STABILITY, STOCHASTIC STATIONARITY, AND GENERALIZED LYAPUNOV EQUATIONS FOR TWO-POINT BOUNDARY-VALUE DESCRIPTOR SYSTEMS[†]

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Abstract

In this paper, we introduce the concept of internal stability for two-point boundary-value descriptor systems (TPBVDSs). Since TPBVDSs are defined only over a finite interval, the concept of stability is not easy to formulate for these systems. The definition which is used here consists in requiring that as the length of the interval of definition increases, the effect of boundary conditions on states located close to the center of the interval should go to zero. Stochastic TPBVDSs are studied, and the property of stochastic stationarity is characterized in terms of a generalized Lyapunov equation satisfied by the variance of the boundary vector. A second generalized Lyapunov equation satisfied by the state variance of a stochastically stationary TPBVDS is also introduced, and the existence and uniqueness of positive definite solutions to this equation is then used to characterize the property of internal stability.

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1. Introduction

Noncausal physical phenomena arise in many fields of science and engineering. These phenomena correspond usually to processes evolving in space, instead of time. To model such processes, the usual state-space models familiar to system theorists are not appropriate, since these models were developed primarily to describe causality, in the sense that the "state" of a system at a given time is a summary of the past inputs sufficient to compute future outputs. One is then led to ask: what is a natural class of models to describe noncausal phenomena in one-dimension? It is the goal of this paper, as well as of earlier papers and reports [1]-[4], to suggest that perhaps the most natural class of discrete-time noncausal models in one-dimension is the class of two-point boundary-value descriptor systems (TPBVDSs). This conclusion is drawn from the observation that the impulse response of a time-invariant descriptor system is *noncausal*, and that the dynamics of these systems are symmetric with respect to forwards and backwards propagation. In addition, for systems defined over a finite interval, two-point boundary-value conditions will also enforce noncausality in the sense that both ends of the interval play a symmetric role in the expression of the boundary conditions.

The noncausality of discrete-time descriptor systems is a well known feature of these systems. It is for example much in evidence in the early work of Luenberger [5]-[6], where it is also pointed out that two-point boundary-value conditions are usually needed to guarantee well-posedness of these systems. In Lewis [7], it was shown that these systems could be decomposed into forwards and backwards propagating subsystems, so that their solution involves recursions in both time directions. However, in spite of these useful observations, it is fair to say that most of the the literature on descriptor systems has focused mainly on issues of structure [8]-[10], and their implication for the control of descriptor systems [11]-[14]. This is primarily due to the fact that in continuous-time, descriptor systems display an impulsive behavior, which until recently has been the focus of most of the attention.

One of the most important influences for the work reported here has been the work by Krener [15]-[18] on the system-theoretic properties of standard (i.e.,

nondescriptor) continous-time boundary-value systems, and on the use of stochastic boundary-value systems to realize reciprocal processes. The results of Krener, as well as the related work of Gohberg, Kaashoek and Lerer [19]-[21], have pointed out that boundary-value linear systems have a rich internal structure, and can be used to model a wide class of non-Markov, i.e. noncausal, stochastic processes. The results presented in this paper, as well those of [1]-[4] combine in some sense the degree of noncausality attributable to the boundary conditions, which was already present in Krener's work, with an additional source of noncausality, namely the noncausal dynamics of discrete-time descriptor systems.

Another important motivation for the study presented here is our own work on linear estimation of noncausal stochastic processes in one or several dimensions [22]-[24]. Since the framework proposed in [22] and [23] for the solution of noncausal estimation problems is totally general, and is applicable to absolutely any model in any dimension, one of our objectives has been to find 1-D models which display as much noncausality as possible, so that estimation results developed for these models will be easy to transpose to higher dimensions. This has led us in [4] to examine estimation problems for TPBVDSs. In this context, it was shown that the TPBVDS smoother was itself a TPBVDS which could be decoupled into forwards and backwards filters through the solution of certain generalized Riccati equations [25]. However, this study raised a number of system-theoretic questions: do reachability and observability guarantee the existence and uniqueness of positive-definite solutions for the generalized Riccati equations that we obtained? Is the estimator stable, and if so, in what sense, since TPBVDSs are defined only over a finite interval? More fundamentally, is it possible to define concepts of reachability, observability, and minimality for purely acausal systems such as TPBVDSs? In other words, we needed to develop a complete *system theory* for TPBVDSs, and the present paper is part of a sequence of papers devoted to the exposition of such a theory.

In [1], the concepts of *outwards and inwards processes*, which were originally introduced by Krener [16] for boundary-value systems, were developed for TPBVDSs, and were then used to define concepts of strong and weak reachability and observability. Several recursive solution schemes for TPBVDSs were also

proposed, which rely on the forwards/backwards and inwards/outwards decompositions of these systems. These results were then specialized to shift-invariant TPBVDSs in [2], and in this context, results linking reachability, observability, and minimality were obtained. Again, these results were closely related to corresponding results obtained by Krener, and by Gohberg and Kaashoek, for boundary value systems. The present paper contains the first significant departure from existing work on boundary value systems in the sense that we introduce a new concept, that of stability, which has not yet been used to study noncausal systems. As will become apparent below, the notion of stability is not easy to formulate for TPBVDSs, since these systems are defined over a finite interval. However, a relatively natural concept is that of *internal stability*, whereby as the length of the interval of definition of a TPBVDS grows, the effect of the boundary conditions on states located close to the center of the interval goes to zero. A theory of stability that parallels the standard theory for causal systems is developed by considering stochastically stationary TPBVDSs, and by showing that stochastic stationarity can be characterized in terms of generalized Lyapunov equations. The existence and uniqueness of positive-definite solutions to these Lyapunov equations is then characterized in terms of the property of internal stability. It turns out that the stability results developed in this paper will play a key role in the study of the stability and asymptotic properties of TPBVDS smoothers, and of the generalized Riccati equations presented in [4] and [25].

This paper is organized as follows. In Section 2, the properties of *displacement* two-point boundary-value descriptor systems, which were studied in [2], are briefly reviewed. Displacement systems are such that their Green's function is invariant under time-shifts, and they play therefore the same role for TPBVDSs as time-invariant systems for causal nondescriptor systems. The results of this paper are restricted to this class of systems. In our study we shall also examine extendible displacement TPBVDSs, which are systems whose Green's function can be extended to a larger interval. In Section 3, two notions of stability, namely *internal stability*, as described previously, and *stable extendibility* are introduced. Stable extendibility corresponds to the ability to extend the Green's function of a displacement TPBVDS defined over a finite interval, in such a way that both the

dynamics and Green's function of the original system are preserved, and the extended Green's function is summable. However, it is shown that this concept of stability is not as fruitful as that of internal stability. In Section 4, we examine stochastic TPBVDSs, and study in particular *stochastically stationary* systems. Two generalized Lyapunov equations which must be satisfied respectively by the state variance, and the boundary variance of the boundary vector are introduced, and the property of stochastic stationarity is characterized in terms of the second of these equations. It is shown in Section 5 that the covariance function of a stochastically stationary TPBVDS satisfies a second-order descriptor equation, with appropriate boundary conditions. Finally, in Section 6 the existence and uniqueness of solutions to the generalized Lyapunov equation satisfied by the state variance is characterized in terms of the property of internal stability. The concluding Section 7 describes the role that the results of this paper are expected to play in the study of the TPBVDS smoothers and generalized Riccati equations of [4] and [25].

2. Displacement Systems and Extendibility

The two-point boundary-value descriptor systems (TPBVDS) considered in this paper satisfy the difference equation

$$Ex(k+1) = Ax(k) + Bu(k) , \quad 0 \leq k \leq N-1 \tag{2.1}$$

with the two-point boundary value condition

$$V_i x(0) + V_f x(N) = v . \tag{2.2}$$

Here E, A, and B are constant matrices, x and v are n-dimensional vectors, and u is an m-dimensional vector. Since the system theoretic properties of this class of systems, such as the displacement, reachability, observability, and minimality properties have been studied in detail in [1]-[3], we review here only the concepts that will be needed in this paper.

It was shown in [1] that, without loss of generality, it can be assumed[†] that

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† A necessary and sufficient condition [1] for (2.1), (2.2) to be well-posed — — i.e., to yield a well-defined map from $\{v,u\}$ to x — — is that, by multiplication on the left only, E, A, V_i, and V_f can be brought to a form satisfying (2.3)-(2.4).

the system (2.1)-(2.2) is in *normalized-form*, i.e., it satisfies the following two properties: (i) there exists some scalars α and β such that

$$\alpha E + \beta A = I \ , \tag{2.3}$$

which implies that E and A commute; and (ii) the boundary matrices V_i and V_f are such that

$$V_i E^N + V_f A^N = I \ . \tag{2.4}$$

A slight generalization of the above normalized form was introduced in [2], and will be of value to us in our development. Specifically, (2.1)-(2.2) is said to be in *block-normalized form* if (2.4) holds and

$$E = \text{diag}(E_1, ..., E_M) \quad , \quad A = \text{diag}(A_1, ..., A_M) \ , \tag{2.5}$$

where: (i) the block sizes of E and A are compatible; (ii) for each j, there exists (α_j, β_j), possibly varying with j , such that

$$\alpha_j E_j + \beta_j A_j = I \ , \tag{2.6}$$

and (iii) the eigenmodes of distinct blocks of the system are different, i.e., for any $(s,t) \neq (0,0)$, $|sE_j - tA_j| = 0$ for at most one value of j. Any well-posed TPBVDS can always be put in normalized or block-normalized form, and we will frequently assume that our system is in one of these two forms.

