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On the Stability of Bilinear Stochastic Systems[1]

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Abstract: We study the stability with probability one of the stochastic bilinear system dX = AX ds + BX dw, where A and B are fixed matrices and w is a Brownian motion. Bounds for the Lyapunov numbers associated with this equation are given.

Bilinear noise models are, after linear ones, the second simplest case of stochastic systems; they may arise in many problems in which linear noise models are inappropriate (many examples are given in [6]).

The aim of this paper is to give a condition for the stability with probability one of the d-dimensional Ito equation which describes the behavior of such a system

$$dY_s = AY_s\,ds + BY_s dw_s \qquad (1)$$
$$Y_0 = y$$

where A and B are two given $d\times d$ matrices and w is a scalar Brownian motion (see also the more general equation (12) below). I.e., we want to find whether or not Y_s tends a.e. to zero as s tends to infinity. Note that in the one dimensional case, we have an explicit solution

$$Y_s = Y_0 \exp\{t(A-B^2/2) + Bw_t\},$$

and the stability with probability one is guaranteed iff $2A-B^2 < 0$. Note also that

$$E[\ (Y_s)^2\ ] = (Y_0)^2 \exp 2t(A+B^2/2)$$

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and the L^2 stability is guaranteed iff $2A+B^2 < 0$. This is a much stronger condition whose generalisation in d dimensions is easy, thanks to the following equations:

$$dY_s \otimes Y_s = (I \otimes A + A \otimes I + B \otimes B) Y_s \otimes Y_s \, ds + (I \otimes B + B \otimes I) Y_s \otimes Y_s dw_s$$

$$dE[Y_s \otimes Y_s] = (I \otimes A + A \otimes I + B \otimes B) \, E[Y_s \otimes Y_s] \, ds$$

(for the definition and the basic properties of $\otimes$, see [1]).The L^2 stability is then governed by $\lambda_{max}(I \otimes A + A \otimes I + B \otimes B)$ (in this paper, $\lambda_{max}(M)$ denotes the largest real part of eigenvalues of the matrix M, and $\lambda_{min}(M) = -\lambda_{max}(-M)$).

We will give an upper bound for the largest Lyapunov number λ_1 of (1) (the smallest γ satisfying the bound of theorem1). Note that the only existing expressions for the largest Lyapunov number of (1) or (12) with general matrices are asymptotic expansions ($B=\varepsilon B_0$, ε tends to 0, A fixed) in dimension 2 ([5]). The following criterion will be proved :

Theorem1. *Setting*

$$\gamma = \frac{1}{2} \lambda_{max}(I \otimes (A-B^2) + (A-B^2) \otimes I + B \otimes B),$$

then, for any value of Y_0,

$$\limsup_{t \to \infty} \frac{1}{t} \log(\|Y_t\|) \leq \gamma \quad \text{a.e.}$$

Considering the matrix equation

$$dP_s = AP_s \, ds + BP_s dw_s$$
(2)
$$P_0 = I,$$

Y_s can be expressed as

$$Y_s = P_s Y_0$$

and the following basic theorem about Lyapunov numbers (the λ_i's below) is true:

Theorem2([2]): *There exist* d *real numbers* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ *such that*

$$\lim_{t\to\infty} \{P_t^T(\omega)P_t(\omega)\}^{1/2t} = O^T(\omega)\Lambda O(\omega) \text{ a.e.}$$

where $O(\omega)$ *is an orthogonal matrix and* Λ *is the diagonal matrix satisfying*

$$\Lambda_{i,i} = \exp(\lambda_i).$$

Furthermore, if

$$E_i = \text{span}\{O^T(\omega)e_i, O^T(\omega)e_{i+1}, \ldots O^T(\omega)e_d\}$$

($(e_i)_{i=1,d}$ *is the canonical basis of* $\mathbb{R}^d$) *then, a.e., we have*

$$\forall u \in E_i(\omega)\backslash E_{i+1}(\omega), \qquad \lim_{t\to\infty} \frac{1}{t} \text{Log}\|P_t(\omega)u\| = \lambda_i.$$

