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STABILITY, STOCHASTIC STATIONARITY AND GENERALIZED LYAPUNOV EQUATIONS FOR TWO-POINT BOUNDARY-VALUE DESCRIPTOR SYSTEMS[†]

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Abstract

In this paper, we introduce the concept of internal stability for two-point boundary-value descriptor systems (TPBVDSs). Since TPBVDSs are defined only over a finite interval, the concept of stability is not easy to formulate for these systems. The definition which is used here consists in requiring that as the length of the interval of definition increases, the effect of boundary conditions on states located close to the center of the interval should go to zero. Stochastic TPBVDSs are studied, and the property of stochastic stationarity is characterized in terms of a generalized Lyapunov equation satisfied by the variance of the boundary vector. A second generalized Lyapunov equation satisfied by the state variance of a stochastically stationary TPBVDS is also introduced, and the existence and uniqueness of positive definite solutions to this equation is then used to characterize the property of internal stability.

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1. Introduction

Noncausal physical phenomena arise in many fields of science and engineering. These phenomena correspond usually to processes evolving in space, instead of time. To model such processes, the usual state-space models familiar to system theorists are not appropriate, since these models were developed primarily to describe causality, in the sense that the "state" of a system at a given time is a summary of the past inputs sufficient to compute future outputs. One is then led to ask: what is a natural class of models to describe noncausal phenomena in one-dimension? It is the goal of this paper, as well as of earlier papers and reports [1]-[4], to suggest that perhaps the most natural class of discrete-time noncausal models in one-dimension is the class of two-point boundary-value descriptor systems (TPBVDSs). This conclusion is drawn from the observation that the impulse response of a time-invariant descriptor system is *noncausal*, and that the dynamics of these systems are symmetric with respect to forwards and backwards propagation. In addition, for systems defined over a finite interval, two-point boundary-value conditions will also enforce noncausality in the sense that both ends of the interval play a symmetric role in the expression of the boundary conditions.

The noncausality of discrete-time descriptor systems is a well known feature of these systems. It is for example much in evidence in the early work of Luenberger [5]-[6], where it is also pointed out that two-point boundary-value conditions are usually needed to guarantee well-posedness of these systems. In Lewis [7], it was shown that these systems could be decomposed into forwards and backwards propagating subsystems, so that their solution involves recursions in both time directions. However, in spite of these useful observations, it is fair to say that most of the literature on descriptor systems has focused mainly on issues of structure [8]-[10], and their implication for the control of descriptor systems [11]-[14]. This is primarily due to the fact that in continuous-time, descriptor systems display an impulsive behavior, which until recently has been the focus of most of the attention.

One of the most important influences for the work reported here has been the work by Krener [15]-[18] on the system-theoretic properties of standard (i.e., nondescriptor) continous-time boundary-value systems, and on the use of stochastic boundary-value systems to realize reciprocal processes. The results of Krener, as well as the related work of Gohberg, Kaashoek and Lerer [19]-[21], have pointed out that boundary-value linear systems have a rich internal structure, and can be used to model a wide class of nonMarkov, i.e. noncausal, stochastic processes. The results presented in this paper, as well those of [1]-[4] combine in some sense the degree of noncausality attributable to the boundary conditions, which was already present in Krener's work, with an additional source of noncausality, namely the noncausal dynamics of discrete-time descriptor systems.

Another important motivation for the study presented here is our own work on linear estimation of noncausal stochastic processes in one or several dimensions [22]-[24]. Since the framework proposed in [22] and [23] for the solution of noncausal estimation problems is totally general, and is applicable to absolutely any model in any dimension, one of our objectives has been to find 1-D models which display as much noncausality as possible, so that estimation results developed for these models will be easy to transpose to higher dimensions. This has led us in [4] to examine estimation problems for TPBVDSs. In this context, it was shown that the TPBVDS smoother was itself a TPBVDS which could be decoupled into forwards and backwards filters through the solution of certain generalized Riccati equations [25]. However, this study raised a number of system-theoretic questions: do reachability and observability guarantee the existence and uniqueness of positive-definite solutions for the generalized Riccati equations that we obtained? Is the estimator stable, and if so, in what sense, since TPBVDSs are defined only over a finite interval? More fundamentally, is it possible to define concepts of reachability, observability, and minimality for purely acausal systems such as TPBVDSs? In other words, we needed to develop a complete *system theory* for TPBVDSs, and the present paper is part of a sequence of papers devoted to the exposition of such a theory.

In [1], the concepts of *outwards and inwards processes*, which were originally introduced by Krener [16] for boundary-value systems, were developed for TPBVDSs, and were then used to define concepts of strong and weak reachability and observability. Several recursive solution schemes for TPBVDSs were also proposed, which rely on the forwards/backwards and inwards/outwards decompositions of these systems. These results were then specialized to deterministically stationary TPBVDSs in [2], and in this context, results linking reachability, observability, and minimality were obtained. Again, these results were closely related to corresponding results obtained by Krener, and by Gohberg and Kaashoek, for boundary value systems. The present paper contains the first significant departure from existing work on boundary value systems in the sense that we introduce a new concept, that of stability, which has not yet been used to study noncausal systems. As will become apparent below, the notion of stability is not easy to formulate for TPBVDSs, since these systems are defined over a finite interval. However, a relatively natural concept is that of *internal stability*, whereby as the length of the interval of definition of a TPBVDS grows, the effect of the boundary conditions on states located close to the center of the interval goes to zero. A theory of stability that parallels the standard theory for causal systems is developed by considering stochastically stationary TPBVDSs, and by showing that stochastic stationarity can be characterized in terms of generalized Lyapunov equations. The existence and uniqueness of positive-definite solutions to these Lyapunov equations is then characterized in terms of the property of internal stability. It turns out that the stability results developed in this paper will play a key role in our subsequent study of the stability of TPBVDS smoothers, and of the generalized Riccati equations presented in [4] and [25].

This paper is organized as follows. The properties of time-invariance and extendibility for two-point boundary-value descriptor systems are described and characterized in Section 2. These properties are then used to define the class of *deterministically stationary* TPBVDSs, to which we restrict our attention in this paper. In Section 3, two notions of stability, namely *stable extendibility* and *internal stability*, are introduced. Stable extendibility corresponds to the ability to extend the Green's function of a TPBVDS defined over a finite interval to an infinite interval, in such a way that both the dynamics and Green's function of the original system are preserved, and the extended Green's function is summable. However, it is shown that this concept of stability is not as fruitful as that of internal stability mentioned above. In Section 4, we examine stochastic TPBVDSs, and study in particular *stochastically stationary* systems. Two generalized Lyapunov equations which must be satisfied respectively by the state variance, and the variance of the boundary vector are introduced. In Section 5, the property of stochastic stationarity is characterized in terms of the second of these generalized Lyapunov equations. Finally, in Section 6 the existence and uniqueness of solutions to the generalized Lyapunov equation satisfied by the state variance is characterized in terms of the property of internal stability. The concluding Section 7 describes the role that the results of this paper are expected to play in the study of the TPBVDS smoothers and generalized Riccati equations of [4] and [25].

2. Time-Invariance and Extendibility

The two-point boundary-value descriptor systems (TPBVDS) considered in this paper satisfy the difference equation

$$Ex(k+1) = Ax(k) + Bu(k)\ , \quad 0 \le k \le N-1 \tag{2.1}$$

with the two-point boundary value condition

$$V_i x(0) + V_f x(N) = v\ . \tag{2.2}$$

Here E, A, and B are constant matrices, x and v are n-dimensional vectors, and u is an m-dimensional vector. Since the system theoretic properties of this class of systems, such as time-invariance, reachability, observability, and minimality have been studied in detail in [1]-[3], we review here only the concepts

It was shown in [1] that, without loss of generality, it can be assumed that the system (2.1)-(2.2) is in *standard-form*, i.e., it satisfies the following two properties: (i) there exists some scalars α and β such that

$$\alpha E + \beta A = I\ , \tag{2.3}$$

which implies that E and A commute; and (ii) the boundary matrices V_i and V_f are such that

$$V_i E^N + V_f A^N = I\ . \tag{2.4}$$

A special class of two-point boundary-value descriptor systems which is of great interest is the class of time-invariant TPBVDSs [2]-[3].

Definition 2.1: A TPBVDS is *time-invariant* if the Green's function $G(k,l)$ appearing in the solution

$$x(k) = A^k E^{N-k} v + \sum_{l=0}^{N-1} G(k,l) B u(l) \tag{2.5}$$

of the TPBVDS (2.1)-(2.2) depends only on the difference between arguments k and l, so that

$$G(k,l) = G(k-l) \ . \tag{2.6}$$

Unlike for causal systems, the fact that the matrices E and A are constant is not sufficient to guarantee that the TPBVDS (2.1)-(2.2) is time-invariant. The matrices E and A must also satisfy some properties in relation to the boundary matrices V_i and V_f. The following characterization of time-invariance was established in [2].

Theorem 2.1: A TPBVDS is time-invariant if and only if the matrices E and A commute with both V_i and V_f, i.e.,

$$[E, V_i] = [E, V_f] = [A, V_i] = [A, V_f] \ , \tag{2.7}$$

where

$$[X, Y] = XY - YX \ . \tag{2.8}$$

In the following, we shall restrict our attention to time-invariant TPBVDSs, and consequently, it will be assumed throughout the remainder of this paper that identity (2.7) is satisfied. In this case, the Green's function $G(k,l)$ can be expressed as (see [2], [3])

$$G(k,l) = G(k-l) = \begin{cases} V_i A^{k-l-1} E^{N-k+l} & k > l \\ -V_f E^{l-k} A^{N-1-l+k} & k \leq l \end{cases} \tag{2.9}$$

which clearly depends only on the difference between arguments k and l.

As was noted above, the reachability, observability, and minimality properties of time-invariant TPBVDSs were previously studied in [2]-[3]. In this paper, we define and characterize the concept of stability for time-invariant TPBVDSs and relate it to the property of stochastic stationarity through the use of a generalized Lyapunov equation. However, an important issue which arises when we attempt to define the concept of stability for system (2.1)-(2.2) is that this system is defined only over a finite interval. It is therefore of interest to see whether a given time-invariant TPBVDS defined over a finite interval is extendible to a larger interval in some appropriate way.

Definition 2.2: A time-invariant TPBVDS given by (2.1)-(2.2) is *extendible* if given any interval $[K,L]$ containing $[0,N]$, i.e., such that $K \leq 0 \leq N \leq L$, there exists a TPBVDS over this larger interval with the same dynamics as in (2.1), but with new boundary matrices $V_i(K,L)$ and $V_f(K,L)$ such that:

(i) The new extended system is time-invariant.

(ii) The Green's function $G(k-l)$ of the original system is the restriction of the Green's function $G_e(k-l)$ of the new extended system:

$$G(k-l) = G_e(k-l) \quad \text{for} \quad |k-l| \leq N \ . \tag{2.10}$$

In [2], it was shown how to ***reduce*** the interval of definition of the TPBVDS (2.1)-(2.2) from $[0,N]$ to a subinterval $[K,L]$, with $0 \leq K \leq L \leq N$, in such a way that the Green's function of the new system defined over the subinterval $[K,L]$ is the restriction to this smaller interval of the Green's function of the original system defined over $[0,N]$. The dynamics (2.1) of the TPBVDS remain the same, but the new matrices $V_i(K,L)$ and $V_f(K,L)$ which specify the boundary condition

$$V_i(K,L)x(K) + V_f(K,L)x(L) = v(K,L) \tag{2.11}$$

over the smaller interval $[K,L]$ can be viewed as obtained by "moving in" the boundary matrices V_i and V_f appearing in (2.2), and are given by

$$V_i(K,L) = V_i E^{N-L+K} \tag{2.12a}$$

$$V_f(K,L) = V_f A^{N-L+K} \ . \tag{2.12b}$$

Conceptually, the problem of *extending* the TPBVDS from interval $[0,N]$ to a larger interval $[K,L]$, in such a way that the Green's function $G(k-l)$ is preserved over the smaller interval $[0,N]$, is just the opposite problem of the one that we have just analyzed. In this case, after the TPBVDS (2.1)-(2.2) has been extended from interval $[0,N]$ to $[K,L]$, when we move back the boundaries from

$[K,L]$ to their original locations, we should recover the original system. This implies that the boundary matrices $V_i(K,L)$ and $V_f(K,L)$ for the larger interval must be such that

$$V_i = V_i(K,L)E^{L-K-N} \tag{2.13a}$$

$$V_f = V_f(K,L)A^{L-K-N} . \tag{2.13b}$$

The constraints (2.13) can be used to obtain the following characterization of extendibility.

