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by

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LATTICE APPROXIMATION IN THE STOCHASTIC QUANTIZATION OF $(\phi^4)_2$ FIELDS[1]

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I. INTRODUCTION

The Parisi-Wu program of stochastic quantization [8] involves construction of a stochastic process which has a prescribed Euclidean quantum field measure as its invariant measure. This program was rigorously carried out for a finite volume $(\phi^4)_2$ measure by G. Jona-Lasinio and P. K. Mitter in [6]. These results were extended in [2], which also proves a finite to infinite volume limit theorem. The aim of this note is to prove a related limit theorem, viz., that of the finite dimensional processes obtained by stochastic quantization of the lattice $(\phi^4)_2$ fields to their continuum limit, i.e., the $(\phi^4)_2$ process of [2], [6]. The proof imitates that of the limit theorem of [2] in broad terms, though the technical details differ. Note that this limit theorem can also be construed as an alternative construction of the $(\phi^4)_2$ process in finite volume.

The next section recalls the finite volume $(\phi^4)_2$ process. Section III summarizes the relevant facts about the lattice approximation to the $(\phi^4)_2$ field from Sections 9.5 and 9.6 of [4]. Section IV proves the limit theorem.

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II. THE $(\phi^4)_2$ PROCESS

Let $\Lambda \subset R^2$ be a finite rectangle which, for simplicity, we take to be the unit cube $x = (x_1, x_2) \mid 0 \le x_1, x_2 \le 1$. Let Δ denote the Dirichlet Laplace operator on Λ. It is diagonalized by the basis $e_k(x) = 2 \sin(k_1 x_1) \sin(k_2 x_2)$, $x = (x_1, x_2)$, $k \varepsilon B = \{(k_1, k_2) \mid k_i = n\pi,\ n \ge 1,\ i = 1, 2\}$. In fact, $-\Delta e_k = k^2 e_k$ where $k^2 = k_1^2 + k_2^2$. For $\alpha \varepsilon R$, let H^α denote the Hilbert space obtained by completing $D(\Lambda)$ with respect to the inner product

$$< f, g >_\alpha = \sum_{k \varepsilon B} (k^2)^\alpha <f, e_k> <g, e_k>$$

where $<\cdot,\cdot>$ is the L_2 scalar product. Topologize $Q = \bigcup H^\alpha$ by the countable family of seminorms $\|\cdot\|_n = <\cdot,\cdot>^{1/2}$ and $Q' = \bigcup_\alpha H^{-\alpha}$ via duality.

Let $C = (-\Delta + 1)^{-1}$, $C(\cdot,\cdot)$ its integral kernel, C^α its α-th operator power, and μ_C the centered Gaussian measure on H^{-1} with covariance C [2], [6]. Let $: :$ denote the Wick ordering with respect to C (see [4], Ch. 8, for a definition). The $(\phi^4)_2$ measure on H^{-1} is defined by

$$\frac{d\mu}{d\mu_C} = \exp\left(-\frac{1}{4} \int :\phi^4: dx\right) / Z \qquad [2.1]$$

where

$$Z = \int \exp\left(-\frac{1}{4} \int_\Lambda :\phi^4: dx\right) d\mu_C < \infty .$$

See [4], Section 8.6, for details.

Let $0 < \varepsilon < 1$ and $\beta_k(\cdot)$, $k \varepsilon B$, a collection of independent standard Brownian motions. Define

$$W(t) = \sum_{k \varepsilon B} (k^2)^{-(1-\varepsilon)/2} \beta_k(t)\, e_k(\cdot), \quad t \ge 0.$$

This defines an H^{-1}-valued Wiener process with covariance $C^{1-\varepsilon}$ [2], [6]. The equation

$$d\phi(t) = -\frac{1}{2} \left(C^{-\varepsilon} \phi(t) + C^{1-\varepsilon} :\phi^3(t):\right) dt + dW(t) \qquad [2.2]$$

with initial law μ can be shown to have a unique stationary weak solution as an H^{-1}-valued process, defining an ergodic process called the $(\phi^4)_2$ process. See [2], [6] for details.