A special class of two-point boundary-value descriptor systems which is of great interest is the class of displacement TPBVDSs [2]-[3].

Definition 2.1: A TPBVDS is a *displacement system* if the Green's function $G(k,l)$ appearing in the solution

$$x(k) = A^k E^{N-k} v + \sum_{l=0}^{N-1} G(k,l) Bu(l) \tag{2.7}$$

of the TPBVDS (2.1)-(2.2) depends only on the difference between arguments k and l, so that

$$G(k,l) = G(k-l) \ . \tag{2.8}$$

Note that the above terminology is consistent with that of Gohberg, Kaashoek and Lerer [20] (see also [21]), who introduced a similar class of

displacement systems in their study of boundary value systems with standard nondescriptor dynamics. Unlike for causal systems, the fact that the matrices E and A are constant is not sufficient to guarantee that the TPBVDS (2.1)-(2.2) is a displacement system. The matrices E and A must also satisfy some properties in relation to the boundary matrices V_i and V_f. The following characterization of displacement systems was established in [2].

Theorem 2.1: A TPBVDS in block-normalized form is a displacement system if and only if the matrices E and A commute with both V_i and V_f, i.e.,

$$[E,V_i] = [E,V_f] = [A,V_i] = [A,V_f] = 0 , \tag{2.9}$$

where

$$[X,Y] = XY - YX . \tag{2.10}$$

Another useful result from [2] is:

Theorem 2.2 Consider a displacement TPBVDS in block-normalized form. Then V_i and V_f are also block diagonal with block sizes compatible with those of E and A.

In the following, we shall restrict our attention to displacement TPBVDSs. For a system of this type, and in block-normalized form, the Green's function $G(k,l)$ can be expressed as (see [2], [3])

$$G(k,l) = G(k-l) = \begin{cases} V_i A^{k-l-1} E^{N-k+l} & k > l \\ -V_f E^{l-k} A^{N-1-l+k} & k \leq l \end{cases} \tag{2.11}$$

which clearly depends only on the difference between arguments k and l.

Note that the TPBVDS (2.1)-(2.2) is defined over an interval of fixed length. In the context of the asymptotic properties studied in this paper , it is of interest to consider the question of changing this interval of definition. In [1]-[2] a natural method for propagating the boundary conditions *inward* was developed. Specifically, for any $[K,L]$ with $0 \leq K \leq L \leq N$ one can construct new matrices $V_i(K,L)$ and $V_f(K,L)$ and a new boundary condition $v(K,L)$, depending only on v and the values of $u(k)$ for $k \in [0,K-1] \bigcup [L,N-1]$, so that the solution of (2.1) together with the boundary condition

$$V_i(K,L)x(K) + V_f(K,L)x(L) = v(K,L) \qquad (2.12)$$

agrees with the solution to (2.1)-(2.2) on $[K,L]$. Note that this implies that the Green's function of (2.1), (2.12) on the smaller interval agrees with the *restriction* of the original Green's function. Also, in [2] it was shown that for a displacement TPBVDS in block-normalized form, we can take

$$V_i(K,L) = V_i E^{N-L+K} \qquad (2.13a)$$

$$V_f(K,L) = V_f A^{N-L+K} . \qquad (2.13b)$$

For the study of asymptotic properties, we are naturally interested in extending rather than restricting the interval of definition. This leads to the following:

Definition 2.2: A displacement TPBVDS given by (2.1)-(2.2) is *extendible* if given any interval $[K,L]$ containing $[0,N]$, i.e., such that $K \leq 0 \leq N \leq L$, there exists a TPBVDS over this larger interval with the same dynamics as in (2.1), but with new boundary matrices $V_i(K,L)$ and $V_f(K,L)$ such that:

(i)   The new extended system is a displacement system.

(ii) The Green's function $G(k-l)$ of the original system is the restriction of the Green's function $G_e(k-l)$ of the new extended system:

$$G(k-l) = G_e(k-l) \quad \text{for} \quad |k-l| \leq N . \qquad (2.14)$$

Using (2.13), it is possible to prove the following [2]:

Theorem 2.3: A displacement TPBVDS in block-normalized form is extendible if and only if the following two conditions are satisfied:

(i) $Ker(E^n) \subset Ker(V_i) \qquad (2.15a)$

(ii) $Ker(A^n) \subset Ker(V_f) . \qquad (2.15b)$

When conditions (2.15a) and (2.15b) are satisfied, it was shown in [2] that a choice of boundary matrices for the extended system over $[K,L]$ is given by

$$V_i(K,L) = V_i E^N (E^D)^{L-K} \qquad (2.16a)$$

$$V_f(K,L) = V_f A^N (A^D)^{L-K} , \qquad (2.16b)$$

where E^D and A^D denote respectively the Drazin inverses of matrices E and A

(see [26], p. 8 for a definition of the Drazin inverse). Note that the extended system is also in block-normalized form.

3. Stability

The extendibility property of displacement TPBVDSs satisfying (2.15) is an important feature that will be useful below to characterize a concept of stability called *stable extendibility*. It turns out, however, that this concept of stability leads to relatively uninteresting results, and in fact there exists a more interesting concept of stability for TPBVDSs, called *internal stability*. Both of these concepts are now defined.

A. Notions of Stability

According to our definition of an extendible displacement TPBVDS, it is always possible to extend the domain of definition of such a system. An interesting question which is related to the issue of stability is under what conditions we can push back the boundaries to $\pm \infty$ in a meaningful way, so that the TPBVDS (2.1)-(2.2) can be viewed as part of a system defined over an infinite interval.

Definition 3.1: An extendible displacement TPBVDS defined over $[0,N]$ admits a *stable extension* if the Green's function $G_e(k)$ of the TPBVDS obtained by extending the interval of definition to the whole real line is summable, i.e.,

$$\sum_{-\infty}^{+\infty} || G_e(k) || < \infty , \tag{3.1}$$

where $||.||$ denotes here the matrix norm induced by the Euclidean norm for vectors of R^n.

The above characterization describes one situation where the issue of stability arises for TPBVDSs. However, there exists a second situation which is actually more meaningful, and which leads to a different concept of stability. In this second situation we examine a displacement, not necessarily extendible, TPBVDS defined over a finite interval, and where the boundary condition (2.2) corresponds to a physical constraint of the problem which cannot be modified. In this case, when the dynamics (2.1) and boundary condition (2.2) are fixed, we would like to

study the effect of increasing the size of the domain $[0,N]$ of definition of the TPBVDS on the state variables $x(k)$ located close to the center of this domain . One issue which arises in this context is that if the TPBVDS (2.1)-(2.2) is originally in bock-normalized form for a length N_0 of the interval of definition, and if we increase the length to N without changing the matrices V_i, V_f and the vector v appearing in (2.2), the system will not remain in block-normalized form, since identity (2.4) is not satisfied for $N > N_0$. Observe however that the boundary condition (2.2) is not affected by a left multiplication by an invertible matrix. Consequently, if we renormalize (2.2) by a left multiplication by $(V_i E^N + V_f A^N)^{-1}$ and change the matrices V_i, V_f and the vector v accordingly, the TPBVDS will be in block-normalized form. In this context, it is possible to describe internal stability as follows.

Definition 3.2: The displacement TPBVDS (2.1)-(2.2) in block-normalized form is *internally stable* if as the length N of the interval of definition tends to infinity, the effect of the boundary value v on any $x(k)$ located near the mid-section of interval $[0,N]$ goes to zero, i.e.,

$$\lim_{N \to \infty} E^{N/2} A^{N/2} (V_i E^N + V_f A^N)^{-1} = 0 . \tag{3.2}$$

To interpret condition (3.2), note that according to (2.7), and taking into account the renormalization described above to put the TPBVDS in bock-normalized form as the interval length N is increased, the effect of the boundary vector v on state $x(k)$ is given by $A^k E^{N-k} (V_i E^N + V_f A^N)^{-1} v$. Thus, for $k = N/2$, which corresponds to a point in the middle of interval $[0,N]$, the effect of v $x(N/2)$ is $E^{N/2} A^{N/2} (V_i E^N + V_f A^N)^{-1} v$.

It is also possible to develop another interpretation of this notion of stability, which we will state without proof. Specifically, as we change N without changing V_i and V_f, except for the renormalization, we actually are changing the entire Green's function of the TPBVDS. Thus, what we have is a *sequence* of Green's functions $G_N(k)$, $1-N \leq k \leq N$, indexed by N. Internal stability is then equivalent to

$$\lim_{N \to \infty} \sum_{k=1-N}^{N} ||G_N(k)|| < \infty , \tag{3.3}$$

which should be contrasted with (3.1).

As an illustration of the above concept of stability, consider a system that describes the heat distribution around a ring. Since the ring is closed, this system has a periodic boundary condition $x(0) = x(N)$, which is independent of the size of the ring. In this case, if a perturbation in heating conditions is applied to one point of the ring, one would expect that as the size of the ring increases, the effect of this perturbation will become smaller and smaller for points which are located on the opposite side of the ring.

As will be shown below, it is possible to obtain necessary and sufficient conditions that characterize the properties of stable extendibility and internal stability for TPBVDSs. However, the conditions that we shall obtain are quite different, and consequently, the two concepts of stability described above do not coincide.

B. Decomposition of a Displacement TPBVDS

The characterizations of stable extendibility and internal stability that will be obtained below rely on a particular decomposition of a displacement TPBVDS. The starting point of this decomposition is the following result, which was already used in [4].