Theorem 2.2: A time-invariant TPBVDS is extendible if and only if the following two conditions are satisfied:

(i) $Ker(E^n) \subset Ker(V_i)$ (2.14a)

(ii) $Ker(A^n) \subset Ker(V_f)$. (2.14b)

Proof: The necessity of the above two conditions is just a consequence of setting $L-K = N+n$ inside constraints (2.13). To prove sufficiency, consider an arbitrary interval $[K,L]$ containing $[0,N]$. Then, note that conditions (2.14a) and (2.14b) are equivalent to requiring that all the generalized eigenvectors of E and A corresponding to the zero eigenvalue should be in the null spaces of V_i and V_f, respectively. In other words, we have

$$Ker(E^s) \subset Ker(V_i) \tag{2.15a}$$

$$Ker(A^s) \subset Ker(V_f) \tag{2.15b}$$

for all integers s. The relations (2.15a) and (2.15b) are equivalent to

$$\mathrm{Im}(V_i^T) \subset \mathrm{Im}((E^s)^T) \tag{2.16a}$$

$$\mathrm{Im}(V_f^T) \subset \mathrm{Im}((A^s)^T) , \tag{2.16b}$$

and setting $s = L-K-N$, this implies that there exists matrices $V_i(K,L)$ and $V_f(K,L)$ satisfying the constraints (2.13). However, these matrices are in general not unique and do not necessarily commute with E and A, so that the extended system over the larger interval $[K,L]$ may not be time-invariant. It turns out, however, that there exists a special choice of boundary matrices $V_i(K,L)$ and $V_f(K,L)$ such that the extended system is time-invariant and is itself extendible, i.e., it satisfies the commutation relations (2.7) as well as conditions (2.14a) and (2.14b), where the matrices V_i and V_f are replaced respectively by $V_i(K,L)$ and $V_f(K,L)$. These boundary matrices can be obtained as follows. Consider the transformation

$$E = XJ_E X^{-1} \tag{2.17a}$$

$$A = XJ_A X^{-1} \tag{2.17b}$$

of E and A into their Jordan forms J_E and J_A, where the fact that E and A admit the same set of generalized eigenvectors is a direct consequence of (2.3). In general, J_E and J_A may contain blocks corresponding to eigenvalues which are equal to 0. Let now

$$\tilde{E} = X\tilde{J}_E X^{-1} \tag{2.18a}$$

$$\tilde{A} = X\tilde{J}_A X^{-1} , \tag{2.18b}$$

where $\tilde{J}_E$ and $\tilde{J}_A$ are matrices obtained by replacing the zero eigenvalues of J_E and J_A by eigenvalues equal to 1. Then, it is easy to check that the boundary matrices given by

$$V_i(K,L) = V_i(\tilde{E}^{L-K-N})^{-1} \tag{2.19a}$$

$$V_f(K,L) = V_f(\tilde{A}^{L-K-N})^{-1} \tag{2.19b}$$

satisfy the constraints (2.13), as well as the time-invariance condition (2.7) and extendibility conditions (2.14). Note also that the new extended system is in standard form, i.e., it satisfies (2.3) and (2.4), where N is replaced by the new interval length $L-K$. □

The conditions (2.14) seem to indicate that if we restrict our attention to extendible TPBVDSs, we may be ignoring interesting systems which are time-invariant, but not extendible. It turns out, however, that given an arbitrary time-invariant TPBVDS defined over $[0,N]$, where it is assumed that $N>2n$, there exists an "almost identical" extendible system. By "almost identical", we mean here that for any input sequence $u(l)$, the states $x(k)$ and $x'(k)$ of the two systems are identical for $k\in[n,N-n]$. In fact, by examining expression (2.9) for the Green's function $G(k-l)$, we see that the almost identical extendible

TPBVDS corresponding to a time-invariant, but not extendible, TPBVDS can be obtained by replacing V_i and V_f by V_i' and V_f' such that

(i) V_i' is the lowest rank matrix satisfying $V_i' E^n = V_i E^n$,

(ii) V_f' is the lowest rank matrix satisfying $V_f' E^n = V_f E^n$.

The above choice has for objective to guarantee that V_i and V_f annihilate all the nilpotent blocks of E and A. Since the n^{th} and higher powers of these blocks are zero in any case, the effect of this modification is seen only near the boundaries 0 and N.

On the basis of the above observations, it is clear that the class of time-invariant, extendible TPBVDSs is in fact quite large, and according to the terminology introduced in [2]-[3] it will be called here the class of *deterministically stationary* TPBVDSs, and most of the results described in this paper will concern this specific class of systems.

3. Stability

The extendibility property of deterministically stationary TPBVDSs is an important feature that will be useful below to characterize a concept of stability called *stable extendibility.* It turns out, however, that this concept of stability leads to relatively uninteresting results, and in fact there exists a more interesting concept of stability for TPBVDSs, called *internal stability.* Both of these concepts are now defined.

A. Notions of Stability

According to our definition of a deterministically stationary TPBVDS, it is always possible to extend the domain of definition of such a system. Another way of looking at this property is that any stationary TPBVDS can be obtained by moving in the boundaries of another stationary system defined over a larger interval. An interesting question which is related to the issue of stability is under what conditions we can push back the boundaries to $\pm\infty$ in a meaningful way, so that the TPBVDS (2.1)-(2.2) can be viewed as part of a system defined over an infinite interval.

Definition 3.1: A deterministically stationary TPVDS defined over $[0,N]$ admits a *stable extension* if the Green's function $G_e(k)$ of the TPVDS obtained by extending the interval of definition to the whole real line is summable, i.e.,

$$\sum_{-\infty}^{+\infty} ||G_e(k)|| < \infty , \tag{3.1}$$

where $||.||$ denotes here the matrix norm induced by the Euclidean norm for vectors of R^n.

The above characterization describes one situation where the issue of stability arises for TPBVDSs. However, there exists a second situation which is actually more meaningful, and which leads to a different concept of stability. In this second situation, we examine a time-invariant, not necessarily extendible TPBVDS defined over a finite interval, and where the boundary condition (2.2) corresponds to a physical constraint of the problem which cannot be modified. In this case, when the dynamics (2.1) and boundary condition (2.2) are fixed, we would like to study the effect of increasing the size of the domain $[0,N]$ of definition of the TPBVDS on the state variables $x(k)$ located close to the center of this domain . One issue which arises in this context is that if the TPBVDS (2.1)-(2.2) is originally in standard form for a length N_0 of the interval of definition, and if we increase the length to N without changing the matrices V_i, V_f and the vector v appearing in (2.2), the system will not remain in standard form, since identity (2.4) is not satisfied for $N>N_0$. Observe however that the boundary condition (2.2) is not affected by a left multiplication by an invertible matrix. Consequently, if we renormalize (2.2) by a left multiplication by $(V_i E^N + V_f A^N)^{-1}$ and change the matrices V_i, V_f and the vector v accordingly, the TPBVDS will be in standard form. In this context, it is possible to describe stability as follows.

Definition 3.2: The time-invariant TPBVDS (2.1)-(2.2) is *internally stable* if as the length N of the interval of definition tends to infinity , the effect of the boundary value v on any $x(k)$ located near the mid-section of interval $[0,N]$ goes to zero, i.e.,

$$\lim_{N\to\infty} E^{N/2} A^{N/2} (V_i E^N + V_f A^N)^{-1} = 0 \ . \tag{3.2}$$

To interpret condition (3.2), note that according to (2.5), and taking into account the renormalization described above to put the TPBVDS in standard form as the interval length N is increased, the effect of the boundary vector v on state $x(k)$ is given by $A^k E^{N-k}(V_i E^N + V_f A^N)^{-1} v$. Thus, for $k = N/2$, which corresponds to a point in the middle of interval $[0,N]$, the effect of v on $x(N/2)$ is $E^{N/2} A^{N/2}(V_i E^N + V_f A^N)^{-1} v$.

As an illustration of the above concept of stability, consider a system that describes the heat distribution around a ring. Since the ring is closed, this system has a periodic boundary condition $x(0) = x(N)$, which is independent of the size of the ring. In this case, if a perturbation in heating conditions is applied to one point of the ring, one would expect that as the size of the ring increases, the effect of this perturbation will become smaller and smaller for points which are located on the opposite side of the ring.

As will be shown below, it is possible to obtain necessary and sufficient conditions that characterize the properties of stable extendibility and internal stability for TPBVDSs. However, the conditions that we shall obtain are quite different, and consequently, the two concepts of stability described above do not coincide.

B. Decomposition of a Time-invariant TPBVDS

The characterizations of stable extendibility and internal stability that will be obtained below rely on the decomposition of a time-invariant TPBVDS into forwards, backwards, and marginally stable components. The starting point of this decomposition is the following result, which was already used in [4].

Lemma 3.1: Given a TPBVDS in standard form, there exists invertible matrices F_D and T such that

$$E_D = F_D E T = \begin{bmatrix} I & 0 & 0 \\ 0 & A_b & 0 \\ 0 & 0 & I \end{bmatrix} \tag{3.3a}$$

$$A_D = F_D A T = \begin{bmatrix} A_f & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U \end{bmatrix}, \tag{3.3b}$$

where A_f and A_b have eigenvalues inside the unit circle, and U has eigenvalues on the unit circle.

The above decomposition is just a modification of the Weierstrass decomposition of a regular matrix pencil (see [26], p. 28). The standard form condition (2.3) guarantees here that the pencil $zE - A$ is regular. Note that the transformation (3.3) can be achieved by left-multiplication of (2.1) by F_D and by performing the state transformation

$$x(k) = Tx_D(k) . \tag{3.4}$$

However, one undesirable aspect of this transformation is that the new TPBVDS is not in standard form, since the matrices E_D and A_D do not satisfy (2.3) for any choice of α and β. This leads us temporarily to rescale the TPBVDS by left multiplication by

$$F_R = (\alpha E_D + \beta A_D)^{-1} , \tag{3.5}$$

for some appropriate choice of α and β, so that

$$E_R = F_R E_D = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \tag{3.6a}$$

$$A_R = F_R A_D = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} , \tag{3.6b}$$

where

$$E_1 = (\alpha I + \beta A_f)^{-1} , E_2 = (\alpha A_b + \beta I)^{-1} A_b ,$$

$$E_3 = (\alpha I + \beta U)^{-1} , \tag{3.7a}$$

and

$$A_1 = (\alpha I + \beta A_f)^{-1} A_f , A_2 = (\alpha A_b + \beta I)^{-1} ,$$

$$A_3 = (\alpha I + \beta U)^{-1} U . \tag{3.7b}$$

Taking into account the fact that the eigenvalues of A_f and A_b are inside the unit circle, and those of U are on the unit circle, it is easy to check that the blocks E_1, E_2 and E_3 do not have eigenvalues in common. Similarly, the blocks A_1, A_2 and A_3 have different eigenvalues.

Combining now transformations (3.3) and (3.5), B becomes $B_R = FB$, where $F = F_R F_D$, and the boundary matrices become

$$V_{Ri} = L_R V_i T \quad , \quad V_{Rf} = L_R V_f T \ , \tag{3.8}$$

where the normalizing matrix L_R is selected here such that relation (2.4) is satisfied by the new TPBVDS. Finally, if the original TPBVDS was time-invariant, the new TPBVDS is also time-invariant since its Green's function is related to the original Green's function through

$$G_R(k-l) = T^{-1} G(k-l) F^{-1} \ . \tag{3.9}$$

In this case, since the TPBVDS specified by (3.6) and (3.8) is both time-invariant and in standard form, we can invoke Theorem 2.1 to conclude that the matrices E_R, A_R, V_{Ri} and V_{Rf} satisfy the commutation relation (2.7).

