Chain and antichain enumeration in posets, and b-ary partitions

by

Edward Fielding Early

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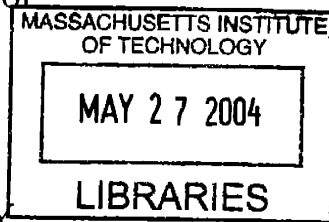
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Author . . . .
Department of Mathematics
April 6, 2004

Certified by . . . .
Richard P. Stanley
Norman Levinson Professor of Applied Mathematics
Thesis Supervisor

Accepted by . . . .
Pavel Etingof
Chairman, Department Committee on Graduate Students

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Abstract

The Greene-Kleitman theorem says that the lengths of chains and antichains in any poset are intimately related via an integer partition, but very little is known about the partition $\lambda(P)$ for most posets P. Our first goal is to develop a method for calculating values of $\lambda_k(P)$ for certain posets. We find the size of the largest union of two or three chains in the lattice of partitions of n under dominance order, and in the Tamari lattice. Similar techniques are then applied to the k-equal partition lattice. We also present some partial results and conjectures on chains and antichains in these lattices.

We give an elementary proof of the rank-unimodality of $L(2, n, m)$, and find a symmetric chain decomposition of $L(2, 2, m)$. We also present some partial results and conjectures about related posets, including a theorem on the size of the largest union of k chains in these posets and a bijective proof of the symmetry of the H-vector for $\mathbf{2 \times n}$.

We answer a question of Knuth about the existance of a Gray path for binary partitions, and generalize to b-ary partitions when b is even. We also discuss structural properties of the posets $R_b(n)$, and compute some chain and antichain lengths in the subposet of join-irreducibles.

Acknowledgments

I have to start by thanking my parents, for everything they've done these past 27 years. You gave me the freedom and encouragement to pursue any life that I wanted, and supported me even when I wanted to travel thousands of miles from home to do math.

There are many people who have helped me in one way or another over the years. If I attempted to thank them all individually, this thesis would be more than twice its current length. So, if you've helped me in any way, even if it was just with a smile when I needed to be reminded that there is happiness in the world, please know that I am thankful for you.

I thank all of my math teachers over the years, especially Sam Baethge; clearly they did something right.

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It's been said that if you are really fortunate, there are a handful of people in your life who mean it when they say, "I'll be there for you." To Alex, Beth, Dylan [2], Kelly, and Matt. I am really fortunate.

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Chapter 1

Introduction

1.1 Posets

A partially ordered set, or *poset*, is a set with an ordering $\leq$ such that $x \leq x$, $x \leq y$ and $y \leq x \Rightarrow x = y$, and $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (for all x, y, and z in the set). Two elements x and y are *incomparable* if $x \not\leq y$ and $y \not\leq x$. A *chain* is a totally ordered subset, and and *antichain* is a subset of pairwise incomparable elements. A chain is *maximal* if it cannot be lengthened by adding any element of the poset. A poset is *ranked* (or *graded*) if all maximal chains have the same length.

We say that y *covers* x if $x < y$ and there is no element z such that $x < z < y$. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that y appears "above" x if $x < y$. For example, the poset of divisors of 12 ordered by divisibility is shown in Figure 1-1.
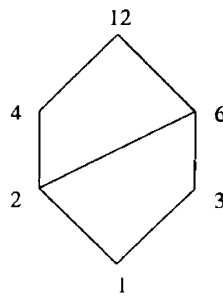

Figure 1-1: Divisors of 12.

Theorem 1 (Dilworth). *For any poset P, the size of the largest antichain in P is the minimum number of chains that cover P, and the size of the largest chain in P is the minimum number of antichains that cover P.*

A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $\Sigma \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots$. The *Young diagram* of λ is a left-justified array of n boxes, with λ_i boxes in row i. The *conjugate* of λ is the partition obtained by reading the lengths of the columns instead of the rows in the Young diagram of λ. For example, the conjugate of (5,2,1) is (3,2,1,1,1), as illustrated in Figure 1-2.
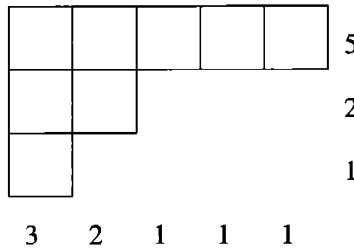
Figure 1-2: A partition and its conjugate.

In 1976, Greene and Kleitman proved the following generalization of Dilworth's theorem. Given any poset P, there exists a partition $\lambda(P)$ such that the sum of the first k parts of $\lambda(P)$ is the maximal number of elements in a union of k chains in P. In fact, the conjugate of λ has the same property with chains replaced by antichains [8, 15, 16]. Let $\lambda_k(P)$ denote the kth part of this partition.

Note that for computing $\lambda_2(P)$, it is not sufficient to find the longest chain in P and the longest chain in the remaining elements. For example, see Figure 1-3. While there is a chain of length 4, and two chains of length 3, one cannot partition the poset into chains of lengths 4 and 2.

The Greene-Kleitman theorem says that the lengths of chains and antichains in any poset are intimately related via an integer partition, but very little is known about the partition $\lambda(P)$ for most posets P. It is interesting to note that there are very few theorems that are true for all posets, so the fact that $\lambda(P)$ exists is remarkable. In this thesis, we develop and apply a method for computing parts of these partitions

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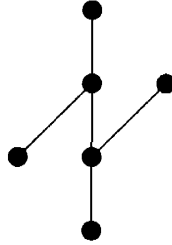
Figure 1-3: A poset P such that the largest chain is not one of the largest two chains. $\lambda(P) = \{4, 2\}$.

for several important families of posets.

If every pair of elements x and y in a poset P has a greatest lower bound (i.e. an element z in P such that any w less than or equal to both x and y is also less than or equal to z) and a least upper bound (defined analogously), then we call P a *lattice*. Most of the posets we will look at have this additional structure.

1.2 Unranked posets

The first two families of posets that we will work with are unranked lattices. We will see the dominance lattice P_n in Chapter 2, and the Tamari lattice T_n in Chapter 3. Each chapter contains an introduction that defines the poset being studied.

The term "ranked" poset comes from the fact that we can define a rank function r on a graded poset such that for any $x \leq y$, $r(x) - r(x)$ is the length of a saturated chain from x to y. When drawing the Hasse diagram of a ranked poset, one can put all of the elements of the same rank on horizontal lines, and arrange those lines vertically in increasing order. Trying to draw the diagram with any element not on its proper level will invariably lead to headaches and frustration. In other words, every element has a well-defined level in the Hasse diagram. But an unranked poset can also have this property, as illustrated in Figure 1-4.

In such a poset, we can draw a Hasse diagram where every element (except those on the top and bottom levels) covers something on the level below and is covered by something on the level above. Call such a poset *leveled*.

Given any poset, it is not difficult to see that the subposet of elements that appear
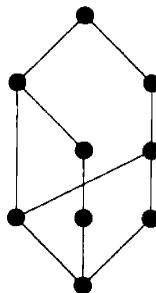
Figure 1-4: An unranked but leveled poset.

in chains of maximal length will be leveled, and that this is the maximal leveled subposet. It is maximal in the sense that it has the largest number of levels possible, and the addition of any other elements would make it no longer leveled. This simple observation turns out to be very useful for computing Greene-Kleitman partitions of certain posets. By temporarily throwing away all the elements that aren't on long chains, we can focus on the elements that really matter for our computations, and know that the chains we're finding really are the longest possible despite the lack of a true rank function.

1.3 Rank-unimodality

A sequence $a_0, a_1, a_2, \ldots, a_n$ is *unimodal* if, for some k, we have $a_0 \leq a_1 \leq \ldots \leq a_k \geq a_{k+1} \geq \ldots \geq a_n$. If a poset is ranked, then we can look at the sequence of sizes of the levels. If this sequence is unimodal, then the poset is *rank-unimodal*. If the sequence is symmetric, then the poset is *rank-symmetric*.

In Chapter 4, we will look at questions of rank-unimodality and chain decompositions in the poset of order ideals of a product of chains. In general, the order ideals of any poset form a lattice. In fact, a lattice is distributive if and only if it arises in this way for some poset.

1.4   Other posets

In Chapter 5, we will explore the problem of partitioning an integer into parts that are all powers of a fixed integer b. In addition to addressing and generalizing a question posed by Knuth, we explore the distributive lattice structure that comes with these partitions.

In Chapter 6, we will look at another kind of partitions, namely partitions of a finite set. This will lead us to another unranked lattice. We conclude with a discussion of open problems.

Chapter 2

The dominance lattice

2.1 Introduction

Let P_n denote the poset of partitions of the positive integer n, ordered by dominance (aka majorization), i.e. $\mu \leq \nu$ if $\mu_1 + \mu_2 \cdots + \mu_k \leq \nu_1 + \nu_2 + \cdots + \nu_k$ for all k. This poset is a lattice, and is self-dual under conjugation. P_n is not graded for $n \geq 7$, since there exist saturated chains from $\{n\}$ to $\{1^n\}$ of all lengths from $2n - 3$ to $cn^{3/2}$ [9, 17].

The length $h(P_n)$ of the longest chain in P_n has been known for some time [17]. If $n = \binom{m+1}{2} + r$, $0 \leq r \leq m$, then $h(P_n) = \frac{m^3 - m}{3} + rm$. In other words, $\lambda_1(P_n) = \frac{m^3 - m}{3} + rm + 1$. Our main results are the following theorems.

Theorem 2. *For $n > 16$, $\lambda_2(P_n) = \lambda_1(P_n) - 6$.*

Theorem 3. *For $n > 135$, $\lambda_3(P_n) = \lambda_2(P_n) - 6$.*

Consider the subposet Q_n of P_n consisting of the partitions that appear in chains of length $h(P_n)$. Clearly Q_n is self-dual under conjugation, since conjugation takes a decreasing chain to an increasing chain of the same length. It seems likely that Q_n is a graded lattice, but for our purposes it will suffice to use a weaker statement, namely: for $\mu \in Q_n$, define $r(\mu)$ to be the length of the longest chain from $\{n\}$ to μ; then $\mu \neq \{1^n\}$, $\{n\}$ is covered by an element ν such that $r(\nu) = r(\mu) - 1$ and covers

an element ν such that $r(\nu) = r(\mu) + 1$. In other words, Q_n is leveled. Note that the top element is level 0, and the levels increase as we move down.

The covering relation in P_n comes in two flavors. Following the methods of [17], we represent Young diagrams with vertical parts, as illustrated in Figure 2-1. We say μ covers ν by an H-step if there exists i such that $\nu_i = \mu_i - 1$, $\nu_{i+1} = \mu_{i+1} + 1$, and $\nu_k = \mu_k$ for $k \neq i, i+1$. In terms of Young diagrams, this corresponds to moving a box horizontally one space to the right (and down some distance). The other flavor is a V-step, which is an H-step on the conjugate, and corresponds to moving a box vertically one space down (and right some distance). Chains from $\{n\}$ to $\{1^n\}$ consisting of H-steps followed by V-steps are maximal.
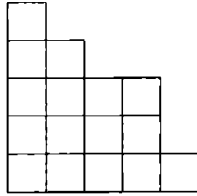
Figure 2-1: The partition $\{5, 4, 3, 3, 1\}$.

In Figure 2-1, the only possible H-step goes to $\{5, 4, 3, 2, 2\}$. This step is also a V-step. The other possible V-step goes to $\{4, 4, 4, 3, 1\}$.

2.2 Down to work

First we focus on Theorem 2. The cases where $n \leq 16$ will be handled separately, so for now assume $n > 16$.

We will prove Theorem 2 by showing that there exist two disjoint chains in Q_n of lengths $h(P_n)$ and $h(P_n) - 6$. Since Q_n is a subposet of P_n, these are also chains in P_n. Since there are six elements of P_n in saturated antichains of size 1, this is clearly the maximum possible number of elements in two chains, thus giving $\lambda_2(P_n)$ exactly.

To that end, we seek two disjoint chains in Q_n from $\{n - 2, 1, 1\}$ and $\{n - 3, 3\}$ to $\{2, 2, 2, 1^{n-6}\}$ and $\{3, 1^{n-3}\}$. Let Q_n^* denote Q_n without the top three and bottom three elements.

Lemma 1. *If Q_n^* has at least two elements on every level, then it has two disjoint chains of maximal length.*

Proof: Clearly we can start two chains with the two elements in the top level, so proceed by induction. The only potential problem is if we reach two elements on level k that both cover only one and the same element on level $k + 1$. In that case, take a second element on level $k + 1$ and a maximal chain ending at it. This chain has a lowest point of intersection with one of the two old chains, so just replace that old chain with the new one from that point on. See Figure 2-2. □
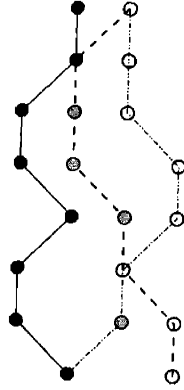
Figure 2-2: Salvaging a dead end.

Since Q_n^* is self-dual, it will suffice to show that the first half of its levels have at least two elements. We do this by explicitly constructing two disjoint chains to the halfway point. As a first approximation of these chains, take the following construction.

The left chain starts at $\{n-2, 1, 1\}$. At every step, we take the right-most possible H-step, e.g. the next partition is $\{n-3, 2, 1\}$. The right chain starts at $\{n-3, 3\}$. At every step, we take the left-most possible H-step, e.g. the next partition is $\{n-4, 4\}$. The names come from the relative positions of the chains when plotted, as in Figure 2-6. Both chains will eventually reach $\{m, m-1, \ldots, r+1, r, r, r-1, \ldots, 2, 1\}$, which is at least the halfway point [17], so the idea is to modify the left chain as little as possible to make it reach the halfway point without intersecting the right chain.

Once we've done that, we can apply Lemma 1 to get two disjoint chains of length $h(P_n) - 6$, then append the top and bottom three elements to one of them two get

the desired chains. The following proposition will be used to prove several lemmas concerning the right chain.

