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VARIATIONAL PROBLEMS IN SBV.[1]

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The aim of these lectures is to introduce a mathematical framework where variational problems modeled on the segmentation problem

$$(1) \quad \text{minimize} \left\{ \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \alpha \mathcal{H}_{n-1}(K) + \beta \int_{\Omega \setminus K} |u-g|^2 \, dx \right\} \qquad K \subset \Omega \text{ closed, } u \in W^{1,2}(\Omega \setminus K)$$

can be stated and have a solution. Here $\Omega \subset \mathbf{R}^n$ is a bounded open set, $\alpha, \beta > 0$ and $\mathcal{H}_{n-1}$ is the Hausdorff $(n-1)$-dimensional outer measure in $\mathbf{R}^n$.

For instance, when $n = 2$ and f represents the 2-dimensional image given by a camera, problem (1) can be seen as the weak formulation of the problem of finding the "best" decomposition of Ω in a finite number of piecewise smooth regions $\Omega_1, \ldots, \Omega_N$, a function u which is harmonic in each Ω_i and close to f. The regions Ω_i correspond to the real objects seen by the camera. The functional (1) takes into account the total length of the boundaries of the sets Ω_i, the L^2 norm of $u - g$, and the $W^{1,2}$ norm on each set Ω_i. The equivalence between the original segmentation problem and its weak formulation (1) has not been proven yet, but it is an highly reasonable conjecture. Actually, a regularity theorem is needed, asserting that for every pair (u, K) minimizing (1), the set K is made up with a finite number of points joined by C^1 arcs. We shall discuss these regularity problems later.

To prove that problem (1) has a solution, basically we use the so-called direct method of the Calculus of Variations, which consists in finding a topology σ such that the functional F is at the same time lower semicontinuous and coercive (i.e., the sublevels $\{F \leq t\}$ of the functional are σ-compact). This will be done first by finding a formulation of problem (1) in a space of functions of bounded variations, and then by proving lower semicontinuity and coercivity with respect to the $L^1_{\text{loc}}(\Omega)$ topology. Moreover, we can prove existence of minimizers for much more general functionals, of the form

$$(2) \qquad \mathcal{F}(u) = \int_{\Omega \setminus K} f(x, u, \nabla u) \, dx + \int_K \varphi(x, u^+, u^-, \nu) \, d\mathcal{H}_{n-1}(x).$$

The techniques we adopt are derived from the early ideas of De Giorgi, Federer and Caccioppoli about sets of finite perimeter in the 50's, therefore the first two lectures are devoted to a recall of the basic notions on this subject and on functions of bounded variation. The subsequent lectures are devoted to the proof of compactness and lower semicontinuity theorems. We shall also deal with the first variation of the functional in (1), with the regularity of solutions, and with the approximation by elliptic functionals. Hence, the scheme will be the following:

1. Sets of finite perimeter.

2. The spaces $BV(\Omega)$, $SBV(\Omega)$.

3. Slicing and main compactness theorem.

4. Lower semicontinuity.

5. Regularity of minimizers and first variation.

6. Approximation with elliptic functionals via Γ-convergence.

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1. Sets of finite perimeter.

First, we introduce some notation. By $\mathbf{B}(\Omega)$ we shall denote the class of Borel sets $B \subset \Omega$. We set also

$$\mathcal{L}_n(B) = |B| = \text{ Lebesgue } n-\text{dimensional measure B},$$

$$\mathcal{H}_{n-1}(B) = \text{ Hausdorff } (n-1)-\text{dimensional measure B},$$

$$\mathbf{S}^{n-1} = \{v \in \mathbf{R}^n : |v| = 1\},$$

$$\omega_n = \text{ Lebesgue measure of the unit ball in } \mathbf{R}^n,$$

$$B_t = \left\{x \in \Omega : \lim_{\rho \to 0^+} \frac{|B \cap B_\rho(x)|}{\omega_n \rho^n} = t\right\} \qquad \left(t \in [0,1]\right).$$

We denote by $\chi_A : \Omega \to \{0,1\}$ the characteristic function of a set $A \subset \Omega$. If $\mu : \mathbf{B}(\Omega) \to [0,+\infty]$ is a measure, and if $h : \Omega \to \mathbf{R}^p$ is a function such that

$$\int_\Omega |h| \, d\mu < +\infty,$$

we denote by $h \cdot \mu$ the measure whose density with respect to μ is h. We set also

$$\mu|_A = \chi_A \cdot \mu,$$

provided A is a Borel set.

From the point of view of Calculus of Variations, we are interested in finding a notion of perimeter for Borel sets $A \subset \Omega$ which fulfils the following two reasonable properties:

(1.1) $\mathcal{P}(A) = \mathcal{H}_{n-1}(\partial A \cap \Omega)$ whenever A is an open set with piecewise C^1 boundary;

(1.2) $\mathcal{P}$ is lower semicontinuous with respect to local convergence in measure, i.e.

$$\mathcal{P}(A) \le \liminf_{h \to +\infty} \mathcal{P}(A_h)$$

for all sequences of Borel sets A_h such that

$$\lim_{h \to +\infty} \Big| [(A_h \setminus A) \cup (A \setminus A_h)] \cap K \Big| = 0$$

for every compact set $K \subset \Omega$. One of the first definitions of perimeter has been given by Caccioppoli, who defined

$$\mathcal{P}_1(A) = \inf\left\{\liminf_{h \to +\infty} \mathcal{H}_{n-1}(\partial A_h \cap \Omega) : A_h \to A \text{ locally in measure, } A_h \text{ piecewise smooth}\right\}. \tag{1.3}$$

In other words, a set $A \subset \Omega$ has finite perimeter (in Ω) if and only if there exists a sequence of piecewise smooth sets A_h converging to A and such that $\mathcal{H}_{n-1}(\partial A_h \cap \Omega)$ is bounded.

A different definition starts from the Gauss-Green formula

$$\int_A \operatorname{div} g \, dx = - \int_{\partial A \cap \Omega} \langle g, \nu \rangle \, d\mathcal{H}_{n-1} \qquad g \in C_0^1(\Omega; \mathbf{R}^n) \tag{1.4}$$

which holds for every open set A with piecewise C^1 boundary. In (1.4), ν is the inner normal to A. We set $\mathcal{P}_2(A) = |\mu|(\Omega)$ if there exists a measure $\mu : \mathbf{B}(\Omega) \to \mathbf{R}^n$ with finite total variation, such that

$$\int_\Omega \operatorname{div} g \, dx = - \sum_{i=1}^n \int_\Omega g^{(i)} \, d\mu_i \qquad \forall g \in C_0^1(\Omega; \mathbf{R}^n). \tag{1.5}$$

If no such measure exists, we set $\mathcal{P}_2(A) = +\infty$. Since in the regular case we have

$$\mu = \nu \cdot \mathcal{H}_{n-1}|_{\partial A}$$

this definition fulfils (1.1). By Riesz's representation theorem, it is easily seen that

$$\mathcal{P}_2(A) = \sup\Bigl\{\int_A \mathrm{div} g\, dx : g \in C_0^1(\Omega;\mathbf{R}^n),\ 0 \le g \le 1\Bigr\} \tag{1.6}$$

for every Borel set $A \subset \Omega$. By using (1.6) it can be shown that $\mathcal{P}_2$ satisfies also condition (1.2).

The third definition of perimeter is due to Federer ([B5], 4.5.6), who defined the essential boundary $\partial^* A$ of a Borel set A as the set of points where A has neither density 0 nor density 1, i.e.,

$$\partial^* A = \Omega \setminus \bigl(A_0 \cup A_1\bigr) = \Bigl\{x \in \Omega : \limsup_{\rho\to 0+} \frac{|B_\rho(x) \cap A|}{\omega_n \rho^n} > 0 \quad \text{and} \quad \limsup_{\rho\to 0+} \frac{|B_\rho(x) \setminus A|}{\omega_n \rho^n} > 0\Bigr\}, \tag{1.7}$$

and

$$\mathcal{P}_3(A) = \mathcal{H}_{n-1}(\partial^* A). \tag{1.8}$$

In a series of papers appeared between 1954 and 1960 ([B2], [B3], [B6]), De Giorgi and Federer proved that all these definitions agree, that is, $\mathcal{P}_1(A) = \mathcal{P}_2(A) = \mathcal{P}_3(A)$ for every Borel set A. Moreover, they proved that the essential boundary $\partial^* A$ of a set of finite perimeter has many interesting properties. The first property is that $\partial^* A$ is equivalent, modulo $\mathcal{H}_{n-1}$-negligible sets, to the set of points where the density is $1/2$, i.e.,

$$\partial^* A \supset A_{1/2} \qquad \text{and} \qquad \mathcal{H}_{n-1}(\partial^* A \setminus A_{1/2}) = 0. \tag{1.9}$$

In particular, the density of sets of finite perimeter exists $\mathcal{H}_{n-1}$-almost everywhere in Ω and belongs to the set $\{0, 1/2, 1\}$. The second property is that $\partial^* A$ is a support for the measure μ in (1.5), and there exist a Borel function $\nu : \partial^* A \to \mathbf{S}^{n-1}$ such that (1.5) holds, that is,

$$\mu(B) = \int_{\partial^* A \cap B} \nu \, d\mathcal{H}_{n-1} \qquad \forall B \in \mathbf{B}(\Omega). \tag{1.10}$$

Finally, the set $\partial^* A$ is countably rectifiable in the sense of Federer ([B5], chapter 3), i.e.,

$$\partial^* A \subset \bigcup_{h\in\mathbf{N}} \Gamma_h \cup N \tag{1.11}$$

for a suitable sequence of C^1 hypersurfaces Γ_h, and $\mathcal{H}_{n-1}(N) = 0$.

By (1.5), (1.10), the classical Gauss-Green formula continues to hold in a weak sense for sets of finite perimeter, provided the topological boundary is replaced by the essentail boundary. Now, we want to show that also the unitary vector ν which occurs in (1.10) can be seen as a "normal" vector to A and $\partial^* A$. Given a set $S \subset \mathbf{R}^n$, one possible way to define a set of tangent vectors to S at x is to consider them as limits of secant vectors, setting

$$\mathrm{Tan}(S, x) = \Bigl\{v \in \mathbf{R}^n : v = \lim_{h\to+\infty} \frac{x_h - x}{\rho_h},\ \rho_h \downarrow 0,\ x_h \in S\Bigr\}.$$

The need to neglect $\mathcal{H}_{n-1}$-negligible sets leads to the definition of approximate tangent set:

$$\mathrm{ap}\,\mathrm{Tan}(S, x) = \bigcap \Bigl\{\mathrm{Tan}(S', x) : S' \subset S,\ \lim_{\rho\to 0+} \frac{\mathcal{H}_{n-1}(S \cap B_\rho(x) \setminus S')}{\rho^{n-1}} = 0\Bigr\}. \tag{1.12}$$

By the De Giorgi theorem (see for instance [B4]) it follows that

$$\text{ap}\,\text{Tan}(\partial^* A, x) = \{v \in \mathbf{R}^n : \langle v, \nu(x)\rangle = 0\} \tag{1.13}$$

for $\mathcal{H}_{n-1}$-almost every $x \in \partial^* A$, provided $\mathcal{H}_{n-1}(\partial^* A) < +\infty$. Moreover, the vector $\nu(x)$ is an "inner" normal to A because

$$\lim_{\rho\to 0^+} \frac{|A \cap \{y \in B_\rho(x) : \langle y - x, \nu(x)\rangle < 0\}|}{\omega_n \rho^n} = 0, \tag{1.14}$$

and

$$\lim_{\rho\to 0^+} \frac{|\{y \in B_\rho(x) : \langle y - x, \nu(x)\rangle > 0\} \setminus A|}{\omega_n \rho^n} = 0. \tag{1.15}$$

Example 1. The topological boundary of a set of finite perimeter may be much greater than the essential boundary, even if the set is open. Let $n > 1$, and let $\mathcal{D} = \{d_i\}_{i\in\mathbf{N}}$ be a countable dense set in $\mathbf{R}^n$. Let $(\rho_i) \subset]0,1]$ be a sequence such that

$$n\omega_n \sum_{i=1}^{\infty} \rho_i^{n-1} \le \epsilon,$$

where $\epsilon > 0$ is a given positive number. The open set

$$A = \bigcup_{i\in\mathbf{N}} B_{\rho_i}(d_i)$$

has finite perimeter in $\mathbf{R}^n$ because

$$\mathcal{P}(A) \le \liminf_{h\to+\infty} \mathcal{P}\big(\bigcup_{i=1}^{h} B_{\rho_i}(d_i)\big) \le n\omega_n \liminf_{h\to+\infty} \sum_{i=1}^{h} \rho_i^{n-1} \le \epsilon.$$

On the other hand, since $|A| \le \epsilon/n < +\infty$, the set $\partial A = \overline{A} \setminus A = \mathbf{R}^n \setminus A$ has infinite Lebesgue measure.

Example 2. If $n = 1$, $\Omega = \mathbf{R}$, the sets of finite perimeter in $\mathbf{R}$ are equivalent (modulo a $\mathcal{L}_1$ negligible set) to a finite union of intervals. This can be easily seen recalling that the characteristic function χ_A of a set of finite perimeter is a real function of bounded variation, hence the nuber of (essential) discontinuities must be finite (see (2.4), §2).

Example 3. Let $u : \Omega \to \mathbf{R}$ be a Lipschitz function. We shall later see that for almost every $t \in \mathbf{R}$ the open set $\{x \in \Omega : u(x) > t\}$ has finite perimeter in Ω, and the following equalities hold ([B5] 3.2.22, 4.5.9)

$$\int_\Omega |\nabla u|\,dx = \int_{-\infty}^{+\infty} \mathcal{H}_{n-1}(\partial^*\{u > t\})\,dt = \int_{-\infty}^{+\infty} \mathcal{H}_{n-1}(\{u = t\})\,dt. \tag{1.16}$$

The following compactness theorem is the main motivation of the introduction of sets of finite perimeter in Calculus of Variations (see for instance [B4]).

Theorem 1.1. *Let $(A_h) \subset \mathbf{B}(\Omega)$ be a sequence of sets of finite perimeter in Ω such that*

$$\sup_{h\in\mathbf{N}} \mathcal{P}(A_h) < +\infty.$$

Then, there exists a subsequence (A_{h_k}) converging locally in measure to a set $A \in \mathbf{B}(\Omega)$ with finite perimeter in Ω. Moreover, if Ω has finite perimeter in $\mathbf{R}^n$, then the subsequence converges in measure in Ω.