III. LATTICE APPROXIMATION

Let $A=\{2^{-n},\ n\ge 1\}$ and pick $\delta\in A$. The finite lattice Λ_δ with spacing δ is defined as follows: Let $\delta Z^2=\{\delta z|z\in Z^2\}$, $\text{int}\,\Lambda_\delta=\text{int}\,\Lambda\cap\delta Z^2$, $\partial\Lambda_\delta=\partial\Lambda\cap\delta Z^2$, $\Lambda_\delta=\text{int}\,\Lambda_\delta\cup\partial\Lambda_\delta=\Lambda\cap\delta Z^2$. $\ell_2(\text{int}\,\Lambda_\delta)$ is the Hilbert space with inner product

$$\langle f,f\rangle_{\text{int}\,\Lambda_\delta}=\sum_{x\in\text{int}\,\Lambda_\delta}\delta^2|f(X)|^2,$$

viewed as a subspace of $\ell_2(\Lambda_\delta)$. On $\ell_2(\delta Z^2)$, define the forward gradient $\partial_{\delta,\alpha}$ in direction α by $(\partial_{\delta,\alpha}f)(x)=\delta^{-1}[f(x+d\mu_\alpha)-f(x)]$ where μ_α is the unit vector in the α-th direction for $\alpha=1,2$. The backward gradient $\partial^*_{\delta,\alpha}$ is its adjoint with respect to the $\ell_2(\delta Z^2)$ inner product.

Let $-\bar{\Delta}_\delta=\partial^*_{\delta,1}\,\partial_{\delta,1}+\partial^*_{\delta,2}\,\partial_{\delta,2}$. Then $(\bar{\Delta}_\delta f)(x)=\delta^{-2}(-4f(x)+\sum f(y))$ where the summation is over the nearest neighbours of x. Let Π be the projection $\ell_2(\delta Z^2)\to\ell_2(\text{int}\,\Lambda_\delta)$. The Dirichlet difference Laplacian Δ_δ is defined as $\Pi\bar{\Delta}_\delta\Pi$ and agrees with $\bar{\Delta}_\delta$ on $\text{int}\,\Lambda_\delta$.

Choose as a basis on $\ell_2(\text{int}\,\Lambda_\delta)$ the $(\delta^{-1}-1)^2$ functions $\{e^\delta_k(x)=e_k(x)|x\in\text{int}\,\Lambda_\delta,\ k_\alpha=\pi,2\pi,\ldots,(\delta^{-1}-1)\pi;\ \alpha=1,2\}$.

<u>Lemma 3.1</u> ([4], p. 221) $\{e^\alpha_k\}$ diagonalize $-\Delta_\delta$ with

$$-\Delta_\delta e^\alpha_k=\lambda^\delta{}_k\,e^\delta_k,\quad \lambda^\delta{}_k=4\delta^{-2}\sum_{i=1}^{2}\sin^2\left(\frac{\delta k_i}{2}\right).$$

Also, $\langle e^\delta_k,e^\delta_\ell\rangle_{\text{int}\,\Lambda_\delta}=1$ if $k=\ell$, $=0$ otherwise

<u>Lemma 3.2</u> ([4], p. 222) The map $i_\delta: e^\delta_k\to e_k$ defines an isometric imbedding of $\ell_2(\text{int}\,\Lambda_\delta)\to L_2(\Lambda)$.

Let Π_δ be the projection operator on $L_2(\Lambda)$ which truncates the Fourier series at $k_\alpha/\pi=\delta^{-1}$, so that