Lemma 3.1: Given a TPBVDS, there exists invertible matrices F and T such that

$$E_D = FET = \begin{bmatrix} I & 0 & 0 \\ 0 & A_b & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (3.4a)$$

$$A_D = FAT = \begin{bmatrix} A_f & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U \end{bmatrix}, \qquad (3.4b)$$

where A_f and A_b have eigenvalues inside the unit circle, and U has eigenvalues on the unit circle.

The above decomposition is just a modification of the Weierstrass decomposition of a regular matrix pencil (see [27], p. 28; note that (2.6) guarantees here that the pencil $zE - A$ is regular). The transformation (3.4) can be achieved by left-

multiplication of (2.1) by F and by performing the state transformation

$$x(k) = Tx_D(k) .\qquad(3.5)$$

Observe that E_D and A_D satisfy (2.5), (2.6) with $(\alpha_1, \beta_1) = (\alpha_3, \beta_3) = (1, 0)$ and $(\alpha_2, \beta_2) = (0, 1)$. Also, by construction the eigenmodes of the three blocks are different.

To complete the transformation of our system, B is replaced by

$$B_D = FB ,\qquad(3.6)$$

and the boundary matrices become

$$V_{Di} = LV_i T \quad , \quad V_{Df} = LV_f T ,\qquad(3.7)$$

where the normalizing matrix L is selected here such that relation (2.4) is satisfied by the new TPBVDS. Finally, if the original TPBVDS was a displacement system, the new TPBVDS is also a displacement system since its Green's function is related to the original Green's function through

$$G_D(k-l) = T^{-1}G(k-l)F^{-1} .\qquad(3.8)$$

In this case, since the TPBVDS specified by (3.4), (3.6), and (3.7) is a displacement system in block-normalized form, we can invoke Theorem 2.2 to conclude that the matrices V_{Di} and V_{Df} are block diagonal, i.e.,

$$V_{Di} = \begin{bmatrix} V_{i1} & 0 & 0 \\ 0 & V_{i2} & 0 \\ 0 & 0 & V_{i3} \end{bmatrix} \text{ and } V_{Df} = \begin{bmatrix} V_{f1} & 0 & 0 \\ 0 & V_{f2} & 0 \\ 0 & 0 & V_{f3} \end{bmatrix} ,\qquad(3.9)$$

which yields the main result of this section:

Theorem 3.1: *(Decomposition of a displacement TPBVDS)*: Through the use of a state transformation T, and by left multiplication of (2.1) and (2.2) by invertible matrices F and L, an arbitrary displacement TPBVDS can be decomposed into three decoupled subsystems of the form

$$x_f(k+1) = A_f x_f(k) + B_f u(k) \quad , \quad V_{i1}x_f(0) + V_{f1}x_f(N) = v_1 ,\qquad(3.10a)$$

$$x_b(k) = A_b x_b(k+1) - B_b u(k) \quad , \quad V_{i2}x_b(0) + V_{f2}x_b(N) = v_2 ,\qquad(3.10b)$$

$$x_m(k+1) = Ux_m(k) + B_m u(k) \quad , \quad V_{i3}x_m(0) + V_{f3}x_m(N) = v_3 \, , \quad (3.10c)$$

where the matrices A_f and A_b have their roots inside the unit circle, and U has its roots on the unit circle. The subsystems (3.10a)-(3.10c) are displacement systems, in normalized form.

In what follows, for convenience only, we will refer to (3.10a)-(3.10c) as the forward, backward, and marginal parts of the system, respectively. Note, for example, that the dynamics of (3.10a) look like forward dynamics, and those of (3.10b) look like backward dynamics, but the boundary conditions in each case can make each of these noncausal.

C. Characterization of Stable Extendibility and Internal Stability

An interesting aspect of the decomposition (3.10) of a displacement TPBVDS is that it reduces the study of stable extendibility and internal stability for a TPBVDS to the study of these properties for each of its components. We consider first the forward component.

Lemma 3.2: Consider a displacement TPBVDS given by

$$x(k+1) = Ax(k) + Bu(k) \qquad (3.11a)$$

$$V_i x(0) + V_f x(N) = v \qquad (3.11b)$$

where A has all its roots inside the unit circle. Then, the system (3.11) is internally stable if and only if the matrix V_i is invertible. If the system (3.11) is extendible, i.e., if $Ker(A^n) \subset Ker(V_f)$, it is stably extendible if and only if $V_f = 0$, in which case the system is causal.

Proof: Taking into account the definition (3.2) of internal stability, we see that (3.11) is internally stable if and only if

$$\lim_{N \to \infty} A^{N/2}(V_i + V_f A^N)^{-1} = 0 \, ,$$

which is clearly equivalent to requiring that V_i should be invertible. To study stable extendibility, it is convenient to note that by using a procedure similar to the one employed to obtain decomposition (3.10), the system (3.11) can be transformed so that

$$A = \begin{bmatrix} J & 0 \\ 0 & M \end{bmatrix},$$

where M is a nilpotent matrix and J is invertible, and

$$V_i = \begin{bmatrix} V_{Ji} & 0 \\ 0 & V_{Mi} \end{bmatrix}, \quad V_f = \begin{bmatrix} V_{Jf} & 0 \\ 0 & V_{Mf} \end{bmatrix}.$$

Then, the extendibility condition $Ker(A^n) \subset Ker(V_f)$ implies that we must have

$$V_{Mf} = 0 . \tag{3.12}$$

Furthermore, by using the extension procedure of equations (2.16), it is easy to check that the Green's function $G_e(k)$ of the system which extends the Green's function of system (3.11) to the whole line is given by

$$G_e(k) = V_i A^{k-1} \quad \text{for } k > 0 , \tag{3.13a}$$

and

$$G_e(k) = \begin{bmatrix} -V_{Jf} \, J^{N-1+k} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for } k \leq 0 . \tag{3.13b}$$

Since P has its roots inside the unit circle, $G_e(k)$ diverges as $k \to -\infty$, unless

$$V_{Jf} = 0 . \tag{3.14}$$

Combining (3.12) and (3.14), we see that the TPBVDS (3.14) is stably extendible if and only if $V_f = 0$, which is the desired result. In this case, the system (3.11) is causal, and the normalized form relation (2.4) implies that $V_i = I$, which is obviously invertible. We can therefore conclude that in this case stable extendibility implies internal stability. □

Lemma 3.2 can then be used to obtain the following characterization of stable extendibility.

Theorem 3.2: An arbitrary extendible displacement TPBVDS is stably extendible if and only if the decomposition (3.10) of this system is such that

$$V_{f1} = V_{i2} = 0 , \tag{3.15}$$

and the system does not have any eigenmode on the unit circle, i.e., it does not

4. Stochastic TPBVDSs and Generalized Lyapunov Equations

In this section, we study the class of stochastic TPBVDSs given by (2.1)-(2.2), where $u(k)$ is a zero-mean white Gaussian noise with unit intensity, and where v is a zero-mean Gaussian random vector independent of $u(k)$ for all k, and with covariance Q. Thus, we have

$$M[u(k)u^T(l)] = I\delta(k-l) \,, \tag{4.1}$$

where $M[z]$ denotes the mean of a random variable z, and $\delta(k)$ is the Kronecker delta function. In addition, it is assumed throughout the remainder of this paper that the TPBVDS (2.1)-(2.2) is a displacement system in normalized form. The displacement assumption is quite important, and all the results of this paper concerning stochastic TPBVDSs are restricted to this class of systems.

In the continuous-time case, and for the usual nondescriptor state-space dynamics, a related class of stochastic boundary-value systems was examined by Krener [17],[18], who studied the relation existing between this class of systems and reciprocal processes. In particular, Krener considered the problem of realizing reciprocal processes with stochastic boundary-value systems. Our goal here is somewhat different, in the sense that we shall seek to obtain a complete set of conditions under which a stochastic TPBVDS of the form (2.1)-(2.2) is stochastically stationary. It turns out that the characterization that will be obtained involves a Lyapunov equation for the boundary variance Q which generalizes the standard Lyapunov equation for stationary Gauss-Markov processes.

Definition 4.1: A TPBVDS is *stochastically stationary* if

$$M[x(k)x^T(l)] = R(k,l) = R(k-l) \,. \tag{4.2}$$

If the TPBVDS (2.1)-(2.2) is stochastically stationary, the *variance matrix* $P(k) = R(k,k)$ of $x(k)$ must be constant, i.e., $P(k) = P$ for all k. Thus, our first step at this point will be to characterize completely the matrix $P(k)$ for a displacement TPBVDS in normalized form. Let

$$\Pi(k) = \sum_{j=0}^{k} A^{k-j} E^j BB^T (A^{k-j} E^j)^T \,. \tag{4.3}$$

Then, using the Green's function solution (2.7), (2.11), multiplying by its

transpose, and taking expected values, we obtain

$$P(k) = A^k E^{N-k} Q (A^k E^{N-k})^T + (V_i E^{N-k}) \Pi(k-1)(V_i E^{N-k})^T$$
$$+ (V_f A^k) \Pi(N-1-k)(V_f A^k)^T . \qquad (4.4)$$

This expression can also be rewritten as

$$P(k) = A^k E^{N-k} Q (A^k E^{N-k})^T + R_w(k) R_w^T(k) , \qquad (4.5)$$

where

$$R_w(k) = \begin{bmatrix} V_i E^{N-k} R_s(k) & V_f A^k R_s(N-k) \end{bmatrix}, \qquad (4.6a)$$

and where $R_s(k)$ is the strong reachability matrix

$$R_s(k) = \begin{bmatrix} A^{k-1}B & EA^{k-2}B & \cdots & E^{k-1}B \end{bmatrix}. \qquad (4.6b)$$