Proposition 1. *If $\mu = \{\mu_1, \mu_2, \ldots, \mu_k\}$ is in the right chain, then $\mu_i - \mu_{i+1} \leq 2$ for $i = 1, 2, \ldots, k - 2$. In other words, only the last difference can be greater than 2. Moreover, excluding the last difference, μ cannot have more than one difference equal to 2.*

Proof: By construction, we are always doing the left-most possible H-step. At first there is nothing to prove, since $k = 2$ through $\{\frac{n}{2}, \frac{n}{2}\}$ or $\{\frac{n+1}{2}, \frac{n-1}{2}\}$. Think in terms of partition diagrams as in the definition of H-steps. If there are no differences greater than 1 (excluding the last one), then push one box from μ_{k-1} to increase the last part (or from μ_k increase the number of parts). Now move to the left, pushing one box at a time until $\mu_i - \mu_{i+1} < 2$ for $i = 1, 2, \ldots, k - 2$ again. Clearly we never get a difference greater than 2 or more than one difference of 2 unless we had one before, so the result follows by induction. $\square$

2.3 Proof of Theorem 2

The proof comes in six cases, depending on r. We begin with general calculations that will be used in multiple cases. If $\mu = \{\mu_1, \mu_2, \mu_3, \ldots\}$ is reachable from $\{n\}$ by only H-steps, such as the elements of the left and right chains, then $r(\mu) = \mu_2 + 2\mu_3 + 3\mu_4 + \cdots$, since each box in μ_i had to be moved horizontally $i - 1$ times.

Note that any μ in the left chain with $\mu_1 - \mu_2 > 2$ is not in the right chain by Proposition 1. This means that the left chain makes it safely to the partition $\{m+r, m-1, m-2, \ldots, 2, 1\}$ at level $\frac{m^3-m}{6}$ for $r \geq 2$. For $r > 2$, we can continue safely to $\{m+2, m-1, m-2, \ldots, r-2, r-2, \ldots, 2, 1\}$ (using both assertions in Proposition 1) for an additional $m(r-2) - \binom{r-2}{2}$ levels. So we're done if $2(m(r-2) - \binom{r-2}{2}) \geq rm$. For $r > 4$, this comes down to $m \geq \frac{r^2-5r+6}{r-4} = r - 1 + \frac{2}{r-4}$. For $r = 5$, this means $m \geq 6$. In fact $m = 5$ also works, since we really just needed $m(r-2) - \binom{r-2}{2} \geq \lfloor \frac{rm}{2} \rfloor$. For $r > 5$, we just need $m \geq r$ (since m must be an integer), but that's as general as possible since $r \leq m$ by definition. Thus we've established Theorem 2 when $r \geq 5$.

If $r = 4$, then the above construction gets us to one level shy of where we need to be, since we only reach $\{m + 2, m - 1, m - 2, \ldots, 3, 2, 2, 1\}$ safely. Since $h(P_n)$ is always even when r is even, the middle level consists of self-conjugate partitions. Note that not all self-conjugate partitions are in Q_n, but one will be if it is covered by an element of Q_n since by duality it covers the conjugate of that element. Now we simply observe that $\{m + 2, m - 1, m - 2, \ldots, 3, 2, 2, 1\}$ covers the self-conjugate partition $\{m + 2, m - 1, m - 2, \ldots, 3, 2, 1, 1, 1\}$. This partition cannot be in the right chain by Proposition 1 (it is also not H-reachable from $\{n\}$ [17]), so this establishes Theorem 6 when $r = 4$. Alternatively, the step to $\{m + 1, m, m - 2, \ldots, 3, 2, 2, 1\}$ is safe for $m > 4$, which will be more useful for proving Theorem 7.

If $r = 0$, then we safely reach $\{m + 1, m - 2, m - 2, \ldots, 2, 1\}$, one level shy again. Once again, we simply observe that this covers the self-conjugate partition $\{m + 1, m - 2, m - 2, \ldots, 3, 1, 1, 1\}$, which is not in the right chain by Proposition 1, so this establishes Theorem 2 when $r = 0$.

The remaining cases each require a lemma to get past the shortfall in the above argument.

If $r = 1$, then we safely reach $\{m + 2, m - 2, m - 2, m - 3, \ldots, 2, 1\}$, but in fact we can go further along the left chain.

Lemma 2. *The partitions $\{m + 1, m - 1, m - 2, \ldots, 2, 1\}$ and $\{m, m - 1, \ldots, k + 1, k, k, k - 2, \ldots, 2, 1\}$, $5 \leq k \leq m$, do not occur in the right chain.*

Proof: If $\{m + 1, m - 1, m - 2, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m + 2, m - 2, m - 2, \ldots, 2, 1\}$ or $\{m + 1, m, m - 3, \ldots, 2, 1\}$ (otherwise we couldn't have done the left-most H-step), both of which violate Proposition 6.

If $\{m, m - 1, \ldots, k + 1, k, k, k - 2, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m, m - 1, \ldots, k + 1, k, k, k - 1, k - 4, k - 4, \ldots, 2, 1\}$ (note this works even for $k = m$) which violates Proposition 1 unless $k - 4 = 0$, hence the need for $k \geq 5$, or by $\{m, m - 1, \ldots, k + 1, k + 1, k - 1, k - 2, \ldots, 2, 1\}$. In this case, we can recursively work our way back to $\{m + 1, m - 1, m - 2, \ldots, 2, 1\}$, which is not in the right chain since it would have to be preceded by $\{m + 2, m - 2, m - 2, \ldots, 2, 1\}$

or $\{m+1, m, m-3, \ldots, 2, 1\}$, both of which violate Proposition 1. $\qquad\square$

Now apply Lemma 2 to extend the left chain safely to $\{m, m-1, \ldots, 5, 5, 3, 2, 1\}$, which occurs at level $\frac{m^3+5m-24}{6}$. Since $h(P_n) = \frac{m^3+2m}{3}$, it suffices if $m^3 + 5m - 24 \geq m^3 + 2m$, or $m \geq 8$. $m = 7$ also works since $h(P_{29}) = 119$ and we reach level 59. The case $m = 6$, $n = 22$ can be dealt with individually. The left chain gets to $\{6, 5, 5, 3, 2, 1\}$ at level 37, but intersects the right chain at level 38 with $\{6, 5, 4, 4, 2, 1\}$. However, the right chain reaches $\{6, 5, 4, 4, 3\}$ at level 37, which also covers the self-conjugate partition $\{5, 5, 5, 4, 3\}$, so this establishes Theorem 2 when $r = 1$.

If $r = 2$, then we safely reach $\{m+2, m-1, m-2, m-3, \ldots, 2, 1\}$, but in fact we can go further along the left chain.

Lemma 3. *The partitions* $\{m+1, m, m-2, m-3, \ldots, 2, 1\}$ *and* $\{m+1, m-1, m-2, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$, $1 \leq k \leq m-1$, *do not occur in the right chain.*

Proof: If $\{m+1, m, m-2, m-3, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m+2, m-1, m-2, m-3, \ldots, 2, 1\}$, $\{m+1, m+1, m-3, m-3, \ldots, 2, 1\}$, or $\{m+1, m, m-1, m-4, m-4, \ldots, 2, 1\}$, all of which violate Proposition 1. Note that we are tacitly assuming that $m > 4$, but that's fine since $n > 16$, so $m \geq 5$.

Since $\{m+1, m-1, m-2, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$ has two differences of size 2 for $k > 2$, Proposition 6 takes care of those cases (note $k = m-1$ means the partition is $\{m+1, m-1, m-1, m-3, \ldots, 2, 1\}$). If $\{m+1, m-1, m-2, \ldots, 3, 2, 2\}$ occurred in the right chain, then it would have to be preceded by $\{m+2, m-2, m-2, \ldots, 3, 2, 2\}$ or $\{m+1, m, m-3, \ldots, 3, 2, 2\}$, both of which violate Proposition 1. $k = 1$ is similar. $\qquad\square$

Now apply Lemma 3 to extend the left chain safely to $\{m+1, m-1, m-2, \ldots, 2, 1, 1\}$, which occurs at level $\frac{m^3+5m}{6}$. Since $h(P_n) = \frac{m^3+5m}{3}$, this establishes Theorem 2 when $r = 2$.

Finally, if $r = 3$, we safely reach $\{m+2, m-1, m-2, \ldots, 2, 1, 1\}$. Now we just modify Lemma 3. Note we could also show that the right chain has no elements ending in 1,1 until it's too late, but this method is cleaner.

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Lemma 4. *The partitions* $\{m + 1, m, m - 2, \ldots, 2, 1, 1\}$ *and* $\{m + 1, m - 1, m - 2, \ldots, k + 1, k, k, k - 2, \ldots, 2, 1, 1\}$, $4 \leq k \leq m - 1$ *do not occur in the right chain.*

Proof: Exactly the same as Lemma 3, since the second 1 at the end never comes into play. $\qquad\square$

Now apply Lemma 4 to extend the left chain safely to $\{m + 1, m - 1, m - 2, \ldots, 5, 4, 4, 2, 1, 1\}$, which occurs at level $\frac{m^3 + 11m - 18}{6}$. Since $h(P_n) = \frac{m^3 + 8m}{3}$, it suffices if $m^3 + 11m - 18 \geq m^3 + 8m$, or $m \geq 6$. When $m = 5$, we get to level 27, and $h(P_5) = 55$, so this case is fine as well. This establishes Theorem 2 when $r = 3$, and thus completes the proof. $\qquad\square$

2.4 Proof of Theorem 3

At all times in this proof, we assume that n is arbitrarily large. Looking back on it, we'll see we never needed more than $n > 135$.

Once again we wish to construct disjoint chains to the middle level. We use the left and right chains constructed in the proof of Theorem 2, plus a middle chain which will start at $\{n - 5, 4, 1\}$. By construction, we can easily tell that some partitions do not occur in the left chain, in analogy with Proposition 1.

Proposition 2. *If* $\mu = \{\mu_1, \mu_2, \ldots, \mu_k\}$ *is in the left chain, then* $\mu_i - \mu_{i+1} \leq 2$ *for* $i = 2, \ldots, k - 1$. *In other words, only the first difference can be greater than 2. Moreover, excluding the first difference, μ cannot have more than one difference equal to 2.*

Thus we will try to keep the middle chain safe by keeping the second difference greater than 2, or having two differences equal to 2 somewhere in the middle. Unfortunately, Lemma 1 does not generalize in the most obvious way for finding three chains, due to posets such as the ones shown in Figure 2-3. Consider the subposet R_n of P_n consisting of the partitions that appear in chains formed from top to bottom by a block of H-steps followed by a block of V-steps (note some steps may be both H-steps and V-steps). This is a subposet of Q_n, and is self-dual under conjugation

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(which switches H-steps and V-steps) [17]. Let R_n^* denote R_n without the top six and bottom six levels.

Lemma 5. *If R_n^* has at least three elements on every level, then it has three disjoint chains of maximal length.*

Proof: We can show this inductively, as in Lemma 1, if we can show that we do not have any two consecutive levels with connecting relations as shown in Figure 2-3. The pairs of bold lines indicate that we could have more lines like them without creating a third chain (e.g. one element covering three or four others, rather than just the two shown). Aside from the bold relations, there must be no other partitions in or relations between the two levels shown.
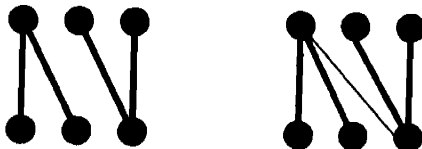
Figure 2-3: Three elements on every level but no three disjoint chains of maximal length.

Proving that these levels do not arise is extremely tedious, so we will not show all the details here. We'll just do a partial case to show how the general argument works; in particular we show that, with a few exceptions that can be ignored, we cannot get either of the subposets in Figure 2-4, where there are no other covering relations between these two levels involving α, β, or γ. This will prove that the first scenario shown in Figure 2-3 does not occur with either part as in Figure 2-4. By duality, it suffices to show that the one with H-steps does not occur.
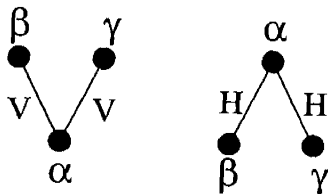
Figure 2-4: Two special cases.

Since an H-step from α is possible only where α has a difference greater than 1, or last part greater than 1, and since any H-step from α will stay in R_n, α must be

of the form $\{a, a-1, \ldots, a-i, b, b-1, \ldots, b-j, c, c-1, \ldots, 1\}$ or $\{a, a-1, \ldots, a-i, b, b-1, \ldots, b-j\}$ where $a - i - b > 1$, $b - j - c > 1$ or $b - j > 1$, and each run $(a, \ldots, a-i, b, \ldots, b-j$, and $c, \ldots, 1)$ has at most one repeat.

If α is $\{a, a-1, a-1, \ldots\}$, $\{a, a-1, \ldots\}$ (but not $\{a, a-1, a-1, \ldots\}$) or $\{a, a, \ldots\}$, then the partition γ we get by taking the right-most possible H-step is also covered by a partition of the form $\{a, a, a-2, \ldots\}$, $\{a+1, a-2, \ldots\}$, or $\{a+1, a-1, \ldots\}$, respectively, where the $\ldots$ ending matches the end of γ. Thus the first run of α must be just a. Similarly the second run must have just one or two numbers in it, and the third run is just $c = 1$ or empty. Thus α is $\{a, b\}$, $\{a, b, 1\}$, $\{a, b, b\}$, $\{a, b, b-1\}$, $\{a, b, b, 1\}$ or $\{a, b, b-1, 1\}$.

If $\alpha = \{a, 2\}$, then $\beta = \{a-1, 3\}$, $\gamma = \{a, 1, 1\}$, and we have an exception to the claim. This exception can be ignored, though, since it happens in the upper levels of R_n that are not in R_n^*. If $\alpha = \{a, b\}$ where $b > 2$, then $\beta = \{a-1, b+1\}$, $\gamma = \{a, b, b-1\}$, and γ is also covered by $\{a+1, b-1, b-1\}$. Working through the other cases, we similarly find only the isolated exceptions $\alpha = \{5, 3, 1\}$, $\{4, 2, 2\}$, $\{5, 3, 3, 1\}$. These are easily ignored since, for such small values of n, R_n^* does not have at least three elements on every level.

If α covered three or more partitions by H-steps, then the above argument would actually be simplified, thanks to the additional runs and large differences. Similarly, it really does suffice just to rule out the subposets in Figure 2-3.

A partition μ is H-reachable (i.e. there exists a chain of H-steps from $\{n\}$ to μ) if the parts of μ that come in runs with differences of at most 1 each have at most one repeated part, and that part appears no more than two times. μ is V-reachable (i.e. there exists a chain of V-steps from μ to $\{1^n\}$) if its conjugate is H-reachable, i.e. μ has no differences greater than 2, and any two differences of size 2 must have a repeated part between them [17].