Example 4. Assume that Ω has finite perimetr in $\mathbf{R}^n$, and let $c \in [0, |\Omega|]$. Let $\varphi : \mathbf{R}^n \to [0, +\infty[$ be a convex function satisfying the conditions

$$\varphi(tp) = t\varphi(p) \qquad \forall t \geq 0,\ p \in \mathbf{R}^n,$$

and

$$\varphi(\nu) > 0 \qquad \forall \nu \in \mathbf{S}^{n-1}.$$

Using theorem 1.1. and a lower semicontinuity result (see theorem 4.2 in §4 or [C27]), it is possible to show that the generalized isoperimetric problem

$$\min\left\{ \int_{\partial^* A} \varphi(\nu)\, d\mathcal{H}_{n-1} : A \in \mathbf{B}(\Omega),\ |A| = c \right\}$$

has a solution in the class of sets of finite perimeter in Ω. The regularity of the solutions is strictly related to the regularity and the strict convexity of φ.

2. The spaces $BV(\Omega)$, $SBV(\Omega)$.

We begin by recalling the definition and the main properties of functions of bounded variation in an open interval $J \subset \mathbf{R}$. We say that $u : J \to \mathbf{R}$ is a function of bounded variation in J if

$$V_J(u) = \sup\left\{\sum_{i=1}^{k-1} |u(t_{i+1}) - u(t_i)| : \inf J < t_1 < \ldots < t_k < \sup J\right\} < +\infty. \tag{2.1}$$

The real number defined in (2.1) is often called total variation of the function u in J. It is not difficult to see that the functions of bounded variation fulfil the following properties:

(2.2) for all $t \in J$ there exist the limits

$$u_+(t) = \lim_{s \downarrow t} u(s), \qquad u_-(t) = \lim_{s \uparrow t} u(s)$$

and these limits are equal outside a countable set;

(2.3) there exists a unique measure $\mu : \mathbf{B}(J) \to \mathbf{R}$ with finite total variation, such that $u(t) = \mu(]0,t[)$ except for a countable set of real numbers t, and moreover $|\mu|(J) \le V_J(u)$. The measure μ is the distributional derivative of u, i.e.

$$\int_J u g'\, dt = -\int_J g\, d\mu \qquad \forall g \in C_0^1(J).$$

By (2.3) and by the Vitali-Lebesgue theorem on differentiation of mesures, we get also

(2.4) the function u is differentiable almost everywhere, and

$$\int_J |u'|\, dt + \int_J |u^+ - u^-|\, d\mathcal{H}_0 \le |\mu|(J) \le V_J(u).$$

In order to gain consistency with the definitions of functions of bounded variation in more than one dimension, we give introduce now a slight generalization of (2.1). We denote by $BV(J)$ the space of Borel functions $u : J \to \mathbf{R}$ such that

$$\text{ess-V}_J(u) = \inf\{V_J(v) : v = u \text{ almost everywhere}\} < +\infty.$$

Of course, properties (2.2), (2.3), (2.4) continue to hold for functions in $BV(J)$, provided a suitable representative in the equivalence class is chosen. Given any open set $\Omega \subset \mathbf{R}$, and given a function $u \in L^1(\Omega)$, we say that $u \in BV(\Omega)$ if $u \in BV(J)$ for every connected component $J \subset \Omega$, and

$$\text{ess-V}_\Omega(u) = \sum\{\text{ess-V}_J(u) : J \subset \Omega \text{ connected component}\} < +\infty.$$

In order to define the functions of bounded variation in domains with 2 dimensions or more, the most natural idea, which goes back to Cesari and Tonelli ([C13], [C32]), is to use a slicing argument. Given a unitary vector $\nu \in \mathbf{S}^{n-1}$, we set

$$\begin{cases} \pi_\nu = \{x \in \mathbf{R}^n : \langle x, \nu\rangle = 0\}; \\ \Omega_x = \{t \in \mathbf{R} : x + t\nu \in \Omega\} \qquad (x \in \pi_\nu); \\ u_x(t) = u(x + t\nu) \qquad (x \in \pi_\nu,\, t \in \Omega_x). \end{cases} \tag{2.5}$$

Then, we say that $u \in L^1(\Omega)$ belongs to $BV(\Omega)$, the space of functions of bounded variation in Ω, if for any unitary vector $\nu \in \mathbf{S}^{n-1}$ we have

$$u_x \in BV(\Omega_x) \qquad \text{for } \mathcal{H}_{n-1} - \text{a.e. } x \in \pi_\nu,$$

and

$$\int_{\pi_\nu} \text{ess-V}_{\Omega_x}(u_x)\,d\mathcal{H}_{n-1}(x) < +\infty, \tag{2.6}$$

(with the convention ess-$V_{\Omega_x} = 0$ if $\Omega_x = \emptyset$). The integral in (2.7) evaluates a sort of "variation along the direction ν", and $u \in BV(\Omega)$ if this variation is finite in any possible direction.

It is possible to give many alternative and useful definitions of the space $BV(\Omega)$. We summarize them in the following theorem (see [B5], theorem 4.5.9, and [B11], [C5], [C20]).

Theorem 2.1. *Let $u \in L^1(\Omega)$. The following conditions are equivalent:*

(i)
$$u \in BV(\Omega);$$

(ii) there exists a measure $Du = (D_1 u, \ldots, D_n u) : \mathbf{B}(\Omega) \to \mathbf{R}^n$ with finite total variation, such that

$$\int_\Omega (\mathrm{div} g) u\, dx = -\sum_{i=1}^n \int_\Omega g^{(i)}\, dD_i u \qquad \forall g \in C_0^1(\Omega; \mathbf{R}^n);$$

(iii) there exists a sequence of $C^\infty(\Omega)$ functions u_h such that

$$\lim_{h\to+\infty} \int_\Omega |u_h - u|\, dx = 0, \qquad \text{and} \qquad L = \limsup_{h\to+\infty} \int_\Omega |\nabla u_h|\, dx < +\infty;$$

(iv) the set $\{x \in \Omega : u(x) > t\}$ has finite perimeter in Ω for almost every $t \in \mathbf{R}$, and

$$\int_{-\infty}^{+\infty} \mathcal{H}_{n-1}(\partial^*\{u > t\})\, dt < +\infty;$$

(v) the set

$$E_u = \{(x,t) \in \Omega \times \mathbf{R} : u(x) < t\}$$

has finite perimeter in $\Omega \times \mathbf{R}$.

Moreover, the total variation in Ω of the measure Du, the least possible L in (iii), and the integral in (iv) are equal.

By any of the three characterization of total variation given in theorem 2.1 it easily follows that the total variation is a lower semicontinuous functional with respect to $L^1_{\mathrm{loc}}(\Omega)$ converegence. We recall a well known compactness theorem concerning sequences of functions of bounded variations. This theorem is the exact counterpart of theorem 1.1.

Theorem 2.2. *Let $\{u_h\} \subset BV(\Omega)$ be a sequence of functions such that*

$$\sup_{h\in\mathbf{N}} \left\{ \int_\Omega |u_h|\, dx + |Du_h|(\Omega) \right\} < +\infty.$$

Then, there exists a subsequence u_{h_k} converging in $L^1_{\mathrm{loc}}(\Omega)$ to $u \in SBV(\Omega)$. Moreover, if Ω has finite perimeter in $\mathbf{R}^n$, then the sequence u_{h_k} converges in $L^1(\Omega)$.

Now, we give some important examples of functions of bounded variation.

Example 1. By (ii), we get the inclusion

$$W^{1,1}(\Omega) \subset BV(\Omega).$$

By (ii), we obtain also that the characteristic function χ_A of any set of finite perimeter A belongs to $BV(\Omega)$, and

$$P(A) = |D\chi_A|(\Omega).$$

Example 2. More generally, by means of (iv) and of the properties of sets of finite perimeter, it is possible to prove that the function

$$w(x) = \begin{cases} u(x) & \text{if } x \in A; \\ v(x) & \text{if } x \in \Omega \setminus A; \end{cases}$$

belongs to $BV(\Omega)$, provided $u,\ v \in BV(\Omega)$ and A has finite perimeter in Ω. In particular, piecewise $W^{1,1}$ functions belong to $BV(\Omega)$. This decomposability property is typical of $BV(\Omega)$, and it is not shared by the Sobolev spaces.

Example 3. Let A be the open set of example 2, §1. It is easy to see that every function equal to χ_A almost everywhere is discontinuous in a set of infinite measure.

Example 3 shows that, even if we choose a particular representative, in more than 1 dimensions we cannot hope for nice continuity and differentiability properties of functions in $BV(\Omega)$. However, we shall see that properties (2.2), (2.4) continue to hold in a weak sense. The basic idea is to neglect not only negligible sets, but also sets with 0 density. Let $u : \Omega \to \mathbf{R}$ be a Borel function. We define the **approximate upper limit** of u at x as the greatest lower bound of all $t \in [-\infty, +\infty]$ such that $\{x \in \Omega : u(x) > t\}$ has 0 density at x, i.e.,

$$u^+(x) = \inf\Big\{t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{|\{u > t\} \cap B_\rho(x)|}{\rho^n} = 0\Big\}. \tag{2.7}$$

Similarly, we define the **approximate lower limit** u^- as

$$u^-(x) = \sup\Big\{t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{|\{u < t\} \cap B_\rho(x)|}{\rho^n} = 0\Big\}. \tag{2.8}$$

The approximate upper and lower limits are Borel functions, and in general $u^-(x) \le u^+(x)$. We set

$$S_u = \{x \in \Omega : u^-(x) < u^+(x)\}. \tag{2.9}$$

We shall consider the points $x \in \Omega \setminus S_u$ as points of "approximate continuity" for the function u, and the value of the limit is the common value of u^+, u^-. This definition is a little bit more general than the definition based on limits of averaged integrals. In fact, for any $z \in \mathbf{R}$ we have the implication

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x)} |u - z|\, dx = 0 \qquad \Longrightarrow \qquad u^+(x) = u^-(x) = z. \tag{2.10}$$

The opposite implication holds only if $u \in L^\infty(\Omega)$, as the following example shows.

Example 4. Let $n = 2$, let $0 \le r < +\infty$, $0 \le \theta < 2\pi$ and let $u = u(r,\theta) : \mathbf{R}^2 \to \mathbf{R}$ be defined in polar coordinates by

$$u(r,\theta) = \begin{cases} 0 & \text{if } r \le \sin(\theta/2); \\ r^{-\alpha} & \text{otherwise.} \end{cases}$$

Then, $u \in L^1_{\text{loc}}(\mathbf{R}^2)$ if $\alpha < 2$, $u^+(0,0) = u^-(0,0) = 0$, but the limit in (2.10) is equal to $+\infty$ if $\alpha > 1$.

We remark that S_u is always a negligible set ([B5], 2.9.13); in the particular case $u = \chi_A$ for some Borel set A, the set S_u is equal to the essential boundary $\partial^* A$. By (2.7), (2.8) it follows also

$$z = u^+ = u^- \in \mathbf{R} \qquad \Longleftrightarrow \qquad x \in \{y \in \Omega : |u(y) - z| > \epsilon\}_0 \qquad \forall \epsilon > 0.$$

Using the above equivalence, we can also define a reasonable generalization of the one-sided limits in (2.2). Given $\nu \in \mathbf{S}^{n-1}$, $z \in \mathbf{R}$, we say that $z = u_+(x,\nu)$ if

$$\lim_{\rho \to 0+} \frac{|\{y \in B_\rho(x) : \langle y - x, \nu \rangle > 0,\ |u(y) - z| > \epsilon\}|}{\omega_n \rho^n} = 0 \tag{2.11}$$

for every $\epsilon > 0$. We set also $u^-(x,\nu) = u^+(x,-\nu)$.

Besides the approximate continuity, we can also define the approximate differentiability. Let $x \in \Omega \setminus S_u$, assume that $u^+(x) = u^-(x) \in \mathbf{R}$, and let $p \in \mathbf{R}^n$. We say that p is the **approximate differential** of u at x if the function

$$v(y) = \frac{|u(y) - u^+(x) - \langle p, y - x \rangle|}{|y - x|}$$

has approximate limit 0 at x. We use for the approximate differential the same notation (i.e., $\nabla u(x)$) of the differential. It can be shown ([A2], Proposition 1.2) that the domain of the approximate differential is a Borel set, and ∇u is a Borel function.

The next theorem summarizes the main properties concerning the approximate differentiability and the approximate continuity of functions of bounded variation (see [B5], 4.5.9 and [C11]).

Theorem 2.3. *Let $u \in BV(\Omega)$. Then u is approximately differentiable almost everywhere, $\nabla u \in L^1(\Omega;\mathbf{R}^n)$, and $\nabla u \cdot \mathcal{L}_n|_\Omega$ is the absolutely continuous part of Du with respect to $\mathcal{L}_n|_\Omega$.*
The set S_u is countably rectifiable in the sense of (1.11), and there exists a Borel function $\nu : S_u \to \mathbf{S}^{n-1}$ such that

$$u_+(x,\nu) = u^+(x) \in \mathbf{R}, \qquad u_-(x,\nu) = u^-(x) \in \mathbf{R}$$

for $\mathcal{H}_{n-1}$-almost every $x \in S_u$. We have also

$$\int_\Omega |\nabla u|\, dx + \int_{S_u} |u^+ - u^-|\, d\mathcal{H}_{n-1} \le |Du|(\Omega),$$

and the vector $\nu(x)$ is related to the approximate tangent space to S_u by the relation

$$\operatorname{ap\,Tan}(K,x) = \{v \in \mathbf{R}^n : \langle v, \nu(x) \rangle = 0\} \qquad \mathcal{H}_{n-1}\text{-a.e. in } K$$

for every $K \subset S_u$ with $\mathcal{H}_{n-1}(K) < +\infty$.