$\Pi_\delta\sum\alpha_k\ e_k=\sum^\delta\alpha_k e_k$ where $\sum^\delta$ denotes the summation over $B_\delta=\{k=(k_1,k_2)|\ 1\le\pi^{-1}k_i\le\delta^{-1}-1,\ i=1,2\}$. Then $i^*_\delta f=\Pi_\delta f|_{\Lambda_\delta}$. We can consider $C_\delta=(-\Delta_\delta+1)^{-1}:\ell_2(\text{int}\,\Lambda_\delta)\to\ell_2(\text{int}\,\Lambda_\delta)$ as an operator on $L_2(\Lambda)$, via the above isometry, i.e., let $C_\delta=i_\delta C_\delta i^*_\delta$ where the C_δ on the right (resp. left) acts on $\ell_2(\text{int}\,\Lambda_\delta)$ (resp. $L^2(\Lambda)$). As an operator on $L_2(\Lambda)$, its kernel is $C_\delta(x,y)=\sum^\delta(\lambda^\delta_k+1)^{-1}e_k(x)e_k(y)$, which when restricted to the lattice points in $\text{int}\,\Lambda_\delta$, coincides with the matrix entries of C_δ as an operator on $\ell_2(\text{int}\,\Lambda_\delta)$.

<u>Lemma 3.3</u> ([4], pp. 222-224) $\|C_\delta-C\|\le(O\ \delta^2)$ as operators on $L_2(\Lambda)$. Moreover, $\sup_{x\in\Lambda}\|C_\delta(x,\cdot)\|_{L_p(\Lambda)}\le O(\delta^\alpha)$ for $\alpha<(2p^{-1},1)$.

If ϕ is a Gaussian field with covariance $\tilde{C}$, $\phi_\delta(x) = (i_\delta^* \phi)(x)$ for $x \varepsilon \operatorname{int} \Lambda_\delta$ defines a Gaussian lattice field with covariance $C_\delta = i_\delta^* C_\delta i_\delta$. The field ϕ_δ can be realized by a Gaussian measure on $L_2(R^{|\operatorname{int}\Lambda_\delta|})$. Explicitly, letting $\prod_{x \varepsilon \operatorname{int} \Lambda_\delta} d\phi_\delta(x)$ denote the Lebesgue measure on $R^{|\operatorname{int} \Lambda_\delta|}$, the above measure is given by

$$d\mu_{\delta C} = (\det C_\delta)^{-\frac{1}{2}} \pi^{-|\operatorname{int} \Lambda_\delta|\frac{1}{2}} \exp\Big(-\frac{\delta^4}{2} \sum_{x,y \varepsilon \operatorname{int} \Lambda_\delta} \phi_\delta(x) C_\delta^{-1}(x,y)\, \phi_\delta(y)\Big) \prod_x d\phi_\delta(x).$$

This is the lattice analog of μ_C. The lattice analog of μ can now be defined as follows: Define for $f \varepsilon \ell_2(\operatorname{int} \Lambda_\delta)$,

$$:\phi_\delta^n:(f) = \delta^2 \sum_{x \varepsilon \operatorname{int} \Lambda_\delta} :\phi_\delta^n(x):_{C_\delta} f(x).$$

The lattice analog μ_δ is given by

$$d\mu_\delta = \exp\Big(-\frac{1}{4} :\phi_\delta^4(x):_\delta(1)\Big)\, d\mu_{\delta C} \Big/ \int \Big(\int \exp\Big(-\frac{1}{4} : \phi_\delta^4 :_\delta(1)\, d\mu_{\delta C}\Big) \qquad [3.1]$$

For $k \varepsilon B_\delta$, let $\{\beta_k(\cdot)\}$ be a collection of independent standard Brownian motions. For $0 < \varepsilon < 1$, define

$$B_\delta(t) = \delta^2 \sum\nolimits^\delta (\lambda_k^\delta + 1)^{-(1-\varepsilon)/2} \beta_k(t)\, e_k(\cdot), \quad t \geq 0.$$

This defines an $L_2(\Lambda)$-valued Wiener process with covariance $C_\delta^{1-\varepsilon}$. The analog of [2.2] in the lattice case is

$$d\phi_\delta(t) = \frac{1}{2}\big(C_\delta^{-\varepsilon} \phi_\delta(t) + C_\delta^{1-\varepsilon} : \phi_\delta^3(t) :_\delta\big)\, dt + dB_\delta(t) \qquad [3.2]$$

where the operators act on $L_2(\Lambda)$. $\phi_\delta(\cdot)$ is viewed here as an $L_2(\Lambda)$-valued process. However, letting $\phi_\delta(t) = \sum^\delta \phi_{\delta k}(t) e_k$, [3.2] translates into an equivalent stochastic differential equation for finitely many scalar processes $\phi_{\delta k}(\cdot)$ with locally Lipschitz (in fact, polynomial) coefficients. This ensures the existence of an a.s. unique strong solution to [3.2] up to an explosion time. That it does not explode a.s. is proved by a standard application of Khasminskii's test for non-explosion exactly as in [G], Section 3.