In addition to Lemma 3.1, we will need the following result.

Lemma 3.2: Let

$$S = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} . \tag{3.10}$$

If $ST = TS$ and no eigenvalue of U equals any eigenvalue of V, then $X = Y = 0$.

Proof: The relation $ST = TS$ implies that $UX = XV$, and thus

$$p(U)X = Xp(V)$$

for any polynomial $p(.)$. Let a be an arbitrary generalized eigenvector of V, and let λ be the corresponding eigenvalue. Then, there exists an integer j such that

$$(\lambda I - V)^j a = 0$$

so that for $p(x) = (\lambda - x)^j$,

$$p(U)Xa = 0 \ .$$

Now, if $Xa \neq \mathbf{0}$, then λ must be an eigenvalue of U, which is a contradiction, since λ is an eigenvalue of V. Thus, we must have $Xa = \mathbf{0}$. However, since the generalized eigenvectors a of V span the whole space, this implies that $X = 0$, which is the desired result. □

When the original TPBVDS (2.1)-(2.2) is time-invariant, we have shown above that the transformed TPBVDS (3.6),(3.8) is such that the boundary matrices V_{Ri} and V_{Rf} commute with E_R and A_R, which have the block structure (3.6), where the blocks along the diagonal of E_R, and along the diagonal of A_R, do not have eigenvalues in common. Consequently, by applying Lemma 3.2, we can conclude that in this case, V_{Ri} and V_{Rf} are also block diagonal, i.e.,

$$V_{Ri} = \begin{bmatrix} V_{i1} & 0 & 0 \\ 0 & V_{i2} & 0 \\ 0 & 0 & V_{i3} \end{bmatrix} \quad \text{and} \quad V_{Rf} = \begin{bmatrix} V_{f1} & 0 & 0 \\ 0 & V_{f2} & 0 \\ 0 & 0 & V_{f3} \end{bmatrix} . \tag{3.11}$$

We are now in position to derive the main result of this section.

Theorem 3.1: *(Decomposition of a time-invariant TPBVDS into forwards, backwards and marginally stable components):* Through the use of a state transformation T, and by left multiplication of (2.1) and (2.2) by invertible matrices F and L, an arbitrary time-invariant TPBVDS can be decomposed into three decoupled subsystems of the form

$$x_f(k+1) = A_f\, x_f(k) + B_f\, u(k) \quad , \quad V_{i1} x_f(0) + V_{f1} x_f(N) = v_1 \, , \tag{3.12a}$$

$$x_b(k) = A_b\, x_b(k+1) - B_b\, u(k) \quad , \quad V_{i2} x_b(0) + V_{f2} x_b(N) = v_2 \, , \tag{3.12b}$$

$$x_m(k+1) = U x_m(k) + B_m\, u(k) \quad , \quad V_{i3} x_m(0) + V_{f3} x_m(N) = v_3 \, , \tag{3.12c}$$

where the matrices A_f and A_b have their roots inside the unit circle, and U has its roots on the unit circle. The subsystems (3.12a)-(3.12c) are time-invariant and in standard form, and correspond respectively to the forwards, backwards and marginally stable components of the original TPBVDS (2.1)-(2.2).

Proof: As was already shown above, an arbitrary time-invariant TPBVDS can be brought to the form (3.6)-(3.8), where the boundary matrices V_{Ri} and V_{Rf} have the block diagonal structure (3.11). The renormalization (3.4) can be undone, and if we denote

$$x_D(k) = \begin{bmatrix} x_f(k) \\ x_b(k) \\ x_m(k) \end{bmatrix} \quad , \quad B_D = F_D B = \begin{bmatrix} B_f \\ B_b \\ B_m \end{bmatrix} , \tag{3.13}$$

we obtain the decomposition (3.12). In this decomposition, since the original system was time-invariant, each subsystem is time invariant and individually in standard form, although in order to guarantee that (2.4) is satisfied for each subsystem, we may need to rescale the boundary matrices V_{ik}, V_{fk} and boundary vector v_k for $k = 1,2,3$, by left multiplication by appropriately selected invertible matrices. □

C. Characterization of Stable Extendibility and Internal Stability

An interesting aspect of the decomposition (3.12) of a time-invariant TPBVDS is that it reduces the study of stable extendibility and internal stability for a TPBVDS to the study of these properties for each of its components. We consider first the forwards stable component.

Lemma 3.3: Consider a time-invariant TPBVDS given by

$$x(k+1) = Ax(k) + Bu(k) \tag{3.14a}$$

$$V_i x(0) + V_f x(N) = v \tag{3.14b}$$

where A has all its roots inside the unit circle. Then, the system (3.14) is internally stable if and only if the matrix V_i is invertible. If the system (3.14) is extendible, i.e., if $Ker(A^n) \subset Ker(V_f)$, it is stably extendible if and only if $V_f = 0$, in which case the system is causal.

Proof: Taking into account the definition (3.2) of internal stability, we see that (3.14) is internally stable if and only if

$$\lim_{N \to \infty} A^{N/2}(V_i + V_f A^N)^{-1} = 0 \ ,$$

which is clearly equivalent to requiring that V_i should be invertible. To study stable extendibility, it is convenient to note that by using a procedure similar to the one employed to obtain decomposition (3.12), the system (3.14) can be transformed so that

$$A = \begin{bmatrix} J & 0 \\ 0 & M \end{bmatrix},$$

where M is a nilpotent matrix and J is invertible, and

$$V_i = \begin{bmatrix} V_{Ji} & 0 \\ 0 & V_{Mi} \end{bmatrix} \ , \quad V_f = \begin{bmatrix} V_{Jf} & 0 \\ 0 & V_{Mf} \end{bmatrix}.$$

Then, the extendibility condition $Ker(A^n) \subset Ker(V_f)$ implies that we must have

$$V_{Mf} = 0 \ . \tag{3.15}$$

Furthermore, by using the procedure described in Theorem 2.1, it is easy to check that the Green's function $G_e(k)$ of the system which extends the Green's function of system (3.14) to the whole line is given by

$$G_e(k) = V_i A^{k-1} \quad \text{for } k > 0 \ , \tag{3.16a}$$

and

$$G_e(k) = \begin{bmatrix} -V_{Jf} J^{N-1+k} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for } k \leq 0 \ . \tag{3.16b}$$

Since P has its roots inside the unit circle, $G_e(k)$ diverges as $k \longrightarrow -\infty$, unless

$$V_{Jf} = 0 \ . \tag{3.17}$$

Combining (3.15) and (3.17), we see that the TPBVDS (3.14) is stably extendible if and only if $V_f = 0$, which is the desired result. In this case, the system (3.14) is causal, and the standard form relation (2.4) implies that $V_i = I$, which is obviously invertible. We can therefore conclude that in this case stable extendibility implies internal stability. □

Lemma 3.3 can then be used to obtain the following characterization of stable extendibility.

Theorem 3.2: An arbitrary deterministically stationary TPBVDS is stably extendible if and only if the decomposition (3.12) of this system is such that

$$V_{f1} = V_{i2} = 0 \ , \tag{3.18}$$

and the system does not have any eigenmode on the unit circle, i.e., it does not contain a marginally stable component of the form (3.12c).

Proof: Condition (3.18) is a direct consequence of applying Lemma 3.3 to the forwards and backwards stable components (3.12a) and (3.12b) of the TPBVDS. Then, if we consider the marginally stable component, we see that its extended Green's function is

$$G_{e3}(k) = \begin{cases} V_{i3}U^{k-1} & \text{for } k > 0 \\ -V_{f3}U^{N-1+k} & \text{for } k \leq 0 \end{cases} .$$

Since U has all its roots on the unit circle, $G_{e3}(k)$ will not be summable for any choice of boundary matrices V_{i3} and V_{f3} satisfying (2.4). Consequently, the TPBVDS will be stably extendible only if it does not have any eigenmode on the unit circle. □

The above characterization shows that the class of stably extendible TPBVDSs is not particularly interesting since it consists of systems which are obtained by combining forwards and backwards causal and stable subsystems. It turns out that the concept of internal stability is more interesting, since it can be characterized as follows.

Theorem 3.3: A time-invariant TPBVDS is internally stable if and only if the decomposition (3.12) of this system is such that boundary matrices V_{i1} and V_{f2} are invertible, and the system does not have any eigenmode on the unit circle.

Proof: The first part of the above characterization is obtained by applying Lemma 3.3 to the forwards and backwards components (3.12a) and (3.12b). The condition concerning the eigenmodes on the unit circle is derived by noting that no choice of boundary matrices V_{i3} and V_{f3} satisfying (2.4) will guarantee that

$$\lim_{N\to\infty} U^{N/2}(V_{i3} + V_{f3}U^N)^{-1} = 0 . \qquad \square$$

Comparing Theorems 3.2 and 3.3, we see that stable extendibility implies internal stability, so that internal stability is the weaker and more interesting of these two properties. In fact, from this point on, we will restrict our attention to internal stability.

4. Stochastic TPBVDSs and Generalized Lyapunov Equations

In this section, we study the class of stochastic TPBVDSs given by (2.1)-(2.2), where $u(k)$ is a zero-mean white Gaussian noise with unit intensity, and where v is a zero-mean Gaussian random vector independent of $u(k)$ for all k, and with covariance Q. Thus, we have

$$M[u(k)u^T(l)] = I\delta(k-l) , \tag{4.1}$$

where $M[z]$ denotes the mean of a random variable z, and $\delta(k)$ is the Kronecker delta function. In addition, it is assumed that the TPBVDS (2.1)-(2.2) is deterministically stationary and in standard form. The assumption of deterministic stationarity is quite important, and all the results of this paper concerning stochastic TPBVDSs are restricted to this class of systems.

In the continuous-time case, and for the usual nondescriptor state-space dynamics, a related class of stochastic boundary-value systems was examined by Krener [17],[18], who studied the relation existing between this class of systems and reciprocal processes. In particular, Krener considered the problem of realizing reciprocal processes with stochastic boundary-value systems. Our goal here is more limited in scope, in the sense that we shall seek only to obtain a complete set of conditions under which a stochastic TPBVDS of the form (2.1)-(2.2) is stochastically stationary. It turns out that the characterization that will be obtained involves a Lyapunov equation for the boundary variance Q which generalizes the standard Lyapunov equation for stationary Gauss-Markov processes.

Definition 4.1: A TPBVDS is *stochastically stationary* if

$$M[x(k)x^T(l)] = R(k,l) = R(k-l) . \tag{4.2}$$

It should be clear that if the TPBVDS (2.1)-(2.2) is stochastically stationary, the *variance matrix* $P(k) = R(k,k)$ of $x(k)$ must be constant, i.e., $P(k) = P$ for all k. Thus, our first step at this point will be to characterize completely the matrix $P(k)$. It can be expressed as follows in terms of the matrices E, A B, V_i, V_f, and Q describing the stochastic TPBVDS (2.1)-(2.2). Let

$$\Pi(k) = \sum_{j=0}^{k} A^{k-j}E^jBB^T(A^{k-j}E^j)^T . \tag{4.3}$$

Then, using the Green's function solution (2.5), (2.9), multiplying by its transpose, and taking expected values, we obtain

$$P(k) = A^k E^{N-k} Q (A^k E^{N-k})^T + (V_i E^{N-k}) \Pi(k-1)(V_i E^{N-k})^T + (V_f A^k)\Pi(N-1-k)(V_f A^k)^T . \quad (4.4)$$

This expression can also be rewritten as

$$P(k) = A^k E^{N-k} Q (A^k E^{N-k})^T + R_w(k) R_w^T(k) , \quad (4.5)$$

where $R_w(k)$ is the weak reachability matrix (see [2]-[3])

$$R_w(k) = \Big[V_i E^{N-k} R_s(k) \quad V_f A^k R_s(N-k) \Big], \quad (4.6a)$$

and where $R_s(k)$ is the strong reachability matrix

$$R_s(k) = \Big[A^{k-1}B \;\; EA^{k-2}B \;\; \cdots \;\; E^{k-1}B \Big]. \quad (4.6b)$$

When the system is weakly reachable, $R_w(k)$ has full rank for $k \in [n, N-n]$ (see [2]-[3]). This means that $R_w(k) R_w^T(k)$ is positive definite, and therefore $P(k)$ is positive definite for $n \leq k \leq N-k$. Thus, when the TPBVDS (2.1)-(2.2) is weakly reachable and has a constant variance P, and $N > 2n$, we can conclude from the above result that P is positive definite.