Thanks to theorem 2.3, functionals of the type

$$\mathcal{F}(u) = \int_\Omega f(x,u,\nabla u)\, dx + \int_{S_u} \varphi(x,u^+,u^-,\nu)\, d\mathcal{H}_{n-1} \qquad (f,\ \varphi \ge 0) \tag{2.12}$$

are well defined in $SBV(\Omega)$, provided f, φ are Borel functions. We recover the functional of segmentation problems in the particular case

$$f(x,u,p) = p^2 + \beta|u - g(x)|^2, \qquad \varphi(x,u,v,\nu) \equiv \alpha.$$

Unfortunately, the space $BV(\Omega)$ may contain very pathological non constant functions which are continuous and have approximate differential 0 almost everywhere. For these functions, the functional (2.12) reduces to

$$\int_\Omega f(x,u,0)\, dx.$$

and does not depend on the (distributional) derivative of u. This is, from the mathematical point of view, a serious drawback, because there is no hope to control by means of the functional (2.12) the total variation $|Du|(\Omega)$, and hence theorem 2.2 cannot be applied. The most famous example of such kind of pathological functions is known in the literature as Cantor-Vitali function.

Example 5. Let us recursively define a sequence of continuous functions $u_h : [0,1] \to \mathbf{R}$ by setting $u_1(t) = t$ and

$$u_{n+1}(t) = \begin{cases} \frac{1}{2}u_n(3t) & \text{if } 0 \le t \le \frac{1}{3}; \\ \frac{1}{2} & \text{if } \frac{1}{3} \le t \le \frac{2}{3}; \\ \frac{1}{2}\left(1 + u_n(3(t - \frac{2}{3}))\right) & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

The sequence u_n converges uniformly to the Cantor-Vitali function u. By the lower semicontinuity of the total variation, $u \in BV(]0,1[)$; by the properties of the approximating functions we get that u is continuous and locally constant in the complement of Cantor's middle third set C. Since C is negligible, $u' = 0$ almost everywhere; the distributional derivative Du is a measure concentrated on C, and it can be shown that $u(t) = \mathcal{H}_\gamma([0,t] \cap C)$ with $\gamma = \ln 2/\ln 3$.

Even for the problem of segmentation, it does not seem natural to consider as admissible solutions Cantor-Vitali like functions. Therefore we need to restrict the domain of functionals (2.9), in order to avoid pathological functions and to gain compactness of the sublevels of the functional. This can be done by a deeper analysis of the distributional derivative Du of a function of bounded variation.

Proposition 2.4. *let $u \in BV(\Omega)$. Then, the distributional derivative splits into three mutually singular measures:*

$$Du = \nabla u \cdot \mathcal{L}_n|_\Omega + Ju + Cu \tag{2.13}$$

The first term in the right hand side of (2.13) corresponds to the absolutely continuous part, accordingly to theorem 2.3. The second term corresponds to the "jump" part of the derivative, and is related to S_u, u^+, u^- by

$$Ju(B) = \int_{S_u \cap B} (u^+ - u^-)\nu \, d\mathcal{H}_{n-1} \qquad \forall B \in \mathbf{B}(\Omega). \tag{2.14}$$

The third part is the "Cantor" part of the derivative, singular with respect to $\mathcal{L}_n|_\Omega$, and such that

$$|Cu|(B) = 0 \qquad \text{whenever} \qquad \mathcal{H}_{n-1}(B) < +\infty. \tag{2.15}$$

Moreover, we have the following implications:

$$u \in W^{1,1}(\Omega) \qquad \Longleftrightarrow \qquad \mathcal{H}_{n-1}(S_u) = 0,\ Cu = 0;$$

and

$$u \in W^{1,p}(\Omega) \qquad \Longrightarrow \qquad \mathcal{H}_{n-p}(S_u) = 0,\ Cu = 0.$$

By proposition 2.4 (for the proof, see [A1], and [C19] for the last implication), it is natural to define the space of **special** function of bounded variation $SBV(\Omega)$ as the functions $u \in BV(\Omega)$ such that $Cu = 0$. Of course, we have the inclusions

$$W^{1,1}(\Omega) \subset SBV(\Omega) \subset BV(\Omega)$$

and a function $u \in SBV(\Omega)$ belongs to $W^{1,1}(\Omega)$ if and only if $\mathcal{H}_{n-1}(S_u) = 0$.

In the next section we shall see that under very natural assumptions on the functions f, φ in (2.12), the sets

$$\{u \in SBV(\Omega) : F(u) \le t\}$$

are relatively compact in $SBV(\Omega)$ with respect to the topology $L^1_{\mathrm{loc}}(\Omega)$. Therefore the direct method of Calculus of Variations can be applied, provided F is $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuous.

3. Slicing and main compactness theorem.

In this section we are interested in finding compactness criteria which ensure the relative compactness of the minimizing sequences for a functional of the form:

$$\mathcal{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx + \int_{S_u} \varphi(x, u^+, u^-, \nu)\, d\mathcal{H}_{n-1}(x). \tag{3.1}$$

In many problems of Calculus of Variations in Sobolev spaces, a very useful condition is a more than linear growth at infinity in the gradient:

$$f(x, u, p) \geq \phi(|p|) \qquad \forall x \in \Omega,\ u \in \mathbf{R},\ p \in \mathbf{R}^n, \tag{3.2}$$

with

$$\phi : [0, +\infty[\to [0, +\infty[\ \text{ non decreasing}, \qquad \lim_{t \to +\infty} \frac{\phi(t)}{t} = +\infty. \tag{3.3}$$

Under this assumption, by Dunford-Pettis criterion ([C18], page 239), we know that the approximating differentials of any minimizing sequence are weakly relatively compact in $L^1(\Omega; \mathbf{R}^n)$, provided the functional in (3.1) is not trivially equal to $+\infty$. Now we need a growth condition on φ; in problems modeled on segmentation, it is reasonable to require that the total length of the discontinuities is uniformly bounded on minimizing sequences. Hence we shall assume that

$$\varphi(x, u, v, \nu) \geq c > 0 \qquad \forall x \in \Omega,\ u, v \in \mathbf{R},\ u > v,\ \nu \in \mathbf{S}^{n-1}. \tag{3.4}$$

Both these conditions are satisfied for the segmentation problem. To avoid technical difficulties and to simplify the statements we assume that there exists a constant $C > 0$ such that

$$\inf\{\mathcal{F}(u) : u \in SBV(\Omega)\} = \inf\{\mathcal{F}(u) : u \in SBV(\Omega) \cap L^\infty(\Omega),\ \|u\|_\infty \leq C\}. \tag{3.5}$$

This condition is satisfied by the functional of segmentation problem

$$\mathrm{F}(u) = \int_\Omega |\nabla u|^2\, dx + \alpha \mathcal{H}_{n-1}(S_u) + \beta \int_\Omega |u - g|^2\, dx$$

provided $g \in L^\infty(\Omega)$, with $C = \|g\|_\infty$. To see this, it is sufficient to replace each function $u \in SBV(\Omega)$ by the function $u_C = (u \wedge C) \vee -C$. Since

$$\nabla u_C = 0 \quad \text{a.e. in } \{|u| > C\}, \qquad \nabla u_C = \nabla u \quad \text{a.e. in } \{|u| \leq C\},$$

$$S_{u_C} \subset S_u, \qquad \int_\Omega |u_C - g|^2\, dx \leq \int_\Omega |u - g|^2\, dx,$$

the value of the functional F decreases on u_C, and (3.5) is satisfied. Thus, we are led to prove the following:

Theorem 3.1. *Let $\{u_h\} \subset SBV(\Omega) \cap L^\infty(\Omega)$ be a sequence such that*

$$\limsup_{h \to +\infty} \left\{ \|u_h\|_\infty + \int_\Omega \phi(|\nabla u_h|)\, dx + \mathcal{H}_{n-1}(S_{u_h}) \right\} < +\infty. \tag{3.6}$$

Then, there exists a subsequence u_{h_k} converging in $L^1_{\mathrm{loc}}(\Omega)$ to $u \in SBV(\Omega)$. Moreover,

$$\nabla u_h \quad \to \quad \nabla u \qquad \text{weakly in } L^1(\Omega; \mathbf{R}^n),$$

$$J u_h \quad \to \quad J u \qquad \text{weakly as Radon measures.}$$

Heuristically, the idea underlying theorem 3.1 is that on one hand it is impossible to approximate Cantor or jump part of derivative by approximate differentials weakly compact in L^1, and on the other hand it is impossible to approximate absolutely continuous or Cantor part of the derivative by means of jumps with controlled length. Therefore the absolutely continuous part and the jump part converge independently, and no Cantor derivative appears.

To show theorem 3.1, we begin to prove that there exists a subsequence (which we still label by u_h) which converges to $u \in BV(\Omega) \cap L^\infty(\Omega)$. This is an easy consequence of theorem 2.2, because

$$(3.7)\quad |Du_h|(\Omega) = \int_\Omega |\nabla u_h|\,dx + \int_{S_{u_h}} |u^+ - u^-|\,d\mathcal{H}_{n-1} \le d|\Omega| + \int_\Omega \phi(|\nabla u_h|)\,dx + \frac{2C}{c}\mathcal{H}_{n-1}(S_{u_h}) \qquad \forall h \in \mathbf{N},$$

where $d > 0$ is such that $\phi(t) \ge t - d$ for all $t \ge 0$.

The main difficulty of theorem 3.1 lies in the proof that $u \in SBV(\Omega)$. In order to prove this we begin with the one dimensional case, i.e. $n = 1$. In this case, the functions u_h are piecewise $W^{1,1}$ and the number of discontinuities is uniformly bounded. Possibly extracting subsequences, we can assume that $S_{u_h} \to K$ in the Kuratowski sense, i.e.,

$$\begin{cases} \forall x \in K\ \exists x_h \in S_{u_h} \text{ such that } x_h \to x; \\ x_h \in S_{u_h},\ x_h \to x \quad \Longrightarrow \quad x \in K \end{cases}$$

and K is of a finite set. Since for every open set $A \subset\subset \Omega \setminus K$ we have $S_{u_h} \cap A = \emptyset$ for h large enough, it is easy to see that $u \in W^{1,1}(\Omega \setminus K)$ and $S_u \subset K$, hence $u \in SBV(\Omega)$. Moreover, we get

$$(3.8)\qquad u_h' \to u' \qquad \text{weakly in } L^1(\Omega).$$

To prove that $u \in SBV(\Omega)$ also in the general case $n \ge 1$ we shall use the slicing properties of functions of bounded variation. By using the one dimensional result, we will be able to show that for every direction ν, $\mathcal{H}_{n-1}$-almost all the slices belong to SBV. Hence, we need the following equivalence: if $u \in BV(\Omega)$, then

$$(3.9)\qquad u \in SBV(\Omega) \quad \Longleftrightarrow \quad \forall \nu \in \mathbf{S}^{n-1},\ u_x \in SBV(\Omega_x) \text{ for } \mathcal{H}_{n-1}\text{-a.e. } x \in \pi_\nu$$

(we use the same notations of §2, see (2.5)). We shall prove much more than (3.9), by showing that the decomposition of the distributional derivative Du given by proposition 2.4 is well preserved under slicing. Before stating our theorem, we need to recall a procedure for constructing measures called fubinisation, a generalisation of the ordinary product of measures.

Assume that for $\mathcal{H}_{n-1}$-a.e. $x \in \pi_\nu$ we are given a measure $\sigma_x : \mathbf{B}(\Omega_x) \to \mathbf{R}^p$ in such a way that the following conditions are satisfied:

$$(i)\qquad x \quad \to \quad \sigma_x(B_x) \text{ is a Borel function } \forall B \in \mathbf{B}(\Omega),$$

(here and in the sequel $B_x = \{t \in \Omega_x : x + t\nu \in B\}$) and

$$(ii)\qquad \int_{\pi_\nu} |\sigma_x|(\Omega_x)\,d\mathcal{H}_{n-1}(x) < +\infty.$$

Then, there exists a unique measure in Ω, which we denote by $\int_{\pi_\nu} \sigma_x\,d\mathcal{H}_{n-1}$ such that

$$\Big(\int_{\pi_\nu} \sigma_x\,d\mathcal{H}_{n-1}\Big)(B) = \int_{\pi_\nu} \sigma_x(B_x)\,d\mathcal{H}_{n-1}(x) \qquad \forall B \in \mathbf{B}(\Omega).$$

Moreover, for every bounded Borel function $h : \Omega \to \mathbf{R}^p$ we have

$$\int_\Omega h\,d\Big(\int_{\pi_\nu} \sigma_x\,d\mathcal{H}_{n-1}\Big) = \int_{\pi_\nu} \sum_{i=1}^p \int_{\Omega_x} h^{(i)}(x + t\nu)\,d\sigma_{xi}\,d\mathcal{H}_{n-1}(x),$$

and

$$\left|\int_{\pi_\nu} \sigma_x d\mathcal{H}_{n-1}\right|(\Omega) = \int_{\pi_\nu} |\sigma_x|(\Omega_x)\, d\mathcal{H}_{n-1}(x). \tag{3.10}$$

Now, we can state our theorem on fubinisation of distributional derivatives of functions of bounded variation.

Theorem 3.2. *Let $u \in BV(\Omega)$, $\nu \in \mathbf{S}^{n-1}$. Then, we have*

$$\langle \nabla u \cdot \mathcal{L}_n|_\Omega, \nu\rangle = \int_{\pi_\nu} u'_x \cdot \mathcal{L}_1|_{\Omega_x}\, d\mathcal{H}_{n-1}, \tag{3.11}$$

$$\langle Ju, \nu\rangle = \int_{\pi_\nu} Ju_x\, d\mathcal{H}_{n-1}, \tag{3.12}$$

$$\langle Cu, \nu\rangle = \int_{\pi_\nu} Cu_x\, d\mathcal{H}_{n-1}. \tag{3.13}$$

Moreover, we have

$$\langle \nabla u(x+t\nu), \nu\rangle = u'_x(t) \qquad \mathcal{L}_1 - \text{a.e. in } \Omega_x$$

and

$$\begin{cases} S_{u_x} = \{t \in \Omega_x : x + t\nu \in S_u\}; \\ u_x^+(t) = u^+(x+t\nu), \quad u_x^-(t) = u^-(x+t\nu) \qquad \forall x \in \Omega_x \setminus S_{u_x} \end{cases}$$

for $\mathcal{H}_{n-1}$-almost every $x \in \pi_\nu$.