To handle cases involving both H-steps and V-steps, we can use the fact that R_n only has H-steps between H-reachable partitions, and V-steps between V-reachable partitions. Of course, some partitions are both, and as long as $r > 0$ the number of such partitions will grow with m. With sufficient patience and brute force, the rest

of the proof is straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Note that this lemma will not apply when $r = 0$, but that's ok since the three chains we construct in that case end on self-conjugate partitions, and hence can be extended to their full length by conjugation. We only need Lemma 5 when r and m are both odd, so that $h(P_n) = \frac{m^3-m}{3} + rm$ is odd, i.e. the number of levels of P_n is even.

Just to take the first few steps along the middle chain without intersecting the right chain, we need $n > 14$ so that we can go through $\{n-5, 4, 1\}$, $\{n-6, 5, 1\}$, $\{n-7, 6, 1\}$, $\{n-7, 5, 2\}$, and $\{n-7, 5, 1, 1\}$, after which we just keep the second difference greater than 2 as long as possible. First we deal with the cases where $r > 4$. We safely reach $\{m + r - 2, m + 1, m - 2, \ldots, 2, 1\}$, and continue on to $\{m + 2, m + 1, m - 2, \ldots, r - 4, r - 4, \ldots, 2, 1\}$ at level $\frac{m^3-m}{6} + 2 + m(r - 4) - \binom{r-4}{2}$. We want this to be at least half of $\frac{m^3-m}{3} + rm$, or $m \geq r - 1 + \frac{8}{r-8}$ for $r > 8$. So if $r \geq 16$, then this works for all m (since $m \geq r$ by definition), and for $8 < r < 16$ we just have to exclude finitely many values of m.

Now we can take another step, to $\{m + 2, m, m - 1, m - 3, m - 4, \ldots, r - 4, r - 4, \ldots, 2, 1\}$. This is not in the right chain by Proposition 1, and one must check that it is not in the left chain, but the usual argument of looking at possible predecessors and seeing they all violate Proposition 2 works. From now on, we will say that such a partition is "safe by the usual methods." From there we continue along to $\{m+1, m+1, m-1, m-3, m-4, \ldots, r-4, r-4, \ldots, 2, 1\}$, $\{m+1, m+1, m-2, m-2, m-4, \ldots, r-4, r-4, \ldots, 2, 1\}$, and on down to $\{m+1, m+1, m-2, m-3, m-4, \ldots, r-3, r-3, \ldots, 2, 1\}$. This is $m - r + 2$ levels beyond where we last computed. One more step to $\{m+1, m, m-1, m-3, m-4, \ldots, r-3, r-3, \ldots, 2, 1\}$ is safe by the usual methods, and on down to $\{m+1, m, m-2, m-3, m-4, \ldots, r-1, r-1, r-3, r-3, r-4, \ldots, 2, 1\}$ (here we need $r - 1 \leq m - 2$, or $m > r$), and finally one more to $\{m+1, m-1, m-1, m-3, m-4, \ldots, r-1, r-1, r-3, r-3, r-4, \ldots, 2, 1\}$. That's another $m - r + 2$ steps, for a grand total of $\frac{m^3-m}{6} + 2 + m(r - 4) - \binom{r-4}{2} + 2m - 2r + 5$. We want this to be at least half of $\frac{m^3-m}{3} + rm$, or $m(r - 4) \geq r^2 - 5r + 4$, or $m \geq r - 1$ for $r > 4$. Since we already assumed $m > r$, this means the only possible bad cases

are where $m = r = 5, 6, \ldots, 15$. Since we only care about large n, we ignore these 11 cases, and we've established Theorem 3 when $r \geq 5$.

For $r = 0$, we safely go through $\{m + 1 + (m - 4), m, m - 3, m - 4, \ldots, 2, 1\}$, and eventually reach $\{m + 1, m, m - 3, m - 4, m - 4, \ldots, 2, 1\}$. One more step to $\{m + 1, m - 1, m - 2, m - 4, m - 4, \ldots, 2, 1\}$ is safe by the usual methods, and then we take two more steps to reach $\{m, m, m - 3, m - 3, m - 4, \ldots, 2, 1\}$ (note the order of these steps actually matters, we must make the m, m first). Now we're just one step away from the middle level of self-conjugate partitions, so we simply observe that this covers the self-conjugate partition $\{m, m, m - 3, m - 3, m - 4, \ldots, 3, 2, 2, 2\}$, and we're done.

The cases $r = 2, 3, 4$ are each straightforward with lemmas such as those used in proving Theorem 2. The case $r = 1$, however, requires something more clever. The trick we use turns out to give quick proofs for the other three cases as well, so we just use one lemma to settle all four cases.

Lemma 6. *The partitions* $\{m, m-1, m-2, m-3, \ldots, r+5, r+4, r+4, r+3, \ldots, 4, 1, 1\}$, $r = 1, 2, 3, 4$, *can be reached safely on the middle chain.*

Proof: We certainly reach $\{m + 1 + (m + r - 4), m, m - 3, m - 4, \ldots, 2, 1\}$ safely. Now the trick is to move the difference of 3 to the right. This will make finding a fourth disjoint chain much more difficult, but fortunately we're only trying to construct three chains. First we continue as before to $\{m + 1 + (m + r - 5), m, m - 3, m - 4, \ldots, 2, 1, 1\}$. Then it's on to $\{m + 1 + (m + r - 6), m, m - 3, m - 4, \ldots, 3, 3, 1, 1\}$ (do not proceed to $\{\ldots, 3, 2, 2, 1\}$), and eventually $\{m + 1 + r, m, m - 3, m - 3, m - 4, \ldots, 4, 3, 1, 1\}$. From there we go to $\{m + r, m + 1, m - 3, m - 3, m - 4, \ldots, 4, 3, 1, 1\}$, then $\{m + r, m, m - 2, m - 3, m - 4, \ldots, 4, 3, 1, 1\}$, then $\{m+r, m-1, m-1, m-3, m-4, \ldots, 4, 3, 1, 1\}$ (safe thanks to two differences of size 2 in the middle), and continue along until reaching $\{m + r, m - 1, m - 2, m - 3, \ldots, 4, 4, 1, 1\}$, and finally on to $\{m, m - 1, m - 2, m - 3, \ldots, r + 5, r + 4, r + 4, r + 3, \ldots, 4, 1, 1\}$. $\qquad\square$

For $r = 1$, we reach $\{m, m - 1, m - 2, m - 3, \ldots, 5, 5, 4, 1, 1\}$ by Lemma 6 at level $\frac{m^3 + 5m}{6} - 5$, which is at least the halfway point, namely $\frac{m^3 + 2m}{6}$, for $m \geq 10$.

For $r = 2$, we reach $\{m, m-1, m-2, m-3, \ldots, 6, 6, 5, 4, 1, 1\}$ by Lemma 6 at level $\frac{m^3+11m}{6} - 10$, which is at least the halfway point, namely $\frac{m^3+5m}{6}$, for $m \geq 10$.

For $r = 3$, we reach $\{m, m-1, m-2, m-3, \ldots, 7, 7, 6, 5, 4, 1, 1\}$ by Lemma 6 at level $\frac{m^3+17m}{6} - 16$, which is at least the halfway point, namely $\frac{m^3+8m}{6}$, for $m \geq 11$.

For $r = 4$, we reach $\{m, m-1, m-2, m-3, \ldots, 8, 8, 7, 6, 5, 4, 1, 1\}$ by Lemma 6 at level $\frac{m^3+23m}{6} - 23$, which is at least the halfway point, namely $\frac{m^3+11m}{6}$, for $m \geq 12$. Other arguments can handle this case for $m \geq 6$, but even $m = 12$, $r = 4$ gives us $n = 76 < 136$.

The largest case we did not settle was $m = r = 15$, or $n = 135$. This completes the proof of Theorem 3. $\qquad\square$

2.5   Smaller cases and related questions

The smaller n for which $\lambda_2(P_n) = \lambda_1(P_n) - 6$ are 10, 13, 14, and 15. Figure 2-5 shows P_{13} and Q_{13}, to give some idea of what's going on. In fact, $R_{13} = Q_{13}$, though this equality does not hold in general. For example, $\{5, 2, 1, 1, 1\}$ is in Q_{10} but not in R_{10}. Figure 2-6 shows Q_{16}. Since there are levels of size 1 in the middle, P_{16} cannot possibly have two chains of the desired lengths. Due to the size of the posets we are working with, we do not attempt to classify all n for which three chains of the desired lengths exist.

More generally, Table 2.1 shows the partitions of chain lengths for P_n, $1 \leq n \leq 14$. It is interesting to note that in all of these cases, the elements added between $\lambda_{k-1}(P_n)$ and $\lambda_k(P_n)$ form a chain that is added to the previous $k-1$ chains (and similarly for antichains). This is not the case for arbitrary posets, such as Figure 1-3. The proofs of Theorems 2 and 3 show that this is the case for every P_n when $k = 2$ or 3; it would be interesting to know if it holds for all k.

We know from Dilworth's theorem that P_n can be covered by $\lambda_1(P_n)$ antichains. What we have shown is that we can do this with six one-element antichains, six two-element antichains, and the remaining antichains of size three or larger. Moreover, there does not exist such a covering with seven one-element antichains, nor with
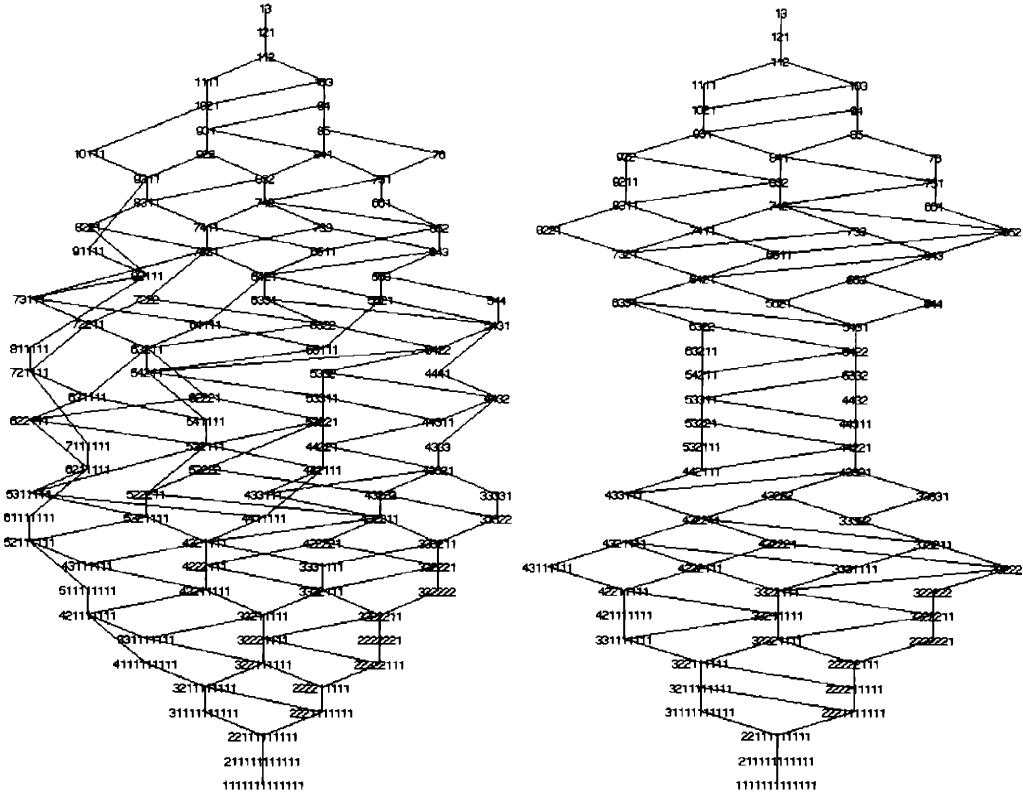
Figure 2-5: P_{13} (left) and Q_{13} (right).

thirteen antichains of at most two elements (for n sufficiently large).

While the proof of Theorem 2 is constructive in the cases where $h(P_n)$ is even, so that the middle level consists of self-conjugate partitions, it is not constructive when $h(P_n)$ is odd, since in those cases the proof relies on Lemma 1. It would be interesting to give an explicit construction of two long chains in those cases, and similarly for three or more chains.

Note that each chain constructed so far was guaranteed to be disjoint from the others by where it had differences greater than 1. We can thus hope to construct arbitrarily many disjoint chains to the middle level for large n, though of course the argument grows more technically difficult with each chain. This idea motivates the following conjecture.

Conjecture 1. *For large n, $\lambda_i(P_n) - \lambda_{i+1}(P_n)$ depends only on i.*

31
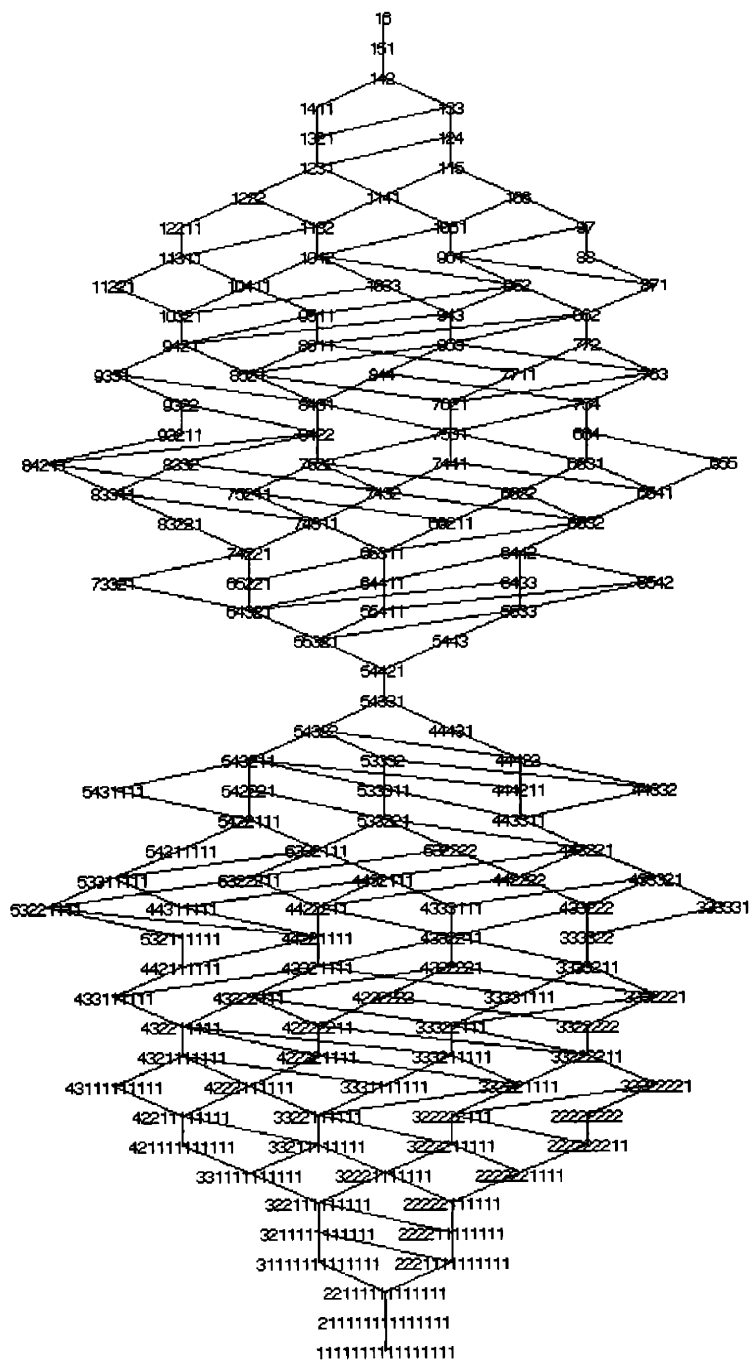
Figure 2-6: Elements of P_{16} on maximal chains.

n	$\lambda(P_n)$
1	$\{1\}$
2	$\{2\}$
3	$\{3\}$
4	$\{5\}$
5	$\{7\}$
6	$\{9,2\}$
7	$\{12,3\}$
8	$\{15,7\}$
9	$\{18,9,3\}$
10	$\{21,15,4,2\}$
11	$\{25,18,10,3\}$
12	$\{29,21,13,10,4\}$
13	$\{33,27,18,14,6,3\}$
14	$\{37,31,24,19,15,6,3\}$

Table 2.1: Known values of $\lambda(P_n)$.

