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EXISTENCE THEORY FOR A NEW CLASS OF VARIATIONAL PROBLEMS[1]

by

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Introduction.

The aim of this paper is to present a mathematical theory where problems of the type

$$(0.1)\qquad \inf\left\{\int_\Omega f(x,u,\nabla u)\,dx+\int_{S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x)\right\}\qquad u:\Omega\subset\mathbf{R}^n\to\mathbf{R}^k$$

can be stated and have a solution. In (0.1), ∇u is a differential, S_u is a set of discontinuities which depends on u and is not prescribed a priori, u^+, u^- are asymptotic values of u near a discontinuity point, ν_u is normal to S_u, and $\mathcal{H}_{n-1}$ is the Hausdorff $(n-1)$-dimensional measure.
The canonical example of functional in (1) is

$$(0.2)\qquad \int_\Omega\left[|\nabla u|^2+\alpha|u-w|^2\right]dx+\beta\mathcal{H}_{n-1}(S_u)\qquad(\alpha,\,\beta>0,\,w\in L^\infty(\Omega)).$$

In the case $n=2$ this functional has been recently proposed by Mumford-Shah (see [32], [33]) to study, by a variational approach, problems of image segmentation. In this case the function w represents an image given by a camera. By the minimization of the functional (0.2) it is possible to detect the "real" discontinuities of w, and the discontinuities due only to the digitalization process are smoothed. We prove that the functional (0.2) admits at least one minimum; a recent result of De Giorgi-Carriero-Leaci (see [15]) shows that any solution u of the minimization problem is C^1 outside $\overline{S}_u$, and $\mathcal{H}_{n-1}(\overline{S}_u\cap\Omega\setminus S_u)=0$.
In order to give a precise mathematical meaning to the minimization problem (0.1), the first problem we face is to specify a class of functions such that ∇u, u^+, u^-, ν_u exist, at least in an approximate sense. To do this we follow the ideas of [18]. Since we are concerned with sets of "jumps" S_u, it is natural to think as domain of the functional in (0.1) the space $BV(\Omega;\mathbf{R}^k)$ of functions u such that all the k components $u^{(j)}$ are functions of bounded variation. Unfortunately, even if the functional in (0.1) is well defined on $BV(\Omega;\mathbf{R}^k)$, it is not coercive in this space. The reason of this phenomenon is the following. The distributional derivative Du of a function $u\in BV(\Omega;\mathbf{R}^k)$ can be split into three parts (see [4]):

$$(0.3)\qquad Du=\nabla u\cdot\mathcal{L}_n+(u^+-u^-)\otimes\nu_u\cdot\mathcal{H}_{n-1}|_{S_u}+Cu.$$

The first term in (0.3) corresponds to the absolutely continuous part of Du with respect to Lebesgue n-dimensional measure $\mathcal{L}_n$. The second term is a $(n-1)$-dimensional measure, because S_u is σ-finite with respect to $\mathcal{H}_{n-1}$. On the contrary, the measure Cu is "diffuse" and singular with respect to $\mathcal{L}_n$, and it may have support on sets which have Hausdorff dimension between $n-1$ and n. Recalling the well known function of Cantor-Vitali, we have called in [18] the measure Cu "Cantor" part of the derivative Du, because for this function $Du=Cu$ is a measure whose support is Cantor's middle third set.
Since the functionals in (0.1) control only the n-dimensional part and the $(n-1)$-dimensional part of the derivative, we have defined in [18] the "special" functions of bounded variation as the functions $u\in BV(\Omega;\mathbf{R}^k)$ such that $Cu=0$ in (0.3). We use the notation $SBV(\Omega;\mathbf{R}^k)$ to denote this space of functions. In many cases the functionals in (0.1) are coercive in $SBV(\Omega;\mathbf{R}^k)$ (see theorem 2.1). In some cases coerciveness may fail (see example 5.3), and an enlargemnet of $SBV(\Omega;\mathbf{R}^k)$ is needed, setting

$$GSBV(\Omega;\mathbf{R}^k)=\left\{u:\Omega\to\mathbf{R}^k:\phi(u)\in SBV_{\mathrm{loc}}(\Omega;\mathbf{R}^k)\ \text{for every}\ \phi\in C^1(\mathbf{R}^k)\ \text{with}\ \mathrm{supp}(\nabla\phi)\subset\subset\mathbf{R}^k\right\}.$$

In §1 we show that the functionals in (0.1) are well defined in $GSBV(\Omega;\mathbf{R}^k)$. In the next section we prove that coerciveness is ensured provided the following conditions are satisfied

$$f(x,u,p)\geq|u|^\alpha+|p|^\beta,\qquad\varphi(x,u,v,\nu)\geq c\wedge|u-v|^\gamma$$

with $\alpha>0$, $\beta>1$, $\gamma<1$, $c>0$. The proof of the coerciveness follows by a compactness theorem proved in [4] in the scalar case $(k=1)$.
In §3 we investigate about the semicontinuity of functionals in (0.1). The main difficulty arises from the term

$$\int_{S_u}\varphi(x,u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x)$$

and we focus our attention mainly on this integral. Following the results of [6], which are relative to the restriction of the functionals to the set

$$\{u \in SBV(\Omega;\mathbf{R}^k) : u(x) \in T \text{ a.e. in } \Omega\}$$

with $T \subset \mathbf{R}^k$ of finite cardinality, we identify two conditions on φ related to semicontinuity. The first one, named BV- ellipticity resembles Morrey quasi-convexity ([12], [31]) and it is necessary and sufficient for semicontinuity, at least if φ is a continuous function. This condition, however, is not easy to be handled, because is given by an integral inequality. To overcome this difficulty, we have found a condition which is sufficient for semicontinuity, and which can be easily checked in many practical cases (see example 5.1). The condition is the following: there exists a convex and weakly* lower semicontinuous function $\tilde{\varphi}$ defined on the space of Radon measures $\mu : \mathbf{B}(\mathbf{R}^k) \to \mathbf{R}^n$ such that

$$\varphi(a, b, \frac{p}{|p|})|p| = \tilde{\varphi}(p(\delta_a - \delta_b))$$

whenever $a, b \in \mathbf{R}^k$, $p \in \mathbf{R}^n \setminus \{0\}$, and $a \neq b$. This condition is trivially satisfied by the integrand in (0.2), the function $\tilde{\varphi}$ being simply equal to $\beta/2$ times the total variation.
In §4, putting together the results of §2 and of §3, we prove by the direct method of the Calculus of Variations some existence theorems for minimization problem (0.1).
The last section is devoted to the discussion of some examples. Firstly, we see how BV-ellipticity and biconvexity can be easily identified and turn out to be equivalent conditions for some special classes of integrands. In particular, we show that our results yield a solution of the segmentation problem (0.2).
We discuss also about the problem of findind the "best" piecewise affine function near a given function $w \in L^2(\Omega)$, by minimizing the functional

$$\int_\Omega |\nabla u|^2\, dx + \beta \mathcal{H}_{n-1}(S_{\nabla u}) + \alpha \int_\Omega |u - w|^2\, dx.$$

Finally, we consider minimization problems of the type

$$\min_{u,\Omega}\left\{ \int_\Omega f(x, u, \nabla u)\, dx + \int_{\partial\Omega} \psi(u^*, \nu)\, d\mathcal{H}_{n-1}(x)\right\}$$

(u^* denotes the trace on the boundary) and we suggest weak formulation of these problems in $SBV(\Omega;\mathbf{R}^k)$ which lead to the study of relaxed functionals. Problems of this type may occur in connection with the static theory of liquid crystals ([18], [13], [21], [26]).

1. Approximate limits, approximate differentials and functions of bounded variation.

We first state the notations frequently used in this paper. We denote by $\mathcal{L}_n$ the Lebesgue n-dimensional measure in $\mathbf{R}^n$ and by $\mathcal{H}_{n-1}$ the Hausdorff $(n-1)$-dimensional measure in $\mathbf{R}^n$. Let $\Omega \subset \mathbf{R}^n$ be an open set; we denote by $\mathbf{B}(\Omega)$ the σ-algebra of Borel subsets of Ω. We set also $|E| = \mathcal{L}_n(E)$ for every Borel set $E \subset \mathbf{R}^n$, and we denote by $\mathcal{L}_{n,k}$ the space of linear mappings $L : \mathbf{R}^n \to \mathbf{R}^k$.
In this section we shall give a precise mathematical definition of functionals of the type

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx + \int_{S_u} \varphi(x, u^+, u^-, \nu)\, d\mathcal{H}_{n-1}(x)$$

where ∇u is a differential, S_u is the set of jumps, u^+, u^- are asymptotic values near the jump points, and ν_u is normal to S_u. To do this, we need first to define a notion of limit "up to negligible sets" for Borel functions.

Let (E,d) be a compact metric space, let $\Omega \subset \mathbf{R}^n$ be an open set, and let $x \in \Omega$, $F \in \mathbf{B}(\Omega)$ such that

$$|B_\rho(x) \cap F| > 0 \quad \forall \rho > 0. \tag{1.1}$$

We say that $z \in E$ is the approximate limit in x of a Borel function $u : \Omega \to E$ in the domain F and we write

$$z = \operatorname{ap} \lim_{\substack{y \to x \\ y \in F}} u(y),$$

if

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x) \cap F} g(u(y))\,dy}{|B_\rho(x) \cap F|} = g(z) \qquad \forall g : E \to \mathbf{R},\ g \text{ continuous.} \tag{1.2}$$

We denote by $C(E)$ the algebra of continuous real valued functions defined in E. Since the algebra $C(E)$ separates points, the approximate limit if exists is unique. We set also

$$S_u = \Big\{x \in \Omega : \text{does not exist} \quad \operatorname{ap} \lim_{\substack{y \to x \\ y \in \Omega}} u(y)\Big\}, \tag{1.3}$$

and we denote for simplicity by $\tilde{u} : \Omega \setminus S_u \to E$ the function

$$\tilde{u}(x) = \operatorname{ap} \lim_{\substack{y \to x \\ y \in \Omega}} u(y). \tag{1.4}$$

In the following proposition we list the main properties of approximate limits (the most important, for our purposes, is given in (v)) and we show the equivalence of our definition with other ones existing in the literature (see [22], [39]).

Proposition 1.1. *Let $x \in \Omega$, $F \in \mathbf{B}(\Omega)$ satisfying (1.1), and let $u : \Omega \to E$ be a Borel function. The following six statements hold:*

(*i*)
$$z = \operatorname{ap} \lim_{\substack{y \to x \\ y \in F}} u(y) \iff \lim_{\rho \to 0^+} \frac{|\{y \in B_\rho(x) \cap F : d(u(y),z) < \epsilon\}|}{|B_\rho(x) \cap F|} = 1 \quad \forall \epsilon > 0;$$

(*ii*)
$$z = \operatorname{ap} \lim_{\substack{y \to x \\ y \in F}} u(y) \iff \lim_{\rho \to 0^+} \frac{\int_{B_\rho(x) \cap F} d(u(y),z)\,dy}{|B_\rho(x) \cap F|} = 0;$$

(*iii*)
$$z = \operatorname{ap} \lim_{\substack{y \to x \\ y \in F}} u(y) \Rightarrow h(z) = \operatorname{ap} \lim_{\substack{y \to x \\ y \in F}} h(u)(y)$$

for every compact metric space $(\tilde{E}, \tilde{d})$ and every continuous function $h : E \to \tilde{E}$;

(*iv*)
$$z = \operatorname{ap} \lim_{\substack{y \to x \\ y \in F}} u(y) \iff z = \operatorname{ap} \lim_{\substack{y \to x \\ y \in F}} v(y)$$

for every Borel function $v : \Omega \to E$ such that

$$\lim_{\rho \to 0^+} \frac{|\{y \in B_\rho(x) \cap F : u(y) = v(y)\}|}{|B_\rho(x) \cap F|} = 1;$$

(v) Let $\mathcal{F} \subset C(E)$ be a set of functions which separates points. Then

$$\exists \ \operatorname{ap}\lim_{\substack{y\to x\\ y\in F}} u(y) \iff \exists \ \operatorname{ap}\lim_{\substack{y\to x\\ y\in F}} \phi(u)(y) \quad \forall \phi\in\mathcal{F};$$

(vi) S_u is a Borel set, $|S_u| = 0$, $\tilde{u} : \Omega\setminus S_u \to E$ is a Borel function and $\tilde{u}(y) = u(y)$ for a.e. $y\in\Omega\setminus S_u$.