The expression (4.4) for $P(k)$ is an *explicit description*, and is valid in general. However, as in the causal case, where $P(k)$ satisfies a time-dependent Lyapunov equation, it is also possible to obtain an *implicit description* for $P(k)$ in the form of a recursion with boundary conditions. Specifically, multiplying both sides of equations (2.1) and (2.2) by their transposes, using the Green's function solution (2.5), (2.9), and taking expected values, it can be shown that $P(k)$ satisfies the TPBVDS

$$EP(k+1)E^T - AP(k)A^T = (V_i E^N) BB^T (V_i E^N)^T - (V_f A^N) BB^T (V_f A^N)^T \quad (4.7a)$$

$$V_i P(0) V_i^T - V_f P(N) V_f^T = (V_i E^N) Q (V_i E^N)^T - (V_f A^N) Q (V_f A^N)^T, \quad (4.7b)$$

which can be viewed as a generalized time-dependent Lyapunov equation for $P(k)$.

Note however that equations (4.7a) and (4.7b) may or may not characterize completely the variance $P(k)$, i.e., they may have several solutions, one of which will be (4.4). This corresponds to situations where (4.7a) and (4.7b) do not completely capture the structure of (4.4), and in this case, additional conditions would have to be imposed to make sure that we obtain a unique solution equal to (4.4). To obtain conditions under which equations (4.7a) and (4.7b) specify $P(k)$ uniquely, these equations can be rewritten in the form of a TPBDS of type (2.1)-(2.2), and we can then apply the well-posedness test for TPBVDSs presented in [1]. This can be done by denoting by $p(k)$, q, and w the vectors obtained by scanning the entries of matrices $P(k)$, Q, and $W = BB^T$ columnwise, and rewriting (4.7a) and (4.7b) as

$$(E \otimes E)p(k+1) - (A \otimes A)p(k) = (V_i E^N \otimes V_i E^N)w - (V_f A^N \otimes V_f A^N)w \quad (4.8a)$$

$$(V_i \otimes V_i)p(0) - (V_f \otimes V_f)p(N) = (V_i E^N \otimes V_i E^N)q - (V_f A^N \otimes V_f A^N)q, \quad (4.8b)$$

where $\otimes$ denotes here the Kronecker product of two matrices [27]. Note that the right-hand sides of the above equations are irrelevant as far as well-posedness is concerned.

The well-posedness condition for the TPBVDS (4.8a)-(4.8b) reduces to the invertibility of the matrix

$$\begin{aligned} F_N &= (V_i \otimes V_i)(E \otimes E)^N - (V_f \otimes V_f)(A \otimes A)^N \\ &= (V_i E^N) \otimes (V_i E^N) - (V_f A^N) \otimes (V_f A^N) \ . \end{aligned} \quad (4.9)$$

We obtain therefore the following result.

Theorem 4.1: Equations (4.7a) and (4.7b) characterize uniquely the variance $P(k)$ if and only if

$$\lambda_j \neq \mu_l \quad \text{for all } j \text{ and } l \ , \quad (4.10)$$

where λ_j and μ_j are the eigenvalues of of matrices $V_i E^N$ and $V_f A^N$, respectively.

Proof: Since matrices V_i^N and $V_f A^N$ satisfy (2.4), they can be brought simultaneously to Jordan form. Furthermore, the eigenvalues λ_j and μ_j corresponding to the same eigenvector z satisfy

$$\lambda_j + \mu_j = 1 \ . \tag{4.11}$$

Then, it is easy to check that the eigenvalues of F_N must have the form $\lambda_j \lambda_l - \mu_j \mu_l$ (assume that $V_i E^N$ and $V_f A^N$ are in Jordan form in (2.4)), so that F_N is invertible as long as

$$\lambda_j \lambda_l \neq \mu_j \mu_l \ .$$

Taking into account (4.11), this gives (4.10). □

Note that in the causal case the eigenvalues λ_j and μ_j are all equal to 1 and 0, respectively. Thus, according to Theorem 4.1, $P(k)$ is uniquely defined. This is expected, since in this case (4.7a) is a forwards recursion for $P(k)$, and (4.7b) is the initial condition $P(0) = Q$.

Theorem 4.1 indicates that, except under very special circumstances, the variance $P(k)$ can be uniquely computed from the generalized time-dependent Lyapunov equations (4.7a) and (4.7b). In addition, when the TPBVDS is stochastically stationary, the matrix $P(k) = P$ is *constant*, and satisfies the two algebraic matrix equations

$$EPE^T - APA^T = (V_i E^N) BB^T (V_i E^N)^T - (V_f A^N) BB^T (V_f A^N)^T \tag{4.12}$$

$$V_i P V_i^T - V_f P V_f^T = (V_i E^N) Q (V_i E^N)^T - (V_f A^N) Q (V_f A^N)^T \ , \tag{4.13}$$

obtained by setting $P = P(k+1) = P(k)$ and $P = P(0) = P(N)$ in (4.7a) and (4.7b), respectively. Equation (4.12) is a generalized algebraic Lyapunov equation, and by analogy with the causal case, it is tempting to think that, if a TPBVDS has a constant positive definite variance matrix P satisfying (4.12), then the TPBVDS is stochastically stationary. Unfortunately, this is not the case, and the correct condition for stochastic stationarity, which is condition (4.14) below, involves the variance Q of the boundary vector v. As a first step, we show that if this condition is satisfied, a stochastic TPBVDS has constant variance.

Theorem 4.2 A stochastic TPBVDS has a constant variance matrix P if Q satisfies the equation

$$EQE^T - AQA^T = V_i BB^T V_i^T - V_f BB^T V_f^T \ . \tag{4.14}$$

Proof: We need to show that $P(k) = P(k+1)$ for all k if Q satisfies (4.14). By using expression (4.4) for $P(k)$ and $P(k+1)$, and noting that

$$\Pi(k) = A\,\Pi(k-1)A^T + E^k BB^T (E^k)^T ,$$

it is easy to check that $P(k) = P(k+1)$ is equivalent to having

$$A^k E^{N-1-k} [EQE^T - AQA^T - V_i BB^T V_i^T + V_f BB^T V_f^T](A^k E^{N-1-k})^T = 0 . \quad (4.15)$$

Clearly, (4.15) is implied by (4.14). □

Note that (4.14) and (4.15) are in fact equivalent if either E or A is invertible. Consequently, if either E or A is invertible, the TPBVDS (2.1)-(2.2) has a constant variance if and only if Q satisfies (4.14). However, this is not true in general, as can be seen from the following example.

Example 4.1: Consider the TPBVDS

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} u(k) \quad (4.16a)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & N \\ 0 & 0 & 1 \end{bmatrix} x(N) = v , \quad (4.16b)$$

where the variance of v is given by

$$Q = \begin{bmatrix} 1 & N & 1 \\ N & N^2+2 & N \\ 1 & N & 1 \end{bmatrix} . \quad (4.17)$$

The system (4.16) is in standard form and is deterministically stationary. Then, it is easy to check that Q satisfies (4.15), but not (4.14), and that (4.16) has a constant variance matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which satisfies both (4.12) and (4.13). This shows therefore that a TPBVDS may have a constant variance matrix even if (4.14) is not satisfied.

At this point, we have introduced two generalized algebraic Lyapunov equations, namely (4.12) and (4.14), for P and Q. These equations have exactly the same form and differ only by their right hand sides. It will be shown in next section that (4.14) is in fact the key equation if we want to characterize stochastic stationarity. However, However, before doing so, we note that since equations (4.12) and (4.14) have the same form, they admit a unique solution under the same condition.

Theorem 4.3: The generalized Lyapunov equations (4.8) and (4.10) have a unique solution if and only if the eigenmodes σ_j of the TPBVDS (2.1), i.e., the values for which $\sigma E - A$ is singular, are such that

$$(i) \quad \sigma_j \sigma_l \neq 1 \quad \text{for all } j \text{ and } l, \tag{4.18}$$

(ii) there does not exist exist simultaneously eigenmodes which are zero, and eigenmodes which are ∞, i.e., the matrices E and A are not both singular.

Proof: The proof is similar to that of Theorem 4.1. Equations (4.12) and (4.14) admit a unique solution if and only if the matrix $M = E \otimes E - A \otimes A$ is invertible. But since E and A satisfy (2.3), they can be brought to Jordan form simultaneously, and we may denote by λ_j and μ_j the eigenvalues of these two matrices appearing in corresponding Jordan blocks. Assuming that E and A are in Jordan form, it is easy to check that the eigenvalues of M are $\lambda_j \lambda_l - \mu_j \mu_l$. Furthermore the eigenmodes $\sigma_j = \mu_j / \lambda_j$. Combining these two observations, and noting from (2.3) that λ_j and μ_j cannot both be zero, we see therefore that M is invertible if and only if conditions (i) and (ii) are satisfied. □

Theorem 4.3 indicates that the class of TPBVDSs such that the generalized Lyapunov equations (4.12) and (4.14) have a unique solution is rather restricted, since either E or A must be invertible. In the causal case, $E = I$ is clearly invertible, and equations (4.12) and (4.14) are identical and correspond to the standard Lyapunov equation. Furthermore, in the causal case, a reachable system has a constant variance matrix if and only if A is stable. This means that the magnitude of eigenmodes σ_j is less than 1, so that condition (4.18) is satisfied, and equation (4.12) characterizes uniquely the variance P.

For general TPBVDSs, it may happen that a TPBVDS has a constant variance matrix P, but yet the generalized Lyapunov equation (4.12) may not specify P completely, i.e., it may have several solutions. In this case, one way to compute P is to use the explicit expression (4.4) for any value of k. However, another way is to use the Lyapunov equation (4.12) in combination with perturbation methods. For example, we can replace A by $A + \epsilon I$ and compute the corresponding $P(\epsilon)$. Then, P is obtained by letting ϵ tend to zero in $P(\epsilon)$. This method can be justified by noting that when (4.4) is used to express $P(\epsilon)$, with A replaced by $A + \epsilon I$, and with V_i, V_f and Q rescaled accordingly to guarantee that the standard form relation (2.4) is satisfied, the entries of $P(\epsilon)$ are rational functions of ϵ, analytic at $\epsilon = 0$. This means of course that $P(\epsilon)$ is a continuous function of ϵ in some neigbourhood of $\epsilon = 0$.

Example 4.2: Consider the anticyclic system

$$x(k+1) = x(k) + bu(k) \tag{4.19a}$$

$$(1/2)(x(0) + x(N)) = 0 , \tag{4.19b}$$

where u(k) is a white noise sequence with variance 1. This system is in standard form. Then, since $q = 0$ satisfies (4.14), (4.19) must have a constant variance. However, equation (4.12) cannot be used directly to compute this variance, since both sides are equal to zero. Thus, it is necessary to use the perturbation technique outlined above. Consider the perturbed system

$$x(k+1) = (1 + \epsilon)x(k) + bu(k) \tag{4.20a}$$

$$m(\epsilon)(x(0) + x(N)) = 0 \tag{4.20b}$$

with $m(\epsilon) = (1 + (1 + \epsilon)^N)^{-1}$, which is also in standard form. Then, to compute the variance p of (4.19), we first compute the solution $p(\epsilon)$ of

$$(1-(1 + \epsilon)^2)p(\epsilon) = m(\epsilon)^2 b^2 (1 - (1 + \epsilon)^{2N}) .$$

This gives

$$p(\epsilon) = Nb^2/4 + O(\epsilon) ,$$

so that

$$p = \lim_{\epsilon \to 0} p(\epsilon) = Nb^2/4 .$$

This result can also be obtained directly from expression (4.4).