The matrix $R_w(k)$ plays a central role for the concept of *weak reachability* of a TPBVDS [1]-[3]. In [1] weak reachability is defined as the condition that the range $\text{Im}(R_w(k))$ of $R_w(k)$ for k sufficiently far from 0 and N (i.e., $k \in [n, N-n]$)is all of $\mathbf{R}^n$. In [2], a somewhat weaker definition is used, namely

$$\bigcup_k \text{Im}(R_w(k)) = \mathbf{R}^n . \qquad (4.7a)$$

Note that this condition is equivalent to the statement

$$v^T R_w(k) = 0 \ \text{ for all } k \ \Rightarrow \ v = 0 . \qquad (4.7b)$$

Examining (4.5), we see that if the TPBVDS is weakly reachable in the sense of [1], $P(k)$ is positive definite for $n \leq k \leq N-n$. If the TPBVDS is weakly reachable in the sense of [2], this need not be the case. However, from (4.5) and (4.7), we can conclude that

$$\bigcup_k \text{Im}(P(k)) = \mathbf{R}^n . \qquad (4.8)$$

Thus, when the TPBVDS is weakly reachable in this sense *and* has a *constant* variance P (so that $\text{Im}(P(k))$ is constant), we can conclude from (4.8) that P is positive definite.

The expression (4.4) for $P(k)$ is an *explicit description*, and is valid in general for displacement TPBVDSs in normalized form. However, as in the causal case, where $P(k)$ satisfies a a time-dependent Lyapunov equation, it is also possible to obtain an *implicit description* for $P(k)$ in the form of a recursion with boundary conditions. Specifically, multiplying both sides of equations (2.1) and (2.2) by their transposes, using the Green's function solution (2.7), (2.11), and taking expected values, it can be shown that $P(k)$ satisfies the TPBVDS

$$EP(k+1)E^T - AP(k)A^T = (V_i E^N)BB^T(V_i E^N)^T$$
$$- (V_f A^N)BB^T(V_f A^N)^T \quad (4.9a)$$

$$V_i P(0) V_i^T - V_f P(N) V_f^T = (V_i E^N)Q(V_i E^N)^T - (V_f A^N)Q(V_f A^N)^T, \quad (4.9b)$$

which can be viewed as a generalized time-dependent Lyapunov equation for $P(k)$.

Note however that equations (4.9a) and (4.9b) may or may not characterize completely the variance $P(k)$, i.e., they may have several solutions, one of which will be (4.4). This corresponds to situations where (4.9a) and (4.9b) do not completely capture the structure of (4.4), and in this case, additional conditions would have to be imposed to make sure that we obtain a unique solution equal to (4.4). To obtain conditions under which equations (4.9a) and (4.9b) specify $P(k)$ uniquely, these equations can be rewritten in the form of a TPBVDS of type (2.1)-(2.2), and we can then apply the well-posedness test for TPBVDSs presented in [1]. This can be done by denoting by $p(k)$, q, and w the vectors obtained by scanning the entries of matrices $P(k)$, Q, and $W = BB^T$ columnwise, and rewriting (4.9a) and (4.9b) as

$$(E \otimes E)p(k+1) - (A \otimes A)p(k) = (V_i E^N \otimes V_i E^N)w - (V_f A^N \otimes V_f A^N)w \quad (4.10a)$$
$$(V_i \otimes V_i)p(0) - (V_f \otimes V_f)p(N) = (V_i E^N \otimes V_i E^N)q - (V_f A^N \otimes V_f A^N)q, (4.10b)$$

where $\otimes$ denotes here the Kronecker product of two matrices [28]. Note that the right-hand sides of the above equations are irrelevant as far as well-posedness is concerned.

The well-posedness condition for the TPBVDS (4.10a)-(4.10b) reduces to the invertibility of the matrix

$$F_N = (V_i \otimes V_i)(E \otimes E)^N - (V_f \otimes V_f)(A \otimes A)^N$$
$$= (V_i E^N) \otimes (V_i E^N) - (V_f A^N) \otimes (V_f A^N) . \tag{4.11}$$

We obtain therefore the following result.

Theorem 4.1: Equations (4.9a) and (4.9b) characterize uniquely the variance $P(k)$ if and only if

$$\lambda_j \neq \mu_l \quad \text{for all } j \text{ and } l , \tag{4.12}$$

where λ_j and μ_j are the eigenvalues of of matrices $V_i E^N$ and $V_f A^N$, respectively.

Proof: Since matrices $V_i E^N$ and $V_f A^N$ satisfy (2.4), they can be brought simultaneously to Jordan form. Furthermore, the eigenvalues λ_j and μ_j corresponding to the same eigenvector z satisfy

$$\lambda_j + \mu_j = 1 . \tag{4.13}$$

Then, it is easy to check that the eigenvalues of F_N must have the form $\lambda_j \lambda_l - \mu_j \mu_l$ (assume that $V_i E^N$ and $V_f A^N$ are in Jordan form in (2.4)), so that F_N is invertible as long as

$$\lambda_j \lambda_l \neq \mu_j \mu_l .$$

Taking into account (4.13), this gives (4.12). □

Note that in the causal case the eigenvalues λ_j and μ_j are all equal to 1 and 0, respectively. Thus, according to Theorem 4.1, $P(k)$ is uniquely defined. This is expected, since in this case (4.9a) is a forwards recursion for $P(k)$, and (4.9b) is the initial condition $P(0) = Q$.

Theorem 4.1 indicates that, except under very special circumstances, the variance $P(k)$ can be uniquely computed from the generalized time-dependent Lyapunov equations (4.9a) and (4.9b). In addition, when the TPBVDS is stochastically stationary, the matrix $P(k) = P$ is *constant*, and satisfies the two algebraic matrix equations

$$EPE^T - APA^T = (V_i E^N)BB^T(V_i E^N)^T - (V_f A^N)BB^T(V_f A^N)^T \tag{4.14}$$

$$V_i PV_i^T - V_f PV_f^T = (V_i E^N)Q(V_i E^N)^T - (V_f A^N)Q(V_f A^N)^T , \tag{4.15}$$

obtained by setting $P = P(k+1) = P(k)$ and $P = P(0) = P(N)$ in (4.9a) and (4.9b), respectively. Equation (4.14) is a generalized algebraic Lyapunov equation, and by analogy with the causal case, it is tempting to think that, if a TPBVDS has a constant positive definite variance matrix P satisfying (4.14), then the TPBVDS is stochastically stationary. Unfortunately, as we shall see, this is not the case, and the correct condition for stochastic stationarity, which is condition (4.16) below, involves the variance Q of the boundary vector v.

Theorem 4.2 A stochastic TPBVDS is stochastically stationary if and only if Q satisfies the equation

$$EQE^T - AQA^T = V_i BB^T V_i^T - V_f BB^T V_f^T . \qquad (4.16)$$

Proof: To prove that (4.16) is a sufficient condition, we need to show that when (4.16) is satisfied, then $R(k+1, l+1) = R(k, l)$ for all $k, l \in [0, N]$. By using the Green's function solution (2.7), (2.11) to evaluate $R(k, l) = M[x(k) x^T(l)]$ for $k \geq l$, we obtain

$$R(k,l) = A^k E^{N-k} Q (A^l E^{N-l})^T$$

$$+ \sum_{j=0}^{l-1} V_i A^{k-j-1} E^{N-k+j} BB^T (V_i A^{l-j-1} E^{N-l+j})^T$$

$$+ \sum_{j=l}^{k-1} V_f A^{k-j-1} E^{N-k+j} BB^T (V_i A^{N-1-j+l} E^{j-l})^T 1(k-l-1)$$

$$+ \sum_{j=k}^{N-1} V_f A^{N-1-j+k} E^{j-k} BB^T (V_f A^{N-1-j+l} E^{j-l})^T , \qquad (4.17)$$

where $1(k)$ is the unit step function, i.e.,

$$1(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases} .$$

From (4.17), it is easy to check that

$$R(k+1, l+1) - R(k, l) = A^k E^{N-1-k} [AQA^T - EQE^T$$

$$+ V_i BB^T V_i^T - V_f BB^T V_f^T] (A^l E^{N-1-l})^T , \qquad (4.18)$$

which indicates clearly that when Q satisfies the generalized Lyapunov equation

(4.16), then $R(k+1,l+1) = R(k,l)$ for all k, l.

Conversely, to prove that (4.16) is a necessary condition for stochastic stationarity, assume that $R(k+1,l+1) = R(k,l)$ for all k, l. Then, the right hand side of (4.18) is zero for all k, l. Thus, if

$$\Delta = AQA^T - EQE^T + V_i BB^T V_i^T - V_f BB^T V_f^T , \qquad (4.19)$$

we have

$$A^k E^{N-1-k} \Delta (A^l E^{N-1-l})^T = 0 \qquad (4.20)$$

for all k, l. Taking into account the normalized form condition (2.3), as well as (4.20), yields

$$\Delta = (\alpha E + \beta A)^{N-1} \Delta (\alpha E + \beta A)^{N-1}$$
$$= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \binom{N-1}{k}\binom{N-1}{l} \alpha^{2(N-1)-k-l} \beta^{k+l} A^k E^{N-1-k} \Delta (A^l E^{N-1-l})^T = 0 , (4.21)$$

which shows that Q must obey the generalized Lyapunov equation (4.16). $\square$

Note that for causal systems — — i.e., when $E = V_i = I$, $V_f = 0$ — — the boundary covariance matrix is simply $P(0)$, and equation (4.16) for Q in this case is *identical* to equation (4.14) for P (which is the usual Lyapunov equation), and (4.15) reduces to $P = Q$. For a general TPBVDS, however, P and Q are different quantities.

