arises from the definition of $\mathbf{GAz}$ and 3.1.2.10. The rest is a consequence of derived Skolem-Noether. Let us make it clear. We omit the adornments for the duration of this proof. Given a section $T \to \mathbf{GAz}$, consider the fiber product $\mathscr{S} = T \times_{\mathbf{GAz}} \mathscr{T}$. We claim that $\mathscr{S} \to \mathrm{Sh}(\mathscr{S}) \to T$ is a $\boldsymbol{\mu}_n$-gerbe over a $\mathrm{Pic}_{X/S}[n]|_T$-torsor. Tensoring by invertible sheaves gives a map $\mathscr{S} \times_T \mathscr{P}ic_{X/S}[n] \to \mathscr{S}$ which descends to a map $a : \mathrm{Sh}(\mathscr{S}) \times_T \mathrm{Pic}_{X/S}[n] \to \mathrm{Sh}(\mathscr{S})$. There is an étale surjection $T' \to T$ and a lift $T' \to \mathbf{Tw}$. The derived Skolem-Noether theorem ensures that when the base is extended to T', the map a is an isomorphism. This shows that $\mathrm{Sh}(\mathscr{S}) \to T$ is a $\mathrm{Pic}_{X/S}[n]$-torsor. Furthermore, again by the derived Skolem-Noether theorem, we see that $\mathscr{S} \to \mathrm{Sh}(\mathscr{S})$ is a $\boldsymbol{\mu}_n$-gerbe. It is easy to see that the formation of Sh (which we emphasize is a big étale sheaf on T) is a local construction in the sense of Laumon and Moret-Bailly [52]; i.e., it defines a sheaf on **GAz**. This yields the stack $\mathscr{M}$. □

Remark 3.2.1.3. By any reasonable definition, we should really be able to see that $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}) \to \mathbf{GAz}_{\mathscr{X}/S}(n)$ is a "strict 2-torsor" under the Picard stack $\mathscr{P}ic_{X/S}[n]$. However, the number of diagrams which intervene far outweighs the payoff for our purposes. ♦

Proposition 3.2.1.4. *If $f : \mathscr{S} \to \mathscr{S}'$ is a map of S-stacks which is representable by fppf morphisms of algebraic stacks then $\mathscr{S}$ is algebraic if and only if $\mathscr{S}'$ is.*

Proof. First, we show that the diagonal of $\mathscr{S}$ is separated, quasi-compact, and representable by algebraic spaces if and only if the same is true for $\mathscr{S}'$. To this end, let $T' \to \mathscr{S}' \times \mathscr{S}'$ be a morphism with T' an affine scheme. Consider the diagram



whose terms we now explain. The sheaf I' is the pullback of T' along the diagonal. By assumption, the fiber product $\mathscr{S} \times \mathscr{S} \times_{\mathscr{S}' \times \mathscr{S}'} T'$ is an algebraic stack over T' with fppf structure morphism. Thus, we may let T be a scheme which gives a smooth cover, and then we let I be the pullback sheaf of T along the diagonal of $\mathscr{S}$. We see that $I \to I'$ is relatively representable by fppf morphisms of algebraic spaces. By a result of Artin [52, 10.1], I is an algebraic space if and only if I' is.

It remains to show that $\mathscr{S}$ has a smooth cover by an algebraic space if and only if $\mathscr{S}'$ does. In fact, it suffices to replace the word "smooth" by "fppf," by Artin's theorem [*ibid.*]. This is left to the reader. □

Corollary 3.2.1.5. *The stack $\mathbf{GAz}_{\mathscr{X}/S}(n)$ is algebraic if and only if $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O})$ is algebraic.*

Proof. This is immediate from 3.2.1.2 and 3.2.1.4. □

3.2.2 An explicit Morita equivalence

We briefly recall a basic fact from [6] which makes use of the theory of fibered Morita equivalence developed in section 2.1.4. This will frequently furnish alternative (non-intrinsic) proofs for certain results later on.

Proposition 3.2.2.1. *Let X be a ringed topos and $\mathscr{X} \to X$ a $\mathbf{G}_m$-gerbe. If there is a faithful locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$, then the functor $\mathscr{H}om(\mathscr{V}, \cdot\,)$ gives an equivalence of abelian categories between the category $\mathscr{X}$-twisted sheaves and the category of $\mathscr{A}$-modules, where $\mathscr{A} = \mathscr{E}nd(\mathscr{V})$ is an Azumaya algebra with Brauer class $[\mathscr{X}] \in \mathrm{H}^2(X, \mathbf{G}_m)$.*

Proof. This follows immediately from the results of section 2.1.4 and the fact that $\mathscr{G}$ is $\mathscr{X}$-twisted if and only if $\mathscr{H}om(\mathscr{V}, \mathscr{G})$ is untwisted. □

Corollary 3.2.2.2. *If X is a scheme admitting an ample invertible sheaf and $\mathscr{X} \to X$ is a $\mathbf{G}_m$-gerbe representing a class in $\mathrm{H}^2(X, \mathbf{G}_m)_{\mathrm{tors}}$, then there is an Azumaya algebra $\mathscr{A}$ on X and an equivalence between the category of $\mathscr{X}$-twisted sheaves and the category of $\mathscr{A}$-modules.*

Proof. This results from 2.2.3.8 and 3.2.2.1. □

When $\mathscr{X}$ does *not* represent a torsion class in $\mathrm{H}^2(X, \mathbf{G}_m)$, there are still plenty of (even coherent) $\mathscr{X}$-twisted sheaves. However, it is no longer the case that such sheaves have natural untwisted avatars as modules over a nice algebra. In general, while it may be tempting to try to reduce the study of twisted sheaves to $\mathscr{A}$-modules, one profits greatly from a more intrinsic approach, as we hope to demonstrate in the remaining chapters.

Chapter 4

Algebraic moduli

In this chapter, we will show that the moduli stacks of twisted sheaves and generalized Azumaya algebras are algebraic. In the process, we will develop a theory of semistable twisted sheaves and generalized Azumaya algebras and study the relation to Geometric Invariant Theory. We also describe the deformation theory of generalized Azumaya algebras.

4.1 Moduli of twisted sheaves

We prove that the stack of twisted sheaves (and hence by 3.2.1 the stack of generalized Azumaya algebras) on a proper morphism $X \to S$ of finite presentation over an excellent (quasi-separated) base is algebraic. We then study the substack of pure twisted sheaves and show that it is open in 4.1.2. This sets the stage for a study of stability of twisted sheaves (in its Mumford-Takemoto and Gieseker forms) when $X \to S$ is projective and its use in producing GIT quotient stacks and corepresenting projective schemes for stacks of semistable twisted sheaves in 4.1.3 and 4.1.4. The work on Gieseker stability will require the definition of a suitable Hilbert polynomial. We define and study this polynomial and state a Riemann-Roch theorem 4.1.3.7 which will be useful at various points throughout this work.

In the special case where S is affine and $X \to S$ is projective of relative dimension 2, we can use 2.2.3.8 to "drastically simplify" the situation by reducing it to work of Simpson on stability of modules for an algebra. Indeed, once there exists a locally free twisted sheaf $\mathscr{V}$, the category of twisted sheaves becomes equivalent with the category of modules for the Azumaya algebra $\mathscr{E}nd(\mathscr{V})$ on X. (This is just fibered Morita equivalence.) Simpson has considered moduli of modules [70] quite generally; being careful, we can choose $\mathscr{V}$ so that the stability condition considered by Simpson agrees with the stability condition defined here. (In fact, in arbitrary dimension such a Morita equivalence will always preserve slope-stability.) We will use this technique to transport Simpson's GIT approach to the twisted setting on a surface (and to prove some boundedness theorems in arbitrary dimension by appeal to classical results after a Morita equivalence).

However, we wish to emphasize that this approach is fundamentally incorrect.

While it is useful to have a Morita equivalence handy for transporting classical theorems, it is always better to work intrinsically when possible. Working directly on the gerbe is also a step toward producing a satisfactory theory of sheaves and bundles on (at least DM) stacks.

4.1.1 Abstract existence

Let $X \to S$ be an algebraic space which is proper of finite presentation over a locally Noetherian scheme, and let $\mathscr{X} \to X$ be a fixed $\boldsymbol{\mu}_n$-gerbe, where n is prime to $\mathrm{char}(X)$. Consider the S-groupoid $\mathscr{T}_{\mathscr{X}/S}$ which assigns to an affine scheme $\operatorname{Spec} R \to S$ over S the category whose objects are R-flat families of coherent $\mathscr{X}$-twisted sheaves. (We reserve the notation $\mathbf{Tw}$ for twisted sheaves without embedded points; 4.1.2 we will show that $\mathbf{Tw} \subset \mathscr{T}$ is an open substack.) Our goal in this section is to apply Artin's Theorem [9] to prove the following.

Proposition 4.1.1.1. *$\mathscr{T}_{\mathscr{X}/S}$ is an algebraic stack locally of finite presentation over S.*

Corollary 4.1.1.2. *The result of 4.1.1.1 is true if and only if it is true when S is excellent and Noetherian.*

Proof. Since 1) X is of finite presentation, 2) being algebraic is local on S and stable under base change, 3) the formation of étale cohomology is compatible with affine limits [25, I.4], and 4) the formation of the stack $\mathscr{T}$ is compatible with base change, we may replace S with a finite type $\mathbf{Z}$-algebra. □

Most of the components necessary to apply Artin's Theorem are described in the deformation theory of 2.2.5.

Lemma 4.1.1.3. *Let R be a complete local Noetherian ring, and suppose $S = \operatorname{Spec} R$ above. Given a compatible system of twisted sheaves $\mathscr{F}_i$ on $\mathscr{X} \otimes R/\mathfrak{m}_R^{i+1}$, there is a twisted sheaf $\mathscr{F}$ on $\mathscr{X}$ whose reduction modulo $\mathfrak{m}_R^{i+1}$ is compatibly isomorphic to $\mathscr{F}_i$.*

Proof. This follows directly from the result of Olsson and Starr for sheaves on DM stacks [58]. If $X \to S$ is a projective morphism of schemes, then by 2.2.3.8 and Morita equivalence, the category of coherent twisted sheaves is equivalent to the category of $\mathscr{E}nd_{\mathscr{X}}(\mathscr{V})$-modules, where $\mathscr{V}$ is a faithful locally free twisted sheaf. But then we are reduced to the obvious analogue of the classical form of Grothendieck's Existence Theorem for modules over a coherent algebra [33, §5]. □

Lemma 4.1.1.4 (Schlessinger). *Suppose $A_1 \to A_0 \leftarrow A_2$ is a diagram of commutative rings such that $A_2 \to A_0$ is a surjection with nilpotent kernel J. Suppose give a diagram of flat modules $M_1 \to M_0 \leftarrow M_2$ over the diagram of rings inducing isomorphisms $M_i \otimes A_0 \cong M_0$. Let $B = A_2 \times_{A_0} A_1$ and $N = M_2 \times_{M_0} M_1$. Then N is a flat B-module and $N \otimes A_i \cong M_i$.*

Proof. The proof of this result given in [66] only treats a special case which does not suffice for our purposes and the reference given there for the general case is not publicly available. Thus, we give a proof which works for Noetherian rings and indicate how to generalize it to arbitrary commutative rings.

To see that N is B-flat, we use the local criterion of flatness [54, §22]. Since $A_2 \to A_0$ is surjective (say with kernel J), we see that $B \to A_1$ is surjective with (nilpotent) kernel $I := J \times_{A_0} 0_{A_1}$. It is easy to see that $N/IN \cong M_1$ as A_1-modules. To show that N is flat over B, it remains to show that the natural map $\varphi : I \otimes_B N \to IN$ is an isomorphism. We may assume (after filtering J and proceeding inductively) that J is generated by a single element t of square 0. (This step of the proof only works in the Noetherian case, but the usual "equational criterion" for flatness [54, 7.6] will work in the general case. We choose to analyze this case for the sake of simplicity, and because it suffices for our purposes.) The statement that $I \otimes N \to IN$ is an isomorphism is then equivalent to the statement that if $n = m_2 \times m_1$ satisfies $(t \times 0)n = 0$ then $(t \times 0) \otimes n = 0$. But if $(t \times 0)n = 0$, then $tm_2 = 0$. As M_2 is flat over A_2, we have $m_2 = tm_2'$, so that $m_2 \mapsto 0 \in A_0$. Thus, $m_1 \mapsto 0 \in A_0$, so $m_1 = \sum k_j m_1^{(j)}$ for some $k_j \in \ker(A_1 \to A_0)$, and so $m_2 \times m_1 = (t \times k_1)(m_2 \times m_1^{(1)}) + (0 \times k_2)(m_2 \times m_2^{(2)}) + \cdots$. Plugging this in, we find that $(t \times 0) \otimes n = 0$ as required. □

Lemma 4.1.1.5. *Given two S-flat $\mathscr{X}$-twisted sheaves $\mathscr{F}$ and $\mathscr{G}$, the functor on S-schemes $T \mapsto \mathrm{Isom}(\mathscr{F}_T, \mathscr{G}_T)$ is representable by a separated algebraic space $\mathrm{Isom}(\mathscr{F}, \mathscr{G})$ of finite type over S with a $\mathbf{G}_m$-action. If $\mathscr{F}$ and $\mathscr{G}$ both have rank n and $\det \mathscr{F}$ and $\det \mathscr{G}$ are given trivializations, then the subspace $\mathrm{Isom}^{\det}$ of isomorphisms preserving the determinant is a $\boldsymbol{\mu}_n$-equivariant closed subspace. When $X \to S$ is locally projective, Isom is locally a closed cone in an affine bundle.*

Proof. The first part of the lemma is an application of Artin's representability theorem, which we omit. It is proven by methods similar to 4.1.1.1. When X is locally projective over S, we can give a more elementary proof, once we admit 2.2.3.8. In this case, we can locally write $\mathscr{F}$ as a cokernel $\mathscr{V}^{-1} \to \mathscr{V}^0$ of locally free twisted sheaves. As it suffices to work locally on S, we may assume that S is affine and that there is a global resolution $\mathscr{V}^{-1} \to \mathscr{V}^0 \to \mathscr{F} \to 0$. Twisting by enough powers of $\mathscr{O}(-1)$ when constructing the resolution $\mathscr{V}^\bullet$, we may further assume that the higher cohomology of the sheaf $\mathscr{H}om(\mathscr{V}^i, \mathscr{G})$ vanishes for $i = 0, -1$ and that it is generated by global sections. Note that $\mathrm{Isom}(\mathscr{F}, \mathscr{G})$ is an open subfunctor of the functor $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ (for a proof of this, see e.g. [52, 4.6.2.1]). Thus, it suffices to show that $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ is a subcone of an affine bundle. The resolution gives rise to a map of sheaves $\mathrm{Hom}(\mathscr{V}^0, \mathscr{G}) \to \mathrm{Hom}(\mathscr{V}^{-1}, \mathscr{G})$ whose kernel is $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$; as the kernel is precisely the fiber product $\mathrm{Hom}(\mathscr{V}^0, \mathscr{G}) \times_{\mathrm{Hom}(\mathscr{V}^{-1}, \mathscr{G})} S$ (the latter map being the zero section), we see that it suffices to prove that $\mathrm{Hom}(\mathscr{V}^0, \mathscr{G})$ is a closed cone in an affine bundle. This last sheaf is the same as $\mathrm{Hom}(\mathscr{O}, \mathscr{H}om(V^0, \mathscr{G}))$, so it suffices to show that given an S-flat coherent sheaf $\mathscr{F}$ on X which is generated by global sections and has no higher cohomology, the functor $T \mapsto \mathscr{F}(X_T)$ is representable by a vector bundle. But now this functor is precisely the functor represented by $\pi_* \mathscr{F}$, which is locally free and commutes with arbitrary base change by standard cohomology and base change arguments [38]. □

Proof of 4.1.1.1. We recall Artin's conditions: let F be the stack of twisted sheaves, $\overline{F}$ the associated presheaf of isomorphism classes. Given a morphism of rings $B \to A$ and an element $a \in F(A)$, we will denote $F_a(B)$ the fiber of $F(B) \to F(A)$ over a (and similarly for $\overline{F}$). The first conditions which must be satisfied to apply Artin's theorem are the Schlessinger-Rim criteria (our versions are slightly more general then are necessary; see [9] for Artin's list):

(S1a) given a diagram $A' \to A \leftarrow B$ with $A' \to A$ surjective with nilpotent kernel, and given $a \in F(A)$, the canonical map

$$\overline{F}_a(A' \times_A B) \to \overline{F}_a(A') \times \overline{F}_a(B)$$

is surjective.

(S1b) If $B \to A$ is a surjection, $b \in F(B)$ with image $a \in F(A)$, and M is a finite A-module then the canonical map

$$\overline{F}_b(B \oplus M) \to \overline{F}_a(A \oplus M)$$

is bijective.

(S2) Given $a \in F(A)$, the A-module $\overline{F}_a(A \oplus M)$ is finite. (The module structure comes about via S1b. See [9, 3, 66] for details.)

(Aut) Given $a \in F(A)$, the module $Aut_a(A \oplus M)$ of infinitesimal automorphisms of a is a finite A-module.

In our case, these are easy to check. (S1a) follows from 4.1.1.4 by an argument similar to [66, 3.1]. (S1b) follows from the cher à Cartan isomorphism 2.2.5.8. (S2) comes from the coherence of derived pushforwards of coherent sheaves on proper morphisms. (Aut) follows from 2.2.5.14.

In addition to the "local versality" conditions, one must check effectivization and constructibility conditions. In particular, one must check that the map $F(\widehat{A}) \to \varprojlim F(\widehat{A}/\mathfrak{m}^n)$ is a 1-isomorphism of groupoids for a local Noetherian A over S. This follows from 4.1.1.3 above or from Olsson's general Grothendieck Existence Theorem for algebraic stacks [57]. The constructibility conditions are the following: the deformation and obstruction theories are compatible with étale localizations and completion (2.2.5.17(1) and (2)), and there is a dense open where they are compatible with base change to fibers (2.2.5.17(3)). One requires that similar conditions hold for the group of infinitesimal automorphisms; this is also subsumed in 2.2.5.17. The last condition to check is that given a reduced finite type S-affine $\operatorname{Spec} A_0 \to S$ and an element $a_0 \in F(A_0)$, any automorphism which induces the identity in the fiber at a dense set of points $A_0 \to k$ of finite type over S is the identity morphism. This is local on $\mathscr{X}$, so it reduces to the case where $\mathscr{X} = X$ is affine and $\mathscr{F}$ is an S-flat coherent sheaf on X. This reduces to showing that a section σ of $\mathscr{F}$ which vanishes in fibers over a dense set of points of $\operatorname{Spec} A_0$ is the zero section. By flatness, the locus of points in $\operatorname{Spec} A_0$ over which σ vanishes is closed under specialization. On the other hand, one easily sees that the set is constructible. (The only non-trivial point comes in checking that if A_0 is integral and σ does not vanish on the generic fiber, then there is an open subset of $\operatorname{Spec} A_0$ consisting of fibers where σ does not vanish. There is an open subset U of X consisting of points $x \in X$ such that $\sigma_{\kappa(x)} \neq 0$, as $\mathscr{F}$ is coherent. But $X \to \operatorname{Spec} A_0$ is of finite type and A_0 is Noetherian, so the image of U contains an open subset of $\operatorname{Spec} A_0$ by Chevalley's theorem.) Thus, the set of fibers where σ vanishes is closed, so if it contains a dense set it is all of $\operatorname{Spec} A_0$, as required. (Alternatively, one can "deduce" this last point from 4.1.1.5, if one is willing to accept a use of Artin's representability theorem for algebraic spaces, where one must check this fact anyway!) □

4.1.2 Purity

The abstract existence 4.1.1.1 yields an enormous and quite unwieldy stack $\mathscr{T}_{\mathscr{X}/S}$. There are certain naturally occurring open substacks with nicer formal properties. In this section, we will study *purity* of twisted sheaves as a precursor to 4.1.4, where we will study various stability conditions on twisted sheaves.

4.1.2.1. *Support of twisted sheaves.* Twisted sheaves may be viewed both as objects on X and as objects on a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$ over X. In other words, putting $\mathscr{X}$ in Căldăraru form yields an étale cover of X where the twisted sheaf "is" an "ordinary" sheaf. This leads to two possible definitions of support for a twisted sheaf, which of course turn out to coincide. (In the sequel, when the gerbe is understood we will often refer to "twisted sheaves on X" for the sake of notational simplicity.)

Definition 4.1.2.2. Given a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$ and an $\mathscr{X}$-twisted coherent sheaf $\mathscr{F}$, the *support* of $\mathscr{F}$ is the closed substack of $\mathscr{X}$ defined by the kernel of the map $\mathscr{O}_{\mathscr{X}} \to \mathscr{E}nd_{\mathscr{X}}(\mathscr{F})$, which is a quasi-coherent sheaf of ideals. The *schematic support* of $\mathscr{F}$ is defined to be the closed subscheme locally determined by $\mathscr{F}$ when $\mathscr{X}$ is put in Căldăraru form.

These two notions of support clearly coincide by 2.1.1.17. In particular, we note that a twisted sheaf $\mathscr{F}$ with schematic support $Y \subset X$ is naturally a $\mathscr{X} \times_X Y$-twisted sheaf with full schematic support (on Y). Thus, considering the support of a sheaf does not nullify its "twistedness."

4.1.2.3. We can use this notion of support to define a torsion filtration on a twisted sheaf. First, we will briefly develop the theory of associated points and torsion subsheaves on an arbitrary Noetherian algebraic stack. (When trying to generalize these results to the non-Noetherian case, certain equivalences will fail, making the theory developed here only one possibility.) Throughout, we systematically work with the underlying topological space $|\mathscr{X}|$ of a Noetherian algebraic stack. The support of a sheaf will be taken to mean simply the underlying set of points of $|\mathscr{X}|$, or the reduced closed substack structure on that set when it is closed (e.g., if $\mathscr{F}$ is coherent). We will not require (as is typical) that $\operatorname{Supp}(\mathscr{F})$ is the closure of the set of points where $\mathscr{F}$ is supported.

Let $\mathscr{F}$ be a quasi-coherent sheaf on $\mathscr{X}$.

Definition 4.1.2.4. A point $p \in |\mathscr{X}|$ is an *associated point* of $\mathscr{F}$ if there is a quasi-coherent subsheaf $\mathscr{G}$ such that $p \in \operatorname{Supp}(\mathscr{G}) \subset \overline{\{p\}}$. The set of associated points of $\mathscr{F}$ will be written $\operatorname{Ass}(\mathscr{F})$.

If $\mathscr{F}$ is *coherent*, this is the same as requiring that $\operatorname{Supp}(\mathscr{G}) = \overline{\{p\}}$. In general, this is not the case, as supports need not be closed for quasi-coherent sheaves.

Remark 4.1.2.5. When $\mathscr{X}$ is a Noetherian scheme, this is the same as the usual notion (essentially because one can extend quasi-coherent subsheaves off of generic points). More generally, if $\mathscr{X}$ is a Noetherian DM stack, one can say that a geometric point $p \to \mathscr{X}$ is associated to $\mathscr{F}$ if p is an associated point for the stalk of $\mathscr{F}$ at p (as a module over $\mathscr{O}^{\mathrm{hs}}_{p,X}$). By an argument similar to 4.1.2.6, a point of $|\mathscr{X}|$ is associated iff some (and hence any) geometric point lying over it is associated, so this yields the same notion as 4.1.2.4. ♦

Proposition 4.1.2.6. *Let $f : X \to \mathscr{X}$ be a flat surjection, with X a Noetherian scheme. If $\mathscr{F}$ is a quasi-coherent sheaf on $\mathscr{X}$, then* $\operatorname{Ass}(\mathscr{F}) = f(\operatorname{Ass}(\mathscr{F}|_X))$.

Proof. Write $\mathscr{F}' = \mathscr{F}|_{X'}$. Given a point $p \in \operatorname{Ass}(\mathscr{F})$, it is easy to see that a generic point of $f^{-1}(\overline{\{p\}})$ will be in $\operatorname{Ass}(\mathscr{F}')$. Conversely, let $q \in \operatorname{Ass}(\mathscr{F}')$ and let $Y = f^{-1}\overline{\{f(q)\}}$ as a reduced closed subscheme of X. Let $\mathscr{G} \subset \mathscr{F}'$ be the maximal subsheaf supported on Y. (It is not true that $\operatorname{Supp}\mathscr{G} = Y$, but we at least know that $q \in \operatorname{Supp}\mathscr{G}$.) We claim that $\mathscr{G}$ descends to a subsheaf of $\mathscr{F}$ with support containing $f(q)$ and contained in $\overline{\{f(q)\}}$. To see this, it is enough to show that the two pullbacks of $\mathscr{G}$ to $X \times_{\mathscr{X}} X$ are equal as subsheaves. In fact, by colimit considerations, we may assume that $\mathscr{F}$ is coherent. We are reduced to showing the following: given a flat surjection $g : Z \to W$ of Noetherian algebraic stacks with W an affine scheme, a closed subspace $Y \subset W$, and a coherent sheaf $\mathscr{F}$ on W, let $\mathscr{G}_Y \subset \mathscr{F}$ denote the maximal subsheaf $\mathscr{G}$ with $\operatorname{Supp}(\mathscr{G}) = Y$. Then $g^*(\mathscr{G}_Y) = \mathscr{G}_{g^{-1}(Y)}$. To prove this, let $\mathscr{I}$ be the ideal cutting out the reduced structure on Y. By flatness, $\mathscr{J} = g^*\mathscr{I}$ is a sheaf of ideals cutting out a substack of Z supported on $g^{-1}(Y)$. To say that $\mathscr{G}_Y$ is maximal is the same as saying that the sheaf $\mathscr{H}om(\mathscr{O}/\mathscr{I}^n, \mathscr{F}/\mathscr{G}_Y)$ vanishes for all $n > 0$. By flat pullback, we conclude that $\mathscr{H}om_{\mathscr{O}_Z}(\mathscr{O}_Z/\mathscr{J}^n, \mathscr{F}_Z/g^*\mathscr{G}_Y) = 0$, whence $g^*\mathscr{G}_Y$ is maximal. □

Corollary 4.1.2.7. *If $f : \mathscr{X}' \to \mathscr{X}$ is a flat surjection of Noetherian algebraic stacks and $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_{\mathscr{X}}$-module, then* $\operatorname{Ass}(\mathscr{F}) = f(\operatorname{Ass}(\mathscr{F}|_{\mathscr{X}'}))$.

Proof. Choosing a smooth cover of $\mathscr{X}'$ reduces this to 4.1.2.6. □

Corollary 4.1.2.8. *If $\mathscr{F}$ is a coherent sheaf on $\mathscr{X}$ then* $\operatorname{Ass}(\mathscr{F})$ *is finite.*

Proof. The stack $\mathscr{X}$ has a smooth cover by a Noetherian scheme X'. By 4.1.2.6, we are reduced to the case of a scheme, where this is a classical result [54, 6.5]. □

Points of a stack are subject to the relations of specialization and generization in the usual way. This gives $\mathrm{Ass}(\mathscr{F})$ the structure of partially ordered set. By 4.1.2.8 there are well-defined *minimal* elements of $\mathrm{Ass}(\mathscr{F})$.

It is easy to check that $\overline{\mathrm{Ass}(\mathscr{F})} = \overline{\mathrm{Supp}(\mathscr{F})}$ and that the minimal points of $\mathrm{Ass}(\mathscr{F})$ coincide with the minimal points of $\mathrm{Supp}(\mathscr{F})$.

Lemma 4.1.2.9. *Suppose $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are three coherent sheaves on $\mathscr{X}$ fitting into an exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$.*

1. $\mathrm{Ass}(\mathscr{F}) \subset \mathrm{Ass}(\mathscr{G}) \subset \mathrm{Ass}(\mathscr{F}) \cup \mathrm{Ass}(\mathscr{H})$. *If the sequence is split exact, the second inclusion is a bijection.*

2. *The minimal points of* $\mathrm{Ass}(\mathscr{H})$ *are contained in* $\mathrm{Ass}(\mathscr{G})$

3. *If $\mathscr{G} \neq 0$, then* $\mathrm{Ass}(\mathscr{G}) \neq \emptyset$.

Proof. This is precisely analogous to the classical proof [54, §6]. □

Definition 4.1.2.10. A *torsion subsheaf* of $\mathscr{F}$ is a subsheaf $\mathscr{F}' \subset \mathscr{F}$ with the property that none of the minimal points of $\mathrm{Ass}(\mathscr{F})$ are contained in $\mathrm{Ass}(\mathscr{F}')$.

Note that any minimal point of $\mathrm{Ass}(\mathscr{F})$ which is also associated to a subsheaf $\mathscr{F}'$ will be minimal in $\mathrm{Ass}(\mathscr{F}')$.

Lemma 4.1.2.11. *The sum of any two torsion subsheaves of $\mathscr{F}$ is a torsion subsheaf. There is a unique maximal coherent torsion subsheaf of $\mathscr{F}$.*

Proof. Suppose $\mathscr{F}'$ and $\mathscr{F}''$ are torsion subsheaves of $\mathscr{F}$. By 4.1.2.9, the minimal points of $\mathrm{Ass}(\mathscr{F}' + \mathscr{F}'')$ are contained in $\mathrm{Ass}(\mathscr{F}') \cup \mathrm{Ass}(\mathscr{F}'')$. This proves the first statement. The second follows by taking the sum of all torsion subsheaves of $\mathscr{F}$ (which is allowable because they form a set). □

The maximal torsion subsheaf of $\mathscr{F}$ will be called *the* torsion subsheaf of $\mathscr{F}$ and denoted $T(\mathscr{F})$.

Lemma 4.1.2.12. *Any non-minimal point of* $\mathrm{Ass}(\mathscr{F})$ *is contained in* $\mathrm{Ass}(T(\mathscr{F}))$.

Proof. Immediate from the definition! □

Remark 4.1.2.13. When $\mathscr{X}$ is a gerbe bound by a diagonalizable group scheme, the decomposition 2.2.1.6 respects torsion subsheaves, so we see that we have also developed a good theory of torsion subsheaves for twisted sheaves.

Definition 4.1.2.14. A coherent sheaf $\mathscr{F}$ is *pure* if $T(\mathscr{F}) = 0$.

Remark 4.1.2.15. By 4.1.2.12, we see that $\mathscr{F}$ is pure if and only if $\mathrm{Ass}(\mathscr{F})$ consists solely of minimal points, i.e., the partial ordering on $\mathrm{Ass}(\mathscr{F})$ is trivial.

Lemma 4.1.2.16. *If $X \to \mathscr{X}$ is a smooth cover, then $\mathscr{F}$ is pure if and only if $\mathscr{F}|_X$ is pure.*

Proof. As in 4.1.2.6, it suffices to show that if $Z \to W$ is a smooth map of schemes then the pullback of a torsion free sheaf is torsion free. As this is a local property on the source and target and is obviously true for arbitrary quasi-finite flat morphisms (hence for étale morphisms), we see that it suffices to prove that the pullback of a torsion free sheaf on W to $\mathbf{A}^n_W$ is torsion free. Again by 4.1.2.6, we see that any torsion subsheaf of $\mathbf{A}^n_W$ must have all associated points lying over minimal (generic) points of W. Thus, we may assume W is the spectrum of an Artinian local ring R and we wish to show that the pullback of any finite R-module to $\mathbf{A}^n_R$ cannot have torsion. Taking a composition series, we may assume that R is a field. The result follows from the fact that $\mathbf{A}^1_K$ is Cohen-Macaulay. □

Let $\pi : \mathscr{X}' \to \mathscr{X}$ be a flat surjection of Noetherian algebraic stacks representable by an open immersion into an integral ring extension and $\mathscr{F}$ a coherent sheaf on $\mathscr{X}$.

Proposition 4.1.2.17. *$\mathscr{F}$ is pure if and only if $\mathscr{F}|_{\mathscr{X}'}$ is pure.*

Proof. By the going-up lemma [54, 9.4] and flatness (which implies the going-down lemma [54, 9.5]), a morphism such as π has the property that $\pi(p)$ is minimal if and only if p is minimal. The result now follows from 4.1.2.6. □

Corollary 4.1.2.18. *If $\mathscr{X}$ is over a field k, then the purity of a coherent sheaf is invariant under finite extensions of k. If $\mathscr{X}$ is finite type over k then purity is geometric.*

The finite type hypothesis in the second statement serves only to ensure that $\mathscr{X} \otimes K$ is Noetherian for any extension $K \supset k$, so that our theory applies.

Remark 4.1.2.19. When X is an integral universally catenary scheme of finite Krull dimension (for example, a projective variety) and $\mathscr{F}$ is an $\mathscr{X}$-twisted sheaf with support of dimension d, we can filter $T(\mathscr{F})$ by the dimension of support: let $T_e(\mathscr{F})$ be the maximal subsheaf of $\mathscr{F}$ whose support is of dimension at most e. Then $T(\mathscr{F}) = T_{d-1}(\mathscr{F}) \supset T_{d-2}(\mathscr{F}) \supset \cdots \supset T_0(\mathscr{F})$. This filtration can be useful when considering various notions of semistability; it will not come up in the sequel.

4.1.2.20. We will now show that the property of being pure is open in flat families of twisted sheaves (on a proper space). Again, we will show this more generally for a proper algebraic stack.

Proposition 4.1.2.21. *Let $\pi : \mathscr{X} \to S$ be a proper morphism of finite presentation from an algebraic stack to an algebraic space. Suppose $\mathscr{F}$ is an S-flat family of coherent sheaves. The locus of points $s \in S$ such that $\mathscr{F}_s$ is pure is open.*

Proof. We may reduce to the case where S is affine, Noetherian, and even excellent (in fact, affine of finite type over $\mathbf{Z}$). Indeed, we may present the stack $\mathscr{X}$ as a groupoid $X_1 \to X_0 \times X_0$ of finite presentation between two schemes of finite presentation over S. Thus, we may descend $\mathscr{X}$ to a Noetherian base. Having done this, note that a coherent sheaf on $\mathscr{X}$ is given by a coherent sheaf on X_0 with an action of the groupoid, i.e., an isomorphism of the pullbacks to X_1 which is compatible with the groupoid structure. By Grothendieck's theory of limits, we can descend these data to a finite level.

Consider the set Ξ of points $x \in |\mathscr{X}|$ with the property "x is contained in the support of $T(\mathscr{F}_{\pi(x)})$." It suffices to show that $\pi(\Xi)$ is constructible and that $\pi(\Xi)$ is closed under specialization when $\mathscr{X}$ is proper over S (when S is Noetherian).

The second statement is immediate: it suffices to check this when S is the spectrum of a discrete valuation ring. By flatness, the torsion subsheaf $T(\mathscr{F})$ of the total family $\mathscr{F}$ is non-zero if and only if the torsion subsheaf of the generic fiber is non-zero. On the other hand, since S is Dedekind, $T(\mathscr{F})$ is also a flat family, and in particular has constant fiber dimension, like $\mathscr{F}$. Finally, the cokernel $\mathscr{F}/T(\mathscr{F})$ is pure, hence S-flat, by definition of the torsion subsheaf. These facts combine to yield the statement about specialization. (Properness is not necessary for this, as long as we assume that the specialization on the base is contained in the image of π.)