5. Characterization of Stochastic Stationarity

In this section, our goal will be to establish the following characterization of stochastic stationarity.

Theorem 5.1 A stochastic TPBVDS is stochastically stationary if and only if the variance Q of the boundary vector v satisfies the generalized Lyapunov equation (4.14).

Before proving Theorem 5.1 in full generality, we consider two special cases.

Lemma 5.1 Theorem 5.1 holds for the class of stochastic TPBVDSs such that either E or A is invertible.

Proof: We have to show that Q satisfies (4.14) if and only if $R(k,l)$ depends only on $k-l$. Observe first that

$$ER(k+1,l) = M[Ex(k+1)x^T(l)] = M[(Ax(k) + Bu(k))x^T(l)] . \qquad (5.1)$$

Then, using the Green's function solution (2.5), (2.9) to compute $M[u(k)x^T(l)]$, (5.1) can be expressed as

$$ER(k+1,l) - AR(k,l) = -BB^T(V_f E^{k-l} A^{N-1-(k-l)})^T \quad \text{for } k \geq l . \quad (5.2a)$$

Similarly, it can be shown that

$$R(k,l+1)E^T - R(k,l)A^T = V_i A^{k-l-1} E^{N-(k-l)} BB^T \quad \text{for } k > l . \quad (5.2b)$$

We prove first that if Q satisfies (4.14), the system is stochastically stationary. Note, according to Theorem 4.2, that in this case $R(k,k) = P$ is constant. We now want to prove that $R(k+s,k)$ does not depend on k. Using (5.2a) and (5.2b), and the fact that $R(k,k) = P$, we obtain

$$ER(k+1,k) = AP - BB^T(V_f A^{N-1})^T$$

$$R(k+1,k)A^T = PE^T - V_i E^{N-1} BB^T$$

More generally, we have

$$E^s R(k+s,k) = A^s P - \sum_{j=0}^{s-1} A^{s-j-1} E^j BB^T (V_f A^{N-1-j} E^j)^T , \tag{5.3a}$$

$$R(k+s,k)(A^s)^T = P(E^s)^T - \sum_{j=0}^{s-1} V_i E^{N-1-j} A^j BB^T (E^{s-j-1} A^j)^T . \tag{5.3b}$$

Since either E or A is invertible, one of equations (5.3a) or (5.3b) completely characterize $R(k+s,k)$, and clearly this matrix does not depend on k, so that $R(k+s,k) = R(s)$.

Conversely, assume that $R(k,l) = R(k-l)$, i.e., the TPBVDS is stochastically stationary. Then, $R(k,k) = P$ is constant, and since either E or A is invertible, according to the comment immediately following the proof of Theorem 4.2, Q must satisfy (4.14), as desired. □

Next, we consider a second special case of Theorem 5.1.

Lemma 5.2: The TPBVDS

$$\begin{bmatrix} I & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{5.4a}$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(N) \\ x_2(N) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{5.4b}$$

is stochastically stationary if and only if Q satisfies (4.14). For this particular system, when Q is partitioned as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix},$$

(4.14) reduces to

$$Q_{11} - A_1 Q_{11} A_1^T = B_1 B_1^T \tag{5.5a}$$

$$Q_{22} - E_2 Q_{22} E_2^T = B_2 B_2^T , \tag{5.5b}$$

and

$$Q_{12} E_2^T = A_1 Q_{12} . \tag{5.5c}$$

Proof: System (5.4) consists of two processes $x_1(k)$ and $x_2(k)$ which are respectively forwards and backwards causal, and are correlated through the noise $u(k)$ and boundary vector v. Clearly, system (5.4) will be stochastically stationary only if each one of these subsystems is stationary. More precisely, if the covariance function of (5.4) is partitioned as

$$R(k,l) = \begin{bmatrix} R_{11}(k,l) & R_{12}(k,l) \\ R_{12}^T(k,l) & R_{22}(k,l) \end{bmatrix},$$

$R_{11}(k,l) = R_{11}(k-l)$ and $R_{22}(k,l) = R_{22}(k-l)$, which corresponds to requiring that subsystems 1 and 2 should be individually stationary, if and only if (5.5a) and (5.5b) are satisfied. Equations (5.5a) and (5.5b) are the usual Lyapunov equations for the causal subsystems 1 and 2. Assuming now that subsystems 1 and 2 are individually stationary, the overall system is stochastically stationary if and only if the cross-correlation between these subsystems is such that $R_{12}(k,l) = R_{12}(k-l)$. From the solutions

$$x_1(k) = A_1^k v_1 + \sum_{j=0}^{k-1} A_1^{k-j-1} B_1 u(j) \tag{5.6a}$$

$$x_2(l) = E_2^{N-l} v_2 - \sum_{j=l}^{N-1} E_2^{j-l} B_2 u(j)\,, \tag{5.6b}$$

it is easy to check that

$$R_{12}(k,l) = A_1^k Q_{12} (E_2^{N-l})^T \quad \text{for } k \leq l\,, \tag{5.7a}$$

and

$$R_{12}(k,l) = A_1^k Q_{12} (E_2^{N-l})^T - \sum_{j=0}^{k-l-1} A_1^{k-l-1-j} B_1 B_2^T (E_2^j)^T \quad \text{for } k > l\,. \tag{5.7b}$$

The second term on the right-hand side of (5.7b) depends only on $k-l$, and consequently we have $R_{12}(k+1,l+1) = R_{12}(k,l)$ for all k, l if and only if

$$A_1^k [Q_{12} E_2^T - A_1 Q_{12}] (E_2^{N-(l+1)})^T\,. \tag{5.8}$$

Clearly (5.8) is implied by (5.5c), and conversely if we set $k=0$ and $l=N-1$ in (5.8), we obtain (5.5c). This establishes that conditions (5.5) are necessary and sufficient for the TPBVDS (5.4) to be stochastically stationary. □

We can now prove Theorem 5.1 in full generality.

Proof of Theorem 5.1: The first step is to use a procedure analog to that of Theorem 3.1 to decompose the TPBVDS (2.1)-(2.2) as

$$\begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & N_E \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} N_A & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u(k) \quad (5.9a)$$

$$\begin{bmatrix} V_{i1} & 0 & 0 \\ 0 & V_{i2} & 0 \\ 0 & 0 & V_{i3} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \begin{bmatrix} V_{f1} & 0 & 0 \\ 0 & V_{f2} & 0 \\ 0 & 0 & V_{f3} \end{bmatrix} \begin{bmatrix} x_1(N) \\ x_2(N) \\ x_3(N) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad (5.9b)$$

where N_E and N_A are nilpotent matrices, and E_1, E_2, A_2, and A_3 are invertible. In addition, observe that since the TPBVDS that we consider was assumed at the beginning of Section 4 to be deterministically stationary, the null space of E^n must be included in the null space of V_i, and the null space of A^n must be included in the null space of V_f. Since N_E and N_A are nilpotent, this implies

$$V_{i3} = 0 \quad \text{and} \quad V_{f1} = 0 \,. \quad (5.10)$$

Then, (5.9) can be simplified by noting that subsystems 1 and 3 are just causal and anticausal nilpotent systems. Indeed, since E_1 is invertible and has the same Jordan structure as N_A, we can multiply subsystem 1 by E_1^{-1}, and the resulting system will be causal and such that the dynamics matrix $E_1^{-1}N_A$ is nilpotent. A similar transformation can be performed on subsystem 3. Thus, without loss of generality, it can be assumed that (5.9) is in the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & N_E \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} N_A & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u(k) \quad (5.11a)$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & V_{i2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & V_{f2} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_1(N) \\ x_2(N) \\ x_3(N) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \quad (5.11b)$$

Note that V_{i1} and V_{f3} are equal to I because the boundary matrices satisfy the standard form relation (2.4).

Then, it is easy to check that the TPBVDS (5.11) is stochastically stationary if and only if the TPBVDSs S_{12}, S_{23}, and S_{13} obtained by combining subsystems 1 and 2, 2 and 3, and 1 and 3, are individually stochastically stationary. This is a consequence of the fact that if we partition $R(k,l)$ as

$$R(k,l) = \begin{bmatrix} R_{11}(k,l) & R_{12}(k,l) & R_{13}(k,l) \\ R_{21}(k,l) & R_{22}(k,l) & R_{23}(k,l) \\ R_{31}(k,l) & R_{32}(k,l) & R_{33}(k,l) \end{bmatrix}$$

and if S_{12}, S_{23}, and S_{13} are stochastically stationary, then everyone of the block entries of $R(k,l)$ will be a function of $k-l$. Similarly, note that the generalized Lyapunov equation (4.14) for the TPBVDS (5.11) is equivalent to the combination of the smaller size generalized Lyapunov equations associated to subsystems S_{12}, S_{23}, and S_{13}. This is seen by noting that if

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix},$$

the Lyapunov equations associated to S_{12}, S_{23}, and S_{13} are obtained by removing respectively the third, first, and second block rows and columns of equation (4.14) for Q. Now, since the matrices

$$E_{12} = \begin{bmatrix} I & 0 \\ 0 & E_2 \end{bmatrix}$$

and

$$A_{23} = \begin{bmatrix} A_2 & 0 \\ 0 & I \end{bmatrix}$$

associated respectively to subsystems S_{12} and S_{23} are *invertible*, we can conclude that Lemma 5.1 is applicable to these two systems. In addition, subsystem S_{13} has precisely the structure considered in Lemma 5.2. Each of the above mentioned subsystems is therefore stochastically stationary if and only if the corresponding smaller size Lyapunov equations are satisfied. This proves therefore Theorem 5.1. □

Assuming now that the TPBVDS (2.1)-(2.2) is stochastically stationary, the covariance $R(k+s,k) = R(s)$ has the following property.

Theorem 5.2: The covariance $R(s)$ of a stochastically stationary TPBVDS satisfies the second-order descriptor recursions

$$ER(s+1)E^T + AR(s+1)A^T = AR(s)E^T + ER(s+2)A^T \ , \tag{5.12}$$

which are *conditionable*, in the sense that there exists boundary conditions involving $R(0)$, $R(1)$, $R(N-1)$ and $R(N)$, which when combined with (5.12) define a well-posed second-order TPBVDS.

The recursions (5.12) are analogous to the second-order differential equation obtained by Krener [17] for the covariance of a continuous-time stationary two-point boundary value process. To obtain equation (5.12), consider (5.2a), and observe that this relation is valid independently of whether E or A are invertible or not. Setting $k-l = s$ inside (5.2a) gives

$$ER(s+1) - AR(s) = L(s) \ , \ 0 \le s \le N-1 \ , \tag{5.13}$$

with

$$L(s) = -BB^T(V_f E^s A^{N-1-s})^T \ . \tag{5.14}$$

Then, noting that

$$L(s+1)A^T - L(s)E^T = 0 \ , \ 0 \le s \le N-2 \ , \tag{5.15}$$

and combining (5.15) with (5.13), we obtain (5.12).

We still need to show the conditionability of (5.12). Recall that the concept of conditionability for TPBVDSs was introduced by Luenberger [5]-[6]. To prove the conditionability of (5.12), we will need the following result.

Lemma 5.3: The mth order descriptor system

$$Q_m x(k+m) + Q_{m-1}x(k+m-1) + \cdots + Q_0 x(k) = Bu(k) \ , \ 0 \le k \le N-m \tag{5.16}$$

is conditionable if and only if the determinant of the polynomial matrix $Q(z) = Q_m z^m + Q_{m-1}z^{m-1} + \cdots + Q_0$ does not vanish identically.