Theorem 3.2 allows to recover all the structure of functions of bounded variation by means of their one dimensional slices. In particular, (3.10) and (3.13) imply (3.9). The proof of theorem 3.2 starts from the following equality, which is easy consequence of Fubini-Tonelli theorem:

$$\langle Du, \nu\rangle = \int_{\pi_\nu} Du_x\, d\mathcal{H}_{n-1}.$$

By proposition 2.4 we get

$$\langle \nabla u \cdot \mathcal{L}_n|_\Omega, \nu\rangle - \int_{\pi_\nu} u'_x \cdot \mathcal{L}_1|_{\Omega_x}\, d\mathcal{H}_{n-1} = -\langle Ju + Cu, \nu\rangle + \int_{\pi_\nu} (Ju_x + Cu_x)\, d\mathcal{H}_{n-1}. \tag{3.14}$$

Now, the measures in the left hand side of (3.14) are absolutely continuous, and the measures in the right hand side are singular with respect to $\mathcal{L}_n|_\Omega$, hence both sides of (3.14) are 0. Therefore (3.11) holds and

$$\langle Ju, \nu\rangle - \int_{\pi_\nu} Ju_x\, d\mathcal{H}_{n-1} = -\langle Cu, \nu\rangle + \int_{\pi_\nu} Cu_x\, d\mathcal{H}_{n-1}. \tag{3.15}$$

A similar argument shows that both sides of (3.15) are 0. The heuristic idea is that the measure in the left hand side of (3.13) is "$(n-1)$-dimensional" and the measure in the right hand side has null total variation on $\mathcal{H}_{n-1}$-finite sets. The arguments involved here are quite technical; the interested reader is referred to [A1], theorem 3.3.

Now, let us come back to the proof of the compactness theorem. We want to show that $u \in SBV(\Omega)$. Let $\nu \in \mathbf{S}^{n-1}$ be a fixed direction; passing eventually to subsequences we can assume that u_{hx} converge in

$L^1_{loc}(\Omega_x)$ to u_x for $\mathcal{H}_{n-1}$-almost every $x \in \pi_\nu$. By Fatou's lemma and theorem 3.2 we get (we recall that $(S_u)_x = \{t \in \Omega_x : x + t\nu \in S_u\}$)

$$\int_{\pi_\nu} \liminf_{h\to+\infty} \left\{ \int_{\Omega_x} \phi(|u'_{hx}|)\,dt + \mathcal{H}_0(S_{u_h x}) \right\} d\mathcal{H}_{n-1}(x) \le \liminf_{h\to+\infty} \int_{\pi_\nu} \left\{ \int_{\Omega_x} \phi(|u'_{hx}|) + \mathcal{H}_0(S_{u_h x}) \right\} d\mathcal{H}_{n-1}(x) \le \tag{3.16}$$

$$\le \liminf_{h\to+\infty} \int_{\pi_\nu} \left\{ \int_{\Omega_x} \phi(|\nabla u_h(x+t\nu)|)\,dt + \mathcal{H}_0\big((S_{u_h})_x\big) \right\} d\mathcal{H}_{n-1}(x) \le \liminf_{h\to+\infty} \int_{\Omega} \phi(|\nabla u_h|)\,dx + \mathcal{H}_{n-1}(S_{u_h}).$$

In the last inequality we have used: $\mathcal{H}_{n-1}(B) \ge \int_{\pi_\nu} \mathcal{H}_0(B_x) d\mathcal{H}_{n-1}$; this inequality is true for every Borel set B ([B5], 3.2.22). Hence, for $\mathcal{H}_{n-1}$-almost every $x \in \pi_\nu$ the hypotheses of theorem 3.1 are satisfied by a subsequence of u_{hx}, with $n = 1$. In particular, $u_x \in SBV(\Omega_x)$ for $\mathcal{H}_{n-1}$-almost every $x \in \pi_\nu$, and (3.9) yields $u \in SBV(\Omega)$, because ν is arbitrary.

Now we prove the last statements of the theorem. For every function $g \in C^1_0(\Omega;\mathbf{R}^n)$ we have

$$\lim_{h\to+\infty} \sum_{i=1}^n \int_\Omega g^{(i)}\,d\,D_i u_h = -\lim_{h\to+\infty} \int_\Omega (\operatorname{div} g)\, u_h\,dx = -\int_\Omega (\operatorname{div} g)\, u\,dx = \sum_{i=1}^n \int_\Omega g^{(i)}\,dD_i u.$$

This, together with (3.7) yields by a standard approximation procedure

$$\lim_{h\to+\infty} \sum_{i=1}^n \int_\Omega g^{(i)}\,dD_i u_h = \sum_{i=1}^n \int_\Omega g^{(i)}\,dD_i u \qquad \forall g \in C_0(\Omega;\mathbf{R}^n)$$

where $C_0(\Omega;\mathbf{R}^n)$ is the space of continuous function $g : \overline{\Omega} \to \mathbf{R}^n$ vanishing on $\partial\Omega$. In other words, $Du_h \to Du$ weakly as Radon measures. If we prove the weak convergence of the differentials, the weak convergence of the jump parts of derivatives will follow by difference. In the one dimensional case, the weak convergence of the differentials is proved (see (3.8)). Unfortunately, (3.16) tells us that the differentials of the slices u_{hx} weakly converge to the differential of u_x only up to subsequences whose choice may depend on x. Hence the slicing argument cannot be directly applied to show the weak convergence of differentials. On the other hand, slicing is a powerful technique to prove lower semicontinuity inequalities. In [A1], by using slicing along the directions $\nu = \mathbf{e}_i$ ($i = \{1,\dots,n\}$), and arguing as in (3.16), it is proved the following inequality:

$$\int_\Omega |w - \frac{\partial u}{\partial x_i}|\,dx \le \liminf_{h\to+\infty} \int_\Omega |w - \frac{\partial u_h}{\partial x_i}|\,dx \qquad \forall w \in L^1(\Omega),\ \forall i \in \{1,\dots,n\}. \tag{3.17}$$

The weak convergence in $L^1(\Omega;\mathbf{R}^n)$ of the approximate differentials implies (3.17), because in every Banach space the norm is sequentially weakly lower semicontinuous; in some Banach spaces E, however, we can reverse the implication:

$$\{e_h\} \text{ weakly compact}, \quad \|w + e\| \le \liminf_{h\to+\infty} \|w + e_h\| \quad \forall w \in E \qquad \Longrightarrow \qquad e_h \to e \text{ weakly in } E. \tag{3.18}$$

This is true for Hilbert spaces, and more generally for Banach spaces whose norm is Frechét differentiable. Even if the norm of $L^1(\Omega)$ is not differentiable, (3.18) is fulfilled in $L^1(\Omega)$. The proof (a personal communication of G.Dal Maso) is the following: it is not restrictive to assume that e_h weakly converges to some $f \in L^1(\Omega)$. We have to show that $f = e$. Let $P = \{f > e\}$, $N = \{f \le e\}$. Since weakly compact sets in $L^1(\Omega)$ are equi-integrable, for every $\epsilon > 0$ it is possible to find $k > 0$ such that

$$\int_{\{|e_h - e| > k\}} |e_h - e|\,dx < \epsilon \qquad \forall h \in \mathbf{N}.$$

Let $w = -e - k\chi_P + k\chi_N$. It is easy to see that

$$|w + e_h| \leq k - (e_h - e)\chi_P + (e_h - e)\chi_N + 2|e_h - e|\chi_{\{|e_h - e| > k\}}.$$

Integrating both members we get

$$k|\Omega| = \int_\Omega |w + e|\,dx \leq \liminf_{h\to+\infty} \int_\Omega |w + e_h|\,dx \leq k|\Omega| - \int_\Omega (f - e)\,dx + \int_\Omega (f - e)\,dx + \epsilon,$$

so that $\int_\Omega |e - f|\,dx < \epsilon$, and since ϵ is arbitrary we conclude that $e = f$. Hence, the weak convergence of differentials follows by (3.18), and the compactness theorem 3.1 is completely proved.

We close this section with the following proposition, which is useful to relate our original formulation of the segmentation problem in terms of (u, K) (see the introduction) with its weak formulation in $SBV(\Omega)$.

Proposition 3.3. *Let $K \subset \Omega$ be a closed set such that $\mathcal{H}_{n-1}(K) < +\infty$, and let $u \in W^{1,1}(\Omega \setminus K) \cap L^\infty(\Omega)$. Then, $u \in SBV(\Omega)$ and $S_u \subset K \cup N$ with $\mathcal{H}_{n-1}(N) = 0$.*

In the one dimensional case the theorem is trivial, and $N = \emptyset$. The general case follows by theorem 3.2. In fact, let $\nu \in \mathbf{S}^{n-1}$ be a fixed direction; by theorem 3.2.22 of [B5] we get

$$+\infty > \mathcal{H}_{n-1}(K) \geq \int_{\pi_\nu} \mathcal{H}_0(\{t \in \Omega_x : x + t\nu \in K\})\,d\mathcal{H}_{n-1}(x).$$

In particular, $\mathcal{H}_0(\{t \in \Omega_x : x + t\nu \in K\}) < +\infty$ for $\mathcal{H}_{n-1}$-almost every $x \in \pi_\nu$, and, as a consequence $u_x \in SBV(\Omega_x)$ for $\mathcal{H}_{n-1}$-almost every $x \in \pi_\nu$. Since

$$|Du_x|(\Omega_x) \leq \int_{\Omega_x} |u_x'|\,dt + \int_{S_{u_x}} |u_x^+ - u_x^-|\,d\mathcal{H}_0 \leq \int_{\Omega_x} |u_x'|\,dt + 2\|u\|_\infty \mathcal{H}_0(\{t \in \Omega_x : x + t\nu \in K\})$$

integrating on π_ν and using again theorem 3.2 we get

$$\int_{\pi_\nu} |Du_x|\,d\mathcal{H}_{n-1}(x) \leq \int_{\Omega\setminus K} |\nabla u|\,dx + 2\|u\|_\infty \mathcal{H}_{n-1}(K),$$

and $u \in SBV(\Omega)$. Moreover, proposition 2.4 implies $\mathcal{H}_{n-1}(\Omega \cap S_u \setminus K) = 0$, which gives the required inclusion. In [A6] proposition 3.3 is proved without using theorem 3.2.

As a consequence, we get (recall also (3.5))

$$\inf\left\{\int_\Omega |\nabla u|^2 + \alpha\mathcal{H}_{n-1}(S_u) + \beta\int_\Omega |u - g|^2\,dx : u \in SBV(\Omega)\right\} \leq \tag{3.19}$$

$$\inf\left\{\int_{\Omega\setminus K} |\nabla u|^2 + \alpha\mathcal{H}_{n-1}(K) + \beta\int_{\Omega\setminus K} |u - g|^2\,dx : u \in W^{1,2}(\Omega \setminus K),\ K \subset \Omega \text{ closed}\right\}.$$

By means of a lower semicontinuity theorem we shall prove in §4 that the first infimum is actually a minimum. A regularity theorem in §5 will imply that both infima are attained, and are equal.

4. Lower semicontinuity.

In this section we look for conditions on f, φ which ensure the lower semicontinuity of the functional

$$\mathcal{F}(u) = \int_\Omega f(x, u, \nabla u)\, dx + \int_{S_u} \varphi(x, u^+, u^-, \nu)\, d\mathcal{H}_{n-1}(x) \tag{4.1}$$

in $SBV(\Omega)$ with respect to the topology $L^1_{\rm loc}(\Omega)$. We shall make assumptions (3.2), (3.4), (3.5). Hence, for sake of simplicity, we look for inequalities

$$\mathcal{F}(u) \le \liminf_{h\to+\infty} \mathcal{F}(u_h), \qquad u_h \to u \text{ in } L^1_{\rm loc}(\Omega),\ \|u_h\|_\infty \le C < +\infty. \tag{4.2}$$

In [A2] it is possible to find more general theorems. By the compactness theorem 3.1, since for the sequences in (4.2) there is weak convergence of approximate differentials and weak convergence of the jump parts of derivatives, we expect that the two terms in (4.1) can be studied separately.

In order to study the first part of the functional it is natural to ask which conditions on f ensure the sequential lower semicontinuity of the functional

$$I(u, v) = \int_\Omega f(x, u, v)\, dx \qquad (u, v) \in L^1(\Omega) \times L^1(\Omega; \mathbf{R}^n) \tag{4.3}$$

with respect to the product topology $L^1_{\rm loc}(\Omega) \times w - L^1(\Omega; \mathbf{R}^n)$. These kind of problem arise in optimal control theory; in this generality the problem has been completely solved by A.D.Ioffe in 1977 (see [C22], [C23]), who proved that I is sequentially lower semicontinuous if and only if

$$(s, p) \;\to\; f(x, s, p) \quad \text{is l.s.c. in } \mathbf{R}^{n+1}, \qquad p \;\to\; f(x, s, p) \quad \text{is convex and l.s.c. in } \mathbf{R}^n\ \forall s \in \mathbf{R} \tag{4.4}$$

for almost every $x \in \Omega$. The lower semicontinuity condition in (s, p) is very reasonable; in order to understand why convexity is necessary for lower semicontinuity with respect to a weak topology, let us consider the simple case $f(x, u, p) = \psi(p)$, and let $p = \lambda p_1 + (1-\lambda)p_2$ with $\lambda \in [0, 1]$. By an elementary procedure, it is possible to construct a sequence of Borel sets $A_h \subset \Omega$ such that $\chi_{A_h} \to \lambda$ weakly* in $L^\infty(\Omega)$, i.e.,

$$\lim_{h\to+\infty} \int_{A_h} z\, dx = \lambda \int_\Omega z\, dx \qquad \forall z \in L^\infty(\Omega).$$

We set $v = p$, $v_h = p_1\chi_{A_h} + p_2\chi_{\Omega\setminus A_h}$. Since v_h weakly converges to v, the lower semicontinuity inequality yields

$$\psi(p)|\Omega| = I(v)\, dx \le \liminf_{h\to+\infty} I(v_h) \le [\lambda\psi(p_1) + (1-\lambda)\psi(p_2)]|\Omega|$$

and dividing by $|\Omega|$ we get the convexity inequality. A similar proof shows that convexity in the gradient is a necessary condition for lower semicontinuity of integral functionals in Sobolev spaces (see [C24]). Heuristically, the sufficiency of the convexity condition is related to the following argument: since ψ is the supremum of affine functions, and since affine functionals are weakly continuous, the functional whose integrand is ψ is weakly lower semicontinuous. Unfortunately the implication

$$\psi = \sup_{h\in\mathbf{N}} \psi_h \qquad \Longrightarrow \qquad \int_\Omega \psi(u)\, dx = \sup_{h\in\mathbf{N}} \int_\Omega \psi_h(u)\, dx \tag{4.5}$$

is true only if ψ_h is monotonically increasing, and in general this is not the case when we approximate convex functions by affine functions. However, if we consider our functionals as measures, and we vary the domain of integration, an implication very likely to (4.5) holds, as the following fundamental lemma shows.