By identifying the vector $\{\phi_\delta(x), x \varepsilon \operatorname{int} \Lambda_\delta\}$ with $\phi_\delta(\cdot) \varepsilon \ell_2(\operatorname{int} \Lambda_\delta)$, μ_δ can be considered as a probability measure on $\ell_2(\operatorname{int} \Lambda_\delta)$ and via the isometry i_δ, as a probability measure on $L_2(\Lambda)$. We retain the notation μ_δ for the latter interpretation, as only this interpretation will be used henceforth. A computation similar to that of [2], Section 3, shows that the generator of the Markov process described by [3.2] is self-adjoint on $L_2(\mu_\delta)$. By Theorem 2.3 of [3], the same holds for the associated transition semigroup of $\{T_t, t \geq 0\}$ of operators on $L_2(\mu_\delta)$. Thus for $f, g \varepsilon L_2(\mu_\delta)$, $\int f T_t g\, d\mu_\delta = \int (T_t f) g\, d\mu_\delta$. Letting $f(\cdot) \equiv 1$, $\int T_t g\, d\mu_\delta = \int g\, d\mu_\delta$, implying that μ_δ is an invariant probability measure for $\phi_\delta(\cdot)$. In fact, the resulting process will be ergodic. We won't need this fact here, so we omit the details. From now on, [3.2] will always be considered with initial law μ_δ.

IV. THE CONTINUUM LIMIT

This section establishes the main result of this paper, viz., the convergence of $\phi_\delta(\cdot)$ to the $(\phi^4)_2$ process as $\delta \to 0$ in A, in the sense of weak convergence of Q'-valued processes. Thus we consider $\phi_\delta(\cdot)$ as a Q'-valued process and μ_δ as a measure on Q' via the injection of $L_2(\Lambda)$ into Q'. From theorem 9.6.4, p.228, [4], it follows that the finite dimensional marginals of the collection $\{\phi_\delta(e_k), k \varepsilon B\}$ under μ_δ converge weakly to the corresponding ones under μ as $\delta \to 0$ in A. Since μ_δ, μ are supported on H^{-1}, it follows that $\mu_\delta \to \mu$ weakly as probability measures on Q'. (A proof of the former assertion would go as follows: Since H^{-1} is Polish, it is homeomorphic to a G_μ subset of $[0,1]^\infty$ whose closure $\overline{H}^{-1}$ can be considered a compactification of H^{-1}. As a measure on $\overline{H}^{-1}$, $\{\mu_\delta\}$ are tight and for any weak limit point υ thereof, its restriction υ' to H^{-1} must yield the same finite dimensional marginals for $\{\phi(e_k), k \varepsilon B\}$ as μ. Thus $\upsilon = \upsilon' = \mu$.) As a first step towards proving the continuum limit, we prove some tightness results.

Let

$$\phi_{\delta_1}(t) = \phi_\delta(t)$$

$$\phi_{\delta_2}(t) = \frac{1}{2}\int_0^t C_\delta^{-\varepsilon}\,\phi_\delta(s)\,ds$$

$$\phi_{\delta_3}(t) = \frac{1}{2}\int_0^t C_\delta^{1-\varepsilon} :\phi_\delta^3(s):\,ds$$

$$\phi_{\delta_4}(t) = B_\delta(t)$$

for $t \leq 0$. Pick $t_1 \leq t_2$ 0 in $[0,T]$, $\infty > T > 0$. In what follows, K denotes a positive constant (not always the same) that may depend on T, but not on δ. Let $f \varepsilon Q$