Clearly, when a TPBVDS is stochastically stationary, it must have a constant variance. Thus, if Q satisfies (4.16), the state variance is constant and satisfies (4.14). The reverse implication is obviously true as well for causal systems, sice the two Lyapunov equations (4.14) and (4.16) are identical in this case. However, constancy of the state variabce matrix does not imply stationarity for all TPBVDSs. for all TPBVDSs. In order to see what happens, set $k = l$ in (4.18) and note that $R(k,k) = P(k)$. This gives

$$P(k+1) - P(k) = A^k E^{N-1-k}[AQA^T - EQE^T + V_i BB^T V_i^T$$
$$- V_f BB^T V_f^T](A^k E^{N-1-k})^T . \qquad (4.22)$$

The relation (4.22) shows that when Q satisfies the Lyapunov equation (4.16), then $P(k+1) = P(k)$ for all k, as expected. Conversely, if $P(k+1) = P(k)$ for all

k, Q must satisfy the equation

$$A^k E^{N-1-k}[AQA^T - EQE^T + V_i BB^T V_i^T$$
$$- V_f BB^T V_f^T](A^k E^{N-1-k})^T = 0 , \quad (4.23)$$

for all k. In the special case when either E or A is invertible, this relation implies that Q must satisfy (4.16). In other words, if either E or A is invertible, the TPBVDS (2.1)-(2.2) is stochastically stationary if and only if it has a constant variance. However, this is not true in general, i.e., (4.16) is not necessarily implied by (4.23), as can be seen from the following example.

Example 4.1: Consider the TPBVDS

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} u(k) \quad (4.24a)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & N \\ 0 & 0 & 1 \end{bmatrix} x(N) = v , \quad (4.24b)$$

where the variance of v is given by

$$Q = \begin{bmatrix} 1 & N & 1 \\ N & N^2+2 & N \\ 1 & N & 1 \end{bmatrix}. \quad (4.25)$$

The system (4.24) is in normalized form and is a displacement system. Then, it is easy to check that Q satisfies (4.23), but not (4.16), and that (4.24) has a constant variance matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which satisfies both (4.14) and (4.15). This shows that a TPBVDS may have a constant variance matrix even if (4.16) is not satisfied, and therefore, the system is not stochastically stationary.

At this point, we have introduced two generalized algebraic Lyapunov equations, namely (4.14) and (4.16), for P and Q. These equations have exactly the

same form and differ only by their right hand sides. Consequently, they admit a unique solution under the same condition.

Theorem 4.3: The generalized Lyapunov equations (4.14) and (4.16) have unique solutions if and only if the eigenmodes σ_j of the TPBVDS (2.1), i.e., the values for which $\sigma E - A$ is singular, are such that

(i) $\sigma_j \sigma_l \neq 1$ for all j and l, $\hspace{4cm}$ (4.26)

(ii) there does not exist exist simultaneously eigenmodes which are zero, and eigenmodes which are ∞, i.e., the matrices E and A are not both singular.

Proof: The proof is similar to that of Theorem 4.1. Equations (4.14) and (4.16) admit unique solutions if and only if the matrix $M = E \otimes E - A \otimes A$ is invertible. But since E and A satisfy (2.3), they can be brought to Jordan form simultaneously, and we may denote by λ_j and μ_j the eigenvalues of these two matrices appearing in corresponding Jordan blocks. Assuming that E and A are in Jordan form, it is easy to check that the eigenvalues of M are $\lambda_j \lambda_l - \mu_j \mu_l$. Furthermore the eigenmodes $\sigma_j = \mu_j / \lambda_j$. Combining these two observations, and noting from (2.3) that λ_j and μ_j cannot both be zero, we see therefore that M is invertible if and only if conditions (i) and (ii) are satisfied. $\square$

Theorem 4.3 indicates that the class of TPBVDSs such that the generalized Lyapunov equations (4.14) and (4.16) have a unique solution is somewhat restricted, since either E or A must be invertible.

Thus, it may happen that a TPBVDS has a constant variance matrix P, but yet the generalized Lyapunov equation (4.14) may not specify P completely, i.e., it may have several solutions. In this case, in order to compute P, instead of using the implicit specification of P provided by the Lyapunov equation (4.14), one should use the explicit expression (4.4) for an arbitrary value of k.

5. Covariance Characterization

In the previous section, it was shown that the variance P of a stochastically stationary TPBVDS satisfies the Lyapunov equation (4.14). As long as the conditions of Theorem 4.3 are satisfied, this provides a simpler method for computing

P than the explicit evaluation of (4.4). To this point, however, the only characterization that we have of the covariance fuction $R(s) = R(l+s,l)$ for a stochastically stationary TPBVDS is (4.17), which we would need to evaluate for every individual value of $s = k-l$. Our goal in this section is to obtain a recursive characterization of $R(s)$ that can be used to compute the covariance in a considerably more efficient fashion. An interesting feature of the recursions that we shall obtain is is that unlike the causal nondescriptor case, where the covariance satisfies first-order causal recursions, for the TPBVDS case, the covariance satisfies *second-order* boundary value recursions. Note however that this result is not totally unexpected, since it was shown by Krener [17] that the covariance of a continuous-time stationary two-point boundary value process with standard dynamics satisfies a second-order differential equation.

The starting point of our derivation is the observation that

$$ER(k+1,l) = M[Ex(k+1)x^T(l)] = M[(Ax(k) + Bu(k))x^T(l)] . \qquad (5.1)$$

Using the Green's function solution (2.7), (2.11) to compute $M[u(k)x^T(l)]$, we find that (5.1) can be expressed as

$$ER(k+1,l) - AR(k,l) = -BB^T(V_f E^{k-l} A^{N-1-(k-l)})^T \quad \text{for } k \geq l . \qquad (5.2a)$$

Similarly, it can be shown that

$$R(k,l+1)E^T - R(k,l)A^T = V_i A^{k-l-1} E^{N-(k-l)} BB^T \quad \text{for } k > l . \qquad (5.2b)$$

Combining (5.2a) and (5.2b), we obtain therefore

$$[ER(k+1,l+1) - AR(k,l+1)]E^T - [ER(k+1,l) - AR(k,l)]A^T = 0 , \qquad (5.3)$$

for $k > l$, which holds independently of whether the TPBVDS (2.1)-(2.2) is stochastically stationary or not.

In the special case when the TPBVDS that we consider is stochastically stationary, by setting $k-l = s+1$ in (5.3), we obtain the following result.

Theorem 5.1: The covariance $R(s)$ of a stochastically stationary TPBVDS satisfies the second-order descriptor recursions

$$ER(s+1)E^T + AR(s+1)A^T = AR(s)E^T + ER(s+2)A^T , \qquad (5.4)$$

which are *conditionable*, in the sense that there exists boundary conditions

involving $R(0)$, $R(1)$, $R(N-1)$ and $R(N)$, which when combined with (5.4) define a well-posed second-order TPBVDS.

The recursions (5.4) are similar to the second-order differential equation obtained by Krener [17] for the covariance of a continuous-time stationary two-point boundary value process with standard dynamics. We still need to show the conditionability of (5.4). Recall that the concept of conditionability for TPBVDSs was introduced by Luenberger [5]-[6]. This will require the following lemma.

Lemma 5.1: The mth order descriptor system

$$Q_m x(k+m) + Q_{m-1}x(k+m-1) + \cdots + Q_0x(k) = Bu(k) , \ 0 \leq k \leq N-m \quad (5.5)$$

is conditionable if and only if the determinant of the polynomial matrix $Q(z) = Q_m z^m + Q_{m-1}z^{m-1} + \cdots + Q_0$ does not vanish identically.

Proof: Using state augmentation, we can rewrite (5.5) as

$$\tilde{E}\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u(k) , \quad (5.6)$$

with

$$\tilde{E} = \begin{bmatrix} I & & & & \\ & I & & & \\ & & \cdot & & \\ & & & I & \\ & & & & Q_m \end{bmatrix}, \tilde{A} = \begin{bmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \cdot & \cdot & \\ & & & 0 & I \\ -Q_0 & \cdots & & & -Q_{m-1} \end{bmatrix}, \text{ and } \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ B \end{bmatrix}.$$

Then, according to [6], p. 474, the descriptor system (5.6) is conditionable if and only if the pencil $z\tilde{E}-\tilde{A}$ is regular, i.e., iff $\det[z\tilde{E}-\tilde{A}] = \det Q(z)$ does not vanish identically. □

Now, using Lemma 5.1, the conditionability of (5.4) becomes equivalent to the invertibility of $-z^2(E \otimes A) + z(E \otimes E + A \otimes A) - A \otimes E)$ for some z. But, this matrix is equal to $(zE - A) \otimes (E - zA)$. Since E and A form a regular pencil, we can always find a z such that $(zE - A)$ and $(E - zA)$ are both invertible, which implies that their Kronecker product is invertible. This completes the proof of Theorem 5.1.