The first statement (that $\pi(\Xi)$) is constructible) is more subtle. It is easily reduced to showing that if S is an integral and Noetherian affine scheme and the generic fiber of Ξ is non-empty, then Ξ is non-empty over an open subscheme of S. By 4.1.2.6, we may assume that $\mathscr{X}$ is in fact a scheme and that π is surjective. Consider the sequence of sheaves $0 \to T(\mathscr{F}) \to \mathscr{F} \to \mathscr{Q} \to 0$ on $\mathscr{X}$. There is an open $U \subset S$ over which all of these sheaves and their supports are flat, and we may in fact assume by further localization that the generic fiber of $T(\mathscr{F})$ is $T(\mathscr{F}_\eta)$. We claim that the fibers of $T(\mathscr{F})$ are torsion subsheaves over any point of U; this will complete the proof. We know that the support of $T(\mathscr{F})$ in the generic fiber has strictly smaller dimension at every point than the support of $\mathscr{F}$ at that point. Let p be a point of the support of $T(\mathscr{F})$ with image $\pi(p) \in S$. The statement about the generic fiber says that the support of $T(\mathscr{F})$ is cut out of the local ring of $\operatorname{Supp}\mathscr{F}$ at p by an ideal I which is not contained in any of the minimal primes of $\mathscr{O}_{\operatorname{Supp}\mathscr{F},p}$. Therefore, any prime $\mathfrak{P} \subset \mathscr{O}_{\operatorname{Supp}T(\mathscr{F}),p}$ has height strictly smaller than the height of its preimage in $\mathscr{O}_{\operatorname{Supp}\mathscr{F},p}$. By the computation of fiber dimension for a flat map of local rings [54, 15.1], we see that the support of $T(\mathscr{F})_s$ has strictly smaller dimension at every point than the support of $\mathscr{F}_s$, where s is any point of U. Thus, $\operatorname{Ass} T(\mathscr{F})_s$ does not contain any of the minimal points of $\operatorname{Ass}\mathscr{F}_s$ for every $s \in U$, i.e., $T(\mathscr{F})_s \subset T(\mathscr{F}_s)$ for every $s \in U$. □

As a consequence of the Proposition, when $\mathscr{X}$ is a $\mathbf{G}_m$-gerbe, there is an open substack of $\mathscr{T}_{\mathscr{X}/S}$ representing families of pure twisted sheaves. Note that since the support of a flat family over a dvr is itself flat over the dvr, the dimension of the fibers of a flat family of coherent $\mathscr{X}$-twisted sheaves over a locally Noetherian base scheme is locally constant.

Corollary 4.1.2.22. *Let $X \to S$ be a proper flat morphism of finite presentation with geometrically integral fibers. There is an open substack of $\mathscr{T}_{\mathscr{X}/S}$ consisting of families of torsion free sheaves, i.e., pure sheaves of maximal dimension.*

Definition 4.1.2.23. If $X \to S$ is a proper flat morphism of finite presentation and $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe, the open substack parametrizing families with torsion free fibers is denoted $\mathbf{Tw}_{\mathscr{X}/S}$.

4.1.2.24. Suppose X is a smooth projective variety over a field k and $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe with $n \in k^\times$. Let $\mathbf{Tw}_{\mathscr{X}/k}(n)$ denote the stack parametrizing torsion free twisted sheaves of rank n. Since $\mathscr{X}$ is smooth, any S-flat family of twisted sheaves $\mathscr{F}$ on $\mathscr{X} \times S$ has finite homological dimension everywhere. In other words, $\mathscr{F}$ is *perfect* as an object of the derived category. As in section 3.2.1, we use the construction of Mumford and Knudsen [45] to define a determinant for $\mathscr{F}$, which will be the pullback to $\mathscr{X} \times S$ of an invertible sheaf on $X \times S$ (as $\mathscr{F}$ has rank n). This yields a morphism of algebraic stacks

$$\det : \mathbf{Tw}_{\mathscr{X}/k}(n) \to \mathscr{P}ic_{X/k}.$$

Given an invertible sheaf $\mathscr{L}$ on X, one can form the fiber of det over the resulting k-point of $\mathscr{P}ic_{X/k}$. This will be a closed substack of $\mathbf{Tw}(n)$. Formally:

Definition 4.1.2.25. $\mathbf{Tw}_{\mathscr{X}/k}(n, \mathscr{L}) := \mathbf{Tw}_{\mathscr{X}/k} \times_{\mathscr{P}ic} k$

Chasing through the definition of the natural 1-fiber product of stacks shows that the objects of $\mathbf{Tw}_{\mathscr{X}/k}(n, \mathscr{L})$ are pairs $(\mathscr{F}, \varphi)$ consisting of a torsion free twisted sheaf $\mathscr{F}$ of rank n and a chosen isomorphism $\det \mathscr{F} \xrightarrow{\sim} \mathscr{L}$. One can develop a good deformation and obstruction theory for $\mathbf{Tw}(n, \mathscr{L})$; we defer this to 4.2.1.10.

4.1.3 Hilbert polynomials and Quot spaces

In this section, we develop a notion of Hilbert polynomial for twisted sheaves which we will ultimately use to define semistability. In later sections, we will show that on a surface (and more generally on a variety which carries a twisted sheaf with enough vanishing Chern classes), our notion agrees with Simpson's notion [70, §3] and thus yields a GIT quotient corepresenting the stack of semistable sheaves. In general, it will be possible to show that the stack of stable twisted sheaves is a gerbe over an algebraic space, but dealing with properly semistable points is difficult.

4.1.3.1. In order to define our semistability condition and for future reference, we briefly recall the basic facts about rational Chow rings of DM stacks over a field. Vistoli [77] and Gillet [29] have defined Chow theories which only work rationally but which are formally identical to the usual Chow theory: in Vistoli's approach, one takes the Chow groups to be generated by integral closed substacks modulo rational equivalence (suitably defined). There is a refined theory due to Edidin and Graham [21] which applies to quotient stacks to yield an integral Chow theory which agrees with Vistoli's theory when tensored with $\mathbf{Q}$. A further refinement of the integral theory for algebraic stacks stratified by quotient stacks was developed by Kresch in his thesis [46]. We will denote the rational Chow groups by $A_{\mathbf{Q}}$. We will write $A^n_{\mathbf{Q}}$ for the group generated by cycles of codimension n. When the underlying stack is smooth, the graded group $\oplus A^n_{\mathbf{Q}}$ has a commutative ring structure. As usual, there is a theory of Chern classes and a splitting principle. The theory admits proper pushforwards, flat pullbacks, and Gysin maps [77].

Given a proper DM stack $\mathscr{X}$ with moduli space X, one can show that the proper pushforward $A_{\mathbf{Q}}(\mathscr{X}) \to A_{\mathbf{Q}}(X)$ is an isomorphism which respects the ring structure when both are smooth. In particular, when $\mathscr{X}$ is of dimension n, there is a rational degree function $\deg : A^n(\mathscr{X})_{\mathbf{Q}} \to \mathbf{Q}$. Given any element α of the graded group $A^*_{\mathbf{Q}}(\mathscr{X})$, we will let α_n denote the part in degree n. Given a class $\beta \in A^*(\mathscr{X})_{\mathbf{Q}}$, we will let $\deg \beta$ denote the degree of β_n.

Let $\mathscr{X}$ be a smooth proper DM stack of dimension n over a field k with projective moduli space X. We will state (without complete proofs) some incredibly naïve Riemann-Roch theorems. The naïveté of the statements will allow us to avoid giving proofs. The reader uncomfortable with this can rest assured that the subset of statements of which we actually make use can be proven by alternative (Morita) methods as described below. It is likely that our statements exist as corollaries of real results in the literature, but the simplicity of these statements and their proofs is so extreme as to preclude any appeal to real work past Fulton's book [27]. It should be noted that approaches to more legitimate Riemann-Roch theorems for stacks may be found in the thesis of Toen [75], and in the equivariant setting in the work of Edidin and Graham [22]. Recall that $K^0(\mathscr{X})$ is the Grothendieck group of vector bundles on $\mathscr{X}$, while $K_0(\mathscr{X})$ is the Grothendieck group of coherent sheaves. When every coherent sheaf on $\mathscr{X}$ admits a finite resolution by locally free sheaves, it is easy to see that $K^0 \cong K_0$. In general, K^0 is a ring and K_0 is a K^0-module (via tensor product). One of the basic problems for arbitrary DM stacks is the fact that K^0 and K_0 are not isomorphic even on smooth DM stacks. For smooth quotient stacks, they are naturally the same, which makes it easier to prove theorems. (This is yet another place where 2.2.3.8 and its corollaries have a large impact.)

Let $\alpha \in K^0(\mathscr{X})$. As usual, $\mathrm{Td}_{\mathscr{X}}$ denotes the Todd class of the tangent sheaf $\mathscr{T}_{\mathscr{X}/k}$ of $\mathscr{X}$.

Definition 4.1.3.2. The *geometric Euler characteristic* of α is

$$\chi^g(\alpha) := \deg(\mathrm{ch}(\alpha) \cdot \mathrm{Td}_{\mathscr{X}}).$$

When X is projective with chosen polarization $\mathscr{O}(1)$, the *geometric Hilbert polynomial* of α is the function

$$n \mapsto P^g_\alpha(n) = \chi^g(\alpha(n)),$$

where $\alpha(n) := \alpha \otimes \mathscr{O}(n)$.

To verify that P^g_α is a polynomial, it suffices to prove it when $\alpha = [\mathscr{E}]$, $\mathscr{E}$ a locally free sheaf on $\mathscr{X}$. This then follows by a simple splitting principle calculation left to the reader.

Remark 4.1.3.3. The geometric Euler characteristic and Hilbert function are clearly additive functions on the category of coherent sheaves. When $\mathscr{F}$ is the pullback to $\mathscr{X}$ of a coherent sheaf on X, they agree with the usual Euler characteristic and Hilbert function by the Grothendieck-Hirzebruch-Riemann-Roch theorem. However, for sheaves which are not pullbacks, they do not agree with the usual cohomologically defined functions. For a trivial example, consider the case of a gerbe $\mathscr{X}$ over a geometric curve X. In this case, there is an invertible sheaf $\mathscr{L}$ on $\mathscr{X}$ which is $\mathscr{X}$-twisted whose nth tensor power $\mathscr{L}^{\otimes n}$ is the pullback of an invertible sheaf $\mathscr{M}$ on X.

The geometric Euler characteristic of $\mathscr{L}$ is easily seen to be

$$\chi^g(\mathscr{L}) = \deg c_1(\mathscr{L}) + \chi(\mathscr{O}_X) = \frac{1}{n}\deg c_1(\mathscr{M}) + \chi(\mathscr{O}_X).$$

Thus, if X is e.g. elliptic, one can easily produce gerbes $\mathscr{X}$ and $\mathscr{X}$-twisted sheaves with non-zero χ^g. On the other hand, if we use coherent cohomology to compute the cohomological Euler characteristic, we find $\chi(\mathscr{L}) = 0$ when $\mathscr{L}$ has non-trivial stabilizer action. (For an even more trivial example, let $\mathscr{X}$ be a gerbe over a point!)

Recent results of Vistoli-Kresch [47], Edidin-Hassett-Kresch-Vistoli [23], and Gabber/de Jong [19] (stated by Gabber and proven by Gabber and independently by de Jong) show that any separated smooth generically tame DM stack over a field with quasi-projective moduli space is a quotient stack, and that such a stack has the "resolution property": any coherent sheaf is a quotient of a locally free sheaf. In these cases, the natural map $K^0 \to K_0$ is thus an isomorphism. We will denote it simply by $K(\mathscr{X})$.

Definition 4.1.3.4. A (smooth) proper generically tame Deligne-Mumford stack $\mathscr{X}$ with projective moduli space will be called a *(smooth) pseudo-projective* stack.

(The reader will note that tame pseudo-projective stacks are locally generating over their moduli spaces, so the results of 2.2.4 apply. This will not be used anywhere in this thesis, but it could be useful for the study of higher K-theory of such stacks.)

Lemma 4.1.3.5. *The category of (smooth) pseudo-projective DM stacks is closed under the formation of (smooth) closed generically tame substacks, products, and blow-ups along (smooth) sub-stacks.*

Proof. We verify the statement about blow-ups, leaving the rest for the reader. Let $\pi : \mathscr{X} \to X$ be a DM stack mapping to its moduli space. Let $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ be a sheaf of ideals. By [47, 2.1], there is a finite flat cover $f : C \to \mathscr{X}$. Let M be the coarse moduli space of $\mathrm{Bl}_{\mathscr{I}}(\mathscr{X})$. Consider the diagram

$$\begin{array}{ccc}
\mathrm{Bl}_{f^*\mathscr{I}}(C) & \longrightarrow & C \\
\downarrow & & \downarrow \\
\mathrm{Bl}_{\mathscr{I}}(\mathscr{X}) & \longrightarrow & \mathscr{X} \\
\downarrow & & \downarrow \\
M & \longrightarrow & X.
\end{array}$$

The center horizontal arrow is a projective morphism, hence there is a relatively ample invertible sheaf $\mathscr{L}$ on $\mathrm{Bl}_{\mathscr{I}}(\mathscr{X})$. It follows that $f^*\mathscr{L}$ and all of its powers are relatively ample for $\mathrm{Bl}_{f^*\mathscr{I}}(C) \to C$. As M is the coarse moduli space, some power of $\mathscr{L}$ descends to M, so we may assume that $\mathscr{L}$ is pulled back from M. Given a very ample invertible sheaf $\mathscr{M}$ on X, the finiteness of $C \to X$ implies that $\mathscr{M}_C$ is very ample. It follows that there is some power N such that $f^*\mathscr{L} \otimes \mathscr{M}^{\otimes N}_{\mathrm{Bl}_{f^*\mathscr{I}}(C)}$ is very ample on $\mathrm{Bl}_{f^*\mathscr{J}}(C)$. As $\mathrm{Bl}_{f^*\mathscr{J}}(C) \to M$ is finite, it follows that $\mathscr{L} \otimes \mathscr{M}_M^{\otimes N}$ is an ample invertible sheaf on M. This implies that $\mathrm{Bl}_{\mathscr{J}}(\mathscr{X})$ is pseudo-projective. □

Proposition 4.1.3.6. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a projective morphism of smooth pseudo-projective DM stacks. Then for all $\alpha \in K(\mathscr{X})$,*

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{Td}_{\mathscr{Y}} = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{Td}_{\mathscr{X}})$$

in $A(\mathscr{Y})_{\mathbf{Q}}$.

Sketch of proof. As in [27, §15.2], this reduces to studying projections $\mathscr{Y} \times P^n \to \mathscr{Y}$ and closed immersions $\mathscr{X} \hookrightarrow \mathscr{Y}$ separately. The case of a projection reduces [*ibid.*] to a computation on $\mathbf{P}^n$. In the case of a closed immersion, one reduces by deformation to the normal cone to the case where f is a section of the projectivization of a vector bundle on $\mathscr{X}$. Since by 4.1.3.5 the total space of the deformation to the normal cone of a closed immersion of smooth pseudo-projective DM stacks is itself smooth pseudo-projective and the Vistoli-Gillet-Chow theory of DM stacks has all of the formal properties (rationally) as the classical Chow theory, one can simply transcribe the proof of Fulton [*ibid.*]. (The reason one has to keep track of pseudo-projectivity is that Fulton's proof uses the resolution property.) □

Corollary 4.1.3.7. *Let $\iota : \mathscr{X} \hookrightarrow \mathscr{Y}$ be a closed immersion of smooth pseudo-projective DM stacks and $\mathscr{F}$ a coherent sheaf on $\mathscr{X}$. Then $\chi^g(\mathscr{X}, \mathscr{F}) = \chi^g(\mathscr{Y}, \iota_*\mathscr{F})$.*

We fix a smooth pseudo-projective DM stack $\mathscr{X}$ with moduli space $\pi : \mathscr{X} \to X$ in what follows. We assume that everything is flat over a fixed locally Noetherian algebraic base space S. For the sake of simplicity, *we will assume that the moduli space X is smooth over the base and the natural map π is étale.* (In fact, we are not certain that the results hold as stated otherwise.) Since we will ultimately apply this to $\boldsymbol{\mu}_n$-gerbes, this will not hinder us in reaching our goal. First suppose $S = \operatorname{Spec} k$, k an arbitrary field.

Remark 4.1.3.8. When $\mathscr{X}$ is a $\boldsymbol{\mu}_n$-gerbe over a smooth projective variety X and there is a locally free $\mathscr{X}$-twisted sheaf with sufficiently many vanishing Chern classes (e.g., X is a surface), then the formula in the last sentence of the proof of 4.1.4.9 gives a much more concrete proof of 4.1.3.7. As we will only use this statement for gerbes on surfaces, we can (in light of this explicit calculation) safely omit the details of 4.1.3.6. ♦

Lemma 4.1.3.9. *The geometric Hilbert function is geometric: if $k \subset K$ is an extension of fields and X is a smooth geometrically connected projective variety over k, then for any coherent $\mathscr{X}$-twisted sheaf $\mathscr{F}$, $P_{\mathscr{F}} = P_{\mathscr{F} \otimes K}$ as functions on $\mathbf{Z}$.*

Proof. This follows from the fact that Chern classes of arbitrary (perfect) coherent sheaves pull back under Tor-independent maps. □

Notation 4.1.3.10. Following the conventions of Huybrechts and Lehn [40, §1.2], we write

$$P_{\mathscr{F}}(m) = \sum_{i=0}^{\dim \mathscr{F}} \alpha_i(\mathscr{F}) \frac{m^i}{i!}.$$

With this definition the coefficients α_i need not be integers (contrary to [40, p. 10]).

Definition 4.1.3.11. Given a coherent sheaf $\mathscr{F}$ of dimension d on $\mathscr{X}$, the *geometric rank* of $\mathscr{F}$ is defined to be

$$\operatorname{rk}\mathscr{F} := \alpha_d(\mathscr{F})/\alpha_d(\mathscr{O}_{\mathscr{X}}).$$

The *geometric degree* of $\mathscr{F}$ is defined to be

$$\deg\mathscr{F} = \alpha_{d-1}(\mathscr{F}) - \operatorname{rk}(\mathscr{F})\cdot\alpha_{d-1}(\mathscr{O}_{\mathscr{X}}).$$

Lemma 4.1.3.12. *Suppose k is infinite. Given a coherent sheaf $\mathscr{F}$ on $\mathscr{X}$, there is a section σ of $\mathscr{O}(1)$ such that $\sigma : \mathscr{F}(-1) \to \mathscr{F}$ is injective.*

Proof. The set $\operatorname{Ass}\mathscr{F}$ is finite and determined by its image in X. Since $\mathscr{O}(1)$ is very ample, there is a section missing these finitely many points. It is easy to see that any associated point of the kernel of σ must then be contained in the zero locus of σ, contradicting the choice of σ and 4.1.2.9. □

Lemma 4.1.3.13. *The geometric rank of $\mathscr{F}$ is non-zero.*

Proof. We show that $\deg P_{\mathscr{F}} = \dim\mathscr{F}$. This is clear by induction and the previous lemma, once we have verified it when $\dim X = 0$. (We have used the fact that a generic section will have smooth pseudo-projective zero locus; this is where we use the assumption that $\mathscr{X} \to X$ is étale. If there were a stacky Bertini theorem, we could drop that requirement. The existence of a stacky Bertini theorem seems to be subtle.) In this case, the geometric Euler characteristic is just the dimension of the fiber of $\mathscr{F}$ over any geometric point of $\mathscr{X}$ by 4.1.3.6. (This reduces one to working on a gerbe over a point, which will be smooth by our assumption that the map to the moduli space is étale.) □

Remark 4.1.3.14. In particular, the geometric Hilbert function of $\mathscr{F}$ vanishes if and only if $\mathscr{F} = 0$. Furthermore, one sees that the geometric rank of $\mathscr{F}$ is precisely the rank of $\mathscr{F}$ as an $\mathscr{O}$-module. Unfortunately, one cannot show this by arguing that $\mathscr{F}$ and $\mathscr{O}^{\operatorname{rk}\mathscr{F}}$ agree on a dense open substack, as this is false. Instead, one must appeal directly to the Hirzebruch-Riemann-Roch formula (and the computation [27, 3.2.2] of Chern classes of a twist). We leave the details to the reader. The geometric degree of $\mathscr{F}$ is related to the degree of $\det(\mathscr{F})$ just as in the case of ordinary sheaves: $\alpha_{d-1} = \deg\det(\mathscr{F})$ (so the geometric degree is arrived at by a linear transformation familiar from [40, 1.6.8ff]). This will aid us in comparing various notions of semistability and slope-semistability to their classical counterparts (as in Simpson's theory for semistability of modules for sheaves of algebras [70, §3]).

4.1.3.15. For the rest of this section, we will consider only the case where $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe with $n \in \mathscr{O}(X)^\times$ and X is a smooth projective scheme over a Noetherian affine base S. It is likely that many of our results generalize, but it is unnecessary for this work and is likely to add complications (especially in positive characteristic).

We start with a refinement of 2.2.3.8 better suited to the study of stability.

Proposition 4.1.3.16. *Given a $\boldsymbol{\mu}_n$-gerbe on a smooth projective morphism $\mathscr{X} \to X \to S$ with Noetherian affine base S, there is a locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$ of rank n. Étale-locally on S, there is a locally free $\mathscr{X}$-twisted sheaf with trivial determinant.*

Proof. The existence of $\mathscr{V}$ is a non-trivial result which holds on any (separated) scheme with an ample invertible sheaf. We refer the reader to the work of de Jong [19] for the (upcoming) details. To make the determinant trivial, first consider $\mathscr{W} := \mathscr{V}^{\oplus n}$. Since $n | \operatorname{rk} \mathscr{W}$, we have $\mathscr{L} = \det \mathscr{W} \in \operatorname{Pic}(X)$. Now $(\mathscr{L}^\vee \oplus \mathscr{O}^{\oplus \operatorname{rk}\mathscr{W}-1}) \otimes \mathscr{W}$ is a locally free twisted sheaf of trivial determinant. □

Recall that $\mathscr{X}$ *has the resolution property* if every coherent sheaf on $\mathscr{X}$ is the quotient of a locally free sheaf [76]. The present virtue of 4.1.3.16 lies in the following corollary.

Corollary 4.1.3.17. *For any affine $T \to S$, the stack $\mathscr{X}_T$ has the resolution property.*

Proof. By 2.2.1.6, the category of coherent sheaves on $\mathscr{X}$ breaks up according to the degree of twisting. It suffices to show that coherent $\mathscr{X}$-twisted sheaves have the resolution property. Applying the fibered Morita equivalence $\mathscr{H}om(\mathscr{V}, \cdot)$ reduces us to showing that $\mathscr{E}nd(\mathscr{V})$-modules on X have the resolution property. This follows from the fact that coherent sheaves on a projective morphism have the resolution property. □

Proposition 4.1.3.18. *If $\mathscr{F}$ on $\mathscr{X}$ is S-flat, then $P^g_{\mathscr{F}_s}$ is constant for all geometric points $s \to S$.*

Proof. It suffices to assume that S is the spectrum of a discrete valuation ring. In that case, the support of $\mathscr{F}$ will be flat over S if $\mathscr{F}$ is, so we may assume that $\mathscr{X}$ and X are S-flat. Thus, we may assume that any locally free coherent sheaf on $\mathscr{X}$ is S-flat.

Let $\mathscr{G}^\bullet \to \mathscr{F}$ be a locally free resolution of $\mathscr{F}$. As $\mathscr{X} \to S$ is smooth, $\mathscr{G}^\bullet$ may be taken to be a finite resolution. If $\mathscr{F}$ is flat, then for any $s \to S$, the complex $\mathscr{G}^\bullet_s$ is a resolution of $\mathscr{F}_s$. Thus, to prove that P^g is constant for $\mathscr{F}$, it suffices by additivity to prove it when $\mathscr{F}$ is assumed locally free. In this case, we may globally apply the splitting principle (noting that the base change which filters the sheaf produces another proper S-flat family). Thus, it is enough to show that given invertible sheaves $L_1, \ldots, L_n$ on $\mathscr{X}$ with n the relative dimension of $\mathscr{X}$ over S, the intersection product $c_1(L_1) \cdot \cdots \cdot c_1(L_n)$ is constant in fibers. As $A(\mathscr{X})_{\mathbf{Q}} = A(X)_{\mathbf{Q}}$, it suffices (by multilinearity) to prove this for invertible sheaves on X. This is now a standard calculation using the fact that Euler characteristics are constant in a flat family. (In other words, we return to Kleiman's definition of intersection product using Snapper's lemma [11, §1], [44], where the intersection number appears as a coefficient in a polynomial Euler characteristic.) □

Thus, given P, the substack $\mathbf{Tw}_{\mathscr{X}/S}(P) \subset \mathbf{Tw}_{\mathscr{X}/S}$ consisting of twisted sheaves with fixed geometric Hilbert polynomial P is open (in fact, a union of connected components). This immediately shows that $\mathbf{Tw}_{\mathscr{X}/S}(P)$ is an algebraic stack.

4.1.3.19. Let P be a fixed polynomial and $\mathscr{E}$ a fixed coherent $\mathscr{X}$-twisted sheaf. We will briefly study the space of quotients of $\mathscr{E}$ with a fixed geometric Hilbert polynomial.

Definition 4.1.3.20. Let $\underline{\mathrm{Quot}}^P_{\mathscr{X}/S}(\mathscr{E})$ denote the functor on affine S-schemes which assigns to $T \to S$ the set of subsheaves $\mathscr{G} \subset \mathscr{E}_T$ such that $\mathscr{E}_T/\mathscr{G}$ is T-flat with geometric Hilbert polynomial P in every fiber.

Proposition 4.1.3.21. *The functor* $\underline{\mathrm{Quot}}^P_{\mathscr{X}/S}(\mathscr{E})$ *is represented by a proper algebraic space* $\mathrm{Quot}^P_{\mathscr{X}/S}(\mathscr{E})$ *over* S.

Proof. It follows by an easy application of Artin's criteria that $\underline{\mathrm{Quot}}$ is representable by an algebraic space which satisfies the valuative criterion of properness. This is checked in great detail in [58] in a slightly different context (which is sufficiently close to ours to be a complete proof in our case as well). The only fact that remains to prove is that the functor $\underline{\mathrm{Quot}}$ is bounded in the sense of [40, 1.7.5]. In other words, we need to show that there is a quasi-compact scheme surjecting onto the functor.

Let $\mathscr{V}$ be a locally free $\mathscr{X}$-twisted sheaf. Given any $\mathscr{G} \subset \mathscr{E}_T$ as above, note that $\mathscr{H}om(\mathscr{V}_T, \mathscr{G}) \subset \mathscr{H}om(\mathscr{V}_T, \mathscr{E}_T)$ is injective with T-flat cokernel $\mathscr{C}$. If we knew that the Hilbert polynomial of the fibers of $\mathscr{C}$ were always the same, we would be done. Unfortunately, this is highly unlikely. However, since we do know the geometric rank and geometric degree of $(\mathscr{E}_T/\mathscr{G})_s$, we know the rank and degree of $\mathscr{C}_s$. Furthermore, we know that the $\mathscr{C}_s$ are quotients of a fixed sheaf $\mathscr{U} := \mathscr{H}om(\mathscr{V}, \mathscr{E})$. By a result of Grothendieck [40, 1.7.9], we know that the set of quotients of $\mathscr{U}_s$ with slope bounded above is bounded. A consideration of the proof of Huybrechts and Lehn [*ibid.*] shows that the set of Hilbert polynomials appearing in such quotients is finite and independent of s. Thus, as the geometric Hilbert polynomial is locally constant, we see that $\mathrm{Quot}^P_{\mathscr{X}}(\mathscr{E})$ is a union of finitely many connected components of finitely many schemes of quotients of $\mathscr{E}nd(\mathscr{V})$-modules which are themselves proper over S. This completes the proof. □

Remark 4.1.3.22. The proof actually works (with slight modification) for any coherent $\mathscr{E}$ on $\mathscr{X}$ (independent of twisting). ♦

4.1.4 Semistability and boundedness

We wish in this section to define a reasonable stability condition for twisted sheaves on smooth projective varieties. Variations on this theme occur throughout the study of moduli of sheaves. The basic goal is to produce a condition which cuts out a well-behaved substack of the stack of pure sheaves. Historically, this has meant two things: from the differential-geometric angle, stability conditions are related to the existence of certain types of metrics on bundles; from the algebro-geometric direction, the choice of a stability condition is influenced by the use of Geometric Invariant Theory to construct the moduli of such sheaves as a quotient stack.

Using the geometric Hilbert polynomial, we define a stability condition for twisted sheaves analogous to the classical definition for untwisted sheaves. As usual, a coarsening of our relation will define μ-semistability (or Mumford-Takemoto semistability).

(In characteristic zero, this condition is probably equivalent to the existence of certain metrics on the associated analytic DM stack (orbifold). We do not pursue this matter here.) On surfaces, we relate our construction to GIT via a Morita equivalence and fundamental work of Simpson on moduli of modules for a sheaf of algebras [70]. (More generally, we make this comparison when there exists a locally free twisted sheaf with enough vanishing Chern classes.)

Definition 4.1.4.1. An $\mathscr{X}$-twisted sheaf $\mathscr{F}$ of dimension d is *semistable* (respectively *stable*) if for any subsheaf $\mathscr{G} \subset \mathscr{F}$ we have $\alpha_d(\mathscr{F})P_{\mathscr{G}} \leq \alpha_d(\mathscr{G})P_{\mathscr{F}}$ (respectively $\alpha_d(\mathscr{F})P_{\mathscr{G}} < \alpha_d(\mathscr{G})P_{\mathscr{F}}$).

Lemma 4.1.4.2. *A semistable coherent $\mathscr{X}$-twisted sheaf $\mathscr{F}$ is pure.*

Proof. If $\mathscr{G} \subset \mathscr{F}$ is a torsion subsheaf, then $\dim \mathscr{G} < \dim \mathscr{F}$, which means that $P_{\mathscr{G}} \leq 0$ (as $\alpha_d(\mathscr{F}) \neq 0$). Thus, $P_{\mathscr{G}} = 0$, and therefore $\mathscr{G} = 0$ by 4.1.3.14. □

Definition 4.1.4.3. The *reduced geometric Hilbert polynomial* of a coherent $\mathscr{X}$-twisted sheaf $\mathscr{F}$ of dimension d is $p_{\mathscr{F}} := (1/\alpha_d)P_{\mathscr{F}}$.

Remark 4.1.4.4. Thus, an $\mathscr{X}$-twisted sheaf $\mathscr{F}$ is semistable if and only if it is pure and for any subsheaf $\mathscr{G} \subset \mathscr{F}$ we have $p_{\mathscr{G}} \leq p_{\mathscr{F}}$.

Definition 4.1.4.5. The *slope* of a coherent $\mathscr{X}$-twisted sheaf $\mathscr{F}$ of dimension d is

$$\mu(\mathscr{F}) := \frac{\deg \mathscr{F}}{\operatorname{rk} \mathscr{F}}.$$

Definition 4.1.4.6. A coherent $\mathscr{X}$-twisted sheaf $\mathscr{F}$ of dimension d is *μ-semistable* (respectively *μ-stable*) if $\mathscr{F}$ is pure and for any subsheaf $\mathscr{G} \subset \mathscr{F}$ we have $\mu(\mathscr{G}) \leq \mu(\mathscr{F})$ (respectively $\mu(\mathscr{G}) < \mu(\mathscr{F})$).

Remark 4.1.4.7. If we define the *modified slope* of an $\mathscr{X}$-twisted sheaf $\mathscr{F}$ of dimension d as $\widehat{\mu}(\mathscr{F}) := \alpha_{d-1}/\alpha_d$, then we get the same notion of slope semistability as above. We will use both notions of slope interchangeably.

Definition 4.1.4.8. Given a sheaf of algebras $\mathscr{A}$ on X, an $\mathscr{A}$-module $\mathscr{F}$ is *Simpson (semi)stable* if the inequality of 4.1.4.1 holds for subsheaves $\mathscr{G}$ which are $\mathscr{A}$-submodules.

Lemma 4.1.4.9. *Let X be a smooth projective variety and $\mathscr{X}$ a $\boldsymbol{\mu}_n$-gerbe on X with a locally free twisted sheaf $\mathscr{V}$ of rank v such that $c_i(V) = 0 \in A_{\mathbf{Q}}(\mathscr{X})$ for all $1 \leq i < n$. Let $\mathscr{A} = \mathscr{E}nd_{\mathscr{X}}(\mathscr{V})$. Then semistability of $\mathscr{X}$-twisted sheaves is equivalent to Simpson-semistability of $\mathscr{A}$-modules via the fibered Morita equivalence $\mathscr{W} \mapsto \mathscr{H}om(\mathscr{V}, \mathscr{W})$.*

Proof. By the Riemann-Roch theorem,

$$\begin{aligned}\chi(\mathscr{H}om(\mathscr{V}, \mathscr{W})) &= \deg(\operatorname{ch}(\mathscr{V}^\vee) \cdot \operatorname{ch}(\mathscr{W}) \cdot \operatorname{Td}_{\mathscr{X}}) \\ &= v \deg(\operatorname{ch}(\mathscr{W}) \cdot \operatorname{Td}_{\mathscr{X}}) + \deg(\operatorname{ch}(\mathscr{V}^\vee)_1 \cdot (\operatorname{ch}(\mathscr{W}) \cdot \operatorname{Td}_{\mathscr{X}})_{n-1}) + \cdots \\ &\qquad \cdots + \operatorname{rk}(\mathscr{W}) \deg(\operatorname{ch}(\mathscr{V}^\vee) \cdot \operatorname{Td}_{\mathscr{X}})\end{aligned}$$

The assumption about the Chern classes of $\mathscr{V}$ kills all of the terms but the first and the last. We see that $\chi(\mathscr{H}om(\mathscr{V}, \mathscr{W})) = v\chi^g(W) + \mathrm{rk}(\mathscr{W}) \cdot \text{constant}$, whence the result follows. □

Proposition 4.1.4.10. *Let X be a smooth projective variety and $\mathscr{X}$ a $\boldsymbol{\mu}_n$-gerbe on X. The category of $\widehat{\mu}$-semistable coherent $\mathscr{X}$-twisted sheaves with fixed geometric Hilbert polynomial is bounded.*

Proof. By 4.1.3.16, there is a locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$ with $\det \mathscr{V} = \mathscr{O}$. Applying the Morita equivalence $\mathscr{H}om(\mathscr{V}, \cdot\,)$ and a result of Simpson [70, 3.3], one sees that the set of $\mathscr{A} := \mathscr{E}nd(\mathscr{V})$-modules resulting from this operation has the property that the slope is fixed and $\mu_{\max}$ is bounded above.

To show boundedness, first consider the subset of reflexive sheaves. Given a reflexive sheaf F on X, temporarily write

$$P_F^g(m) = \sum_{i=0}^{\dim X} a_i(F)\binom{m + \dim X - i}{\dim X - i}.$$

By a result of Langer (proving a theorem of Maruyama in arbitrary characteristic), the set of coherent reflexive sheaves F on X with a fixed upper bound on $\mu_{\max}(F)$, $a_0(F) = r$, $a_1(F) = a_1$, and $a_2(F) \geq a_2$ for fixed a_0, a_1, a_2 is bounded [51, 4.3]. Thus, to apply this to our situation, it remains to show that the "codimension 2" coefficient a_2 is bounded below. Looking at the formula in 4.1.4.9 and using the formula for the Chern character, we see that if $\det \mathscr{V} = \mathscr{O}$ then the correction to the codimension 2 term of the Hilbert polynomial of $\mathscr{H}om(\mathscr{V}, \mathscr{W})$ coming from higher terms (= after the first term) is given by $-\kappa(c_2(\mathscr{V}) \cdot c_1(\mathscr{O}(1))^{d-2})t^{d-2}$, where κ is a coefficient which depends only on d (the dimension of X) and the degree of $\mathscr{O}(1)$ on X. In particular, after dualizing $\mathscr{V}$ if necessary, we may assume that this correction term is always non-negative. Thus, we see that the Morita equivalence we apply will yield Hilbert polynomials with bounded below codimension 2 terms. Applying Langer's theorem [*ibid.*], we are done for reflexive twisted sheaves.