Lemma 4.1. *Let $\mu : \mathbf{B}(\Omega) \to [0,+\infty[$ be a measure, and let $f_h : \Omega \to [0,+\infty[$ be a sequence of Borel functions. Then*

$$\int_\Omega \sup_{h\in\mathbf{N}} f_h\, d\mu = \sup_{k\in\mathbf{N}}\left\{\sum_{i=1}^k \int_{A_i} f_i\, d\mu : A_i \subset \Omega \text{ open and disjoint}\right\}.$$

The proof of the lemma is very simple. By Beppo-Levi's monotone convergence theorem, we need only to show that

$$\int_\Omega \sup_{1\le h\le k} f_h\, d\mu = \sup\left\{\sum_{i=1}^k \int_{A_i} f_i\, d\mu : A_i \subset \Omega \text{ open and disjoint}\right\}$$

for every $k \in \mathbf{N}$. By the inner regularity of μ we get

$$\int_\Omega \sup_{1\le h\le k} f_h\, d\mu = \sup\left\{\sum_{i=1}^k \int_{B_i} f_i\, d\mu : B_i \subset \Omega \text{ Borel and disjoint}\right\} =$$

$$= \sup\left\{\sum_{i=1}^k \int_{K_i} f_i\, d\mu : K_i \subset \Omega \text{ compact and disjoint}\right\} = \sup\left\{\sum_{i=1}^k \int_{A_i} f_i\, d\mu : A_i \subset \Omega \text{ open and disjoint}\right\}.$$

Now we can give a sketch of a recent new proof of Ioffe's theorem (see [C3]). We first approximate the function f; in [C3] it is proved that for any Borel function f satisfying (4.4) there exists a sequence of Borel functions $a_h : \Omega\times\mathbf{R} \to \mathbf{R}$, $b_h : \Omega\times\mathbf{R} \to \mathbf{R}^n$ such that

$$f(x,s,p) = \sup_{h\in\mathbf{N}} a_h(x,s) + \langle b_h(x,s), p\rangle \qquad \forall (s,p) \in \mathbf{R}\times\mathbf{R}^n \tag{4.6}$$

and

$$s \to a_h(x,s), \qquad s \to b_h(x,s) \qquad \text{are continuous in } \mathbf{R} \tag{4.7}$$

for almost every $x \in \Omega$. The idea of this approximation is that for fixed (x,s), since $f(x,u,\cdot)$ is convex and lower semicontinuous, we can find a_h, b_h satisfying (4.6); and we need only to choose them in such a way that (4.7) is satisfied. This can be done by means of measurable and continuous selection theorems. In the approximation we have gained a continuous dependence on u (instead of lower semicontinuous) and a linear dependence on p (instead of a convex dependence). Since the sum of lower semicontinuous functionals is still lower semicontinuous, in order to prove the lower semicontinuity of $\mathcal{I}$ we need, by lemma 4.1, only to show that the functional

$$(u,v) \to \int_A [a_h(x,u) + \langle b_h(x,u), v\rangle]^+\, dx \tag{4.8}$$

is lower semicontinuous in $L^1(\Omega;\mathbf{R}^{n+1})$ for fixed open set $A \subset \Omega$ and $h \in \mathbf{N}$ ($[g]^+ = g \vee 0$ is the positive part of g). Replacing possibly a_h, b_h by

$$a_{h,k} = a_h \chi_{\{|a_h|+|b_h|<k\}}, \qquad b_{h,k}\chi_{\{|a_h|+|b_h|<k\}} \qquad (k \in \mathbf{N})$$

we easily see that it is not restrictive to assume that a_h, b_h are bounded. We get rid of the positive part by multiplying by a function g; we have infact

$$\int_A [a_h(x,u) + \langle b_h(x,u), v\rangle]^+\, dx = \sup\left\{\int_A (a_h(x,u) + \langle b_h(x,u), v\rangle) g\, dx : g \in C_0^1(A),\ 0 \le g \le 1\right\}$$

and each one of the functionals in the right hand side is continuous with respect to the product topology. In particular, all the functionals (4.8) are lower semicontinuous, and the Ioffe theorem is proved. A typical example of function f satisfying (4.4) is a strictly positive definite quadratic form in p whose coefficients are measurable in x and continuous in u (see (4.11) below).

In the proof of Ioffe's theorem we have seen that varying the domain of integration may be a powerful technique. Now, we see that this technique, combined with a slicing argument, yields a proof of the lower semicontinuity (in the sense of (4.2)) of the functional

$$\int_\Omega |\nabla u|^2\,dx + \alpha\mathcal{H}_{n-1}(S_u) \tag{4.9}$$

which occurs in the segmentation problems. We neglect the term $\beta\int_\Omega |u-g|^2\,dx$ because it is trivially lower semicontinuous, by the Fatou lemma. Let $\{u_h\} \subset SBV(\Omega)$ be a sequence converging to $u \in SBV(\Omega)$ in $L^1_{\rm loc}(\Omega)$, and let $A \subset \Omega$ be a fixed open set, $\nu \in \mathbf{S}^{n-1}$. We can assume (with the notations of (2.5)) that u_{hx} converges to u_x in $L^1_{\rm loc}(\Omega_x)$ for $\mathcal{H}_{n-1}$-almost every $x \in \pi_\nu$. By the elementary considerations made in the proof of the one dimensional version of the compactness theorem 3.1 we get

$$\int_{A_x} |u_x'|^2\,dt + \alpha\mathcal{H}_0(A_x \cap S_{u_x}) \le \liminf_{h\to+\infty}\int_{A_x} |u_{hx}'|^2\,dt + \alpha\mathcal{H}_0(A_x\cap S_{u_{hx}})$$

for $\mathcal{H}_{n-1}$-almost every $x \in \pi_\nu$. To avoid confusion with fixed the direction ν, we denote by ν_u the "normal" to S_u given by proposition 2.3. Integrating both members and using Fatou's lemma and theorem 3.2 we get

$$\int_A |\langle\nabla u,\nu\rangle|^2\,dx + \alpha\int_{A\cap S_u} |\langle\nu_u,\nu\rangle|\,d\mathcal{H}_{n-1} \le \liminf_{h\to+\infty}\int_A |\langle\nabla u_h,\nu\rangle|^2\,dx + \alpha\int_{A\cap S_{u_h}} |\langle\nu_{u_h},\nu\rangle|\,d\mathcal{H}_{n-1} \le$$

$$\le \liminf_{h\to+\infty}\int_A |\nabla u_h|^2\,dx + \alpha\mathcal{H}_{n-1}(A\cap S_{u_h}).$$

In the first inequality we have used the formula

$$\int_B |\langle\nu_B,\nu\rangle|\,d\mathcal{H}_{n-1} = \int_{\pi_\nu}\mathcal{H}_0(t\in\Omega_x : x+t\nu\in B\})\,d\mathcal{H}_{n-1}(x) \tag{4.10}$$

which is true for every countably rectifiable set B ([B5], 3.2.22). In (4.10), the unitary vector ν_B is related to B by the formula

$$\operatorname{ap\,Tan}(K,x) = \{v : \langle v,\nu_B(x)\rangle = 0\} \qquad \mathcal{H}_{n-1}\text{−a.e. in K}$$

for every set $K \subset B$ with $\mathcal{H}_{n-1}(K) < +\infty$. Let $D \subset \mathbf{S}^{n-1}$ be a countable dense set. By the previous inequality and by lemma 4.1 we get

$$\int_\Omega |\nabla u|^2\,dx + \alpha\mathcal{H}_{n-1}(S_u) = \sup_{k\in\mathbf{N}}\left\{\sum_{i=1}^k\int_{A_i}|\langle\nabla u,\nu_i\rangle|^2\,dx + \alpha\int_{A_i\cap S_u}|\langle\nu_u,\nu_i\rangle|\,d\mathcal{H}_{n-1} : \nu_i\in D,\ A_i \text{ open disjoint}\right\} \le$$

$$\le \sup_{k\in\mathbf{N}}\left\{\sum_{i=1}^k \liminf_{h\to+\infty}\int_{A_i}|\nabla u_h|^2\,dx + \alpha\mathcal{H}_{n-1}(A_i\cap S_{u_h})\,d\mathcal{H}_{n-1} : A_i \text{ open disjoint}\right\} \le$$

$$\le \sup_{k\in\mathbf{N}}\left\{\liminf_{h\to+\infty}\sum_{i=1}^k\int_{A_i}|\nabla u_h|^2\,dx + \alpha\mathcal{H}_{n-1}(A_i\cap S_{u_h}) : A_i \text{ open disjoint}\right\} \le \liminf_{h\to+\infty}\int_\Omega|\nabla u_h|^2\,dx + \alpha\mathcal{H}_{n-1}(S_{u_h}),$$

and the lower semicontinuity is proved.

We want to find criteria of lower semicontinuity for functionals more general than (4.9), for instance

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx + \int_{S_u} \delta(u^+,u^-)\psi(\nu)\,d\mathcal{H}_{n-1}. \tag{4.11}$$

For the gradient part of the functional in (4.11), by Ioffe's theorem, it is sufficient to require that the functions $a_{ij}(x,s)$ are measurable in x, continuous in s, and

$$\sum_{i,j=1}^n a_{ij}(x,s)p_ip_j \ge c|p|^2 \qquad \forall (x,s)\in\Omega\times\mathbf{R},\ p\in\mathbf{R}^n$$

with $c > 0$. The slicing argument cannot be applied to the functionals (4.11), mainly because of the dependence of the jump part on ν. We have seen that convexity in the gradient is a necessary condition of semicontinuity in Sobolev spaces. In our case, the jump part of derivative is representable in the form (recall (2.14))

$$Ju = (u^+ - u^-)\nu\cdot\mathcal{H}_{n-1}|_{S_u},$$

therefore we must look for a sort of "joint convexity" of $\varphi(u,v,\nu)$ in ν and in the pair (u,v). To state this property conveniently, we need an auxiliary space, the space $\mathcal{M}_0$ of measures $\mu:\mathbf{B}(\mathbf{R})\to\mathbf{R}^n$ of finite total variation such that $\mu(\mathbf{R})=0$. We endow $\mathcal{M}_0$ with the weakest topology σ such that the mappings

$$L_h(\mu)=\sum_{i=1}^n\int_{\mathbf{R}} h^{(i)}\,d\mu_i \qquad h:\mathbf{R}\to\mathbf{R}^n \text{ continuous and bounded}$$

are continuous. We denote by $\Delta\subset\mathcal{M}_0$ the set

$$\Delta=\{\mu\in\mathcal{M}_0:\mu=p(\delta_u-\delta_v),\ p\in\mathbf{R}^n,\ u,\,v\in\mathbf{R}\}$$

and by $\mathrm{co}(\Delta)$ the convex hull of Δ. By the Hahn-Banach theorem, $\mathrm{co}(\Delta)$ is dense in $\mathcal{M}_0$, because each continuous linear functional which vanishes on Δ corresponds to a constant function. To each integrand $\varphi(u,v,\nu)$ satisfying the condition

$$\varphi(u,v,\nu)=\varphi(v,u,-\nu) \tag{4.12}$$

we can associate a function $\hat\varphi:\Delta\to\mathbf{R}$ by

$$\hat\varphi(\mu)=\varphi(u,v,\frac{p}{|p|})|p| \qquad \text{if } \mu=p(\delta_u-\delta_v),\ p\neq 0.$$

We say that φ is a **biconvex integrand** if it satisfies (4.12) and the function $\hat\varphi$ admits a convex, positively 1-homogeneous and σ-lower semicontinuous extension to $\mathcal{M}_0$. We remark that (4.12) is not restrictive: since in (4.1) $u^+>u^-$, only the values of φ in the domain

$$\{(u,v,\nu)\in\mathbf{R}\times\mathbf{R}\times\mathbf{S}^{n-1}:u>v\}$$

are important; hence (4.12) can be considered a definition of $\varphi(u,v,\nu)$ when $u<v$.

Now we can state the main theorem of these lectures.

Theorem 4.2. *Let $f:\Omega\times\mathbf{R}\times\mathbf{R}^n\to[0,+\infty[$ be a Borel function satisfying conditions (3.2), (4.4), and let $\varphi:\Omega\times\mathbf{R}\times\mathbf{R}\times\mathbf{S}^{n-1}\to[0,+\infty[$ be a bounded continuous function satisfying (3.4), (4.12) and*

$$\varphi(x,\cdot,\cdot,\cdot) \qquad \text{is biconvex for every } x\in\Omega.$$

Then, the functional $\mathcal{F}$ in (4.1) satisfies the lower semicontinuity property (4.2). In particular, if $\mathcal{F}$ satisfies also condition (3.5), then has a minimizer in $SBV(\Omega)$.

The lower semicontinuity of the jump part of functionals in theorem 4.2 is proved by an approximation procedure similar to (4.6). Assuming for simplicity that φ does not depend on x, by using the Hahn-Banach theorem in $\mathcal{M}_0$, we can find a sequence of bounded continuous functions $V_h : \mathbf{R} \to \mathbf{R}^n$ such that

$$\varphi(u,v,\nu) = \sup_{h\in\mathbf{N}} \langle V_h(u) - V_h(v), \nu\rangle \qquad \forall (u,v,\nu) \in \mathbf{R}\times\mathbf{R}\times\mathbf{S}^{n-1},\ u \neq v.$$

Then, using lemma 4.1 we are led to prove the lower semicontinuity of functionals

$$u \quad\to\quad \int_{A\cap S_u} \langle V_h(u^+) - V_h(u^-), \nu\rangle^+ \, d\mathcal{H}_{n-1}. \tag{4.13}$$

The functional in (4.13) is closely related to the jump part of derivative of $V_h(u)$, if V_h is smooth. Then, by smoothing V_h and integrating by parts in very small neighbourhoods A of the negligible set

$$\bigcup_{h\in\mathbf{N}} S_{u_h} \cup S_u$$

(in this way we have a very small contribution of the approximate differentials in A) we conclude the proof.

We remark that the integrand φ of segmentation problems is trivially biconvex, because the extension of $\hat{\varphi}$ is simply the total variation. Now, let us see how the biconvexity condition, which is very abstract, can be checked for particular class of integrands, namely the integrands in (4.11).