Theorem 5.1 indicates that there exists a set of boundary conditions involving $R(0)$, $R(1)$, $R(N-1)$ and $R(N)$, which when combined with (5.4) define a

well-posed TPBVDS. However, as one might expect, there is in fact a wide choice of boundary conditions which will work. In order to exhibit a specific set of boundary conditions, instead of examining (5.4) directly, it is convenient to start from (5.2a), where the TPBVDS is now assumed to be stochastically stationary. Setting $k-l = s$ inside (5.2a) gives

$$ER(s+1) - AR(s) = L(s) \quad , \quad 0 \le s \le N-1 \ , \tag{5.7}$$

with

$$L(s) = -BB^T(V_f E^s A^{N-1-s})^T \ . \tag{5.8}$$

Then, noting that

$$L(s+1)A^T - L(s)E^T = 0 \quad , \quad 0 \le s \le N-2 \ , \tag{5.9}$$

it is easy to check that the coupled system of first-order descriptor equations (5.7), (5.9) is equivalent to (5.4). A set of boundary conditions for this system will therefore be also applicable to (5.4).

Suppose for the moment that the function $L(s)$ appearing on the right-hand side of (5.7) has already been computed, with either the analytical expression (5.8), or through recursions (5.9). Then, a boundary condition which when combined with (5.7) defines a well-posed first-order TPBVDS is given by

$$V_i R(0) + V_f R(N) = Q(E^N)^T \ . \tag{5.10}$$

This boundary condition is obtained by multiplying (2.2) on the right by $x^T(0)$, taking expected values, and using the Green's function expression (2.7). Note that the TPBVDS (5.7), (5.10) has exactly the same dynamics and boundary matrices as (2.1)-(2.2) and is therefore guaranteed to be well-posed. This leaves us with the problem of computing $L(s)$ for $0 \le s \le N-1$ from the first-order recursions (5.9). However, we already know that the solution must be given by (5.8). This implies in particular that

$$ER(1) - AR(0) = L(0) = -BB^T(V_f A^{N-1})^T \tag{5.11a}$$

and

$$ER(N) - AR(N-1) = L(N-1) = -BB^T V_i(E^{N-1})^T \ . \tag{5.11b}$$

Note that, as expected, boundary conditions (5.11a) and (5.11b) involve only $R(0)$, $R(1)$, $R(N-1)$, and $R(N)$. Also, the TPBVDS defined by (5.9) and (5.11a), (5.11b) is clearly well-posed. In fact, it is *overdetermined* since the boundary conditions (5.11a) and (5.11b) are redundant. This redundancy can be eliminated by considering the smaller-size boundary condition

$$L(0)(V_i A)^T + L(N-1)(V_f E)^T = -BB^T V_f^T \qquad (5.12)$$

which is obtained by combining (5.11a) and (5.11b), and checking that the TPBVDS (5.9), (5.12) is well-posed. Note that to some extent, the problem of finding boundary conditions which guarantee that the first-order recursions (5.9) for $L(s)$ are well-posed is purely academic, since the closed-form solution (5.8) is already available. However, if we consider the second-order recursions (5.4), the above discussion shows that boundary conditions (5.10) and (5.12) will guarantee well-posedness and therefore provide us with an implicit set of equations whose solution yields the entire covariance function (including, of course, $P = R(0)$).

As was already mentioned, these boundary conditions are not the only ones which will guarantee well-posedness. For example, if we use (5.2b) as starting point, we obtain the coupled first-order descriptor equations

$$R(s)E^T - R(s+1)A^T = M(s) \ , \quad 0 \le s \le N-1 \qquad (5.13a)$$

$$EM(s+1) - AM(s) = 0 \ , \quad 0 \le s \le N-2 \qquad (5.13b)$$

where $M(s)$ is in fact given by the closed-form expression

$$M(s) = V_i A^s E^{N-1-s} BB^T \ . \qquad (5.14)$$

Then, these equations are well-posed for the boundary conditions

$$R(0)V_f^T + R(N)V_i^T = A^N Q \qquad (5.15a)$$

$$V_i EM(0) + V_f AM(N-1) = V_i BB^T \ , \qquad (5.15b)$$

where (5.15a) is obtained by multiplying (2.2) on the right by $x^T(N)$ and taking expected values, and where (5.15b) is a direct consequence of analytical expression (5.14). Substituting (5.13a) inside (5.15b), it is also easy to check that the boundary conditions (5.15) for (5.4) involve only $R(0)$, $R(1)$, $R(N-1)$, and $R(N)$, as desired.

There are in fact many valid choices of boundary conditions for the second-order system (5.4). For example, one obvious boundary condition is given by $R(0) = P$, where P can be found either by solving the algebraic Lyapunov equation (4.14) or by using analytic expression (4.4).

Example 5.1: Consider the anticyclic system

$$x(k+1) = x(k) + bu(k) \tag{5.16a}$$

$$(\tfrac{1}{2})(x(0) + x(N)) = v , \tag{5.16b}$$

where the variance of v is q. In this case, both sides of the generalized Lyapunov equation (4.16) are equal to 0, so that the TPBVDS (5.16) is stochastically stationary independently of the choice of q. The Lyapunov equation (4.14) for the state variance p also reduces to zero on both sides, and cannot therefore be used to compute p. However, by direct evaluation of (4.4), it is easy to verify that

$$p = r(0) = Nb^2/4 + q . \tag{5.17}$$

We now compute the covariance function $r(k)$ of (5.16) for $k \in [0,N]$. We use the second-order recursion (5.4), which here takes the form

$$r(k+2) = 2r(k+1) - r(k) . \tag{5.18}$$

Since $r(0)$ is already known, we only need $r(1)$ to be able to solve (5.18) in the forward direction. But according to (5.11a), we have

$$r(1) - r(0) = -b^2/2 ,$$

so that

$$r(1) = (N-2)b^2/4 + q ,$$

and then using (5.18), we find

$$r(k) = (N-2k)b^2/4 + q . \tag{5.19}$$

6. Characterization of Internal Stability

For causal systems, the relationship between the existence of a positive definite solution to the standard Lyapunov equation and stability is well known. Specifically, for a causal and reachable system, the Lyapunov equation has a

positive definite solution if and only if the system is strictly stable. In this section, for the class of displacement TPBVDSs, we study the relation existing between the existence and uniqueness of positive definite solutions to the generalized Lyapunov equation (4.14) for the state variance P, and the property of internal stability. Note that, whereas the generalized Lyapunov equation (4.16) for Q was the key to the characterization of stochastic stationarity derived in the previous section, equation (4.14) for P plays the main role in our study of internal stability. An important feature of this equation, which was not present in the causal case, is that it depends on the interval length N. It turns out that this dependence on interval length is in fact very useful in characterizing internal stability, since this last concept relies also on increasing the interval length to study the effect of the boundary conditions on states close to the center of the interval.

More precisely, to see why interval length plays an important role in studying the generalized Lyapunov equation (4.14), consider the anticyclic system (5.16) of Example 5.1. This system is clearly unstable, since its only mode is on the unit circle. Yet, for an arbitrary value of the variance q of the boundary condition, the system is stochastically stationary, and has a constant positive state variance $p = Nb^2/4 + q$. Thus, the existence of a positive definite solution to the generalized Lyapunov equation (4.14) for a fixed interval length is clearly not sufficient to guarantee that a TPBVDS is internally stable. However, in this particular case the variance p, viewed as a function of the interval length N, diverges as $N \to \infty$, which is an indication that the system is actually unstable.

Another useful observation is that for TPBVDSs, the generalized Lyapunov equation (4.14) for P may admit a nonnegative definite solution even when the system cannot be made stationary by any choice of boundary vector variance Q, i.e., there may be a nonnegative solution to (4.14) when there is no nonnegative solution to equation (4.16) for Q. This is illustrated by the following example.

Example 6.1: Consider the system

$$x(k+1) = (1/2)x(k) + u(k) \qquad (6.1a)$$

$$m(x(0) + 2x(N)) = v , \qquad (6.1b)$$

where $m = (1 + 2(1/2)^N)^{-1}$, and $u(k)$ is a white noise sequence with unit

variance. System (6.1) is in normalized form and internally stable. The generalized Lyapunov equation (4.16) for q takes the form

$$(3/4)q = -3m^2 , \qquad (6.2)$$

which yields a negative value of q, so that the system cannot be made stationary over any interval $[0,N]$. Yet, the Lyapunov equation (4.14) is given by

$$(3/4)p = m^2(1 - 4(1/4)^N) , \qquad (6.3)$$

and its solution p is positive provided that N is larger than 1. However, this solution is *not* the state variance of the TPBVDS (6.1), which in this case is not even constant. This can be seen by noting from (4.3)-(4.4) that the state variance $p(k)$ is given by

$$p(k) = \frac{q}{4^k} + \frac{4}{3}m^2 \left[1 - \frac{4}{4^N} + \frac{3}{4^k} \right], \qquad (6.4)$$

which is clearly not constant.

Example 6.1 shows that the generalized Lyapunov equation (4.14) may admit a unique positive definite solution P even when the TPBVDS (2.1)-(2.2) cannot be made stochastically stationary for any choice of boundary vector variance Q, but in general this matrix P bears no relation whatsoever with the state variance. However, it will be shown below in Theorem 6.3 that, for an internally stable displacement TPBVDS, independently of the choice of boundary matrix Q, as the interval length $N \rightarrow \infty$, the variance matrices $P(k)$ of states near the center of the interval approach a constant matrix P^* which *is* the solution to the generalized Lyapunov equation (4.12) with N set equal to ∞.