Given a torsion free twisted sheaf, taking its reflexive hull fixes a_0 and a_1 and increases a_2. Thus, we have just shown that the set of reflexive hulls of the sheaves we are interested in is bounded. In particular, only finitely many geometric Hilbert polynomials occur. We can now apply 4.1.3.21 finitely many times to yield the desired result. □

Corollary 4.1.4.11. *The category of semistable $\mathscr{X}$-twisted sheaves with fixed geometric Hilbert polynomial is bounded.*

Proof. It is elementary that any semistable sheaf is $\widehat{\mu}$-semistable. □

Remark 4.1.4.12. In the future, we will only be interested in $\boldsymbol{\mu}_n$-classes on a smooth surface. In this case, the hypothesis of 4.1.4.9 (which is simply that $\mathscr{V}$ have trivial determinant) will always be satisfied for us. In fact, as is shown below in 4.1.4.14 below, when X is a smooth projective surface over a strictly Henselian local ring, every $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$ admits a locally free $\mathscr{X}$-twisted sheaf of rank n and determinant

$\mathscr{O}$. We include the proof as it yields the only case of 2.2.3.8 which is strictly necessary for our work and is much easier to prove. ♦

Corollary 4.1.4.13. *Let $X \to S$ be a smooth projective morphism, $\mathscr{X}$ a $\boldsymbol{\mu}_n$-gerbe on X, and $\mathscr{F}$ an S-flat family of coherent $\mathscr{X}$-twisted sheaves. The locus of $\widehat{\mu}$-semistable (resp. semistable, resp. geometrically $\widehat{\mu}$-stable, resp. geometrically stable) fibers of $\mathscr{F}$ is open in S.*

Proof. It suffices to prove this when S is affine, whence we may assume that $\mathscr{X}$ is a quotient stack (2.2.3.8 again!) and that the theory developed above applies. Now one can apply [40, 2.3.1] verbatim, with the additional remark that their proof also works for $\widehat{\mu}$-semistability (even though they do not state this explicitly). □

4.1.4.14. Let us indicate how to locally produce an Azumaya algebra as mentioned at the end of 4.1.4.12 for a smooth projective morphism of relative dimension 2. The argument given here marginally resembles a sketched argument of [40, 5.2.6]. In the present context, this result was first proven by Artin and de Jong [10, 8.4.2] using elementary transforms of Azumaya algebras. Our approach gives a different "sheaf-theoretic" proof of the result in the same spirit.

Lemma 4.1.4.15. *Let X be a quasi-compact regular scheme, $C \subset X$ a regular Cartier divisor, and n a positive integer prime to the characteristics of X. Suppose $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe possessing a faithful locally free $\mathscr{X}$-twisted sheaf and which has a section when restricted to C. The determinant (on $\mathscr{X}$) of a locally free $\mathscr{X} \times_X C$-twisted sheaf of rank r is $\mathscr{O}(rC)$.*

Proof. It is clear by regularity that any two $\mathscr{X}_C$-twisted sheaves of the same rank have the same determinant on $\mathscr{X}$ (as they differ on a subset of X of codimension at least 2). Furthermore, as any sheaf on $\mathscr{X}_C$ has rank zero on $\mathscr{X}$, the determinant will be 0-fold twisted and therefore be the pullback of a unique invertible sheaf from X. By the hypothesis that $\mathscr{X}_C$ has a section, we see that there is a locally free $\mathscr{X}_C$-twisted sheaf of rank 1. Call its determinant L (which we view as an invertible sheaf on X by abuse of notation). Let $\mathscr{V}$ be a locally free $\mathscr{X}$-twisted sheaf of positive rank t. The sequence $0 \to \mathscr{V}(-C) \to \mathscr{V} \to \mathscr{V}_C \to 0$ shows that $L^{\otimes t} \cong \mathscr{O}(tC)$. On the other hand, there is a surjection $\mathscr{V} \to \mathscr{P}$, where $\mathscr{P}$ is a torsion free $\mathscr{X}_C$-twisted sheaf of rank 1. (Indeed, there is such a map generically, and we can extend the quotient as X is Noetherian.) The kernel $\mathscr{W}$ of such a surjection will be isomorphic to $\mathscr{V}$ away from C. Thus, $\det \mathscr{W}$ will differ from $\det \mathscr{V}$ by a power of $\mathscr{O}(C)$ (by the usual facts about Picard groups and the fact that X has normal integral connected components), which implies that $\det \mathscr{P} = \mathscr{O}(sC)$ for some s. Combining the two statements, we see that $s = 1$ and $L = \mathscr{O}(C)$. □

Remark 4.1.4.16. I am not sure if this holds in the absence of the hypothesis that there is a locally free $\mathscr{X}$-twisted sheaf or a global section of $\mathscr{X} \times_X C$.

Lemma 4.1.4.17. *Let C be a projective variety over an algebraically closed field k with fixed very ample invertible sheaf $\mathscr{O}(1)$. Suppose $\mathscr{V}$ is a locally free sheaf of rank*

$n > 1$ on C, $\mathscr{L}$ and $\mathscr{K}$ are invertible sheaves on C, and $\sigma : \mathscr{V} \to \mathscr{V} \otimes \mathscr{K}$ is a given non-zero map of sheaves. Then for a generic map $\pi : \mathscr{V} \to \mathscr{L}(n)$, the composite $(\pi \otimes \mathscr{K})\sigma$ is non-zero.

Proof. To see the non-vanishing of generic $(\pi \otimes \mathscr{K})\sigma$, it suffices to show the existence of a single such π. Indeed, the locus of maps which do not kill σ is easily shown to be open. To find such a map, note that there is trivially one over the generic point of C, and this extends upon twisting enough. □

Corollary 4.1.4.18. *Let $\mathscr{V}$ be a locally free $\mathscr{X}$-twisted sheaf as in 4.1.4.15, $\mathscr{P}$ an invertible $\mathscr{X}_C$-twisted sheaf, and K an invertible sheaf on X. Given a map $\sigma : \mathscr{V} \to \mathscr{V} \otimes K$, there is a map $\pi : \mathscr{V} \to \mathscr{P}(n)$ for sufficiently large n such that $(\pi \otimes K)\sigma \neq 0$.*

Proof. Twisting everything by $\mathscr{P}^\vee$ reduces this to 4.1.4.17. □

Proposition 4.1.4.19. *Let $\mathscr{X} \to X \to S$ be a $\boldsymbol{\mu}_n$-gerbe on a smooth surface over a strictly Henselian local scheme S, with $n \in \mathscr{O}(S)^\times$. There is a locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$ of rank n and trivial determinant.*

Proof. We may assume that S is the strict localization of an excellent scheme and is therefore excellent. By the Grothendieck Existence Theorem on DM stacks [58, 57] and Popescu's Theorem, it suffices to show that there is a totally unobstructed locally free twisted sheaf on the special fiber of rank n and trivial determinant. By this we mean a twisted sheaf $\mathscr{V}$ of rank n and trivial determinant such that the space of traceless homomorphisms $\mathrm{Hom}(\mathscr{V}, \mathscr{V} \otimes K_X)_0$ (which is the space of obstructions to deformation with constant determinant 4.2.1.10) vanishes entirely. (For the uninitiated, the argument then proceeds by deforming the sheaf to the completion of S at the closed point and then using the two hammers – the Existence Theorem and Popescu's theorem – to effectivize the formal deformation and descend it to the Henselization. This technique is the algebraists' version of local analytic constructions.)

Thus, let X be a smooth projective surface over an algebraically closed field k, $\mathscr{X}$ a $\boldsymbol{\mu}_n$-gerbe on X (with n prime to the characteristic of k), $\mathscr{V}$ a locally free $\mathscr{X}$-twisted sheaf (of rank n), $L = \det \mathscr{V}$. Let C be a smooth member of the linear system of sections of $L^\vee(mn)$ for large m and $\mathscr{P}$ an invertible $\mathscr{X}_C$-twisted sheaf. The kernel $\mathscr{W}$ of a general map $\pi : \mathscr{V} \to \mathscr{P}(N)$ will have determinant $\mathscr{O}(-mn)$, so that $\mathscr{W}(m)$ will have determinant $\mathscr{O}$. For large enough N, we claim that $\mathrm{Hom}(\mathscr{W}, \mathscr{W} \otimes K_X)_0$ has strictly smaller dimension than $\mathrm{Hom}(\mathscr{V}, \mathscr{V} \otimes K_X)_0$ (the subscript denoting traceless homomorphisms). This will imply by induction on the number of independent traceless maps that there is some locally free $\mathscr{X}$-twisted sheaf $\mathscr{W}$ with trivial determinant and $\mathrm{Hom}(\mathscr{W}, \mathscr{W} \otimes K)_0 = 0$.

Choose N large enough that $\mathrm{Ext}^1(\mathscr{V}, \mathscr{P} \otimes K(N)) = 0$. Then any map $\mathscr{W} \to \mathscr{W} \otimes K$ will extend to a map $\mathscr{V} \to \mathscr{V} \otimes K$. Furthermore, any traceless map must extend to a traceless map because $\mathscr{W} \hookrightarrow \mathscr{V}$ is an isomorphism at the generic point. Given such an extension, restriction defines a map $\mathscr{P}(N) \to \mathscr{P}(N) \otimes K$ over the open on C where π is a surjection; if the map is traceless to begin with, this restriction must also be traceless, which implies that it vanishes as $\mathscr{P}$ is invertible. If N is sufficiently large, we may apply 4.1.4.18 to conclude the following: if $\mathscr{V}$ admits a traceless map σ to $\mathscr{V} \otimes K$, then no traceless map $\mathscr{W} \to \mathscr{W} \otimes K$ extends to yield σ. Thus, $\hom(\mathscr{W}, \mathscr{W} \otimes K)_0 < \hom(\mathscr{V}, \mathscr{V} \otimes K)_0$. □

4.1.5 Applications of GIT

Let $(X, \mathscr{O}(1))$ be a (polarized) smooth projective variety over an algebraically closed field k and $\mathscr{X} \to X$ a $\boldsymbol{\mu}_n$-gerbe with $n \in k^\times$. According to 4.1.4.13, there is an algebraic stack $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, P)$ of semistable twisted sheaves of fixed rank r and geometric Hilbert polynomial P containing an open substack $\mathbf{Tw}^{s}_{\mathscr{X}/k}(n, P)$ of geometrically stable points (which contains a further open substack of geometrically μ-stable points). We consider only sheaves of rank n in this section. Recall that in this case there is a determinant 1-morphism

$$\mathbf{Tw}_{\mathscr{X}/k}(n) \to \mathscr{P}ic_{X/k}.$$

Let L be an invertible sheaf on X, corresponding to a 1-morphism $\varphi_L : k \to \mathscr{P}ic_{X/k}$.

Definition 4.1.5.1. With the above notation the *stack of semistable twisted sheaves of rank n, determinant L and geometric Hilbert polynomial P*, is defined to be

$$\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P) := \mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, P) \times_{\mathscr{P}ic_{X/k}, \varphi_L} k.$$

The open substack of geometrically stable sheaves will be denoted $\mathbf{Tw}^{s}_{\mathscr{X}/k}(n, L, P)$.

The usual computation [52, 2.2.2] of the 1-fiber product of stacks shows that $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P)$ has as objects over $T \to \operatorname{Spec} k$ pairs $(\mathscr{V}, \varphi)$, where $\mathscr{V}$ is a flat family of torsion free semistable $\mathscr{X}$-twisted sheaves parametrized by T, $\varphi : \det \mathscr{V} \xrightarrow{\sim} L_T$ is a chosen isomorphism, and for all points $t \to T$ one has $P^g_{\mathscr{V}_t} = P$. As usual, isomorphisms in the groupoid are given by isomorphisms of the sheaves $\mathscr{V}$ which respect the trivializations φ. Combining 4.1.1.1 with 4.1.4.13 shows that $\mathbf{Tw}^{ss}$ and $\mathbf{Tw}^{s}$ are algebraic stacks, locally of finite presentation over k, hence the same is true for $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P)$ and $\mathbf{Tw}^{s}_{\mathscr{X}/k}(n, L, P)$.

Suppose X is a surface. Given L, fixing the geometric Hilbert polynomial of $\mathscr{V}$ is the same as fixing $\deg c_2(\mathscr{V})$ by the Riemann-Roch formula. In this case, we will often write $\mathbf{Tw}(n, L, c)$ in place of $\mathbf{Tw}(n, L, P)$ in order to align ourselves with the classical literature on surfaces. When all of the adornments are clear from context (or irrelevant), we will omit them from the notation.

Lemma 4.1.5.2. *The stack* $\mathbf{Tw}^{ss}(n, P)$ *(resp.* $\mathbf{Tw}^{s}(n, L, P)$*) is quasi-compact and universally closed over* k*. The substack* $\mathbf{Tw}^{s}(n, P)$ *(resp.* $\mathbf{Tw}^{s}(n, L, P)$*) is quasi-compact and separated over* k*.*

Proof. The numerical properties of the geometric Hilbert polynomial allow for a transcription of Langton's proof [40, §2.B]. The uncomfortable reader may use the Morita equivalence of 4.1.4.9 to reduce this to [70, §4] (but only when there exists a locally free twisted sheaf with enough vanishing Chern classes, e.g., if X is a surface). □

Historically, moduli of semistable sheaves (and more generally modules) were studied using the tools of Geometric Invariant Theory, as developed in Mumford's thesis

[55]. The basic consequence of these methods is a proof that $\mathbf{Tw}^{ss}$ is corepresented by a projective scheme; in fact, one can say quite a bit more about the corepresenting object using the full theory. The philosophy adopted in this thesis is that the stack is really a more fundamental object. (It is galling that the semistability of a sheaf still lacks a convincing explanation in intrinsic terms without recourse to GIT. However, as we remind the reader, stable sheaves *do* have a convincing description in terms of unitary connections in characteristic 0. In fact, these bundles arose independently of GIT and it was only discovered later that they solve a GIT problem [64].) We will apply some of the classical results in this section to deduce GIT-like properties of our own moduli problem. When the underlying projective variety is a surface, techniques of Simpson will yield the result for all of $\mathbf{Tw}^{ss}$. In general, even without GIT, one can show that $\mathbf{Tw}^{s}$ has a coarse moduli space.

Lemma 4.1.5.3. *Let $\mathscr{X}$ be an algebraic stack, and suppose $\mathscr{I}(\mathscr{X}) \to \mathscr{X}$ is fppf. Then the big étale sheaf $\mathrm{Sh}(\mathscr{X})$ associated to $\mathscr{X}$ is an algebraic space and $\mathscr{X} \to \mathrm{Sh}(\mathscr{X})$ is a coarse moduli space.*

Proof. Martin Olsson pointed out that this is essentially the content of [9, appendix, item 2]. Here is a sketch of a more explicit proof:

We show that $\mathrm{Sh}(\mathscr{X})$ has an fppf cover. Let $X \to \mathscr{X}$ be a smooth cover; we will show that $X \to \mathrm{Sh}(\mathscr{X})$ is fppf. To this end, let $f : Y \to \mathrm{Sh}(\mathscr{X})$ be a map from an affine scheme. Consider the map of sheaves $\pi : X \times_{\mathscr{X}} Y \to X \times_{\mathrm{Sh}(\mathscr{X})} Y$. By the definition of $\mathrm{Sh}(\mathscr{X})$, we see that π has a section after pulling back by an étale cover $Y' \to Y$. If $U \to X \times_{\mathrm{Sh}(\mathscr{X})} Y$ has a lift along π with projection $\rho : U \to \mathscr{X}$, then one can see that $(X \times_{\mathscr{X}} Y) \times_{X \times_{\mathrm{Sh}(\mathscr{X})} Y} U \to U$ is a torsor under $\mathscr{A}ut(\rho)$. Thus, π is a representable fppf surjection of sheaves. As $\mathscr{A}ut(\rho)$ is quasi-compact, we in fact see using Artin's theorems on fppf algebraic spaces [52, 10.4.1] that $X \times_{\mathrm{Sh}(\mathscr{X})} Y$ is a quasi-separated algebraic space. It is easy to check that $X \times_{\mathrm{Sh}(\mathscr{X})} Y \to Y$ is fppf. Thus, we have shown that $X \to \mathrm{Sh}(\mathscr{X})$ is an fppf cover. The fact that $\mathrm{Sh}(\mathscr{X})$ is quasi-separated is similar and is left to the reader. □

Thus, we see that $\mathbf{Tw}^{s} \to \mathrm{Sh}(\mathbf{Tw}^{s})$ is a $\boldsymbol{\mu}_n$-gerbe on an algebraic space of finite type over k. The class of this $\boldsymbol{\mu}_n$-gerbe in $\mathrm{H}^2(X, \mathbf{G}_m)$ is the famous "Brauer obstruction" to the existence of a tautological twisted sheaf on $\mathrm{Sh}(\mathbf{Tw}^{s}) \times \mathscr{X}$. (In other words, a flat family of stable twisted sheaves on $\mathrm{Sh}(\mathbf{Tw}^{s}) \times \mathscr{X}$ such that any flat family of stable twisted sheaves on $T \times \mathscr{X}$ is given by pulling back along a morphism $T \to \mathrm{Sh}(\mathbf{Tw}^{s})$ and tensoring with the pullback of an invertible sheaf on T.)

Definition 4.1.5.4. The algebraic space $\mathrm{Tw}^{s}_{\mathscr{X}/k}(n, L, c) := \mathrm{Sh}\,\mathbf{Tw}^{s}_{\mathscr{X}/k}(n, L, c)$ is the *moduli space of stable twisted sheaves*.

Thus, that Tw^{s} is an algebraic space, as we have seen above, is quite easy to prove using the proper abstract foundations. The interesting challenge is to show the existence of an ample invertible sheaf on Tw^{s}. This really does seem like a difficult problem. Of course, by 4.1.4.11 if we apply a Morita equivalence there are only finitely many possible Hilbert polynomials occurring, so we see that it suffices to prove quasi-projectivity under the assumption that both the geometric Hilbert polynomial and the Morita-Simpson-Hilbert polynomial is constant in fibers. The standard techniques (making a linearized invertible sheaf on a Quot scheme) can be seen to work only when we know that the moduli space is a GIT quotient. I have no idea how to show abstractly that there is an ample invertible sheaf.

Proposition 4.1.5.5. *Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe on a smooth projective variety of dimension d. Suppose there is a twisted sheaf $\mathscr{V}$ such that all Chern classes but (possibly) $c_d(\mathscr{V})$ are zero in $A(\mathscr{X})_{\mathbf{Q}}$. Then $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P)$ is a GIT quotient stack with stable sublocus $\mathbf{Tw}^{s}_{\mathscr{X}/k}(n, L, P)$.*

Proof. By 4.1.4.9, this reduces to work of Simpson [70, §4]. ☐

Corollary 4.1.5.6. *Given the hypotheses of 4.1.5.5, there is a morphism to a projective scheme $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P) \to \mathrm{Tw}^{ss}_{\mathscr{X}/k}(n, L, P)$ corepresenting $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P)$ in the category of schemes and an open subscheme $U \subset \mathrm{Tw}^{ss}$ such that the restriction of the morphism $\mathbf{Tw}^{ss} \to \mathrm{Tw}^{ss}$ to U yields an isomorphism $\mathbf{Tw}^{s} \to \mathrm{Tw}^{s} \xrightarrow{\sim} U$.*

Question 4.1.5.7. In the absence of a $\mathscr{V}$ with enough vanishing Chern classes, is it still true that the coarse moduli space Tw^s is quasi-projective? Attempting to prove this in various naïve ways always leads one back to GIT. If the space is quasi-projective, can one find a projective scheme corepresenting $\mathbf{Tw}^{ss}$ by taking a projective closure of Tw^s?

4.1.6 Essentially trivial gerbes

In this section we describe the situation for a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$ on a projective variety X which is the gerbe of nth roots of an invertible sheaf. These correspond to the kernel of the natural map $\mathrm{H}^2(X, \boldsymbol{\mu}_n) \to \mathrm{H}^2(X, \mathbf{G}_m)$. If one chooses the "correct" polarization of X, then the stack of semistable twisted sheaves is canonically isomorphic to the stack of semistable sheaves on the underlying variety X. These spaces have essentially been studied by Ellingsrud-Göttsche, Thaddeus, Yoshioka, and Matsuki-Wentworth, in the guise of "twisted stability." These authors did not think in terms of gerbes, but rather investigated what happens when instead of computing the Hilbert polynomial of a torsion free sheaf F one computes the Hilbert polynomial of $F \otimes \mathscr{O}(\alpha)$, where α is some $\mathbf{Q}$-divisor (with stability now being called "α-twisted stability"). We refer the reader to their work ([79] and the references therein) for a detailed description of the situation (in characteristic 0); we will only use a small bit of the theory in what follows. At the end of the section we will spend a few moments considering what happens when the base field is not algebraically closed. Let $X \to \operatorname{Spec} k$ be a geometrically connected smooth projective variety over a field.

Definition 4.1.6.1. A $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$ is *(geometrically) essentially trivial* if the class $[\mathscr{X}]$ has trivial image in $\mathrm{H}^2(X, \mathbf{G}_m)$ (respectively, $\mathrm{H}^2(X \otimes_k \overline{k}, \mathbf{G}_m)$).

A gerbe $\mathscr{X}$ is essentially trivial if and only if there exists an invertible $\mathscr{X}$-twisted sheaf $\mathscr{L}$. As usual, $\mathscr{L}^{\otimes n}$ will be the pullback of an invertible sheaf on X. There is another way of identifying $\mathscr{L}^{\otimes n}$, using the Kummer sequence.

Definition 4.1.6.2. Let $\mathscr{M}$ be an invertible sheaf on X. The *gerbe of nth roots* of $\mathscr{M}$, denoted $[\mathscr{M}]^{1/n}$, is the stack whose objects over T are pairs $(\mathscr{L}, \varphi)$, where $\mathscr{L}$ is an invertible sheaf on $X \times T$ and $\varphi : \mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{M}$ is an isomorphism.

It is immediate that $[\mathscr{M}]^{1/n}$ is a $\boldsymbol{\mu}_n$-gerbe.

Proposition 4.1.6.3. *The cohomology of the Kummer sequence* $1 \to \boldsymbol{\mu}_n \to \mathbf{G}_m \to \mathbf{G}_m \to 1$ *yields an exact sequence*

$$0 \to \operatorname{Pic}(X)/n\operatorname{Pic}(X) \to \mathrm{H}^2(X, \boldsymbol{\mu}_n) \to \operatorname{Br}(X)[n] \to 0.$$

Under this identification and with the notation preceding 4.1.6.2, the cohomology class of $\mathscr{X}$ *in* $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ *equals the image of the class of* $\mathscr{L}^{\otimes n}$.

Proof. The construction of the (moderately) long exact sequence in non-abelian cohomology shows that given $\mathscr{M} \in \operatorname{Pic}(X)$, the coboundary $\delta(\mathscr{M}) \in \mathrm{H}^2(X, \boldsymbol{\mu}_n)$ is just $[\mathscr{M}]^{1/n}$. Up to isomorphism, this gerbe depends only on the residue of $\mathscr{M}$ modulo $n\operatorname{Pic}(X)$. The sequence shows that any essentially trivial gerbe has the form $\delta(\mathscr{M})$ for some $\mathscr{M}$. On $\delta(\mathscr{M})$, there is a universal nth root $\mathscr{L}$ with $\mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{M}$. If $\mathscr{N}$ is any other invertible twisted sheaf then $\mathscr{L} \otimes \mathscr{N}^\vee$ is an untwisted invertible sheaf, say $\mathscr{M}'$, and one has $\mathscr{L}^{\otimes n} \otimes (\mathscr{N}^{\otimes n})^\vee \cong (\mathscr{M}')^{\otimes n}$. Thus, the nth tensor power of any invertible twisted sheaf lies in the same class as $\mathscr{M}$ modulo $n\operatorname{Pic}(X)$. ☐

Corollary 4.1.6.4. *Any essentially trivial* $\boldsymbol{\mu}_n$*-gerbe is the gerbe of nth roots of a very ample invertible sheaf on* X.

Proof. Since X is projective, there is some very ample invertible sheaf $\mathscr{O}(1)$. But for all $m \geq 1$, $[\mathscr{M}]^{1/n} \cong [\mathscr{M}(mn)]^{1/n}$ by 4.1.6.3. Thus, as $\mathscr{M}(mn)$ is very ample for sufficiently large m, we are done. ☐

Proposition 4.1.6.5. *Suppose* $\mathscr{X}$ *is essentially trivial. Then there exists a polarization* $\mathscr{O}(1)$ *of* X *such that there is an isomorphism of* $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P)$ *with the stack* $\mathbf{Sh}^{ss}_{X/k}(n, L(-1), Q)$ *of semistable sheaves on* X *of rank* n*, determinant* $L(-1)$*, and Hilbert polynomial* $Q(t) = P(t - \frac{1}{n})$.

Proof. By 4.1.6.4, we may assume that $\mathscr{X} = [\mathscr{O}(1)]^{1/n}$. Let $\mathscr{L}$ be the universal twisted nth root of $\mathscr{O}(1)$. We claim that the functor $\mathscr{V} \mapsto \mathscr{V} \otimes \mathscr{L}^\vee = \mathscr{W}$ gives the desired isomorphism. The rank and determinant of $\mathscr{W}$ are clearly as stated. The Hilbert polynomial follows from the multiplicativity of the Chern character and a trivial calculation. It now follows that the functor preserves semistability. ☐

Definition 4.1.6.6. Given an essentially trivial $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$ on X, a polarization $\mathscr{O}(1)$ of X is *suitable* to $\mathscr{X}$ if $\mathscr{X} \cong [\mathscr{O}(1)]^{1/n}$.

A suitable polarization is unique up to nth powers.

Suppose now that k is a perfect field and that $\mathscr{X}$ is only geometrically essentially trivial. In this case, there is a polarization of $X \otimes \overline{k}$ which is suited to the isomorphism of 4.1.6.5, but this polarization may not descend to X.

Corollary 4.1.6.7. *Suppose $\mathscr{X}$ is geometrically essentially trivial. If there is a suitable polarization for $\mathscr{X} \otimes \overline{k}$ which descends to X, then with respect to this polarization there is an isomorphism*

$$\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P) \xrightarrow{\sim} \mathbf{Sh}^{ss}_{X/k}(n, L(-1), Q)$$

as in 4.1.6.5. In general (without changing the polarization), there is an isomorphism

$$\mathbf{Tw}^{\mu}_{\mathscr{X}/k}(n, L) \otimes \overline{k} \hookrightarrow \mathbf{Sh}^{\mu}_{X/k}(n, L') \otimes \overline{k},$$

where the superscript μ denotes the open substacks of μ-stable sheaves and $\mathscr{X} = [L \otimes (L')^{\vee}]^{1/n}$. If X is a surface, there is an isomorphism

$$\mathbf{Tw}^{\mu}_{\mathscr{X}/k}(n, L, P) \otimes \overline{k} \hookrightarrow \mathbf{Sh}^{\mu}_{X/k}(n, L', Q) \otimes \overline{k},$$

with Q an appropriate polynomial.

Proof. The first statement follows just as in 4.1.6.5. To prove the second statement, we may assume that k is algebraically closed and $\mathscr{X} = [\mathscr{N}]^{1/n}$. If $\mathscr{M}$ is a twisted invertible sheaf such that $\mathscr{M}^{\otimes n} \cong \mathscr{N}$, then the functor $\mathscr{F} \mapsto \mathscr{F} \otimes \mathscr{M}$ defines an isomorphism of stacks $\mathrm{Sh}(n, L \otimes \mathscr{N}^{\vee}) \to \mathbf{Tw}(n, L)$. When X is a surface, one can see that in fact this functor will also preserve the disriminant of $\mathscr{F}$ (defined in 5.2.1.1) and therefore fixing a Hilbert polynomial on one side will also fix it on the other. It remains to understand how this functor behaves with respect to μ-stability. But one has $\mu(\mathscr{F} \otimes \mathscr{M}) = \mu(\mathscr{F}) + \mu(\mathscr{M})$, so this functor in fact respects the μ-stable loci. (As discussed at the beginning of this section, it is not true in general that it will respect full-blown stability.) □

In general (still assuming k perfect), we may describe the obstruction to the descent of a suitable polarization. There is a natural map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})/n\,\mathrm{Pic}(\overline{X}) = \mathrm{NS}(\overline{X})/n\,\mathrm{NS}(\overline{X}).$$

The suitable polarization corresponds to a class on the right, and the obstruction to descent is precisely the failure of the map to be surjective. We can consider first the map $\mathrm{Pic}(X) \to \mathrm{NS}(\overline{X})$. Suppose $\mathrm{Br}(k) = 0$, so that $\mathrm{Pic}(\overline{X})^{\mathrm{Gal}_k} = \mathrm{Pic}(X)$. The sequence

$$0 \to \mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to \mathrm{NS}(\overline{X}) \to 0$$

gives rise to a sequence in cohomology

$$\mathrm{Pic}(X) \to \mathrm{NS}(\overline{X})^{\mathrm{Gal}_k} \to \mathrm{H}^1(\operatorname{Spec} k, \mathrm{Pic}^0(\overline{X})).$$

We see that if in addition $\mathrm{H}^1(\operatorname{Spec} k, \mathrm{Pic}^0) = 0$ (for example, if k is finite, by Lang's Theorem [68, Cor. 1 to Prop. 3, Chap. VI]), the obstruction to the descent of a suitable polarization is in the failure of a Galois-invariant class in $\mathrm{NS}(\overline{X})$ to map to the suitable polarization in $\mathrm{NS}(\overline{X})/n\,\mathrm{NS}(\overline{X})$. (The difference between this and the Galois-invariance of the class of the suitable polarization in $\mathrm{NS}(\overline{X})/n\,\mathrm{NS}(\overline{X})$ lies in

$\mathrm{H}^1(\operatorname{Spec} k, n\,\mathrm{NS}(\overline{X}))$.) Thus, if all ample components of the Néron-Severi group of $\overline{X}$ are k-rational (and k is perfect with vanishing Brauer group, e.g., k is finite) or $n\,\mathrm{NS}(\overline{X})_{\mathrm{tors}} = 0$, then we recover a k-rational version of 4.1.6.5.

4.2 Moduli of generalized Azumaya algebras

Using the work of previous section, we can prove analogous results for generalized Azumaya algebras. We include a section on their deformations and obstructions as a subject of independent interest, and with an eye toward the numerical estimates of 5.2.1 and ultimately construction of a virtual fundamental class (which will be written out in future work). We then define semistability for generalized Azumaya algebras and compare it to the existing notions for PGL_n-bundles. Finally, we show that on a surface there is a projective scheme corepresenting **GAz**.

4.2.1 Deformations and obstructions redux

The complex definition of generalized Azumaya algebras makes understanding their deformation theory seem daunting. As we will show in this section, this fear is easily allayed. Let $I \to A \to A_0$ be a small extension of Noetherian rings over S. Suppose given a generalized Azumaya algebra $\mathscr{A}$ of degree n on X_{A_0}. We assume that n is invertible in A_0 in what follows.

Lemma 4.2.1.1. *Let $(X, \mathscr{O})$ be a ringed topos and A, B, and C complexes of $\mathscr{O}$-modules. There is a natural isomorphism*

$$\mathbf{R}\mathscr{H}om(A \overset{\mathbf{L}}{\otimes} B, C) \xrightarrow{\sim} \mathbf{R}\mathscr{H}om(A, \mathbf{R}\mathscr{H}om(B, C))$$

and a natural isomorphism

$$\mathrm{RHom}(A \overset{\mathbf{L}}{\otimes} B, C) \xrightarrow{\sim} \mathrm{RHom}(A, \mathbf{R}\mathscr{H}om(B, C)).$$

Proof. This is a close relative of 2.2.5.8 and can proven similarly using the techniques of Neeman and Spaltenstein: replace A and B by K-flat resolutions F_A, F_B, and C by a K-injective resolution I_C. Then $\mathscr{H}om(F_B, I_C)$ is weakly K-injective, hence $\mathbf{R}\mathscr{H}om(A, \mathbf{R}\mathscr{H}om(B, C))$ is computed by

$$\mathscr{H}om(F_A, \mathscr{H}om(F_B, F_C)).$$

Using the hom-tensor adjunction on modules, this is naturally isomorphic to

$$\mathscr{H}om(F_A \otimes F_B, I_C),$$

which computes $\mathbf{R}\mathscr{H}om(A \overset{\mathbf{L}}{\otimes} B, C)$ as usual. The last formula follows upon taking derived global sections of the sheafified version. □

If $\mathscr{F}$ is *perfect*, then there is a natural isomorphism $\mathscr{F} \xrightarrow{\sim} \mathscr{F}^{\vee\vee}$. Applying 4.2.1.1 we see that to the identity in $\mathrm{End}(\mathscr{F})$ corresponds some morphism $\mathrm{Hom}_{\mathbf{D}}(\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{F}^{\vee}, \mathscr{O})$. This gives rise to a morphism $\mathbf{R}\mathscr{H}om(\mathscr{F}, \mathscr{F}) \to \mathscr{O}$, called the *trace morphism*, which we will denote Tr.

Definition 4.2.1.2. The homotopy fiber of $\mathrm{Tr} : \mathscr{A} \to \mathscr{O}$ in $\mathbf{D}(X)$ is the *traceless part* of $\mathscr{A}$ and denoted $s\mathscr{A}$.

Lemma 4.2.1.3. *Under the natural isomorphisms $\mathscr{A}^{\vee} \xrightarrow{\sim} \mathscr{A}$ and $\mathscr{O}^{\vee} \xrightarrow{\sim} \mathscr{O}$, the trace is dual to the unit $\mathscr{O} \to \mathscr{A}$.*

Proof. By functoriality, we can localize and assume that $\mathscr{F}$ is a strict perfect complex, where this is just a computation. □

Lemma 4.2.1.4. *If $\mathscr{F}$ is a perfect complex of $\mathscr{O}$-modules, the composition*

$$\mathscr{O} \to \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathscr{F}) \xrightarrow{\mathrm{Tr}} \mathscr{O}$$

is equal to multiplication by the rank of $\mathscr{F}$.

Proof. If $\mathscr{F}$ is a strict perfect complex, i.e., there is a quasi-isomorphism $\mathscr{V} \xrightarrow{\sim} \mathscr{F}$ with $\mathscr{V}$ a finite complex of locally free modules, this comes down to checking that the adjunction is induced by the obvious maps. As every perfect complex is locally quasi-isomorphic to such a complex, this will prove the general case by functoriality. □

Definition 4.2.1.5. The *reduced trace* of $\mathscr{A}$ is the map $\tau = \frac{1}{n}\mathrm{Tr} : \mathbf{R}\mathscr{E}nd(\mathscr{F}) \to \mathscr{O}$. This is independent of the choice of realization of $\mathscr{A}$ as derived endomorphism algebras.

Proposition 4.2.1.6. *Let $f : A \to B$ be a map in the derived category $\mathbf{D}(\mathscr{C})$ of an abelian category. If f has a section $g : B \to A$ then there is an isomorphism* $\mathrm{holim}(g) \cong \mathrm{hocolim}(f)$.

Proof. In other words, the homotopy fiber of g is isomorphic to the homotopy cofiber ("mapping cone") of f. This is a straightforward exercise which works in any triangulated category. □

Corollary 4.2.1.7. *The third vertex $p\mathscr{A}$ of the unit $\mathscr{O} \to \mathscr{A}$ is isomorphic to the traceless part $s\mathscr{A}$.*

Proof. This is an application of 4.2.1.6 to 4.2.1.4 and 4.2.1.3. □

The main result of this section is that the traceless part of a generalized Azumaya algebra governs its deformation theory. In the general case (when the rank is not invertible on the base), a more subtle analysis is called for. It is not especially difficult, and may be found in [10, §8.4], but we will not make use of it here.