All the λ_i *are equal if and only if there exist a matrix* M *such that the matrices* $M(A - \frac{1}{d}\text{trace}(A))M^{-1}$ and $M(B - \frac{1}{d}\text{trace}(B))M^{-1}$ *are skew-symmetric.*

Note that the Lyapunov numbers of $Q_s=P_s^{-T}$ are $(-\lambda_d,\ldots,-\lambda_1)$.
To prove theorem1, instead of trying to find upper bounds on the Lyapunov numbers of P_s, we will get lower bounds for the ones of Q_s. Theorem2 shows that the smallest Lyapunov number is generally attained on a random 1-dimensional space; this fact is at the origin of all the difficulty of step1 below. The motivation of this method will appear in step2, when the (non-quadratic) term $(u^TFu)^2$ in (3) will be bounded from below by u^TF^TFu (dealing with P_s, we would have to find an upper bound for $(u^TBu)^2$).

Proof of theorem1
c denotes a constant depending only on d and is allowed to change, finitely often, during the calculation.

Note that $Q_s=P_s^{-T}$ satisfies the equation

$$dQ_s = EQ_s\, ds + FQ_s dw_s$$

where

$$E = -A^T + B^{2T}$$
$$F = -B^T.$$

By virtue of theorem2 (and the following remark), the largest Lyapunov number of P_s is a.e.

$$-\lim_{t\to\infty} \inf_x \frac{1}{t}\log(\|Q_t x\|).$$

For any X_0, $X_t = Q_t X_0$ satisfies the equation

$$dX_s = EX_s\, ds + FX_s dw_s$$

Using Ito's formula, and setting $u_s = X_s/\|X_s\|$, we obtain

$$dX_s^T X_s = X_s^T X_s(\ 2u_s^T E u_s\, ds + 2u_s^T F u_s\, dw_s + u_s^T F^T F u_s\, ds\ )$$

$$d\log(\|X_s\|) = \{\ u_s^T E u_s + \frac{1}{2} u_s^T F^T F u_s - (u_s^T F u_s)^2\ \}ds + u_s^T F u_s\, dw_s.$$

This equation, due to Khas'minski ([3]), is basic for the study of Lyapunov numbers, since it may be written

$$t^{-1}\ \{\log(\|X_t\|) - \log(\|X_0\|)\} = t^{-1} \int_0^t \{u_s^T E u_s + \frac{1}{2} u_s^T F^T F u_s - (u_s^T F u_s)^2\} ds) + t^{-1} \int_0^t u_s^T F u_s dw \qquad (3)$$

<u>step1</u>: $E[\sup_{X_0} | \int_0^t u_s^T F u_s dw_s |] = o(t)$, $\quad$ t integer >0.

The sup may be taken a over a dense subset of $\mathbb{R}^d$ (because the stochastic integral is a.e. a continuous function of X_0) so that the expectation is well defined.

This is the most difficult step of the proof. All our efforts will be focused on getting rid of the sup. We will suppose that the λ_i are not all equal (if they are, using theorem2, the expression above is w_t times a constant and the result is obvious).

The following lemma is needed

Lemma *Let* σ *be a uniform measure on the the unit sphere* S_{d-1} *of* $\mathbb{R}^d$, x *any vector of* $\mathbb{R}^d$, $f \in L^1(\sigma)$, P *an invertible matrix, and* $Q=P^{-T}$,*the following equalities are true:*

$$\frac{x}{\|x\|} = c_1 \int \text{sign}(\langle x,y\rangle)\, y\, \sigma(dy) \qquad c_1 = (\sigma(|y_1|))^{-1}$$

(4)

$$\int f(y)\sigma(dy) = \det(P) \int f\left(\frac{Py}{\|Py\|}\right) \|Py\|^{-d}\, \sigma(dy)$$

(5)

$$\frac{x^T Q^T F Q x}{\|Qx\|^2} = c_1^2 \int \text{sign}(\langle x,y\rangle)\text{sign}(\langle x,z\rangle)(z^T P^T F P y)\, \|Py\|^{-d-1}\|Pz\|^{-d-1} \det(P)^2\, \sigma(dy)\sigma(dz)$$

(6)

Proof of the lemma:

Observe that identity (4) has only to be proved for $\|x\|=1$, and that, because of the rotational invariance of σ, we can suppose $x=(1,0,0....0))$, and we get c_1.