Proposition 4.3. *Assume that δ is symmetric and continuous, and $\delta(u,u) = 0$ $\forall u \in \mathbf{R}$. Assume also that $\psi : \mathbf{R}^n \to [0,+\infty[$ is a positively 1-homogeneous function satisfying the condition $\psi(-p) = \psi(p)$. Assuming that neither ψ nor δ are identically 0, the integrand*

$$\varphi(u,v,\nu) = \delta(u,v)\psi(\nu)$$

in (4.11) is biconvex if and only if δ is a distance in $\mathbf{R}$, i.e.,

$$\delta(u,v) \le \delta(u,w) + \delta(w,v) \qquad \forall u,\ v,\ w \in \mathbf{R} \tag{4.14}$$

and ψ is convex, i.e.,

$$\psi(p_1 + p_2) \le \psi(p_1) + \psi(p_2) \qquad \forall p_1,\ p_2 \in \mathbf{R}^n. \tag{4.15}$$

Moreover, condition (4.14), (4.15) are necessary for lower semicontinuity.

If φ is biconvex, then (4.14), (4.15) are necessarily true, because $\hat{\varphi}$ is convex in Δ. Conversely, if φ fulfils (4.14), (4.15) we get

$$\varphi(u,v,\nu) = \sup_{c,z} \varphi_{c,z}(u,v,\nu) = \sup_{c,z} \big(\delta(u,c) - \delta(v,c)\big)\langle z, \nu\rangle \tag{4.16}$$

where c varies in the rational numbers and z varies in the subdifferential of ψ

$$\{z \in \mathbf{R}^n : \psi(p) \ge \langle z, p\rangle \quad \forall p \in \mathbf{R}^n\}.$$

It can be easily seen that

$$L_{c,z}(\mu) = \sum_{i=1}^n \int_{\mathbf{R}} \delta(c,v) z_i \, d\mu_i(v)$$

is a convex σ-continuous extension of $\hat{\varphi}_{c,z}$ to $\mathcal{M}_0$. In particular, φ is biconvex. Now we picture briefly the proof of the necessity of conditions (4.14), (4.15) for lower semicontinuity. Assume that $0 \in \Omega$. Let ν such that $\psi(\nu) \neq 0$; since any function w jumping between u and v along the hyperplane π_ν can be approximated by functions w_h jumping between u and w, w and v along π_ν, $\pi_\nu + 2^{-h}\nu$ respectively, writing the lower semicontinuity inequality we get (4.14). Let $p = p_1 + p_2$, and let ν_1, ν_2 be the unitary vectors corresponding to p_1, p_2 respectively. Let w be jumping between u and v as before (we assume that $\delta(u,v) \neq 0$), and let w_h be jumping between u and v along a polyhedral set whose faces are normal to ν_1, ν_2 alternately. The inequality (4.15) follows by the lower semicontinuity inequality on this particular sequence.

It is to be noted that the integrands in proposition 4.3 fail to satisfy growth condition (3.4) because they are continuous and $\delta(u,u) = 0$. However, many function φ satisfying (3.4) are monotonically approximable by this class of integrands, for example

$$\varphi(u,v,\nu) = \Theta(|u-v|)\psi(\nu)$$

with $\Theta : [0,+\infty[\to]0,+\infty[$ concave and non decreasing (we simply approximate Θ by $\Theta_k(t) = \Theta(t) \wedge kt$).

For the class of integrands in proposition 4.3 necessary and sufficient conditions of lower semicontinuity have been found. In [A2] a characterization of integrands φ whose functional is lower emicontinuous has been identified. This condition, called BV-ellipticity, is very close to the ellipticity conditions of Geometric Measure Theory, is not easy to be checked and it is formulated by an integral inequality. The problem of equivalence between biconvexity and BV- ellipticity is still completely open.

Regularity of minimizers and first variation.

We have shown in §4 that the problem

$$(5.1)\qquad \min\left\{\mathcal{F}(u)=\int_\Omega |\nabla u|^2\,dx+\alpha\mathcal{H}_{n-1}(S_u)+\beta\int_\Omega |u-g|^2\,dx : u\in SBV(\Omega)\right\}\qquad (g\in L^\infty(\Omega))$$

has at least one solution. In this section we shall be concerned with the properties of minimizers of (5.1). Since the regularity of minimizers is trivial in the case $n=1$, we assume that $n\geq 2$. We begin with the following result, recently proved by De Giorgi-Carriero-Leaci (see [A6]).

Theorem 5.1. *Let $u\in SBV(\Omega)$ be a solution of problem (5.1). Then,*

$$(i)\qquad u\in L^\infty(\Omega),\qquad \|u\|_\infty\leq\|g\|_\infty;$$

$$(ii)\qquad u\in W^{2,p}_{\mathrm{loc}}(\Omega\setminus\overline{S_u})\ \forall p\in[1,+\infty[\quad \textit{and}\quad \Delta u=\beta(u-g)\quad \textit{in } \Omega\setminus\overline{S_u};$$

$$(iii)\qquad \textit{the function}\quad \tilde u(x)=u^+(x)=u^-(x)\quad \textit{belongs to } C^1(\Omega\setminus\overline{S_u});$$

$$(iv)\qquad \mathcal{H}_{n-1}(\Omega\cap\overline{S_u}\setminus S_u)=0.$$

The proof of the first three statements is not very difficult; the hardest part of theorem 5.1 is (iv). The first statement can be proved by the same truncation argument of §3, (3.5). Let $\varphi\in C^1_0(\Omega\setminus\overline{S_u})$ be an arbitrary function; the inequality

$$\liminf_{\epsilon\to 0}\frac{\mathcal{F}(u+\epsilon\varphi)-\mathcal{F}(u)}{|\epsilon|}\geq 0$$

implies

$$\int_\Omega\langle\nabla u,\nabla\varphi\rangle\,dx=-\beta\int_\Omega(u-g)\varphi\,dx$$

so that u is a weak solution of the equation in (ii). By the regularity theory for the Dirichlet problem (see [C1]), it follows that $u\in W^{2,p}_{\mathrm{loc}}(\Omega\setminus\overline{S_u})$ for every $p\in[1,+\infty[$. By the Sobolev embedding theorems, it suffices to take any $p>n$ to show that u has a C^1 representative which must coincide with the approximate limit $\tilde u$.

Now, we sketch the main ideas involved in the proof of (iv). Some examples coming from the theory of elliptic problems in domains with corners suggest that S_u is not necessarily a closed set (see [A10], section 3). The idea of (iv) consists mainly in the proof of the following statements:

$$(5.2)\qquad \text{the sets } \Omega_0=\left\{x\in\Omega:\limsup_{\rho\to0^+}\rho^{1-n}\Big[\int_{B_\rho(x)}|\nabla u|^2\,dx+\mathcal{H}_{n-1}(S_u\cap B_\rho(x))\Big]=0\right\}\text{ and } S_u \text{ are disjoint}$$

for all $u\in SBV(\Omega)$, and

$$(5.3)\qquad \text{for each minimizer } u \text{ of (5.1) the set } \Omega_0 \text{ is open.}$$

Let us first see how (iv) follows by (5.2), (5.3). Let $\Gamma=\Omega\setminus\Omega_0$; the set Γ is relatively closed in Ω. We can write this set as

$$\Gamma=\bigcup_{\epsilon>0}\Gamma_\epsilon$$

where the sets Γ_ϵ are defined by

$$\Gamma_\epsilon = \{x \in \Omega : \limsup_{\rho \to 0^+} \rho^{1-n} [\int_{B_\rho(x)} |\nabla u|^2 \, dx + \mathcal{H}_{n-1}(S_u \cap B_\rho(x))] \geq \epsilon\}.$$

Since

$$\Gamma_\epsilon \subset \{x \in \Omega : \limsup_{\rho \to 0^+} \rho^{-n} [\int_{B_\rho(x)} |\nabla u|^2 \, dx + \mathcal{H}_{n-1}(S_u \cap B_\rho(x))] = +\infty\},$$

the set Γ_ϵ is negligible; moreover, by a general lemma concerning Hausdorff measures (see for instance [B5], 2.10.19(3)), it follows that

$$\int_B |\nabla u|^2 \, dx + \mathcal{H}_{n-1}(B \cap S_u) \geq \epsilon \mathcal{H}_{n-1}(B \cap \Gamma_\epsilon) \qquad \forall \epsilon > 0, \ \forall B \in \mathbf{B}(\Omega).$$

In particular, by taking $B = \Gamma_\epsilon \setminus S_u$ we get

$$\mathcal{H}_{n-1}(\Gamma_\epsilon \setminus S_u) = 0,$$

and since $\epsilon > 0$ is arbitrary, $\mathcal{H}_{n-1}(\Gamma \setminus S_u) = 0$. Hence, S_u contains almost all of Γ; since $S_u \subset \Gamma$, and since Γ is relatively closed, we get also

$$\Omega \cap \overline{S_u} \setminus S_u \subset \Gamma \setminus S_u,$$

which implies (iv). The proof of statement (5.2) is based on a very nice generalisation to $SBV(\Omega)$ of Poincaré-Wirtinger inequality. It is also possible to show that in each point $x \in \Omega_0$ the approximate limit is finite. Statement (5.2) is true for all the funcions in $SBV(\Omega)$, not necessarily minimizers. On the other hand, statement (5.3) heavily depends on minimality. To better describe the main lemma involved in the proof of (5.3) we need some notation. For every compact set $K \subset \Omega$, we set

$$F(u, \alpha, K) = \int_K |\nabla u|^2 \, dx + \alpha \mathcal{H}_{n-1}(S_u \cap K)$$

and

$$\Phi(u, \alpha, K) = \inf \Big\{ F(u, \alpha, K) : v \in SBV(\Omega), \ v = u \ \in \Omega \setminus K \Big\}.$$

We have of course $\Phi \leq F$; the quantity

$$\Psi(u, \alpha, K) = F(u, \alpha, K) - \Phi(u, \alpha, K)$$

is often called deviation from minimality, or excess. In the proof of regularity theorems, the deviation from minimality plays an essential role, and it is also an useful device to state regularity properties which are not affected by perturbations of the functional not depending on derivative. It is important to have a fast decay of the deviation from minimality as the diameter of K goes to 0: for instance, let us assume that Ω is smooth and let $v \in SBV(\mathbf{R}^n \setminus \overline{\Omega}) \cap L^\infty(\mathbf{R}^n \setminus \overline{\Omega})$. It is not difficult to see that the "Dirichlet" problem

$$\min \Big\{ \int_\Omega |\nabla u|^2 \, dx + \alpha \mathcal{H}_{n-1}(\Omega \cap S_u) : u \in SBV_{\text{loc}}(\mathbf{R}^n), \ u = v \text{ in } \mathbf{R}^n \setminus \overline{\Omega} \Big\}$$

has a solution, and the deviation from minimality is 0 for sets K inside Ω. For the segmentation problem, we have a perturbation $\beta \int_\Omega |u - g|^2 \, dx$. Since u, $g \in L^\infty(\Omega)$, it is not difficult to show the following result:

Proposition 5.2. *Let $u \in SBV(\Omega)$ be a minimizer of (5.1). Then, we have*

$$\Psi(u, \alpha, \overline{B}_\rho(x)) \leq 2\|g\|_\infty^2 \omega_n \rho^n \qquad \forall x \in \Omega, \ \forall \rho \in]0, \text{dist}(x, \partial\Omega)[.$$

The following fundamental result shows that if F, Ψ are small enough, then F has a decay faster than ρ^{n-1} as $\rho \downarrow 0$.

Theorem 5.3. *Let $\gamma \in]0,1[$. Then, there exist two universal constants $\epsilon(n,\alpha,\gamma)$, $\theta(n,\alpha,\gamma)$ such that if $\rho > 0$, $\overline{B}_\rho(x) \subset \Omega$, $u \in SBV(\Omega)$ with*

$$F(u,\alpha,\overline{B}_\rho(x)) \leq \epsilon\rho^{n-1}, \qquad \Psi(u,\alpha,\overline{B}_\rho(x)) \leq \theta F(u,\alpha,\overline{B}_\rho(x))$$

then

$$F(u,\alpha,\overline{B}_{\rho/2}) \leq \left(\frac{1}{2}\right)^{n-\gamma} F(u,\alpha,\overline{B}_\rho(x)).$$

The theorem is proved by contradiction: if the theorem is not true it is possible to find $\gamma \in]0,1[$, $\alpha > 0$ and sequences ϵ_h, θ_h, ρ_h, x_h, u_h such that $\epsilon_h \downarrow 0$, $\theta_h \downarrow 0$, $\overline{B}_{\rho_h}(x_h) \subset \Omega$,

$$F(u_h,\alpha,\overline{B}_{\rho_h}(x_h)) = \epsilon_h\rho_h^{n-1}, \qquad \Psi(u_h,\alpha,\overline{B}_{\rho_h}(x_h)) \leq \theta_h F(u_h,\alpha,\overline{B}_{\rho_h}(x_h),$$

and

$$F(u_h,\alpha,\overline{B}_{\rho_h/2}(x_h)) > (\frac{1}{2})^{n-\gamma} F(u_h,\alpha,\overline{B}_{\rho_h}(x_h)).$$

Rescaling and translating we obtain a sequence v_h of functions in $SBV(B_1)$ such that $F(v_h,\alpha/\epsilon_h,\overline{B}_1) = 1$, $\psi(v_h,\alpha/\epsilon_h,\overline{B}_1) \leq \theta_h$, and

$$F(v_h,\frac{\alpha}{\epsilon_h},\overline{B}_{1/2}) > (\frac{1}{2})^{n-\gamma}.$$

The idea is that since $\alpha/\epsilon_h \uparrow +\infty$ and the excesses in B_1 go to 0, in the limit the functions v_h behave as an harmonic function v; since

$$\int_{B_{1/2}} |\nabla v|^2\,dx \leq \left(\frac{1}{2}\right)^n \int_{B_1} |\nabla v|^2\,dx$$

for each harmonic function (see for instance [C21], page 80), a contradiction is found.

Now, it can be seen that in the points x where theorem 5.3 can be applied iteratively, we have a fast decay of F, hence $x \in \Omega_0$; the conditions to begin this iteration are a decay of Ψ faster than ρ^{n-1} and the existence of ρ such that

$$F(u,\alpha,\overline{B}_\rho(x)) < \frac{\epsilon}{2}\rho^{n-1}.$$

If $x \in \Omega_0$, the above condition holds for x and for all the points in a small neighbourhood of x, hence the small neighbourhood is contained in Ω_0. Since in our case the decay of Ψ is ρ^n, the set Ω_0 is open for each minimizer of the functional (5.1).