Proof. (i) Assume that $z\in E$ is the approximate limit, and let $g_\epsilon \in C(E)$ such that $0\le g_\epsilon\le 1, g_\epsilon(z) = 1$ and $g_\epsilon = 0$ outside $B(z,\epsilon)$. We have

$$\liminf_{\rho\to 0^+} \frac{|\{y\in B_\rho(x)\cap F : d(u(y),z)<\epsilon\}|}{|B_\rho(x)\cap F|} \ge \liminf_{\rho\to 0^+} \frac{\int\limits_{B_\rho(x)\cap F} g_\epsilon(u(y))\,dy}{|B_\rho\cap F|} = g_\epsilon(z) = 1.$$

Conversely, let $z\in E$ such that

$$\lim_{\rho\to 0^+} \frac{|\{y\in B_\rho(x)\cap F : d(u(y),z)<\epsilon\}|}{|B_\rho(x)\cap F|} = 1 \quad \forall \epsilon>0,$$

and let $g\in C(E)$. Since

$$\frac{\int\limits_{B_\rho(x)\cap F} |g(u(y)) - g(z)|\,dy}{|B_\rho(x)\cap F|} \le 2\|g\|_\infty \frac{|\{y\in B_\rho(x)\cap F : d(u(y),z)>\epsilon\}|}{|B_\rho(x)\cap F|} + \sup\{|g(w)-g(z)| : d(z,w)\le\epsilon\},$$

by letting first $\rho\to 0^+$ and then $\epsilon\to 0^+$ we get

$$\lim_{\rho\to 0^+} \frac{\int\limits_{B_\rho(x)\cap F} g(u(y))\,dy}{|B_\rho(x)\cap F|} = g(z).$$

(ii) follows by (i), and (iii), (iv) are straightforward consequence of the definitions.

(v) By (iii), the implication $\Rightarrow$ is trivial. Since the approximate limit commutes with sums and products, there is no loss of generality if we assume that $\mathcal{F}$ is an algebra of functions. Assume that in $x\in\Omega$ every function $\phi(u)$ with $\phi\in\mathcal{F}$ has an approximate limit t_ϕ and set for $g\in C(E), \rho>0$

$$L(g,\rho) = \frac{\int\limits_{B_\rho(x)\cap F} g(u(y))\,dy}{|B_\rho(x)\cap F|}.$$

Since $\mathcal{F}$ is dense in $C(E)$, the function $L(g,\rho)$ admits a limit as $\rho\to 0^+$ for every $g\in C(E)$, because this is true for every function $g\in\mathcal{F}$. By Riesz's representation theorem, there exists a probability measure μ in E such that

$$\lim_{\rho\to 0^+} L(g,\rho) = \int\limits_E g\,d\mu \qquad \forall g\in C(E) \tag{1.5}$$

By (1.5), our statement will be proved if μ is a Dirac measure δ_z for some $z\in E$. To prove this, we put in (1.5) $g = \psi(\phi)$ for some $\psi\in C(\mathbf{R}), \phi\in\mathcal{F}$ and we get

$$\psi(t_\phi) = \int_E \psi(\phi)\,d\mu. \tag{1.6}$$

Equality (1.6) remains true by approximation even for semicontinuous bounded functions ψ. In particular, taking $\psi_\phi(t) = 1$ if $t = t_\phi$ and $\psi_\phi(t) = 0$ otherwise, we obtain $\mu(\phi^{-1}(t_\phi)) = 1$ for every $\phi\in\mathcal{F}$. If $\mathcal{F}'$ is a countable dense subfamily of $\mathcal{F}$, setting

$$K = \bigcap_{\phi\in\mathcal{F}'} \phi^{-1}(t_\phi)$$

we find that $\mu(K) = 1$; on the other hand, since $\mathcal{F}'$ separates points in E, K contains exactly one point $z \in E$, that is, μ is a Dirac measure.

(vi) We prove first the statement under the assumption that E is a closed interval of the real line $\mathbf{R}$ endowed with the Euclidean distance. Under this assumption, it can be seen that (see [22] , 2.9.12)

$$S_u = \{x \in \Omega : u_-(x) < u_+(x)\}$$

where $u_-, u_+ : \Omega \to \mathbf{R}$ are the Borel functions ([22], 4.5.9) defined by

$$u_-(x) = \sup\Big\{t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{|B_\rho(x) \cap \{y \in \Omega : u(y) < t\}|}{|B_\rho(x)|} = 0\Big\},$$

$$u_+(x) = \inf\Big\{t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{|B_\rho(x) \cap \{y \in \Omega : u(y) > t\}|}{|B_\rho(x)|} = 0\Big\}.$$

In particular, S_u is a Borel set. In addition, it is well known that

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x)} |u(x) - u(y)|\, dy = 0 \quad \text{for a.e. } x \in \Omega,$$

and by (ii) we obtain that $|S_u| = 0$ and $\tilde{u} = u$ almost everywhere. Now we turn to the general case. By (iii) and (v) one gets

$$S_u = \bigcup_{\phi \in \mathcal{F}} S_{\phi(u)};$$

$$\phi(\tilde{u}) = \phi(u) \quad \text{a.e. in } \Omega \setminus S_u \quad \forall \phi \in \mathcal{F};$$

for every countable dense family of functions $\mathcal{F}$. In particular, S_u is a negligible Borel set and $\tilde{u} = u$ almost everywhere. Finally, $\tilde{u}$ is a Borel function because

$$\big\{x \in \Omega \setminus S_u : \tilde{u}(x) \in B(z, r)\big\} = \Big\{x \in \Omega \setminus S_u : \liminf_{\rho \to 0^+} \frac{|B_\rho(x) \cap \{y \in \Omega : d(u(y), z) < r\}|}{|B_\rho(x)|} = 1\Big\}$$

for every $z \in E$, $r > 0$, and

$$x \quad \longrightarrow \quad \liminf_{\rho \to 0^+} \frac{|B_\rho(x) \cap B|}{|B_\rho(x)|}$$

is a Borel function for every set $B \in \mathbf{B}(\Omega)$. **q.e.d.**

In the following sections we shall deal with functions $u : \Omega \to \mathbf{R}^k$. We take $\tilde{\mathbf{R}}^k = \mathbf{R}^k \cup \{\infty\}$ the one point compactification of $\mathbf{R}^k$ and we consider these functions as functions with values in $\tilde{\mathbf{R}}^k$. The set S_u and $\tilde{u}$ are defined as in (1.3), (1.4); in particular, the function $\tilde{u}$ is allowed to take the value ∞, but the set $\{\tilde{u} = \infty\}$ is negligible.

Using the same ideas, it is possible to define approximate differentials. Let $u : \Omega \to \mathbf{R}^k$ be a Borel function, and let $x \in \Omega \setminus S_u$ such that $\tilde{u}(x) \neq \infty$. We say that a linear mapping $L \in \mathcal{L}_{n,k}$ is the approximate differential of u at x if

$$\operatorname*{ap\,lim}_{\substack{y \to x \\ y \in \Omega}} \frac{|u(y) - \tilde{u}(x) - \langle L, y - x\rangle|}{|y - x|} = 0.$$

The approximate differential if exists is unique, and we shall denote it by $\nabla u(x)$. The following proposition lists the most useful properties of approximate differentials.

Proposition 1.2. *Let $u : \Omega \to \mathbf{R}^k$ be a Borel function and let $x \in \Omega \setminus S_u$ with $\tilde{u}(x) \neq \infty$. The following four statements hold:*

(i) if u is approximately differentiable at x then for every function $\phi \in C^1(\mathbf{R}^k)$ the function $\phi(u)$ is approximately differentiable at x and

$$\nabla(\phi(u))(x) = \nabla\phi(\tilde{u}(x))\nabla u(x);$$

(ii) if $v : \Omega \to \mathbf{R}^k$ is a Borel function and

$$\lim_{\rho\to 0+} \frac{|B_\rho(x) \cap \{y \in \Omega : u(y) = v(y)\}|}{|B_\rho(x)|} = 1,$$

then $x \in \Omega \setminus S_v$, $\tilde{u}(x) = \tilde{v}(x)$, u is approximately differentiable at x if and only if v is approximately differentiable at x and the approximate differentials are equal;

(iii) the function u is approximately differentiable at y if and only if all the k components $u^{(i)}$ are differentiable;

(iv) the set $\nabla_u = \{y \in \Omega \setminus S_u : \tilde{u}(x) \neq \infty, \exists\, \nabla u(y)\}$ belongs to $\mathbf{B}(\Omega)$ and $\nabla u : \nabla_u \to \mathcal{L}_{n,k}$ is a Borel function.

Proof. (i) It is formally similar to the proof of the classical chain rule for derivatives.

(ii) It is a straightforward consequence of proposition 1.1(iv).

(iii) Trivial.

(iv) By (iii), we can assume that $k = 1$. By [20], theorem 19 and theorem 21, both the statements are true if the set

$$G = \big\{(x,\omega) \in (\Omega \setminus S_u) \times \mathcal{L}_{n,k} :\ \tilde{u}(x) \neq \infty, \text{ there exists the approximate differential in } x \text{ and } \omega = \nabla u(x)\big\}$$

belongs to $\mathbf{B}(\Omega \times \mathcal{L}_{n,k})$. Let $x \in \Omega \setminus S_u$, $\omega \in \mathcal{L}_{n,k}$, $g \in C(\tilde{\mathbf{R}})$, $\rho > 0$ and let $W(x,\omega,g,\rho)$ be defined by

$$W(x,\omega,g,\rho) = \int_\Omega \phi\Big(\frac{|x-y|}{\rho}\Big) g\Big(\frac{|u(y) - \tilde{u}(x) - \omega(y-x)|}{|y-x|}\Big)\, dy.$$

where $\phi(t) = 1$ if $0 \le t < 1$ and $\phi(t) = 0$ otherwise. By the definition of approximate limit we get

$$G = \bigcap_{g\in\mathcal{F}} \bigcap_{n\in\mathbf{N}} \bigcup_{p\in\mathbf{N}} \bigcap_{\rho\in]0,\frac{1}{p}[\cap\mathbf{Q}} \{(x,\omega) \in (\Omega \setminus S_u) \times \mathcal{L}_{n,k} : \tilde{u}(x) \neq \infty,\ W(x,\omega,g,\rho) < \frac{1}{n}\}$$

for every countable dense family of functions $\mathcal{F} \subset C^+(\tilde{\mathbf{R}})$. Since for every $g \in C(\tilde{\mathbf{R}})$ and every $\rho > 0$ the mapping $W(\cdot,\cdot,g,\rho)$ is a Borel function, G is a Borel set. **q.e.d.**

We denote by $BV(\Omega)$ the space of functions $u \in L^1(\Omega)$ such that the distributional derivative is representable by means of a measure $Du : \mathbf{B}(\Omega) \to \mathbf{R}^n$ of finite total variation. The functions $u \in BV(\Omega)$ are called functions of bounded variation; for the main properties of these functions we refer to [22], [25], [39], [40]. The sets $E \in \mathbf{B}(\Omega)$ such that the characteristic function χ_E belongs to $BV(\Omega)$ are called sets of finite perimeter, or Caccioppoli sets. If $u = \chi_E$, then the set S_u defined by (1.3) coincides with the essential boundary $\partial^* E$ of E, i.e.,

$$\partial^* E = \big\{x \in \Omega : \limsup_{\rho\to 0+} \frac{|B_\rho(x) \cap E|}{\rho^n} > 0 \quad\text{and}\quad \limsup_{\rho\to 0+} \frac{|B_\rho(x) \setminus E|}{\rho^n} > 0\big\}.$$

For every set E of finite perimeter in Ω we have (see [17], [39])

$$|D\chi_E|(B) = \mathcal{H}_{n-1}(B \cap \partial^* E) \qquad \forall B \in \mathbf{B}(\Omega). \tag{1.7}$$

Moreover, Fleming-Rishel (see [23]) proved that the set

$$\{t \in \mathbf{R} : \{u > t\} \quad \text{has not finite perimeter in } \Omega\}$$

is negligible for every function $u \in BV(\Omega)$, and

$$|Du|(B) = \int_{-\infty}^{+\infty} |D\chi_{\{u>t\}}|(B)\,dt = \int_{-\infty}^{+\infty} \mathcal{H}_{n-1}(B \cap \partial^*\{u > t\})\,dt \quad \forall B \in \mathbf{B}(\Omega). \tag{1.8}$$