Note that $\lambda_i(P_n) - \lambda_{i+1}(P_n)$ need not always be 6. It appears that the fourth chain starts just one level further down, so we conjecture that $\lambda_3(P_n) - \lambda_4(P_n) = 2$ for large n. Indeed, we can show this for large r by taking a chain of partitions where the third difference is greater than 2; the technical difficulties arise when r is small.

Let M be the transition matrix from the bases $\{e_\mu\}$ to $\{m_{\mu'}\}$ of homogeneous symmetric functions of degree n. Since $M_{\mu\nu} > 0$ iff $\nu \leq \mu'$, it is a theorem of Gansner and Saks (independently) that a generic matrix with the same 0 entries will have jordan blocks whose sizes are exactly the parts of $\lambda(P_n)$ (see [8, 13, 27]). Using Table 2.1 and Maple, one can verify that M is sufficiently generic at least for $n \leq 13$.

Another open problem is to find the size $a(n)$ of the largest antichain in P_n. Let $p(n)$ be the number of partitions of n. There is the obvious upper bound $a(n) \leq p(n)$. By Dilworth's theorem, $a(n) \geq p(n)/(h(P_n) + 1)$, so we have $\Omega(n^{-5/2}e^{\pi\sqrt{2n/3}}) \leq a(n) \leq O(n^{-1}e^{\pi\sqrt{2n/3}})$. It would be interesting to find a constructive proof that $a(n)$ is at least as large as the lower bound. In addition to the values of $a(n)$ implied by Table 2.1, we can see that $a(15) = 9$. Moreover, $\lambda_9(P_{15}) = 2$, with the long antichains being $71^8, 6221^5, 541^6, 53221^3, 52^5, 4431^4, 442221, 433311, 3^5$ and their conjugates. One can also verify that $a(16) = 10$, with $\lambda_{10}(P_{16}) = 5$. The sequence of $a(n)$'s is number A076269 in [28].

One construction that shows $a(n)$ has a lower bound of the form $e^{c\sqrt{n}}$ is as follows. Begin with the antichain 7321^4, 722221, 651^5, 642211, 63322, 553111, 55222, 54421, 4444 in P_{16}. Let $\nu + 7n$ denote a partition ν from the list with $7n$ added to each part. Consider ν to have 7 parts, so some of them might be 0. Then $\{\nu + 7n, \nu + 7(n-1), \ldots, \nu + 7, \nu\}$ is a partition of $N = 16(n+1) + 49\frac{n^2+n}{2} = \frac{49}{2}n^2 + O(n)$. There are 9^{n+1} choices for the ν's, yielding an antichain of size 9^{n+1} in P_N. This yields a lower bound for $a(n)$ of $e^{c\sqrt{n}}$ where $c = \ln 9\sqrt{2/49} = 0.4439\ldots$ By starting with a 28-element antichain in P_{27} where each ν has at most 9 parts, and largest part at most 8, one can similarly get $c = \frac{\ln 28}{6} = 0.555\ldots$ This is still a long way from $\pi\sqrt{2/3} = 2.565\ldots$, but at least it's constructive.

Chapter 3

The Tamari lattice

3.1 Introduction

The *Tamari lattice* T_n is defined as the set of all binary bracketings on $n + 1$ symbols ordered by applying the associativity rule in one direction, i.e. $(ab)c < a(bc)$. For the products $x_0 x_1 \cdots x_n$, applying transformations of that form gives us larger elements. In paticular, $((\cdots (x_0 x_1) x_2) \cdots x_n)$ is minimal and $(x_0(x_1(\cdots (x_{n-1} x_n))) \cdots)$ is maximal. We can take the ordered n-tuple $(v_1, \ldots, v_n)$ of integers from 1 to n, inclusive, where the opening bracket before x_i closes after x_{v_i}. In particular, $(1, 2, \ldots, n)$ is minimal and $(n, n, \ldots, n)$ is maximal. In fact, T_n is the set of these n-tuples that satisfy $i \leq v_i$ for all i, and $i \leq j \leq v_i$ implies $v_j \leq v_i$ for all i, ordered by componentwise comparison [18].

The number of elements in T_n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. T_n is self-dual, Cohen-Macaulay, and has many other interesting properties [1, 5, 7, 14, 18, 22].

Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, then b covers a in T_n iff there exists j such that $a_i = b_i$ for all $i \neq j$, $a_j < b_j$, and $b_j = b_k = a_k$ where $k = a_j + 1$ [22].

It is not difficult to see that $\lambda_1(T_n) = \frac{n(n-1)}{2} + 1$ [22]. A chain of maximal length is one where each cover comes from just increasing one part by 1. Our main results are the following theorems.

Theorem 4. *For $n > 5$, $\lambda_2(T_n) = \lambda_1(T_n) - 4$.*

Theorem 5. *For $n > 6$, $\lambda_3(T_n) = \lambda_2(T_n) - 2$.*

The proof of each theorem is by explicit construction of disjoint chains of lengths $\lambda_1(T_n)$, $\lambda_2(T_n)$, and $\lambda_3(T_n)$ as needed. To prove that we can't do any better, we show that T_n can be decomposed into antichains of appropriate sizes.

While T_n is not ranked for $n > 2$, we can extract the subposet U_n of elements that appear in chains of maximal length, as shown in Figure 3-1. It seems likely that U_n is graded, but for our purposes it will suffice to use the fact that U_n is leveled. For $\mu \in U_n$, define $r(\mu)$ to be the length of the longest chain from $(1, 2, \ldots, n)$ to μ; then $\mu \neq (1, 2, \ldots, n), (n, n, \ldots, n)$ is covered by an element ν such that $r(\nu) = r(\mu) + 1$ and covers an element η such that $r(\eta) = r(\mu) - 1$. In other words, every element of U_n is on a fixed level. Note that the bottom element is level 0, and the levels increase as we move up. The bulk of the construction of the two or three disjoint chains will take place in U_n.

Lemma 7. *The elements of U_n are the ordered n-tuples $(v_1, \ldots, v_n) \in T_n$ such that $v_{i+1} - v_i \leq 1$ for $i = 1, \ldots, n - 1$.*

Proof: If we ever have $v_{i+1} - v_i > 1$ for some i, then v_i will increase by more then 1 when it is the place that changes in a cover, hence it cannot be on a chain of maximal length. For the remaining n-tuples, we can get to them from $(1, 2, \ldots, n)$ by increasing the leftmost possible part at all times (i.e. increment the first part until it reaches v_1, then start incrementing v_2, etc.). We then continue along to $(n, n, \ldots, n)$ by starting over incrementing the leftmost possible part. $\square$

3.2   Proof of Theorem 4

First we show that there are indeed two disjoint chains that use up $n(n - 1) - 2$ elements. The longer chain will start at $(1, 2, \ldots, n)$, and by Lemma 7 must proceed to $(2, 2, 3, \ldots, n)$. Since $n > 5$, for illustrating the chain we will just write $v_1 v_2 v_3 v_4 v_5 v_6$ to denote $(v_1, v_2, v_3, v_4, v_5, v_6, 7, \ldots, n)$. Our long chain starts 123456, 223456, 323456,
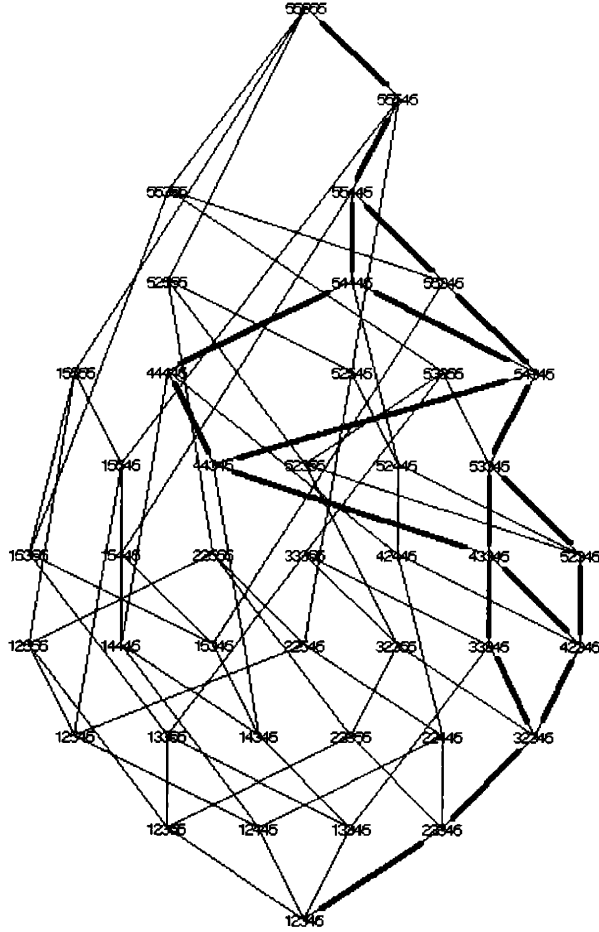
Figure 3-1: T_5, with U_5 highlighted.

423456, 523456, 623456, 633456, 643456, 644456. The chain continues by increasing the rightmost possible part that keeps us in U_n, all the way up to $(n, n, \ldots, n)$.

The second chain starts 133456 (not in U_n!), 333456, 433456, 533456, 543456, 553456, 653456. The chain continues by increasing the leftmost possible part (which will always keep us in U_n) until we reach $(n, n, \ldots, n, n-2, n-1, n)$, and then ends at $(n, n, \ldots, n, n-2, n, n)$. This chain has four fewer elements than the long one, as desired, and it is not hard to see that the two chains are disjoint. The key idea here was to take advantage of the crossing covers highlighted in Figure 3-2. Note that it would have been easier to construct two chains of length $\frac{n(n-1)-2}{2}$ by not having them cross, but it is interesting to see that we really can get disjoint chains of length $\lambda_1(T_n)$ and $\lambda_2(T_n)$.

66666

66656

666556

665556   666456

655556   665456

555556   655456   664456

555456   654456   663456

554456   644456   653456

544456   553456   643456

444456   543456   633456

443456   533456   623456

433456   523456

333456   423456

323456

223456

123456
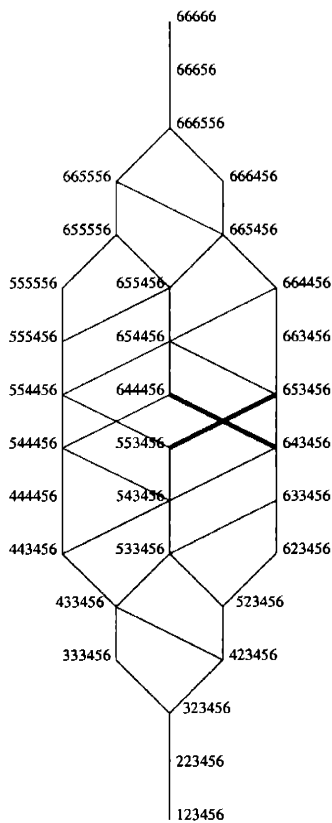
Figure 3-2: The poset U_6.

To show that this does indeed give us $\lambda_2(T_n) = \lambda_1(T_n) - 4$, we prove that T_n can be decomposed into $N = \lambda_1(T_n) = \frac{n(n-1)}{2} + 1$ antichains, four of which consist of a single element. To this end, we start with the most obvious decomposition into N antichains.

Draw the Hasse diagram of T_n by starting with $(1, 2, \ldots, n)$ at the bottom, as level 0. Now each subsequent level consists of the minimal elements of what's left of T_n. It is not hard to see that level i will consist of the elements of T_n whose components sum to $\frac{n(n+1)}{2} + i$. These levels give us a decomposition of T_n into N antichains, but unfortunately only the top two levels $((n, n, \ldots, n)$ and $(n, n, \ldots, n, n - 1, n))$ and the bottom level have just one element. Note, however, that level 1 of U_n consists of only one element, namely $(2, 2, 3, \ldots, n)$, so everything in $T_n - U_n$ can be shifted up one level in the diagram. This makes level 1 have only one element, but adds $(n, n, \ldots, n, n - 2, n, n)$ to level $N - 1$. However, $(n, n, \ldots, n, n - 2, n, n)$ only covers $(n, n, \ldots, n, n-2, n-1, n)$ (which is in U_n) and things that were originally at least two

levels below it ($(n, n, \ldots, n, n-2, n-2, n, n)$ comes the closest), so we can safely push it back down one level. Now we have just one element in each of levels 1, 2, $N-1$, and N. Thus any two disjoint chains in T_n can contain at most $2N-4$ elements, as desired.

$\square$

3.3 Proof of Theorem 5

First we show that there are indeed three disjoint chains that use up $\frac{3n(n-1)}{2} - 8$ elements. This time we record just the first seven numbers in each element. The longest chain starts 1234567, 2234567, 3234567, 4234567, 5234567, 6234567, 6334567, 6444567, 6544567, 6644567, 6654567. The chain continues by increasing the rightmost possible part that keeps us in U_n, all the way up to $(n, n, \ldots, n)$.