Proof: Using state augmentation, we can rewrite (5.16) as

$$\tilde{E}\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u(k) , \tag{5.17}$$

with

$$\tilde{E} = \begin{bmatrix} I & & & & \\ & I & & & \\ & & \cdot & & \\ & & & I & \\ & & & & Q_m \end{bmatrix} , \tilde{A} = \begin{bmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \cdot & \cdot & \\ & & & 0 & I \\ -Q_0 & \cdot & \cdot & \cdot & -Q_{m-1} \end{bmatrix} , \text{ and } \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ B \end{bmatrix} .$$

Then, according to [6], p. 474, the descriptor system (5.17) is conditionable if and only if the pencil $z\tilde{E}-\tilde{A}$ is regular, i.e. iff $\det[z\tilde{E}-\tilde{A}] = \det Q(z)$ does not vanish identically. □

Now, using Lemma 5.3, the conditionability of (5.12) becomes equivalent to the invertibility of $-z^2(E\otimes A) + z(E\otimes E + A\otimes A) - A\otimes E)$ for some z. But, this matrix is equal to $(zE - A)\otimes(E - zA)$. Since E and A form a regular pencil, we can always find a z such that $(zE - A)$ and $(E - zA)$ are both invertible, which implies that their Kronecker product is invertible. This completes the proof of Theorem 5.2.

Theorem 5.2 indicates that there exists a set of boundary conditions involving $R(0)$, $R(1)$, $R(N-1)$ and $R(N)$, which when combined with (5.12) define a well-posed TPBVDS. However, as one might expect, there is in fact a wide choice of boundary conditions which will work. One possible choice can be obtained by considering the two coupled first order descriptor equations (5.13) and (5.15), instead of the second order system (5.12). Suppose that $L(s)$ has already been computed, yielding solution (5.14). Then, a boundary condition which when combined with (5.13) defines a well-posed first-order TPBVDS is given by

$$V_i R(0) + V_f R(N) = Q(E^N)^T . \tag{5.18}$$

This boundary condition is obtained by multiplying (2.2) on the right by $x^T(0)$, taking expected values, and using the Green's function expression (2.5). Note that the TPBVDS (5.13), (5.18) has exactly the same dynamics and boundary matrices as (2.1)-(2.2) and is therefore guaranteed to be well-posed. This leaves us with the problem of computing $L(s)$ for $0 \leq s \leq N-1$ from the first-order recursions (5.15). However, we already know that the solution must be given by (5.14). This implies in particular that

$$ER(1) - AR(0) = L(0) = -BB^T(V_f A^{N-1})^T \tag{5.19a}$$

and

$$ER(N) - AR(N-1) = L(N-1) = -BB^T V_i (E^{N-1})^T \ . \tag{5.19b}$$

Note that, as expected, boundary conditions (5.19a) and (5.19b) involve only $R(0)$, $R(1)$, $R(N-1)$, and $R(N)$. Also, the TPBVDS defined by (5.15) and (5.19a), (5.19b) is clearly well-posed. In fact, it is ***overdetermined*** since the boundary conditions (5.19a) and (5.19b) are redundant. This redundancy can be eliminated by considering the smaller-size boundary condition

$$L(0)(V_i A)^T + L(N-1)(V_f E)^T = -BB^T V_f^T \tag{5.20}$$

which is obtained by combining (5.19a) and (5.19b), and checking that the TPBVDS (5.15), (5.20) is well-posed. Note that to some extent, the problem of finding boundary conditions which guarantee that the first-order recursions (5.15) for $L(s)$ are well-posed is an exercise in futility, since the closed-form solution (5.14) is already available. However, if we consider the second-order recursions (5.12), the above discussion shows that boundary conditions (5.18) and (5.20) will guarantee well-posedness.

As was already mentioned, these boundary conditions are not the only ones which will guarantee well-posedness. For example, if we use (5.2b) as starting point, we obtain the coupled first-order descriptor equations

$$R(s)E^T - R(s+1)A^T = M(s) \ , \ 0 \leq s \leq N-1 \tag{5.21a}$$

$$EM(s+1) - AM(s) = 0 \ , \ 0 \leq s \leq N-2 \tag{5.21b}$$

where $M(s)$ is in fact given by the closed-form expression

$$M(s) = V_i A^s E^{N-s-1} BB^T \ . \tag{5.22}$$

Then, these equations are well-posed for the boundary conditions

$$R(0)V_f^T + R(N)V_i^T = A^N Q \tag{5.23a}$$

$$V_i EM(0) + V_f AM(N-1) = V_i BB^T \ , \tag{5.23b}$$

where (5.23a) is obtained by multiplying (2.2) on the right by $x^T(N)$ and taking expected values, and where (5.23b) is a direct consequence of analytical expression (5.22). Substituting (5.21a) inside (5.23b), it is also easy to check that the boundary conditions (5.23) for (5.12) involve only $R(0)$, $R(1)$, $R(N-1)$, and $R(N)$, as desired.

There are in fact many valid choices of boundary conditions for the second-order system (5.12). For example, one obvious boundary condition is given by $R(0) = P$, where P can be found by solving the algebraic Lyapunov equation (4.12) either directly or by the perturbation technique described in Example 4.2.

Example 5.1: Consider system (4.19). In Example 4.2 it was shown that its variance matrix is given by

$$p = r(0) = Nb^2/4 \ .$$

We shall now seek to compute its covariance function $r(k)$ for $k \in [0,N]$. We use the second-order recursions (5.12), which here take the form

$$r(k+2) = 2r(k+1) - r(k) \ . \tag{5.24}$$

Since $r(0)$ is already known, we only need $r(1)$ to be able to solve (5.24) in the forward direction. But according to (5.19a), we have

$$r(1) - r(0) = -b^2/2 \ ,$$

so that

$$r(1) = (N-2)b^2/4 \ ,$$

and then using (5.24), we find

$$r(k) = (N-2k)b^2/4 \ .$$

6. Characterization of Internal Stability

For causal systems, the relationship between the existence of a positive definite solution to the standard Lyapunov equation and stability is well known. Specifically, for a causal and reachable system, the Lyapunov equation has a positive definite solution if and only if the system is strictly stable. In this section, for the class of deterministically stationary TPBVDSs, we will study the relation existing between the existence and uniqueness of positive definite solutions to the generalized Lyapunov equation (4.12) for the state variance P, and the property of internal stability. Note that, whereas the generalized Lyapunov equation (4.14) for Q was the key to the characterization of stochastic stationarity derived in the previous section, equation (4.12) for P plays the main role in our study of internal stability. An important feature of this equation, which was not present in the causal case, is that it depends on the interval length N. It turns out that this dependence on interval length is in fact very useful to characterize internal stability, since this last concept relies also on increasing the interval length to study the effect of the boundary conditions on states close to the center of the interval.

More precisely, to see why interval length plays an important role in studying the generalized Lyapunov equation (4.12), consider the anticyclic system (4.19) of Example 4.2. This system is clearly unstable, since its only mode is on the unit circle. Yet, the choice $q = 0$ for the variance of the boundary condition guarantees that the system is stochastically stationary, and has a constant positive state variance $p = Nb^2/4$. Thus, the existence of a positive definite solution to the generalized Lyapunov equation (4.12) for a fixed interval length is clearly not sufficient to guarantee that a TPBVDS is internally stable. However, in this particular case the variance p, viewed as a function of the interval length N, diverges as $N \rightarrow \infty$, which is an indication that the system is actually unstable.

Another useful observation is that for TPBVDSs, the generalized Lyapunov equation (4.12) for P may admit a nonnegative definite solution even when the system cannot be made stationary by any choice of boundary vector variance Q, i.e., there may be a nonnegative solution to (4.12) when there is no nonnegative solution to equation (4.14) for Q. This is illustrated by the following example.

Example 6.1: Consider the system

$$x(k+1) = (1/2)x(k) + u(k) \tag{6.1a}$$

$$m(x(0) + 2x(N)) = v \ , \tag{6.1b}$$

where $m = (1 + 2(1/2)^N)^{-1}$, and $u(k)$ is a white noise sequence with unit variance. System (6.1) is in standard form and internally stable. The generalized Lyapunov equation (4.14) for q takes the form

$$(3/4)q = -3m^2 \ , \tag{6.2}$$

which yields a negative value of q, so that the system cannot be made stationary over any interval $[0,N]$. Yet, the Lyapunov equation (4.12) is given by

$$(3/4)p = m^2(1 - 4(1/4)^N) \ , \tag{6.3}$$

and its solution p is positive provided that N is larger than 1. However, this solution is *not* the state variance of the TPBVDS (6.1), which in this case is not even constant. This can be seen by noting from (4.3)-(4.4) that the state variance $p(k)$ is given by

$$p(k) = \frac{q}{4^k} + \frac{4}{3}m^2\left(1 - \frac{4}{4^N} + \frac{3}{4^k}\right), \tag{6.4}$$

which is clearly not constant.

Example 6.1 shows that the generalized Lyapunov equation (4.12) may admit a unique positive definite solution P even when the TPBVDS (2.1)-(2.2) cannot be made stochastically stationary for any choice of boundary vector variance Q, but in general this matrix P bears no relation whatsoever with the state variance. However, it will be shown below in Theorem 6.3 that, for an internally stable deterministically stationary TPBVDS, independently of the choice of boundary matrix Q, as the interval length $N \to \infty$, the variance matrices $P(k)$ of states near the center of the interval approach a constant matrix P^* which *is* the solution to the generalized Lyapunov equation (4.12) with N set equal to ∞.

The main objective of this section is to characterize the property of internal stability in terms of positive definite solutions of (4.12), regardless of whether such solutions correspond to the variance of a stochastically stationary TPBVDS or not. Specifically, it will be shown that if for any N, the generalized Lyapunov equation (4.12) has a nonnegative definite solution P whose main-diagonal elements are unique in the coordinate system where E and A are both in Jordan form, then the system (2.1)-(2.2) is internally stable.

To see why the main diagonal elements of P come into the picture, consider the proof of Theorem 4.3, where the existence and uniqueness of solutions to equations (4.12) and (4.14) was discussed. By examining these equations in the coordinate system where both E and A are in Jordan form, it is easy to check that the main diagonal elements are unique if and only if the pencil $\sigma E - A$ does not have any eigenmode σ_j located on the unit circle, whereas as was observed in Theorem 4.3, the off-diagonal elements are unique iff there does not exist eigenmodes σ_j and σ_l such that either $\sigma_j \sigma_l = 1$ or such that $\sigma_j = 0$ and $\sigma_l = \infty$. Note that there is no contradiction between the above conditions for uniqueness of the diagonal and off-diagonal elements of P, respectively, since when σ_j is on the unit circle, then σ_j^* is also an eigenmode, and $\sigma_j \sigma_j^* = 1$. The reason why it is important to distinguish between the case when P does not have unique diagonal elements, and the case when the off-diagonal elements are not unique, is that in the first case, the system has eigenmodes on the unit circle, and is therefore unstable, whereas in the second case, the TPBVDS may be internally stable. As an illustration of this last fact, consider system (5.4), and assume that A_1 and E_2 are *nilpotent matrices.* Since this system is constituted of two decoupled forward and backward causal and stable subsystems, (5.4) is clearly internally stable. Yet, since A_1 and E_2 do not have full rank, there exists eigenmodes $\sigma_j = 0$ and $\sigma_l = \infty$, so that the off-diagonal elements of P are not unique in (4.12).

Before presenting the main results of this section, we need to prove the following lemma.

Lemma 6.1: Let A and V be two square matrices which commute, i.e.,

$$AV = VA \ . \tag{6.5}$$

Then, if V is singular, there exists a right (left) eigenvector of A in the right (left) null space of V.

Proof: We will prove this result for the case of a right eigenvector of A. Let $x \in Ker(V)$. Then,

$$Vx = 0$$

so that

$$VAx = AVx = 0 \ ,$$

and consequently $Ax \in Ker(V)$. Thus $Ker(V)$ is A invariant, which implies that A has at least one eigenvector in the null space of V. □

We can now prove the following result.