We are particularly interested to properties concerning the approximate continuity and the approximate differentiability of such functions. By an early result of De Giorgi ([17]) it follows that for every function $u \in BV(\Omega)$ the set S_u is countably $(n-1)$-rectifiable, i.e.,

$$S_u = \bigcup_{h \in \mathbf{N}} K_h \cup N \tag{1.9}$$

where $\mathcal{H}_{n-1}(N) = 0$ and (K_h) is a sequence of compact sets, each contained in a C^1 hypersurface Γ_h. Moreover, in $\mathcal{H}_{n-1}$ almost every $x \in S_u$ there exists a triplet $(u^+(x), u^-(x), \nu_u(x)) \in \mathbf{R} \times \mathbf{R} \times \mathbf{S}^{n-1}$ such that

(1.10)

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{\{y \in B_\rho(x) : \langle y-x, \nu_u(x)\rangle > 0\}} |u(y) - u^+(x)|\,dy = \lim_{\rho \to 0^+} \rho^{-n} \int_{\{y \in B_\rho(x) : \langle y-x, \nu_u(x)\rangle < 0\}} |u(y) - u^-(x)|\,dy = 0.$$

In particular, setting

$$\pi^+(x, \nu_u(x)) = \{y \in \mathbf{R}^n : \langle y - x, \nu_u(x)\rangle > 0\}, \quad \pi^-(x, \nu_u(x)) = \{y \in \mathbf{R}^n : \langle y - x, \nu_u(x)\rangle < 0\},$$

there exist the approximate limits

$$u^+(x, \nu_u(x)) = \operatorname{ap} \lim_{\substack{y \to x \\ y \in \pi^+(x, \nu_u(x))}} u(y), \qquad u^-(x, \nu_u(x)) = \operatorname{ap} \lim_{\substack{y \to x \\ y \in \pi^-(x, \nu_u(x))}} u(y). \tag{1.11}$$

The condition (1.11) means that in $\mathcal{H}_{n-1}$-almost every $x \in S_u$ the function u jumps between two asymptotic values $u^+(x)$, $u^-(x)$. The triplet (u^+, u^-, ν_u) is uniquely determined by (1.10) up to a change of sign of ν_u and and interchange of u^+, u^- ([40]). In addition, the versor $\nu_u(x)$ is normal to S_u in the following sense ([4], proposition 3.2, [22]): for every representation of S_u as in (1.9) the versor $\nu_u(x)$ is normal $\mathcal{H}_{n-1}$-almost everywhere in K_h to the surface Γ_h. Since

$$\nu_\Gamma(x) = \pm\,\nu_{\Gamma'}(x) \quad \mathcal{H}_{n-1}\text{-a.e. in } \Gamma \cap \Gamma'$$

for every pair of C^1 hypersurfaces Γ, Γ', we obtain the following remarkable property:

$$\nu_u(x) = \pm\,\nu_v(x) \quad \mathcal{H}_{n-1}\text{-a.e. in } S_u \cap S_v \tag{1.12}$$

for every pair of functions u, $v \in BV(\Omega)$.

Calderon-Zygmund (see [14]) proved that every function $u \in BV(\Omega)$ is approximately differentiable almost everywhere, and the approximate differential ∇u belongs to $L^1(\Omega; \mathbf{R}^n)$. In addition (see [4], proposition 3.1) the distributional derivative Du can be written as

$$Du = \nabla u \cdot \mathcal{L}_n + (u^+ - u^-)\nu_u \cdot \mathcal{H}_{n-1}|_{S_u} + Cu \tag{1.13}$$

(note that $(u^+ - u^-)\nu_u$ does not depend on the choice of the sign of ν_u) where Cu is a measure singular with respect to $\mathcal{L}_n$, such that

$$|Cu|(B) = 0 \qquad \forall B \in \mathbf{B}(\Omega) \text{ with } \mathcal{H}_{n-1}(B) < +\infty. \tag{1.14}$$

In (1.13) the n-dimensional part of Du is given by $\nabla u \cdot \mathcal{L}_n$, the $(n-1)$-dimensional part is given by $(u^+ - u^-)\nu_u \mathcal{H}_{n-1}|_{S_u}$, and the "intermediate" part is Cu. Thinking to the well known Cantor-Vitali function, we call Cu the Cantor part of the derivative Du; for this function, in fact, $\nabla u = 0$ almost everywhere, $S_u = \emptyset$ and $Du = Cu$. We recall also that the total variation $|Du|$ can't assign positive measure to sets with Hausdorff dimension less than $(n-1)$, because (1.8) implies $|Du|(B) = 0$ for every $\mathcal{H}_{n-1}$-negligible Borel set.
In this paper we shall consider functions of bounded variation which have null Cantor part of derivative. We denote by $SBV(\Omega)$ this space of functions. However, some of the problems we shall study may not be coercive in $SBV(\Omega)$ (the reasons of this will be clear in §3, see also example 5.3). This is the motivation of the following definition (see also [18]).
Let $u : \Omega \to \mathbf{R}^k$ be a Borel function. We say that u is a generalized function of bounded variation in Ω if

$$\phi(u) \in BV_{\mathrm{loc}}(\Omega) \qquad \forall \phi \in C^1(\mathbf{R}^k) \text{ with } \mathrm{supp}(\nabla\phi) \subset\subset \mathbf{R}^k. \tag{1.15}$$

We denote by $GBV(\Omega;\mathbf{R}^k)$ such class of functions. The class of functions $GSBV(\Omega;\mathbf{R}^k)$ is defined similarly, by requiring $\phi(u) \in SBV_{\mathrm{loc}}(\Omega)$. In the case $k = 1$, it can be easily seen that

$$u \in GBV(\Omega;\mathbf{R}) \qquad \Longleftrightarrow \qquad (u \wedge N) \vee -N \in BV_{\mathrm{loc}}(\Omega) \quad \forall N \in \mathbf{N}$$

and a similar equivalence is true for $GSBV(\Omega;\mathbf{R})$. In the case $k > 1$ the space of test functions ϕ can be taken equal to $C_0^1(\mathbf{R}^k)$, because no compact set disconnects $\mathbf{R}^k$ at infinity. If $u \in L^\infty(\Omega;\mathbf{R}^k)$, then

$$u \in BV_{\mathrm{loc}}(\Omega;\mathbf{R}^k) \quad \Longleftrightarrow \quad u \in GBV(\Omega;\mathbf{R}^k)$$

and the corresponding equivalence holds for $GSBV(\Omega;\mathbf{R}^k)$.
The generalized functions of bounded variation inherit many properties of the ordinary functions of bounded variation. We state the most important in the following two propositions.

Proposition 1.3. *Let $u \in GBV(\Omega;\mathbf{R}^k)$. Then, the set S_u is countably $(n-1)$-rectifiable. In addition, there exists a Borel function $\nu_u : S_u \to \mathbf{S}^{n-1}$ such that the approximate limits (1.11) exist $\mathcal{H}_{n-1}$-almost everywhere on S_u.*

Proof. Let $\mathcal{F} \subset C_0^1(\mathbf{R}^k) \subset C(\tilde{\mathbf{R}}^k)$ be a countable set of functions which separates points in $C(\tilde{\mathbf{R}}^k)$. By proposition 1.1(v) we get

$$S_u = \bigcup_{\phi \in \mathcal{F}} S_{\phi(u)}, \tag{1.16}$$

which implies, by (1.9), that S_u is countably $(n-1)$-rectifiable. Using (1.9), (1.12), (1.16), we can construct a Borel function $\nu_u : S_u \to \mathbf{S}^{n-1}$ such that

$$\nu_u = \pm\,\nu_{\phi(u)} \quad \mathcal{H}_{n-1}\text{-a.e. in } S_{\phi(u)}$$

for every function $\phi \in \mathcal{F}$. By the quoted properties of functions of bounded variation, the approximate limits

$$\mathrm{ap} \lim_{\substack{y \to x \\ y \in \pi^+(x,\nu_u(x))}} \phi(u)(y), \qquad \mathrm{ap} \lim_{\substack{y \to x \\ y \in \pi^-(x,\nu_u(x))}} \phi(u)(y)$$

exist $\mathcal{H}_{n-1}$-almost everywhere on $S_{\phi(u)}$. On the other hand, in the set $S_u \setminus S_{\phi(u)}$ both the approximate limits exist and are equal to $\widetilde{\phi(u)}(x)$. In conclusion, it is possible to find a $\mathcal{H}_{n-1}$-negligible Borel set $N \subset S_u$ such that the above approximate limits exist for every $x \in S_u \setminus N$ and every $\phi \in \mathcal{F}$. Thus, the statement follows by proposition 1.2(v). **q.e.d.**

Given a Borel function $\nu_u : S_u \to \mathbf{S}^{n-1}$ as in the statement of proposition 1.3, by the same techniques exploited in proposition 1.2(iv) it is possible to prove that the domains of the approximate limits (1.11) belong to $\mathbf{B}(\Omega)$. Moreover, u^+, u^- defined by (1.11) are Borel functions in their domains.

If $u \in BV(\Omega)$, then (1.10) implies that $u^+, u^- \in \mathbf{R}$ for $\mathcal{H}_{n-1}$-almost every $x \in S_u$. On the contrary, for GBV functions it may happen that

$$\mathcal{H}_{n-1}(\{x \in S_u : u^+(x) = \infty \quad \text{or} \quad u^-(x) = \infty\}) > 0.$$

proposition 1.4. *Let $u \in GBV(\Omega;\mathbf{R}^k)$. Then ∇u exists almost everywhere in Ω.*

Proof. Let $(\phi_h) \subset C_0^1(\mathbf{R}^k;\mathbf{R}^k)$ be a sequence of functions such that $\phi_h(x) = x$ for every $x \in B_h(0)$. By the Calderon-Zygmund theorem, all the functions $\phi_k(u)$ are approximately differentiable almost everywhere in Ω. Since almost every $x \in \Omega$ is a point of density 1 for one of the sets $\{|u| < h\}$, the statement follows by proposition 1.2(ii). **q.e.d.**

2. Compactness.

In this section we shall state some compactness theorems. Since we deal with functions which are not necessarily summable (GBV functions), the most natural topology is given by (local) convergence in measure. We recall that a sequence of Borel functions $u_h : \Omega \to \mathbf{R}^k$ converges in measure to a Borel function $u : \Omega \to \mathbf{R}^k$ if

$$\lim_{h \to +\infty} |\{x \in K : |u_h(x) - u(x)| > \epsilon\}| = 0$$

for every compact set $K \subset \Omega$ and every $\epsilon > 0$. Every sequence converging almost everywhere converges in measure, and every sequence converging in measure admits a subsequence converging almost everywhere to the same limit.

Let $\phi : [0,+\infty[\to [0,+\infty]$ be a convex non decreasing function satisfying the condition

$$\lim_{t \to +\infty} \frac{\phi(t)}{t} = +\infty, \tag{2.1}$$

and let $\Theta : [0,+\infty] \to [0,+\infty]$ be a concave non decreasing function such that

$$\lim_{t \to 0^+} \frac{\Theta(t)}{t} = +\infty. \tag{2.2}$$

The following compactness theorem is a straightforward consequence of theorem 2.1 of [4], which deals with the case $k = 1$.

Theorem 2.1. *Let $K \subset \mathbf{R}^k$ be a compact set, and let $(u_h) \subset SBV_{\text{loc}}(\Omega;\mathbf{R}^k)$ be a sequence such that*

$$\sup_{h \in \mathbf{N}} \left\{ \int_\Omega \phi(|\nabla u_h|)\, dx + \int_{S_{u_h}} \Theta(|u_h^+ - u_h^-|)\, d\mathcal{H}_{n-1}(x) \right\} < +\infty$$

and $u_h(x) \in K$ almost everywhere. Then there exists a subsequence (u_{h_k}) converging in measure to a function $u \in SBV_{\text{loc}}(\Omega;\mathbf{R}^k)$. Moreover, $u(x) \in K$ almost everywhere and (∇u_{h_k}) weakly converges to ∇u in $L^1(A;\mathcal{L}_{n,k})$ for every open set $A \subset \Omega$ such that $|A| < +\infty$.