The second chain starts 1334567 (not in U_n), 3334567, 4334567, 5334567, 5434567, 5534567, 6534567, 7634567, 7734567, 7744567, 7754567, 7755567. The chain then continues by increasing the leftmost possible part (so we stay in U_n) that keeps it disjoint from the third chain (defined below), until we reach $(n, \ldots, n, n-2, n-1, n)$, and then ends at $(n, \ldots, n, n-2, n, n)$.

The third chain starts 1244567, 2244567, 4244567, then enters U_n at 4444567, followed by 5444567, 5544567, 5554567, 6554567, 7554567, 7654567, 7664567, 7764567. The chain then continues by increasing the leftmost possible part (which will always keep us in U_n) until we reach $(n, \ldots, n, n-3, n-2, n-1, n)$, and then leaves U_n to end with $(n, \ldots, n, n-3, n-1, n-1, n)$, $(n, \ldots, n, n-3, n, n-1, n)$, $(n, \ldots, n, n-3, n, n, n)$.

These chains have the desired lengths by construction. To show that this does indeed give us $\lambda_3(T_n) = \lambda_2(T_n) - 2 = \lambda_1(T_n) - 6$, we prove that T_n can be decomposed into N antichains, six of which together comprise only eight elements. To this end, we start with the Hasse diagram that we had at the end of the proof of Theorem 6. Shift everything possible up one more level. This will leave U_n alone, as well as anything that was covered by an element of U_n where the covering just increased one part by 2. The bottom two levels still have one element each, and level 3 now has two elements, namely $(3, 2, 3, \ldots, n)$ and $(1, 3, 3, \ldots, n)$. Now we have the top three levels as shown

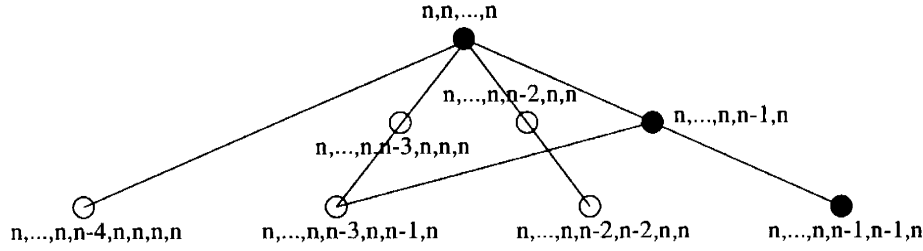in Figure 3-3, so we have to move a few things back down again.

Figure 3-3: Top three levels of T_n after second upward shift.

Of course the elements of U_n are left alone, so we need to get rid of four of the elements marked by hollow circles. All we need is to move each of them down one level, then convince one of the two that remain to go away. Take a few deep breaths, and here we go...

To push $(n, n, \ldots, n, n - 4, n, n, n, n)$ down one level, we have to lower a chain of elements below it, namely $(n, \ldots, n, n - 4, n, n, n - 1, n)$, $(n, \ldots, n, n - 4, n, n - 1, n - 1, n)$ $(n, \ldots, n, n - 4, n - 1, n - 1, n - 1, n)$, $(n, \ldots, n, n - 4, n - 1, n - 2, n - 1, n)$ $(n, \ldots, n, n - 4, n - 2, n - 2, n - 1, n)$, and $(n, \ldots, n, n - 4, n - 3, n - 2, n - 1, n)$. We also have to throw in $(n, \ldots, n, n - 4, n, n - 2, n - 1, n)$. Each of these only covers other elements that are either in U_n (and hence more than one level down after shifting) or were more than one level down in the first place, so it is safe to move all of these elements down one level.

The elements $(n, \ldots, n, n - 2, n, n)$ and $(n, \ldots, n, n - 2, n - 2, n, n)$ go down with much less of a fight, they only take $(n, \ldots, n, n - 3, n - 2, n, n)$ with them. For $(n, \ldots, n, n - 3, n, n, n)$ and $(n, \ldots, n, n - 3, n, n - 1, n)$, we also have to move the chain of elements $(n, \ldots, n, n - 3, n - 1, n - 1, n)$, $(n, \ldots, n, n - 1, n - 3, n - 1, n - 1, n)$, $(n, \ldots, n, n - 1, n - 1, n - 3, n - 1, n - 1, n)$, $\ldots$, $(n - 1, \ldots, n - 1, n - 3, n - 1, n - 1, n)$. It is easy to check that all of these only cover other elements that are in U_n or were originally more than one level down, so moving this chain down works.

We're almost there. $(n, \ldots, n, n - 2, n, n)$ can't move down any further, since it covers $(n, \ldots, n, n - 2, n - 1, n) \in U_n$, so we focus on $(n, \ldots, n, n - 3, n, n, n)$. Of course we have to move the whole chain of elements that we had before with it, and now we have to also move anything not in U_n that was originally covered by one of

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those by a move that increased one part by 2 (those that increased by just 1 are already included). The only such element is $(n, \ldots, n, n - 3, n - 2, n, n)$, which we already moved down one level in the previous paragraph, so we're good to go.

The levels of this drawing of the Hasse diagram now show that three chains can use at most $3N - 10$ elements of T_n, as desired. □

3.4   Smaller cases and related questions

Some partial values of $\lambda(T_n)$ for small n are shown in Table 3.1. Note that $\lambda_1(T_n) - \lambda_2(T_n)$ can be larger or smaller than 4 for small values of n.

n	$\lambda(T_n)$
1	$\{1\}$
2	$\{2\}$
3	$\{4, 1\}$
4	$\{7, 3, 3, 1\}$
5	$\{11, 6, 6, ?\}$
6	$\{16, 12, ?\}$
7	$\{22, 18, 16, ?\}$

Table 3.1: Known values of $\lambda(T_n)$.

There is no reason to believe that similar reasoning can't be applied to unions of more chains, the only obstacle is the difficulty in carrying out the explicit constructive proof.

Conjecture 2. *For large n, $\lambda_i(T_n) - \lambda_{i+1}(T_n)$ depends only on i.*

Also unknown is the size of the largest antichain in T_n. Clearly it is smaller than C_n, and by Dilworth's theorem it is at least $C_n/\lambda_1(T_n)$. Asymptotically, this tells us that the size of the largest antichain is between $\Omega(n^{-7/2}4^n)$ and $O(n^{-3/2}4^n)$.

Chapter 4

$L(n_1, \ldots, n_k)$

4.1 Introduction

The *direct product* of posets P and Q is the poset $P \times Q$ on the set $\{(x,y) \| x \in P \text{ and } y \in Q\}$ such that $(x,y) \leq (x',y')$ if $x \leq x'$ in X and $y \leq y'$ in Q. An *order ideal* in a poset P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$.

Let $L(n_1, n_2, \ldots, n_k)$ denote the lattice of order ideals of the poset $\mathbf{n_1} \times \mathbf{n_2} \times \cdots \times \mathbf{n_k}$, where $\mathbf{n}$ is an n-element chain. When $k = 2$, this is the lattice of integer partions with at most n_1 parts, and maximum part at most n_2, ordered by inclusion of Young diagrams. When $k = 3$, we get the lattice of plane partitions that fit inside an $n_1 \times n_2 \times n_3$ box. This poset is rank-symmetric, and is known to be Peck (i.e. rank-symmetric, rank-unimodal, and strongly Sperner) for $k < 4$ [24, 25, 30], with a combinatorial proof of rank-unimodality known only for $k < 3$ [23, 35]. In a few cases, it is even known to have a symmetric chain decomposition (SCD) [21, 26, 34].

Our main result on rank-unimodality is a new proof of the following theorem.

Theorem 6. $L(2, n, m)$ *is rank-unimodal.*

The idea is patch together several other results with a nice combinatorial argument. Let $F(P)$ denote the rank-generating function of a poset P. $F(L(m, n))$ is the

q-binomial coefficient (or Gaussian polynomial)

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{(1-q^{m+n})(1-q^{m+n-1})\cdots(1-q^{m+1})}{(1-q^n)(1-q^{n-1})\cdots(1-q)}.$$

This polynomial is symmetric and unimodal, centered around $\frac{mn}{2}$ [23, 25]. We will also use a special case of Proposition 8.2 in [29], namely that the rank-generating function for $L(n_1, n_2, \ldots, n_k)$ is a sum of q-binomial coefficients multiplied by certain polynomials in q. The general result is

$$U_m(P,\omega) = \sum_{s=0}^{p-1} \begin{bmatrix} p+m-s \\ p \end{bmatrix}_q W_s(P,\omega). \tag{4.1}$$

For our purposes, P will be $\mathbf{n_1} \times \mathbf{n_2} \times \cdots \times \mathbf{n_{k-1}}$, $m = n_k$, ω is a natural labeling, so $U_m(P,\omega)$ is just $F(L(n_1, n_2, \ldots, n_k))$. On the right-hand side, p is the size of P, namely $n_1 n_2 \cdots n_{k-1}$. $W_s(P,\omega) = \sum_\pi q^{maj(\pi)}$, where the sum is over all linear extensions π of P (labeled by ω) with s descents, and $maj(\pi)$ is the sum of the descents. From now on we will omit ω from the notation, since we are only dealing with a natural labeling as in Figure 4-1, obtained by labeling the elements of P from bottom to top, left to right.

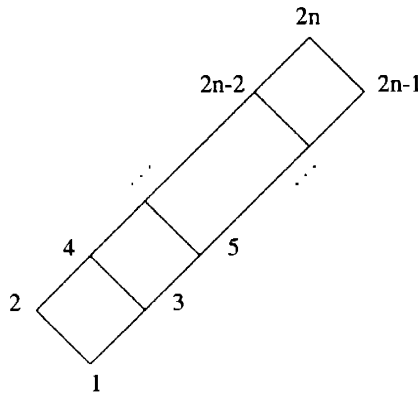Figure 4-1: A natural labeling of $\mathbf{2} \times \mathbf{n}$.

Now $\begin{bmatrix} p+m-s \\ p \end{bmatrix}_q$ is symmetric and unimodal, centered around degree $\frac{p(m-s)}{2}$.

If $W_s(P)$ is symmetric and unimodal, centered around degree $ps/2$, then the product

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is symmetric and unimodal, centered around degree $pm/2$ [31], and hence the sum of these products, namely $F(L(n_1, n_2, \ldots, n_k))$, is symmetric and unimodal, centered around degree $pm/2$.

Our goal is to show that $W_s(P)$ is indeed symmetric and unimodal, centered around degree $ps/2$, at least for $P = \mathbf{2} \times \mathbf{n}$ ($p = 2n$). The proof is by induction on n.

4.2 Proof of Theorem 6

For symmetry and degree, suppose π is any extension of $P = \mathbf{2} \times \mathbf{n}$. Replace each number x in π by $2n + 1 - x$ and reverse the order to get a new extension π' (of the dual of P) with the same number of descents, but with $maj(\pi') = 2ns - maj(\pi)$. Since P is self-dual, π' is also an extension of P.

If $n = 1$, then $L(2, 1, m)$ is isomorphic to $L(2, m)$, so we're done. For $n > 1$, a linear extension of $2 \times n$ can have anywhere from 0 to $n - 1$ descents.

Our strategy is to put a poset structure on the set of linear extensions of P with s descents. We will get a poset $W^s(P)$ with rank-generating function $W_s(P)$ (after factoring out the lowest power of q) for each s, and we can then attempt to prove statements about $W^s(P)$. To this end, given an extension π with s descents, replace the increasing subsequences from the beginning with 0's, 1's, up to s. Thus 123546879 becomes 000011122. Now label P with these smaller numbers in the order of the extension (remember we've fixed a natural labeling of P). Since $P = \mathbf{2} \times \mathbf{n}$, this labeling is a $2 \times n$ array of numbers 0 to s, weakly increasing in rows and columns. But this is precisely an element of $L(2, n, s)$.

Now replace each row with a row that counts the number of entries in the original row that are at least s, $s - 1$, ..., 1. This will be an element of $L(2, s, n)$. The point is that now we have a minimal element, shown below.

$$
\begin{array}{ccccccc}
0 & \leq & 1 & \leq & \cdots & \leq & s-1 \\
\text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & & & \text{\rotatebox{90}{$\leq$}} \\
2 & \leq & 3 & \leq & \cdots & \leq & s+1
\end{array}
$$

This minimal element can be subtracted from all of the others to bring us down to $L(2, s, n - s - 1)$.

Lemma 8. *The map described above shows that $W^s(P) \cong L(2, s, n - s - 1)$.*

Proof: To see this, we verify that each step is an order-preserving bijection. The map from the set of extensions to the subset of $L(2, n, s)$ is clearly a bijection, and we use the ordering on $L(2, n, s)$ to define the ordering on $W^s(P)$. The elements of $L(2, n, s)$ that we get are those where for each i, $1 \leq i \leq s$, the first i in the bottom row occurs to the left of the last $i - 1$ in the top row.

The next step is just the obvious bijection from $L(2, n, s)$ to $L(2, s, n)$. After that we subtract off the minimal element, which is possible since back in $L(2, n, s)$ we had to have at least the element shown below, which maps to the minimal element in $L(2, s, n)$.

$$0 \leq \cdots \leq 0 \leq 0 \leq 1 \leq \cdots \leq s-2 \leq s-1$$
$$\wedge| \qquad\qquad \wedge| \quad \wedge| \quad \wedge| \qquad\qquad \wedge| \qquad \wedge|$$
$$0 \leq \cdots \leq 1 \leq 2 \leq 3 \leq \cdots \leq s \leq s$$

Thus we clearly have an injection into $L(2, s, n - s - 1)$. To see that it is surjective, consider the inverse map. Given any element of $L(2, s, n - s - 1)$, we add the minimal element to it, work our way back to $L(2, n, s)$, and observe that for each i, $1 \leq i \leq s$, the first i in the bottom row occurs to the left of the last $i - 1$ in the top row, as desired. □

Since $s < n$, we know $L(2, s, n - s - 1)$ is rank-unimodal by induction, so we're done. □

4.3 Related questions

There is more structure that we can exploit here. If $n = 2$, then we just had $F(L(2, 2, m)) = F(L(4, m)) + q^2 F(L(4, m - 1))$, which suggests a way to prove the following theorem.

Theorem 7. $L(2, 2, m)$ *has a symmetric chain decomposition.*

Proof: We find a bijection from $L(2,2,m)$ to the disjoint union of $L(4,m)$ and $L(4,m-1)$, whose inverse is order-preserving, where the rank gets shifted down by 2 in the second case. Since an explicit SCD is known for these posets [34], that is all we need.