<u>Lemma 4.1</u> $$E\left[\left(\int_{t_1}^{t_2} C_\delta^{-\varepsilon}\,\phi_\delta(t)(f)\,dt\right)^4\right] \leq K|t_2 - t_1|^2 \qquad [4.1]$$

<u>Proof</u> Using Jensen's inequality and stationarity of $\phi_\delta(\cdot)$, one obtains

$$E\left[\left(\int_{t_1}^{t_2} C_\delta^{-\varepsilon}\,\phi_\delta(t)(f)\,dt\right)^4\right] \leq K\,|t_2 - t_1|^2\; E\left[|C_\delta^{-\varepsilon}\,\phi_\delta(0)(f)|^4\right].$$

Letting $\Lambda_\delta = d\mu_\delta / d\mu_{\delta C}$, the expectation on the right is bounded by

$$\left[\int |C_\delta^{-\varepsilon}\phi(f)|^8\,d\mu_{\delta C}(\phi)\right]^{1/2}\left[\int \Lambda_\delta^2\,d\mu_{\delta C}\right]^{1/2}.$$

By Lemma 9.6.2, p. 227, [4], the second term above is bounded uniformly in δ. Using Feynman graph calculations, as in Theorem 8.5.3, p. 191, [4], one has

$$\int |C_\delta^{-\varepsilon}\phi(f)|^2\, d\mu_{\delta C}(\phi) \le K \, \| C_\delta^{-\varepsilon} f \|_2^2 .$$

Now

$$\| C_\delta^{-\varepsilon} f - C^{-\varepsilon} f \|_2^2 = C \sum_{k\in B_\delta} <f, e_k>^2 \left((\lambda_k^\delta + 1)^\varepsilon I\{k\varepsilon B_\delta\} - (\lambda_k + 1)^\varepsilon \right)^2 .$$

The summand on the right can be dominated in absolute value by $K < f, e_k >^2 \lambda_k^2$ which is summable for $f\varepsilon Q$. By the dominated convergence theorem,

$$\lim \| C_\delta^{-\varepsilon} f - C^{-\varepsilon} f \|_2 = 0,$$

implying $\sup_\delta \| C_\delta^{-\varepsilon} f \|_2 < \infty$. [4.1] follows. QED

<u>Lemma 4.2</u> $E[(\int_{t_2}^{t_2} C_\delta^{1-\varepsilon} :\phi_\delta^3(t):(f)\, dt)^4] \le K|t_2 - t_1|^2$. [4.2]

This follows along similar lines.

<u>Lemma 4.3</u> $E[(|B_\delta(t_2)(f) - B_\delta(t_1)(f)|^4] \le K|t_2 - t_1|^2$. [4.3]

<u>Proof</u> The lefthand side equals

$$3|C_\delta^{-\varepsilon}(f,f)|^2\, |t_2 - t_1|^2 \le 3 \sup_\delta \| C_\delta^{(1-\varepsilon)/2} f \|_2^2 \, |t_2, t_1|^2 .$$

As in the proof of Lemma 4.1, one can prove

$$\lim_{\delta\to 0} \| C_\delta^{(1-\varepsilon)/2} f - C^{(1-\varepsilon)/2} f \|_2 = 0.$$

Thus $\sup_\delta \| C_\delta^{(1-\varepsilon)} f \|_2 < \infty$ and the claim follows. QED

<u>Corollary 4.1</u> $E[|\phi(t_2)(f) - \phi(t_1)(f)|^4 \le K|t_2 - t_1|^2$ [4.4]

<u>Proof</u> Follows from [3.2] and [4.1] - [4.3]. QED

<u>Lemma 4.4</u> The laws of the processes $[\phi_{\delta_1}(\cdot), \phi_{\delta_2}(\cdot), \phi_{\delta_3}(\cdot), \phi_{\delta_4}(\cdot)]$ viewed as $(C(0,\infty); Q'))^4$-valued random variables remain tight as δ varies over A.