The growth condition (2.1) on ϕ is very natural in Calculus of Variations and guarantees compactness, in the weak L^1 topology, of the approximate differentials. The condition (2.2) is necessary: if we take for instance $\Theta(t) = t$, then it is possible to approximate the Cantor-Vitali function (which is not in SBV) by step functions u_h such that

$$\sup_{h \in \mathbf{N}} \int_{S_{u_h}} \Theta(|u_h^+ - u_h^-|)\, d\mathcal{H}_0 < +\infty.$$

Under assumption (2.2), the integral

$$\int_{S_u} \Theta(|u^+ - u^-|)\, d\mathcal{H}_{n-1}$$

has a fast growth when the jumps are small, and this guarantess compactness in $SBV_{\mathrm{loc}}(\Omega;\mathbf{R}^k)$.
To deal with problems where no constraint $\{u \in K\}$ exists, the most natural domain seems to be the class of functions $GSBV(\Omega;\mathbf{R}^k)$ (see also [18]). In fact, if for instance $\Theta = 1$, there is no possibility to control

$$\int_{S_u} |u^+ - u^-|\, d\mathcal{H}_{n-1}$$

by $\mathcal{H}_{n-1}(S_u)$, so that limit of functions u_h which satisfy the integral condition of theorem 2.1 may have not finite total variation.
On the other hand, under very mild assumptions on (u_h), the sequence converges to a function $u \in GSBV(\Omega;\mathbf{R}^k)$, as the following theorem shows.

Theorem 2.2. *Let $g(x,u) : \Omega \times \mathbf{R}^k \to [0,+\infty]$ be a Borel function, lower semicontinuous in u and satisfying the condition*

$$\lim_{|u|\to+\infty} g(x,u) = +\infty \qquad \text{for a.e. } x \in \Omega. \tag{2.3}$$

Let $(u_h) \subset GSBV(\Omega;\mathbf{R}^k)$ be a sequence such that

$$\sup_{h\in\mathbf{N}} \left\{ \int_\Omega \phi(|\nabla u_h|)\, dx + \int_{S_{u_h}} \Theta(|u_h^+ - u_h^-|)\, d\mathcal{H}_{n-1}(x) + \int_\Omega g(x,u_h)\, dx \right\} < +\infty.$$

Then, there exists a subsequence (u_{h_k}) converging in measure to a function $u \in GSBV(\Omega;\mathbf{R}^k)$, and (∇u_{h_k}) weakly converges to ∇u in $L^1(A;\mathcal{L}_{n,k})$ for every open set $A \subset \Omega$ such that $|A| < +\infty$.

Proof. Let $\mathcal{F} \subset C_0^1(\mathbf{R}^k)$ be as in the proof of proposition 1.3. Applying a diagonal argument, we can find a subsequence (h_k) and functions $v_\phi \in SBV_{\mathrm{loc}}(\Omega)$ such that $\phi(u_{h_k})$ converges almost everywhere to v_ϕ for every $\phi \in \mathcal{F}$. Since $\tilde{\mathbf{R}}^k$ is compact, and since $\mathcal{F}$ separates points in $\tilde{\mathbf{R}}^k$, it can be easily seen that necessarily u_{h_k} converges almost everywhere to a Borel function $u : \Omega \to \tilde{\mathbf{R}}^k$ such that $\phi(u) = v_\phi$ almost everywhere for every $\phi \in \mathcal{F}$. Let $\tilde{g}(x,u) : \Omega \times \tilde{\mathbf{R}}^k \to [0,+\infty]$ be the extension of g obtained setting $\tilde{g}(x,\infty) = +\infty$. By our hypothesis, $\tilde{g}(x,u)$ is lower semicontinuous in u for almost every $x \in \Omega$, and by Fatou's lemma we get

$$\int_\Omega \tilde{g}(x,u)\, dx \le \liminf_{k\to+\infty} \int_\Omega \tilde{g}(x,u_{h_k})\, dx < +\infty,$$

so that, $u(x) \in \mathbf{R}^k$ almost everywhere. Let $\phi \in C^1(\mathbf{R}^k)$ be a functions such that $\nabla\phi$ has compact support. Since the functions $\phi(u_h)$ converge almost everywhere to $\phi(u)$, by theorem 2.1 we get that $\phi(u) \in SBV_{\mathrm{loc}}(\Omega)$. Since ϕ is arbitrary, the function u belongs to $GSBV(\Omega;\mathbf{R}^k)$. By (2.1), the approximate differentials are weakly compact in $L^1(A;\mathcal{L}_{n,k})$ for every open set $A \subset \Omega$ with $|A| < +\infty$. The weak convergence of the approximate differentials can be easily proved using theorem 2.1 and test functions ϕ as in proposition 1.4. **q.e.d.**

3. Semicontinuity.

Let $f : \Omega \times \mathbf{R}^k \times \mathcal{L}_{n,k} \to [0,+\infty]$, $\varphi : \Omega \times \tilde{\mathbf{R}}^k \times \tilde{\mathbf{R}}^k \times \mathbf{S}^{n-1} \to [0,+\infty]$ be functions, and let us consider the functional

$$F(u) = \int_\Omega f(x,u,\nabla u)\, dx + \int_{S_u} \varphi(x,u^+,u^-,\nu_u)\, d\mathcal{H}_{n-1}(x) \qquad u \in GSBV(\Omega;\mathbf{R}^k).$$

By proposition 1.3 and proposition 1.4 the functional is well defined, provided f, φ are Borel functions, and

$$\varphi(x,u,v,\nu) = \varphi(x,v,u,-\nu). \tag{3.1}$$

because the triplet (u^+, u^-, ν_u) is not uniquely determined. We shall always tacitly assume in this section that all the integrands φ satisfy this condition.
We are interested in finding necessary or sufficient conditions on f, φ which ensure the lower semicontinuity of the functional F with respect to convergence in measure. Since for the first integral many semicontinuity criteria are available ([1], [27], [28], [11]), we shall study in particular the integral depending on φ.
Let $T \subset \mathbf{R}^k$ be a finite set, and let $BV(\Omega; T)$ be the set of all functions $u \in BV(\Omega; \mathbf{R}^k)$ such that $u(x) \in T$ almost everywhere. The functional F is equal on $BV(\Omega; T)$ to the functional

$$\int_\Omega f(x, u, 0)\, dx + \int_{S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x).$$

If $f(x, \cdot, 0)$ is lower semicontinuous for almost every $x \in \Omega$, Fatou's lemma implies that

$$\int_\Omega f(x, u, 0)\, dx \leq \liminf_{h \to +\infty} \int_\Omega f(x, u_h, 0)\, dx$$

for every sequence (u_h) converging in measure to u. Hence, on $BV(\Omega; T)$ the problem reduces to the semicontinuity of functionals of the type

$$\int_{S_u} \psi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \tag{3.2}$$

where $\psi : \Omega \times T \times T \times \mathbf{S}^{n-1} \to [0, +\infty]$ is a Borel function. The functionals in (3.2) act on partitions of Ω in card(T) sets of finite perimeter labeled by the elements of T, and of course the condition $T \subset \mathbf{R}^k$ is not essential. This kind of functionals occur in many problems of mathematical physics and in particular in problems of phase transitions (we refer to [29], [30], [10] for a wide bibliography on the subject).
In a joint paper with A.Braides the author has studied several conditions which are necessary and sufficient for the semicontinuity of functionals of type (3.2). We recall briefly the main definitions and results of the paper.
Definition. Let $T \subset \mathbf{R}^k$ be a finite set, and let $\psi : T \times T \times \mathbf{S}^{n-1} \to [0, +\infty]$ be a function. We say that ψ is BV-elliptic if for every triplet $(i, j, \nu) \in T \times T \times \mathbf{S}^{n-1}$ we have

$$\int_{S_u} \psi(u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1} \geq \int_{\Omega \cap H_\nu} \psi(i, j, \nu)\, d\mathcal{H}_{n-1} \qquad \forall u \in BV(\Omega; T) \text{ with } u^* = u^*_{ij}\ \mathcal{H}_{n-1}\text{-a.e. on } \partial\Omega, \tag{3.3}$$

where Ω is a smooth open set containing 0, $H_\nu = \{x \in \mathbf{R}^n : \langle x, \nu \rangle = 0\}$, the function u_{ij} is defined by

$$u_{ij}(x) = \begin{cases} i & \text{if } \langle x, \nu \rangle > 0; \\ j & \text{if } \langle x, \nu \rangle \leq 0, \end{cases}$$

and u^*, u^*_{ij} denote the inner traces on the boundary $\partial\Omega$ of the functions u, u_{ij} respectively ([25], Chapter 2). It is not difficult to see that (3.3) does not depend on the choice of Ω. The condition means that, among all partitions u with the same boundary trace of u_{ij}, the minimal one is u_{ij}.
As its name suggests, this condition is closely related with the ellipticity conditions of geometric measure theory (see for instance [22]). In [6] we have proved the following theorem:
Theorem 3.1. *Let $c > 0$, and let $\psi : \Omega \times T \times T \times \mathbf{S}^{n-1} \to [c, +\infty[$ be a continuous function. Then, a necessary and sufficient condition for the functional in (3.2) to be lower semicontinuous in $BV(\Omega; T)$ with respect to convergence in measure is the BV-ellipticity of $\psi(x, \cdot, \cdot, \cdot)$ for every $x \in \Omega$.*

Even if theorem 3.1 solves the problem of characterization of integrands ψ which define lower semicontinuous functionals, the condition (3.3) is not completely satisfactory, because it is of integral type, and in general it is not easy to be checked.

In [6] various algebraic conditions on ψ related to BV- ellipticity are studied, and they are also compared with other definitions already existing in the literature (see [8]). The most important is given below.

Definition. Let $T = \{z_1, \dots, z_m\} \subset \mathbf{R}^k$, let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ be the canonical basis of $\mathbf{R}^m$ and let $\psi : T \times T \times \mathbf{S}^{n-1} \to]-\infty, +\infty]$ be a function. We say that ψ is biconvex if there exists a convex and positively 1-homogeneous function $\theta : \mathcal{L}_{n,m} \to]-\infty, +\infty]$ such that

$$\psi(z_i, z_j, \frac{p}{|p|})|p| = \theta\big((\mathbf{e}_i - \mathbf{e}_j) \otimes p\big) \qquad \forall i, j \in \{1, \dots, m\},\ i \neq j,\ p \in \mathbf{R}^n \setminus \{0\}. \tag{3.4}$$

We want to emphatize that (3.4) is an algebraic condition. Infact, since (3.4) determines θ only on vectors $z \in \mathcal{L}_{n,m}$ of the form $(\mathbf{e}_j - \mathbf{e}_i) \otimes p$, the function θ exists if and only if, denoting by $\hat{\psi}$ the 1-homogeneous extension of ψ to $T \times T \times \mathbf{R}^n$, we have

$$\hat{\psi}(z_{i_0}, z_{j_0}, p_0) \leq \sum_{\lambda=1}^{N} \hat{\psi}(z_{i_\lambda}, z_{j_\lambda}, p_\lambda) \tag{3.5}$$

whenever

$$(\mathbf{e}_{i_0} - \mathbf{e}_{j_0}) \otimes p_0 = \sum_{\lambda=1}^{N} (\mathbf{e}_{i_\lambda} - \mathbf{e}_{j_\lambda}) \otimes p_\lambda \qquad \text{in } \mathcal{L}_{n,m}, \quad i_0 \neq j_0.$$

In [6], by using the Jensen inequality, we have proved also the following result.