Theorem 5.1 shows that any solution of the segmentation problem in SBV is also a solution of the formulation of the same problem in terms of closed sets and $W^{1,2}$ functions, hence both infima in (3.19) are equal. The set K can be taken exactly equal to the closure of S_u in Ω. The natural question is now: what can we say about the regularity of K? This problem is still completely open, however some conjectures can be done. Some regularity results for free boundary problems proved by Alt-Caffarelli (see [C10]), and all the regularity theory for minimizing currents suggest that it might exist a "singular" closed set $N \subset K$ such that $\mathcal{H}_{n-1}(N) = 0$ and K is locally a C^1 hypersurface outside N. A stronger conjecture is that actually $\mathcal{H}_{n-2}(K) < +\infty$. If this stronger conjecture were true, in the case $n = 2$ we would have a jump set K made by a finite number of points joined by C^1 arcs, and this set would effectively give a segmentation of the domain Ω.

We recall also a paper by D.Mumford and J.Shah were, under some regularity assumptions on K, the kind of singularity which can occur are classified, and an asymptotic expansion of the minimizers near the points in N is given. In the same paper it is also investigated the behaviour of the solutions of the problems (5.1) as $\beta \downarrow 0$ or $\beta \uparrow +\infty$.

Now, we want to compute the first variation of the functional in (5.1). The idea is to get informations about the geometry of K by considering variations $u_\epsilon(x) = u(x+\epsilon\eta)$, where $\eta \in C_0^1(\Omega;\mathbf{R}^n)$ and

$$x \quad \to \quad \Phi_\epsilon(x) = x + \epsilon\eta(x)$$

is a smooth diffeomorphism of Ω for $|\epsilon|$ small enough. We shall assume also for simplicity that $g \in C^1(\Omega)$; weaker conditions as $g \in BV(\Omega)$ can be considered, but there are technical difficulties mainly due to the intersection (which certainly might be non empty) of S_u and S_v. Arguing essentially as in [A4] it is possible to show the following result:

Proposition 5.4. *Let $u \in SBV(\Omega)$ be a minimizer of (5.1), and let $\eta \in C_0^1(\Omega;\mathbf{R}^n)$. Then*

$$(5.4) \qquad \int_\Omega \Big[\big(|\nabla u|^2 + \beta u(u-2g)\big)\mathrm{div}\eta - 2\beta u\langle\eta,\nabla g\rangle - 2\langle\nabla u, \nabla\eta\nabla u\rangle\Big]\,dx + \alpha\int_{S_u}\mathrm{div}_{S_u}(\eta)\,d\mathcal{H}_{n-1} = 0$$

where

$$\mathrm{div}_{S_u}(\eta) = \sum_{i=1}^n \delta_i\eta^{(i)} = \sum_{i=1}^n\left(\frac{\partial\eta^{(i)}}{\partial x_i} - \nu_i\langle\nabla\eta^{(i)},\nu\rangle\right)$$

is the tangential divergence of η.

Equality (5.4) is a direct consequence of the minimality condition

$$\liminf_{\epsilon\to 0}\frac{\mathcal{F}(u_\epsilon) - \mathcal{F}(u)}{|\epsilon|} \geq 0.$$

The non trivial part of the proof is the estimate

$$\mathcal{H}_{n-1}(S_{u_\epsilon}) = \mathcal{H}_{n-1}(\Phi_\epsilon^{-1}(S_u)) = -\int_{S_u}\delta_i\eta^{(i)}\,d\mathcal{H}_{n-1} + o(|\epsilon|),$$

whose proof, based on area formulas, is available for instance in [B9]. Even if the functional $\mathcal{F}$ is not smooth or convex, equation (5.4) can be regarded as the Euler equation of $\mathcal{F}$. Since we have made no regularity assumptions on S_u, this equation embodies information about the geometry of S_u and its relation with ∇u in an implicit form. Now, we see how, under some regularity assumptions on S_u, equation (5.4) implies an explicit relation between the mean curvature of S_u and the jump of $|\nabla u|^2 + \beta(u-g)^2$.

Theorem 5.5. *Let $B \subset \Omega$ be a ball such that $S_u \cap B$ is the graph of a C^2 function, and let B_+, B_- be its open epigraph and hypograph respectively. Denoting by the subscripts $+$, $-$ the traces on S_u on the side corresponding to B_+ and the side corresponding to B_-, we have*

$$(5.5) \qquad \left(\frac{\partial u}{\partial\nu}\right)_+ = \langle(\nabla u)_+,\nu\rangle = 0, \qquad \left(\frac{\partial u}{\partial\nu}\right)_- = \langle(\nabla u)_-,\nu\rangle = 0 \qquad \mathcal{H}_{n-1}\text{-a.e. in } B\cap S_u,$$

and

$$(5.6) \qquad \left(|\nabla u|_+^2 + \beta(u_+-g)^2\right) - \left(|\nabla u|_-^2 + \beta(u_- - g)^2\right) = -\alpha(n-1)H \qquad \mathcal{H}_{n-1}\text{-a.e. in } B\cap S_u,$$

$H = \sum_{i=1}^n \delta_i\nu^{(i)}/(n-1)$ being the mean curvature of S_u.

By localizing, we can assume that $u \in H^2(B_+\cup B_-)$ (see for instance [C7]), and there exists a C^1 mapping $\nu : B \to \mathbf{S}^{n-1}$ normal to S_u in $B\cap S_u$ and pointing in B_-. Let $\varphi : B \to \mathbf{R}$ be a function such that $\varphi = 0$ in B_- and $\varphi \in C^1(\overline{B}_+)$. By the minimality of u, arguing as in the proof of theorem 5.1(ii) we get

$$\int_{B_+}\langle\nabla u,\nabla\varphi\rangle\,dx = -\beta\int_{B_+}\varphi(u-g)\,dx$$

and using the equation $\Delta u = \beta(u-g)$ and the Gauss-Green theorem we obtain

$$\int_{B\cap S_u} \varphi\langle(\nabla u)_+, \nu_+\rangle \, d\mathcal{H}_{n-1} = 0,$$

where ν_+ is the outer normal to B_+ in B. Since φ is arbitrary, we get the first of (5.5); the second one can be proved by the same argument.

Let us prove (5.6). Let $\xi \in C_0^1(B)$, $\eta = \xi\nu$. By (5.5) and the Gauss-Green formula we get

$$0 = 2\int_{B_+} \operatorname{div}\big(\nabla u\langle\nabla u, \eta\rangle\big)\, dx.$$

Expanding the divergence, and using the equality $\Delta u = \beta(u-g)$ we have

$$0 = 2\beta\int_{B_+} (u-g)\langle\nabla u, \eta\rangle\, dx + \int_{B_+} \langle\eta, \nabla(|\nabla u|^2)\rangle\, dx + 2\int_{B_+} \langle\nabla u, \nabla\eta\nabla u\rangle\, dx.$$

Now, we use the identity $\nabla\big(u(u-2g)\big) = 2(u-g)\nabla u - 2u\nabla g$ and we get

$$0 = \int_{B_+} \langle\nabla\big(\beta u(u-2g) + |\nabla u|^2\big), \eta\rangle\, dx + 2\int_{B_+} \langle\nabla u, \nabla\eta\nabla u\rangle\, dx + 2\beta\int_{B_+} u\langle\nabla g, \eta\rangle\, dx.$$

The first integral can be evaluated using the divergence theorem in B_+:

$$0 = \int_{B\cap S_u} \big(\beta u_+(u_+-2g) + |\nabla u|_+^2\big)\xi\, d\mathcal{H}_{n-1} - \int_{B_+} \Big[\big(\beta u(u-2g) + |\nabla u|^2\big)\operatorname{div}\eta - 2\langle\nabla u, \nabla\eta\nabla u\rangle\Big]\, dx + 2\beta\int_{B_+} u\langle\nabla g, \eta\rangle\, dx.$$

Similarly, we have also

$$0 = -\int_{B\cap S_u} \big(\beta u_-(u_--2g) + |\nabla u|_-^2\big)\xi\, d\mathcal{H}_{n-1} - \int_{B_-} \Big[\big(\beta u(u-2g) + |\nabla u|^2\big)\operatorname{div}\eta - 2\langle\nabla u, \nabla\eta\nabla u\rangle\Big]\, dx + 2\beta\int_{B_-} u\langle\nabla g, \eta\rangle\, dx.$$

Adding these two equations and using proposition 5.4 we obtain the following equality

$$\int_{B\cap S_u} \Big(\big(|\nabla u|_+^2 + \beta(u_+-g)^2\big) - \big(|\nabla u|_-^2 + \beta(u_--g)^2\big)\Big)\xi\, d\mathcal{H}_{n-1} = -\alpha\int_{B\cap S_u} \operatorname{div}_{S_u}(\eta)\, d\mathcal{H}_{n-1}.$$

Since

$$\operatorname{div}_{S_u}(\eta) = \sum_{i=1}^n \delta_i\eta^{(i)} = \xi\sum_{i=1}^n \delta_i\nu^{(i)} + \langle\nu, \big(\nabla\xi - \langle\nabla\xi, \nu\rangle\nu\big)\rangle = \xi\sum_{i=1}^n \delta_i\nu^{(i)} = (n-1)\xi H,$$

we get

$$\int_{B\cap S_u} \Big(\big(|\nabla u|_+^2 + \beta(u_+-g)^2\big) - \big(|\nabla u|_-^2 + \beta(u_--g)^2\big)\Big)\xi\, d\mathcal{H}_{n-1} = -\alpha(n-1)\int_{B\cap S_u} H\xi\, d\mathcal{H}_{n-1}.$$

Since $\xi \in C_0^\infty(B)$ is arbitrary, (5.6) is true $\mathcal{H}_{n-1}$-almost everywhere in $B\cap S_u$, and this concludes the proof of the theorem.

6. Approximation with elliptic functionals via Γ-convergence.

In this section we want to show a way to approximate the functional of segmentation

$$\int_\Omega |\nabla u|^2\,dx + \alpha \mathcal{H}_{n-1}(S_u) + \beta \int_\Omega |u-g|^2\,dx \tag{6.1}$$

by means of elliptic functionals defined in Sobolev spaces. In order to do this we first need a precise definition of approximation for functionals. We shall first deal with this problem in an abstract setting.

Let X be a separable metric space, and let

$$\mathcal{S}(X) = \{f\colon X \to [0,+\infty]: \quad \text{such that } f \text{ is lower semicontinuous}\}.$$

We want to define in $\mathcal{S}(X)$ a variational convergence satisfying the condition

$$f_h \ \to\ f \quad \Longrightarrow \quad \inf_{x\in X} f_h(x) \ \to\ \inf_{x\in X} f \tag{6.2}$$

in case the functions f_h are equicoercive, in the sense that for every $t \geq 0$ there exists $K_t \subset X$ compact such that

$$\{x \in X : f_h(x) \leq t\} \subset K_t \qquad \forall h \in \mathbf{N}. \tag{6.3}$$

Condition (6.2), very natural from the point of view of Calculus of Variations, shows that pointwise convergence topology, that is, the topology inherited from $[0,+\infty]^X$ is not enough; moreover, $\mathcal{S}(X)$ is not closed in $[0,+\infty]^X$. On the other hand, it turns out that for the major part of the applications uniform convergence is a too strong topology, even if it fulfils (6.2).

In the 70's (see [C9], [C15], [C16], [C17]) De Giorgi has introduced in $\mathcal{S}(X)$ a variational convergence, called **Γ-convergence**, which is particularly useful in problems of convergence of integral functionals. In the French literature, Γ-convergence is also called epi-convergence (see [C6]). The definition is the following. Let $\{f_h\} \subset \mathcal{S}(X)$, and let $f \in \mathcal{S}(X)$; we say that f_h Γ-converges to f, and we write

$$f_h \ \underset{\Gamma}{\to}\ f, \qquad f = \Gamma - \lim_{h\to+\infty} f_h$$

if

$$\begin{cases} \forall\, x_h \to x & \liminf\limits_{h\to+\infty} f_h(x_h) \geq f(x); \\ \exists\, x_h \to x & \text{such that } \limsup\limits_{h\to+\infty} f_h(x_h) \leq f(x), \end{cases} \tag{6.4}$$

for all $x \in X$. In the next proposition we summarize the main properties of Γ-convergence.

Proposition 6.1. *Let $\{f_h\} \subset \mathcal{S}(X)$. Then,*

(*i*) $$f = \Gamma - \lim_{h\to+\infty} f_h \qquad \textit{if exists is unique;}$$

(*ii*) *there exists a subsequence $f_{h_k}, f \in \mathcal{S}(X)$ such that $f = \Gamma - \lim\limits_{h\to+\infty} f_{h_k}$;*

(*iii*) $$f = \Gamma - \lim_{h\to+\infty} f_h \quad \Longrightarrow \quad f + g = \Gamma - \lim_{h\to+\infty} f_h + g \qquad \textit{for all } g : X \to \mathbf{R} \textit{ continuous;}$$

(*iv*) *if $f = \Gamma - \lim\limits_{h\to+\infty} f_h$ and f_h satisfy (6.3), then (6.2) is satisfied; let $\{x_h\} \subset X$ be a sequence such that*

$$\lim_{h\to+\infty} f_h(x_h) = \lim_{h\to+\infty} \min_{x\in X} f_h(x).$$

Then every point in the compact set

$$K = \bigcap_{h \in \mathbf{N}} \overline{\bigcup_{k \geq h} \{x_k\}}$$

is a minimizer of f;
(v) If X is compact, there exists a distance δ in $\mathcal{S}(X)$ such that $(\mathcal{S}(X), \delta)$ is a compact metric space, and

$$\lim_{h \to +\infty} \delta(f_h, f) = 0 \qquad \Longleftrightarrow \qquad f_h = \Gamma - \lim_{h \to +\infty} f.$$

Now we briefly describe some of the best applications of the theory of Γ-convergence.