<u>Proof</u> By Theorem 3.1 of [7], it suffices to establish the tightness of $[\phi_{\delta_1}(\cdot)(f), \phi_{\delta_2}(\cdot)(f), \phi_{\delta_3}(\cdot)(f), \phi_{\delta_4}(\cdot)(f)]$ on $[0,T]$ as $(C([0,T]; R))^4$-valued random variables for arbitrary $T > 0$ and $f\varepsilon Q$. This, however, is immediate from the tightness of $\{\mu_\delta\}$ (since $\mu_\delta \to \mu$ weakly as a measure on H^{-1}), the estimates [4.1] - [4.4] and the criterion of [1], p. 95. QED

Recall that a family of probability measures on a product of Polish spaces is tight if and only if its images under projection onto each factor space are. Letting $\{\bar{e}_i\}$ denote an enumeration of $\{e_k\}$. This implies, in view of the foregoing, that $[\phi_{\delta_1}(\cdot)(\bar{e}_1), \ldots\ldots, \phi_{\delta_4}(\cdot)(\bar{e}_1), \phi_{\delta_1}(\cdot)(\bar{e}_2), \ldots\ldots, \phi_{\delta_4}(\cdot)(\bar{e}_2), \phi_{\delta_1}(\cdot)(\bar{e}_3), \ldots\ldots]$ are tight as $(C([0,\infty]; R))^\infty$-valued random variables. By dropping to a subsequence of A, denoted by A again, we may assume that they converge in law as $\delta \to 0$ along A. Then for any finite subset $\{t_1, \ldots, t_k\}$ of $[0,\infty]$ and a collection $\{g_1, \ldots, g_k\}$ of finite linear combinations of $\{\bar{e}_i\}$, the joint laws of $\{\phi_{\delta 1}(t_j)(g_j),\ 1 \le i \le 4,\ 1 \le j \le k\}$ converge. Consider a collection $f_1, \ldots, f_k$ in Q. Using the kind of estimates used in the proofs of Lemmas 4.1–4.3, we have

$$E[|\phi_{\delta 1}(t_j)(f_j - g_j)|^2] \le M\|f_j - g_j\|_2^2 \qquad [4.5]$$

$$E[|\phi_{\delta 2}(t_j)(f_j - g_j)|^2] \le M\|C_\delta^{-\varepsilon}(f_j - g_j)\|_2^2 \qquad [4.6]$$

$$E[|\phi_{\delta 3}(t_j)(f_j - g_j)|^2] \le M\|C_\delta^{1-\varepsilon}(f_j - g_j)\|_2^2 \qquad [4.7]$$

$$E[|\phi_{\delta 4}(t_j)(f_j - g_j)|^2] \le M\|C_\delta^{(1-\varepsilon)/2}(f_j - g_j)\|_2^2 \qquad [4.8]$$

for a suitable constant M depending on $\max(t_1, \ldots, t_k)$. As $\delta \to 0$ in A, the righthand sides of [4.6] - [4.8] converge to the corresponding quantities with C replacing C_δ. Since g_j can be obtained by suitably truncating the Fourier series of f_j in $\{e_i\}$, each of these limiting expressions and the righthand side of [4.5] can be made smaller than any prescribed $\eta > 0$ uniformly in $1 \le j \le k$ by a suitable choice of $\{g_j\}$. It follows that the righthand sides of [4.5] - [4.8] can be made smaller than any prescribed $\eta \to 0$ uniformly in $\delta \varepsilon A$ and $1 \le j \le k$ by a suitable choice of $\{g_j\}$.