Theorem 3.2. *Every biconvex integrand $\psi(u, v, \nu)$ is BV-elliptic.*

It is possible to find many interesting examples of biconvex functions (see §5). There is a close similarity between BV-ellipticity and Morrey quasi convexity on one hand, biconvexity and rank 1 convexity on the other hand (see [12], [31] for the definitions of Morrey quasi-convexity and rank 1 convexity). A long standing conjecture of nonlinear elasticity is the equivalence between quasi convexity and rank 1 convexity. We also conjecture that BV-ellipticity and biconvexity are equivalent. One implication is given by theorem 3.2. The idea to prove the opposite implication would be to show that each condition listed in (3.5) is necessary for semicontinuity. This has been done for some of these conditions, but no general procedure has been found. For instance, in the case

$$(\mathbf{e}_i - \mathbf{e}_j) \otimes \nu = (\mathbf{e}_i - \mathbf{e}_k) \otimes \nu + (\mathbf{e}_k - \mathbf{e}_j) \otimes \nu$$

or

$$(\mathbf{e}_i - \mathbf{e}_j) \otimes (p_1 + p_2) = (\mathbf{e}_i - \mathbf{e}_j) \otimes p_1 + (\mathbf{e}_i - \mathbf{e}_j) \otimes p_2$$

the corresponding conditions

$$\begin{cases} \psi(z_i, z_j, \nu) \leq \psi(z_i, z_k, \nu) + \psi(z_k, z_j, \nu) & \forall \nu \in \mathbf{S}^{n-1}; \\ p \quad \to \quad \hat{\psi}(z_i, z_j, p) \text{ is convex in } \mathbf{R}^n, \end{cases} \tag{3.6}$$

have been proved to be necessary for lower semicontinuity. In [6] it is possible to find a more detailed discussion on this subject, enriched with examples and conjectures.

The proof of theorem 3.1 is rather technical, and follows closely the proof of similar results in geometric measure theory ([22], 5.1.5). In contrast, we shall see in theorem 3.6 a much simpler proof of the sufficiency of biconvexity for lower semicontinuity, not based on theorem 3.2. The proof relies on the following approximation scheme

$$\psi(x, z_i, z_j, \nu) = \sup_{h \in \mathbf{N}} \langle V_h(x, z_i) - V_h(x, z_j), \nu \rangle \qquad x \in \Omega,\ i \neq j,\ \nu \in \mathbf{S}^{n-1}, \tag{3.7}$$

where $V_h : \Omega \times T \to \mathbf{R}^n$ is a suitable sequence of continuous functions . By a standard technique (see [19], [2], [3]), the semicontinuity of the functional in (3.2) can be desumed by the semicontinuity of functionals

$$\int_{A \cap S_u} \langle V_h(x, u^+) - V_h(x, u^-), \nu_u \rangle^+ \, d\mathcal{H}_{n-1} \qquad A \subset \Omega \text{ open},\ h \in \mathbf{N}$$

and this is done using chain rule for derivates of compositions of BV functions with Lipschitz functions ([7], [39], [40]) and integrating by parts (see also the proof of theorem 3.6).
We say that an integrand $\psi : \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{S}^{n-1} \to [0, +\infty]$ is BV-elliptic (respectively, biconvex) if the restriction to $T \times T \times \mathbf{S}^{n-1}$ is BV-elliptic (respectively, biconvex) for every finite set $T \subset \mathbf{R}^k$.
The following theorem shows that BV-ellipticity, together with continuity of the integrand and with a growth condition is sufficient for semicontinuity.

Theorem 3.3. *Let $c > 0$, and let $\varphi : \Omega \times \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{S}^{n-1} \to [c, +\infty[$ be a continuous function such that $\varphi(x, \cdot, \cdot, \cdot)$ is BV-elliptic for every $x \in \Omega$. Then, for every sequence $(u_h) \subset SBV(\Omega; \mathbf{R}^k) \cap L^\infty(\Omega; \mathbf{R}^k)$ converging in measure to $u \in SBV(\Omega; \mathbf{R}^k)$ and satisfying the conditions*

$$(i) \qquad M = \sup_{h \in \mathbf{N}} \|u_h\|_\infty < +\infty;$$

$$(ii) \qquad \nabla u_h \text{ is an equi-integrable sequence in } L^1(A; \mathcal{L}_{n,k}) \text{ for every open set } A \subset\subset \Omega;$$

we have

$$(3.8) \qquad \int_{S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \leq \liminf_{h \to +\infty} \int_{S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x).$$

Proof. It is not restrictive to assume that u_h converges to u almost everywhere and

$$L = \liminf_{h \to +\infty} \int_{S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x) = \lim_{h \to +\infty} \int_{S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x) < +\infty,$$

hence

$$(3.9) \qquad \limsup_{h \to +\infty} \mathcal{H}_{n-1}(S_{u_h}) \leq \frac{L}{c} < +\infty.$$

Moreover, we shall assume for simplicity that $M < 1$, so that all the functions u_h and u take their values in the set $]0, 1[^k$. Since Ω can be approximated by an increasing sequence of open sets $A_h \subset\subset \Omega$, we assume that φ is uniformly continuous in $\Omega \times]0, 1[^k \times]0, 1[^k \times \mathbf{S}^{n-1}$ and we denote by $\omega : [0, +\infty[\to [0, +\infty[$ a continuous function such that $\omega(0) = 0$ and

$$|\varphi(x, u, v, \nu) - \varphi(x, u', v', \nu)| \leq \omega(|u - u'| + |v - v'|) \qquad \forall x \in \Omega,\ u,\ v,\ u',\ v' \in]0, 1[^k,\ \nu \in \mathbf{S}^{n-1}.$$

We also assume that Ω is bounded and (∇u_h) is equi-integrable in $L^1(\Omega; \mathcal{L}_{n,k})$.
We approximate the functions u_h by step functions, dividing the set $]0, 1[^k$ in small cubes. Since we need step functions whose singular set is not too large, a careful choice of the sides of the cubes is needed, and this choice depends on the index h. Let $p \in \mathbf{N}$, $p \geq 2$ and let $A_p \subset \Omega$ be an open set such that

$$\sup_{h \in \mathbf{N}} \int_{A_p} |\nabla u_h|\, dx < 2^{-p}, \qquad A_p \supset \bigcup_{h \in \mathbf{N}} S_{u_h} \cup S_u.$$

We set

$$B_h = A_p \setminus S_{u_h}.$$

By Fleming-Rishel formula (1.8), it is possible to find real numbers $\xi_{i,h}^j$ such that

$$(3.10) \qquad \{x \in \Omega : u_h^{(j)}(x) > \xi_{i,h}^j\} \text{ has finite perimeter in } \Omega \text{ and } |\{x \in \Omega : u_h^{(j)}(x) = \xi_{i,h}^j\}| = 0;$$

(3.11) $$\xi^j_{i,h} \in]\frac{2i-2}{2p}, \frac{2i-1}{2p}];$$

(3.12) $$\mathcal{H}_{n-1}(B_h \cap \partial^*\{x \in \Omega : u_h^{(j)}(x) > \xi^j_{i,h}\}) \le 2p \int_{(i-1)/p}^{i/p} \mathcal{H}_{n-1}(B_h \cap \partial^*\{x \in \Omega : u_h^{(j)}(x) > t\})\, dt;$$

for every $i \in \{1, \dots, p\}$, $j \in \{1, \dots, k\}$, $h \in \mathbf{N}$. We also set $\xi^j_{0,h} = 0$ and $\xi^j_{p+1,h} = 1$. We denote by $\mathcal{S}$ the set of functions $\sigma : \{1, \dots, k\} \to \{0, \dots, p\}$, and we set

$$Q_{\sigma,h} = \{z \in \mathbf{R}^k : \xi^j_{\sigma(j),h} < z^{(j)} < \xi^j_{\sigma(j)+1,h}\ \forall j \in \{1, \dots, k\}\};$$

$$E_{\sigma,h} = \{x \in \Omega : u_h(x) \in Q_{\sigma,h}\}, \qquad \eta_\sigma^{(j)} = \frac{\sigma(j)}{p}.$$

By (3.10), the sets $\{E_{\sigma,h}\}_{\sigma \in \mathcal{S}}$ are mutually disjoint, have finite perimeter in Ω and $\eta_\sigma \in \overline{Q}_{\sigma,h}$. By (3.11), (3.12) the functions

$$v_h(x) = \begin{cases} \eta_\sigma & \text{if } x \in E_{\sigma,h} \text{ for some } \sigma \in \mathcal{S} \\ 0 & \text{otherwise.} \end{cases}$$

belong to $BV(\Omega; \mathbf{R}^k)$ and $\|u_h - v_h\|_\infty \le 2k/p$. Since $A_p \setminus B_h \subset S_{u_h}$, we have (recall (1.12))

$$\int_{A_p \cap S_{v_h} \setminus B_h} \varphi(x, v_h^+, v_h^-, \nu_{v_h})\, d\mathcal{H}_{n-1}(x) \le \int_{S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x) + \omega(\frac{4k}{p}) \mathcal{H}_{n-1}(S_{u_h}).$$

Moreover, by our choice of $\xi^j_{i,h}$ and of the open set A_p we get by (3.12)

$$\int_{S_{v_h} \cap B_h} \varphi(x, v_h^+, v_h^-, \nu_{v_h})\, d\mathcal{H}_{n-1}(x) \le C\mathcal{H}_{n-1}(S_{v_h} \cap B_h) \le C\mathcal{H}_{n-1}(\bigcup_{\sigma \in \mathcal{S}} \partial^* E_{\sigma,h} \cap B_h) \le$$

$$\le Cp \sum_{j=1}^{k} \frac{1}{p} \sum_{i=1}^{p} \mathcal{H}_{n-1}(\partial^*\{x \in \Omega : u_h^{(j)}(x) > \xi^j_{i,h}\} \cap B_h) \le Cp \sum_{j=1}^{k} |Du_h^{(j)}|(B_h) \le Ckp2^{-p},$$

where $C = \sup\{\varphi(x, u, v, \nu) : x \in \Omega,\ |u| \le 1,\ |v| \le 1,\ \nu \in \mathbf{S}^{n-1}\}$. Adding these two inequalities, we get

(3.13) $$\int_{A_p \cap S_{v_h}} \varphi(x, v_h^+, v_h^-, \nu_{v_h})\, d\mathcal{H}_{n-1}(x) \le \int_{S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x) + \omega(\frac{4k}{p}) \mathcal{H}_{n-1}(S_{u_h}) + Ckp2^{-p}.$$

In particular, we get

$$\limsup_{h \to +\infty} \mathcal{H}_{n-1}(A_p \cap S_{v_h}) \le \frac{1}{c}[L + \omega(\frac{4k}{p})\frac{L}{c} + Ckp2^{-p}] < +\infty$$

so that, by theorem 2.1, the sequence (v_h) is relatively compact with respect to convergence in measure in A_p. Let $T = \{\eta^\sigma\}_{\sigma \in \mathcal{S}}$. Possibly passing to a subsequence, we can assume that

$$\xi^j_{i,h} \to \xi^j_i \in [\frac{2i-2}{2p}, \frac{2i-1}{2p}] \qquad \forall i \in \{0, \dots, p+1\},\ j \in \{1, \dots, k\},$$

and (v_h) converges almost everywhere in A_p to a function $w_p \in SBV(A_p; \mathbf{R}^k)$ with values in T. Let $Q_\sigma \subset [0,1]^k$ be the set of vectors $(z^{(1)}, \dots, z^{(k)})$ such that

$$\xi^j_{\sigma(j)} < z^{(j)} < \xi^j_{\sigma(j)+1} \quad \forall j \in \{1, \dots, k\},$$

and let

$$w_p = \sum_{\sigma \in S} \eta^\sigma \chi_{E_\sigma}, \qquad E_\sigma \in \mathbf{B}(A_p).$$

Since u_h converges almost everywhere to u, it is easy to see that $|E_\sigma \setminus u^{-1}(\overline{Q}_\sigma)| = 0$, so that

$$E_\sigma \subset A_p \cap u^{-1}(\overline{Q}_\sigma)$$

up to negligible sets. Applying theorem 3.1 we get

$$\int_{A_p \cap S_{w_p}} \varphi(x, w_p^+, w_p^-, \nu_{w_p})\, d\mathcal{H}_{n-1}(x) \le \liminf_{h \to +\infty} \int_{A_p \cap S_{v_h}} \varphi(x, v_h^+, v_h^-, \nu_{v_h})\, d\mathcal{H}_{n-1}(x),$$

which, together with (3.13) gives

$$\int_{A_p \cap S_{w_p}} \varphi(x, w_p^+, w_p^-, \nu_{w_p})\, d\mathcal{H}_{n-1}(x) \le L + \omega\left(\frac{4k}{p}\right)\frac{L}{c} + Ckp2^{-p}. \tag{3.14}$$