We return to our situation: $\mathscr{A}$ is a generalized Azumaya algebra on X_{A_0}. To study possible deformations of $\mathscr{A}$, we first study the possible deformations of the class $\mathrm{cl}(\mathscr{A}) \subset \mathrm{H}^2(X_{A_0}, \boldsymbol{\mu}_n)$.

Lemma 4.2.1.8. *The natural map* $\mathrm{H}^2(X_A, \boldsymbol{\mu}_n) \to \mathrm{H}^2(X_{A_0}, \boldsymbol{\mu}_n)$ *is an isomorphism.*

Proof. By assumption, the group scheme $\boldsymbol{\mu}_{n,X_A}$ is étale, so the result follows from the usual topological invariance for constructible étale cohomology. □

Remark 4.2.1.9. Note that a similar analysis of the Brauer class may be carried out since $\operatorname{Spec} A_0$ is affine. This result is independent of the characteristics but is slightly more complicated, as we cannot invoke the topological invariance of constructible cohomology; if one wants to lift to a class of same order in the Brauer group and n is prime to characteristics, the same result is achieved. However, things are more complicated with the characteristic divides n. We leave this analysis to the reader. ♦

Thus, if we choose a gerbe $\mathscr{X} \to X_A$ such that $\mathscr{X}_{A_0}$ carries a twisted sheaf $\mathscr{F}$ with $\mathscr{A} \cong \mathbf{R}\mathscr{E}nd(\mathscr{F})$, we see that the deformation theory of $\mathscr{A}$ as a generalized Azumaya algebra admits a map to the deformation theory of $\mathscr{F}$ as a twisted sheaf. Given a deformation $\widetilde{\mathscr{A}} = \mathbf{R}\mathscr{E}nd(\mathscr{G})$ on X_A, we see by 3.1.2.11 that $\mathscr{G}_{A_0} \cong \mathscr{L} \otimes \mathscr{F}$ for some invertible sheaf $\mathscr{L}$ on X_{A_0}. As $\det \mathscr{G}$ and $\det \mathscr{F}$ are both trivialized, we see that $\mathscr{L} \in \operatorname{Pic}(X_{A_0})[n]$, whence it deforms to X_A (as $\operatorname{Pic}_{X/S}[n]$ is finite étale over S by the assumption that n is invertible on S). Changing $\mathscr{G}$ by an invertible sheaf does not change $\widetilde{\mathscr{A}}$, so we see that the deformation theory of $\mathscr{A}$ is the same as the *equideterminantal* deformation theory of $\mathscr{F}$.

Proposition 4.2.1.10. *Let* $\mathscr{F}$ *be an* A_0*-flat* $\mathscr{X}_{A_0}$*-twisted sheaf with torsion free fibers of rank* n *and trivial determinant* $\mathscr{O}_{X_{A_0}} \xrightarrow{\sim} \det \mathscr{F}$*. Let* $\mathscr{A} = \mathbf{R}\mathscr{E}nd(\mathscr{F})$*.*

1. *The obstruction to deforming* $\mathscr{F}$ *while preserving the determinant lies in the hypercohomology* $\mathbf{H}^2(I \overset{\mathbf{L}}{\otimes} s\mathscr{A}) = \operatorname{Ext}^2(\mathscr{F}, I \otimes \mathscr{F})_0$.

2. *The isomorphism classes of equideterminantal deformations of* $\mathscr{F}$ *are a principal homogeneous space under the hypercohomology*

$$\mathbf{H}^1(I \overset{\mathbf{L}}{\otimes} s\mathscr{A}) = \operatorname{Ext}^1(\mathscr{F}, I \otimes \mathscr{F})_0.$$

3. *The determinant-preserving infinitesimal automorphisms of a deformation are equal to* $\mathbf{H}^0(I \overset{\mathbf{L}}{\otimes} s\mathscr{A}) = \operatorname{Hom}(\mathscr{F}, I \otimes \mathscr{F})_0$.

Proof. According to the standard deformation theory of sheaves in topoi (recalled in 2.2.5.16), we have only to show that the trace of the obstruction of $\mathscr{F}$ is the obstruction of $\det \mathscr{F}$. As we will only use this in the case where $X \to A_0$ is smooth and projective, we will only treat this case. By 2.2.3.8, we may assume that $\mathscr{X}$ has enough locally free twisted sheaves. The argument one can use to prove this is practically identical to the argument of Artamkin [8] and proceeds by induction on the homological dimension of $\mathscr{F}$. If $\mathscr{F}$ is locally free, the statement is quite easy. The inductive step works as follows: choose a surjection $0 \to \mathscr{K} \to \mathscr{V} \to \mathscr{F} \to 0$ with $\mathscr{V}$ a locally free twisted sheaf whose deformation is unobstructed. Then the obstruction to deforming $\det \mathscr{F}$ is the same as the obstruction to deforming $\det \mathscr{K}$. Furthermore,

$\mathscr{K}$ has smaller homological dimension, hence the obstruction of $\det \mathscr{K}$ is the trace of the obstruction of $\mathscr{K}$. A simple argument shows that the trace of the obstruction of $\mathscr{F}$ equals the trace of the obstruction of $\mathscr{K}$.

The second statement works in a similar way and uses 4.2.1.7. The last statement is left to the reader. □

Remark 4.2.1.11. It is important to emphasize that the fact that the deformation and obstruction theory of the generalized Azumaya algebra $\mathscr{A} = \mathbf{R}\mathscr{E}nd(\mathscr{F})$ is the same as the equideterminantal deformation and obstruction theory of $\mathscr{F}$ hinges on the fact that $\operatorname{rk} \mathscr{F}$ is invertible on the base. In fact, the natural deformation theory of $\mathscr{A}$ as a generalized Azumaya algebra lies in the hypercohomology of $I \overset{\mathbf{L}}{\otimes} p\mathscr{A}$, which is dual (in the derived category of X_{A_0}) to $I \overset{\mathbf{L}}{\otimes} s\mathscr{A}$ only under the hypothesis on the degree. When $\mathscr{A}$ is an Azumaya algebra, the infinitesimal form of the Skolem-Noether theorem says that $p\mathscr{A}$ is the sheaf of autoderivations of $\mathscr{A}$; that this sheaf carries the deformation theory of $\mathscr{A}$ is a fun exercise in Čech cohomology (i.e., étale local deformations are easy to make, and then one simply tries to glue them together). ♦

4.2.2 On the existence of a universal object

The abstract nature of the definition of a generalized Azumaya algebra makes the existence of a universal object on $\mathbf{GAz}_{\mathscr{X}/k}(n) \times X$ difficult to understand. Throughout this section, we will write $\mathbf{GAz}$ in place of $\mathbf{GAz}_{\mathscr{X}/k}(n)$, and we will write $\mathscr{Y} = \mathbf{GAz} \times X$.

Let $\varphi : Y \to \mathscr{Y}$ be a map from an affine scheme to $\mathscr{Y}$. This yields a generalized Azumaya algebra $\mathscr{A}$ on $Y \times X$ and a section of $\pi_1 : Y \times X \to Y$, hence by pullback defines a generalized Azumaya algebra $\mathscr{A}_\varphi$ on Y. To $\mathscr{A}_\varphi$ is associated a $\boldsymbol{\mu}_n$-gerbe of trivializations τ_φ and we may define a fibered category $\mathscr{G}$ over the site of $\mathscr{Y}$ by sending φ to the global objects of τ_φ. The construction of τ_φ also yields a twisted sheaf $\mathscr{F}_\varphi$ on τ_φ such that $\mathscr{A}_\varphi \cong \mathbf{R}\mathscr{E}nd(\mathscr{F}_\varphi)$.

Lemma 4.2.2.1. *With the preceding notation, $\mathscr{G}$ is a $\boldsymbol{\mu}_n$-gerbe over $\widetilde{\mathscr{Y}}$ whose restriction via φ is τ_φ.*

Proof. This follows by functoriality. □

Proposition 4.2.2.2. *There is a $\mathscr{G}$-twisted perfect coherent sheaf $\mathscr{F}$ of rank n and trivial determinant on $\widetilde{\mathscr{Y}}$ such that for all φ, $\mathbf{R}\mathscr{E}nd(\varphi^* \mathscr{F}) \cong \mathscr{A}_\varphi$.*

Proof. A map $\eta : T \to \mathscr{G}$ is the same thing as a global object b of the gerbe τ_φ associated to a map $\varphi : T \to \mathscr{Y}$. Assigning to η the sections of $\mathscr{F}_\varphi$ over b yields the desired twisted sheaf. □

Pulling back to a smooth cover of $\mathbf{GAz}$ shows that the "universal" twisted sheaf is $\mathbf{GAz}$-flat and of locally finite homological dimension when X is smooth. Thus, $\mathbf{R}\mathscr{E}nd(\mathscr{F}) \in \mathbf{D}(\mathscr{Y})$ is a perfect complex. As in 4.2.1, we can produce the "universal deformation theory" for generalized Azumaya algebras by taking the traceless part of the universal algebra. This gives a perfect obstruction theory (on a surface) which is yields the virtual fundamental class by the construction of Behrend and Fantechi [13]. (Ongoing work with Olsson and de Jong should help to say this correctly even when **GAz** is not a Deligne-Mumford stack.)

4.2.3 Semistability

The structure of the surjection $\mathbf{Tw}_{\mathscr{X}/k}(n, \mathscr{O}) \to \mathbf{GAz}_{\mathscr{X}/k}(n)$ (where the first term means the union over all geometric Hilbert polynomials P) has been well described in 3.2.1.2. We can use that description to define semistability for generalized Azumaya algebras.

Lemma 4.2.3.1. *A geometric point of* **Tw** *is semistable (resp. stable, μ-semistable, μ-stable) if and only if every geometric point in the fiber over its image in* **GAz** *is semistable (resp. stable, μ-semistable, μ-stable).*

Proof. It suffices to check that if $\mathscr{F}$ is semistable (resp. ...) and L is an n-torsion invertible sheaf, then $\mathscr{F} \otimes L$ is semistable (etc.). But this is clear, as all of the semistability properties are numerical, hence are insensitive to tensoring with a numerically trivial invertible sheaf. □

Definition 4.2.3.2. The *semistable locus* of **GAz**, denoted $\mathbf{GAz}^{ss}$, is the image of $\mathbf{Tw}^{ss}$ by the natural map $\mathbf{Tw} \to \mathbf{GAz}$. Similarly for the geometrically stable locus, μ-semistable locus, and geometrically μ-stable locus.

Note that it is essential that we keep the determinant of the twisted sheaves fixed for this to work. We will always take the stability condition induced by twisted sheaves with trivial determinant (as this comports well with existing definitions for PGL_n-torsors, as we will see in a moment).

Using the description of the map $\mathbf{Tw} \to \mathbf{GAz}$ in 3.2.1.2, we also see that the semistable (geometrically stable, μ-semistable, geometrically μ-stable) locus is an open substack of **GAz**. The lemma above shows that it is quite easy to tell when a point is in it: a generalized Azumaya algebra $\mathbf{R}\mathscr{E}nd_{\mathscr{X}}(\mathscr{F})$, $\det \mathscr{F} \cong \mathscr{O}$, is semistable (etc.) precisely when $\mathscr{F}$ is.

It is worthwhile to compare our notion to the classical notion when one thinks of Azumaya algebras as associated to PGL_n-bundles. We recall a definition of Hyeon [41], giving a "Gieseker" form of a classical situation first considered by Ramanathan in his thesis [61, 62] and subsequently by many authors, including Friedman, Morgan, and Witten [26]. In the following, G denotes a reductive algebraic group.

Definition 4.2.3.3. A principal G-bundle $E \to X$ is *semistable* if the vector bundle $\operatorname{ad} E$ induced by the adjoint representation $G \to \mathrm{GL}(\mathrm{Lie}(G))$ is semistable.

If $G = \mathrm{PGL}_n$, we can be explicit about this. The Lie algebra of PGL_n is the same as the Lie algebra of SL_n (when n is prime to the characteristic of the base field!), namely the algebra of traceless $n \times n$ matrices. The adjoint representation is precisely acting by conjugation. Thus, if E is the bundle associated to an Azumaya algebra

$\mathscr{A}$, the adjoint bundle $\operatorname{ad} E$ is just the traceless part $s\mathscr{A}$. On the other hand, the reduced trace gives a splitting $\mathscr{A} = \mathscr{O} \oplus s\mathscr{A}$, which shows that $s\mathscr{A}$ is semistable if and only if $\mathscr{A}$ is semistable as a sheaf. The same holds for slope semistability. The situation for stability is a bit more complicated, and we will satisfy ourselves for the moment with semistability.

On surfaces and higher-dimensional varieties, these notions are still being worked out, and there has been a great deal of recent work on the subject [67], [31]. Our goal for the moment is just to show that our space contains Hyeon's and is compact, so that we have added a small set of points and made the moduli quasi-proper. We have not yet tried to compare our method with Gomez and Sols or Schmitt's methods. Our methods constantly exploit the fact that there is a central extension of PGL_n whose bundle theory is fairly well understood in low dimensions, namely SL_n. This fact has no hope of generalizing to other groups G!

Fix a class $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_n)$ and a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$ in the class α.

Lemma 4.2.3.4. *Given $[\mathscr{V}] \in \mathbf{Tw}_{\mathscr{X}/k}(n, \mathscr{O}, P)$, if $\mathscr{E}nd(\mathscr{V})$ is semistable as in 4.2.3.3 then $\mathscr{V}$ is semistable.*

Proof. This is an immediate consequence of 4.1.4.9. □

Thus, we have (inadvertently) constructed a compactification of Hyeon's space.

Remark 4.2.3.5. It is worth pointing out that it is quite likely that the definitions of semistable principal bundles now in fashion are wrong in positive characteristic. In particular, when $G = \mathrm{GL}_n$ or SL_n, the adjoint representation does not preserve semistability (as $\mathscr{H}om(\mathscr{V}, \mathscr{V})$ need not be semistable any longer); it follows that this is also true for PGL_n. On the other hand, we will see below that the moduli spaces of semistable PGL_n-bundles are asymptotically irreducible. As Hyeon's condition is open in ours, any compactification which can be constructed of his space will be birational to ours, and in particular will need to account for points where the adjoint bundle is not semistable. This leaves some big questions to be resolved which have not been seriously approached to date. We feel that our approach gives a satisfactory theory for PGL_n, but there is no hope of going further in the zoo of algebraic groups with our methods. ♦

4.2.4 Corepresenting $\mathbf{GAz}^{ss}(n)$

Since any generalized Azumaya algebra $\mathscr{A}$ of degree n with class $\mathscr{X}$ has an associated weak algebra, which has an underlying perfect complex, we can define a geometric Hilbert polynomial for $\mathscr{A}$. (The easiest way to do this is to replace $\mathscr{A}$ by a quasi-isomorphic strict perfect complex and then work in K^0. Since the geometric Hilbert polynomial factors through K^0, this is immediately independent of the resolution. When $\mathscr{A}$ does not have a global resolution, things get more complicated. Illusie has defined and studied Chern classes for arbitrary perfect complexes in a topos [42], but we will not make use of his theory.)

Definition 4.2.4.1. The stack $\mathbf{GAz}_{\mathscr{X}/k}(n, P)$ has as objects over a k-scheme T generalized Azumaya algebras $\mathscr{A}$ on $X \times T$ such that for every geometric point $t \to T$, the fiber $\mathscr{A}_t$ has class $[\mathscr{X}]$, degree n, and geometric Hilbert polynomial P.

As usual, the isomorphisms are isomorphisms of generalized Azumaya algebras. This stack is easily seen to be an open substack of $\mathbf{GAz}(n)$. When the underlying variety X is a surface, it is easy to understand what it means to fix the geometric Hilbert polynomial of $\mathscr{A}$.

Lemma 4.2.4.2. *If X is a surface, $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe, and $\mathscr{V}$ is a torsion free twisted sheaf of rank n with trivial determinant, then $c_2(\mathbf{R}\mathscr{E}nd(\mathscr{V})) = 2nc_2(\mathscr{V})$. Thus, the geometric Hilbert polynomial of $\mathscr{V}$ determines the geometric Hilbert polynomial of $\mathbf{R}\mathscr{E}nd(\mathscr{V})$.*

Proof. This follows from the splitting principle (as $\mathscr{X}$ has the resolution property) and the usual computations. □

As with twisted sheaves, when X is a surface we will use $\mathbf{GAz}_{\mathscr{X}/k}(n, c)$ to denote the stack of generalized Azumaya algebras with class $[\mathscr{X}]$, degree n, and second Chern class of degree c.

The GIT description of $\mathbf{Tw}$ gives a projective scheme corepresenting $\mathbf{GAz}$.

Corollary 4.2.4.3. *There is a morphism $\mathbf{GAz}^{ss} \to \mathrm{Tw}^{ss} / \mathrm{Pic}_{X/k}[n]$ which corepresents $\mathbf{GAz}^{ss}$ in the category of schemes. The image of $\mathbf{GAz}^{s}$ is the open subscheme $\mathrm{Tw}^{s} / \mathrm{Pic}_{x/k}[n]$ and the induced map $\mathbf{GAz}^{s} \to \mathrm{Tw}^{s} / \mathrm{Pic}_{X/k}[n]$ is a coarse moduli space.*

Proof. Write $G = \mathrm{Tw}^{ss} / \mathrm{Pic}[n]$. Given any object $T \to \mathbf{GAz}^{ss}$, we have by 3.2.1.2 an étale surjection $T' \to T$ and a lift $T' \to \mathbf{Tw}^{ss}$. To get a map $\mathbf{GAz}^{ss} \to G$, it thus suffices to show that the composition $T' \to \mathrm{Tw}^{ss} / \mathrm{Pic}[n]$ factors through T, i.e., that the étale local maps $T' \to G$ agree on overlaps. To do this, it suffices to show that if $T'' \rightrightarrows \mathbf{Tw}$ are two maps with isomorphic images in $\mathbf{GAz}$ then they have the same image in G. But any two such maps differ (up to isomorphism) by an element of $\mathrm{Pic}[n]$, so this is clear. We have thus produced a map $\mathbf{GAz} \to G$. On the other hand, suppose $\mathbf{GAz} \to T$ is any map to a scheme. The composition $\mathbf{Tw} \to \mathbf{GAz} \to T$ yields a map $\mathrm{Tw} \to T$, and we want to show that this map is $\mathrm{Pic}[n]$-equivariant. But the map $\mathbf{Tw} \to \mathbf{GAz}$ is $\mathscr{P}ic[n]$-invariant, so this is immediate. □

Chapter 5

Curves and surfaces

In this section, we develop the theory of $\mathbf{Tw}^{ss}$ when the underlying variety X is a curve or a surface. Over an algebraically closed field, there is a guiding meta-theorem: *Anything which happens in the theory of* $\mathbf{Sh}^{ss}$ *happens in the theory of* $\mathbf{Tw}^{ss}$. For curves, this is not just a meta-theorem: as we will show in section 5.1, $\mathbf{Sh}^{ss}$ and $\mathbf{Tw}^{ss}$ are isomorphic (with the proper adornments added to the symbols). For surfaces, there is not a similar direct comparison, but the classical structure theory for $\mathbf{Sh}^{ss}$ carries over to $\mathbf{Tw}^{ss}$. In particular, as the second Chern class grows, $\mathbf{Tw}^{ss}$ becomes irreducible. One can further compute examples on K3 surfaces, but we have unfortunately not included these examples in this thesis. Thus, we will show that despite the excessively abstract foundations, we have a reasonable understanding of the geometry of these moduli spaces for low-dimensional varieties over algebraically closed fields. There are many gems from the untwisted world waiting to be properly twisted which we have not been able to include in this thesis. They will hopefully appear in future work.

When the base field is allowed to be non-algebraically closed, things get more interesting, and the stacks $\mathbf{Tw}^{ss}$ carry arithmetic information which $\mathbf{Sh}^{ss}$ knows nothing about. The straightforward geometry of the moduli spaces can now be brought to bear on arithmetic problems. We will exploit this extra information in section 6 when we study the Brauer group of a surface over an algebraically closed field, a finite field, and a local field. The work here also appears to be just the beginning of a possibly fruitful line of investigation.

5.1 Twisted sheaves on curves

We illustrate the theory developed up to this point with the example of semistable twisted sheaves on curves. This serves two purposes: first, twisted sheaves are easy to understand. Second, we will use the results mentioned in this section when we study semistable twisted sheaves on surfaces.

Let C be a proper curve over a field k. By this we mean a proper scheme of equidimension 1 over k, not necessarily smooth. For the sake of simplicity, we will assume that a curve is irreducible and generically reduced. Even the classical theory of sheaves has not been very well worked out for severely pathological curves. Allowing the curve to become reducible should be relatively straightforward, but we do not pursue this here. The reason to consider such a general type of curve is that in the relative case it is nice to be able to handle degenerate fibers. For example, if a surface X carries a generically nice pencil $\widetilde{X} \to \mathbf{P}^1$, it is likely (usually necessary) that there will be singular fibers in the pencil. We would still like to relate the space of semistable twisted sheaves on $\widetilde{X}$ to the relative space of twisted sheaves of X viewed as a family of curves over $\mathbf{P}^1$.

5.1.1 A curve over a point

Let $C \to \operatorname{Spec} k$ be a curve over an algebraically closed field, and let $\mathscr{C} \to C$ be a $\boldsymbol{\mu}_n$-gerbe over C with $n \in k^\times$.

Lemma 5.1.1.1. $\operatorname{Br}(C) = 0$.

Proof. We sketch the proof. One first reduces to the case where C is reduced. (E.g., consider $0 \to \mathscr{I} \to \mathscr{O}_C \to \mathscr{O}_{C_{\mathrm{red}}} \to 0$. Then $1 \to 1 + I \to \mathscr{O}_C^* \to \mathscr{O}_{C_{\mathrm{red}}}^* \to 1$ is exact, and taking cohomology we find $0 = \mathrm{H}^2(C, I) \to \mathrm{H}^2(C, \mathbf{G}_m) \to \mathrm{H}^2(C_{\mathrm{red}}, \mathbf{G}_m) \to \mathrm{H}^3(C, I) = 0$ is exact.) It then suffices to show that the Brauer group of any irreducible component vanishes (see 5.1.5.5ff for the type of reasoning used in this argument). This follows from Tsen's theorem. □

In other words, there exists an invertible $\mathscr{C}$-twisted sheaf, say $\mathscr{L}$. Recall that for any torsion free coherent sheaf $\mathscr{G}$ on C, one defines $\deg \mathscr{G} := \chi(\mathscr{G}) - \operatorname{rk} \mathscr{G} \chi(\mathscr{O}_C)$.

Definition 5.1.1.2. Given a $\mathscr{C}$-twisted sheaf $\mathscr{F}$, the *degree* of $\mathscr{F}$ is

$$\deg(\mathscr{F}) := \deg(\mathscr{F} \otimes \mathscr{L}^\vee) + \frac{\operatorname{rk} \mathscr{F}}{n} \deg \mathscr{L}^{\otimes n} \in \mathbf{Q}.$$

Thus, for example,

$$\deg(\mathscr{L}) = \deg \mathscr{O} + \frac{1}{n} \deg \mathscr{L}^{\otimes n} = \frac{1}{n} \deg \mathscr{L}^{\otimes n}.$$

Lemma 5.1.1.3. *Given $\mathscr{M} \in \operatorname{Pic}(X)$, $\deg \mathscr{F} \otimes \mathscr{M} = \deg \mathscr{F} + \operatorname{rk} \mathscr{F} \deg \mathscr{M}$.*

Proof. It is enough to prove this for very ample $\mathscr{M}$. Thus, we may assume that $\mathscr{M}$ has a section with zeroes in the locally free locus of $\mathscr{F}$. There rest is straightforward and left to the reader. □

Definition 5.1.1.4. The *slope* of $\mathscr{F}$, denoted $\mu(\mathscr{F})$, is $\deg \mathscr{F} / \operatorname{rk} \mathscr{F}$. The twisted sheaf $\mathscr{F}$ is *(semi-)stable* if for every twisted subsheaf $\mathscr{G} \subset \mathscr{F}$, we have $\mu(\mathscr{G}) (\leq) \mu(\mathscr{F})$.

It is easy to see that tensoring with $\mathscr{L}^\vee$ creates a bijection between the semistable $\mathscr{C}$-twisted sheaves of rank r and degree d and the semistable sheaves on C with rank r and degree $d - r \deg \mathscr{L}$. Note that this last number must be an integer. In fact, $\deg \mathscr{L} \in \frac{1}{n}\mathbf{Z}$, so $d \in \frac{1}{\gcd r,n}\mathbf{Z}$.

It is easy to see that the stack of semistable sheaves on C of rank r and degree d is non-canonically isomorphic to the stack of semistable sheaves of rank r and degree $d+nr$ for any n. In the case of twisted sheaves, a similar statement will hold. Writing δ for the image of $[\mathscr{C}]$ in $\mathbf{Z}/n\mathbf{Z}$ under the map induced by the degree map on $\mathrm{Pic}(C)$, we see (by definition) that any invertible $\mathscr{C}$-twisted sheaf $\mathscr{L}$ will have degree $q+\overline{\delta}$ for some integer q, where $\overline{\delta}$ is the fraction with denominator n which corresponds to δ under the natural identification of $\mathbf{Z}/n\mathbf{Z}$ with $(1/n)\mathbf{Z}/\mathbf{Z}$.

We have proven the following proposition.

Proposition 5.1.1.5. *The moduli stack of semistable $\mathscr{C}$-twisted sheaves of rank r and degree d is non-canonically isomorphic to the moduli stack of semistable sheaves on C of rank r and degree $d-r\overline{\delta}$.*

In particular, it is a GIT stack (hence corepresented by a projective variety).

The usual structure theory for moduli spaces of semistable sheaves on smooth curves developed by Seshadri, Ramanan, Ramanathan, Narasimhan, Mumford, Newstead, etc., now carries over to the twisted setting. (See [55, Appendix 5C] for a relatively exhaustive list of references.) We omit proofs for the sake of brevity.

Corollary 5.1.1.6. *If C is smooth of genus $g \geq 1$, the moduli space of semistable $\mathscr{C}$-twisted sheaves of rank r and any fixed determinant is unirational of dimension $(r^2-1)(g-1)$. The stack of semistable $\mathscr{C}$-twisted sheaves of rank r and fixed degree d is integral and smooth over k of dimension $r^2(g-1)+1$ at stable points.*

Note that, as usual, even though the stack is smooth, its corepresenting GIT quotient need not be smooth away from the stable locus (over which the stack is a gerbe).

Proposition 5.1.1.7. *Suppose $d-r\overline{\delta} \in \mathbf{Z}$ and r are relatively prime. The open immersion $\mathbf{Tw}^{s}_{\mathscr{C}/k}(r,d) \hookrightarrow \mathbf{Tw}^{ss}_{\mathscr{C}/k}(r,d)$ is an isomorphism. In this case, Tw^{ss} is a smooth rational projective variety isomorphic to $\mathrm{Sh}(\mathbf{Tw}^{ss})$. There is a tautological sheaf $\mathscr{F}$ on $\mathrm{Tw}^{ss}\times\mathscr{C}$, and $\mathrm{Pic}(\mathrm{Tw}^{ss}) \cong \mathbf{Z}$.*

5.1.2 The relative case

When C is allowed to move over a base (or descend over a non-algebraically closed base field) things get more interesting. In this section, we let $\pi : C \to S$ denote a proper morphism of finite presentation whose geometric fibers are curves as above, and we let $\mathscr{C} \to C$ be a $\boldsymbol{\mu}_n$-gerbe with $n \in \mathscr{O}_S(S)^\times$. We continue to assume that all geometric fibers are irreducible. (However, a more careful analysis of the general case should not be too difficult.) We fix a rank r and a rational number d, the degree.

Note that the degree map on the relative Picard scheme induces a morphism $\varphi : \mathrm{H}^0(S, \mathbf{R}^2\pi_*\mu_n) \to \mathrm{H}^0(S, \mathbf{Z}/n\mathbf{Z})$ which is a relative version of the map considered above: The image over a connected component $S' \subset S$ is equal to n times the constant value for the minimal degree of an invertible $\mathscr{C}$-twisted sheaf on a fiber. If S is connected, write $\overline{\delta} := (1/n)\varphi([\mathscr{C}])$ as above (where $\varphi([\mathscr{C}])$ is chosen to lie between 0 and $n-1$).

Proposition 5.1.2.1. *Suppose S is connected. The moduli stack of semistable S-flat $\mathscr{C}$-twisted sheaves of rank r and degree d is an étale form of the moduli stack of semistable S-flat sheaves of rank r and degree $d - r\overline{\delta}$.*

Proof. We simply need to note that one can étale-locally on the base find an invertible twisted sheaf $\mathscr{L}$ on $\mathscr{C}$ of degree $\overline{\delta}$. (The obstruction to the gluing of these local invertible sheaves is the image of $[\mathscr{C}]$ in $\mathrm{H}^1(S, \mathbf{R}^1\pi_*\mathbf{G}_m)$). The comparison is made by tensoring with $\mathscr{L}^\vee$; this will not change the S-flatness of the sheaf because it will not change its local structure. Note that deformation theory still provides a family of such $\mathscr{L}$ étale-locally on π even though it is not flat because the universal obstruction space vanishes. In this case, as $\mathscr{L}$ is locally free, it is easy to deform it explicitly to a family of invertible $\mathscr{C}$-twisted sheaves on the completions of S. Descent to the Henselization follows from Popescu's theorem (after reducing to the case of an excellent base). □

5.1.2.2. Consider the case $S = \operatorname{Spec} k$ for some possibly non-algebraically closed field k. There is another way to describe the spaces $\mathrm{Tw}^{ss}_{\mathscr{C}/k}(r, d)$ using the theory of Galois twists. Suppose C is a smooth projective curve over k with a rational point p. The Leray spectral sequence for $\mathbf{G}_m$ yields a natural isomorphism $\mathrm{H}^2(C, \mathbf{G}_m) = \mathrm{H}^2(S, \mathbf{G}_m) \oplus \mathrm{H}^1(S, \mathrm{Pic}_{C/k})$. Since C has a point, it follows that $\mathrm{H}^1(S, \mathrm{Pic}_{C/k}) = \mathrm{H}^1(S, \mathrm{Pic}^0_{C/k})$. Similarly, there is a decomposition

$$\mathrm{H}^2(C, \boldsymbol{\mu}_n) = \mathrm{H}^2(S, \boldsymbol{\mu}_n) \oplus \mathrm{H}^1(S, \mathrm{Pic}_{C/k}[n]) \oplus \mathrm{H}^0(S, \mathbf{R}^2 f_*\boldsymbol{\mu}_n).$$

The sheaf $\mathbf{R}^2 f_*\boldsymbol{\mu}_n$ is in fact isomorphic to $\mathbf{Z}/n\mathbf{Z}$, and a splitting of the natural map $\mathrm{H}^2(X, \boldsymbol{\mu}_n) \to \mathrm{H}^0(S, \mathbf{R}^2 f_*\boldsymbol{\mu}_n) = \mathbf{Z}/n\mathbf{Z}$ is given by sending $1 \in \mathbf{Z}/n\mathbf{Z}$ to the gerbe $[\mathscr{O}(p)]^{1/n}$.

In particular, the gerbe $\mathscr{C}$ gives rise to an element $\tau \in \mathrm{H}^1(S, \mathrm{Pic}_{C/k}[n])$ by projection. Note that tensoring yields an injection $\mathrm{Pic}_{C/k}[n] \hookrightarrow \mathrm{Aut}(\mathrm{Sh}^{ss}_{C\otimes\overline{k}/\overline{k}}(n, \mathscr{O}(p))$. By descent theory we see that varieties V over k which are geometrically isomorphic to $\mathrm{Sh}^s_{C/k}(n, \mathscr{O}(p))$ are classified (up to isomorphism) by $\mathrm{H}^1(S, \mathrm{Aut}(\mathrm{Sh}^s_{C/k}(n, \mathscr{O}(p))))$. (Such varieties are called *twists* of Sh^s.) In particular, to any class $\tau \in \mathrm{H}^1(S, \mathrm{Pic}_{C/k}[n])$ is associated a twist M_τ of $\mathrm{Sh}^s_{C/k}(n, \mathscr{O}(p))$.

Proposition 5.1.2.3. *With notation as above, let r be the projection of $[\mathscr{C}]$ in $\mathbf{Z}/n\mathbf{Z} = \mathrm{H}^0(S, \mathbf{R}^2 f_*\boldsymbol{\mu}_n)$, and fix an invertible sheaf $\mathscr{M} \in \mathrm{Pic}(C)$. Then $\mathrm{Tw}^{ss}_{\mathscr{C}/k}(n, \mathscr{M})$ is the Galois twist M_τ of $\mathrm{Sh}^{ss}_{C/k}(n, \mathscr{M}(-rp))$ associated to τ. When $r = 0$, the projection of $[\mathscr{C}]$ into $\mathrm{H}^2(\operatorname{Spec} k, \mathbf{G}_m)$ is trivial, and n is prime to $\deg \mathscr{M}$, there is a tautological sheaf on $\mathrm{Tw}^s_{\mathscr{C}/k}(n, \mathscr{M})$.*

Proof. Over the separable closure $\overline{k}$, there is an invertible $\mathscr{C}$-twisted sheaf $\mathscr{L}$ such that $\mathscr{L}^{\otimes n} \cong \mathscr{O}(rp)$. Thus, there is some finite separable extension $L \supset k$ over which such an $\mathscr{L}$ is defined. If we let p_0 and p_1 denote the two projection maps $\operatorname{Spec}(L \otimes L) \to \operatorname{Spec} L$, then we find that $(p_0^*\mathscr{L}) \otimes (p_1^*\mathscr{L})^\vee \in \mathrm{Pic}_{C/S}[n](L \otimes L)$. Moreover, it is immediate that this gives a 1-cocycle representing an element of $\mathrm{H}^1(S, \mathrm{Pic}_{C/S}[n])$; this element is in fact τ. (To actually prove this is not entirely trivial; at some point one must explicitly compute an edge map in a spectral sequence.)

On the other hand, the invertible sheaf $\mathscr{L}$ induces an isomorphism

$$\mathbf{Sh}^{ss}(n, \mathscr{M}(-rp)) \to \mathbf{Tw}^{ss}(n, \mathscr{M})$$

by sending $\mathscr{V}$ to $\mathscr{V} \otimes \mathscr{L}$. Chasing everything through now shows that the isomorphism class of $\mathrm{Tw}^{ss}(n, \mathscr{O}(p))$ in $\mathrm{H}^1(S, \mathrm{Pic}_{C/S}[n])$ is τ.

