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Time-Varying Compensation Can Yield No Improvement For ℓ_1-Sensitivity Minimization

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Abstract

It is shown that time-varying compensation gives no improvement over time-invariant compensation for the ℓ_1-sensitivity minimization problem.

I Introduction

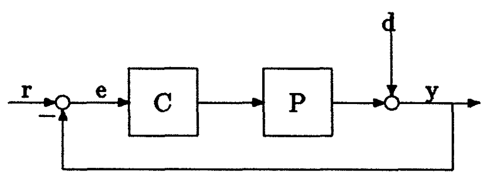

Figure 1: Basic Feedback System

Consider the feedback control system shown above in Figure 1 where P and C are causal linear operators mapping some normed linear space $\mathcal{B}$, with norm $\|\cdot\|_{\mathcal{B}}$, into itself; we assume that the system is well-posed. A *sensitivity minimization problem* for this system is a problem in which one aims to pick the compensator, C, to stabilize the closed-loop and minimize $\mu(H_{yd})$ where

H_{yd} denotes the transfer function from d to y and μ is some given norm. The optimum "μ-sensitivity" is given by

$$\alpha \triangleq \inf_{C \text{ stabilizes } P} \mu\left((I+PC)^{-1}\right) \tag{1}$$

A problem that falls into this framework that has received much attention ([1,2,3,4,5] et. al.) is the H_∞- sensitivity problem where $\mathcal{B} = \ell_2$ (with its usual norm) and μ is given by:

$$\mu(H_{yd}) \triangleq \sup_{d \in \ell_2} \frac{\|H_{yd}(d)\|_2}{\|d\|_2}$$

Here the optimum sensitivity is the minimum energy gain from disturbance to output of the closed-loop system. It might be suspected that the achievable minimum energy gain increases when only a restricted type of stabilizing controller — e.g. finite-dimensional and time-invariant (FDLTI)— is considered. However, when P is itself FDLTI, it has been shown ([4,9]) that the optimum H_∞-sensitivity cannot be improved by considering time-varying or distributed controllers.

Though H_∞-sensitivity minimization has received much attention, it does not directly control *peak* deviation in the output, and hence the ℓ_1-sensitivity minimization problem was formulated by Vidyasagar in [6]. For this problem, $\mathcal{B}$ is chosen to be ℓ_∞ with its usual norm and μ is selected to be

$$\mu(H_{yd}) \triangleq \sup_{d \in \ell_\infty} \frac{\|H_{yd}(d)\|_\infty}{\|d\|_\infty}$$

as this gives the peak gain from disturbance to output.

Once again, one might suspect that excluding time-varying or distributed controllers from consideration would worsen the achievable optimum ℓ_1-sensitivity. Dahleh and Pearson ([7]) showed that excluding distributed compensators does not worsen the achievable optimum, and gave a procedure for computing a (rational) compensator, optimal in the class of time-invariant compensators. The question of improvement using time-varying compensation remained open. It might be pointed out that results in the H_∞ case on time-varying compensation *do not* directly lead one to believe that a similiar result would be true in the ℓ_1 case. Firstly, the rational optimal solutions to the two problems have vastly different properties ([12]), and secondly the techniques used in H_∞ proofs are not generally applicable in the ℓ_1 case.

In this paper we show that, for FDLTI plants, the optimum ℓ_1-sensitivity achievable with time-invariant linear compensation is the same as that achievable with possibly time-varying linear compensation. Hence in fact the ℓ_1-sensitivity minimization problem shares with the H_∞ problem the amazing property that over the class of time-varying distributed compensators, an FDLTI compensator is optimal.

For brevity and clarity, we shall prove the result only in the single-input, single-output (SISO) case. The generalization to the multi-input, multi-output case is straightforward.