The main objective of this section is to characterize the property of internal stability in terms of positive definite solutions of (4.14), regardless of whether such solutions correspond to the variance of a stochastically stationary TPBVDS or not. Specifically, it will be shown that for a displacement TPBVDS with no eigenvalues on the unit circle, if for any N, the generalized Lyapunov equation (4.14) has a nonnegative definite solution P then the system (2.1)-(2.2) is internally stable. The assumption that there are no roots on the unit circle is introduced here to rule out a situation such as that of Example 5.1, where as was indicated

above, (4.14) has positive definite solutions which grows unbounded as $N \rightarrow \infty$.

Our results will require the following lemma.

Lemma 6.1: Let A and V be two square matrices which commute, i.e.,

$$AV = VA .\qquad (6.5)$$

Then, if V is singular, there exists a right (left) eigenvector of A in the right (left) null space of V.

Proof: We will prove this result for the case of a right eigenvector of A. Let $x \in Ker(V)$. Then,

$$Vx = 0$$

so that

$$VAx = AVx = 0 ,$$

and consequently $Ax \in Ker(V)$. Thus $Ker(V)$ is A invariant, which implies that A has at least one eigenvector in the null space of V. $\square$

We can now prove the following result.

Theorem 6.1: Assume that the TPBVDS (2.1)-(2.2) is a weakly reachable displacement system wih no eigenvalues on the unit circle. Then, if for some N, the generalized Lyapunov equation (4.14) has a nonnegative definite solution P, the TPBVDS is internally stable.

Proof: Since the TPBVDS that we consider has no eigenmodes on the unit circle, the decomposition of Theorem 3.1 takes the form

$$E = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix} , \quad A = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} , \quad B = \begin{bmatrix} B_f \\ B_b \end{bmatrix} , \qquad (6.6a)$$

where the eigenvalues of A_f and A_b are inside the unit circle, and

$$V_i = \begin{bmatrix} V_{i1} & 0 \\ 0 & V_{i2} \end{bmatrix} , \quad V_f = \begin{bmatrix} V_{f1} & 0 \\ 0 & V_{f2} \end{bmatrix} . \qquad (6.6b)$$

To prove stability, we need to show that V_{i1} and V_{f2} are invertible. Using the above decomposition, the generalized Lyapunov equation (4.14) can be expressed as

$$P_f - A_f P_f A_f^T = V_{i1} B_f B_f^T V_{i1}^T - (V_{f1} A_f^N) B_f B_f^T (V_{f1} A_f^N)^T \qquad (6.7a)$$

$$A_b P_b A_b^T - P_b = (V_{i2} A_b^N) B_b B_b^T (V_{i2} A_b^N)^T - V_{f2} B_b B_b^T V_{f2}^T \qquad (6.7b)$$

$$P_{fb} A_b^T - A_f P_{fb} = V_{i1} B_f B_b^T (V_{i2} A_b^N)^T - (V_{f1} A_f^N) B_f B_b^T V_{f2}^T , \qquad (6.7c)$$

where

$$P = \begin{bmatrix} P_f & P_{fb} \\ P_{fb}^T & P_b \end{bmatrix} . \qquad (6.8)$$

Clearly, if P is nonnegative definite, so is P_f. Since we also know that A_f is strictly stable, from (6.7a) we can conclude that if x^T is an arbitrary left eigenvector of A_f, then

$$x^T (V_{i1} B_f B_f^T V_{i1}^T - (V_{f1} A_f^N) B_f B_f^T (V_{f1} A_f^N)^T) x \geq 0 . \qquad (6.9)$$

We would like to show that V_{i1} is invertible. To do so, assume that V_{i1} is not invertible. Then, according to Lemma 6.1, there exists a left eigenvector x^T of A_f, i.e.,

$$x^T A_f = \lambda x^T , \qquad (6.10a)$$

such that

$$x^T V_{i1} = 0 . \qquad (6.10b)$$

We also know that the system is weakly reachable, and from the characterization of weak reachability presented in [2], we have

$$x^T [V_{i1} B_f \quad V_{f1} B_f] \neq 0 ,$$

so that

$$x^T V_{f1} B_f \neq 0 . \qquad (6.11)$$

Now, taking (6.10b) into account in (6.9), and observing that A_f and V_{f1} commute, we find that

$$0 = x^T V_{f1} A_f^N B_f = \lambda^N x^T V_{f1} B_f , \qquad (6.12)$$

where λ is the eigenvalue appearing in (6.10a). But (6.12) is compatible with (6.11) only if we have $\lambda = 0$, so that x^T must be in the left null space of both A_f and

V_{i1}. However, in this case the matrix

$$V_{i1} + V_{f1}A_f^N$$

characterizing the well-posedness of the forward stable subsystem is not invertible, which contradicts our assumptions. Thus V_{i1} must be invertible. Similarly, it can be proved that V_{f2} is invertible. $\square$

As in the causal case, the above result has also a converse, i.e., given an internally stable TPBVDS, there exists a positive definite solution to the Lyapunov equation (4.14). However, this result is only valid for large N, and it requires stronger conditions than those of Theorem 6.1. First, the conditions of Theorem 4.3 on the eigenmodes of the TPBVDS must be satisfied, so that (4.14) will be guaranteed to have a solution independently of the choice of of input matrix B and of boundary matrices V_i and V_f, in which case this solution will in fact be unique. The second condition is that the TPBVDS must be strongly reachable, instead of weakly reachable as in Theorem 6.1. This is due to the fact that we need to make sure that as $N \rightarrow \infty$, the solution of (4.14) is positive definite, instead of merely nonnegative definite.

Theorem 6.2: Consider a displacement TPBVDS which is internally stable, strongly reachable, and whose eigenmodes σ_j satisfy the conditions of Theorem 4.3 for the existence of a unique solution P_N to the generalized Lyapunov equation (4.14) Here the interval length N is allowed to vary, and the dependence of P on N is denoted by the subscript N of P_N. Then, there exists $N^* > 0$ such that P_N is positive definite for all $N \geq N^*$. Furthermore, as $N \rightarrow \infty$,

$$P_N \rightarrow P^* = \begin{bmatrix} P_f^* & 0 \\ 0 & P_b^* \end{bmatrix}, \tag{6.13}$$

where P_f^* and P_b^* are respectively the solutions of the usual algebraic Lyapunov equations for the forward and backward stable subsystems, i.e.,

$$P_f^* - A_f P_f^* A_f^T = B_f B_f^T, \tag{6.14a}$$

$$P_b^* - A_b P_b^* A_b^T = B_b B_b^T. \tag{6.14b}$$

Proof: First, observe that since the interval length N varies, the boundary matrices V_{i1}, V_{f1}, and V_{i2}, V_{f2} associated respectively to the forward and backward stable subsystems need to be rescaled in order to satisfy the normalized form identity (2.4) for all N. The rescaled boundary matrices are given by

$$V_{i1}(N) = (V_{i1} + V_{f1}A_f^N)^{-1}V_{i1} \quad , \quad V_{f1}(N) = (V_{i1} + V_{f1}A_f^N)^{-1}V_{f1} \quad (6.15a)$$

$$V_{i2}(N) = (V_{i2}A_b^N + V_{f2})^{-1}V_{i2} \quad , \quad V_{f2}(N) = (V_{i2}A_b^N + V_{f2})^{-1}V_{f2} \quad , \quad (6.15b)$$

and since the TPBVDS is internally stable, the matrices V_{i1} and V_{f2} are invertible, so that as $N \to \infty$,

$$V_{i1}(N) \to I \ , \ V_{f1}(N) \to V_{i1}^{-1}V_{f1} \ , \ V_{i2}(N) \to V_{f2}^{-1}V_{i2} \ , \ V_{f2}(N) \to I \quad (6.16)$$

Consider now the matrix P_N given by (6.8), whose entries satisfy the Lyapunov equations (6.7a-c), where the boundary matrices on the right hand side are replaced by the scaled matrices (6.15). We want to show that for N large enough, the solutions $P_{f,N}$ and $P_{b,N}$ of (6.7a) and (6.7b) are positive definite and tend to P_f^* and P_b^* given by (6.14), and that the solution $P_{fb,N}$ of (6.7c) goes to zero as N goes to infinity.

The first step is to observe that, as $N \to \infty$, since the scaled boundary matrices tend to finite limits given by (6.16), the right-hand side of (6.7c) tends to zero. But the eigenmodes of the system are such that the solution P_N is unique, and therefore the solution $P_{fb,N}$ of equation (6.7c) is unique and tends to zero as N goes to infinity.

Next, consider Lyapunov equation (6.7a), and observe that since the TPBVDS is strongly reachable, the matrix pair (A_f, B_f) is reachable in the usual sense for causal systems. But since the system is internally stable, $V_{i1}(N)$ given by (6.15a) is invertible, and noting that it commutes with A_f, we can conclude that the pair $(A_f, V_{i1}(N)B_f)$ is also reachable in the usual sense. Then, the solution $P_{f,N}$ of (6.7a) can be expressed as

$$P_{f,N} = P_{f,N}^+ - P_{f,N}^- \quad (6.17)$$

where $P_{f,N}^+$ and $P_{f,N}^-$ are respectively the solutions of

$$P_{f,N}^+ - A_f P_{f,N}^+ A_f^T = V_{i1}(N)B_f B_f^T V_{i1}(N)^T \quad (6.18a)$$

$$P_{f,N}^- - A_f P_{f,N}^- A_f^T = (V_{f1}(N)A_f^N)B_f B_f^T(V_{f1}(N)A_f^N)^T .$$ (6.18b)

Since $(A_f,\ V_{i1}(N)B_f)$ is reachable, $P_{f,N}^+$ is positive definite for all N, and since $V_{i1}(N)\to I$ as $N\to\infty$, $P_{f,N}^+\to P_f^*$, where P_f^* is the unique positive definite solution of (6.14a). Furthermore, as $N\to\infty$, the right-hand side of (6.18b) tends to zero, so that $P_{f,N}^-$ tends to zero. From (6.17), we can therefore conclude that there exists an integer N^* such that $P_{f,N}$ is positive definite for all $N\geq N^*$. Similarly, it can be shown that the solution $P_{b,N}$ of (6.7b) is positive definite for large enough N and tends to P_b^*, which is the unique positive definite solution of (6.14b).