An element of $L(2,2,m)$ is a diagram of the form below.

$$
\begin{array}{ccccc}
0 & \leq & a & \leq & c \\
& & \wedge| & & \wedge| \\
& & b & \leq & d & \leq & m
\end{array}
$$

This element can be thought of as an ordered 4-tuple (a,b,c,d) of numbers from 0 to m, where $a \leq b$, $a \leq c$, $b \leq d$, and $c \leq d$. If $b \leq c$, then this is just an element of $L(4,m)$. If $b > c$, then we take $(a,c,b-1,d-1) \in L(4,m-1)$. It is not difficult to see that this is the desired bijection. □

We can prove that $F(L(n_1, n_2, \ldots, n_k))$ is unimodal for fixed $n_1, n_2, \ldots, n_{k-1}$ and arbitrary n_k with a straightforward (though lengthy) calculation. For example, we have the following new results.

Proposition 3. $L(2,2,2,m)$ *is rank-unimodal for all* m.

Proof: Let $P = \mathbf{2} \times \mathbf{2} \times \mathbf{2}$, labeled as shown in Figure 4-2.
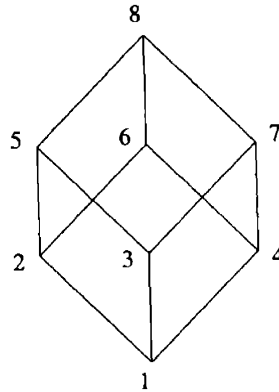

Figure 4-2: A natural labeling of $\mathbf{2} \times \mathbf{2} \times \mathbf{2}$.

By computing all 48 linear extensions, we can see the following.

$$W_0(P) = 1$$

$$W_1(P) = 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6$$

$$W_2(P) = q^5 + 3q^6 + 4q^7 + 8q^8 + 4q^9 + 3q^{10} + q^{11}$$

$$W_3(P) = 2q^{10} + 2q^{11} + 3q^{12} + 2q^{13} + 2q^{14}$$

$$W_4(P) = q^{16}$$

Since each of these is symmetric and unimodal, centered around degree $4s$, the result follows. $\square$

Only a few other products of more than two chains can be done with Maple in a reasonable amount of time.

Proposition 4. $L(2, 2, 3, m)$ and $L(2, 2, 4, m)$ are rank-unimodal for all m.

Proof: Once again, by computing all the linear extensions for $P = \mathbf{2} \times \mathbf{2} \times \mathbf{3}$ or $\mathbf{2} \times \mathbf{2} \times \mathbf{4}$, we find that all of the polynomials $W_s(P)$ are symmetric and unimodal, centered around the appropriate degree. In particular,

$$W_0(P) = 1$$

$$W_1(P) = 2q^2 + 3q^3 + 5q^4 + 5q^5 + 7q^6 + 5q^7 + 5q^8 + 3q^9 + 2q^{10}$$

$$W_2(P) = q^5 + 5q^6 + 9q^7 + 20q^8 + 26q^9 + 36q^{10} + 39q^{11} + 43q^{12} + $$
$$39q^{13} + 36q^{14} + 26q^{15} + 20q^{16} + 9q^{17} + 5q^{18} + q^{19}$$

$$W_3(P) = 3q^{10} + 7q^{11} + 17q^{12} + 23q^{13} + 50q^{14} + 70q^{15} + 94q^{16} + 107q^{17} + 119q^{18} + $$
$$107q^{19} + 94q^{20} + 70q^{21} + 55q^{22} + 29q^{23} + 17q^{24} + 7q^{25} + 3q^{26}$$

$$W_4(P) = 3q^{16} + 7q^{17} + 17q^{18} + 23q^{19} + 50q^{20} + 70q^{21} + 94q^{22} + 107q^{23} + 119q^{24} + $$
$$107q^{25} + 94q^{26} + 70q^{27} + 55q^{28} + 29q^{29} + 17q^{30} + 7q^{31} + 3q^{32}$$

$$W_5(P) = q^{23} + 5q^{24} + 9q^{25} + 20q^{26} + 26q^{27} + 36q^{28} + 39q^{29} + 43q^{30} + $$
$$39q^{31} + 36q^{32} + 26q^{33} + 20q^{34} + 9q^{35} + 5q^{36} + q^{37}$$

$$W_6(P) = 2q^{32} + 3q^{33} + 5q^{34} + 5q^{35} + 7q^{36} + 5q^{37} + 5q^{38} + 3q^{39} + 2q^{40}$$

$$W_7(P) = q^{42}$$

so the result follows. □

The big question is whether the polynomials $W_s(P)$ will behave so nicely for all products of chains, even just for all products of two chains. There are ranked lattices for which they are neither unimodal nor symmetric, but of course products of chains have much more structure (such as self-duality). The proof of Theorem 6 does not generalize in the most obvious way, since we cannot always find a minimal element to subtract off. Nevertheless, we conjecture that these polynomials will always be unimodal, and hence that $L(n_1, n_2, \ldots, n_k)$ will always be rank-unimodal.

For example, $W^1(3 \times 3)$ is isomorphic to $L(3, 3)$ with one maximal chain removed, and hence is rank-unimodal. It also possible to prove that some of the polynomials $W_s(P)$ have the desired properties by brute force.

Proposition 5. $W_1(3 \times n)$ *is symmetric and unimodal, centered around degree* $3n/2$.

Proof: The symmetry and degree follow from self-duality as before. Let $3 \times n$ be labeled as shown in Figure 4-3.
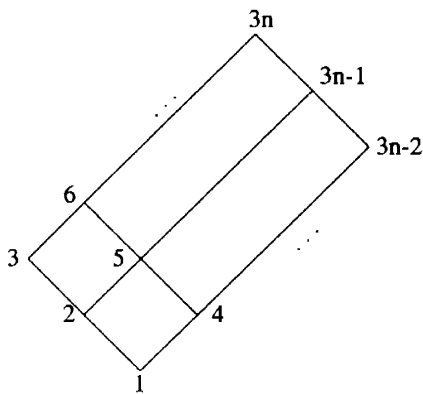
Figure 4-3: A natural labeling of $3 \times n$.

A descent must be of the form $3a + 2 > 3b$, $3a + 1 > 3b - 1$, or $3a + 1 > 3b$, where $n - 1 \geq a \geq b \geq 1$. These occur respectively at positions $2a + b + 1$, $a + 2b - 1$, and $a + 2b + k$ for $k = 0, 1, \ldots, a - b$. Thus, for fixed a and b, we get exactly one descent at position r for r from $a + 2b - 1$ to $2a + b + 1$ inclusive. Thus the coefficient of q^r in $W_1(3 \times n)$ is just the number of pairs (a, b), $n - 1 \geq a \geq b \geq 1$, such that $a + 2b - 1 \leq r \leq 2a + b + 1$. The only potential problem for unimodality is when

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$r = 2a + b + 1$, $r < \left\lfloor \frac{3n}{2} \right\rfloor$. We must show that there are at least as many pairs (a, b), $n - 1 \geq a \geq b \geq 1$, such that $r + 1 = a + 2b - 1$.

The solutions to $r = 2a + b + 1$ in positive integers when r is even are of the form $a = \frac{r-2i}{2}$, $b = 2i - 1$. We need $a \geq b$, or $i \leq \frac{r+2}{6}$, so there are $\left\lfloor \frac{r+2}{6} \right\rfloor$ in all. If r is odd, then the solutions are of the form $a = \frac{r-2i-1}{2}$, $b = 2i$. We need $a \geq b$, or $i \leq \frac{r-1}{6}$, so there are $\left\lfloor \frac{r-1}{6} \right\rfloor$ in all.

The solutions to $r + 1 = a + 2b - 1$ in positive integers when r is even are of the form $a = 2i$, $b = \frac{r+2-2i}{2}$. We need $n - 1 \geq a \geq b$, or $\frac{r+2}{6} \leq i \leq \frac{n-1}{2}$, so there are $\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lceil \frac{r+2}{6} \right\rceil + 1$ in all. If r is odd, then the solutions are of the form $a = 2i - 1$, $b = \frac{r+3-2i}{2}$. We need $n - 1 \geq a \geq b$, or $\frac{r+5}{6} \leq i \leq \frac{n}{2}$, so there are $\left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{r+5}{6} \right\rceil + 1$ in all.

Thus the result will follow from the following two statements.

$$\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lceil \frac{r+2}{6} \right\rceil + 1 \geq \left\lfloor \frac{r+2}{6} \right\rfloor \quad \text{where r} < \left\lfloor \frac{3n}{2} \right\rfloor \text{ is even}$$

$$\left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{r+5}{6} \right\rceil + 1 \geq \left\lfloor \frac{r-1}{6} \right\rfloor \quad \text{where r} < \left\lfloor \frac{3n}{2} \right\rfloor \text{ is odd}$$

These claims are straightforward to verify by just computing all 12 possible cases depending on $n \bmod 2$ and $r \bmod 6$, to get rid of the floor and ceiling functions. As an example, we do the case where n is even and r is 0 mod 6, which is the one that comes closest to failing. In this case, we need $\frac{n-2}{2} - \frac{r+6}{6} + 1 \geq \frac{r}{6}$, or $\frac{3n-6}{2} \geq r$. Since n is even, $\frac{3n}{2}$ is an integer. While we were only given $r < \left\lfloor \frac{3n}{2} \right\rfloor$, we also have that r is a multiple of 3 (as is $\frac{3n}{2}$), so in fact we do have that $\frac{3n-6}{2} \geq r$, as desired. $\square$

While the posets $W^s(P)$ are nice, they do leave something to be desired. For starters, they depend on the natural labeling of P. There is a simple bijection between $L(n_1, n_2, \ldots, n_k, m)$ and the disjoint union of $W^s \times L(p, m - s)$ (which proves the special case of (4.1) that we used), but in general it is not order-preserving in either direction, and the posets W^s do not always possess a SCD.

Let $h_k(P)$ denote the number of linear extensions of P with k descents. For P a ranked poset, it is known that the sequence $h_0, h_1, h_2, \ldots$ is symmetric, but without a bijective proof. In the special case where $P = \mathbf{2} \times \mathbf{n}$, we get this bijection for free

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from our proof of Theorem 6, since $W^s(P) \cong L(2, s, n-1-s) \cong L(2, n-1-s, s) \cong W^{n-1-s}(P)$. If we use self-duality along the way, then we get another bijection, so at least one of these will be different from a bijection found by J. Farley [12]. For products of more or larger chains, however, the analogous posets W^s are not isomorphic (though the polynomials W_s will be the same up to a power of q factor, so ideally a bijection would prove this as well), and the problem of finding a simple bijection is still open.

4.4   Chain Lengths

The longest chain in $L(n_1, \ldots, n_k)$ clearly consists of $n_1 \cdots n_k + 1$ elements. If a poset P has a SCD, then $\lambda_i(P)$ would just be the length of the i longest chains in the SCD, which is easily computed by looking at the levels of the poset up to the first level with at most i elements. Since the bottom levels of $L(n_1, \ldots, n_k)$ remain the same for large $n_1, \cdots, n_k$, this motivates the following theorem.

Theorem 8. *For fixed k and large $n_1, \ldots, n_k$,*

$\lambda_i(L(n_1, \ldots, n_k)) - \lambda_{i+1}(L(n_1, \ldots, n_k))$ *depends only on i.*

Proof: We start with a detailed proof for $k = 2$, so we're working with $L(m, n)$ where m and n are as large as we need them to be. The idea is to construct i disjoint chains of the maximum possible lengths. The elements of this poset are partitions with m parts of size at most n. To do this, the ith chain will go from some partition $\mu = \{\mu_1, \mu_2, \ldots\}$ (where m is assumed to be larger than the number of parts of μ) to $\{n, n, \ldots, n - \mu_2, n - \mu_1\}$. For the ith chain, the partition μ is selected from the lowest level that isn't covered by the first $i - 1$ chains. If we have more than one choice, it doesn't really matter, but for clarity say we take the one that occurs last in lexicographic order. The difficulty is keeping the chains disjoint, which we will do by passing through $\{n - i, n - i, \ldots, n - i\}$.

More precisely, we start our chain by incrementing the first part from μ_1 up to $n - i$ (since we can assume that $n > \mu_1 + i$), then the second part, and so on. Once

we've reached $\{n-i, n-i, \ldots, n-i\}$, then we increment the first part until it reaches n, then the second part, and so on, where the jth part from the end goes up to $n - \mu_j$. We just need to check that this will never give us the same partition in more than one chain. The only potential difficulty comes when incrementing the first two parts. Suppose μ and ν agree after their first two parts, but $\mu_1 > \nu_1$ (and $\mu_2 < \nu_2$). If μ is in the ith chain, then that chain will go through $\{n-i, \nu_2 \ldots\}$, so we need to ensure that ν is not in the first i chains. The partition $\{a, b, \ldots\}$ is on the chain that started at $\{b, b, \ldots\}$, so μ (the partition with the smaller second part) will indeed come first in our construction.

For larger k, we use the same trick, but since now we have some $k-1$-dimensional array of numbers up to n_k we just need to pick a natural labeling of $\mathbf{n_1} \times \cdots \times \mathbf{n_{k-1}}$ to tell us in what order to increment the parts. $\qquad\square$

The same proof works for a more general result, with Theorem 8 as the $j = 0$ case.

Corollary 1. *For fixed k, $j < k$, and $n_1, \ldots, n_j$, and large $n_{j+1}, \ldots, n_k$,*
$$\lambda_i(L(n_1, \ldots, n_k)) - \lambda_{i+1}(L(n_1, \ldots, n_k)) \text{ depends only on } i.$$

We can also consider the poset $M(n)$ of partitions into distinct parts not exceeding n. This poset is also known to have nice properties [25, 30], but a SCD or even a combinatorial proof of unimodality are still open. We can, however, prove that the analogue of Theorem 8 holds here as well. Actually, it follows from the fact that $M(n)$ is Peck, but here we give a proof by explicit construction of the chains of length $\lambda_1(M(n))$, $\lambda_2(M(n))$, etc.

Theorem 9. *For large n, $\lambda_i(M(n)) - \lambda_{i+1}(M(n))$ depends only on i.*

Proof: We start the ith chain at some unused partition in the first level with at least i elements. We increase the largest possible part until we reach $\{\lfloor \frac{n}{2} \rfloor - i, \lfloor \frac{n}{2} \rfloor - i - 1, \ldots, 1\}$ (i.e. we stop incrementing the first part once it reaches $\lfloor \frac{n}{2} \rfloor - i$, and so on). From there we start over incrementing the largest part until we reach $\{n, n - 1, \ldots, \lceil \frac{n}{2} \rceil + i + 1\}$.