The left-hand side of (5) is equal to

$$c\int f\left(\frac{y}{\|y\|}\right) e^{-\|y\|}\, dy = c\int f\left(\frac{Px}{\|Px\|}\right) e^{-\|Px\|}\det(P)\, dx$$

$$= c\det(P) \int f\left(\frac{Py}{\|Py\|}\right) e^{-r\|Py\|}\, r^{d-1}\, \sigma(dy)\, dr$$

$$= c\det(P) \int f\left(\frac{Py}{\|Py\|}\right) \|Py\|^{-d}\, \sigma(dy)$$

the constant c, independent of P, is identified by taking P=I.

In order to prove (6), use (4) and (5) to obtain

$$\frac{Qx}{\|Qx\|} = c\int \text{sign}(\langle x,Q^T y\rangle)\, y\, \sigma(dy)$$

$$= \det(P) \int \text{sign}(\langle x,y\rangle) Py\, \|Py\|^{-d-1}\, \sigma(dy)$$

which implies (6).

This lemma gives us

$$u_s^T F u_s = \frac{X_0^T Q_s^T F Q_s X_0}{\|Q_s X_0\|^2} = \int \Phi(X_0,y,z)\Psi(P_s,y,z)\sigma(dy)\sigma(dz)$$

where

$$\Phi(x,y,z) = \mathrm{sign}(\langle x,y\rangle)\ \mathrm{sign}(\langle x,z\rangle)$$

$$\Psi(P,y,z) = c\ (z^T P^T F P y)\ \|Py\|^{-d-1}\|Pz\|^{-d-1}\ \det(P)^2.$$

And

$$\int_0^t u_s F u_s^T dw_s = \int_0^t \int \Phi(X_0,y,z)\Psi(P_s,y,z)\sigma(dy)\sigma(dz)\ dw_s$$

$$= \int \Phi(X_0,y,z) \int_0^t \Psi(P_s,y,z) dw_s\ \sigma(dy)\sigma(dz)$$

$$\sup_{X_0} \left| \int_0^t u_s F u_s^T dw_s \right| \le \int \left| \int_0^t \Psi(P_s,y,z) dw_s \right| \sigma(dy)\sigma(dz)$$

$$E[\sup_{X_0} \left| \int_0^t u_s F u_s^T dw_s \right|] \le \int E[\{ \int_0^t \Psi^2(P_s,y,z)ds\ \}^{1/2}\ ]\ \sigma(dy)\sigma(dz)$$

$$E[\sup_{X_0} \left| \int_0^t u_s F u_s^T dw_s \right|] \le E[\int \{ \int_0^t \Psi^2(P_s,y,z)ds\ \}^{1/2}\ \sigma(dy)\sigma(dz))\ ] \quad (7)$$

we will prove that for almost all (ω,y,z),

$$\int_0^\infty \Psi^2(P_s,y,z)ds < \infty \quad (8)$$

and that the integrals

$$\frac{1}{t}\left(\int_0^t \Psi^2(P_s,y,z)ds\right)^{1/2} \tag{9}$$

are uniformly (on t) integrable (over (ω,y,z)). And then (7), (8), and (9) will imply the bound of step1.

To prove (8), note that

$$|\Psi(P_s,y,z)| = c\,(z^TP_s^TFP_sy)\ \|P_sy\|^{-d-1}\|P_sz\|^{-d-1}\ \det(P_s)^2$$
$$\leq c\ \|F\|\ \|P_sy\|^{-d}\|P_sz\|^{-d}\ \det(P_s)^2$$
$$\leq c\ \|F\|\ \exp\{(-2\lambda_1 d+2\sum\lambda_i+\varepsilon(s))s\}$$

where $\varepsilon(s)\to 0$ as $s\to\infty$ and the λ_i are the Lyapounov numbers associated with P. Since we are in a case where the λ_i are not all equal, we get the exponential decrease of the expression above, and (8) is true.