The statement about the existence of the tautological sheaf follows from the fact that such a sheaf exists geometrically (in the untwisted case) and is left to the reader. □

This shows that the geometry of the moduli space $\mathrm{Tw}^s_{\mathscr{C}/k}(n, \mathscr{M})$ is determined by the image of $[\mathscr{C}]$ in $\mathrm{H}^1(S, \mathrm{Pic}_{C/k}[n]) \oplus \mathbf{Z}/n\mathbf{Z}$ (and the geometry of the classical spaces of semistable sheaves). It is natural to wonder about the relation between the other projection of $[\mathscr{C}]$ in $\mathrm{H}^2(S, \boldsymbol{\mu}_n)$ and the properties of the moduli space Tw^s. This is actually rather mysterious. For example, when the class in $\mathrm{H}^2(C, \boldsymbol{\mu}_n)$ is the pullback of $\chi \in \mathrm{H}^2(S, \boldsymbol{\mu}_n)$, the existence of a rational point on the stack $\mathbf{Tw}^s_{\mathscr{C}/k}(n, \mathscr{O}(p))$ is related to the period-index problem for χ over k (see 6). As k is an arbitrary field, this is utterly unknowable *a priori*. Thus, for example, it is difficult to see the relationship between χ and the Brauer obstruction for Tw^s, i.e., the class of the $\boldsymbol{\mu}_n$-gerbe $\mathbf{Tw}^s_{\mathscr{C}/k}(n, \mathscr{O}(p)) \to \mathrm{Tw}^s_{\mathscr{C}/k}(n, \mathscr{O}(p))$. For in this case, $\mathrm{Tw}^s \cong \mathrm{Sh}^s$ and thus has many rational points, but the obstruction to lifting these points into the stack $\mathbf{Tw}^s$ depends upon the arithmetic of the arbitrary base field k. We leave this as an imprecisely formulated question:

Question 5.1.2.4. Given $\mathscr{C}$ with projection $\gamma \in \mathrm{H}^2(S, \boldsymbol{\mu}_n)$, (how) can γ be recovered from the stack $\mathbf{Tw}^{ss}_{\mathscr{C}/k}(n, \mathscr{O}(p))$?

5.1.3 Rank 1: twisted Picard schemes

Let k be a field, C (resp. X) a smooth geometrically connected proper k-variety of dimension 1 (resp. 2), and $\pi : X \to C$ a generically smooth surjection. We assume for the sake of simplicity that π has a section $\sigma : C \to X$ and that all geometric fibers are generically reduced and irreducible. In this case, $\pi_* \mathbf{G}_{m,X} = \mathbf{G}_{m,C}$, as π is universally cohomologically flat in dimension 0.

Consider the Leray spectral sequence for $\mathbf{G}_m$. It is well-known that $\mathbf{R}^2\pi_*\mathbf{G}_m = 0$. This is due to Artin; it is equivalent to the statement that the Brauer group of a relative curve over a strictly Henselian ring is trivial, and this in turn becomes a problem in the deformation theory of Azumaya algebras. The argument is essentially contained in Grothendieck's article [36]. Thus, we are left with a sequence

$$0 \to \mathrm{Br}(C) \to \mathrm{Br}(X) \to \mathrm{H}^1(C, \mathbf{R}^1\pi_*\mathbf{G}_m) = \mathrm{H}^1(C, \mathrm{Pic}_{X/C}) \to 0$$

which is split-exact owing to the section σ. The left-hand map is easily interpretable: one takes a Brauer class on C and pulls it back via π to X! Using the theory of twisted sheaves, we can also identify the right-hand map: if $\alpha \in \mathrm{Br}(X)$, then the corresponding element of $\mathrm{H}^1(\mathrm{Pic})$ is the space of α-twisted invertible sheaves in the fibers. In fact, there is a very simple way to describe this space.

Given α, let $\mathscr{X} \to X$ be a $\mathbf{G}_m$-gerbe in the class α. By 2.1.3.9, $\mathscr{X}$ is naturally isomorphic to the stack of twisted left $\mathbf{G}_m$-torsors. We can push forward $\mathscr{X}$ to C, yielding a stack $\pi_* \mathscr{X} \to C$ whose fiber category over $T \to C$ is the category of invertible α-twisted sheaves on $T \times_C X$. Thus, for example, $\pi_* \mathrm{B}\mathbf{G}_m \cong \mathscr{P}ic_{X/C}$, the relative Picard stack. In general, we find that the inertia stack of $\pi_* \mathscr{X}$ is the constant sheaf $\mathbf{G}_{m,\pi_*\mathscr{X}}$. By 4.1.5.3, $\pi_* \mathscr{X} \to \mathrm{Sh}(\pi_* \mathscr{X})$ is a $\mathbf{G}_m$-gerbe over an algebraic space which we will denote $\mathrm{Pic}_{X/C,\alpha}$. The obvious action of $\mathrm{B}\mathbf{G}_m$ on $\mathscr{X}$ gives rise to an action of $\mathrm{Pic}_{X/C}$ on $\mathrm{Pic}_{X/C,\alpha}$. It is easy to see that this makes $\mathrm{Pic}_{X/C}(\alpha)$ into a torsor. Furthermore, an explicit Čech computation shows that the class of this torsor is the class of the image of α.

Note that there is an exact sequence given by the degree map $0 \to \mathrm{Pic}^0_{X/C} \to \mathrm{Pic}_{X/C} \to \mathbf{Z} \to 0$. Furthermore, since there is a section σ, this sequence remains exact after taking global sections. Thus, there is an exact sequence $\mathrm{H}^1(C, \mathrm{Pic}^0) \to \mathrm{H}^1(C, \mathrm{Pic}) \to \mathrm{H}^1(C, \mathbf{Z}) = 0$. (That the last group is zero follows from the fact that $\mathbf{Z}_C$ is the pushforward of $\mathbf{Z}_{k(C)}$, combined with the case of a field and the Leray spectral sequence.) This has the interesting consequence that $\mathrm{Pic}_{X/C,\alpha}$ not only splits into components corresponding to the components of Pic, but that it is possible to choose a distinguished component corresponding to the image class of the Pic^0-torsor. As we have defined $\mathrm{Pic}_{X/C,\alpha}$, it is not obvious how to find this component. However, we claim that one may do the following: if $n\alpha = 0$, choose a lift $\widetilde{\alpha} \in \mathrm{H}^2(X, \boldsymbol{\mu}_n)$ and a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}_n$ representing $\widetilde{\alpha}$. Adding a multiple of $[\mathscr{O}_X(\sigma(C))]^{1/n}$, we may assume that the restriction of $\widetilde{\alpha}$ to any geometric fiber is trivial. Any two such classes $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ must differ by the gerbe of nth roots of $\pi^* \mathscr{M}$ for some $\mathscr{M} \in \mathrm{Pic}(C)$ (as an invertible sheaf on X which is trivial in every geometric fiber of π must be a pullback from C). On $\mathscr{X}_n$ we may consider just those invertible twisted sheaves of degree 0. We claim that this construction is independent of the choice of $\mathscr{X}_n$ (with trivial geometric fibers, as above) in the sense that the resulting torsor $\mathrm{Pic}^0_{X/C,\alpha}$ is well-defined up to isomorphism, hence that its class in $\mathrm{H}^1(C, \mathrm{Pic}^0_{X/C})$ is well-defined. To see this, note that in general given two $\boldsymbol{\mu}_n$-gerbes $\mathscr{X}_n$ and $\mathscr{X}'_n$, differing by $[\mathscr{N}]^{1/n}$, the space of invertible $\mathscr{X}_n$-twisted sheaves of degree 0 in each fiber is identified with the space of invertible $\mathscr{X}'_n$-twisted sheaves of degree $\deg \mathscr{N}/n$ in each fiber. (This follows from the fact that $\mathscr{X}'_n$ is identified with the space of invertible $\mathscr{X}$-twisted sheaves whose nth powers are identified with $\mathscr{M}$.) Thus, if $\mathscr{N} = \pi^* \mathscr{M}$, then the two spaces are isomorphic.

Now consider the case when k is finite. The twisted Picard spaces just constructed in fact lie in the Tate-Shafarevich group. Before we prove this, we first recall the definition of the Tate-Shafarevich group.

Definition 5.1.3.1. If F is an étale sheaf on $\mathrm{Spec}\, k(C)$, then $\text{Ш}(k(C), F)$ is defined to be the kernel of the map $\mathrm{H}^1(\mathrm{Spec}\, k(C), F) \to \prod_\nu \mathrm{H}^1(\mathrm{Spec}\, \widehat{k(C)}_\nu, F)$, where ν runs over all closed points of C and $\widehat{k(C)}_\nu$ denotes the completion of the function field with respect to the equivalence class of discrete valuations corresponding to ν.

Lemma 5.1.3.2. *If X is a geometrically irreducible curve over a finite field k then* $\mathrm{Br}(X) = 0$.

Proof. We may reduce to the case where X is irreducible and reduced (by the methods of 5.1.5.5ff and deformation theory). By the Leray spectral sequence for $\mathbf{G}_m$, Tsen's theorem, and Wedderburn's theorem, it suffices to show that $\mathrm{H}^1(\operatorname{Spec} k, \mathrm{Pic}_{x/k}) = 0$. Since $\mathrm{H}^1(\operatorname{Spec} k, \mathbf{Z}) = 0$, this will follow if $\mathrm{H}^1(\operatorname{Spec} k, \mathrm{Pic}^0_{X/k}) = 0$. But since X is irreducible, $\mathrm{Pic}^0_{X/k}$ is a connected group variety over a finite field, hence the vanishing of $\mathrm{H}^1(\operatorname{Spec} k, \mathrm{Pic}^0)$ follows from Lang's theorem (see e.g. [68, Cor. 1 to Prop. 3, Chap. VI]). □

Proposition 5.1.3.3. *Suppose the base field k is finite. Let $\alpha \in \mathrm{Br}(X)$. The space $\mathrm{Pic}^0_{X/C}(\alpha)$ has a section over the spectrum of the complete local ring of C at every closed point.*

Proof. This is a standard argument. By 5.1.3.2, given a closed point $c \in C$, one has that $\mathrm{Pic}^0_{X/C,\alpha}(c) \neq \emptyset$. Furthermore, since every fiber of π has dimension 1, the deformation theory of twisted sheaves 2.2.5.16 shows that $\mathrm{Pic}^0_{X/C,\alpha}$ is formally smooth over C. Thus, any section over the closed point c admits a formal deformation over $\operatorname{Spf} \widehat{\mathscr{O}}_{c,C}$. By the Grothendieck Existence Theorem for twisted sheaves 4.1.1.3, this comes from an effective formal deformation, i.e., an element of $\mathrm{Pic}^0_{X/C,\alpha}(\widehat{\mathscr{O}}_{c,C})$. □

Corollary 5.1.3.4. *Given a fibration $X \to C$ with a section σ over a finite base field as above,* $\mathrm{H}^1(C, \mathrm{Pic}^0_{X/C}) = Ш(k(C), \mathrm{Pic}^0_{X_\eta/k(C)})$.

Proof. This follows from the definition of the Tate-Shafarevich group and 5.1.3.3. □

As a result of all of this, when k is finite we recover Artin's isomorphism $\mathrm{Br}(X) \cong Ш(k(C), \mathrm{Pic}^0_{X_\eta/k(C)})$, giving a link between the Brauer group of a fibered surface and the Tate-Shafarevich group of the generic Jacobian. This allows one to study, e.g., the Tate-Shafarevich conjecture for an elliptic curve over a function field by instead studying the Brauer group of a minimal model of the curve. There are problems known as "period-index problems" in both contexts, but this translation does not respect the meaning of that term, as will be discussed below in section 6.

In general, in the absence of the section σ, one can carry out most of the analysis of this section, as long as one is willing to forget about trying to narrow the problem down to Pic^0 and work with Pic-torsors throughout. However, in this case it is no longer true that $\mathrm{Br}(X) \to \mathrm{H}^1(C, \mathrm{Pic}_{X/C})$ is surjective. In fact, reasoning as above will show that the image of this map lies in $Ш(k(C), \mathrm{Pic}_{X_\eta/k(C)})$. A more subtle analysis carried out by Artin and Grothendieck [36] yields a greater understanding of the cokernel of the map $\mathrm{Br} \to Ш$, the relation between $Ш(\mathrm{Pic})$ and $Ш(\mathrm{Pic}^0)$, and the state of things when the fibers are reducible. We refer the reader to [36] for more recent results in this direction.

5.1.4 Moving twisted sheaves on curves

Given a divisor moving in a surface and a twisted sheaf on the divisor, we can push it along the moving curve. This gives us a way of connecting two stable twisted sheaves on linearly equivalent smooth divisors in a family. Throughout this section, X is a smooth projective surface over an algebraically closed field k and $\mathscr{X} \to X$ is a fixed $\boldsymbol{\mu}_n$-gerbe on X.

Proposition 5.1.4.1. *Let C_0 and C_1 be smooth curves in X and let $\mathscr{P}_i$ be the push-forward to $\mathscr{X}$ of a stable locally free twisted sheaf on $C_i \times_X \mathscr{X}$. If C_0 is linearly equivalent to C_1 and $P^g_{\mathscr{X},\mathscr{P}_0} = P^g_{\mathscr{X},\mathscr{P}_1}$ then there is an irreducible k-variety S, two points $s_0, s_1 \in S(k)$, and an S-flat family of $\mathscr{X}$-twisted sheaves $\mathscr{F}$ on $\mathscr{X} \times S$ such that $\mathscr{F}_{s_i} \cong \mathscr{P}_i$.*

Proof. The idea is to push $\mathscr{P}_0$ along an embedded deformation of C_0 into C_1 and then move the image through the moduli space of twisted twisted sheaves on C_1. We can actually do both simultaneously (which is more likely to yield an *irreducible* parameter space for the family).

Since C_0 and C_1 are linearly equivalent, there is a map $X \to \mathbf{P}^1$ and a flat Cartier divisor $\mathcal{C} \subset X \times \mathbf{P}^1 \to \mathbf{P}^1$ such that $\mathcal{C}_0 = C_0$ and $\mathcal{C}_1 = C_1$. (E.g., one can take the total space of the pencil of sections of $\mathscr{O}(C_0)$ generated by C_0 and C_1.) Passing to an open subset $U \subset \mathbf{P}^1$ if necessary, we may assume $\mathcal{C} \to U$ is smooth. Consider the stack $\mathscr{M} := \mathbf{Tw}^{ss}_{\mathscr{X}\times\mathcal{C}/U}(n, P)$. It is a classical result that the stack $\mathrm{Sh}^{ss}_{C/k}(n,d)$ is an irreducible GIT quotient stack [55, Appendix 5C]. Thus, applying 5.1.2.1 and using quasi-properness, we see that $\mathscr{M}$ is irreducible and smooth over U (and thus smooth over k).

Let $M \to \mathscr{M}$ be a smooth cover. Write $M_i, i = 1, \ldots, t$ for the connected components of M. Then each M_i is an open irreducible subspace of M, hence has open image in $\mathscr{M}$. Since $\mathscr{M}$ is irreducible, there is some i such that $M_i \to \mathscr{M}$ is surjective. In other words, $\mathscr{M}$ has an irreducible smooth cover. Choosing points $m_0, m_1 \in M(k)$ mapping to $\mathscr{P}_0$ and $\mathscr{P}_1$ respectively, we see that we can make a family of semistable sheaves on $\mathcal{C} \times_U M$ base containing $\mathscr{P}_0$ and $\mathscr{P}_1$. Since $\mathcal{C} \subset X \times U$, we see that $\mathcal{C} \times_U M \subset X \times M$. Pushing forward the family yields the result. □

Corollary 5.1.4.2. *The conclusion of 5.1.4.1 holds when $\mathscr{P}_i$ are invertible twisted sheaves, without explicit stability hypotheses.*

Proof. This follows from the fact that any invertible sheaf is stable and the fact that $\mathbf{Sh}^s_{C/k}(1,d) = \mathrm{Pic}^d_{C/k}$ is smooth and irreducible. □

5.1.5 Moduli of restrictions

We use the above machinery to study what happens when restricting stable twisted sheaves on a surface X to a very ample smooth curve D. In particular, we show that there are no positive-dimensional complete families of locally free stable twisted sheaves on X which all restrict to the same stable twisted sheaf (up to isomorphism) on D. This will ultimately be used to show that asymptotically, the irreducible components of the stack of semistable twisted sheaves on X contain both locally free and non-locally free points.

Throughout this section, X is a smooth projective surface (with fixed very ample invertible sheaf $\mathscr{O}(1)$) over an algebraically closed field k and $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe on X, with $n \in k^{\times}$. The proof written here is based on an approach of de Jong using the relative moduli stack of the incidence correspondence of a very ample linear system. A different proof follows from more general theorems analogous to those of Li [40, §8.1] describing certain line bundles on the moduli stack. Those line bundles can also be used to blow down the Gieseker compactification to arrive at an "Uhlenbeck compactification", which differs non-trivially from the compactification constructed here even in the optimal case. We defer a discussion (verification!) of this construction and its properties to future work.

Let $D \in |\mathscr{O}(1)|$ be a general member.

Situation 5.1.5.1. Let C be a smooth projective curve over k and $\varphi : C \to \mathbf{Tw}^s_{\mathscr{X}/k,lf}$ a 1-morphism to the locally free locus (which will be denoted by a subscript lf in the future) corresponding to $\mathscr{F}$ on $C \times X$. Suppose that every object $\varphi(c)$, $c \in C(k)$, restricts to a fixed stable locally free $\mathscr{X}_D$-twisted sheaf $\mathscr{F}_0$.

This condition on D will be described as $R(D)$.

Proposition 5.1.5.2. *φ is essentially constant (i.e., isotrivial).*

(Since φ lands in the stable locus, being isotrivial is equivalent to the map to the coarse moduli space Tw^s being constant. Indeed, if φ is isotrivial, then there is a finite étale extension $C' \to C$ such that the induced map $C' \to \mathrm{Tw}^s$ is constant, whence the original map must be constant. Conversely, if $C \to \mathrm{Tw}^s$ is constant, then φ lands in the fiber $\mathscr{T}$ of $\mathbf{Tw}^s \to \mathrm{Tw}^s$ over a point. Since $\mathscr{T}$ is a $\boldsymbol{\mu}_n$-gerbe, the map $pt \to \mathscr{T}$ is finite étale, and pulling back by this map yields a finite étale cover $C' \to C$ such that the restriction of φ to C' is constant.)

Lemma 5.1.5.3. *There is an open subset of $|\mathscr{O}(1)|$ consisting of smooth divisors D' such that $R(D')$ holds.*

Proof. Write P for $|\mathscr{O}(1)|$. Let $I \subset X \times P$ be the incidence correspondence of $\mathscr{O}(1)$; the fiber of the second projection over a point $p \in |\mathscr{O}(1)|$ is the divisor corresponding to p. The family $\mathscr{F}$ on $C \times X$ corresponding to φ pulls back to give a flat family $\mathbf{F}$ of twisted sheaves on $C \times I \to C \times P$. (The sheaf $\mathbf{F}$ is flat by e.g. a Hilbert polynomial calculation after applying a Morita equivalence.) The condition $R(D)$ says that the locus Ψ of stable fibers contains all of $C \times \{[D]\}$. By openness of stability and properness of C, we conclude that there is an open $U \subset P$ such that $C \times U \subset \Psi$. $\square$

Lemma 5.1.5.4. *Suppose $R(D)$ holds. The twisted sheaf $\mathscr{F}_{C\times D}$ has the form $\mathrm{pr}_1^*(\mathscr{M}) \times \mathrm{pr}_2^*(\mathscr{F}_0)$, where $\mathscr{M}$ is an invertible sheaf of $\mathscr{O}_C$-modules and $\mathscr{F}_0$ is a stable twisted sheaf on D.*

Proof. Write $\mathscr{D} := \mathscr{X} \times_X D$. By $R(D)$, the family $\mathscr{F}_{C\times D}$ gives rise to a diagram

$$\begin{array}{ccc} C & \xrightarrow{\tilde{\varphi}} & \mathbf{Tw}^s_{\mathscr{D}/k} \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \pi} \\ & & \mathrm{Tw}^s_{\mathscr{D}/k} \end{array}$$

such that φ is constant with value $[\mathscr{F}_0]$. There is also a constant lift ψ of φ given by the family $\mathrm{pr}_2^* \mathscr{F}_0$ on $C \times D$. Since π is a $\mathbf{G}_m$-gerbe, we see that ψ and $\widetilde{\varphi}$ are identified with two sections of a trivial $\mathbf{G}_m$-gerbe. Using one of them to trivialize the gerbe, they differ by a map $C \to \mathrm{B}\mathbf{G}_m$, which gives the invertible sheaf $\mathscr{M}$. □

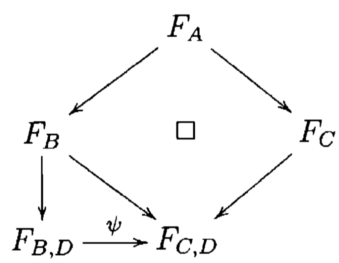
Given a (possibly singular) divisor E, we will say $R'(E)$ holds if there is an invertible sheaf $\mathscr{M}$ on C and a fixed twisted sheaf $\mathscr{F}_0$ on E such that $\mathscr{F}_{C\times E} \cong \mathrm{pr}_1^* \mathscr{M} \otimes \mathrm{pr}_2^* \mathscr{F}_0$. We just showed that for a smooth divisor D, $R(D)$ implies $R'(D)$, and that if $R(D)$ holds for one smooth very ample divisor, then it holds for an open set of them. Using these two facts, we now provide an inductive procedure for enlarging the divisor D satisfying $R'(D)$.

Lemma 5.1.5.5. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be a Cartesian diagram of surjections of sheaves of groups in a topos T. The natural map $\mathrm{B}A \to \mathrm{B}B \times_{\mathrm{B}D} \mathrm{B}C$ is a 1-isomorphism of classifying stacks.

Proof. The natural map $\mathrm{B}A \to \mathrm{B}B \times_{\mathrm{B}D} \mathrm{B}C$ is given by sending a right A-torsor F_A to the triple $(F_A \times^A B, F_A \times^A C, \varphi)$, where $\varphi : (F_A \times^A B) \times^B D \xrightarrow{\sim} (F_A \times^A C) \times^C D$ is the natural isomorphism arising from the associativity of the contracted product (i.e., $(F_A \times^A B) \times^B D \xrightarrow{\sim} F_A \times^A (B \times^B D) \xrightarrow{\sim} F_A \times^A D$ and similarly for C). There is a 1-morphism in the other direction arising as follows. An object of $\mathrm{B}B \times_{\mathrm{B}D} \mathrm{B}C$ is given by a triple (F_B, F_C, ψ), where $\psi : F_B \times^B D \xrightarrow{\sim} F_C \times^C D$ is an isomorphism of right D-torsors. Given such an object, one can produce a right A-torsor by forming the commutative diagram



where $F_{B,D} := F_B \times^B D$, etc. That F_A is in fact an A-torsor follows from the surjectivity of $B \to D$. We leave the check that these maps of stacks are 2-inverse to one another as an exercise. □

We can use 5.1.5.5 to prove a (twisted) classical result about vector bundles on a union of curves meeting transversely. Let D and D' be curves with transverse intersection $D \cap D' = \{q_1, \ldots, q_r\}$. Let X be a k-scheme. The transversality of the intersection of D and D' says that the diagram of surjections of sheaves of rings on

$X \times (D \cup D')$

$$\begin{array}{ccc} \mathcal{O}_{X\times(D\cup D')} & \longrightarrow & \mathcal{O}_{X\times D'} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X\times D} & \longrightarrow & \mathcal{O}_{X\times(D\cap D')} \end{array}$$

is Cartesian, where all schemes are given their reduced structures. (More generally, given a ring $\mathcal{O}$ in a topos and two ideals I and I' such that $I \cap I' = 0$, one has a corresponding diagram. For non-CM schemes, there can be complex information at embedded intersection points.) Here we write (by abuse of notation) $\mathcal{O}_D$ for the pushforward of the structure sheaf of D and similarly for D' and $D \cap D'$. It follows that given any k-scheme X the diagram

$$\begin{array}{ccc} \mathrm{GL}_n\, \mathcal{O}_{X\times(D\cup D')} & \longrightarrow & \mathrm{GL}_n\, \mathcal{O}_{X\times D'} \\ \downarrow & & \downarrow \\ \mathrm{GL}_n\, \mathcal{O}_{X\times D} & \longrightarrow & \mathrm{GL}_n\, \mathcal{O}_{X\times(D\cap D')} \end{array}$$

is a Cartesian diagram of surjections of sheaves of groups on $X \times (D \cup D')$.

Suppose $\mathcal{V}$ and $\mathcal{V}'$ are locally free sheaves of rank n on D and D', respectively. Our goal is to describe the space of locally free sheaves $\mathcal{W}$ on $D \cup D'$ which restrict to $\mathcal{V}$ on D and $\mathcal{V}'$ on D'.

Define a stack Σ on k-schemes as follows. Given a k-scheme X, the fiber category Σ_X is the groupoid of triples $(\mathcal{W}, \alpha, \beta)$ where $\mathcal{W}$ is locally free of rank n on $X\times(D\cup D')$ and $\alpha : \mathcal{W}|_{X\times D} \xrightarrow{\sim} \mathcal{V}_{X\times D}$ and $\beta : \mathcal{W}|_{X\times D'} \xrightarrow{\sim} \mathcal{V}'_{X\times D'}$ are isomorphisms.

Proposition 5.1.5.6. *With the above notation, the sheaf* $\mathrm{Sh}(\Sigma)$ *is representable by* $\mathrm{Isom}_{D\cap D'}(\mathcal{V}|_{D\cap D'}, \mathcal{V}'|_{D\cap D'})$.

Proof. By 5.1.5.5 and transversality (as discussed above), we have a 1-isomorphism of stacks

$$\mathrm{BGL}_n\, \mathcal{O}_{X\times(D\cup D')} \xrightarrow{\sim} \mathrm{BGL}_n\, \mathcal{O}_{X\times D} \times_{\mathrm{BGL}_n \mathcal{O}_{D\cap D'}} \mathrm{BGL}_n\, \mathcal{O}_{X\times D'}.$$

On the other hand, there is a natural 1-morphism

$$\mathrm{BGL}_n\, \mathcal{O}_{X\times D} \times_{\mathrm{BGL}_n \mathcal{O}_{D\cap D'}} \mathrm{BGL}_n\, \mathcal{O}_{X\times D'} \to \mathrm{BGL}_n\, \mathcal{O}_{X\times D} \times \mathrm{BGL}_n\, \mathcal{O}_{X\times D'}$$

given by forgetting the isomorphism of the restrictions. The choice of $\mathcal{V}$ and $\mathcal{V}'$ gives a map $X \to \mathrm{BGL}_n\, \mathcal{O}_D \times \mathrm{BGL}_n\, \mathcal{O}_{D'}$ (which comes by base change from a map defined when $X = k$). Taking the fiber product of the diagram

$$\begin{array}{ccc} \mathrm{BGL}_n\, \mathcal{O}_{X\times(D\cup D')} & \longrightarrow & \mathrm{BGL}_n\, \mathcal{O}_{X\times D} \times_{\mathrm{BGL}_n \mathcal{O}_{D\cap D'}} \mathrm{BGL}_n\, \mathcal{O}_{X\times D'} \\ & & \downarrow \\ & & \mathrm{BGL}_n\, \mathcal{O}_{X\times D} \times \mathrm{BGL}_n\, \mathcal{O}_{X\times D'} \end{array}$$

with X over $\mathrm{BGL}_n\, \mathcal{O}_{X\times D} \times \mathrm{BGL}_n\, \mathcal{O}_{X\times D'}$ and writing out the isomorphism class categories yields the result. (In fact, one checks that the fiber products are representable.) □

Corollary 5.1.5.7. *Suppose $\mathscr{V}$ and $\mathscr{V}'$ are locally free simple sheaves of rank n. The moduli space of locally free sheaves $\mathscr{W}$ of rank n on $D \cup D'$ such that $\mathscr{W}|_D \cong \mathscr{V}$ and $\mathscr{W}|_{D'} \cong \mathscr{V}'$ is isomorphic to $\mathrm{GL}_n^r / \mathbf{G}_m$, where $\mathbf{G}_m$ is embedded along the diagonal. Moreover, this scheme is affine.*

Proof. That the scheme is affine follows from the fact that the quotient is the complement of a hypersurface (cut out by the product of the determinants) in a projective space. Since $\mathscr{V}$ and $\mathscr{V}'$ are simple, it is easy to see that the moduli space M parametrizing $\mathscr{W}$ restricting to $\mathscr{V}$ and $\mathscr{V}'$ exists as an algebraic space. Furthermore, there is a surjection $\mathrm{Sh}(\Sigma) \to M$ which is a $\mathbf{G}_m$-bundle, and in fact M is identified with $\mathrm{Sh}(\Sigma)/\mathbf{G}_m$, where $\mathbf{G}_m$ acts in the natural way on the isomorphism β. Applying 5.1.5.6 completes the proof. □

Corollary 5.1.5.8. *Suppose $\mathscr{C} \to D \cup D'$ is a $\boldsymbol{\mu}_n$-gerbe and $\mathscr{V}$ and $\mathscr{V}'$ are locally free simple twisted sheaves of rank n on $\mathscr{C}_D$ and $\mathscr{C}_{D'}$. The moduli space $M(\mathscr{V}, \mathscr{V}')$ of locally free twisted sheaves $\mathscr{W}$ of rank n on $\mathscr{C}$ such that $\mathscr{W}|_{\mathscr{C}_D} \cong \mathscr{V}$ and $\mathscr{W}|_{\mathscr{C}_{D'}} \cong \mathscr{V}'$ is (non-canonically) isomorphic to $\mathrm{GL}_n^r / \mathbf{G}_m$.*

Proof. This follows from 5.1.5.7 after twisting down by a $\mathscr{C}$-twisted invertible sheaf. □

Lemma 5.1.5.9. *Suppose D and D' are (not necessarily smooth) elements of $|\mathscr{O}(1)|$ which intersect transversely such that $R'(D)$ and $R'(D')$ hold. Then $R'(D \cup D')$ holds.*

Proof. In the decompositions $\mathscr{F}_{C\times D} \cong \mathrm{pr}_1^* \mathscr{M} \otimes \mathrm{pr}_2^* \mathscr{F}_0$ and $\mathscr{F}_{C\times D'} \cong \mathrm{pr}_1^* \mathscr{M}' \otimes \mathrm{pr}_2^* \mathscr{F}_0'$, we claim that $\mathscr{M} \cong \mathscr{M}'$. Indeed, let $q \in D \cap D'$ be a point. Restricting $\mathscr{F}$ to $C \times \{q\}$ and using the two decompositions, we find that $\mathscr{M} \otimes (\mathscr{F}_0 \otimes \kappa(q)) \cong \mathscr{M}' \otimes (\mathscr{F}_0' \otimes \kappa(q))$. Both $\mathscr{F}_0 \otimes \kappa(q)$ and $\mathscr{F}_0' \otimes \kappa(q)$ are non-zero finite-dimensional $\kappa(q) = k$-vector spaces. Thus, we conclude that both $\mathscr{M} \otimes \mathscr{M}'^{-1}$ and $\mathscr{M}' \otimes \mathscr{M}^{-1}$ have non-zero global sections, whence $\mathscr{M} \cong \mathscr{M}'$. Choosing such an isomorphism and twisting down by $\mathrm{pr}_1^* \mathscr{M}$, there results a map from C to the moduli space $M(\mathscr{V}, \mathscr{V}')$ of 5.1.5.8. Since $M(\mathscr{V}, \mathscr{V}')$ is affine and C is proper, the map $C \to M(\mathscr{V}, \mathscr{V}')$ must be constant. As moduli of simple sheaves are a $\mathbf{G}_m$-gerbe over moduli and C is a curve over an algebraically closed field, Tsen's theorem shows that the family $\mathrm{pr}_1^* \mathscr{M}^\vee \otimes \mathscr{F}_{C\times(D\cup D')}$ is constant. Thus, $R'(D \cup D')$ holds. □

Proof of 5.1.5.2. Note that $\mathbf{Tw}^s$ is a $\mathbf{G}_m$-gerbe over its moduli space Tw^s. This means that any curve C in T admits a 1-morphism $C \to \mathrm{Tw}^s(\mathscr{X})$ lifting the inclusion $C \hookrightarrow T$. Replacing C by the normalization of the lift of its image in T, we may assume that the map $C \to \mathbf{Tw}^s$ is separably generated. Thus, to show that it is essentially constant, it suffices to show that the map on tangent spaces is the zero map, i.e., that the first-order deformations of any point in C induce the trivial deformation of the image point in moduli. We will do this by showing that they induce the trivial deformation on a sufficiently ample divisor. It is easy to see that given a locally free twisted sheaf $\mathscr{G}$ on X, the space of first-order infinitesimal deformations of $\mathscr{G}$ which restrict to the trivial deformation on an effective divisor D is principal homogeneous under the kernel of the restriction map $\mathrm{H}^1(X, \mathscr{E}nd(\mathscr{G})) \to \mathrm{H}^1(D, \mathscr{E}nd(\mathscr{G}_D))$; in the case where $\mathscr{G}$ and $\mathscr{G}_D$ are simple, this is precisely $\mathrm{H}^1(X, \mathscr{E}nd(\mathscr{G})(-D))$. Thus, if D is sufficiently ample, the deformations of $\mathscr{G}$ inject into the deformations of $\mathscr{G}_D$. By 5.1.5.3 and 5.1.5.9, we see that 1) $R(D)$ holds for some D in $\mathscr{O}(1)$, and 2) when $R(D)$ holds for some $D \in |\mathscr{O}(1)|$, there is an arbitrarily ample divisor $D^{(n)} = D_1 \cup D_2 \cup \cdots \cup D_n \in |\mathscr{O}(n)|$ such that $R'(D^{(n)})$ holds. But $R'(D^{(n)})$ says precisely that the infinitesimal deformation of $\mathscr{F}_{D^{(n)}}$ induced by a tangent vector t of C is trivial. As $D^{(n)}$ is arbitrarily ample, we see that the deformation of $\mathscr{F}$ induced by t is also trivial. □

5.2 Twisted sheaves on surfaces

In this section, we discuss the moduli of twisted sheaves on surfaces. In the process, we develop tools to reduce certain twisted statements to their classical counterparts. This should be viewed as a preliminary survey of a theory which is certainly amenable to significant further development. In particular, ongoing work of Langer (mentioned in [50]) should help clarify the classical situation in positive characteristic (and therefore, in our view, in characteristic 0 as well), and we believe that his methods will ultimately prove useful in the twisted case.

Throughout, we focus on moduli of twisted sheaves of rank n. This is technically simpler, as then determinants naturally take values in the Picard group of X itself. This is also the case one is naturally led to consider when approaching the classification of (generalized) Azumaya algebras of degree n in a Brauer class of order n, which is the most natural (and naïve) thing to do on a surface. In general, if one wants to consider rank r twisted sheaves on a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$, then there is a $\boldsymbol{\mu}_r$-gerbe $\mathscr{X}_r$ carrying them all with the same Brauer class as $\mathscr{X}$. However, the stability conditions on $\mathscr{X}$ and $\mathscr{X}_r$ are not identical. It seems likely that the resulting moduli spaces are (at least asymptotically) birational (related by flips). A special case of this has been worked out by several authors including Ellingsrud-Götsche, Yoshioka, ..., when comparing classical Gieseker semistability with what they call "twisted semistability" (which is our stability condition on a gerbe of roots of a divisor, i.e., every sheaf is twisted by a $\mathbf{Q}$-divisor before the Hilbert polynomial is computed). This work was mentioned (with references) in section 4.1.6.

The reader will observe throughout this section evidence for our meta-theorem ("All phenomena which occur for moduli spaces of semistable sheaves on surfaces also occur for moduli spaces of semistable twisted sheaves"). Unlike the case of curves, the evidence in this case is purely behavioral and not attributable to any direct comparison of the twisted and untwisted situations.