II Main Results

Let $\mathcal{L}$ denote the normed ring of all bounded causal linear operators, H, from ℓ_∞ to itself that possess a pulse response[1] $h(l,k)$, and let $\mathcal{L}_{\text{TI}}$ denote the subring of $\mathcal{L}$ consisting of time-invariant operators. For $H \in \mathcal{L}$ it is easy to show that

$$\sup_{x \in \ell_\infty} \frac{\|H(x)\|_\infty}{\|x\|_\infty} = \sup_k \sum_{l=0}^{\infty} |h(l,k)| \triangleq \|H\|_1$$

We can think of the operators in $\mathcal{L}$ as infinite matrices, $\mathcal{H}$, that are lower triangular, have each column in ℓ_1, and the sequence of absolute column sums in ℓ_∞. Of course, an operator in $\mathcal{L}_{\text{TI}}$ is marked by the fact that its matrix is Toeplitz. We may also identify $\mathcal{L}_{\text{TI}}$ with the space ℓ_1. An isomorphism of $\mathcal{L}_{\text{TI}}$ onto ℓ_1, for instance, takes an infinite Toeplitz matrix to its first column.

The *dual* of $\mathcal{L}$, $\mathcal{L}^*$, is also representable as a space of infinite matrices, $\mathcal{R}$, upper triangular, with rows in ℓ_∞ and the sequence of absolute row supremums in ℓ_1.

These observations are summed up in:

Lemma 1

1. *The subring $\mathcal{L}_{\text{TI}}$ may be identified with ℓ_1. The dual of ℓ_1 may be identified with ℓ_∞.*
2. *Any bounded linear functional, ρ, on $\mathcal{L}$ can be identified with a doubly indexed sequence, $\rho(l,k)$ such that*

$$\sum_{k=0}^{\infty} \left(\sup_l |\rho(l,k)| \right) = M < \infty$$

the action of ρ on H can be expressed as

$$\rho[H] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho(j,i) h(i,j) \tag{2}$$

[1] Trace class is sufficient.

3. Any sequence in ℓ_∞, $\tilde{\rho}(k)$, gives rise to a bounded linear functional, ρ on $\mathcal{L}$ whose doubly indexed sequence is

$$\rho(l,k) = \begin{cases} \rho(k) & \textit{for } l = 0; \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, ρ's action on $\mathcal{L}_{\text{TI}}$ agrees with $\rho(k)$'s action on ℓ_1.

PROOF:

Easy. Left to the reader. ■

Note that the action (2) can be interpreted via matrices as Trace($\mathcal{R}\mathcal{H}$).

Lemma 2 **Let $\frac{N}{D}$ be a stable coprime factorization of P. Let X and Y be stable and satisfy**

$$XN + YD = I$$

Then any stabilizing compensator for P in $\mathcal{L}$ has the form

$$C = \frac{(X + DQ)}{(Y - NQ)} \qquad Q \in \mathcal{L} \tag{3}$$

With C as given in (3), the closed-loop map from d to y in Figure 1 is given by

$$H_{yd} = YD - NDQ \tag{4}$$

and thus all disturbance to output maps achievable with a stabilizing time-varying compensator are given by (4) with $Q \in \mathcal{L}$; all disturbance to output maps achievable with a time-invariant compensator are given by (4) with $Q \in \mathcal{L}_{\text{TI}}$.

PROOF:

This result can be derived from work in [1]. It appears in [10, Theorem 2.7] and [8]. ■

Recalling (1) and using Lemma 2 the optimum ℓ_1-sensitivity achievable with time-varying compensation is given by

$$\alpha = \inf_{Q \in \mathcal{L}} \|(YD - NDQ)\|_1 \tag{5}$$

whereas the optimum ℓ_1-sensitivity achievable with time-invariant compensation is given by

$$\tilde{\alpha} = \inf_{Q \in \mathcal{L}_{\text{TI}}} \|(YD - NDQ)\|_1 \tag{6}$$

Our main result is the following.

Theorem 3 $\alpha = \tilde{\alpha}$.