Now we shall achieve the proof by letting $p \to +\infty$. Let $\epsilon > 0$ be given, and let $p_0 \in \mathbf{N}$ such that all the diameters of the sets Q_σ with $p \ge p_0$ are less than $\epsilon/3$. By (1.16), in $\mathcal{H}_{n-1}$ almost every $x \in S_u$ such that $|u^+(x) - u^-(x)| > \epsilon$ the sets

$$\{y \in \Omega : |u(y) - u^+(x)| < \frac{\epsilon}{3}\}, \qquad \{y \in \Omega : |u(y) - u^-(x)| < \frac{\epsilon}{3}\},$$

have density 1/2 at x, hence, since E_σ is essentially contained in $u^{-1}(\overline{Q}_\sigma)$, we have

$$S_\epsilon = \{x \in S_u : |u^+(x) - u^-(x)| > \epsilon\} \subset \left(A_p \cap S_{w_p}\right) \cup N$$

for every $p \ge p_0$, for a suitable Borel set N with $\mathcal{H}_{n-1}(N) = 0$. Moreover, by (1.12) we get

$$\nu_{w_p} = \pm \nu_u \quad \mathcal{H}_{n-1}\text{-almost everywhere in } S_\epsilon.$$

Since $|w_p - u|$ is essentially bounded by $2k/p$ in A_p, the functions

$$v_p^+(x) = \begin{cases} w_p^+(x) & \text{if } \nu_{w_p}(x) = \nu_u(x); \\ w_p^-(x) & \text{otherwise,} \end{cases} \qquad v_p^-(x) = \begin{cases} w_p^-(x) & \text{if } \nu_{w_p}(x) = \nu_u(x); \\ w_p^+(x) & \text{otherwise,} \end{cases}$$

converge uniformly to u^+, u^- on S_ϵ, hence

$$\int_{S_\epsilon} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \le \liminf_{p \to +\infty} \int_{S_\epsilon} \varphi(x, v_p^+, v_p^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \le$$

$$\le \liminf_{p \to +\infty} \int_{S_\epsilon} \varphi(x, w_p^+, w_p^-, \nu_{w_p})\, d\mathcal{H}_{n-1}(x) \le \liminf_{p \to +\infty} \int_{A_p \cap S_{w_p}} \varphi(x, w_p^+, w_p^-, \nu_{w_p})\, d\mathcal{H}_{n-1}(x) \le L.$$

Since $S_\epsilon \uparrow S_u$ as $\epsilon \downarrow 0$, the inequality follows. q.e.d.

Theorem 3.3 is not completely satisfactory, because we do not allow φ to take the value $+\infty$, thus excluding the possibility to include some constraints of the variational problem into the functional to be minimized. We shall improve theorem 3.3 by assuming on φ biconvexity instead of ellipticity. To do this, we first need to give some definitions.

Definition. We denote by $\mathcal{M}_0$ the space of measures $\mu : \mathbf{R}^k \to \mathbf{R}^n$ of finite total variation such that $\mu(\mathbf{R}^k) = 0$, and we endow it by the weak* topology given by the duality

$$\langle \mu, V\rangle = \sum_{i=1}^{n} \int_{\mathbf{R}^k} V^{(i)}(x)\, d\mu^{(i)}(x) \qquad V \in C_0(\mathbf{R}^k;\mathbf{R}^n)$$

where $C_0(\mathbf{R}^k;\mathbf{R}^n)$ is the space of continuous functions vanishing at infinity. Let $\Delta \subset \mathcal{M}_0$ be defined by

$$\Delta = \{p(\delta_a - \delta_b) : p \in \mathbf{R}^n,\ a, b \in \mathbf{R}^k\}$$

where $p\delta_z$ denotes the Dirac measure

$$p\delta_z(B) = \begin{cases} p & \text{if } z \in B; \\ 0 & \text{otherwise,} \end{cases} \qquad B \in \mathbf{B}(\mathbf{R}^k)$$

and let $\mathrm{co}(\Delta) \subset \mathcal{M}_0$ be the convex hull of Δ, i.e.,

$$\mathrm{co}(\Delta) = \left\{\sum_{i=1}^{N} \mu_i : N \in \mathbf{N},\ \mu_1, \ldots, \mu_N \in \Delta\right\}.$$

By the Hahn-Banach theorem, $\mathrm{co}(\Delta)$ is w*- dense in $\mathcal{M}_0$, because any linear continuos functional vanishing on Δ corresponds to a constant function. To each function $\varphi : \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{S}^{n-1} \to [0, +\infty]$ satisfying (3.1) there corresponds in a natural way a positively 1-homogeneous function $\hat{\varphi} : \Delta \to [0, +\infty]$ by setting

$$\hat{\varphi}\big(p(\delta_a - \delta_b)\big) = \varphi(a, b, \frac{p}{|p|})|p| \qquad \forall a, b \in \mathbf{R}^k,\ a \neq b,\ p \in \mathbf{R}^n \setminus \{0\}.$$

It is now easy to see that φ is biconvex if and only if there exists a convex function $\tilde{\varphi}$ which extends $\hat{\varphi}$ to $\mathrm{co}(\Delta)$. If condition (3.5) is satisfied, the function $\tilde{\varphi}$ can be defined by

$$\tilde{\varphi}(\mu) = \inf\left\{\sum_{i=1}^{N} \hat{\varphi}(\mu_i) : N \in \mathbf{N},\ \mu_1, \ldots, \mu_N \in \Delta\right\}$$

for every measure $\mu \in \mathrm{co}(\Delta)$. We say that φ is a regular biconvex function if a stronger condition is satisfied: there exists a convex and weakly* lower semicontinuous function $\tilde{\varphi} : \mathcal{M}_0 \to [0, +\infty]$ such that $\tilde{\varphi} = \hat{\varphi}$ on Δ. A simple characterization of regular biconvex functions is given by the following lemma.

Lemma 3.4. *Let $\varphi : \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{S}^{n-1} \to [0, +\infty]$ be a function. Then, φ is regular biconvex if and only if*

$$\varphi(a, b, \nu) = \sup_{h \in \mathbf{N}} \langle V_h(a) - V_h(b), \nu\rangle \qquad \forall a, b \in \mathbf{R}^k,\ a \neq b,\ \nu \in \mathbf{S}^{n-1}. \tag{3.15}$$

for a suitable sequence of functions $V_h \in C_0(\mathbf{R}^k;\mathbf{R}^n)$.

Proof. If φ satisfies condition (3.15), the function

$$\tilde{\varphi}(\mu) = \sup_{h \in \mathbf{N}} \langle V_h, \mu\rangle \tag{3.16}$$

is the required extension of $\hat{\varphi}$ to $\mathcal{M}_0$. Conversely, if a convex and lower semicontinuous extension $\tilde{\varphi}$ exists, we can assume with no loss of generality that $\tilde{\varphi}$ is positively 1-homogeneous. Then by the Hahn-Banach theorem it is possible to find a sequence $(V_h) \subset C_0(\mathbf{R}^k;\mathbf{R}^n)$ such that (3.16) holds. In particular, if $\mu = \nu(\delta_a - \delta_b)$, with $b \neq a$, we find

$$\varphi(a, b, \nu) = \hat{\varphi}(\mu) = \tilde{\varphi}(\mu) = \sup_{h \in \mathbf{N}} \langle V_h, \mu\rangle = \sup_{h \in \mathbf{N}} \langle V_h(a) - V_h(b), \nu\rangle$$

and the statement is proved. q.e.d.

We shall see in §5 some examples of regular biconvex functions. By lemma 3.4, regular biconvexity implies the lower semicontinuity of the function in the set

$$\{(a,b,\nu) \in \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{S}^{n-1} : a \neq b\}.$$

It is not clear whether the opposite implication is true, that is, if biconvexity and lower semicontinuity in the above set imply regular biconvexity.
A well known technique exploited to prove lower semicontinuity theorems in spaces of functions which are, in some weak sense, differentiable goes back to the pioneeristic papers of L. Tonelli ([37]) and J. Serrin ([36]). This technique is based on the integration by parts. Recently, by using this method, De Giorgi, Buttazzo and Dal Maso and the author ([2], [3], [19]) have proved general semicontinuity results for functionals of the type

$$\int_\Omega f(x,u,\nabla u)\,dx \qquad u \in W^{1,1}(\Omega)$$

where the integrand $f(x,u,p)$ may also be very discontinuous in (x,u).
Also in this case we want to use the same ideas, and we need a rule to compute the distributional derivative Dv, where $u \in SBV(\Omega;\mathbf{R}^k)$, $f \in C^1(\mathbf{R}^k;\mathbf{R}^m)$ and $v = f(u)$. This problem has been studied by A.I. Vol'pert in [39], even in the case $u \in BV(\Omega;\mathbf{R}^k)$. For functions $u \in SBV(\Omega;\mathbf{R}^k)$ his result can be summarized as follows:

$$Dv(B\cap S_u) = \int_{B\cap S_u} \big(f(u^+) - f(u^-)\big)\otimes \nu_u \, d\mathcal{H}_{n-1} \qquad \forall B \in \mathbf{B}(\Omega). \tag{3.17}$$

where $p\otimes q \in \mathcal{L}_{n,k}$ is the tensor product of $p \in \mathbf{R}^k$ and $q \in \mathbf{R}^n$, and

$$\nabla v = \nabla f(u)\nabla u \qquad \text{a.e. in } \Omega. \tag{3.18}$$

In a recent joint paper with G. Dal Maso (see [7]), the author has proved that (3.17) remains valid even if the function f is only Lipschitz continuous. In this case, (3.18) may be meaningless, because ∇f may not exist on the range of u. Our result shows that for almost every $x \in \Omega$ the restriction of the function f to the tangent space

$$T_x^u = \{z \in \mathbf{R}^k : z = u(x) + \langle \nabla u(x), p\rangle,\ p \in \mathbf{R}^n\}$$

is differentiable at $u(x)$, and

$$\nabla v = \nabla\big(f|_{T_x^u}\big)\nabla u \qquad \text{a.e. in } \Omega.$$

The following lemma is a straightforward consequence of (3.17), (3.18).
Lemma 3.5. *Let $V \in C^1(\mathbf{R}^k;\mathbf{R}^n)$, and let $u \in SBV(\Omega;\mathbf{R}^k)$. We have*

$$\int_{A\cap S_u} \langle V(u^+) - V(u^-), \nu_u\rangle g(x)\,d\mathcal{H}_{n-1}(x) + \int_A g(x)\sum_{i=1}^n\sum_{j=1}^k \frac{\partial V^{(i)}}{\partial u_j}\frac{\partial u^{(j)}}{\partial x_i}\,dx = -\int_A \langle V(u), \nabla g\rangle\,dx$$

for every open set $A \subset \Omega$, for every function $g \in C_0^1(A)$.