5.2.1 Discriminants and dimension estimates

Definition 5.2.1.1. Let X be a smooth projective surface and $\mathscr{X} \to X$ a $\boldsymbol{\mu}_n$-gerbe. Given a coherent $\mathscr{X}$-twisted sheaf $\mathscr{F}$ of rank r, the *discriminant* of $\mathscr{F}$ is the quantity

$$\Delta(\mathscr{F}) := \deg(2rc_2(\mathscr{F}) - (r-1)c_1(\mathscr{F})^2) \in \mathbf{Z}.$$

Proof that $\Delta(\mathscr{F}) \in \mathbf{Z}$. Since X is smooth, 2.2.3.8 shows that $\mathscr{F}$ has a global resolution by locally free twisted sheaves. A formal calculation (in K^0) shows that

$$\deg \operatorname{ch}^\vee(\mathscr{F}) \operatorname{ch}(\mathscr{F}) \operatorname{Td}_X = \sum_{i=0}^{2} (-1)^i \operatorname{ext}^i(\mathscr{F}, \mathscr{F}),$$

where $\operatorname{ch}^\vee(\mathscr{F})_i = (-1)^i \operatorname{ch}(\mathscr{F})_i$. Another formal calculation shows that

$$\operatorname{ch}^\vee(\mathscr{F}) \operatorname{ch}(\mathscr{F}) = \operatorname{rk}(\mathscr{F})^2 - \Delta(\mathscr{F}).$$

We thus conclude that $\chi(\mathscr{F}, \mathscr{F}) = \Delta(\mathscr{F}) - (\operatorname{rk}(\mathscr{F})^2 - 1)\chi(\mathscr{O}_X)$. (Such "formal calculations" show at least that the Chern character and all related results – Grothendieck-Hirzebruch-Riemann-Roch, discriminant calculations, etc. – can be extended to K^0, hence to the homotopy category of strict perfect complexes.) □

When $\mathscr{F}$ is locally free, $\Delta(\mathscr{F}) = c_2(\mathscr{E}nd(\mathscr{F}))$. (More generally, one has $\Delta(\mathscr{F}) = c_2(\mathbf{R}\mathscr{E}nd(\mathscr{F}))$, where $\mathbf{R}\mathscr{E}nd(\mathscr{F})$ here should be taken to mean the complex of sheaves $\mathscr{H}om(\mathscr{V}^\bullet, \mathscr{V}^\bullet)$ coming from any finite locally free resolution $\mathscr{V}^\bullet \to \mathscr{F}$.) The discriminant plays an important role in the behavior of the moduli space.

Lemma 5.2.1.2. *The discriminant is locally constant in flat families: given an S-flat family of coherent twisted sheaves $\mathscr{F}$ on $X \times S$ with S connected, the number $\Delta(\mathscr{F}_s)$ is constant for all (geometric) points $s \in S$.*

Proof. Implicit is the statement that $\Delta(\mathscr{F})$ may be computed after making any base field extension, which is clear. One easy way to see that $\Delta(\mathscr{F}_s)$ is locally constant in our case is to (locally on S) resolve $\mathscr{F}$ by a complex of locally free twisted sheaves $\mathscr{V}^\bullet \to \mathscr{F}$, use the fact that $\Delta(\mathscr{F}) = c_2(\mathscr{H}om^\bullet(\mathscr{V}^\bullet, \mathscr{V}^\bullet))$ and then use the fact that intersection products and geometric Hilbert polynomials are constant in a flat family (4.1.3.18). Another proof is based on the equality $\Delta(\mathscr{F}) = \chi(\mathscr{F}, \mathscr{F}) + (\operatorname{rk}(\mathscr{F})^2 - 1)\chi(\mathscr{O}_X)$ and the semicontinuity theorems for higher Exts (whose methods are demonstrated somewhat in 2.2.5.17(3)). □

In fact, when the determinant is fixed, it is equivalent to specify Δ, P^g, or c_2. Since we will usually fix a determinant in what follows, this means we can use any of these surrogates to divide the moduli problem into clusters of connected components.

Recall that the deformation theory of $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, P)$ at a point $[\mathscr{F}]$ is governed by $\operatorname{Ext}^1(\mathscr{F}, \mathscr{F})$ and $\operatorname{Ext}^2(\mathscr{F}, \mathscr{F})$, while the deformation theory with fixed determinant is determined by $\operatorname{Ext}^1(\mathscr{F}, \mathscr{F})_0$ and $\operatorname{Ext}^2(\mathscr{F}, \mathscr{F})_0$ (2.2.5), where the subscript 0 denotes traceless elements (4.2.1). We can use this to estimate the dimension of

$\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, P)$. We start with a well-known lemma, whose proof we feel comfortable recording because we know that no one will ever read this. There is a proof in e.g., [40, 2A.11], but we feel that the proof given here is slightly simpler. We borrow the notation of Huybrechts and Lehn out of sloth.

Lemma 5.2.1.3. *Let k be a field and $F : Art_k \to \mathrm{Set}$ a functor with a hull R. If the embedding dimension $\dim_k \mathfrak{m}_R/\mathfrak{m}_R^2 = d$ and F has an obstruction theory with values in an r-dimensional vector space $\mathcal{O}$, then $d \geq \dim R \geq d - r$.*

Proof. As a consequence of Schlessinger's proof [66, 2.11], it is easy see that R has the form $k[\![x_1, \ldots, x_d]\!]/J$ for some closed ideal J. It is enough by the Krull Hauptidealsatz to show that J is generated by at most r elements. Let $b_1, \ldots, b_r$ be a basis for the obstruction space $\mathcal{O}$. Let $A = k[\![x_1, \ldots, x_d]\!]$ and $\mathfrak{m} = (x_1, \ldots, x_d)$. Given $N \geq 2$, consider the infinitesimal extension

$$A_1 := A/\mathfrak{m}J + \mathfrak{m}^N \to A/J + \mathfrak{m}^N =: A_0.$$

By construction, there is a map $\varphi_0 : R \twoheadrightarrow A/J + \mathfrak{m}^N$. By definition, there is some obstruction $\beta \in (J + \mathfrak{m}^N/\mathfrak{m}J + \mathfrak{m}^N) \otimes_k \mathcal{O}$ to lifting φ_0 to a map

$$\begin{array}{ccc} R & \dashrightarrow & A/\mathfrak{m}J + \mathfrak{m}^N \\ & \searrow & \downarrow \\ & & A/J + \mathfrak{m}^N. \end{array}$$

Writing $\beta = \overline{\beta_i^N} \otimes b_i$ for some $\beta_i^N \in J$, we see that there is a lift of φ_0 into $A/\mathfrak{m}J + \mathfrak{m}^N + (\beta_1^N, \ldots, \beta_r^N)$. Furthermore, by linearity of the obstruction theory, we see that the β_i^{N+1} reduces to β_i^N modulo $J \cap \mathfrak{m}^N$. Thus, taking the limit over N, we can define elements $\beta_1, \ldots, \beta_r \in J$ which reduce to β_i^N for all N. On the other hand, we know that $A \to A/J$ induces an isomorphism on tangent spaces. Thus, $A/\mathfrak{m}J + (\beta_1, \ldots, \beta_r) + \mathfrak{m}^N \to A/J + \mathfrak{m}^N$ induces an isomorphism on tangent spaces. By the construction of the hull [66], $A/J + \mathfrak{m}^N$ is universal for maps from Artinian k-algebras with maximal ideal annihilated by $\mathfrak{m}^N$ to F. Thus the lift $A/J \to A/\mathfrak{m}J + \mathfrak{m}^N$ induces a splitting of the surjection $A/\mathfrak{m}J + (\{\beta_i\}) + \mathfrak{m}^N \to A/J + \mathfrak{m}^N$. Since this surjection is an isomorphism on tangent spaces, we see that $J + \mathfrak{m}^N = \mathfrak{m}J + (\{\beta_i\}) + \mathfrak{m}^N$. By Nakayama's lemma, we conclude that $(\beta_1, \ldots, \beta_r)$ generates $J/\mathfrak{m}^N \cap J$ for all $N \geq 2$. Since A is complete, we see that $J = (\beta_1, \ldots, \beta_r)$. □

We will systematically use the phrase "miniversal deformation space" for what was called by Schlessinger "hull" and originally called "versal deformation space" by Artin. The term "versal deformation" seems now to denote a more general smooth map to a stack without any hypotheses on the tangent spaces.

Proposition 5.2.1.4. *Suppose $\mathscr{F}$ is a semistable $\mathscr{X}$-twisted sheaf of rank n, geometric Hilbert polynomial P, and determinant L. Given an algebraic stack $\mathscr{M}$ containing $\mathscr{F}$ as a point, write $\dim_{\mathscr{F}} \mathscr{M}$ for the dimension of the miniversal deformation space of $\mathscr{F}$.*

(i) $\mathrm{ext}^1(\mathscr{F},\mathscr{F}) \geq \dim_{\mathscr{F}} \mathbf{Tw}^{ss}_{\mathscr{X}/k}(n,P) \geq \mathrm{ext}^1(\mathscr{F},\mathscr{F}) - \mathrm{ext}^2(\mathscr{F},\mathscr{F})$;

(ii) $\mathrm{ext}^1(\mathscr{F},\mathscr{F})_0 \geq \dim_{\mathscr{F}} \mathbf{Tw}^{ss}_{\mathscr{X}/k}(n,L,P) \geq \mathrm{ext}^1(\mathscr{F},\mathscr{F})_0 - \mathrm{ext}^2(\mathscr{F},\mathscr{F})_0$.

In both cases, the moduli stack is a local complete intersection at $\mathscr{F}$ if the lower bound is achieved and formally smooth at $\mathscr{F}$ if and only if the upper bound is achieved.

Proof. This is an application of the results of 2.2.5 and 4.2.1 along with 5.2.1.3. □

Definition 5.2.1.5. Given a semistable twisted sheaf of rank n, geometric Hilbert polynomial P, and determinant L, the *expected dimension* of $\mathbf{Tw}^{ss}(n,L,P)$ at $\mathscr{F}$, denoted $\mathrm{expdim}_{\mathscr{F}} \mathbf{Tw}^{ss}(n,L,P)$, is the quantity

$$\mathrm{expdim}_{\mathscr{F}} \mathbf{Tw}^{s}(n,L,P) := \mathrm{ext}^1(\mathscr{F},\mathscr{F})_0 - \mathrm{ext}^2(\mathscr{F},\mathscr{F})_0.$$

Lemma 5.2.1.6. *The expected dimension at stable points is independent of the choice of $\mathscr{F} \in \mathbf{Tw}^s(n,L,P)$ and is equal to $\Delta(\mathscr{F}) - (n^2-1)\chi(\mathscr{O}_X)$. The expected dimension jumps at properly semistable points. There is a constant β_∞ such that for all points $\mathscr{F} \in \mathbf{Tw}^s_{\mathscr{X}/k}(n,L,P)_k$,*

$$\mathrm{expdim}\, \mathbf{Tw}^s(n,L,P) \leq \dim_{\mathscr{F}} \mathbf{Tw}^s(n,L,P) \leq \mathrm{expdim}\, \mathbf{Tw}^s(n,L,P) + \beta_\infty.$$

Proof. The formula for the expected dimension follows from the identity

$$-\mathrm{hom}(\mathscr{F},\mathscr{F})_0 + \mathrm{ext}^1(\mathscr{F},\mathscr{F})_0 - \mathrm{ext}^2(\mathscr{F},\mathscr{F})_0 = \chi(\mathscr{O}_X) - \sum_{i=0}^{2}(-1)^i\,\mathrm{ext}^i(\mathscr{F},\mathscr{F})$$

and formal calculations. One uses the fact that stable sheaves $\mathscr{F}$ are simple ($\mathrm{End}(\mathscr{F}) = k$), which immediately implies that $\mathrm{Hom}(\mathscr{F},\mathscr{F})_0 = 0$, and the rest follows. (The given identity also uses the trace map splitting 4.2.1.4 and thus requires that the rank of $\mathscr{F}$ be relatively prime to the characteristic of X.) Details of this type of calculation may be found in [40, 4.5, 6.1, 8.3]. Since $\Delta(\mathscr{F})$ is determined by the determinant and Hilbert polynomial, we see that this is independent of the stable twisted sheaf $\mathscr{F}$.

The jumping of the expected dimension at properly semistable points comes from the fact that $\mathrm{Hom}(\mathscr{F},\mathscr{F})_0$ need not be zero. The identity above shows that

$$\mathrm{expdim}_{\mathscr{F}} \mathbf{Tw}^{ss}_{\mathscr{X}/k}(n,L,P) - \mathrm{Hom}(\mathscr{F},\mathscr{F})_0$$

is constant, so the expected dimension jumps whenever there are traceless endomorphisms (i.e., infinitesimal automorphisms acting trivially on the determinant).

The last inequality follows immediately from the fact that there is a constant β_∞ such that for all semistable twisted sheaves of rank r with fixed discriminant (and no restrictions on Chern classes if $\mathrm{char}\, X = 0$), $\mathrm{ext}^2(\mathscr{F},\mathscr{F})_0 \leq \beta_\infty$. In characteristic 0, this follows easily (using the methods of section 5.2.3) from the Le Potier-Simpson estimate and the fact that the endomorphism sheaf of a semistable sheaf is semistable [40, 4.5.7], a fact which does not hold in positive characteristic. In general, this is slightly subtle (whence the restriction on the discriminant, which is not present in characteristic 0) and will be proven in 5.2.3.8 below. □

5.2.2 Preparation for restriction theorems

Let X be a smooth projective surface over an algebraically closed field k and $P \to X$ a Brauer-Severi variety of relative dimension n. Note that the Brauer class of P is split by P, hence by any subscheme of P. Choose a projective embedding of P. Let $D \subset X$ be a smooth divisor. We start with a lemma about generic hyperplane sections of a Brauer-Severi scheme, which is essentially a refinement of a special case of a lemma of Vistoli and Kresch [47].

Lemma 5.2.2.1. *Let $P \to X$ be a surjective map of smooth projective varieties with fibers of equidimension n which is generically smooth over D. Let $P \hookrightarrow \mathbf{P}^N$ be a closed immersion. A generic hyperplane section P_H of P has the following properties: P_H is smooth and irreducible, $P_H \times_X D \subset P_H$ is an irreducible smooth divisor, $P_H \to X$ is surjective and generically smooth over D with fibers of equidimension $n-1$.*

Proof. Let Ξ be the projective space parametrizing hyperplane sections of P. The smoothness of the hyperplane section of P and its intersection with the pre-image of D defines an open subset $U \subset \Xi$. Let $d \in D(k)$ be a smooth point, with smooth fiber $P_d \subset P$. The condition that a hyperplane $H \in \Xi$ intersect P_d in a smooth variety of dimension $n-1$ defines an open subset V in Ξ. Let $W = U \cap V$. We claim that the hyperplane sections parametrized by W have the properties of the lemma. Indeed, if $H \in W$, then P_H and $P_H \times_X D$ are smooth and irreducible since $W \subset U$. Furthermore, the fiber of $P_H \to X$ over d is smooth of dimension $n-1$ (and hence also irreducible, incidentally) since $H \in V$. We claim that this forces $P_H \to X$ be be surjective, generically smooth, with equidimensional fibers. Indeed, we have $\dim P - \dim X = n$, hence $\dim P_H - \dim X = n-1$. If $\operatorname{im} P_H = I$, then the usual inequalities [54, §15] show that $\dim P_H - \dim I \leq n-1$ (as $\dim(P_H)_d = n-1$). Thus, $I = X$ and $P_H \to X$ is surjective. Applying the identity once more shows that any closed fiber has dimension at least $n-1$ at any closed point. Thus, every closed fiber is equidimensional of dimension $n-1$. □

Lemma 5.2.2.2. *Let $f : C \to \operatorname{Spec} K$ be a normal curve over a field. If $S \subset C$ is a closed subscheme which is finite étale over K and f is smooth along $C \setminus S$, then f is smooth.*

Proof. The scheme C is Noetherian and reduced. Thus, to show that the sheaf $\Omega^1_{C/K}$ is locally free of rank 1, it suffices to show that for every point $P \in C$, the $\kappa(P)$-vector space $\Omega^1_{C/K} \otimes_C \kappa(P)$ is 1-dimensional. For points $P \in C \setminus S$, this holds by assumption. On the other hand, given a point $Q \in S$, there is a canonical sequence

$$\mathfrak{m}_Q/\mathfrak{m}_Q^2 \to \Omega^1_{C/K} \otimes \kappa(Q) \to \Omega^1_{\kappa(Q)/K} \to 0.$$

Since Q is a Weil divisor on a normal separated scheme, it is a Cartier divisor and therefore the left-most term is 1-dimensional over $\kappa(Q)$. Since $\kappa(Q)$ is separable over K, the right-most term vanishes. □

Lemma 5.2.2.3. *Suppose Y is a smooth surface over an algebraically closed field k and $D, D' \in |\mathscr{O}(1)|$ are very ample divisors such that D is at worst nodal and D and*

D' intersect transversely. Then the general member of the pencil spanned by D and D' is smooth.

Proof. Write $\widetilde{Y} \to \mathbf{P}^1$ for the total space of the pencil. The non-smooth locus of $\widetilde{Y} \to \mathbf{P}^1$ has the property that it is unramified over $\mathbf{P}^1$ at $[D]$. Indeed, the fiber over $[D]$ is a nodal curve, so this follows from the standard construction of the scheme structure on the non-smooth locus using Fitting ideals [18, 2.21]. (Really, this just comes down to showing that the relative differentials of a node are supported precisely on the node with length 1.) Thus, all components of the non-smooth locus which intersect the generic fiber must be generically étale over $\mathbf{P}^1$. This implies that any non-smooth points of the generic fiber have separable residue fields over $k(\mathbf{P}^1)$. The result follows by 5.2.2.2.

An alternative (well-known) argument (rather than exploit the scheme structure of the non-smooth locus) comes from the versal deformation space of a node. Completing $\widetilde{Y}$ with respect to the uniformizing parameter of $[D_n]$ at one of the nodes over $[D_n]$ yields an effective formal deformation of the node over $k[\![t]\!]$ with the property that the total space is regular. On the other hand, the versal deformation of a node is isomorphic to $k[\![\xi, X_0, X_1]\!]/(X_0X_1 - \xi)$ parametrized by $k[\![\xi]\!]$. (In other words, given any family of curves $\mathscr{C} \to S$ with a node c in a closed fiber $\mathscr{C}_s$, there is a map $k[\![\xi]\!] \to \widehat{\mathscr{O}}_{S,s}$ such that $\widehat{\mathscr{O}}_{\mathscr{C},c} = \widehat{\mathscr{O}}_{S,s} \widehat{\otimes}_{k[\![\xi]\!]} k[\![\xi, X_0, X_1]\!]/(X_0X_1 - \xi)$.) Thus, there is some map $k[\![\xi]\!] \to k[\![t]\!]$ giving rise to $\widehat{\widetilde{Y}}$, and the condition of regularity forces ξ to map to ut where u is a unit of $k[\![t]\!]$. This shows that the generic fiber is smooth in the generizations of the node. (Indeed, the compatibility properties of Ω^1 allow us to assume that the base is $k[\![t]\!]$. Now the map from $\widetilde{Y}_{\text{node}}$ to its completion is regular as $\widetilde{Y}$ is excellent. Thus, the map from the generic fiber of $\widetilde{Y}_{\text{node}}$ to the generic fiber of the completion is regular. But given a regular map $A \to B$ of Noetherian rings over a field, it follows that A is geometrically over the field if and only if B is geometrically regular over the field. This applies to our situation to show that $\widetilde{Y}_{\text{node}}$ is smooth over $k((t))$.) □

Proposition 5.2.2.4. *There exists a smooth subvariety $Y \subset P$ which is finite flat generically étale over X such that for every n, the pullback of a general member of $|\mathscr{O}_X(n)|$ to Y is smooth.*

Proof. By 5.2.2.1 and induction, we may carry this out for $n = 1$. (Indeed, once a single smooth member pulls back to a smooth divisor, it will hold for a general smooth member. This follows from a consideration of the pullback of the incidence correspondence for $\mathscr{O}(1)$ on X to Y and the standard results about generization of smoothness in a flat family.) Let $f : Y \to X$ be the restriction of the projection $P \to X$. We will show that once it holds for $n = 1$, it holds for all n. Indeed, once it holds for $n = 1$, there is a dense open in $|\mathscr{O}_X(1)|$ of smooth members whose preimages in Y are smooth. Given n, we may choose n such general members which intersect transversely away from the branch curve of f. Call such a resulting nodal divisor D_n. Choose $D'_n \in |\mathscr{O}_X(n)|$ which is at worst nodal and intersects D_n transversely away from the branch curve. Then the pencil generated by $D_n \times_X Y$ and $D'_n \times_X Y$

satisfies the conditions of 5.2.2.3, hence has smooth general member. (So does the pencil generated by D_n and D'_n on X.) □

5.2.2.5. It is likely that the cover produced by 5.2.2.4 is not ideal, in the sense that the degree of the map $Y \to X$ is far too large. (We can know this "abstractly" because the proof of 5.2.2.4 is so easy.) In fact, if $P \to X$ is a Brauer-Severi variety of relative dimension $d-1$ representing a Brauer class of index d (defined below), the lowest degree for the map $Y \to X$ arising in 5.2.2.4 will be d^{d-1}. On the other hand, we know by results of Artin and de Jong [10, §8.1] that there will be a finite flat surjection $Y' \to X$ from a smooth surface to X of degree d. Here is a sketch of their method, paying slightly more attention to the pullback of an ample divisor. We will also use (a weaker form of) this below in our study of period-index phenomena.

Proposition 5.2.2.6. *Let $f : Y \to X$ be a finite map of smooth surfaces. Let $B \subset X$ be the (reduced) branch curve and $R \subset Y$ the ramification curve with its reduced structure. If the restriction $R \to B$ is generically unramified over every component of B, then the general member of $|\mathscr{O}_X(1)|$ has smooth preimage in Y for any very ample invertible sheaf $\mathscr{O}_X(1)$.*

Proof. By the openness of smoothness in flat families, it is enough to prove the following: let $D, D' \in |\mathscr{O}(1)|$ be smooth members intersecting transversely in $X \setminus B$ such that D intersects B transversely (in the smooth locus). Then the general member of the pencil spanned by D and D' has smooth preimage in Y. To prove this, consider the total space $\widetilde{X}$ of the pencil $\langle D, D' \rangle$. There is a natural diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \downarrow & & \\ \mathbf{P}^1 & & \end{array}$$

realizing $\widetilde{X}$ as a blow-up of X along the base points of the pencil. Since $D \pitchfork B$ and $D \cap D' \cap B = \emptyset$, the preimage $\widetilde{B}$ of B in $\widetilde{X}$ is isomorphic to B and is generically unramified over $\mathbf{P}^1$. Thus, the algebra $k(\mathbf{P}^1) \subset k(B)$ is finite separable. (Here we use $k(B)$ to denote the generic algebra of B, which is the product of the function fields of the components of B.) We can pullback the diagram by the map $f : Y \to X$ to yield the total space $\widetilde{Y}$ of the pullback pencil $f^{-1}\langle D, D' \rangle$. Since $D \cap D' \subset X \setminus B$, we see that $\widetilde{Y} \to Y$ is the blow-up at a finite set of reduced points not on R (namely the preimages of the base locus under f). Thus, the preimage of R in $\widetilde{Y}$ is isomorphic to R. Since $R \to B$ is separable, we see that $k(\mathbf{P}^1) \subset k(R)$ is finite separable. (Indeed, $\operatorname{Spec} k(R) \to \operatorname{Spec} k(B)$ is finite étale by the assumption that $R \to B$ is generically unramified over each component of B.) We may now apply 5.2.2.2. □

We now review the construction of Artin and de Jong, using 5.2.2.6 to keep track of the preimages of divisors. Let X be a smooth surface and $\mathscr{A}$ an Azumaya algebra of degree d on X. Let $C \subset X$ be a very ample curve. Given any section of $\mathscr{A}$, we can associate to it a monic characteristic polynomial of degree d. This gives a map over X

$$\chi : \mathscr{A} \to \mathbf{A}^d \times X$$

from $\mathscr{A}$ to the constant bundle on X with fiber the space $\mathbf{A}^d$ of monic polynomials of degree d.

The space $\mathbf{A}^d$, interpreted as the space of polynomials, comes with a finite cover $Z \to \mathbf{A}^d$, with Z a closed subscheme of $\mathbf{A}^d[t]$, such that the fiber of Z over $[f]$ is the locus of zeroes of the polynomial f. This situation may be explicitly realized as follows: let $Z = \mathbf{A}^d$ and let the map $\mathbf{A}^d \to \mathbf{A}^d$ send $(a_1, \ldots, a_d)$ to the tuple whose ith coordinate is $(-1)^i \sigma_i(a_1, \ldots, a_d)$, where σ_i is the ith elementary function of the a_i. In other words, the map sends $(a_1, \ldots, a_n)$ to the coefficients of the polynomial $(x - a_1) \cdots (x - a_d)$. The resulting map $\pi : \mathbf{A}^d \to \mathbf{A}^d$ is just the quotient map for the natural permutation action of Σ_d on $\mathbf{A}^d$.

Lemma 5.2.2.7. *With the above notation, the ramification divisor R is the "multi-diagonal" – the divisor where at least two coordinates agree. The branch divisor B is the discriminant hypersurface. The map $R \to B$ is generically unramified.*

Proof. This is mostly straightforward. The statement about ramification follows from the fact that the map $\mathbf{A}^2 \to \mathbf{A}^2$ sending (a, b) to $(a + b, ab)$ kills precisely the tangent direction $\partial/\partial a - \partial/\partial b$ (in any characteristic). □

Let L be an (ample) invertible sheaf on X such that $\mathscr{A} \otimes L$ and $\mathscr{A}_C \otimes L_C$ are generated by global sections and the map $\mathrm{H}^0(X, \mathscr{A} \otimes L) \to \mathrm{H}^0(C, \mathscr{A}_C \otimes L_C)$ is surjective. The map χ gives rise to

$$\chi \otimes L : \mathscr{A} \otimes L \to P_d(L) := L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes d}.$$

There is a map $\pi : L^{\oplus d} \to P_d(L)$ representing the zero locus of a monic polynomial section of L of degree d. Pulling back by a section σ of $\mathscr{A} \otimes L$ yields a finite map $Z_\sigma \to X$.

Proposition 5.2.2.8 (Artin, de Jong). *For a general section $\sigma \in \mathrm{H}^0(X, \mathscr{A} \otimes L)$, the scheme Z_σ is a smooth surface of degree d over X, and the map $Z_\sigma \to X$ has the property that the ramification curve is generically unramified over every component of the branch curve.*

Proof. For smoothness, see [10, 8.1.11]. The statement about the ramification is proven as follows: by the hypothesis on L, a general section over X restricts to a general section over C. In particular, a Bertini-like analysis shows that we may assume that $Z_\sigma|_C \to C$ has everywhere tame ramification of order at most 2. (In other words, the characteristic polynomial of the section at all points in C has at most two roots coming together.) Thus, the characteristic polynomial of σ over the generic point on each component of the branch curve for $Z_\sigma \to X$ must have precisely 2 roots coming together (as C is ample). As 5.2.2.7 shows, this means that $R \to B$ is generically unramified over each component of B. □

Corollary 5.2.2.9. *Given a smooth surface X and a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$, if there is a locally free $\mathscr{X}$-twisted sheaf of rank d then there is a finite flat surjection of smooth surfaces $Y \to X$ of degree d such that*

1. *there exists an invertible $\mathscr{X} \times_X Y$-twisted sheaf, and*
2. *for every very ample invertible sheaf $\mathscr{O}(1)$ on X, a general member has smooth preimage in Y.*

Remark 5.2.2.10. The method of 5.2.2.1 and 5.2.2.4 seems likely to generalize to higher dimensional varieties X. The only difficulty in the argument is in ensuring that general members of $\mathscr{O}(n)$ have smooth preimages once it is true for $n = 1$. For the applications envisioned, it is in fact sufficient that such divisors have normal preimages, which may be easier to arrange. In either case, it seems likely that a similar (more subtle) analysis of the behavior of a pencil with a fiber consisting of a divisor with sufficiently transverse crossings will yield a geometrically normal generic fiber, which is enough for applications. In other words, there would result a finite flat cover $Y \to X$ by a smooth variety such that the general member of $\mathscr{O}(n)$ has normal integral preimage.

On the other hand, the method of Artin and de Jong seems harder to generalize directly, because it appears possible that the zero loci Z_σ can acquire singularities in codimension 3. Nevertheless, if one is willing to let Y be normal as well, it seems possible that a refinement of their method could yield a finite flat covering with a better degree and all of the properties necessary to carry out analogues of our proofs below. This of course has the advantage of yielding a better numerical answer, hence more effective bounds, but at the present time it is not clear if having a non-smooth cover Y is compatible with the methods used here. We leave this investigation to future work.

5.2.3 Restriction theorems and the Bogomolov inequality

Classically, Mehta and Ramanathan proved that the restriction of a slope-semistable sheaf to a general sufficiently ample divisor is again slope-semistable. An effective version (which specifies what "sufficiently" means) was first proven in characteristic 0 by Bogomolov; a recent paper of Langer [51] gives a much more general statement, valid in all characteristics. Using Langer's results, we will give twisted versions of these theorems in this section. One of the (future) uses of these theorems is to construct the Uhlenbeck compactification of the space of twisted sheaves (and then, hopefully, the space of PGL_n-bundles). We also use the work of Langer to provide a twisted Bogomolov inequality, recovering earlier work of Artin and de Jong [10, §7.2] in the context of Azumaya algebras. Throughout, X is a smooth projective surface over an algebraically closed field k.

Definition 5.2.3.1. Given a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$, the *index* of $\mathscr{X}$, denoted $\mathrm{ind}(\mathscr{X})$ is the minimal rank of a locally free $\mathscr{X}$-twisted sheaf. The *period* of $\mathscr{X}$, denoted $\mathrm{per}(\mathscr{X})$, is the order of the image of $[\mathscr{X}]$ in $\mathrm{Br}(X)$. The gerbe $\mathscr{X}$ is *optimal* if the period is n.

As we will see in section 6, one has $\mathrm{per}(\mathscr{X}) | \mathrm{ind}(\mathscr{X})$ and $\mathrm{ind}(\mathscr{X}) | \mathrm{per}(\mathscr{X})^m$ for some m. When X is a surface (over an algebraically closed field), a theorem of de Jong (which we will re-prove below) shows that $\mathrm{per}(\mathscr{X}) = \mathrm{ind}(\mathscr{X})$. Thus, *on a surface*, the index of a $\boldsymbol{\mu}_n$-gerbe divides n. It is easy to show that the rank of any locally free $\mathscr{X}$-twisted sheaf is divisible by $\mathrm{ind}(\mathscr{X})$.

5.2.3.2. We first study restriction theorems. Fix a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$.

Lemma 5.2.3.3. *Let $f : Y \to X$ be a finite separable morphism of smooth surfaces. A torsion free coherent twisted sheaf $\mathscr{F}$ on X is μ-semistable if and only if $f^*\mathscr{F}$ is μ-semistable.*

Proof. This may be found in [40, 3.2.2]. □

Lemma 5.2.3.4. *Let $f : Y \to X$ be a finite flat map of smooth surfaces of degree d, $\mathscr{O}_X(1)$ a very ample invertible sheaf on X, $\mathscr{X} \to X$ a $\boldsymbol{\mu}_n$-gerbe, $n \in k^\times$. Write $\mathscr{Y} = \mathscr{X} \times_X Y$. The diagram*

$$\begin{array}{ccc} A^2(\mathscr{X})_{\mathbf{Q}} & \xrightarrow{f^*} & A^2(\mathscr{Y})_{\mathbf{Q}} \\ {\scriptstyle \deg}\downarrow & & \downarrow{\scriptstyle \deg} \\ \mathbf{Q} & \xrightarrow{d} & \mathbf{Q} \end{array}$$

commutes. In particular, given a torsion free $\mathscr{X}$-twisted sheaf $\mathscr{F}$, one has

$$\mu_{f^*\mathscr{O}_X(1)}(f^*\mathscr{F}) = d\mu_{\mathscr{O}_X(1)}(\mathscr{F})$$

and $\Delta(f^*\mathscr{F}) = d\Delta(\mathscr{F})$.

Proof. It suffices to show that the similar diagram with X and Y in place of $\mathscr{X}$ and $\mathscr{Y}$ commutes. That can be seen easily on the level of 0-cycles. □

By 5.2.2.9, we may fix a finite map $f : Y \to X$ of smooth surfaces of degree $d = \mathrm{ind}(\mathscr{X})$ with the property that a general member of *any* very ample linear system on X has smooth preimage in Y, and such that there is an invertible twisted sheaf $\mathscr{L}$ on Y. Fix a very ample linear system $\mathscr{O}_X(1)$ on X, with associated divisor class H. Following Langer [51], we choose a nef divisor A on Y such that $\mathscr{T}_Y(A)$ is globally generated, and we set

$$\beta_r = \left(\frac{r(r-1)}{p-1} AH \right)^2,$$

where we assume that $\mathrm{char}\, X = p$. This depends upon A, and it is slightly unfortunate that this fact is not recorded in the notation. (When $\mathrm{char}\, X = 0$, set $\beta_r = 0$.) Our method has the perverse consequence that effective restriction theorems are easier to prove than generic restriction theorems.

Proposition 5.2.3.5 (Twisted Langer). *Let $\mathscr{E}$ be a torsion free $\mathscr{X}$-twisted sheaf of rank r. Let $D \in |kH|$ be a smooth divisor such that $\mathscr{E}_D$ is torsion free and $D \times_X Y$ is smooth.*

(i) If $\mathscr{E}$ is μ-stable and

$$k > \frac{r-1}{r}\operatorname{ind}(\mathscr{X})\Delta(\mathscr{E}) + \frac{1}{\operatorname{ind}(\mathscr{X})\deg_H(X)(r-1)} + \frac{(r-1)\beta_r}{\operatorname{ind}(\mathscr{X})\deg_H(X)}$$

then $\mathscr{E}_D$ is μ-stable.

(ii) If $\mathscr{E}$ is μ-semistable and all of the Jordan-Hölder factors of $\mathscr{E}$ have torsion free restrictions to D, and the inequality of (i) holds, then $\mathscr{E}_D$ is μ-semistable.

Proof. After twisting by $\mathscr{L}^\vee$, the pullback of $\mathscr{E}$ to Y is naturally identified with a torsion free coherent untwisted sheaf $\mathscr{F}$, satisfying the stability conditions of (i) or (ii). Furthermore, $\Delta(\mathscr{F}) = \operatorname{ind}(\mathscr{X})\Delta(\mathscr{E})$ and $\deg_{f^*H}(Y) = \operatorname{ind}(\mathscr{X})\deg_H(X)$. The inequalities reduce to those of Langer's effective restriction theorems [51, 5.2 and 5.4], whence $\mathscr{F}_D$ is (i) μ-stable or (ii) μ-semistable. Applying 5.2.3.3, we see that $\mathscr{E}_D$ is (i) μ-stable or (ii) μ-semistable, as required. □

Remark 5.2.3.6. It is irritating to have to pay attention to $D \times_X Y$, as this makes the result quite a bit less effective. One might be tempted to see 5.2.3.5 (as we have proven it) as an "effective generic restriction theorem," as the integer k is effectively bounded, whereas by 5.2.2.9 we know that a general member of $|kH|$ will have smooth preimage in Y. This state of affairs is the unfortunate consequence of what is a fundamentally unsatisfying proof. The right way to proceed would be to analyze twisted sheaves, their behavior under Frobenius, and the properties of connections on them, as Langer has done in the untwisted case. This is made especially difficult by the inability to use sheaf cohomology.