Example 1. Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and let Λ, λ be strictly positive constants. Let $\mathcal{E} \subset \mathcal{S}(L^2(\Omega))$ be the set of functionals $F : L^2(\Omega) \to [0, +\infty]$ such that

$$F(u) = \begin{cases} \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx & \text{if } u \in H_0^1(\Omega); \\ +\infty & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega), \end{cases}$$

with the Borel matrix a_{ij} simmetric and such that

$$\Lambda |p|^2 \geq \sum_{i,j=1}^n a_{ij}(x) p_i p_j \geq \lambda |p|^2 \qquad \forall x \in \Omega,\ p \in \mathbf{R}^n.$$

The set $\mathcal{E}$ is sequentially compact in $\mathcal{S}(L^2(\Omega))$ with respect to Γ-convergence. In particular, problems modeled on the homogeneisation, that is, find the "limit" as ϵ goes to 0 of the functionals

$$F_\epsilon(u) = \int_\Omega \sum_{i,j=1}^n a_{ij}\left(\frac{x}{\epsilon}\right) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx \qquad a_{ij}(x) \text{ periodic}$$

can be studied by the tools of Γ-convergence. There is also a close relation betweeen Γ-convergence of functionals in $\mathcal{E}$ and convergence of the linear operators corresponding to the Euler equation.
Let $F \in \mathcal{E}$, $\{F_h\} \subset \mathcal{E}$, and let us denote by a_{ij}, $a_{ij}^{(h)}$ the coefficients corresponding to F, F_h respectively. Given $w \in H^{-1}$, we denote by $L_h(w)$, $L(w) \in H_0^1(\Omega)$ the solutions of the elliptic problem

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}^{(h)}(x) \frac{\partial u}{\partial x_j} \right) = w, \qquad \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = w,$$

Then, we have the implication:

$$F_h = \Gamma - \lim_{h \to +\infty} F \qquad \Longleftrightarrow \qquad L_h(w) \to L(w) \quad \text{weakly in } H_0^1(\Omega) \text{ for all } w \in H^{-1}.$$

The implication $\Rightarrow$ is straightforward consequence of proposition 6.1 (iii), (iv): $L_h(w)$ are the solutions of the problems

$$\min \left\{ \int_\Omega \sum_{i,j=1}^n a_{ij}^{(h)}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx + \langle w, u \rangle : u \in H_0^1(\Omega) \right\}.$$

The opposite implication is shown in [C15]. A convergence theory for the functionals in $\mathcal{E}$ based on the convergence of Euler's equations has been developed by S.Spagnolo some years before Γ-convergence ([C30], [C31]). The major advantage of Γ-convergence is that it makes sense even if the Euler equation is missing, as it happens for instance in the case of non-smooth functionals. In our case, the functional in (6.1) is not convex and the Euler's conditions of §5 don't give us complete information about the functional, therefore Γ-convergence provides the right way of formulating the approximation problem.

Example 2. (See [C25]) The free energy of some fluids (called Cahn-Hilliard fluids) depends on the density u of the fluid, and it has the form

$$W(u) = \Theta(u) + L(u)$$

with $\Theta \geq 0$, $L(t) = at + b$ affine function,

$$\Theta(u) = 0 \qquad \Longleftrightarrow \qquad u = \alpha \quad \text{or} \quad u = \beta$$

and $\alpha < \beta$. The configurations of minimal energy with a mass constraint

$$\min\left\{\int_\Omega W(u)\,dx : u:\Omega \to [0,+\infty[\quad \text{Borel,} \quad \int_\Omega u\,dx = K\right\}$$

($\alpha|\Omega| < K < \beta|\Omega|$) are all the Borel functions $u : \Omega \to \{\alpha, \beta\}$ such that

$$\alpha|\{u=\alpha\}| + \beta|\{u=\beta\}| = K \tag{6.5}$$

because

$$\int_\Omega W(u)\,dx = \int_\Omega \Theta(u)\,dx + aK + b|\Omega|$$

for each admissible function u. On the other hand the observed equilibrium positions of the fluid satisfy condition (6.5), but there is also a smooth interface with constant mean curvature betweeen the regions $\{u = \alpha\}$, $\{u = \beta\}$. To explain this phenomenon, it has been suggested the introduction of a small smoothing term to the energy functional to be minimized:

$$F_\epsilon(u) = \begin{cases} \int_\Omega \left[\Theta(u) + \epsilon^2|\nabla u|^2\right]dx & \text{if } u \in W^{1,2}(\Omega),\ u \geq 0 \text{ a.e., } \int_\Omega u\,dx = K; \\ +\infty & \text{otherwise.} \end{cases}$$

Recently (see [C26]), L.Modica and S.Mortola have proved that the the functionals F_ϵ/ϵ Γ-converge in $S(L^1(\Omega))$ to the functional

$$F(u) = \begin{cases} c(\alpha,\beta)\mathcal{H}_{n-1}(\partial^*\{u=\alpha\}\cap\Omega) & \text{if } u \in \{\alpha,\beta\} \text{ a.e., } \int_\Omega u\,dx = K; \\ +\infty & \text{otherwise,} \end{cases}$$

where $c(\alpha,\beta) = 2\int_\alpha^\beta \sqrt{\Theta}(s)\,ds$. By the mass constraint, minimizing F is equivalent to minimize

$$\mathcal{H}_{n-1}(\partial^* A \cap \Omega)$$

over all sets $A \in \mathbf{B}(\Omega)$ such that

$$|A| = \frac{\beta|\Omega| - K}{\beta - \alpha} \in]0, |\Omega|[$$

and the minimizers of this functionals (recall example 4, §1) are really observed.

Example 2 has been the starting point of our research: in fact, it shows that the Γ-limit of elliptic functionals may be a completely different functional, namely, an area-like functional. Now, let us turn to our original problem. For reasons which will be clear later, we need an extra variable $v \in [0,1]$; it can be considered, as we shall see, a sort of control variable on the gradient of u. We fix an arbitrary constant $C > 0$ and we set

$$X = \left\{(u,v) \in L^\infty(\Omega;\mathbf{R}^2) : 0 \leq v \leq 1 \text{ a.e. in } \Omega,\ \|u\|_\infty \leq C\right\},$$

and

$$\mathcal{F}(u,v)=\begin{cases}\int\limits_\Omega |\nabla u|^2\,dx+\mathcal{H}_{n-1}(S_u) & \text{if } u\in SBV(\Omega),\ v=0;\\ +\infty & \text{otherwise.}\end{cases}$$

We endow X with the $L^1_{\rm loc}$ topology. By theorem 4.2, $\mathcal{F}\in\mathcal{S}(X)$. Now we define the approximating functionals $\mathcal{F}_\epsilon$; formally they are defined by

$$\mathcal{F}_\epsilon(u,v)=\int\limits_\Omega\left[\left(|\nabla u|^2+|\nabla v|^2\right)(1-v^2)^{2/\epsilon}+\frac{\alpha^2}{4\epsilon^2}v^2\right]dx.$$

These functional are well defined in $W^{1,2}(\Omega)\times W^{1,2}(\Omega)\cap X$; unfortunately, they are not coercive in this space, because the term multiplying the gradients can approach 0. Let $\varphi_\epsilon:[0,1]\to\mathbf{R}$ be the function

$$\varphi_\epsilon(t)=\int_0^t(1-s^2)^{1/\epsilon}\,ds.$$

If $w=\varphi_\epsilon(v)$, the chain rule for approximate differentials implies that the equality

$$\nabla w=(1-v^2)^{1/\epsilon}\nabla v \tag{6.6}$$

makes sense if $w\in W^{1,2}(\Omega)$, and this condition is weaker than $v\in W^{1,2}(\Omega)$. By a similar argument, we see that

$$|\nabla u|(1-v^2)^{1/\epsilon}\quad\text{belongs to}\quad L^2(\Omega)$$

if $u(1-v^2)^{1/\epsilon+1}\in W^{1,2}(\Omega)$. Hence, we set

$$X_\epsilon=\left\{(u,v)\in X:\varphi_\epsilon(v)\in W^{1,2}(\Omega),\ u(1-v^2)^{1/\epsilon+1}\in W^{1,2}(\Omega)\right\}$$

and

$$\mathcal{F}_\epsilon(u,v)=\begin{cases}\int\limits_\Omega\left[\left(|\nabla u|^2+|\nabla v|^2\right)(1-v^2)^{2/\epsilon}+\frac{\alpha^2}{4\epsilon^2}v^2\right]dx & \text{if } (u,v)\in X_\epsilon;\\ +\infty & \text{otherwise.}\end{cases}$$

By using (6.6) and similar chain rule for differentials, it can be seen that the functionals $\mathcal{F}_\epsilon$ are coercive; moreover, by the slicing techniques of §4 it is possible to show that these functionals belong to $\mathcal{S}(X)$. Now, we can state our approximation result, proved in a joint work with V.M.Tortorelli (see [A3]):

Theorem 6.2. *Let $\epsilon_h\subset]0,+\infty[$ be any sequence converging to 0. The functionals $\mathcal{F}_{\epsilon_h}$ Γ-converge in $\mathcal{S}(X)$ to $\mathcal{F}$ and satisfy the equi-coercivity condition (6.3). In particular, let $g\in L^\infty(\Omega)$ and let $(u_{\epsilon_h},v_{\epsilon_h})$ be solutions of the problems*

$$\min\left\{\int\limits_\Omega\left[\left(|\nabla u|^2+|\nabla v|^2\right)(1-v^2)^{2/\epsilon_h}+\frac{\alpha^2}{4\epsilon_h^2}v^2\right]dx+\beta\int\limits_\Omega|u-g|^2\,dx\right\}.$$

Then, all the function u in the (non-empty) set

$$K=\bigcap_{h\in\mathbf{N}}\overline{\bigcup_{k\geq h}\{u_{\epsilon_k}\}}$$

minimize the functional

$$\mathcal{F}(u)=\int\limits_\Omega|\nabla u|^2\,dx+\alpha\mathcal{H}_{n-1}(S_u)+\beta\int\limits_\Omega|u-g|^2\,dx.$$

We sketch very briefly the proof of theorem 6.2. The inequality

$$\liminf_{\epsilon\to 0^+} \mathcal{F}_\epsilon(u_\epsilon) \geq \mathcal{F}(u)$$

can be proved first in the one-dimensional case, and then, by the same arguments of §4, in the general case. To prove that the second condition in (6.4) is fulfilled, we begin with the following approximation lemma:

Lemma 6.3. *Let $u \in SBV(\Omega)\cap L^\infty(\Omega)$ be a function such that $|\nabla u| \in L^2(\Omega)$ and $\mathcal{H}_{n-1}(S_u) < +\infty$. Then, there exists a sequence of functions $u_h \in SBV(\Omega)\cap L^\infty(\Omega)$ such that*

$$(i) \qquad \lim_{h\to+\infty} \int_\Omega |u-u_h|^2\,dx = 0;$$

$$(ii) \qquad \lim_{h\to+\infty} \int_\Omega |\nabla u_h - \nabla u|^2\,dx = 0, \qquad \lim_{h\to+\infty} \mathcal{H}_{n-1}(S_{u_h}) = \mathcal{H}_{n-1}(S_u);$$

$$(iii) \qquad \mathcal{H}_{n-1}(S_{u_h}) = \mathcal{H}_{n-1}(\overline{S}_{u_h}\cap\Omega) \qquad \forall h \in \mathbf{N}.$$

This lemma is a straightforward consequence of the regularity theorem of §5: it suffices to take the solution u_h of the problem

$$\min\left\{\int_\Omega |\nabla w|^2\,dx + \mathcal{H}_{n-1}(S_w) + h\int_\Omega |w-u|^2 : w \in SBV(\Omega)\right\}.$$

By this lemma and by a diagonal argument it can be seen that we need only to construct the sequence (u_ϵ, v_ϵ) for the functions u such that $\mathcal{H}_{n-1}(\Omega\cap\overline{S}_u\setminus S_u) = 0$. Since the functions u_ϵ must approximate u, the integral

$$\int_\Omega |\nabla u_\epsilon|^2\,dx$$

cannot be controlled near S_u. Therefore we need the control variable v_ϵ equal to 1 near S_u, so that no cost is paid by $|\nabla u_\epsilon|$ in that region. On the other hand, since

$$\mathcal{F}_\epsilon(u_\epsilon, v_\epsilon) \geq \frac{1}{4\epsilon^2}\int_\Omega v_\epsilon^2\,dx,$$

the functions v_ϵ must converge to 0 very fastly; hence, far from S_u the function v_ϵ can be equal to 0. We need to specify how v_ϵ moves between 0 and 1. Even in this case, the idea is to solve first the problem in one dimension, and after in the general case. Let $n=1$; since $\mathcal{F}_\epsilon$ can be, by Cauchy's inequality, minorized by

$$\frac{\alpha}{\epsilon}\int_0^1 |v||v'|(1-v^2)^{1/\epsilon}\,dt,$$

the functions v_ϵ must behave as approximate solutions γ_ϵ of the problem

$$\inf\left\{\frac{\alpha}{\epsilon}\int_0^a |\gamma||\gamma'|(1-\gamma^2)^{1/\epsilon}\,dt : a>0,\ \gamma\in C^1([0,a]),\ \gamma(0)=0,\ \gamma(a)=1\right\}.$$

The infimum above is not attained and is equal to $\alpha/2$. Since for every jump of u, the function v_ϵ must go between 0 and 1 two times, the cost is exactly α. In more than one dimension the argument is similar: the functions v_ϵ are defined as

$$v_\epsilon(x) = \gamma_\epsilon\big(d(x,\overline{S}_u)\big)$$

where γ_ϵ are the functions of the one-dimensional case, and the cost paid in the transition between 0 and 1 is $\mathcal{H}_{n-1}(\overline{S}_u\cap\Omega) = \mathcal{H}_{n-1}(S_u)$.

A:References on segmentation problems.

[A1] **L.Ambrosio:** *A compactness theorem for a special class of functions of bounded variation.* To appear in: "Boll. Un. Mat. It."

The main compactness theorem and existence of weak solutions in $SBV(\Omega)$ for the segmentation problem:

$$\min\left\{\int_\Omega |\nabla u|^2\,dx + \alpha\mathcal{H}_{n-1}(S_u) + \beta\int_\Omega |u-g|^2\,dx\right\}.$$

[A2] **L.Ambrosio:** *Existence theory for a new class of variational problems.* To appear.

Compactness and lower semicontinuity for functionals

$$\int_\Omega f(x,u,\nabla u)\,dx + \int_{S_u} \varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1} \tag{1}$$

defined in $SBV(\Omega;\mathbf{R}^k)$, and equivalence, using [A6], of the weak formulation of segmentation problems and the problem

$$\min\left\{\int_{\Omega\setminus K} |\nabla u|^2\,dx + \alpha\mathcal{H}_{n-1}(K) + \beta\int_{\Omega\setminus K} |u-g|^2\,dx\right\}. \tag{2}$$

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