Now we have at our disposal all the tools to prove:
Theorem 3.6. *Let $c > 0$ and let $\varphi : \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{S}^{n-1} \to [c,+\infty]$ be a regular biconvex function. Then, for every sequence $(u_h) \subset SBV(\Omega;\mathbf{R}^k) \cap L^\infty(\Omega;\mathbf{R}^k)$ converging in measure to $u \in SBV(\Omega;\mathbf{R}^k)$ and satisfying the conditions*

$$M = \sup_{h\in\mathbf{N}} \|u_h\|_\infty < +\infty; \tag{i}$$

(*ii*) ∇u_h *is an equi-integrable sequence in* $L^1(A;\mathcal{L}_{n,k})$ *for every open set* $A \subset\subset \Omega$;

we have

$$\int_{S_u} \varphi(u^+,u^-,\nu_u)\,d\mathcal{H}_{n-1}(x) \le \liminf_{h\to+\infty} \int_{S_{u_h}} \varphi(u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}_{n-1}(x). \tag{3.19}$$

Proof. It is not restrictive to assume that

$$L = \liminf_{h\to+\infty} \int_{S_{u_h}} \varphi(u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}_{n-1}(x) = \lim_{h\to+\infty} \int_{S_{u_h}} \varphi(u_h^+,u_h^-,\nu_{u_h})\,d\mathcal{H}_{n-1}(x) < +\infty,$$

so that

$$\limsup_{h\to+\infty} \mathcal{H}_{n-1}(S_{u_h}) \le \frac{L}{c} < +\infty. \tag{3.20}$$

By lemma 3.4, we can find a sequence of continuous functions $V_h : \mathbf{R}^k \to \mathbf{R}^n$ such that

$$\varphi(a,b,\nu) = \sup_{h\in\mathbf{N}} \langle V_h(a)-V_h(b),\nu\rangle \qquad \forall a,b\in\mathbf{R}^k,\ a\neq b,\ \nu\in\mathbf{S}^{n-1}. \tag{3.21}$$

It is not difficult to see that (3.21) implies (see for instance [19])

$$\int_{S_v} \varphi(v^+,v^-,\nu_v)\,d\mathcal{H}_{n-1}(x) = \sup\left\{\sum_{i=1}^{N} \int_{A_i\cap S_v} \langle V_i(v^+)-V_i(v^-),\nu_v\rangle^+\,d\mathcal{H}_{n-1}(x)\right\}$$

for every function $v\in SBV(\Omega;\mathbf{R}^k)$, where the supremum is taken over all $N\in\mathbf{N}$ and over all the families $A_1,\ldots,A_N$ of mutually disjoint open sets with closure compact in Ω. Hence, we are led to prove that for every open set $A\subset\subset\Omega$ and for every continuous function $V:\mathbf{R}^k\to\mathbf{R}^n$ we have

$$\int_{A\cap S_u} \langle V(u^+)-V(u^-),\nu_u\rangle^+\,d\mathcal{H}_{n-1}(x) \le \liminf_{h\to+\infty} \int_{A\cap S_{u_h}} \langle V(u_h^+)-V(u_h^-),\nu_{u_h}\rangle^+\,d\mathcal{H}_{n-1}(x). \tag{3.22}$$

By (3.20), since V can be uniformly approximated in $\{x:|x|\le M\}$ by smooth functions, there is no loss of generality if we assume that $V\in C^1(\mathbf{R}^k;\mathbf{R}^n)$.
Let $\epsilon>0$ be given; by assumption (ii) on (∇u_h), we can find an open set $B\subset A$ such that

$$\sup_{h\in\mathbf{N}} \int_B |\nabla u_h|\,dx < \epsilon, \qquad B \supset \bigcup_{h\in\mathbf{N}} S_{u_h}\cup S_u.$$

Let $\mathcal{F}=\{g\in C_0^1(B): 0\le g\le 1\}$. By lemma 3.2 we get

$$\int_{A\cap S_u} \langle V(u^+)-V(u^-),\nu_u\rangle^+ d\mathcal{H}_{n-1}(x) = \sup\{\int_{A\cap S_u} \langle V(u^+)-V(u^-),\nu_u\rangle g(x)d\mathcal{H}_{n-1}(x) : g\in\mathcal{F}\}$$

$$\le C\epsilon + \sup\{\int_A \langle V(u),\nabla g\rangle\,dx : g\in\mathcal{F}\} \le C\epsilon + \liminf_{h\to+\infty} \sup\{\int_A \langle V(u_h),\nabla g\rangle\,dx : g\in\mathcal{F}\} \le$$

$$\le 2C\epsilon + \liminf_{h\to+\infty} \int_{A\cap S_{u_h}} \langle V(u_h^+)-V(u_h^-),\nu_{u_h}\rangle^+\,d\mathcal{H}_{n-1}(x)$$

where $C > 0$ is a constant depending only on V, A, M. Since $\epsilon > 0$ is arbitrary, the proof of the theorem is achieved. **q.e.d.**

By a standard localization argument, it is possible to prove theorem 3.6 also for integrands $\psi(x,u,v,\nu)$ depending on x, provided the functions $\psi(x,\cdot,\cdot,\cdot)$ are regular biconvex for every $x \in \Omega$ and the functions $\psi(\cdot,u,v,\nu)$ are equicontinuous in Ω when u, v vary in compact sets, and $\nu \in \mathbf{S}^{n-1}$. In such a case, one can approximate the function $\psi(x,u,v,\nu)$ by the sums

$$\sum_{i=1}^{N} \chi_{U_i}(x)\psi(x_i,u,v,\nu)$$

with $N \in \mathbf{N}$, U_i mutually disjoint open sets, and $x_i \in U_i$.
Using the property

$$u \in GSBV(\Omega;\mathbf{R}^k) \qquad \Rightarrow \qquad \phi(u) \in SBV_{\rm loc}(\Omega) \quad \forall \phi \in C_0^1(\mathbf{R}^k)$$

it is possible to prove also lower semicontinuity theorems in $GSBV(\Omega;\mathbf{R}^k)$. Let $\psi : \mathbf{R}^n \to [0,+\infty[$ be a convex and positively 1- homogeneous functions such that

$$0 < c \le \psi(\nu) \le C < +\infty, \quad \psi(\nu) = \psi(-\nu) \quad \forall \nu \in \mathbf{S}^{n-1},$$

and let $\Theta : [0,+\infty[\to [c,+\infty[$ be a concave, non decreasing function. We set $\Theta(+\infty) = \sup\{\Theta(t) : t > 0\}$ and $|\infty - z| = +\infty$ for every $u \in \mathbf{R}^k$.

Theorem 3.7. *Let $(u_h) \subset GSBV(\Omega;\mathbf{R}^k)$ be a sequence converging in measure to $u \in GSBV(\Omega;\mathbf{R}^k)$, and assume that (∇u_h) is equi-integrable in $L^1(A;\mathcal{L}_{n,k})$ for every open set $A \subset\subset \Omega$. Then*

$$\int_{S_u} \Theta(|u^+ - u^-|)\psi(\nu_u)\, d\mathcal{H}_{n-1}(x) \le \liminf_{h\to+\infty} \int_{S_{u_h}} \Theta(|u_h^+ - u_h^-|)\psi(\nu_{u_h})\, d\mathcal{H}_{n-1}(x).$$

Proof. The function

$$\varphi(u,v,\nu) = \Theta(|u-v|)\psi(\nu) \qquad u,\, v \in \mathbf{R}^k,\ \nu \in \mathbf{S}^{n-1},$$

is regular biconvex (see §5), because $\Theta(s+t) \le \Theta(s) + \Theta(t)$. We set $\varphi(\infty,u,\nu) = \Theta(+\infty)\psi(\nu)$ for every $u \in \mathbf{R}^k$, $\nu \in \mathbf{S}^{n-1}$. Let $\mathcal{F} \subset C(\mathbf{R}^k \cup \{\infty\})$ be a countable set of functions such that $\mathrm{supp}(\nabla\phi) \subset\subset \mathbf{R}^k$ for every $\phi \in \mathcal{F}$, and

$$|u - v| = \sup_{\phi\in\mathcal{F}} |\phi(u) - \phi(v)| \quad \forall u,\, v \in \mathbf{R}^k \cup \{\infty\},\ u \ne v, \tag{3.23}$$

(one can take for instance $f_{pq}(x) = p \wedge |x - q|$ with $p \in \mathbf{N}$ and $q \in \mathbf{Q}^k$). Proposition 1.1 and proposition 1.2 yield

$$\nabla(\phi(u)) = \nabla\phi(u)\nabla u \qquad \text{a.e. in } \Omega, \qquad S_u = \bigcup_{\phi\in\mathcal{F}} S_{\phi(u)},$$

and

$$|(\phi(u))^+ - (\phi(u))^-| = |\phi(u^+) - \phi(u^-)| \quad \mathcal{H}_{n-1}\text{-a.e. in } S_{\phi(u)} \subset S_u$$

for every function $u \in GSBV(\Omega;\mathbf{R}^k)$ and every function $\phi \in \mathcal{F}$. Applying theorem 3.6 and recalling (2.12) we get

$$\int_{A\cap S_{\phi(u)}} \Theta(|\phi(u^+) - \phi(u^-)|)\psi(\nu_u)\, d\mathcal{H}_{n-1}(x) \le \liminf_{h\to+\infty} \int_{A\cap S_{u_h}} \Theta(|\phi(u_h^+) - \phi(u_h^-)|)\psi(\nu_{u_h})\, d\mathcal{H}_{n-1}(x) \le$$

$$\leq \liminf_{h\to+\infty} \int_{A\cap S_{u_h}} \Theta(|u_h^+ - u_h^-|)\psi(\nu_{u_h})\, d\mathcal{H}_{n-1}(x)$$

for every open set $A \subset \Omega$. To obtain the statement, we have only to remark that (3.23) implies

$$\int_{S_u} \Theta(|u^+ - u^-|)\psi(\nu_u)\, d\mathcal{H}_{n-1}(x) = \sup\left\{\sum_{i=1}^{N} \int_{A_i\cap S_{\phi_i(u)}} \Theta(|\phi_i(u^+) - \phi_i(u^-)|)\psi(\nu_u)\, d\mathcal{H}_{n-1}(x)\right\}$$

where $N \in \mathbf{N}$, $A_1, \ldots, A_N$ are mutually disjoint open subsets of Ω and $\phi_1, \ldots, \phi_N \in \mathcal{F}$. **q.e.d.**

4. Existence theorems.

Our existence theorem is a straightforward consequence of theorem 2.1 and theorem 3.3.

Theorem 4.1. *Let $\varphi : \Omega \times \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{S}^{n-1} \to [0, +\infty[$ be a function satisfying the hypotheses of theorem 3.3. Let $f(x, s, p) : \Omega \times \mathbf{R}^k \times \mathcal{L}_{n,k} \to [0, +\infty[$ be a Borel function, lower semicontinuous in (s, p) and convex in p, such that*

$$f(x, s, p) \geq \phi(|p|) \qquad \forall x \in \Omega,\ s \in \mathbf{R}^k,\ p \in \mathcal{L}_{n,k},$$

with $\phi(t)$ as in (2.1). Then, for every compact set $K \subset \mathbf{R}^k$, the problem

$$\min\left\{ \int_\Omega f(x, u, \nabla u)\, dx + \int_{S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) : u \in SBV_{\mathrm{loc}}(\Omega; \mathbf{R}^k),\ u(x) \in K \text{ a.e. in } \Omega \right\}$$

has a solution in $SBV_{\mathrm{loc}}(\Omega; \mathbf{R}^k)$

Proof. Let $(u_h) \subset SBV_{\mathrm{loc}}(\Omega; \mathbf{R}^k)$ be a minimizing sequence for the problem. By theorem 2.1, we can assume that (u_h) converges in measure to a function $u \in SBV_{\mathrm{loc}}(\Omega; \mathbf{R}^k)$. Of course, the function u still satisfies the constraint $u \in K$ almost everywhere. The conditions (i), (ii) of theorem 3.3 are satisfied, because ϕ has a more than linear growth at infinity. Hence

$$\int_{S_u} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) \leq \liminf_{h\to+\infty} \int_{S_{u_h}} \varphi(x, u_h^+, u_h^-, \nu_{u_h})\, d\mathcal{H}_{n-1}(x). \tag{4.1}$$

On the other hand, since the approximate differentials weakly converge in $L^1(A; \mathcal{L}_{n,k})$ for every bounded open set $A \subset \Omega$, the Ioffe lower semicontinuity theorem (see [27], [28]) yields

$$\int_\Omega f(x, u, \nabla u)\, dx \leq \liminf_{h\to+\infty} \int_\Omega f(x, u_h, \nabla u_h)\, dx.$$

This inequality and (4.1) imply that u is the required solution of the minimization problem. **q.e.d.**

Many variants of theorem 4.1 are possible. For instance, if one needs to consider unbounded integrands $\psi(u, v, \nu)$, one requires regular biconvexity instead of BV-ellipticity, and using theorem 3.4 proves the existence of a solution of the problem

$$\min\left\{ \int_\Omega f(x, u, \nabla u)\, dx + \int_{S_u} \psi(u^+, u^-, \nu_u)\, d\mathcal{H}_{n-1}(x) : u \in SBV(\Omega; \mathbf{R}^k),\ u(x) \in K \text{ a.e. in } \Omega \right\}.$$

A different formulations of theorem 4.1 can be given in $GSBV(\Omega; \mathbf{R}^k)$, when there is no constraint $u \in K$. In this case, one can apply lower semicontinuity theorem 3.7 for the "jump" part of the functional. For the first part, it can be exploited the extension of Ioffe's theorem proved by Balder (see [11]).

5. Examples.

In this section we shall discuss about possible applications of our existence theorems to various recent variational problems.

5.1: Some examples of regular biconvex integrands.