Corollary 5.2.3.7 (Twisted Mehta-Ramanathan). *If $\mathscr{F}$ is a torsion free μ-semistable $\mathscr{X}$-twisted sheaf then the restriction of $\mathscr{F}$ to a general sufficiently ample curve $C \subset X$ is μ-semistable.*

Proof. This is immediate from 5.2.3.5 and the properties of preimages of divisors ensured by 5.2.2.9 (or 5.2.2.4, which will just change the estimates in 5.2.3.5). □

As promised in 5.2.1.6, we prove the existence of the universal constant β_∞ such that $\operatorname{ext}^2(\mathscr{F},\mathscr{F})_0 \leq \beta_\infty$ for all μ-semistable $\mathscr{F}$ with rank n and fixed discriminant Δ. The notation grates slightly with the notation β_r of this section, but we have chosen to retain the notation of both Huybrechts and Lehn (β_∞) and Langer (β_r). In future sections, we will not return to the restriction theorems, so β_r will vanish, which makes this annoyance temporary.

Lemma 5.2.3.8. *There exists a constant β_∞ depending only on $X, \mathscr{X}, Y, n, H$ and Δ such that for any μ-semistable twisted sheaf $\mathscr{F}$ of rank n and discriminant Δ, one has* $\operatorname{ext}^2(\mathscr{F},\mathscr{F})_0 \leq \beta_\infty$.

Proof. It suffices to prove this after pulling back to Y. (Indeed, by the obvious twisted Serre duality, one can see that $\operatorname{ext}^2(\mathscr{F},\mathscr{F})_0 = \hom(\mathscr{F},\mathscr{F}\otimes\omega_X)_0$. Furthermore, $f^*\omega)_X \hookrightarrow \omega_Y$, so

$$\operatorname{Hom}(\mathscr{F},\mathscr{F}\otimes\omega_X)_0 \leq \operatorname{Hom}_Y(\mathscr{F}_Y,\mathscr{F}_Y\otimes f^*\omega_X)_0 \leq \operatorname{Hom}_Y(\mathscr{F}_Y,\mathscr{F}_Y\otimes\omega_Y)_0$$

and we may apply Serre duality again on Y.) Thus, we may assume that $\mathscr{F}$ is a semistable untwisted sheaf. We can then suppress Y from the notation; the dependence of β_∞ on Y only comes in the form of a β_r in the formula. Pushing the formulas given here back down to X will result in multiplying each $\deg_H(X)$ and each $\Delta(\mathscr{F})$ by $\operatorname{ind}(\mathscr{X})$.

In general, we have $\Delta(\mathscr{E}nd(\mathscr{F})) \leq 2n^2\Delta(\mathscr{F})$. Indeed, $\mathscr{F}$ injects into its reflexive hull $\mathscr{F}^{\vee\vee}$, yielding an injection $\mathscr{E}nd(\mathscr{F}) \hookrightarrow \mathscr{E}nd(\mathscr{F}^{\vee\vee})$. It is not hard to see that

$$\ell(\mathscr{E}nd(\mathscr{F}^{\vee\vee})/\mathscr{E}nd(\mathscr{F})) \leq n\ell(\mathscr{F}^{\vee\vee}/\mathscr{F}). \tag{5.1}$$

On the other hand [40, 3.4.1], we have

$$\Delta(\mathscr{F}) = \Delta(\mathscr{F}^{\vee\vee}) + 2n\ell(\mathscr{F}^{\vee\vee}/\mathscr{F}) \tag{5.2}$$

and similarly for $\mathscr{E}nd(\mathscr{F})$. Combining (5.1) with (5.2) for $\mathscr{F}$ and for $\mathscr{E}nd(\mathscr{F})$ shows that $\Delta(\mathscr{E}nd(\mathscr{F})) \leq 2n^2\Delta(\mathscr{F})$. Now a theorem of Langer [51, 5.1] combined with the inequality $\Delta(\mathscr{E}nd(\mathscr{F})) \leq 2r^2\Delta(\mathscr{F})$ and the fact that $\mu(\mathscr{E}nd(\mathscr{F})) = 0$ shows that

$$\mu_{\max}(\mathscr{E}nd(\mathscr{F})) \leq 2n\deg_H(X)\Delta(\mathscr{F}) + \beta_r.$$

Another theorem of Langer [50, 3.3] says (in the case of surfaces) that for any torsion free sheaf of rank n on X,

$$h^0(X,E) \leq n\deg_H(X)\binom{\frac{\mu_{\max}(E)}{\deg_H(X)} + f(n) + 2}{2},$$

where $f(n) = -1 + \sum_{i=1}^{n} 1/i$. Combining this with the estimate for $\mu_{\max}(\mathscr{E}nd(\mathscr{F}))$ yields a bound for $\operatorname{Hom}(\mathscr{F},\mathscr{F})$. Similarly, we get a bound for $\operatorname{Hom}(\mathscr{F},\mathscr{F}\otimes\omega_X)$ which differs from the first by a constant depending only on X. By Serre duality,

$$\operatorname{ext}^2(\mathscr{F},\mathscr{F})_0 = \hom(\mathscr{F},\mathscr{F}\otimes\omega_X) - \mathrm{h}^0(\omega_X),$$

so we are done. □

Remark 5.2.3.9. Note that bounding the discriminant does not suffice to bound the Hilbert polynomial when the determinant is not fixed. Thus, 5.2.3.8 is non-trivial. Of course, when working with a fixed determinant and therefore a bounded set of sheaves, some constant β_∞ will exist by virtue of the boundedness and the usual semicontinuity theorems for Ext sheaves. In characteristic 0 (or for strongly semistable sheaves in general, which we will briefly describe below), the dependence upon the discriminant disappears; it is not clear to me whether or not this should still be true in positive characteristic.

5.2.3.10. We can also use the work of Langer and the coverings of 5.2.2.9 to produce a version of the Bogomolov inequality for twisted sheaves. After defining a notion of Frobenius pullback and strict semistability for twisted sheaves, we can use these methods to recover a Bogomolov-like inequality first proven by Artin and de

Jong in the context of Azumaya algebras. This inequality will be important at one point during the study of asymptotic properties of the moduli spaces.

We begin by defining a Frobenius map which is appropriate for our situation. First, note that the (absolute) Frobenius can be defined for stacks of characteristic p. If $\mathscr{S} \to S$ is such a stack (with $\operatorname{char}(S) = \{p\}$), which we may assume split as a fibered category, then the Frobenius 1-morphism $F_{\mathscr{S}} : \mathscr{S} \to \mathscr{S}$ sends a 1-morphism $T \to \mathscr{S}$ to the composition $T \xrightarrow{F_T} T \longrightarrow \mathscr{S}$ (and fixes all morphisms in fiber categories).

Lemma 5.2.3.11. *If $\mathscr{X} \to X$ is any stack and $\chi : \mathscr{I}(\mathscr{X}) \to \mathbf{G}_m$ is any character, then the Frobenius map $F_{\mathscr{X}}$ pulls back χ-twisted sheaves to p-fold χ-twisted sheaves. In particular, if $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe, then the Frobenius map $F_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}$ pulls back $\mathscr{X}$-twisted sheaves to p-fold twisted sheaves.*

Proof. Note that the map on the site of $\mathscr{X}$ induced by the Frobenius is the identity. In particular, there is a natural isomorphism

$$F_{\mathscr{X}}^*(\mathscr{I}(\mathscr{X})) \xrightarrow{\sim} \mathscr{I}(\mathscr{X})$$

(as this is true for any sheaf). It is not hard to see that the composition

$$\mathscr{I} \xrightarrow{\sim} F^*\mathscr{I} \xrightarrow{\sim} \mathscr{I}$$

is equal to the identity, where the left-hand map in the composition is the natural map 2.1.1.9. Under this identity, given any sheaf $\mathscr{F}$ on $\mathscr{X}$, the action $\mathscr{F} \times \mathscr{I} \to \mathscr{F}$ pulls back under $F_{\mathscr{X}}$ to be the *same* action $\mathscr{F} \times \mathscr{I} \to \mathscr{F}$. On the other hand, given any $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{M}$, the $\mathscr{O}$-structure on $F^*\mathscr{M}$ is given by $\mathscr{M} \otimes_{\mathscr{O}, F_{\mathscr{O}}} \mathscr{O}$, with the map $F_{\mathscr{O}} : \mathscr{O} \to \mathscr{O}$ given by sending a section s to s^p. Thus, if a section of $\mathscr{I}$ acts by χ on $\mathscr{M}$, when pulled back it acts by χ^p.

The second sentence is just a restatement of the first one for readers who cleverly skipped section 2.1! □

Definition 5.2.3.12. Let ℓ be the order of p in $(\mathbf{Z}/n\mathbf{Z})^\times$. The power $F_{\mathscr{X}}^{\ell}$ is called the *twisted Frobenius* of $\mathscr{X}$, denoted $F_{\mathscr{X},\tau}$. The resulting map

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{F_{\mathscr{X}}^{\ell}} & \mathscr{X} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

is an isomorphism of $\boldsymbol{\mu}_n$-gerbes which pulls back twisted sheaves to twisted sheaves.

Definition 5.2.3.13. An $\mathscr{X}$-twisted sheaf $\mathscr{F}$ is *strictly (μ-) semistable* if $(F_{\mathscr{X},\tau}^q)^*\mathscr{F}$ is (μ-) semistable for all $q \geq 0$.

As with untwisted sheaves, it is the strictly μ-semistable twisted sheaves which have the best properties.

Proposition 5.2.3.14 (Twisted Langer-Bogomolov Inequality). *Let $\mathscr{E}$ be a torsion free $\mathscr{X}$-twisted sheaf, $Y \to X$ a cover as in 5.2.2.9 and β_r as in 5.2.3.5.*

(i) If $\mathscr{E}$ is μ-semistable then $\operatorname{ind}(\mathscr{X})^2\Delta(\mathscr{E}) + \beta_r \geq 0$.

(ii) If $\mathscr{E}$ is strongly μ-semistable then $\Delta(\mathscr{E}) \geq 0$

(iii) If $\operatorname{rk}(\mathscr{E}) = \operatorname{ind}(\mathscr{X})$ *then* $\Delta(\mathscr{E}) \geq 0$.

Proof. Parts (i) and (ii) follow immediately from Langer's version of the Bogomolov inequality [51, 3.2] (which is our statement if $\operatorname{ind}(\mathscr{X}) = 1$) and 5.2.3.3. Part (iii) follows from the fact that if $\operatorname{rk}(\mathscr{E}) = \operatorname{ind}(\mathscr{X})$, then $\mathscr{E}$ has no proper torsion free submodules of strictly smaller rank, so $\mathscr{E}$ is μ-stable. Thus, since the rank of $\mathscr{E}$ is unchanged by Frobenius pullback, $\mathscr{E}$ is strongly μ-stable and we may apply (ii). □

Remark 5.2.3.15. The fact that n is prime to the characteristic figures essentially into part (iii). We see from (i) that in general there is still a lower bound for the second Chern class of any Azumaya algebra of class $[\mathscr{X}]$, depending only upon $\mathscr{X}$ (and possibly the choice of covering $Y \to X$). ♦

Corollary 5.2.3.16 (Artin, de Jong [10, 7.2.1]). *Let X be a smooth projective surface with function field K, and let A be an Azumaya algebra over X such that A_K is a division ring of degree prime to the characteristic. Then $c_2(A) \geq 0$.*

Proof. Let $\deg A = d$. There is a $\boldsymbol{\mu}_d$-gerbe $\mathscr{X} \to X$ and a locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$ of rank d and trivial determinant such that $\mathscr{E}nd(\mathscr{V}) = A$. It is easy to see that $c_2(A) = 2rc_2(\mathscr{V})$, so we are done by 5.2.3.14(iii). □

Remark 5.2.3.17. Artin and de Jong's original proof of 5.2.3.16 is not very difficult, but in their approach positive characteristic and characteristic 0 are treated in completely different ways. Our method "explains" what is going on in a characteristic free manner. They must also bound the second Chern class from below by a different method before showing it is 0, while both things happen at once in our approach (which also applies to more general Azumaya algebras with possibly non-division generic points). Finally, our proof gives a reason for the failure of 5.2.3.16 when the characteristic divides the degree, namely the failure of strict stability of $\mathscr{V}$. We feel that this is another demonstration of the usefulness of working with twisted sheaves (and thus thinking sheaf-theoretically). ♦

5.2.4 Asymptotic properties for optimal classes

In this section we study the behavior of $\mathbf{Tw}^{ss}_{\mathscr{X}/k}(n, L, c_2)$ as $\Delta \to \infty$. We will always work with spaces of twisted sheaves with a fixed determinant. Due to inadequacies in the classical theory of semistable sheaves on surfaces in positive characteristic (currently being ameliorated by Langer), we only prove these theorems in the optimal case in all characteristics. For the arithmetic applications of the next section, this is the only case that we will need.

The approach is essentially that of O'Grady, as presented by Huybrechts and Lehn in [40, Chapter 9]. The biggest difference between the approach here and their approach is 5.2.4.23, which is an alternative ending step in the proof of asymptotic irreducibility. The idea behind this proof was worked out jointly with Johan de Jong. Other than this, the rest of the proof is essentially identical to the classical proof. In the optimal case, certain better numerical estimates can be made, which we present here. Otherwise, we quote the book of [40] for certain proofs. While they were written in an untwisted context, they carry over verbatim (as indicated) to the twisted (arbitrary characteristic) context. I believe (but have not checked) that in the non-optimal characteristic 0 case, one can carry out a similar transcription of the classical proofs. However, I have avoided dealing with e-stability and related numerical estimates in this work, so the reader should take this belief with a grain of salt. It is likely that the current characteristic-free work of Langer (alluded to in [50]) will prove amenable to a twisted transcription.

Throughout this section, $\mathscr{X} \to X$ is an optimal $\boldsymbol{\mu}_n$-gerbe with n prime to the characteristic of the base field k. Thus, any rank n torsion free twisted sheaf will be μ-stable. We will continue to use the notation $\mathbf{Tw}^{ss}$, even though in this case there are equalities $\mathbf{Tw}(n, L, P) = \mathbf{Tw}^{s}(n, L, P) = \mathbf{Tw}^{ss}(n, L, P)$. Furthermore, all of these stacks are Deligne-Mumford and are gerbes over their moduli spaces. We are therefore free to conflate their closed substacks and closed subspaces of their coarse moduli spaces; in particular, the dimension theory does not change.

We write $\mathbf{Tw}$ for $\mathbf{Tw}_{\mathscr{X}/k}$, etc. We will also use the notation $\mathbf{Tw}(n, L, c)$, where $c = c_2$, rather than $\mathbf{Tw}(n, L, P)$, where P is the geometric Hilbert polynomial. By the Riemann-Roch theorem, these are equivalent sets of data. Finally, as we will always work with fixed rank and determinant, we will write $\mathbf{Tw}^{ss}(\Delta)$ for $\mathbf{Tw}^{ss}(n, L, c)$, where Δ is the discriminant.

5.2.4.1. We first outline the asymptotic properties and their proofs. The statements will be proven in 5.2.4.12 below.

Definition 5.2.4.2. The closed subspace in $\mathbf{Tw}^{ss}(n, L, c)$ parametrizing non-locally free twisted sheaves is the *boundary*, denoted $\partial\mathbf{Tw}^{ss}(n, L, c)$.

For any map $T \to \mathbf{Tw}^{ss}(n, L, c)$ corresponding to a family of twisted sheaves on $T \times X$, the preimage of $\partial\mathbf{Tw}^{ss}$ in T equals a closed subspace ∂T, which we will also call the boundary of T.

Definition 5.2.4.3. A (μ-stable) point $\mathscr{F} \in \mathbf{Tw}^{ss}$ is *good* if $\mathscr{F}$ is locally free and $\operatorname{ext}^2(\mathscr{F}, \mathscr{F})_0 = 0$.

(We include the μ-stability so that the reader is aware of the general definition.) In general, we will write $\beta(\mathscr{F}) = \operatorname{ext}^2(\mathscr{F}, \mathscr{F})_0$ and $\beta(Z) = \max\{\beta(\mathscr{F}) | \mathscr{F} \in Z\}$ for a substack $Z \subset \mathbf{Tw}^{ss}(\Delta)$. The good locus is the vanishing set for β.

Lemma 5.2.4.4. *There is an open substack of good points $\mathbf{Tw}^{ss}_g(\Delta) \subset \mathbf{Tw}^{ss}(\Delta)$ which is smooth over k with smooth moduli space.*

Proof. The openness follows from the semicontinuity properties of higher Exts (see [12] and 2.2.5.17(3) for an example of the method involved in the twisted case). Smoothness of the stack is well known and comes from 5.2.1.3 (which shows that the universal deformation space of a point is formally smooth). Smoothness of the moduli space follows from the fact that $\mathbf{Tw}^s(\Delta) \to \mathrm{Tw}^s(\Delta)$ is a $\boldsymbol{\mu}_n$-gerbe. □

The asymptotic properties of $\mathbf{Tw}^{ss}(\Delta)$ come from an analysis of the substacks $\partial\mathbf{Tw}^{ss}(\Delta)$ and $\mathbf{Tw}^{ss}_g(\Delta)$. We can first show that sufficiently large irreducible closed substack of $\mathbf{Tw}^{ss}(\Delta)$ must intersect $\partial\mathbf{Tw}^{ss}(\Delta)$.

Proposition 5.2.4.5. *There are constants A_1, C_1, and C_2 such that if $\Delta \geq A_1$ and if Z is an irreducible closed substack of $\mathbf{Tw}^{ss}(\Delta)$ such that*

$$\dim Z > \left(1 - \frac{1}{n+2}\right)\Delta + C_1\sqrt{\Delta} + C_2$$

then $\partial Z \neq \emptyset$.

Using 5.2.4.5, we will then show that as Δ grows, so does the codimension of the complement of $\mathbf{Tw}^{ss}_g(\Delta)$. More precisely, we have the following. Let $W = \mathbf{Tw}^{ss}(\Delta) \setminus \mathbf{Tw}^{ss}_g(\Delta)$ (as a reduced closed substack).

Proposition 5.2.4.6. *There is a constant $C_3 \geq C_2$ and a constant $A_2 \geq A_1$ such that for all $\Delta \geq A_2$,*

$$\dim W \leq \left(1 - \frac{1}{2n}\right)\Delta + C_1\sqrt{\Delta} + C_3.$$

Thus, the stack will asymptotically become generically smooth and everywhere l.c.i. of the expected dimension, hence normal.

Proposition 5.2.4.7. *Suppose Δ satisfies*

(1) $\Delta > A_1$

(2) $\Delta - (n^2-1)\chi(\mathscr{O}_X) \geq \left(1 - \frac{1}{2n}\right)\Delta + C_1\sqrt{\Delta} + C_3 + 2.$

Then every irreducible component of $\mathbf{Tw}^{ss}(\Delta)$ intersects $\mathbf{Tw}^{ss}_g(\Delta)$. In particular, it is generically smooth of the expected dimension. Furthermore, $\mathbf{Tw}^s(\Delta)$ is normal and a local complete intersection.

Proof. The two properties and the fact that $\operatorname{expdim}\mathbf{Tw}^{ss}(\Delta) = \Delta - (n^2-1)\chi(\mathscr{O}_X)$ (at any point, hence on any irreducible component) shows that the locus of good points $\mathbf{Tw}^{ss}_g(\Delta)$ is dense in every component of $\mathbf{Tw}^{ss}(\Delta)$. When $\operatorname{ext}^2(\mathscr{F},\mathscr{F})_0 = 0$, one then has

$$\dim \mathbf{Tw}^{ss}_g(\Delta) = \operatorname{ext}^1(\mathscr{F},\mathscr{F})_0 = \operatorname{expdim}\mathbf{Tw}^{ss}_g(\Delta),$$

so the stack $\mathbf{Tw}^{ss}(\Delta)$ is generically smooth of the expected in every irreducible component, hence at every point. This implies by 5.2.1.3 that $\mathbf{Tw}^s(\Delta)$ is a local complete intersection. Furthermore, by condition (2) and 5.2.4.6, $\mathbf{Tw}^{ss}(\Delta)$ is regular in codimension 1. By Serre's theorem, $\mathbf{Tw}^{ss}(\Delta)$ is normal. □

Another use of 5.2.4.5 is in proving that $\mathbf{Tw}^{ss}(\Delta)$ is irreducible for sufficiently large Δ. Suppose $\mathscr{F} \in \mathbf{Tw}^{ss}(\Delta)$ is good. This implies that $\mathscr{F}$ lies on a unique irreducible component of $\mathbf{Tw}^{ss}(\Delta)$. Any subsheaf $\mathscr{F}' \subset \mathscr{F}$ of finite colength ℓ (i.e., such that the quotient $\mathscr{F}/\mathscr{F}'$ has finite length ℓ) must also be good. Indeed, by Serre duality (which carries over to the twisted category) and compatibility with trace, $\mathrm{ext}^2(\mathscr{F}', \mathscr{F}')_0 = \mathrm{hom}(\mathscr{F}', \mathscr{F}' \otimes \omega_X)_0$, and similarly for $\mathscr{F}$. Furthermore, taking the reflexive hull gives a natural injection $\mathrm{Hom}(\mathscr{F}', \mathscr{F}' \otimes \omega_X)_0 \hookrightarrow \mathrm{Hom}(\mathscr{F}, \mathscr{F} \otimes \omega_X)_0$.

Lemma 5.2.4.8. $\Delta(\mathscr{F}') = \Delta(\mathscr{F}) + 2n\ell$.

Proof. This reduces to showing that $c_2(\mathscr{F}/\mathscr{F}') = \ell$, which itself reduces to showing that a twisted sheaf $\mathscr{Q}$ of length 1 has $c_2(\mathscr{Q}) = 1$. This follows from the twisted Hirzebruch-Riemann-Roch theorem 4.1.3.6 (or from the classical Hirzebruch-Riemann-Roch theorem after applying a Morita equivalence $\mathscr{H}om(\mathscr{V}, \cdot\,)$ for some twisted progenerator with trivial determinant, which is why we did not need to feel guilty about not including the proof of 4.1.3.6!) applied to the inclusion of $\mathrm{Supp}(\mathscr{Q})$ in $\mathscr{X}$, along with a trivial calculation when $\mathscr{X}$ is a $\boldsymbol{\mu}_n$-gerbe over a geometric point. □

Thus, $\mathscr{F}'$ lies on a unique irreducible component of $\mathbf{Tw}^{ss}(\Delta + 2n\ell)$. It is trivial that every locally free twisted sheaf $\mathscr{F}$ contains a colength 1 subsheaf $\mathscr{F}_1$. Let Λ_Δ denote the set of irreducible components of $\mathbf{Tw}^{ss}(\Delta)$.

Lemma 5.2.4.9. *Suppose Δ satisfies the conditions of 5.2.4.7. Then map sending a good twisted sheaf $\mathscr{F}$ to $\mathscr{F}_1$ yields a well-defined map $\varphi : \Lambda_\Delta \to \Lambda_{\Delta+2n}$.*

Proof. We will show that for a locally free twisted sheaf, the space of quotients of fixed finite length is irreducible 5.2.4.16. Thus, the irreducible component containing $\mathscr{F}_1$ is independent of the choice of $\mathscr{F}_1$. □

The idea behind the proof of irreducibility of $\mathbf{Tw}^{ss}(\Delta)$ for large Δ is to show that φ is eventually surjective, and that any two points are eventually brought together under an iterate of φ.

Proposition 5.2.4.10. *There is a constant A_3 such that for all $\Delta \geq A_3$, the following hold.*

1. *(1) Every irreducible component of $\mathbf{Tw}^{ss}(\Delta)$ contains a locally free good twisted sheaf.*
2. *(2) Every irreducible component of $\mathbf{Tw}^{ss}(\Delta)$ contains a point $\mathscr{F}$ such that both $\mathscr{F}$ and $\mathscr{F}^{\vee\vee}$ are good and $\ell(\mathscr{F}^{\vee\vee}/\mathscr{F}) = 1$.*

Theorem 5.2.4.11. *There is a constant A_4 so that for all $\Delta \geq A_4$, the stack $\mathbf{Tw}^{ss}(\Delta)$ is irreducible.*

Proof. By 5.2.4.10(2), for $\Delta \geq A_3$ the map $\varphi : \Lambda_{\Delta-2n} \to \Lambda_\Delta$ is surjective. We wish to show that this implies that Λ_Δ is eventually a singleton. In the twisted case, there is a slight wrinkle, as c_2 need not be an integer. Thus, not all discriminants are congruent modulo $2n$. However, we do know that Δ is always an integer. Consider the sequences of surjections

$$\Lambda_\Delta \longrightarrow \Lambda_{\Delta+2n} \longrightarrow \Lambda_{\Delta+4n} \longrightarrow \cdots$$

$$\Lambda_{\Delta+1} \longrightarrow \Lambda_{\Delta+1+2n} \longrightarrow \Lambda_{\Delta+1+4n} \longrightarrow \cdots$$

$$\vdots$$

$$\Lambda_{\Delta+2n-1} \longrightarrow \Lambda_{\Delta+2n-1+2n} \longrightarrow \Lambda_{\Delta+2n-1+4n} \longrightarrow \cdots .$$

For any sufficiently large discriminant Δ', one of the sequences above will contain $\Lambda_{\Delta'}$. If we show that any two components in the first set of the sequence eventually map to the same point, then we see that each sequence is eventually singletons, and hence that any $\Lambda_{\Delta'}$ is eventually a singleton (for large enough Δ').

We claim that it is enough to show that given locally free $\mathscr{V}$ and $\mathscr{W}$ of rank n with the same determinant and discriminant, there are finite colength subsheaves $\mathscr{V}' \subset \mathscr{V}$ and $\mathscr{W}' \subset \mathscr{W}$ and an irreducible flat family containing both $\mathscr{V}'$ and $\mathscr{W}'$. This is not obviously the same as making colength 1 subsheaves of locally free good sheaves in each stage. To see that these are the same, note that the irreducibility of the twisted Quot scheme shows that we may assume that the supports of $\mathscr{V}/\mathscr{V}'$ and $\mathscr{W}/\mathscr{W}'$ are finite sets of distinct reduced points. Now suppose given a family of twisted sheaves $\mathscr{F}$ on $X \times S$. The S-scheme of quotients of $\mathscr{F}$ of length ℓ with supports distinct reduced points disjoint from the singular locus of $\mathscr{F}$ in each fiber is easily seen to be irreducible when S is irreducible (see e.g., the proof of 5.2.4.16 below). Thus, if S is irreducible, so is this scheme of quotients. So as we let a point move in it, it will end up in the same irreducible component of $\mathbf{Tw}^{ss}(\Delta + 2n)$.

We will prove the existence of $\mathscr{V}'$ and $\mathscr{W}'$ below in 5.2.4.20. ☐

5.2.4.12. We now prove everything! First comes 5.2.4.5.

Lemma 5.2.4.13. *Let $C \in |\mathscr{O}(N)|$ be a smooth member (for any N) and let $\mathscr{C} = \mathscr{X} \times_X C$. Let $Z \subset \mathbf{Tw}^{ss}(\Delta)$ be a closed irreducible substack with $\partial Z = \emptyset$. If $\dim Z > \dim \mathbf{Tw}^{ss}_{\mathscr{C}/k}(n, \mathscr{Q}_C)$ then there is a point of Z parametrizing an $\mathscr{X}$-twisted sheaf $\mathscr{F}$ whose restriction to C is unstable.*

Proof. By 5.1.5.2, we see that if it is defined the restriction map $Z \to \mathbf{Tw}^{ss}_{\mathscr{C}/k}(n, \mathscr{Q}_C)$ is finite. Thus, if every restriction of a point of Z to C is stable, we see that $\dim Z \leq \dim \mathbf{Tw}^{ss}_{\mathscr{C}/k}(n, \mathscr{Q}_C)$. ☐

Proposition 5.2.4.14. *Let $Z \subset \mathbf{Tw}^{ss}(\Delta)$ be a closed irreducible substack. Let $C \in |\mathscr{O}(N)|$ be smooth. Suppose Z contains a point $[\mathscr{F}]$ such that $\mathscr{F}_C$ is unstable. If*

$$\dim Z > \operatorname{expdim} \mathbf{Tw}^{ss}(\Delta) + \beta_\infty + \frac{n^2}{4} - \frac{n-1}{2} C(C-K)$$

then $\partial Z \neq \emptyset$.

Proof. This may be copied almost verbatim from [40, 9.5.4], but omit the part about e-stability. □

Proof of 5.2.4.5. This is an application of 5.2.4.13 and 5.2.4.14. Indeed, these show that if Z is an irreducible component such that

$$\dim Z > \dim \mathbf{Tw}^{ss}_{\mathscr{C}/k}(n, L_C) = \frac{n^2-1}{2}(N^2H^2 + NKH)$$

and

$$\dim Z > \Delta - (n^2-1)\chi(\mathscr{O}_X) + \beta_\infty + \frac{n^2}{4} - \frac{n-1}{2}C(C-K)$$

then $\partial Z \neq \emptyset$. We seek a function of Δ which is greater than both right-hand sides for large Δ (and some choice of N) but which is smaller than $\Delta - (n^2-1)\chi(\mathscr{O}_X)$ by an amount which grows without bound as Δ increases. (The second condition becomes necessary when trying to make the codimension of W high.) For the purposes of the present work, we do not make any attempt to be especially effective; this will make things easier. Letting $N \sim c\sqrt{\Delta}$ and examining the resulting inequalities for that value of N leads one to choose c with

$$c^2 < \frac{2}{(n+1)(n-1)H^2}$$

to ensure that the "leading term" (coefficient of Δ) of the top line is larger than that of the bottom line and less than Δ. As we let Δ grow, this will eventually produce positive integers for N, and working through the arithmetic shows that there will be a function $f(\Delta) = C_1\sqrt{\Delta} + C_2$ such that for $N \sim C_1\Delta$, the inequalities are satisfied and $f(\Delta) < \Delta - (n^2-1)\chi(\mathscr{O}_X)$. Then any Z with $\dim Z > f(Z)$ will satisfy both 5.2.4.13 and 5.2.4.14 and have dimension strictly smaller than the expected dimension of $\mathbf{Tw}^{ss}(\Delta)$. (This step is harder to write down than it is to understand. A reader confused by the poor writing of this paragraph is encouraged to think this through for a few moments.) For a similar argument, see [40, pp. 209-210]. □

5.2.4.15. Next come 5.2.4.6 and 5.2.4.10. We begin with some preparatory lemmas.

Lemma 5.2.4.16. *If* $\mathscr{E}$ *is a locally free* $\mathscr{X}$*-twisted sheaf of rank* r*, then* $\mathrm{Quot}(\mathscr{E}, \ell)$ *is irreducible of dimension* $\ell(r+1)$

Proof. We use the highly non-trivial fact that this is true when $\mathscr{X}$ is trivial [40, 6.A.1]. First, note that there is an open subspace of the Quot corresponding to length ℓ quotients which are just ℓ distinct points. This open subspace is isomorphic to an étale $(\mathbf{P}^{r-1})^\ell$-bundle over $\mathrm{Sym}^\ell(X) \setminus \Delta$, where Δ is the multidiagonal, hence is irreducible (and has the right dimension). It is thus enough to show that the entire Quot is the closure of this open, which is the same as showing that any quotient may be deformed into a quotient with reduced support. Let $\mathscr{E} \to \mathscr{Q}$ be any quotient of length ℓ. Write the support (with its natural scheme structure) of $\mathscr{Q}$ as Z (which will be the preimage of a closed subscheme of X). The quotient map is the same as a quotient $\mathscr{E}_Z \to \mathscr{Q}$. Since Z is a scheme of finite length over an algebraically closed field, we have $\mathrm{Br}(Z) = 0$. Let $\mathscr{L}$ be a twisted invertible sheaf on Z; any two invertible twisted sheaves are in fact mutually isomorphic. Twisting down by $\mathscr{L}$, we see that $\mathscr{E}_Z \to \mathscr{Q}$ is the same thing as a surjection $\mathscr{O}_Z^r \to Q$. (In other words, $\mathscr{E} \otimes \mathscr{L}^\vee \cong \mathscr{O}^r$.) By the irreducibility of $\mathrm{Quot}(\mathscr{O}^r, \ell)$, we know that there is a complete discrete valuation ring R containing k and a flat family of quotients $\mathscr{O}^r_{X\otimes R} \to \widetilde{Q}$ on $X \otimes R$ whose special fiber is $\mathscr{O}^r \to Q$. The support S of $\widetilde{Q}$ will be finite over R, and hence will be strictly Henselian. Thus, $\mathrm{Br}(S) = 0$, and we may choose an invertible twisted sheaf $\widetilde{\mathscr{L}}$ on S (for the pullback of $\mathscr{X}$ to $X \otimes R$). Since S is semilocal, it follows that $\widetilde{\mathscr{L}}^r \cong (\mathscr{E} \otimes R)_S$. Thus, twisting the quotient $\widetilde{Q}$ by $\widetilde{\mathscr{L}}$, we find an effective deformation of $\mathscr{Q}$ into a quotient with reduced support. □

Lemma 5.2.4.17. *If* $\partial\mathbf{Tw}^{ss}(\Delta) \neq \emptyset$ *then* $\mathrm{codim}(\partial\mathbf{Tw}^{ss}(\Delta), \mathbf{Tw}^{ss}(\Delta)) \leq n-1$.

Proof. The statement is local on the stack. Locally on $\mathbf{Tw}^{ss}$, one may choose a locally free resolution of the universal object on $\mathbf{Tw}^{ss}(\Delta) \times X$ by two sheaves $\varphi : L_1 \to L_0 \to \mathscr{F}_{\mathrm{univ}}$ (as surfaces have homological dimension 2). The result follows from studying the locus where the rank of φ drops, which is known from standard theorems about determinant schemes. See [40, 9.2.2] for more details. Note that while the reference given for determinantal loci is written over $\mathbf{C}$, the estimates are independent of the characteristic. □

We need one more lemma, which is well known. Given a torsion free twisted sheaf $\mathscr{F}$, there is a reflexive hull $\mathscr{F}^{\vee\vee} := \mathscr{H}om(\mathscr{H}om(\mathscr{F}, \mathscr{O}), \mathscr{O})$. As usual, there is an injection $\mathscr{F} \hookrightarrow \mathscr{F}^{\vee\vee}$ such that the cokernel T has finite length.

Lemma 5.2.4.18. *If $\mathscr{F}$ is an S-flat family of torsion free twisted sheaves then the function $s \mapsto \ell(\mathscr{F}_s^{\vee\vee}/\mathscr{F}_s)$ is upper semicontinuous. If S is reduced and the function is constant than the formation of the reflexive hull commutes with base change and $\mathscr{F}^{\vee\vee}$ is locally free.*

Proof. See e.g. [40, 9.6.1]. One uses the fact that a surface has homological dimension 2 and that there are locally free resolutions (which is true in the twisted setting as well). □

Definition 5.2.4.19. The *double-dual stratification* of $\mathbf{Tw}^{ss}(\Delta)$ is given by subsets

$$\mathbf{Tw}^{ss}(\Delta)_\nu = \{\mathscr{F} | \ell(\mathscr{F}^{\vee\vee}/\mathscr{F}) \geq \nu\}.$$

These are closed subsets by 5.2.4.18. For any family of torsion free twisted sheaves over S, there is an induced stratification S_ν by pullback along the classifying map $S \to \mathbf{Tw}^{ss}(\Delta)$.