Let $\{h_\ell\}$ be an enumeration of finite linear combinations of $\{\bar{e}_i\}$ with rational coefficients. By a well-known theorem of Skorohod ([5], p. 9), we can construct on some probability space random variables $X_{\delta ij\ell}$, $Y_{ij\ell}$, $\delta \varepsilon A$, $1 \le i \le 4$, $1 \le j \le k$, $\ell \ge 1$, such that $\{X_{\delta ij\ell}\}$ agrees in law with $\{\phi_{\delta i}(t_j)(h_\ell)\}$ for each fixed δ and $X_{\delta ij\ell} \to Y_{ij\ell}$ a.s. as $\delta \to 0$ in A. By augmenting this probability space, if necessary, we may construct on it random variables $Z_{\delta ij}$, (δ, i, j) as above, such that the joint law of $[\phi_{\delta i}(t_j)(f_j), \phi_{\delta i}(t_j)(h_1), \phi_{\delta i}(t_j)(h_2), \ldots]$ agrees with that of $[Z_{\delta ij}, X_{\delta ij_1}, X_{\delta ij_2}, \ldots\ldots]$ for each δ, i, j. Since $X_{\delta ij\ell} \to Y_{ij\ell}$ a.s. and $E[|X_{\delta ij\ell}|^4] = E[|\phi_{\delta i}(t_j)(h_\ell)|^4]$ can be bounded uniformly in δ for each i, j, ℓ by estimates analogous to [4.5] - [4.8], we have $E[|X_{\delta ij\ell} - Y_{ij\ell}|^2] \to 0$ as $\delta \to 0$ in A for each i, j, ℓ. On the other hand, given $\eta \to 0$, we can pick $\ell(j)$, $1 \le j \le k$, such that setting $g_j = h_{\ell(j)}$ in [4.5] - [4.8] makes all the quantities on the righthand side there less than η. Thus

$$\lim_{\substack{\delta, \alpha \to 0 \\ \delta, \alpha \,\varepsilon\, A}} E[|Z_{\delta ij} - Z_{\alpha ij}|^2] \le 2\eta + \lim_{\substack{\delta, \alpha \to 0 \\ \delta, \alpha \,\varepsilon\, A}} E[|X_{\delta ij\ell(i)} - X_{\alpha ij\ell(i)}|^2] = 2\eta.$$

Thus $Z_{\delta ij}$ converge in mean square for each i, j as $\delta \to 0$ in A. It follows that the joint laws of $\{\phi_{\delta i}(t_j)(f_j),\ 1 \le i \le 4,\ 1 \le j \le k\}$ converge. Theorem 5.3, [7], now implies that $[\phi_{\delta 1}(\cdot), \ldots\ldots, \phi_{\delta 4}(\cdot)]$ converge as $(C([0,\infty]; Q'))^4$-valued random variables. Let $[\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot), \phi_4(\cdot)]$ denote its limit in law (abbreviated as "l.i.l" henceforth). By taking the l.i.l. in [3.2] along an appropriate subsequence,

$$\phi_1(t) = \phi_1(0) + \sum_{i=2}^{4} \phi_i(t) \quad \text{a.s.} \qquad [4.9]$$

<u>Theorem 4.1</u> $\phi_1(\cdot)$ is the $(\phi^4)_2$ process.

<u>Proof</u> We prove the theorem by identifying each term of [4.9]. Let $f \in Q$. By Jensen's inequality and stationarity, $E[|\int_0^t \phi_\delta(s)(C_\delta^{-\varepsilon}f)\,ds - \int_0^t \phi_\delta(s)(C^{-\varepsilon}f)\,ds|^2] \le t\,E[|\phi_\delta(0)(C_\delta^{-\varepsilon}f - C^{-\varepsilon}f)|^2] \le t K \|C_\delta^{-\varepsilon}f - C^{-\varepsilon}f\|_2^2$. The righthand side tends to zero as $\delta \to 0$ by arguments similar to those employed in the proof of Lemma 4.1. Thus

$$\begin{aligned}
\underset{\delta\to 0}{\text{l.i.l.}}\ (\phi_{\delta 1}(\cdot), \phi_{\delta 2}(t)(f)) &= (\phi_1(\cdot), -2\phi_2(t)(f)) \\
&= \underset{\delta\to 0}{\text{l.i.l.}}\ \left(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C_\delta^{-\varepsilon}f)\,ds\right) \\
&= \underset{\delta\to 0}{\text{l.i.l.}}\ \left(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C^{-\varepsilon}f)\,ds\right) \\
&= \left(\phi_1(\cdot), \int_0^t \phi_1(s)(C^{-\varepsilon}f)\,ds\right).
\end{aligned}$$