We have therefore shown that as $N\to\infty$, $P_{f,N}$ and $P_{b,N}$ approach positive definite matrices P_f^* and P_b^*, and that $P_{fb,N}$ tends to zero. Consequently, the matrix P_N is positive definite for sufficiently large N and has for limit P^* given by (6.13). $\square$

Example 6.2 Consider system (6.1), which is both internally stable and stongly reachable. Then, the solution of the generalized Lyapunov equation (6.3) is

$$p_N = \frac{4}{3}m^2(1 - \frac{4}{4^N}) ,$$

which is positive definite for $N\geq 2$. Furthermore, as $N\to\infty$,

$$p_N \to p^* = 4m^2/3 ,$$ (6.19)

where p^* is the solution of the generalized Lyapunov equation (6.3) with $N = \infty$.

It is worth noting that when $N = \infty$, if the TPBVDS is internally stable, in the coordinate system corresponding to decomposition (6.6), the generalized Lyapunov equation (4.14) takes the form

$$EPE^T - APA^T = W ,$$ (6.20)

with

$$W = \begin{bmatrix} B_f B_f^T & 0 \\ 0 & -B_b B_b^T \end{bmatrix} .$$ (6.21)

Then, independently of whether eigenmodes σ_j satisfy the conditions of Theorem 4.3, *one* solution of (6.20) is P^* given by (6.13)-(6.14), which is nonnegative

definite regardless of the reachability properties of the TPBVDS (2.1)-(2.2). In other words, for $N = \infty$, the conditions of Theorem 6.2 can be weakened, thus giving the following result.

Corollary 6.1 Let displacement TPBVDS (2.1)-(2.2) be internally stable. Then the generalized Lyapunov equation (4.14) with $N = \infty$ has a nonnegative definite solution P^*. This solution is positive definite if the system is strongly reachable.

For an internally stable TPBVDS, the solution P^* of the generalized Lyapunov equation (4.14) with $N = \infty$ has also the following stochastic interpretation.

Theorem 6.3 Let displacement system (2.1)-(2.2) be internally stable. Then, for *any* choice of boundary variance Q, as N goes to infinity, the variance matrix of states located close to the center of interval $[0,N]$ converges to the solution P^* of the generalized Lyapunov equation with $N = \infty$.

Proof: Let $P_N(k)$ be the variance matrix of the state $x(k)$ of system (2.1)-(2.2) defined over interval $[0,N]$. Then, if l is an arbitrary but fixed integer, we want to show that

$$\lim_{N \to \infty} P_N((N/2)+l) = P^* , \tag{6.22}$$

where for simplicity it has been assumed that N is even. Our starting point is expression (4.4) for the state variance, i.e.,

$$P_N((N/2)+l) = A^{(N/2)+l} E^{(N/2)-l} Q (A^{(N/2)+l} E^{(N/2)-l})^T$$
$$+ (V_i(N)E^{(N/2)-l})\Pi((N/2)+l-1)(V_i(N)E^{(N/2)-l})^T$$
$$+ (V_f(N)A^{(N/2)+l})\Pi((N/2)-l-1)(V_f(N)A^{(N/2)+l})^T ,$$

where $\Pi(k)$ is given by (4.3), and boundary matrices $V_i(N)$ and $V_f(N)$ are obtained by rescaling V_i and V_f so that the normalized form identity (2.4) is satisfied for all N. Then, in the coordinate system corresponding to decomposition (6.6) of the TPBVDS in its forward and backward stable components, by using expressions (6.16) for the limit of $V_i(N)$ and $V_f(N)$ as $N \to \infty$, and taking into account the fact that A_f and A_b are stable matrices, we find that

$$\lim_{N \to \infty} P_N((N/2)+l) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Pi(\infty) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \Pi(\infty) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \quad (6.23)$$

But since

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

commute with both E and A, (6.23) can be rewritten as

$$\lim_{N \to \infty} P_N((N/2)+l) = \lim_{k \to \infty} \sum_{j=0}^{k} A^{k-j} E^j \begin{bmatrix} B_f B_f^T & 0 \\ 0 & B_b B_b^T \end{bmatrix} (A^{k-j} E^j)^T. \quad (6.24)$$

Thus,

$$\lim_{N \to \infty} P_N((N/2)+l) = \begin{bmatrix} \sum_{j=0}^{\infty} A_f^j B_f B_f^T (A_f^j)^T & 0 \\ 0 & \sum_{j=0}^{\infty} A_b^j B_b B_b^T (A_b^j)^T \end{bmatrix}$$

$$= \begin{bmatrix} P_f^* & 0 \\ 0 & P_b^* \end{bmatrix} = P^*. \quad (6.25)$$

This completes the proof of Theorem 6.3. □

Example 6.3 Consider the TPBVDS (6.1) of Example 6.1. According to (6.19), for this example the solution of the generalized Lyapunov equation (4.14) with $N = \infty$ is $p^* = 4m^2/3$. Then, setting $k = (N/2)+l$ in expression (6.4) for the state variance, we obtain

$$\lim_{N \to \infty} p_N((N/2)+l) = 4m^2/3 = p^*,$$

as expected.

Theorem 6.3 shows that, regardless of the boundary variance Q, the state variance of an internally stable displacement TPBVDS converges to the constant matrix P^* given by (6.13)-(6.14). However an even more interesting observation is that under the above assumptions, the TPBVDS will converge to a *stochastically*

stationary system as $N \to \infty$. More precisely, if we denote by

$$R_N((N/2)+k,(N/2)+l) = M[x((N/2)+k)x^T((N/2)+l)] \qquad (6.26)$$

the correlation matrix of states $x((N/2)+k)$ and $x((N/2)+l)$, where k and l are fixed integers, by using the analytic expression (4.17) for the correlation matrix, and following steps similar to those used in the proof of Theorem 6.3, it can be shown that in the coordinate system corresponding to the forward and backward stable decomposition (6.6), we have

$$\lim_{N \to \infty} R_N((N/2)+k,(N/2)+l) = R^*(k-l)$$

$$= \begin{bmatrix} A_f^{k-l}P_f^* & -\sum_{j=0}^{k-l-1} A_f^{k-l-j-1}B_f\,B_b^T(A_b^j)^T \\ 0 & P_b^*(A_b^{k-l})^T \end{bmatrix}, \qquad (6.27)$$

where for convenience it has been assumed that $k \geq l$. Since the limit obtained in (6.27) depends only on $k-l$, we can therefore conclude that independently of the choice of boundary variance Q, an internally stable TPBVDS converges to a stochastically stationary system as $N \to \infty$. This stochastically stationary system is separable into forward and backward causal components, which are however correlated through the input noise $u(k)$. This last fact can be seen from (6.27), where if we denote by $x^*(k)$ the limiting process obtained by letting $N \to \infty$, and by shifting the left boundary of the interval of definition to $-\infty$, the cross-correlation $R_{fb}^*(k-l)$ between the forward component $x_f^*(k)$ and the backward component $x_b^*(l)$ is nonzero for $k \geq l$, since both of these processes depend on the noise over interval $[l,k]$, whereas the cross-correlation between $x_b^*(k)$ and $x_f^*(l)$ is zero, since they depend on the noise over disjoint intervals.

7. Conclusions

In this paper, in spite of the fact that two-point boundary-value descriptor systems are defined only over a finite interval, we have been able to introduce a concept of internal stability for these systems. According to the definition that was selected, a TPBVDS is internally stable if the effect of boundary conditions on states close to the center of the interval goes to zero as the interval length goes

to infinity. Stochastic TPBVDSs have also been examined, and the property of stochastic stationarity was characterized in terms of a generalized Lyapunov equation for the variance of of the boundary vector. It was also shown that the state variance satisfies another generalized Lyapunov equation which can be used to characterize the property of internal stability. Specifically, it was shown that for a weakly reachable TPBVDS over a finite interval, with no eigenvalues on the unit circle, if the generalized Lyapunov equation for the state variance admits a nonnegative solution, then the TPBVDS is internally stable. Conversely, it was shown that for an internally stable TPBVDS, the generalized Lyapunov equation for the state variance admits a positive definite solution when the interval length N is sufficiently large. It was also proved that, independently of the boundary matrix variance, an internally stable stochastic TPBVDS converges to a stochastically stationary process as the interval length $N \rightarrow \infty$.

As was already mentioned in the introduction, this paper is part of a larger effort devoted to the study of the system properties, and the development of estimation algorithms for TPBVDSs. In particular, the smoothing problem for TPBVDSs was examined in [4], where it was shown that the smoother itself is a TPBVDS which can then be decoupled into forward and backward stable components through the introduction of generalized Riccati equations that were studied in [25]. An interesting question which arises in this context is whether for a strongly reachable and observable TPBVDS, the smoother is internally stable in the sense discussed in this paper. It turns out that this is the case, and the proof of this fact will appear in [29]. In other words, the concept of internal stability developed here for a class of noncausal systems appears to be the natural generalization of the corresponding notion for standard causal state-space models and leads to just as rich a set of system-theoretic results.

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