Theorem 6.1: Assume that TPBVDS (2.1)-(2.2) is deterministically stationary and weakly reachable. Then, if for some N, the generalized Lyapunov equation (4.12) has a nonnegative definite solution P whose main diagonal elements are unique in the coordinate system where E and A are both in Jordan form, the TPBVDS is internally stable

Proof: The uniqueness of the main diagonal elements of P guarantees that there are no eigenmodes on the unit circle. Thus, the TPBVDS decomposition of Theorem 3.1 takes the form

$$E = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix} , \ A = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} , \ B = \begin{bmatrix} B_f \\ B_b \end{bmatrix} , \tag{6.6a}$$

where the eigenvalues of A_f and A_b are inside the unit circle, and

$$V_i = \begin{bmatrix} V_{i1} & 0 \\ 0 & V_{i2} \end{bmatrix} , \ V_f = \begin{bmatrix} V_{f1} & 0 \\ 0 & V_{f2} \end{bmatrix} . \tag{6.6b}$$

To prove stability, we need to show that V_{i1} and V_{f2} are invertible. Using the above decomposition, the generalized Lyapunov equation (4.12) can be expressed as

$$P_f - A_f P_f A_f^T = V_{i1} B_f B_f^T V_{i1}^T - (V_{f1} A_f^N) B_f B_f^T (V_{f1} A_f^N)^T \tag{6.7a}$$

$$A_b P_b A_b^T - P_b = (V_{i2} A_b^N) B_b B_b^T (V_{i2} A_b^N)^T - V_{f2} B_b B_b^T V_{f2}^T \tag{6.7b}$$

$$P_{fb} A_b^T - A_f P_{fb} = V_{i1} B_f B_b^T (V_{i2} A_b^N)^T - (V_{f1} A_f^N) B_f B_b^T V_{f2}^T , \tag{6.7c}$$

where

$$P = \begin{bmatrix} P_f & P_{fb} \\ P_{fb}^T & P_b \end{bmatrix} . \tag{6.8}$$

Clearly, if P is nonnegative definite, so is P_f. Since we also know that A_f is strictly stable, from (6.7a) we can conclude that if x^T is an arbitrary left eigenvector of A_f, then

$$x^T (V_{i1} B_f B_f^T V_{i1}^T - (V_{f1} A_f^N) B_f B_f^T (V_{f1} A_f^N)^T) x \geq 0 . \tag{6.9}$$

We would like to show that V_{i1} is invertible. To do so, assume that V_{i1} is not invertible. Then, according to Lemma 6.1, there exists a left eigenvector x^T of A_f, i.e.,

$$x^T A_f = \lambda x^T , \tag{6.10a}$$

such that

$$x^T V_{i1} = 0 . \tag{6.10b}$$

We also know that the system is weakly reachable, and from the characterization of weak reachability presented in [2], we have

$$x^T [V_{i1} B_f \quad V_{f1} B_f] \neq 0 ,$$

so that

$$x^T V_{f1} B_f \neq 0 . \tag{6.11}$$

Now, taking (6.10b) into account in (6.9), and observing that A_f and V_{f1} commute, we find that

$$0 = x^T V_{f1} A_f^N B_f = \lambda^N x^T V_{f1} B_f , \tag{6.12}$$

where λ is the eigenvalue appearing in (6.10a). But (6.12) is compatible with

(6.11) only if we have $\lambda = 0$, so that x^T must be in the left null space of both A_f and V_{i1}. However, in this case the matrix

$$V_{i1} + V_{f1} A_f^N$$

characterizing the well-posedness of the forward stable subsystem is not invertible, which contradicts our assumptions. Thus V_{i1} must be invertible. Similarly, it can be proved that V_{f2} is invertible. □

As in the causal case, the above result has also a converse, i.e., given an internally stable TPBVDS, there exists a positive definite solution to the Lyapunov equation (4.12). However, this result is only valid for large N, and it requires stronger conditions than those of Theorem 6.1. First, the conditions of Theorem 4.3 on the eigenmodes of the TPBVDS must be satisfied, so that (4.12) will be guaranteed to have a solution independently of the choice of of input matrix B and of boundary matrices V_i and V_f, in which case this solution will in fact be unique. The second condition is that the TPBVDS must be strongly reachable, instead of weakly reachable as in Theorem 6.1. This is due to the fact that we need to make sure that as $N \rightarrow \infty$, the solution of (4.12) is positive definite, instead of merely nonnegative definite.

Theorem 6.2: Consider a deterministically stationary TPBVDS which is internally stable, strongly reachable, and whose eigenmodes σ_j satisfy the conditions of Theorem 4.3 for the existence of a unique solution P_N to the generalized Lyapunov equation (4.12). Here the interval length N is allowed to vary, and the dependence of P on N is denoted by the subscript N of P_N. Then, there exists $N^* > 0$ such that P_N is positive definite for all $N \geq N^*$. Furthermore, as $N \rightarrow \infty$,

$$P_N \rightarrow P^* = \begin{bmatrix} P_f^* & 0 \\ 0 & P_b^* \end{bmatrix}, \tag{6.13}$$

where P_f^* and P_b^* are respectively the solutions of the usual algebraic Lyapunov equations for the forward and backward stable subsystems, i.e.,

$$P_f^* - A_f P_f^* A_f^T = B_f B_f^T , \tag{6.14a}$$

$$P_b^* - A_b P_b^* A_b^T = B_b B_b^T \ . \tag{6.14b}$$

Proof: First, observe that since the interval length N varies, the boundary matrices V_{i1}, V_{f1}, and V_{i2}, V_{f2} associated respectively to the forward and backward stable subsystems need to be rescaled in order to satisfy the standard form identity (2.4) for all N. The rescaled boundary matrices are given by

$$V_{i1}(N) = (V_{i1} + V_{f1}A_f^N)^{-1}V_{i1} \ \ , \ \ V_{f1}(N) = (V_{i1} + V_{f1}A_f^N)^{-1}V_{f1} \tag{6.15a}$$

$$V_{i2}(N) = (V_{i2}A_b^N + V_{f2})^{-1}V_{i2} \ \ , \ \ V_{f2}(N) = (V_{i2}A_b^N + V_{f2})^{-1}V_{f2} \ , \tag{6.15b}$$

and since the TPBVDS is internally stable, the matrices V_{i1} and V_{f2} are invertible, so that as $N \rightarrow \infty$,

$$V_{i1}(N) \rightarrow I \ , \ V_{f1}(N) \rightarrow V_{i1}^{-1}V_{f1} \ , \ V_{i2}(N) \rightarrow V_{f2}^{-1}V_{i2} \ , \ V_{f2}(N) \rightarrow I \tag{6.16}$$

Consider now the matrix P_N given by (6.8), whose entries satisfy the Lyapunov equations (6.7a-c), where the boundary matrices on the right hand side are replaced by the scaled matrices (6.15). We want to show that for N large enough, the solutions $P_{f,N}$ and $P_{b,N}$ of (6.7a) and (6.7b) are positive definite and tend to P_f^* and P_b^* given by (6.14), and that the solution $P_{fb,N}$ of (6.7c) goes to zero as N goes to infinity.

The first step is to observe that, as $N \rightarrow \infty$, since the scaled boundary matrices tend to finite limits given by (6.16), the right-hand side of (6.7c) tends to zero. But the eigenmodes of the system are such that the solution P_N is unique, and therefore the solution $P_{fb,N}$ of equation (6.7c) is unique and tends to zero as N goes to infinity.

Next, consider Lyapunov equation (6.7a), and observe that since the TPBVDS is strongly reachable, the matrix pair (A_f, B_f) is reachable in the usual sense for causal systems. But since the system is internally stable, $V_{i1}(N)$ given by (6.15a) is invertible, and noting that it commutes with A_f, we can conclude that the pair $(A_f, V_{i1}(N)B_f)$ is also reachable in the usual sense. Then, the solution $P_{f,N}$ of (6.7a) can be expressed as

$$P_{f,N} = P_{f,N}^{+} - P_{f,N}^{-} \tag{6.17}$$

where $P_{f,N}^{+}$ and $P_{f,N}^{-}$ are respectively the solutions of

$$P_{f,N}^{+} - A_f P_{f,N}^{+} A_f^T = V_{i1}(N) B_f B_f^T V_{i1}(N)^T \tag{6.18a}$$

$$P_{f,N}^{-} - A_f P_{f,N}^{-} A_f^T = (V_{f1}(N) A_f^N) B_f B_f^T (V_{f1}(N) A_f^N)^T \ . \tag{6.18b}$$

Since $(A_f,\ V_{i1}(N)B_f)$ is reachable, $P_{f,N}^{+}$ is positive definite for all N, and since $V_{i1}(N) \rightarrow I$ as $N \rightarrow \infty$, $P_{f,N}^{+} \rightarrow P_f^*$, where P_f^* is the unique positive definite solution of (6.14a). Furthermore, as $N \rightarrow \infty$, the right-hand side of (6.18b) tends to zero, so that $P_{f,N}^{-}$ tends to zero. From (6.17), we can therefore conclude that there exists an integer N^* such that $P_{f,N}$ is positive definite for all $N \geq N^*$. Similarly, it can be shown that the solution $P_{b,N}$ of (6.7b) is positive definite for large enough N and tends to P_b^*, which is the unique positive definite solution of (6.14b).

We have therefore shown that as $N \rightarrow \infty$, $P_{f,N}$ and $P_{b,N}$ approach positive definite matrices P_f^* and P_b^*, and that $P_{fb,N}$ tends to zero. Consequently, the matrix P_N is positive definite for sufficiently large N and has for limit P^* given by (6.13). □

Example 6.2 Consider system (6.1), which is both internally stable and stongly reachable. Then, the solution of the generalized Lyapunov equation (6.3) is

$$p_N = \frac{4}{3} m^2 (1 - \frac{4}{4^N}) \ ,$$

which is positive definite for $N \geq 2$. Furthermore, as $N \rightarrow \infty$,

$$p_N \rightarrow p^* = 4m^2/3 \ , \tag{6.19}$$

where p^* is the solution of the generalized Lyapunov equation (6.3) with $N = \infty$.

It is worth noting that when $N = \infty$, if the TPBVDS is internally stable, in the coordinate system corresponding to decomposition (6.6), the generalized Lyapunov equation (4.12) takes the form

$$EPE^T - APA^T = W \ , \tag{6.20}$$

with

$$W = \begin{bmatrix} B_f B_f^T & 0 \\ 0 & -B_b B_b^T \end{bmatrix}. \tag{6.21}$$

Then, independently of whether eigenmodes σ_j satisfy the conditions of Theorem 4.3, *one* solution of (6.20) is P^* given by (6.13)-(6.14), which is nonnegative definite regardless of the reachability properties of the TPBVDS (2.1)-(2.2). In other words, for $N = \infty$, the conditions of Theorem 6.2 can be weakened, thus giving the following result.

Corollary 6.1 Let TPBVDS (2.1)-(2.2) be internally stable. Then the generalized Lyapunov equation (4.12) with $N = \infty$ has a nonnegative definite solution P^*. This solution is positive definite if the system is strongly reachable.

For an internally stable TPBVDS, the solution P^* of the generalized Lyapunov equation (4.12) with $N = \infty$ has also the following stochastic interpretation.

Theorem 6.3 Let system (2.1)-(2.2) be internally stable. Then, for *any* choice of boundary variance Q, as N goes to infinity, the variance matrix of states located close to the center of interval $[0,N]$ converges to the solution P^* of the generalized Lyapunov equation with $N = \infty$.