PROOF :

For clarity in the proof, we will consider $\mathcal{L}_{TI}$ to be a normed ring in its own right and will denote elements of $\mathcal{L}_{TI}$ with a tilde. The same operator considered as an element of $\mathcal{L}$ will be denoted without the tilde. Since $\mathcal{L}_{TI}$ is a subring of $\mathcal{L}$ we have a canonical imbedding (isometric isomorphism) $\phi: \mathcal{L}_{TI} \to \mathcal{L}$. We will write

$$\tilde{\alpha} = \inf_{\tilde{Q}\in\mathcal{L}_{TI}} \|(\tilde{H} - \tilde{U}\tilde{Q})\|_1$$

and

$$\alpha = \inf_{Q\in\mathcal{L}} \|(H - UQ)\|_1$$

Where $\tilde{H}, \tilde{U}$ are defined appropriately from (6) and H, and U are the images under $\phi(\cdot)$ of $\tilde{H}$, and $\tilde{U}$ respectively.

Define the sets

$$\begin{aligned}
\tilde{I} &\triangleq \{\tilde{U}\tilde{Q} \mid \tilde{Q} \in \mathcal{L}_{TI}\} \subseteq \mathcal{L}_{TI} \\
I &\triangleq \{UQ \mid Q \in \mathcal{L}\} \subseteq \mathcal{L} \\
E &\triangleq \{\rho \in I^{\perp} \mid |\rho[G]| \leq \|G\|_1 \; \forall G \in \mathcal{L}\} \subseteq \mathcal{L}^* \\
\tilde{E} &\triangleq \{\tilde{\rho} \in \tilde{I}^{\perp} \mid |\tilde{\rho}[\tilde{G}]| \leq \|\tilde{G}\|_1 \; \forall \tilde{G} \in \mathcal{L}_{TI}\} \subseteq \mathcal{L}^*_{TI}
\end{aligned}$$

An application of Fenchel's Theorem ([11, Theorem 1, p. 119]) then gives

$$\tilde{\alpha} = \max_{\tilde{\rho}\in\tilde{E}} \tilde{\rho}[\tilde{H}] \tag{7}$$

$$\alpha = \max_{\rho\in E} \rho[H] \tag{8}$$

Now, ϕ induces a map, $\phi^*: \mathcal{L}^* \to \mathcal{L}^*_{TI}$ by

$$\phi^*(\rho)[\tilde{G}] \triangleq \rho[\phi(\tilde{G})] \tag{9}$$

Morover, by the Hahn-Banach Theorem, ϕ^* is onto $\mathcal{L}^*_{TI}$ and is norm non-increasing.

By the results in [7] or [13], we have that $\tilde{E}$ is spanned by m linear functionals representable as elements in ℓ_∞ by

$$\begin{aligned}
\tilde{\rho}_{\Re} &= 1, \Re(z_0^{-1}), \Re(z_0^{-2}), \ldots \\
\tilde{\rho}_{\Im} &= 0, \Im(z_0^{-1}), \Im(z_0^{-2}), \ldots
\end{aligned}$$

where z_0 is a non-minimum phase zero of $\tilde{U}$'s z-transform. Now, for each such z_0 there are, by Lemma 1, elements $\rho_{\Re}$ and $\rho_{\Im}$ in $\mathcal{L}^*$ that are mapped to $\tilde{\rho}_{\Re}$ and

$\tilde{\rho}_{\Im}$ under ϕ^*. The action of ϕ^* given in (9) and the observation that $\phi(\tilde{I}) \subset I$, shows $\rho_{\Re}$ and $\rho_{\Im}$ annihilate I and hence are in E.[2] Thus $\phi^*(E) = \tilde{E}$.

Then

$$
\begin{aligned}
\alpha &= \max_{\rho \in E} \rho[H] = \max_{\rho \in E} \rho[\phi(\tilde{H})] \\
&= \max_{\rho \in E} \phi^*(\rho)[\tilde{H}] \qquad (10) \\
&= \max_{\tilde{\rho} \in \tilde{E}} \tilde{\rho}[\tilde{H}] \\
&= \tilde{\alpha}
\end{aligned}
$$

■

III Conclusions

We have shown that ℓ_1-sensitivity cannot be improved by the use of time-varying compensation. This means the ℓ_1 problem and the H_∞ problem both share the property of having the compensator optimal over the class of all linear compensators be FDLTI when the plant is FDLTI.

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