To prove the uniform integrability of (9), we will show that

$$E\left[\int a\left(\frac{1}{t}\left(\int_0^t \Psi^2(P_s,y,z)ds\right)^{1/2}\right)\sigma(dy)\sigma(dz)\right] < K$$

(10)

where K is a constant independent of t, and

$$a(x) = |x|\ \log^{1/2}(1+x^2).$$

Denoting

$$\psi(t,y,z) = \int_0^t |\Psi(P_s,y,z)|^2 ds,$$

equation (10) reduces to

$$E\left[\int \frac{1}{t}\psi(t,y,z)^{1/2}\ \log^{1/2}(1+\psi(t,y,z)/t^2)\ \sigma(dy)\sigma(dz)\right] < K.$$

Step1 will be finished by proving

$$E\left[\int \psi(t,y,z)^{1/2}\ \log^{1/2}(1+\psi(t,y,z))\ \sigma(dy)\sigma(dz)\right] < Kt.$$

(11)

An elementary calculation shows that the function $x^{1/2}\log^{1/2}(1+x)$ is concave; this implies

$$(x+y)^{1/2}\log^{1/2}(1+x+y) \le x^{1/2}\log^{1/2}(1+x) + y^{1/2}\log^{1/2}(1+y) \quad \text{for any } x,y>0$$

Defining ψ_1 by

$$\psi_1(t,y,z) = \psi(t+1,y,z) - \psi(1,y,z) = \int_1^{t+1} |\Psi(P_s,y,z)|^2 ds$$

we obtain

$$E[\int \psi(t+1,y,z)^{1/2} \log^{1/2}(1+\psi(t+1,y,z))\, \sigma(dy)\sigma(dz)\,]$$
$$\le E[\int \psi(1,y,z)^{1/2} \log^{1/2}(1+\psi(1,y,z))\, \sigma(dy)\sigma(dz)\,] +$$
$$E[\int \psi_1(t,y,z)^{1/2} \log^{1/2}(1+\psi_1(t,y,z))\, \sigma(dy)\sigma(dz)\,].$$

We will now get the index 1 out of the last formula. The basic tool is the following identity resulting from Markov property

$$P_{s+1} = (\theta_1 P_s)\, P_1$$

where θ_1 is the shift operator on the trajectories of Brownian motion. Using this identity in

$$\psi_1(t,y,z) = \int_0^t |\Psi(P_{s+1},y,z)|^2 ds$$

and

$$\Psi(P,y,z) = c\,(z^T P^T F P y)\, \|Py\|^{-d-1}\|Pz\|^{-d-1}\, \det(P)^2,$$

we get

$$\psi_1(t,y,z) = \int_0^t |\Psi(\theta_1 P_s, P_1 y, P_1 z)|^2 ds \quad \det(P_1)^2.$$

The second term of the right-hand side may be rewritten as

$$E[\int \psi(t,P_1 y,P_1 z)^{1/2} \log^{1/2}(1+\psi(t,P_1 y,P_1 z)\det(P_1)^4)$$
$$\det(P_1)^2 \sigma(dy)\sigma(dz)]$$

where P_1 and $\psi(t,y,z)$ are independent(i.e. constructed from two independent Brownian motions and the expectation is taken over the product space). Using lemma1 we obtain, after some reductions ($Q_1 = P_1^{-1}$)

$$E[\int \psi(t,y,z)^{1/2} \log^{1/2}(1+\psi(t,y,z)\det(P_1)^4 \|Q_1 y\|^{-2d} \|Q_1 z\|^{-2d})\ \sigma(dy)\sigma(dz)]$$

and with the inequalities ($a,b>0$)

$$\log(1+ab) \leq \log(1+a) + \log(1+b)$$
$$(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$$

we get the upper bound

$$E[\int a(\psi(t,y,z)^{1/2})\ \sigma(dy)\sigma(dz)\ ] + E[\int \log^{1/2}(1+\det(P_1)^4 \|Q_1 y\|^{-2d} \|Q_1 z\|^{-2d})\ \sigma(dy)\sigma(dz)]$$

The last term is finite since $\log(\|Q_s y\|)$ satisfies equation (3) and $\det(P_s)$ satisfies

$$d\log(\det(P_s)) = \{\mathrm{trace}(A) - 1/2\ \mathrm{trace}(B^2)\}\ ds + \mathrm{trace}(B)\ dw_s\ ).$$

Finally, we have

$$E[\int a(\psi(t+1,y,z)^{1/2})\ \sigma(dy)\sigma(dz)\ ] \leq E[\int a(\psi(t,y,z)^{1/2})\ \sigma(dy)\sigma(dz)\ ] + K$$

where K is a constant. This ends the proof of (10), and the first step.