The most important fact about this stratification is that formation of the double dual induces a map

$$(\partial\mathbf{Tw}^{ss}(\Delta)_\nu \setminus \partial\mathbf{Tw}^{ss}(\Delta)_{\nu+1})_{\mathrm{red}} \to \mathbf{Tw}^{ss}_{lf}(\Delta - 2n\ell).$$

The fiber over a (locally free) point $\mathscr{F}$ is just (set-theoretically, at least) $\operatorname{Quot}(\mathscr{F}, \ell)$. Let $Z \subset \mathbf{Tw}^{ss}(\Delta)$ be a closed irreducible subspace with $\partial Z \neq \emptyset$ and $\beta(Z) > 0$. Following Huybrechts and Lehn, we define a sequence of triples

$$Y_i \subset Z_i \subset \mathbf{Tw}^{ss}(\Delta_i)$$

as follows: $\Delta_0 = \Delta$, $Z_0 = Z$, and $Y_i \subset \partial Z_i$ is an irreducible component of the maximal open stratum of the double-dual stratification of ∂Z. If ℓ is the constant colength on this stratum, then, as we just remarked, there is an induced map $Y_i \to \mathbf{Tw}^{ss}(\Delta_i - 2n\ell)$. Set $\Delta_{i+1} = \Delta_i - 2n\ell$ and Z_{i+1} equal to the closure of the image of Y_i. There is some index m such that $\partial Z_m = \emptyset$ by the twisted Langer-Bogomolov inequality $\Delta \geq 0$ 5.2.3.14(iii) (which applies since $\operatorname{ind}(\mathscr{X}) = n$).

Using 5.2.4.16 and 5.2.4.17, one finds $\dim Z_i \geq \dim Y_{i-1} - \ell_i(n+1)$ and $\dim Y_{i-1} \geq \dim Z_{i-1} - (n-1)$, whence $\dim Z_i \geq \dim Z_{i-1} - (2n-1)\ell_i - 1$. A careful analysis of when equality can hold between $\dim Z_i$ and $\dim Z_{i-1} - (2n-1)\ell_i - 1$ (which may be found in [40, pp. 211-212]) yields an inequality

$$\dim Z_m - \left(1 - \frac{1}{2n}\right) \Delta_m \geq \dim Z - \left(1 - \frac{1}{2n}\right) \Delta - \beta_\infty.$$

It is now clear what is going to happen: if $\dim Z$ is too large, then $\dim Z_m$ is too large, i.e., satisfies 5.2.4.5, contradicting the fact that $\partial Z_m = \emptyset$. The numerical details may be found in [40, p. 212-213], where it is shown that

$$C_3 := \max\{C_2 + \beta_\infty, A_1/2n + 2\beta_\infty - (n^2-1)\chi(\mathscr{O}_X)\}$$

works in the statement of 5.2.4.6.

Finally, the proof of 5.2.4.10 may be copied verbatim from [40, p. 213].

5.2.4.20. As promised in 5.2.4.11, we show that given two (good) locally free twisted sheaves $\mathscr{V}$ and $\mathscr{W}$ with the same rank, determinant, and discriminant, there are finite colength subsheaves $\mathscr{V}' \subset \mathscr{V}$ and $\mathscr{W}' \subset \mathscr{W}$ which belong to a common irreducible family of (good) twisted sheaves.

Lemma 5.2.4.21. *A general map $\mathscr{V} \to \mathscr{W}(N)$ is injective with cokernel supported on a divisor where it has rank 1 in every fiber.*

Proof. This is a Bertini type theorem. Over any field, the space of $n \times n$-matrices which have rank at most $n-1$ is a divisor in $\mathrm{M}_n(k)$ with singular locus of codimension 3 (in the divisor) given by matrices of rank at most $n-2$. Thus, the cone of matrices of rank at most $n-2$ has codimension 4 in each fiber, and the usual argument (e.g., the simpler version of the argument found in [19] for 1-dimensional bases) shows that on a surface a generic section (for N large enough that $\mathscr{H}om(\mathscr{V}, \mathscr{W}(N))$ is globally generated) will avoid this locus. As the rank drops on a divisor, we are done. □

Corollary 5.2.4.22. *A general map $\mathscr{V} \to \mathscr{W}(N)$ is injective with cokernel an invertible twisted sheaf supported on a smooth curve in $|\det \mathscr{W}(nN) \otimes \det \mathscr{V}^{\otimes -n}|$.*

Proof. This involves a similar Bertini argument with the second jet bundle of a matrix algebra. At a point p with local coordinates x and y, an element of the fiber of this bundle is a matrix $M_0 + xM_1 + yM_2$. Taking the determinant yields a function $f_0+xf_1+yf_2$ (as $x^2 = y^2 = xy = 0$ in the jet bundle). In order for the determinant to vanish to order at least 2 at the point, all three functions f_i must vanish. This defines a "forbidden cone" of codimension 3 in every fiber (see [10, 8.1.1.6] for a verification that these conditions are independent), which is greater than the dimension of X. The usual argument shows that once the jet bundle is globally generated, a general section will miss the forbidden cone in each fiber. □

Proposition 5.2.4.23. *Let $\mathscr{V}$ and $\mathscr{W}$ be two locally free twisted sheaves of rank n with the same determinant and discriminant. Then there exist torsion free twisted sheaves and finite colength inclusions $\mathscr{V}' \subset \mathscr{V}$ and $\mathscr{W}' \subset \mathscr{W}$ (of the same colength) and an irreducible flat family of twisted sheaves containing $\mathscr{V}'$ and $\mathscr{W}'$. If $\mathscr{V}$ and $\mathscr{W}$ are both (μ-)(semi)stable, then there exists an irreducible family consisting of (good) (μ-)(semi)stable sheaves.*

Proof. For N sufficiently large, there are extensions

$$0 \to \mathscr{V}(-N) \to \mathscr{V} \to \mathscr{P} \to 0$$

and

$$0 \to \mathscr{V}(-N) \to \mathscr{W} \to \mathscr{Q} \to 0,$$

where $\mathscr{P}$ and $\mathscr{Q}$ are invertible twisted sheaves on smooth curves in the linear system $|\mathscr{O}(nN)|$. Furthermore, $\mathscr{P}$ and $\mathscr{Q}$ have the same geometric Hilbert polynomial. By 5.1.4.2, there is an irreducible variety S (which we may assume is affine) and an S-flat family of twisted sheaves $\mathscr{D}$ on $X \times S$ supported on an S-flat Cartier divisor which interpolates between $\mathscr{P}$ (the fiber over $s_0 \in S(k)$) and $\mathscr{Q}$ (the fiber over $s_1 \in S(k)$).

We will use the usual Grauert semicontinuity results for Ext spaces to make a connected family interpolating between finite colength subsheaves of $\mathscr{P}$ and $\mathscr{Q}$. We can do this explicitly quite easily as follows. Let $\mathscr{F}^\bullet \to \mathscr{D}$ be a finite resolution by a complex of locally free twisted sheaves. (In fact, it will have length at most 2!) Twisting $\mathscr{F}^\bullet$ by a very negative power of $\mathscr{O}(1)$, we see that the perfect complex $\mathfrak{C} = \mathrm{Hom}^\bullet(\mathscr{F}^\bullet(-m), \mathscr{V}(-N))$ on S universally computes relative Ext spaces. In other words, for any $T \to S$, $\mathrm{H}^i(\mathfrak{C} \otimes_S T) \cong \mathrm{Ext}^i_{X\times_S T}(\mathscr{D}(-m)X \times_S T, \mathscr{V}(-N)_{X\times_S T})$. Moreover, for large enough m, it is the case that the function

$$s \in S \mapsto \dim \mathrm{H}^0(\mathfrak{C} \otimes \kappa(s))$$

is constant and the function

$$s \in S \mapsto \dim \mathrm{H}^2(\mathfrak{C} \otimes \kappa(s))$$

is the 0 function (by Serre duality). Standard methods (see [38, III.12] for example) now show that $\mathrm{H}^1(\mathfrak{C})$ is a locally free sheaf and that for all $f : T \to S$ the natural map $f^* \mathrm{H}^1(\mathfrak{C}) \to \mathrm{H}^1(\mathfrak{C} \otimes_S T)$ is an isomorphism. Let $\mathbf{V} \to S$ be the vector bundle whose sections are $\mathrm{H}^1(\mathfrak{C})$. Then we have shown that $\mathbf{V}$ represents the functor $T \to S \mapsto \mathrm{Ext}^1(\mathscr{D}(-m), \mathscr{V}(-N))$ (for sufficiently large m). (In fact, we could have easily shown that $\mathrm{H}^2(\mathfrak{C} \otimes T)$ is universally 0 to begin with, by a trivial homological dimension calculation, but the method here generalizes slightly to higher dimensional ambient varieties.) As such, there is a universal extension

$$0 \to \mathscr{V}(-N)_{\mathbf{V}\times X} \to \mathscr{E} \to \mathscr{D}(-m)_{\mathbf{V}\times X} \to 0.$$

Furthermore, once the existence of a vector bundle representing $\mathrm{Ext}^1(\mathscr{D}(-m), \mathscr{V}(-N))$ is true for m, it will be true for all $m' > m$. Thus, to get $\mathbf{V}$ to have nice properties, we can keep enlarging m.

We claim that for sufficiently large m, given any $s \in S$ there is a non-empty open subset $U_S \subset \mathbf{V}_s$ parametrizing torsion free extensions. It is enough to prove that there is a single torsion free extension by the openness of purity in families. Furthermore, the existence of such a point is stable under increases of m: if $\mathscr{E}$ is torsion free element of $\mathrm{Ext}^1(\mathscr{D}(-m), \mathscr{V}(-N))$, then the preimage of $\mathscr{D}_s(-m-m_0)$ in $\mathscr{E}$ gives a torsion free element in $\mathrm{Ext}^1(\mathscr{D}_s(-m-m_0), \mathscr{V}(-N))$. Let s be a point of S, so that we are considering extensions $\mathrm{Ext}^1_X(\mathscr{D}_s(-m), \mathscr{V}(-N))$. We are reduced to proving that if m is large enough, there is a point of this space representing a torsion free twisted sheaf. Let $\mathscr{E}$ be any extension with torsion subsheaf $T(\mathscr{E})$. Since $\mathscr{V}(-N)$ is torsion free, the intersection $\mathscr{V}(-N) \cap T(\mathscr{E}) = 0$, so $T(\mathscr{E}) \hookrightarrow \mathscr{D}_s(-m)$. Now consider the situation generically. Over the local ring at the generic point of $\operatorname{Supp}\mathscr{D}$ there is certainly a torsion free extension, so over the complement $W := X \setminus D$ of some sufficiently ample hyperplane section $D \in |\mathscr{O}(m_0)|$ there is a torsion free extension

$$0 \to \mathscr{V}(-N)_W \to \mathscr{E}_W \to \mathscr{D}_s(-m)_W \to 0.$$

Since W is affine, this may be realized as a map $\varphi_W : \mathscr{F}^{-1}(-m)_W \to \mathscr{V}(-N)_W$ whose composition $\mathscr{F}^{-2}(-m)_W \to \mathscr{F}^{-1}(-m)_W \to \mathscr{V}(-N)_W$ is 0. Twisting by a high power c of D we find an extension $\varphi : \mathscr{F}^{-1}(-m-cm_0) \to \mathscr{V}(-N)$ of φ_W. It is immediate that φ satisfies the cocycle condition, hence gives rise to an extension $\mathscr{E}$ which restricts to $\mathscr{E}_W$ on W. Since W contains the generic point of C and $\mathscr{D}_s(-m-cm_0)$ is torsion free, the inclusion $T(\mathscr{E}) \hookrightarrow \mathscr{D}_s(-m-cm_0)$ implies that $T(\mathscr{E}) = 0$.

Therefore, by the openness of the torsion free locus and Noetherian induction, we may choose a large m so that the torsion free locus of every fiber of $\mathbf{V} \to S$ is open and dense. This implies that the locus $\mathbf{U} \subset \mathbf{V}$ parametrizing torsion free sheaves is irreducible.

Now consider the original points s_0 and s_1 over which lie $\mathscr{P}$ and $\mathscr{Q}$. Choosing a section of $\mathscr{O}(m)$, we find finite colength subsheaves $\mathscr{V}' \subset \mathscr{V}$ and $\mathscr{W}' \subset \mathscr{W}$ parametrized by points of $\mathbf{U}$, hence lying in an irreducible family of torsion free twisted sheaves. If $\mathscr{V}$ and $\mathscr{W}$ are (good) (μ-)(semi)stable, then the same is true of $\mathscr{V}'$ and $\mathscr{W}'$, and we are done by the openness of these loci in families and irreducibility. □

Remark 5.2.4.24. This is the key step to proving that the stack of semistable twisted sheaves is asymptotically irreducible for non-optimal classes as well. Our proof is sufficiently general to work in the general case. However, some of the other foundations (notably a study of e-stability) cannot be carried out in positive characteristic yet. The general characteristic 0 case is likely to work precisely as it does in the classical case, but we have not yet checked the details. ♦

5.2.4.25. We can give a relative version of all of the constructions here. Stack-theoretically, this extension is trivial. The GIT construction of Simpson also gives a good global projective corepresenting scheme (although in positive characteristic it is no longer clear whether or not this is universal on the base). In the case of an optimal class, all points will be stable, so $\mathbf{Tw}^{s}_{\mathscr{X}/S}$ is a gerbe over a projective scheme, which shows that in this case the formation of the coarse moduli space is universal on S. (Universal means "compatible with all base change." It is always true that the GIT quotient is *uniform*, which means that it is compatible with flat base change.)

For our purposes, we only remark on one aspect of this story.

Proposition 5.2.4.26. *Let $\mathscr{X} \to X \to S$ be a $\boldsymbol{\mu}_n$-gerbe on a smooth proper morphism over a regular Noetherian scheme with geometrically connected fibers of dimension 2, and assume that n is invertible on S. Suppose $\mathscr{X}$ has optimal geometric fibers. The stack $\mathbf{Tw}^{ss}_{\mathscr{X}/S}(\Delta) \to S$ is a proper flat local complete intersection morphism for large Δ.*

Proof. (This result can also be finagled when the fibers are either geometrically optimal or geometrically essentially trivial, and is likely to hold completely generally in characteristic 0 by a simple extension of our methods. As above, the general positive characteristic case is still in progress.) This follows from 5.2.4.7 which shows that $\mathbf{Tw}^{ss}_{\mathscr{X}/S}(\Delta)$ is Cohen-Macaulay, combined with the fact that the dimension of the fibers is constant. The usual criterion of flatness for maps from a Cohen-Macaulay scheme to a regular scheme [54, 2.3.1ff] then yields flatness. □

Question 5.2.4.27. Is 5.2.4.26 still true when S is no longer regular?

Chapter 6

Period-Index results

In this section we apply the above theory to study period-index phenomena for Brauer groups of function fields. Our results range from a new proof of the period-index theorem of de Jong for surfaces over algebraically closed fields to a proof of the period-index theorem for unramified classes on a geometrically connected smooth surface over a finite field. Before we embark on the details, we provide a brief introduction to the problem.

6.1 Background and definitions

The period-index problem has several manifestations. The modern way of describing the problem uses Galois cohomology. Let K be a field and F an étale sheaf on $\operatorname{Spec} K$. Let $\alpha \in \mathrm{H}^i(\operatorname{Spec} K, F)$ be a Galois cohomology class with $i > 0$.

Definition 6.1.0.28. The *period* of α, denoted $\operatorname{per}(\alpha)$, is the order of α in the group $\mathrm{H}^i(\operatorname{Spec} K, F)$. The *index* of α, denoted $\operatorname{ind}(\alpha)$, is $\gcd\{\deg L/K : \alpha|_L = 0\}$ with L a separable extension.

In certain instances, the index of α is actually the minimal degree of a field extension killing α (for example, if $i = 2$ and $F = \mathbf{G}_m$). In the context of elliptic curves, this was considered by Lang-Tate [48] and Lichtenbaum [53].

Lemma 6.1.0.29. *Given F, i, and α as above,* $\operatorname{per}(\alpha) | \operatorname{ind}(\alpha)$ *and both have the same prime factors.*

Proof. This is well-known, but we review the proof for the sake of completeness. That $\operatorname{per}(\alpha) | \operatorname{ind}(\alpha)$ is a consequence of the existence, for any finite separable extension $f : \operatorname{Spec} L \to \operatorname{Spec} K$, of a trace map $f_* f^* F \to F$ (called the "corestriction") such that the composition with the restriction is multiplication by $\deg L/K$. Applying this to L/K such that $\alpha_L = 0$ shows that the degree of any such extension kills α, whence the gcd kills α. To show that the prime factors are the same, suppose L/K kills α and let p be a prime number not dividing $\operatorname{per}(\alpha)$. We may suppose L is Galois with group G. Let G_p be a p-Sylow subgroup of G and L_p the fixed field. Thus, the degree of L_p over K has no factors of p. We claim that L_p kills α. Indeed, α_{L_p} is still per(α)-torsion and p still acts invertibly on the subgroup generated by α. Thus, restricting further to L and corestricting back to L_p, we see that α is trivial over L_p if and only if it is trivial over L. This removes p from ind(α). □

Thus, there is some minimal ℓ_α so that $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{\ell_\alpha}$.

Definition 6.1.0.30. With the preceding notation, ℓ_α is the *period-index factor* of α.

The period-index problem is to understand the period-index factor. In particular, one can study the situation for $F = \mathbf{G}_m$ and $i = 2$, where this problem exists in the guise of the Brauer group. To rephrase it, suppose $\alpha = [D]$, with D a central division algebra. In this case, the classical theory of division algebras shows that the index of α is the degree of D, while the period is the order of α in $\mathrm{Br}(K)$. Thus, the period-index problem for the Brauer group is about deciding how large a division algebra is necessary to represent a given Brauer class. In this form, it was considered by Brauer in his foundational paper and by Albert in subsequent work (see [48] for the reference).

Definition 6.1.0.31. A field K has *(unramified) period-index factor n (prime to p)* if every class α in the (unramified) Brauer group of K (of order prime to p) has period-index factor at most n.

Our goal throughout this section will be to demonstrate that certain function fields of low transcendence degree over constant fields of low homological dimension have low period-index factor. In general, one expects [16] that the period-index factor of a field should be related to the Galois cohomological dimension. Our results below show that for unramified classes, the global geometry of a variety can intervene to lower the expected period-index factor of classes over the function field.

Another natural source of period-index problems is the category of Abelian varieties. This was first considered by Tate and Lang, inspired by Brauer and Albert's work for division algebras. A basic result in this direction concerns elliptic surfaces. Let $\pi : E \to \mathbf{P}^1$ be an elliptic surface over a finite field (with a section σ). Then it is a classical result [69, Exercise 10.11] that given an element $\alpha \in Ш(k(\mathbf{P}^1), E_\eta)$ of period n prime to the characteristic of k, there is a field extension of degree n of $k(\mathbf{P}^1)$ killing α. In other words, for these particular classes of $\mathrm{H}^1(\mathrm{Spec}\, k(\mathbf{P}^1), E_\eta)$, the period equals the index. Using Artin's isomorphism $\mathrm{Br}(E) \xrightarrow{\sim} Ш(k(\mathbf{P}^1), E_\eta)$ (see section 5.1.3), we see that this inadvertently also proves that E has period-index factor 1. We will generalize this result to all surfaces over a finite field below using a different approach.

Using our methods, we can translate the period-index problem into a series of rationality questions on twists of well-known moduli spaces.

6.2 One method: reduction to a curve

By birationally fibering a variety of dimension d as a curve over a projective space, we can reduce the period-index problem to the study of rationality properties of varieties over $k(x_1, \ldots, x_{d-1})$. We sketch the method and give as an application a theorem originally proved by de Jong and refined by Starr.

6.2.1 Rationality questions

In this section, we transform the period-index problem into a rationality question on twists of the moduli spaces of stable sheaves.

Lemma 6.2.1.1. *Let X be a smooth geometrically connected projective variety over a field k which is either algebraically closed or finite. Given any n, there is an extension $k' \supset k$ of degree prime to n and a birational equivalence of $X \otimes k'$ with a fibration $\widetilde{X} \to \mathbf{P}^{d-1}_{k'}$ with smooth generic fiber of dimension 1 with a rational point over $k'(\mathbf{P}^{d-1})$.*

Proof. Suppose first k is algebraically closed and X is $\mathbf{P}^d$. Fixing a point p and taking the linear system of hyperplanes through it, one sees that the blowup $\mathrm{Bl}_p \mathbf{P}^d$ fibers of $\mathbf{P}^{d-1}$ with a section and with generic fiber $\mathbf{P}^1_{k(\mathbf{P}^{d-1})}$. Given any variety X over k of dimension d, we can apply Noether normalization to yield a finite generically étale map $\nu : X \to \mathbf{P}^d$. Choosing p to lie in the locus over which ν is étale, we can simply pull back the picture from $\mathbf{P}^n$. The space $\widetilde{X}$ will then be the blowup of the fiber, which is just a finite set of reduced points. Geometrically, this is the same as taking a general linear system of dimension $\mathbf{P}^{d-1}$ in any very ample complete linear system on X.

When k is finite, the description in terms of general linear subsystems shows that the space parametrizing good fibrations is open in a projective space. Such a set must contain a point rational over an extension of degree prime to n (by e.g., the Lang-Weil estimates [49]). □

Lemma 6.2.1.2. *Let $X \leftarrow \widetilde{X} \to \mathbf{P}^{d-1}$ be as in 6.2.1.1. Given a class $\alpha \in \mathrm{Br}(X)$, one has*

$$\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha_{\widetilde{X}}) = \mathrm{ind}(\alpha_{\widetilde{X}_\eta}),$$

where $\widetilde{X}_\eta$ is the generic fiber curve.

Proof. This follows from the fact that $\widetilde{X} \to X$ is birational and the index is determined at the generic point. □

By construction, $\widetilde{X}_\eta$ has a rational point p.

Proposition 6.2.1.3. *Let $\pi : \widetilde{X}_\eta \to \eta = \operatorname{Spec} K$ be a smooth curve over a field with a rational point p. Given $\alpha \in \mathrm{Br}(\widetilde{X}_\eta)[n]$, there is a lift $[\mathscr{C}] \in \mathrm{H}^2(\widetilde{X}_\eta, \boldsymbol{\mu}_n)$ whose projection in $\mathrm{H}^0(\eta, \mathbf{R}^2 f_* \boldsymbol{\mu}_n)$ is trivial. Suppose the projection of α in $\mathrm{H}^2(\eta, \mathbf{G}_m)$ is zero. The following are equivalent:*

(i) α has the period-index property;

(ii) there is a locally free $\mathscr{C}$-twisted sheaf of rank n and determinant of degree 1;

(iii) the twist of $\mathrm{Sh}^s(n,1)$ *determined by the image of* α *in* $\mathrm{H}^1(\eta, \mathrm{Pic}_{\widetilde{X}_\eta/\eta}[n]^0)$ *has a rational point, where* $\mathrm{Sh}^s(n,1)$ *is the space of stable sheaves of rank* n *and degree* 1.

If K *is* C_1 *then in (ii) we may assume the determinant is* $\mathscr{O}(p)$ *and in (iii) we may consider the twist of* $\mathrm{Sh}^s(n, \mathscr{O}(p))$.

Proof. Suppose there is a locally free twisted sheaf $\mathscr{F}$ of rank n on $\mathscr{C}$, and let $\mathscr{M} = \det \mathscr{F}$. The assumption that the projection of α into $\mathrm{H}^2(\eta, \mathbf{G}_m)$ vanishes says precisely that $\alpha|_p = 0 \in \mathrm{H}^2(\mathrm{Spec}\, \kappa(p), \mathbf{G}_m)$. Thus, there is a quotient of $\mathscr{F}|_p$ with any rank at most n. Taking an elementary transformation shows that we may assume that the determinant has degree 1. When K is C_1, we can apply this argument at any Cartier divisor on $\widetilde{X}_\eta$ and therefore ensure that the determinant is precisely $\mathscr{O}(p)$. Thus, (i) and (ii) are equivalent. By 5.1.2.3 and the existence of a tautological sheaf when the degree and rank are coprime and the class is trivial at p (5.1.2.3), (ii) is equivalent to (iii). □

6.2.2 An application: period and index on a geometric surface

We give a proof of de Jong's theorem on the period-index property for surfaces over algebraically closed fields. For the sake of simplicity (and lack of earlier definitions in this thesis), we focus on the case of unramified classes. The proof we give adapts verbatim (upon adding the phrase "intersects the ramification locus transversely") to the case of classes over the function field of order prime to the characteristic, which is de Jong's original version of the theorem. This restriction was subsequently removed by Starr. We believe that our methods can also be used to re-prove Starr's result, but we have not checked this completely.

Proposition 6.2.2.1. *Let* K *be a function field in one variable (i.e., a field of transcendence degree 1 over an algebraically closed field) and* $C \to \mathrm{Spec}\, K$ *a smooth proper curve over* K *with a rational point* p. *Given any cocycle* $\xi \in \mathrm{H}^1(\mathrm{Spec}\, K, \mathrm{Pic}^0_{C/K}[n])$ *and any invertible sheaf* $\mathscr{L} \in \mathrm{Pic}^1(C)$, *the twist of* $\mathrm{Sh}^{ss}(n, \mathscr{L})$ *induced by* ξ *has a* K*-rational point.*

Proof. This is just an application of the powerful theorem of de Jong and Starr [20] (generalizing to positive characteristic a result [32] of Graber-Harris-Starr) which finds rational points on strongly rationally connected varieties over function fields in one variable, once we note that $\mathrm{Sh}^s(n, \mathscr{O}(p))$ is a smooth rational variety. □

Theorem 6.2.2.2 (de Jong). *A field of transcendence degree* 2 *over an algebraically closed field* k *of characteristic* p *has the period-index property prime to* p.

Proof. Let X be a smooth projective surface modelling the given function field and let $\alpha \in \mathrm{Br}(k(X))[n]$ with n prime to $\mathrm{char}\, k$. The fibration of 6.2.1.1 is just the total space of a general pencil of hyperplane sections. (If D is the ramification curve of α, we may choose general hyperplane sections intersecting D transversely. It then follows that the pullback of α to $\widetilde{X}$ is unramified along the generic fiber [10, 3.1.4].) To finish the proof we simply apply 6.2.2.1 and 6.2.1.3. □

Remark 6.2.2.3. The result of Starr and de Jong used in the proof of 6.2.2.1 is itself quite non-trivial, so this is not truly a simplification of de Jong's proof. However, we feel that it is illuminating to put the problem in the general framework of a natural rationality question. This shows where the fact that X has dimension 2 is used: rationality questions are significantly easier over fields of transcendence degree 1. In the next section, we will see how a different problem about rational points can be used for surfaces over finite fields. There the fundamental necessity that $\dim X = 2$ arises from the fact that the moduli spaces of sheaves are quite well behaved, something which is highly unlikely in higher dimensions. ♦

6.3 Another method: using moduli on a surface

Another way to reduce the period-index problem to a rationality question is to use the known structure of the moduli spaces of twisted sheaves along with classical estimates on the existence of points (e.g., the Lang-Weil estimates for geometrically integral varieties over finite fields).

6.3.1 A cheap trick

We mention here a simple trick which can be used to make certain base field extensions. For the sake of generality, we slightly extend the definition of the index of a Brauer class on a scheme.

Definition 6.3.1.1. Given a scheme X, a Brauer class $\alpha \in \mathrm{H}^2(X, \mathbf{G}_m)$ and a point $p \in X$, the *local index* of α at p, denoted $\mathrm{ind}_p(\alpha)$, is the index of the restriction of α to $\operatorname{Spec} \kappa(p)$.

Proposition 6.3.1.2. *Let X be a scheme and $\alpha \in \mathrm{H}^2(X, \mathbf{G}_m)$. If $f : Y \to X$ is a locally free morphism of schemes of degree d and n is prime to d, then α has index dividing n at a point $p \in X$ if and only if $\alpha_{f^{-1}(p)}$ has index dividing n.*

Proof. This reduces to the case where X is the spectrum of a field κ. If there is a locally free α_Y-twisted sheaf on Y of rank n then pushing it forward to X yields a locally free α-twisted sheaf of rank nd. Taking endomorphisms yields a central simple κ-algebra A of degree nd with $[A] = \alpha$. If D is the (unique up to isomorphism) central division κ-algebra with class α, we may write $A = \mathrm{M}_r(D)$ for some r. Since $\mathrm{per}(\alpha)|n$, we see that $(\mathrm{ind}(\alpha), d) = 1$, so the degree of D is prime to d. Since $nd = r \deg D$, we see that $d|r$. Thus, $n = \ell \deg D$ for some ℓ, and we conclude that $\mathrm{M}_\ell(D)$ is a central simple algebra of degree n with class α. (When $\mathrm{per}(\alpha) = n$, we in fact conclude that $r = d$ and that $\deg(D) = n$.) □

Corollary 6.3.1.3. *If X is a k-scheme and $\alpha \in \mathrm{H}^2(X, \mathbf{G}_m)$ has period n, then* $\mathrm{ind}(\alpha) = n$ *if and only if there is a field extension $k' \supset k$ of degree prime to n such that* $\mathrm{ind}(\alpha_{k'})|n$.

6.3.2 Period and index on a surface over a finite field

In this section, we will prove the following theorem.

Theorem 6.3.2.1. *If X be a proper smooth geometrically connected surface over $\mathbf{F}_q$ then X has the period-index property prime to q.*

For classes of period divisible by the characteristic, the methods employed here cannot be applied (even though the moduli theory can be developed). Thus, (sadly) we can give no insight into the problem for these classes.

Due to the inadequacy of our proofs of asymptotic properties (e.g., applying only to geometrically optimal classes and geometrically essentially trivial classes), we are forced to be slightly clever. Let $PIP_n(k)$ denote the phrase "classes of period dividing n on any geometrically connected proper sooth surface over k have the period-index property."

Definition 6.3.2.2. A Brauer class $\alpha \in \mathrm{Br}(X)$ is *geometrically optimal* if $\mathrm{per}(\alpha) = \mathrm{per}(\alpha \otimes \overline{k})$.

This is the same as saying that given a (necessarily optimal) $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$ with Brauer class α, the gerbe $\mathscr{X} \otimes \overline{k} \to X \otimes \overline{k}$ is optimal.

Proposition 6.3.2.3. *If k is any field and $PIP_\ell(k)$ for all primes ℓ dividing n then $PIP_n(k)$.*

Proof. First, we may assume from the beginning that k is infinite. Indeed, if $k = \mathbf{F}_q$ then by the cheap trick 6.3.1, to prove the period-index property over $\mathbf{F}_q$ it suffices to prove it after replacing $\mathbf{F}_q$ by its maximal prime to n extension $k = \mathbf{F}_q^{\text{non-}n}$.

We show that PIP_ℓ for all primes ℓ in a set P of primes implies PIP_n for any n in the submonoid of $\mathbf{N}^{>0}$ generated by P. We proceed by induction on the number of primes occurring in the product expansion. Thus, let α be a class with period n and let ℓ be a prime factor of n, so $n = \ell n'$. The class $n'\alpha$ has period ℓ, hence has the period-index property by assumption. By 5.2.2.8 and the fact that k is infinite, there is a finite map $f : Y \to X$ of degree ℓ with Y a smooth proper geometrically connected surface such that $f^* n'\alpha = 0 \in \mathrm{Br}(Y)$. Thus, $f^*\alpha$ has period dividing n', whence by induction $f^*\alpha$ has the period-index property. Pushing forward to X shows that α has the period-index property.

Instead of using 5.2.2.8, one can use a different (sloppier) argument: the generic section of $\mathscr{A} \otimes L$ (in the notation of 5.2.2.8) is easily seen to define a geometrically connected geometrically normal surface $Y \to X$ with a finite map of degree ℓ. Now we can simply resolve the singularities of Y (as the dimension is 2) and thus arrive at a map $Y \to X$ between smooth proper geometrically connected surfaces which is *generically* finite of degree ℓ, which suffices for our argument (by the injection $\mathrm{Br}(Z) \hookrightarrow \mathrm{Br}(k(Z))$ for any regular integral algebraic space Z over k). □

Corollary 6.3.2.4. *To prove 6.3.2.1 it suffices to prove it for classes of prime order prime to q. Thus, it suffices to prove it for classes which are either geometrically optimal or geometrically essentially trivial.*

Proof of 6.3.2.1 for geometrically optimal classes. Let $\mathscr{X} \to X$ be a geometrically optimal $\boldsymbol{\mu}_n$-gerbe. By 5.2.4.11 and 5.2.4.7 the stack $\mathbf{Tw}^s(\Delta)$ is geometrically integral for sufficiently large Δ and is a $\boldsymbol{\mu}_n$-gerbe over its moduli space. Thus, $\mathrm{Tw}^s(\Delta)$ is a geometrically irreducible (even projective) variety over $\mathbf{F}_q$, and furthermore since $\mathrm{H}^2(\operatorname{Spec}\mathbf{F}_q, \boldsymbol{\mu}_n) = 0$ we see that a rational point of Tw^s lifts to a point of $\mathbf{Tw}^s$, i.e., an object. Thus, by the cheap trick 6.3.1, it suffices to find a rational point of Tw^s over $\mathbf{F}_q^{\text{non-}n}$, the maximal extension of degree prime to n. By the Lang-Weil estimates [*ibid.*], any geometrically integral variety over $\mathbf{F}_q$ has rational points over $\mathbf{F}_q^{\text{non-}n}$, *if it is non-empty*. By de Jong's theorem 6.2.2.2, $\mathbf{Tw}^s(\Delta)$ is asymptotically non-empty, so we are done. □

Proof of 6.3.2.1 for geometrically essentially trivial classes. In this case $\mathscr{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe such that $\mathscr{X} \otimes \overline{\mathbf{F}}_q$ admits an invertible twisted sheaf (has trivial Brauer class). There are two ways to proceed. First, we can use the moduli theory developed here. By 4.1.6.7, the stack $\mathbf{Tw}^\mu(\Delta)$ is geometrically isomorphic to $\mathrm{Sh}^\mu(\Delta)$. On the other hand, the stack of semistable sheaves on X is asymptotically geometrically irreducible, with Sh^μ as a dense open substack. The proof in characteristic 0 is contained in [40]; Langer claims in [50] to have proven this in arbitrary characteristic. In either case, the method is closely related to the method used in section 5.2.4 for geometrically optimal classes. Applying the Lang-Weil estimates to the substack of μ-stable points of $\mathbf{Tw}^s$ completes the proof.

There is a better, elementary proof due to de Jong which also applies to more general ambient varieties. This will appear written up in future work. □

6.3.3 Period and index on a surface over a local field

Using 6.3.2.1 and 5.2.4.26, we can prove a partial result on the period-index problem for surfaces over local fields. Throughout this section, K denotes a local field with integer ring R and (finite) residue field k.

Proposition 6.3.3.1. *Let X be a proper smooth geometrically connected surface over K which extends to a proper smooth surface $\mathfrak{X} \to \operatorname{Spec} R$. If $\alpha \in \mathrm{Br}(X)$ has period prime to* $\mathrm{char}(k)$, *then* $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^2$. *If α is unramified on $\mathfrak{X}$, then* $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$.

Proof. First suppose α extends to all of $\mathfrak{X}$. By 5.2.4.26, for large Δ the stack $\mathbf{Tw}^s_{\mathscr{X}/R}(\Delta)$ and its moduli space are proper flat generically smooth local complete intersections over R. There is an unramified extension $R' \supset R$ of degree prime to n such that $\mathrm{Tw}^s(\Delta)$ has a rational point in the smooth locus of the special fiber. Since R' is complete, this extends to a section, and we see furthermore that this section lifts into the stack $\mathbf{Tw}^s$ because $\mathrm{Br}(R') = 0$. Passing to the generic fiber yields a locally free twisted sheaf of rank n.

If α is ramified along the special fiber, then extracting the $\mathrm{per}(\alpha)$th root of a uniformizer of R will kill the ramification [10, 2.3.4] (which uses the main purity result of [28]). Thus, after making a finite free extension $R' \supset R$ of degree $\mathrm{per}(\alpha)$, we are reduced to the unramified case. This is easily seen to imply the result. □

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