Proof: Let $P_N(k)$ be the variance matrix of the state $x(k)$ of system (2.1)-(2.2) defined over interval $[0,N]$. Then, if l is an arbitrary but fixed integer, we want to show that

$$\lim_{N \to \infty} P_N((N/2)+l) = P^* , \tag{6.22}$$

where for simplicity it has been assumed that N is even. Our starting point is expression (4.4) for the state variance, i.e.,

$$\begin{aligned} P_N((N/2)+l) &= A^{(N/2)+l} E^{(N/2)-l} Q (A^{(N/2)+l} E^{(N/2)-l})^T \\ &+ (V_i(N) E^{(N/2)-l}) \Pi((N/2)+l-1) (V_i(N) E^{(N/2)-l})^T \\ &+ (V_f(N) A^{(N/2)+l}) \Pi((N/2)-l-1) (V_f(N) A^{(N/2)+l})^T , \end{aligned}$$

where $\Pi(k)$ is given by (4.3), and boundary matrices $V_i(N)$ and $V_f(N)$ are obtained by rescaling V_i and V_f so that the standard form identity (2.4) is satisfied for all N. Then, in the coordinate system corresponding to decomposition (6.6) of the TPBVDS in its forward and backward stable components, by using expressions (6.16) for the limit of $V_i(N)$ and $V_f(N)$ as $N \to \infty$, and taking into account the fact that A_f and A_b are stable matrices, we find that

$$\lim_{N\to\infty} P_N((N/2)+l) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Pi(\infty) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \Pi(\infty) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} . \qquad (6.23)$$

But since

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

commute with both E and A, (6.23) can be rewritten as

$$\lim_{N\to\infty} P_N((N/2)+l) = \lim_{k\to\infty} \sum_{j=0}^{k} A^{k-j} E^j \begin{bmatrix} B_f B_f^T & 0 \\ 0 & B_b B_b^T \end{bmatrix} (A^{k-j} E^j)^T . \qquad (6.24)$$

Thus,

$$\lim_{N\to\infty} P_N((N/2)+l) = \begin{bmatrix} \sum_{j=0}^{\infty} A_f^j B_f B_f^T (A_f^j)^T & 0 \\ 0 & \sum_{j=0}^{\infty} A_b^j B_b B_b^T (A_b^j)^T \end{bmatrix}$$

$$= \begin{bmatrix} P_f^* & 0 \\ 0 & P_b^* \end{bmatrix} = P^* . \qquad (6.25)$$

This completes the proof of Theorem 6.3. □

Example 6.3 Consider the TPBVDS (6.1) of Example 6.1. According to (6.19), for this example the solution of the generalized Lyapunov equation (4.12) with $N = \infty$ is $p^* = 4m^2/3$. Then, setting $k = (N/2)+l$ in expression (6.4) for the state variance, we obtain

$$\lim_{N\to\infty} p_N((N/2)+l) = 4m^2/3 = p^* ,$$

as expected.

Theorem 6.3 shows that, regardless of the boundary variance Q, the state variance of an internally stable, deterministically stationary TPBVDS converges to the constant matrix P^* given by (6.13)-(6.14). However an even more interesting observation is that under the above assumptions, the TPBVDS will converge to a *stochastically stationary* system as $N \rightarrow \infty$. More precisely, if we denote by

$$R_N((N/2)+k,(N/2)+l) = M[x((N/2)+k)x^T((N/2)+l)] \tag{6.26}$$

the correlation matrix of states $x((N/2)+k)$ and $x((N/2)+l)$, where k and l are fixed integers, by using the Green's function solution (2.5),(2.9) to evaluate the correlation matrix, and following steps similar to those used in the proof of Theorem 6.3, it can be shown that in the coordinate system corresponding to the forward and backward stable decomposition (6.6), we have

$$\lim_{N \rightarrow \infty} R_N((N/2)+k,(N/2)+l) = R^*(k-l)$$

$$= \begin{bmatrix} A_f^{k-l}P_f^* & -\sum_{j=0}^{k-l-1} A_f^{k-l-j-1}B_f\, B_b^T(A_b^j)^T \\ 0 & P_b^*(A_b^{k-l})^T \end{bmatrix}, \tag{6.27}$$

where for convenience it has been assumed that $l \leq k$. Since the limit obtained in (6.27) depends only on $k-l$, we can therefore conclude that independently of the choice of boundary variance Q, an internally stable TPBVDS converges to a stochastically stationary system as $N \rightarrow \infty$. This stochastically stationary system is separable into forward and backward causal components, which are however correlated through the input noise $u(k)$. This last fact can be seen from (6.27), where if we denote by $x^*(k)$ the limiting process obtained by letting $N \rightarrow \infty$, and by shifting the left boundary of the interval of definition to $-\infty$, the cross-correlation $R_{fb}^*(k-l)$ between the forward component $x_f^*(k)$ and the backward component $x_b^*(l)$ is nonzero for $l \leq k$, since both of these processes depend on the noise over interval $[l,k]$, whereas the cross-correlation between $x_b^*(k)$ and $x_f^*(l)$ is zero, since they depend on the noise over disjoint intervals.

7. Conclusions

In this paper, in spite of the fact that two-point boundary-value descriptor systems are defined only over a finite interval, we have been able to introduce a concept of internal stability for these systems. According to the definition that was selected, a TPBVDS is internally stable if the effect of boundary conditions on states close to the center of the interval goes to zero as the interval length goes to infinity. Stochastic TPBVDSs have also been examined, and the property of stochastic stationarity was characterized in terms of a generalized Lyapunov equation for the variance of of the boundary vector. It was also shown that the state variance satisfies another generalized Lyapunov equation which can be used to characterize the property of internal stability. Specifically, it was shown that for a weakly reachable TPBVDS defined over a finite interval, if the generalized Lyapunov equation for the state variance admits a nonnegative solution with unique diagonal elements in the coordinate system where the dynamics are in Jordan form, then the TPBVDS is internally stable. Conversely, it was shown that for an internally stable TPBVDS, the generalized Lyapunov equation for the state variance admits a positive definite solution when the interval length N is sufficiently large. It was also proved that, independently of the boundary matrix variance, an internally stable stochastic TPBVDS converges to a stochastically stationary process as the interval length $N \to \infty$.

As was already mentioned in the introduction, this paper is part of a larger effort devoted to the study of the system properties, and the development of estimation algorithms for TPBVDSs. In particular, the smoothing problem for TPBVDSs was examined in [4], where it was shown that the smoother itself is a TPBVDS which can then be decoupled into forward and backward stable components through the introduction of generalized Riccati equations that were studied in [25]. An interesting question which arises in this context is whether for a strongly reachable and observable TPBVDS, the smoother is internally stable in the sense discussed in this paper. It turns out that this is the case, and the proof of this fact will appear in [28]. In other words, the concept of internal stability developed here is expected to have the same far ranging applications as for standard causal systems.

References

1. R. Nikoukhah, A.S. Willsky, and B.C. Levy, "Boundary-value descriptor systems: well-posedness, reachability and observability," *Int. J. Control*, vol. 46, no. 5, pp. 1715-1737, Nov. 1987.
2. R. Nikoukhah, A.S. Willsky, and B.C. Levy, "Reachability, observability, and minimality for stationary two-point boundary value descriptor systems," Technical report, Lab. for Information and Decision Systems, M.I.T., Cambridge, MA, 1988.
3. R. Nikoukhah, "System theory for two-point boundary value descriptor systems," M.S. thesis, Dept. of Elec. Eng. and Comp. Science, and Report LIDS-TH-1559, Lab. for Information and Decision Systems, M.I.T., Cambridge, MA, June 1986.
4. R. Nikoukhah, M.B. Adams, A.S. Willsky, and B.C. Levy, "Estimation for boundary-value descriptor systems," Report LIDS-P-1600, Lab. for Information and Decision Systems, M.I.T., Cambridge, MA, Aug. 1986; also submitted to *Circuits, Systems, and Signal Processing.*
5. D.G. Luenberger, "Dynamic systems in descriptor form," *IEEE Trans. Automat. Control*, vol. AC-22, no. 3, pp. 312-321, June 1977.
6. D.G. Luenberger, "Time-invariant descriptor systems," *Automatica*, vol. 14, no. 5, pp. 473-480, Sept. 1978.
7. F.L. Lewis, "Descriptor systems: Decomposition into forwards and backwards subsystems," *IEEE Trans. Automat. Control*, vol. AC-29, no. 2, pp. 167-170, Feb. 1984.
8. G.C. Verghese, B.C. Levy, and T. Kailath, "A generalized state-space for singular systems," *IEEE Trans. Automat. Control*, vol. AC-26, no. 4, pp. 811-831, Aug. 1981.
9. A.J.J. Van der Weiden and O.H. Bosgra, "The determination of structural properties of a linear multivariable system by operations of system similarity; 2. Non-proper systems in generalized state-space form," *Int. J. Control*, vol. 32, no. 3, pp. 489-537, 1980.
10. D. Cobb, "Controllability, observability, and duality in singular systems," *IEEE Trans. Automat. Control*, vol. AC-29, no. 12, pp.1076-1082, Dec. 1984.
11. D. Cobb, "Descriptor variable systems and optimal state regulation," *IEEE Trans. Automat. Control*, vol. AC-28, no. 5, pp. 601-611, May 1983.

12. V. Kucera, "Stationary LQG control of singular systems," *IEEE Trans. Automat. Control*, vol. AC-31, no. 1, pp. 31-39, Jan. 1986.

13. D.J. Bender and A.J. Laub, "The linear-quadratic optimal regulator for descriptor systems: discrete-time case," *Automatica*, vol. 23, no. 1, pp. 71-85, Jan. 1987.

14. Z. Zhou, M.A. Shayman, and T.-J. Tarn, "Singular systems: a new approach in the time-domain," *IEEE Trans. Automat. Control*, vol. AC-32, no. 1, pp. 42-50, Jan. 1987.

15. A.J. Krener, "Boundary value linear systems," *Astérisque*, vol. 75-76, pp. 149-165, Oct. 180.

16. A.J. Krener, "Acausal realization theory, Part 1: Linear deterministic systems," *SIAM J. Control and Optimization*, vol. 25, no. 3, pp. 499-525, May 1987.

17. A.J. Krener, "Reciprocal processes and the stochastic realization problem for acausal systems, in *Modeling, Identification and Control*, C.I. Byrnes and A. Lindquist eds., pp. 197-211, North-Holland, Amsterdam, The Netherlands, 1986.

18. A.J. Krener, "Realizations of reciprocal processes," in *Proceedings IIASA Conf. on Modeling and and Adaptive Control," Sopron, 1986.*

19. I. Gohberg and M.A. Kaashoek, "On minimality and stable minimality of time-varying linear systems with well-posed boundary conditions," *Int. J. Control*, vol. 43, no. 5, pp. 1401- 1411, 1986.

20. I. Gohberg and M.A. Kaashoek, and L. Lerer, "Minimality and irreducibility of time-invariant linear boundary-value systems," *Int. J. Control*, vol. 44, no. 2, pp. 363-379, 1986.

21. I. Gohberg and M.A. Kaashoek, "Similarity and reduction for time-varying linear systems with well-posed boundary conditions," *SIAM J. Control and Optimization*, vol. 24, no. 5, pp. 961-978, Sept. 1986.

22. M.B. Adams, "Linear estimation of boundary value stochastic processes," Sc. D. dissertation, Dept. of Aeron. and Astr., M.I.T., Cambridge, MA, April 1983.

23. M.B. Adams, A.S. Willsky, and B.C. Levy, "Linear estimation of boundary-value stochastic processes, Part 1: The role and construction of complementary models," *IEEE Trans. on Automat. Control*, vol. AC-29, no. 9, pp. 803-811, Sept. 1984.

24. M.B. Adams, A.S. Willsky, and B.C. Levy, "Linear estimation of boundary-value stochastic processes, Part 2: 1-D smoothing problems," *IEEE Trans. on Automat. Control*, vol. AC-29, no. 9, pp. 811-821, Sept. 1981.

25. R. Nikoukhah, A.S. Willsky, and B.C. Levy, "Generalized Riccati equations for two-point boundary-value descriptor systems," in *Proc. 26th IEEE Conf. on*

Decision and Control, pp. 1140-1141, Los Angeles, CA, Dec. 1987.

26. F.R. Gantmacher, *The Theory of Matrices*, vol. 2, Chelsea Publishing Co., New York, N.Y., 1960.
27. J.W. Brewer, "Kronecker products and matrix calculus in system theory," *IEEE Trans. on Circuits and Systems*, vol. CAS-25, no. 9, pp. 772-781, Sept. 1978.
28. R. Nikoukhah, "A deterministic and stochastic theory of two-point boundary-value descriptor systems," Ph.D. dissertation, Dept. of Elec. Eng. and Comp. Science, M.I.T., expected date of completion: Aug. 1988.