It follows that

$$\phi_2(t)(f) = \frac{1}{2}\int_0^t \phi_1(s)(C^{-\varepsilon}f)\,ds \quad \text{a.s.}$$

Similarly

$$E\left[\left|\int_0^t :\phi_\delta^3(s):_\delta (C_\delta^{1-\varepsilon}f)\,ds - \int_0^t :\phi_\delta^3(s):_\delta (C^{1-\varepsilon}f)\,ds\right|^2\right]$$

$\le t\,E[|:\phi_\delta^3(0):(C_\delta^{1-\varepsilon}f - C^{1-\varepsilon})|^2] \to 0$ as in $\delta \to 0$ in A, by arguments analogous to those above. Hence

$$\begin{aligned}
\underset{\delta\to 0}{\text{l.i.l.}}\ \left(\phi_\delta(\cdot), \int_0^t :\phi_\delta^3(s):_\delta (C_\delta^{1-\varepsilon}f)\,ds\right) &= (\phi_1(\cdot), -2\phi_3(t)(f)) \\
&= \underset{\delta\to 0}{\text{l.i.l.}}\ \left(\phi_\delta(\cdot), \int_0^t :\phi_\delta^3(s):_\delta (C^{1-\varepsilon}f)\,ds\right) \qquad [4.10]
\end{aligned}$$

Let $\alpha > \delta$ in A. Then

$$E\left[\left|\int_0^t :\phi_\delta^3(s):_\delta (C^{1-\varepsilon}f)\,ds - \int_0^t :\phi_\alpha^3(s):_\delta (C^{1-\varepsilon}f)\,ds\right|^2\right]$$

$\le t\,E[|:\phi_\delta^3(0):_\delta(C^{1-\varepsilon}f) - :\phi_\alpha^3(0):_\delta(C^{1-\varepsilon}f)|^2] \le O(\alpha^\beta)$ for a suitable $\beta > 0$ uniformly in δ as $\delta \to 0$, by virtue of (9.6.9), p. 228, [4]. Thus the righthand side of [4.10] equals

$$\begin{aligned}
&\underset{\alpha\to 0}{\text{l.i.l.}}\ \underset{\delta\to 0}{\text{l.i.l.}}\ \left(\phi_\delta(\cdot), \int_0^t :\phi_\alpha^3(s):_\delta (C^{1-\varepsilon}f)\,ds\right) \\
&\qquad = \underset{\alpha\to 0}{\text{l.i.l.}}\ \left(\phi_1(\cdot), \int_0^t :\bar{\phi}_\alpha^3(s):(C^{1-\varepsilon}f)\,ds\right)
\end{aligned}$$

where $\bar{\phi}_\alpha(\cdot)$ is defined by

$$\bar{\phi}_\alpha(t)(h) = \sum^{\alpha} \phi_1(t)(e_k)\langle e_k, h\rangle, \quad h \in Q.$$

The above limit equals

$$(\phi_1(\cdot), \int_0^t :\phi_1^3(s): (C^{1-\varepsilon} f)\, ds),$$

Thus

$$\phi_3(t)(f) = -\frac{1}{2}\int_0^t :\phi_1^3(s): (C^{1-\varepsilon} f)\, ds \text{ a.s.}$$

Finally, it is easy to check that $\phi_4(\cdot)$ will be a Wiener process with covariance $C^{1-\varepsilon}$. Thus $\phi_1(\cdot)$ satisfies [3.2] with initial law μ. By the uniqueness in law of this equation (proved in [2], Section IV), we conclude that $\phi_1(\cdot)$ is the $(\phi^4)_2$ process. QED

<u>Corrollary 4.2</u> $\phi_\delta(\cdot)$ converge in law to $\phi(\cdot)$ as $C([0,\infty]; Q')$-valued random variables as $\delta \to 0$ in A, as defined originally.

<u>Proof</u> A careful look at the foregoing shows that any subsequence of A will have a further subsequence along which the above convergence holds. QED

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