<u>step</u>2: end of the proof.

Using step1, one can find a sequence t_n such that

$$\lim_{n\to\infty} \frac{1}{t_n} \sup_{X_0} | \int_0^{t_n} u_s F u_s^T dw_s | \ ] = 0 \quad \text{a.e.}$$

and obtain, with equation (3):

$$\lim_{n\to\infty} \inf_{X_0} \frac{1}{t_n}\log(\|X_{t_n}\|) \geq \inf_{u\in S_{d-1}} u^TEu+\frac{1}{2}u^TF^TFu-(u^TFu)^2.$$

But

$$(Fu - u(u^TFu))^T(Fu - u(u^TFu)) \geq 0$$

implies

$$u^TF^TFu \geq (u^TFu)^2,$$

which gives the bound

$$\lim_{t\to\infty} \inf_{X_0} \frac{1}{t}\log(\|X_t\|) \geq \inf_{u\in S_{d-1}} u^TEu - \frac{1}{2}u^TF^TFu$$

$$\geq \lambda + \inf_{u\in S_{d-1}} u^T(E-\lambda I)u - \frac{1}{2}u^TF^TFu$$

for any real λ.

Actually, we have still the choice of the basis of $\mathbb{R}^d$ we are working in. A change of basis with a matrix R changes E into RER^{-1} and F into RFR^{-1} and then for any invertible matrix R, we have

$$\lim_{t\to\infty} \inf_{X_0} \frac{1}{t}\log(\|X_t\|) \geq \lambda + \inf_{u\in S_{d-1}} u^TR(E-\lambda I)R^{-1}u - \frac{1}{2}u^TR^{-T}F^TR^TRFR^{-1}u$$

$$\geq \lambda - \sup_{Rv\in S_{d-1}} v^TS(-E+\lambda I)v + \frac{1}{2}v^TF^TSFv$$

where $S=R^TR$ ($u=Rv$). One can prove ([4]) that, if

$$\lambda_{max}(I \otimes(-E+\lambda I) + (-E+\lambda I)\otimes I + F\otimes F) < 0,$$

then the equation

$$S(-E+\lambda I) + (-E^T+\lambda I)S + F^TSF = -I$$

has a unique solution which is positive definite. Then, for any λ smaller than $\lambda_{min}(I \otimes E + E\otimes I - F\otimes F)/2$, we have

$$\lim_{t\to\infty} \inf_{X_0} \frac{1}{t}\log(\|X_t\|) \geq \lambda$$

The smallest Lyapunov number of Q_s is then larger than $\lambda_{max}(I \otimes E+E\otimes I-F\otimes F)/2$. But

$$I \otimes E + E\otimes I - F\otimes F = -\{ I \otimes(A-B^2) +(A-B^2)\otimes I + B\otimes B \}^T$$

This ends the proof of the theorem.

We have just proved the following result

Theorem3: *The smallest Lyapunov number of equation* (1) *is larger than*

$$\delta = \frac{1}{2}\lambda_{min}(I \otimes A + A\otimes I - B\otimes B).$$

If equation (1) is replaced by

$$dY_s = AY_s\, ds + \sum_{i=1}^{n} B_i Y_s dw_s^i \qquad (12)$$

where $B_1,\ldots B_n$, are n matrices and $w^1,\ldots,w^n$ are n independent Brownian motions, the proof can still be carried out in the same way and we obtain

Theorem1(general form): *If equation* (12) *is subsituted to equation* (1), *theorem1 and 3 are still valid with*

$$\gamma = \frac{1}{2}\lambda_{max}(I\otimes A+A\otimes I -\sum_{i=1}^{n} I \otimes B_i^2+B_i^2\otimes I-B_i\otimes B_i)$$

$$\delta = \frac{1}{2}\lambda_{min}(I\otimes A+A\otimes I -\sum_{i=1}^{n} B_i\otimes B_i)$$

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