Given a partition in $M(n)$, define the complementary partition to be the one that uses precisely the parts from 1 to n not used in the original partition. Now there is some partition near the top we're trying to reach, which will be complementary to one in the level where we started. Thus we can consider the chain from that complementary partition to $\{\left\lceil \frac{n}{2} \right\rceil + i, \left\lceil \frac{n}{2} \right\rceil + i - 1, \ldots, 1\}$ by increasing the largest possible part. But this is complementary to the point we've reached from the bottom, so taking complements tells us how to finish the chain. Again, it is easy to check that the i chains thus constructed are disjoint. $\square$

Chapter 5

b-ary partitions

5.1 Introduction

Let b and n be positive integers. A *b-ary* partition of n is an integer partition all of whose parts are powers of b. When $b = 2$, we call them *binary* partitions. The problem of enumerating binary partitions was considered as far back as 1750 by Euler. For more about the history of this problem, see [20] and its references.

Let $R_b(n)$ denote the set of all b-ary partitions of n. We define a partial order on $R_b(n)$ by saying α covers β if β can be obtained from α just by splitting a b^k into b b^{k-1}'s for some $k > 0$. The partial order is then the transitive closure of this covering relation. Latapy has shown that this ordering gives $R_b(n)$ the structure of a distributive lattice [20]. For an introduction to distributive lattices, see [32]. We will take a closer look at this structure, focusing on the subposet of join-irreducibles.

Knuth has asked if the set of binary partitions of n has a Gray path, i.e. a Hamiltonian path for the Hasse diagram of $R_2(n)$ [19]. This question was answered affirmatively by Colthurst and Kleber [10] independently of this work, though we construct the same Gray path. We will see that in fact the same is true for $R_b(n)$ whenever b is even, but not for all n when b is odd.

Let $e_b(n)$ denote the exponent of the highest power of b that divides n, e.g. $e_2(20) = 2$ and $e_3(81) = 4$.

5.2 Gray paths for b-ary partitions

The critical observation here is that $R_b(n)$, when n is a multiple of b, is just a copy of $R_b(n-b)$ (with b 1's appended to every partition) and a copy of $R_b(\lfloor n/b \rfloor)$ with every part multiplied by b. When n is not a multiple of b, say $n = bk + r$, where $0 < r < b$, then $R_b(n)$ is just $R_b(bk)$ with r ones appended to every partition.

Note that a Gray path does not always exist when b is odd. Let B denote the square of b so that we can use the notation b^i to denote i copies of b in a partition. Then $R_b(2B)$ does not have a Gray path. This is easily seen by examining the Hasse diagram shown in Figure 5-1, keeping in mind that b and B are odd, since we cannot hit all four of the elements that make up the top and bottom levels of size 2. It is unknown whether or not $R_b(n)$ has a Gray path for b odd and n large.
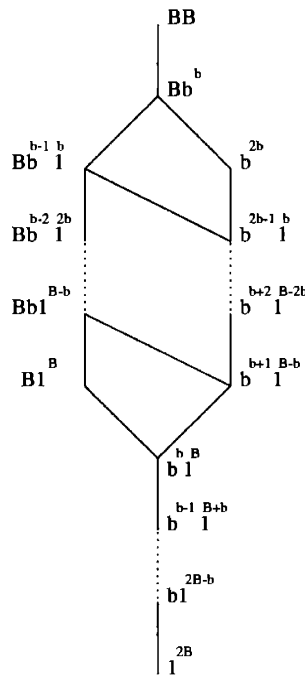

Figure 5-1: $R_b(2B)$.

Theorem 10. *If b is even, then $R_b(n)$ has a Gray path.*

Proof: The proof is by recursive construction. Suppose that b is even, and that n is a multiple of b (the result follows easily from this case when n is not a multiple of b since $R_b(bk + r)$, where $0 \leq r < b$, is isomorphic to $R_b(bk)$).

The subposet of b-ary partitions of n with exactly k 1's is isomorphic to $R_b(\frac{n-k}{b})$, with every part multiplied by b and then the k 1's appended. The idea is to go through these subposets via previously constructed Gray paths that either start or end at the partition $1^{(n-k)/b} \in R_b(\frac{n-k}{b})$. As we go up (i.e. as k decreases), we alternate whether we start or end at $1^{(n-k)/b}$, and the fact that b is even will make this construction work.

More precisely, we prove that there is a Gray path that starts at 1^n and ends at b-ary partition of n whose only (repeated) part is the largest power of b that divides n, namely $(b^{e_b(n)})^{n/b^{e_b(n)}}$. One can easily check that this works for small values of n. For example, with $b = 2$ and $n = 2, 4, 6, 8, 10, 12, 14, 16, 18$, and 20, the respective ending partitions are $2, 4, 2^3, 8, 2^5, 4^3, 2^7, 16, 2^9$, and 4^5.

The subpaths start at $1^{(n-k)/b}$ when $(n-k)/b$ is even, and end there when $(n-k)/b$ is odd. In particular, we start at 1^n, where $(n-k)/b = 0$ is even, then go to $b1^{n-b}$, where $(n-k)/b = 1$ is odd, and so on.

The subpath we want when b divides $(n-k)/b$ starts at $1^{(n-k)/b}$ in $R_b(\frac{n-k}{b})$, and ends at

$$\left(b^{e_b\left(\frac{n-k}{b}\right)}\right)^{\frac{n-k}{b \cdot b^{e_b((n-k)/b)}}}.$$

In $R_b(n)$, it starts at $b^{(n-k)/b}1^k$ and ends at

$$\left(b^{1+e_b\left(\frac{n-k}{b}\right)}\right)^{\frac{n-k}{b \cdot b^{e_b((n-k)/b)}}} 1^k.$$

For other even values of $(n - k)/b$, we append the appropriate number of 1's to the endpoints from the next largest multiple of b to get the endpoints in $R_b(\frac{n-k}{b})$, equivalently we insert the same number of b's in $R_b(b \cdot \lfloor \frac{n}{b} \rfloor)$ to get to $R_b(n)$. For the subpath when $1 + \frac{n-k}{2}$ is odd, we have the same endpoints as for $\frac{n-k}{2}$ but with a 1 appended in $R_b(1 + \frac{n-k}{b})$, equivalently a b inserted in $R_b(n)$, and we travel in the opposite direction.

Now the big question is whether these paths actually join together to form a Gray path for $R_b(n)$. In other words, we want to make sure that each path starts at something that covers the end of the previous path, and ends at something covered

by the start of the next one. Of course the answer is yes, the endpoints were chosen precisely to make this step work. An example of how this works is shown in Figure 5-2.
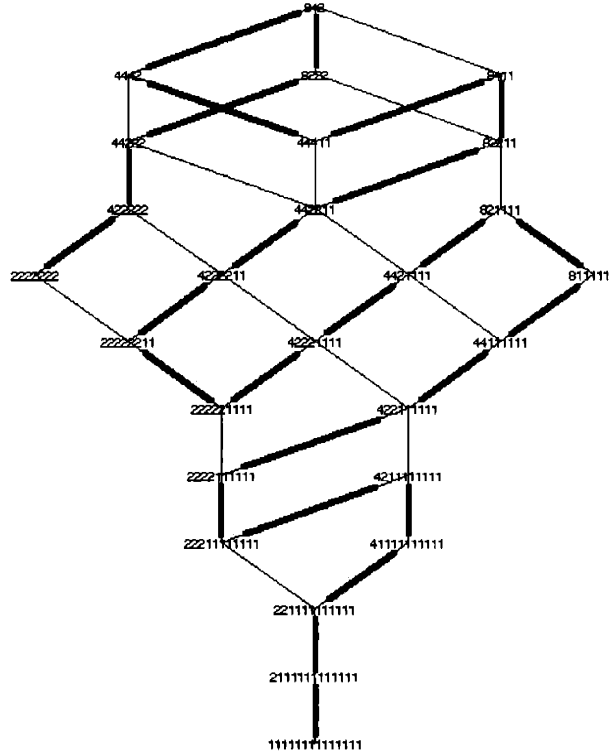

Figure 5-2: $R_2(14)$, with a Gray path highlighted.

Going from $\frac{n-k}{b}$ odd to even is easy, since the odd subpath ends at $1^{(n-k)/b}$ in $R_b(\frac{n-k}{b})$, or $b^{(n-k)/b}1^k$ in $R_b(n)$, which is covered by $b^{1+(n-k)/b}1^{k-b}$ in $R_b(n)$, or $1^{1+(n-k)/b}$ in $R_b(1 + \frac{n-k}{b})$, the start of the even subpath.

Going from even to odd is just as simple, but looks more technical. First consider the case where b divides $\frac{n-k}{b}$. The even subpath ends at

$$(b^{e_b(\frac{n-k}{b})})^{\frac{n-k}{b \cdot b^{e_b((n-k)/b)}}} \in R_b(\frac{n-k}{b}),$$

or

$$(b^{1+e_b(\frac{n-k}{b})})^{\frac{n-k}{b \cdot b^{e_b((n-k)/b)}}} 1^k \in R_b(n),$$

which is covered by

$$(b^{1+e_b(\frac{n-k}{b})})^{\overline{\frac{n-k}{b \cdot b^{e_b((n-k)/b)}}}} b1^k \in R_b(n),$$

or

$$(b^{e_b(\frac{n-k}{b})})^{\overline{\frac{n-k}{b \cdot b^{e_b((n-k)/b)}}}} 1 \in R_b(1 + \frac{n-k}{b}),$$

the start of the odd subpath. If $\frac{n-k}{b}$ is even but not a multiple of b, then going from the end in $R_b(\frac{n-k}{b})$ to the start in $R_b(1 + \frac{n-k}{b})$ we again just appended a 1. Thus we have indeed constructed a Gray path for $R_b(n)$. □

Note that the Gray path is not unique in general. For example, $R_2(10)$ has 6 different Gray paths starting at 1^{10}. The enumeration of these paths is an open problem. The one we have constructed was chosen because of its recursive structure. When $b = 2$, it is the same Gray path that was found by Colthurst and Kleber [10].

Suppose we want to find the successor or predecessor of a particular b-ary partition, say λ^0, with k 1's. Let $\lambda^i = \left\lfloor \frac{\lambda^{i-1}}{b} \right\rfloor$ for $i = 1, 2, \ldots$, i.e. we throw away the 1's and divide each remaining part of λ^{i-1} by b. In particular, $\lambda^1 \in R_2(\frac{n-k}{b})$. Whether we want to move forward or backward along the Gray path in $R_b(\frac{n-k}{b})$ starting at $1^{(n-k)/b}$ is determined by the sign of $(-1)^{|\lambda^1|}$, with positive meaning forward and negative meaning backward. Thus if we recursively work our way to λ^i, then we want to move according to the sign of $\epsilon = (-1)^{|\lambda^1|+|\lambda^2|+\cdots+|\lambda^i|}$.

Eventually we get down to $\lambda^k = (b^a)^x 1^y$, whose successor or predecessor (depending on ϵ) μ^k can be computed easily by hand. We can then define μ^{i-1} (for $1 \le i \le k$) by multiplying every part of μ^i by b and appending the appropriate number of 1's so that $|\mu^{i-1}| = |\lambda^{i-1}|$. Then μ^0 is the successor of λ^0. If instead we want the predecessor of λ^0, then just switch the sign of ϵ. A detailed description of this rule when $b = 2$ is given in [10].

5.3 Interval structure

Our first result of this section concerns the Möbius function of $R_b(n)$. While it follows from the fact that $R_b(n)$ is a distributive lattice, we give an explicit proof that

illuminates the structure at work.

Proposition 6. *The Möbius function of $R_b(n)$ only takes the values, 0, 1, and -1.*

Proof: Consider the interval $[x, y]$ in $R_b(n)$. If $x \not\leq y$, then of course $\mu(x, y) = 0$. If $x \leq y$, then x can be obtained from y by some sequence of breaking larger powers of b into smaller ones. We claim that the non-zero values of $\mu(x, y)$ will be when x comes from breaking at most one of each distinct part of y. For example, if $b = 2$ and $y = 8842221$, then the values of x such that $\mu(x, y) \neq 0$ are 88422111, 88222221, 84442221, 882222111, 844422111, 844222221, and 8442222111. Note that these make up the interval $[8442222111, 8842221]$, which is isomorphic to the boolean algebra B_3. In general, the set of such x will make up an interval isomorphic to B_i, where i is the number of distinct parts greater than 1 in y. The isomorphism is simple, just number the distinct parts greater than 1 in y 1 through i, then map x to the subset of $\{1, 2, \ldots, i\}$ of parts that haven't been broken yet. In the example above, if we call the 8's 1, the 4's 2, and the 2's 3, then 84442221 maps to $\{2, 3\}$. Thus any x of this form will have $\mu(x, y) = \pm 1$. Call the set of b-ary partitions of this form $B(y)$.

Suppose x is less than y but not in $B(y)$, then some distinct part of y was broken more than once to reach x. Suppose just one part is broken twice, and the others at most once, equivalently x is covered by some $z \in B(y)$. This z is unique, since the only way to get to $B(y)$ from x is to mend the part that was broken twice. There must be some maximal level at which such an x exists, so for the partitions x on that level we have $\mu(x, y) = 0$. We can now use induction to conclude that all $x < y$, $x \notin B(y)$, have $\mu(x, y) = 0$ since they are covered by at most one partition z with $\mu(z, y) \neq 0$. $\qquad\square$

A poset is a distributive lattice iff it is the set of order ideals $J(P)$ of a poset P, and P is isomorphic to the subposet of join-irreducibles in $J(P)$. The more general result that implies Proposition 6 is that for any distributive lattice $J(P)$, $\mu(I, I') = (-1)^{l(I, I')} = (-1)^{|I - I'|}$ if $[I, I']$ is a boolean algebra (i.e. if $I' - I$ is an antichain of P), and 0 otherwise. In order for this to be useful for computational purposes, we need to know something about the P such that $R_b(n) = J(P)$. Let

$Q_b(n)$ denote the subposet of join-irreducibles in $R_b(n)$, as shown in Figure 5-3 (the 1's at the end of each partition have been omitted for aesthetics), so $R_b(n) = J(Q_b(n))$ [32].
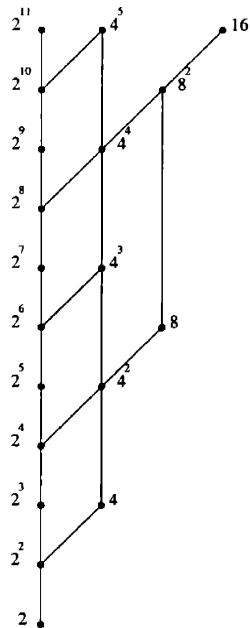

Figure 5-3: $Q_2(22)$.

Proposition 7. *The subposet $Q_b(n)$ of join-irreducibles in $R_b(n)$ is the set of partitions that have exactly one (possibly repeated) part greater than 1. Moreover, the meet-irreducibles are the partitions that have exactly one repeated part.*

Proof: Partitions with only one (possibly repeated) part greater than 1 only cover one other element, and hence are join-irreducible. If a partition has two distinct parts greater than 1, then we can break either one. Thus it covers (at least) two elements on the level below, and hence is their join. The reasoning for meet-irreducibles is similar. □

There are strong connections between a poset P and the distributive lattice $J(P)$. For example, the number of chains $\hat{0} = I_0 < I_1 < \ldots < I_m = \hat{1}$ of length m in $J(P)$ is the number of surjective order-preserving maps from P to $\mathbf{m}$ (an m-element chain), and the number of multichains $\hat{0} = I_0 \leq I_1 \leq \ldots \leq I_m = \hat{1}$ is the number of order-preserving maps [32]. In particular, antichains in P are in 1-1 correspondence

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with order ideals in P, order-preserving maps to $\{1, 2\}$, and elements of $J(P)$. Thus the problem of enumerating the elements of $R_b(n)$ is equivalent to enumerating the antichains of $Q_b(n)$.

Proposition 8. *For n even, $\lambda_1(Q_2(n)) = \frac{n}{2} + \alpha(n)$, where*

$$\alpha(n) = \max_i \{e_2(n - 2i) - (i + 1)\}.$$

Equivalently, $\alpha(n) = \max\{\alpha(n - 2) - 1, e_2(n) - 1\}$.

Proof: If we draw $Q_2(n)$ with every element on the lowest possible level, then we get a Hasse diagram like Figure 5-3. The chain that starts at 21^{n-2} and goes up to $2^i 1^{n-2i}$, then takes a turn to $4^{i/2} 1^{n-2i}$, $8^{i/4} 1^{n-2i}$, etc., contains $\frac{n}{2} + e_2(n - 2i) - (i + 1)$ elements, and it is easy to see that a chain of this form will hit every level of the Hasse diagram, and hence have maximal length. This proves the first statement. For the second statement, just observe that $\max\{\alpha(n - 2) - 1, e_2(n) - 1\} = \max\{e_2(n - 2i) - (i + 1)\}_{i>0} \cup \{e_2(n) - 1\} = \max_i \{e_2(n - 2i) - (i + 1)\}$. In other words, the $\alpha(n - 2) - 1$ term is the $i > 0$ case, and $e_2(n) - 1$ is the $i = 0$ case. $\qquad\square$

We can similarly prove a far more general result.

Theorem 11. *For $n \geq b^k$, we have*

$$\lambda_1(Q_b(n)) + \lambda_2(Q_b(n)) + \cdots + \lambda_k(Q_b(n)) = \left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{b^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{b^k} \right\rfloor + \gamma(n),$$

where

$$\gamma(n) = \max_{i_1, i_2, \cdots, i_k} \left\{ \sum_{j=1}^{k} (e_b(b^j \left\lfloor \frac{n}{b^j} \right\rfloor - b^j i_j) - (i_j + j)) \right\}.$$

The maximum is taken over all non-negative integers $i_1, i_2, \cdots, i_k$ such that

$$b \left\lfloor \frac{n}{b} \right\rfloor - b i_1 > b^2 \left\lfloor \frac{n}{b^2} \right\rfloor - b^2 i_2 > \cdots > b^k \left\lfloor \frac{n}{b^k} \right\rfloor - b^k i_k.$$

Proof: Use the same idea for drawing the Hasse diagram, putting everything on the lowest possible level. The tricky thing is that the largest union of two chains will not

in general be the union of the longest chain and one other chain. One example where this fails is $b = 2$, $n = 10$. The unique longest chain is $2 < 22 < 222 < 2222 < 44 < 8$, leaving the antichain $\{22222, 4\}$, while the largest union of two chains contains everything, e.g. $2 < 22 < 222 < 2222 < 22222$ and $4 < 44 < 8$.

To see that the formula for γ is correct, observe that the longest k chains will start off as chains of b's, b^2's, ..., and b^k's, and that the chain of b^2's won't use $(b^2)^{i+1}$ if the chain of b's branched off to use $(b^2)^i$ (since we're no worse off giving that branch to the b^2-chain and letting the b-chain branch higher up), etc. The condition $b\frac{n}{b} - bi_1 > b^2 \left\lfloor \frac{n}{b^2} \right\rfloor - b^2 i_2 > \cdots > b^k \left\lfloor \frac{n}{b^k} \right\rfloor - b^k i_k$ just says that the branches occur in such an order. The formula $e_b(b^j \left\lfloor \frac{n}{b^j} \right\rfloor - b^j i_j) - (i_j + j)$ is the length of the b^j-chain if it branches off at $(b^j)^{i_j}$, minus the length $\left\lfloor \frac{n}{b^j} \right\rfloor$ of the b^j-chain that doesn't branch. $\square$

We can also compute that the size of the largest antichain, using the proof of Proposition 6. Note each open interval of $R_b(n)$ is contractible or homotopy equivalent to a sphere of some dimension, and this number is the highest of those dimensions. It is also $\lambda_1'(Q_b(n))$, and the largest k for which $\lambda_k(Q_b(n)) > 0$.

Proposition 9. *The largest antichain in $Q_b(n)$ has size*

$$\lfloor \log_b(nb - n + b) \rfloor - 1.$$

Proof: By the proof of Proposition 6, the size of the largest antichain in $R_b(n)$ is the largest number of distinct parts greater than 1 in any b-ary partition of n. The smallest n which was a partition with k such parts is $b^k + b^{k-1} + \cdots + b^2 + b = \frac{b^{k+1} - b}{b-1}$. For any larger n, we can get this partition with 1's appended, so the largest antichain has size k for $\frac{b^{k+1} - b}{b-1} \leq n < \frac{b^{k+2} - b}{b-1}$. Rearrange to get $k + 1 \leq \log_b(nb - n + b) < k + 2$, and the result follows. $\square$

5.4 Related questions

The posets $R_b(n)$ do have some nice properties, such as being distributive lattices, but they lack others. In general, they are not self-dual, nor even rank-symmetric.

While $R_2(n)$ is rank-unimodal for small n, it is not in general. The first counterexample comes when $n = 30$. The sizes of the levels of $R_2(30)$, starting from the top, are 1, 4, 6, 7, 10, 11, 10, 11, ... To compute these level sizes, we use the fact that if $f_k(n, b)$ denotes the number of b-ary partitions of n with $n - k$ parts (hence 0 if k is negative), then we have a nice recursion.

Proposition 10. *If n is a multiple of b, then*

$$f_k(n, b) = f_k(n - b, b) + f_{k - \frac{(b-1)n}{b}}(n/b, b).$$

Proof: Given a b-ary partition of n with $n - k$ parts, there are two possibilities. If it has any parts equal to 1, then it has at least b of them (since $b|n$), so we can subtract those off to get a b-ary partition of $n - b$ into $n - b - k$ parts, counted by $f_k(n - b, b)$. Otherwise we have no parts equal to 1, so we can divide every part by b to get a b-ary partition of n/b into $n - k = \frac{n}{b} - (k - \frac{(b-1)n}{b})$ parts, counted by $f_{k - \frac{(b-1)n}{b}}(n/b, b)$. $\square$

In particular, for n even we have

$$f_k(n, 2) = f_k(n - 2, 2) + f_{k - \frac{n}{2}}(n/2, 2).$$

It seems reasonable to expect that $R_b(n)$ will similarly not be rank-unimodal for any fixed b and arbitrary n. We can still ask if these posets are Sperner, but unfortunately we don't have an answer at this time.

Chapter 6

And beyond...

6.1 Intersection lattices

A *partition* of a finite set S is a collection $\pi = \{B_1, \ldots, b_k\}$ of disjoint, nonempty subsets whose union is S. These subsets are called *blocks*.

Let Π_n denote the poset of all partitions of $[n] = \{1, 2 \ldots, n\}$, ordered by refinement. Thus the partition $\{\{1\}, \{2\}, \ldots, \{n\}\}$ is minimal and $\{\{1, 2, \ldots, n\}\}$ is maximal. This is a ranked lattice that is not rank-unimodal in general. Let $\Pi_{n,k}$ denote the poset of all partitions of $[n]$ with no blocks of size $2, 3, \ldots$, or $k - 1$. Thus $\Pi_{n,2} = \Pi_n$. For $k > 2$, we get an unranked lattice.

These lattices have geometric origins. $\Pi_{n,k}$ is the intersection lattice of a hyperplane arrangement called the k-equal arrangement [3, 4, 6].

We can see that Π_n is ranked, where the rank function $r(\pi)$ is n minus the number of blocks of π. Thus the bottom level is 0, and the top level is $n - 1$.

If we take the maximal leveled subposet of $\Pi_{n,k}$, for $k > 2$, then we get the partitions with at most one non-singleton block. These partitions occur on chains of $n - (k - 2)$ elements, since every non-singleton block requires us to skip $k - 2$ levels in Π_n while getting there from the bottom. But this subposet is clearly isomorphic to B_n, the poset of all subsets of $[n]$ ordered by inclusion, with levels 1 through $k - 1$ removed.

Let $\Pi^1_{n,k}$ denote the maximal leveled subposet of $\Pi_{n,k}$. Let $\Pi^2_{n,k}$ denote the maximal

leveled subposet of the poset that remains if we remove $\Pi^1_{n,k}$ from $\Pi_{n,k}$, and so on. For $k > 2$, $\Pi^m_{n,k}$ consists of the elements of $\Pi_{n,k}$ with exactly m non-singleton blocks (plus the all-singleton partition when $m = 1$). See Figure 6-1 for the decomposition of $\Pi_{10,3}$ into $\Pi^1_{10,3}$, $\Pi^2_{10,3}$, and $\Pi^3_{10,3}$, shown as three boxes. Since there are far too many elements to draw, we simply show the non-singleton block sizes on each level. For the elements not in $\Pi^1_{10,3}$, we put them on the lowest possible level.
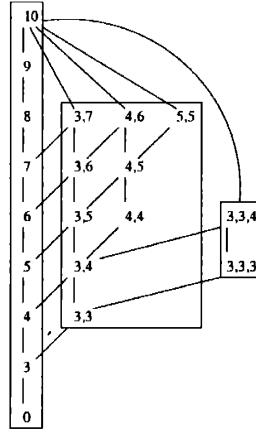

Figure 6-1: $\Pi_{10,3}$, in three leveled subposets.

B_n, also known as the *Boolean algebra* or *Boolean lattice* of rank n, has a symmetric chain decomposition [11]. Thus we can exploit that SCD to find long chains in $\Pi_{n,k}$. For stating our main result, we will use the notation i^j in a partition to denote the part i repeated j times.

Theorem 12. *Let* $\lambda(\Pi_{n,k}) = 1^{m_1} 2^{m_2} \ldots$, *and* $k > 2$, *then:*

(1) $m_i = 0$ *for* $i > n - k + 2$

(2) $m_{n-k+2} = 1$

(3) $m_{n-k+1} = 0$

(4) *For* $k \leq i \leq 2k - 3$,

$$\sum_{j=0}^{i} m_{n-j} = \binom{n}{i - k + 1}$$

In particular, $m_{n-k} = n - 1$

(5) *For* $n > 2k$, $\sum_{j=0}^{2k-2} m_{n-j} = \binom{n}{k}$

Beyond that, we can calculate any particular m_i *for* n *sufficiently large.*

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Proof: The longest chains have length $n - k + 2$, and $\Pi_{n,k}$ has a top and a bottom element, so (1), (2), and (3) follow immediately.

To get (4), we just use the SCD for B_n. Working just within $\Pi^1_{n,k}$, we can decompose the part from block size k to $n - k$ into $\binom{n}{k}$ chains. $\binom{n}{k-1}$ of these can be extended to the $n - k + 1$ blocks, and so on. Since $\Pi^m_{n,k}$ starts on the same level as the $k + m - 1$ blocks, and ends on the same level as the $n - (m - 1)(k - 1)$ blocks (remember we're putting everything on the lowest possible level), for $m > 1$ these elements do not matter yet.

For (5), the limiting level starts to be at the bottom instead of the top. Chains starting with a block of size k and containing $n - 2k + 2$ elements would end with a block of size $k + n - 2k + 2 - 1 = n - k + 1$, but there are more subsets of size k than size $n - (k - 1)$. However, the level that contains blocks of size $n - k + 1$ also contains the two-block partitions with block sizes k and $n - k$, $k + 1$ and $n - k - 1$, etc. The chains that end before this level make it to a block of size $n - k$, just one level lower, and there is an obvious bijection between these partitions and those with blocks of size $k, n - k$ for $n > 2k$, so we simply branch off into $\Pi^2_{n,k}$ for our final element.

Beyond that, we have to start looking at chains that start with two non-singleton blocks, so the numbers get ugly, but the key idea of exploiting injections from B_n and letting n be sufficiently large will still work. $\square$

6.2 More questions

A wise man (named Dan Kleitman) once said that the worst thing you can do to a problem is to solve it completely. With that in mind, we end with a discussion of things we don't know.

Of course there are the questions and conjectures that came up in each chapter, so we just have a couple more questions to ask here.

For the dominance lattice, all of the known values of $\lambda_i(P_n) - \lambda_{i+1}(P_n)$ are less than or equal to the difference we get for large n, but this is not the case for the Tamari lattice. Other than that, these lattices seem to have a surprising amount in

common. Do they share some property that makes the difference in chain lengths stabilize? Note that this stabilization does not happen for $\Pi_{n,k}$, at least not in the same way.

Is there a better method for computing $\lambda_4(P)$ for the dominance lattice and the Tamari lattice? Computing it isn't the hard part so much as proving that one can't do any better. The antichain decomposition we got from the levels when everything is placed on the lowest possible level only got us so far. Is there a better antichain decomposition for either of these two lattices that provides a better starting point?

While binary partitions have been around for over 250 years, general b-ary partitions have not received very much attention despite their nice structural properties, so there is still plenty of exploring to be done.

Except in the cases where an SCD is known to exist, $\lambda(P)$ has not been even partially computed for most posets P. While we've made some progress here, the problem of computational Greene-Kleitman theory is still far from being solved completely. Thus, at the very least, we have not done the worst possible thing for it.

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