Let us consider the functions $\varphi(u,v,\nu):\mathbf{R}^k\times\mathbf{R}^k\times\mathbf{S}^{n-1}\to[0,+\infty[$ of the form

$$\varphi(u,v,\nu)=\Theta(u,v)\psi(\nu)$$

with $\psi:\mathbf{R}^n\to[0,+\infty[$ convex and positively 1- homogeneous function such that $\psi(p)=\psi(-p)$ and $\Theta:\mathbf{R}^k\times\mathbf{R}^k\to[0,+\infty[$ such that $\Theta(a,b)=\Theta(b,a)$. For this class of integrands, BV- ellipticity and biconvexity are equivalent and can be easily checked.

Proposition 5.1. *(i) If φ is BV-elliptic, $\varphi\not\equiv 0$, then*

$$\Theta(a,b)\le\Theta(a,c)+\Theta(c,b)\qquad\forall a,\,b,\,c\in\mathbf{R}^k. \tag{5.1}$$

(ii) If Θ satisfies condition (5.1), then φ is biconvex.

Proof. (i) Follows easily by the definition of BV-ellipticity (see also (3.6)).
(ii) Let $\tilde\Theta(a,b)=\Theta(a,b)$ if $a\neq b$ and $\tilde\Theta(a,b)=0$ if $a=b$. Let $\partial^-\psi$ be the set of subdifferentials of ψ, i.e.,

$$\partial^-\psi=\big\{z\in\mathbf{R}^n:\psi(p)\ge\langle z,p\rangle\ \forall p\in\mathbf{R}^n\big\}.$$

It is easy to see that (5.1) yields

$$\varphi(a,b,\nu)=\sup\big\{\langle\tilde\Theta(a,c)z-\tilde\Theta(b,c)z,p\rangle:c\in\mathbf{R}^k,\ z\in\partial^-\psi\big\}. \tag{5.2}$$

On the other hand, all the function of the type

$$\langle V(a)-V(b),\nu\rangle$$

with $V:\mathbf{R}^k\to\mathbf{R}^n$ are biconvex, because equality holds in (3.5). Hence, φ is biconvex. q.e.d.

Let $\Theta:[0,+\infty[\to[0,+\infty[$ be a concave function. Then the integrand

$$\varphi(u,v,\nu)=\Theta(|u-v|)\psi(\nu)$$

of theorem 3.7 is regular and biconvex. The biconvexity follows directly by proposition 5.1. Let $\Theta_k(t)=\Theta(t)\wedge kt$. Since

$$\Theta_k(|u-v|)\psi(\nu)=\sup\big\{\langle\Theta_k(|u-c|)z-\Theta_k(|v-c|)z,\nu\rangle:z\in\partial^-\psi,\ c\in\mathbf{R}^k\big\}$$

all the integrands $\Theta_k(|u-v|)\psi(\nu)$ are regular and biconvex. The equality

$$\varphi(u,v,\nu)=\sup_{k\in\mathbf{N}}\Theta_k(|u-v|)\psi(\nu)\qquad u,\,v\in\mathbf{R}^k,\ u\neq v,\ \nu\in\mathbf{S}^{n-1}$$

implies the regularity of φ.
A remarkable example of regular biconvex integrands is given by functions of the form

$$\varphi(u,v,\nu)=\psi(u,\nu)+\psi(v,-\nu)$$

with $\psi(u,p):\mathbf{R}^k\times\mathbf{R}^n\to[0,+\infty]$ lower semicontinuous function, convex and positively 1-homogeneous in p. In this case, the convex and weak* lower semicontinuous extension of $\hat\varphi$ is given by (see [34], and [3], theorem 4.4)

$$\int_{\mathbf{R}^k}\psi(z,\frac{d\mu}{d|\mu|}(z))\,d|\mu|(z).$$

Example 5.2: A functional of pattern recognition problems.

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set, α, $\beta > 0$ and let $w : \Omega \to \mathbf{R}$ be a bounded Borel function. Let us consider the following problem:

$$\inf\left\{ \int_{\Omega\setminus K} |\nabla u|^2\, dx + \beta \mathcal{H}_{n-1}(K) + \alpha \int_{\Omega\setminus K} |w-u|^2\, dx \right\}, \tag{5.3}$$

where the minimization is made over all pairs (u, K) such that K is a closed subset of Ω and $u \in W^{1,2}(\Omega\setminus K)$. Let us call L_1 the infimum in (5.3). We consider now a weak formulation of problem (5.3) in $SBV(\Omega)$:

$$\min\left\{ \int_{\Omega} |\nabla u|^2\, dx + \beta \mathcal{H}_{n-1}(S_u) + \alpha \int_{\Omega} |w-u|^2\, dx : u \in SBV(\Omega) \right\}. \tag{5.4}$$

By theorem 4.1, problem (5.4) has a solution. Infact, let $c = \|w\|_\infty$; since $S_{c\wedge u\vee -c} \subset S_u$ and

$$\int_{\Omega} |\nabla(c\wedge u \vee -c)|^2\, dx + \alpha \int_{\Omega} |w - c\wedge u\vee -c|^2\, dx \le \int_{\Omega} |\nabla u|^2\, dx + \alpha \int_{\Omega} |w-u|^2\, dx$$

we restrict the minimization to functions u essentially bounded by c. Let us call L_2 the minimum value of the functional. We claim that $L_1 = L_2$. The inequality $L_1 \ge L_2$ is not very difficult, and it is a consequence of the following proposition (for a different proof, see [15]):

Proposition 5.2. *Let $K \subset \Omega$ be a closed set such that $\mathcal{H}_{n-1}(K) < +\infty$ and let $u \in W^{1,1}(\Omega\setminus K)\cap L^\infty(\Omega)$. Then, $u \in SBV(\Omega)$ and $\mathcal{H}_{n-1}(S_u\setminus K) = 0$.*

Proof. We give only a sketch of the proof, leaving the details to the reader. Let $\pi \subset \mathbf{R}^n$ be an arbitrary hyperplane normal to $\nu \in \mathbf{S}^{n-1}$, and let

$$\Omega_x = \{t \in \mathbf{R} : x + t\nu \in \Omega\}, \quad u_x(t) = u(x+t\nu)$$

for every $x \in \pi$. By theorem 3.3 of [4], it will be sufficient to show that $u_x \in SBV(\Omega_x)$ for $\mathcal{H}_{n-1}$-almost every $x \in \pi$, and

$$\int_{\pi} |Du_x|(\Omega_x)\, d\mathcal{H}_{n-1}(x) < +\infty, \qquad \mathcal{H}_{n-1}(\{y \in \Omega : y = x + t\nu,\ x\in\pi,\ t \in S_{u_x}\}\setminus K) = 0.$$

This can be easily done recalling that $\mathcal{H}_{n-1} \ge \mathcal{H}_{n-1}|_\pi \times \mathcal{H}_0$ (see [22], 2.10.27), hence $\Omega_x \cap \{t : x + t\nu \in K\}$ has finite cardinality for $\mathcal{H}_{n-1}$-almost every $x \in \pi$. **q.e.d.**

The opposite inequality $L_2 \ge L_1$ is much more difficult. Let $u \in SBV(\Omega)$ be a minimizer of (5.4). A partial regularity theorem recently proved by De Giorgi-Carriero-Leaci (see [15]) shows that $u \in C^1(\Omega\setminus \overline{S}_u)$ and $\mathcal{H}_{n-1}(\overline{S}_u \cap \Omega \setminus S_u) = 0$. Therefore, if we take $K = \overline{S}_u \cap \Omega$, then $u \in W^{1,2}(\Omega\setminus K)$ and

$$\int_{\Omega\setminus K} |\nabla u|^2\, dx + \beta\mathcal{H}_{n-1}(K) + \alpha \int_{\Omega\setminus K} |u-w|^2\, dx = \int_{\Omega} |\nabla u|^2\, dx + \beta \mathcal{H}_{n-1}(S_u) + \alpha\int_{\Omega} |u-w|^2\, dx.$$

In the particular case $n = 2$, this type of variational problem seems to be suitable to model pattern recognition problems in Computer Vision Theory (see for instance [32], [33]). In this case w is the image, tipically given by a camera and corrupted by noise, and the set K represents the edges of the "real" image.

Example 5.3: Approximation by piecewise affine functions.

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set. We say that a function $u \in W^{1,2}(\Omega)$ is piecewise affine if u is continuous and

$$\nabla u = \sum_{i=1}^{N} \alpha_i \chi_{V_i}(x) \qquad \text{a.e. in } \Omega,$$

with $\alpha_1, \ldots, \alpha_N \in \mathbf{R}^n$ and $V_1, \ldots, V_N$ open polyhedral sets such that

$$|\Omega \setminus \bigcup_{i=1}^{N} V_i| = 0.$$

Let $w \in L^2(\Omega)$, α, $\beta > 0$ and let us consider the problem

$$\inf\Big\{\alpha \int_\Omega |u-w|^2\,dx + \int_\Omega |\nabla u|^2\,dx + \beta \mathcal{H}_{n-1}(\gamma_u) : u \in W^{1,2}(\Omega),\ u \text{ piecewise affine}\Big\}, \tag{5.5}$$

where γ_u is the set of discontinuities of ∇u. The functional in (5.5) takes into account the distance (in $L^2(\Omega)$) of w and u, and too large derivatives and too irregular level sets of ∇u are penalized by $\int_\Omega |\nabla u|^2\,dx$ and $\beta\mathcal{H}_{n-1}(\gamma_u)$ respectively.

Since $\nabla u \in SBV(\Omega; \mathbf{R}^n)$ for every piecewise affine function u, it is natural to give to problem (5.5) the following weak formulation: we set

$$\mathcal{G} = \Big\{(u,v) \in W^{1,2}(\Omega) \times GSBV(\Omega; \mathbf{R}^n) : \nabla u = v,\ \nabla v = 0 \text{ a.e. in } \Omega\Big\}$$

and

$$\mathcal{F}(u,v) = \alpha \int_\Omega |u-w|^2\,dx + \int_\Omega |v|^2\,dx + \beta\mathcal{H}_{n-1}(S_v).$$

Using theorem 2.2 and theorem 3.7 we easily see that there exists $(u_0, v_0) \in \mathcal{G}$ such that

$$\mathcal{F}(u_0, v_0) = \min\{\mathcal{F}(u,v) : (u,v) \in \mathcal{G}\}.$$

The function $u_0 \in W^{1,2}(\Omega)$ can be condidered a weak solution of problem (5.5). A reasonable conjecture is that $\mathcal{F}(u_0, \nabla u_0)$ is equal to the infimum in (5.5), and it would be interesting to know whether u_0 is a piecewise affine function or not. It is to be noted that the functional $\mathcal{F}$ is not coercive in $\mathcal{G} \cap W^{1,2}(\Omega) \times SBV(\Omega; \mathbf{R}^n)$, because there is no possibility to control the integral

$$\int_{S_v} |v^+ - v^-|\,d\mathcal{H}_{n-1}(x).$$

Example 5.4: A class of free boundary problems.

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set, let $f : \Omega \times \mathbf{R}^k \times \mathcal{L}_{n,k} \to [0, +\infty]$ be as in theorem 4.1, and let $\psi(x,p) : \mathbf{R}^k \times \mathbf{R}^n \to [0, +\infty]$ be a lower semicontinuous function, convex and positively 1-homogeneous in p. Let $\Gamma \subset \mathbf{R}^k$ be an arbitrary compact set; to fix the ideas, we shall assume that $0 \notin \Gamma$. Let us consider the problem

$$\inf\Big\{\int_D f(x,u,\nabla u)\,dx + \int_{\partial D} \psi(u^*, \nu)\,d\mathcal{H}_{n-1}(x) : |D| = c,\ u \in W^{1,1}(D; \mathbf{R}^k)\Big\} \tag{5.6}$$

where D varies in the open subsets of Ω with piecewise C^1 boundary, u^* is the trace on the boundary ∂D, ν is the inner normal to D, and $u(x) \in \Gamma$ almost everywhere.
A natural weak formulation of problem (5.6) in $SBV(\Omega;\mathbf{R}^k)$ is the following: we set $\Gamma' = \Gamma \cup \{0\}$ and we identify the pair (u, D) with the function $v \in SBV(\Omega;\mathbf{R}^k)$ defined by

$$v(x) = \begin{cases} u(x) & \text{if } x \in D; \\ 0 & \text{otherwise.} \end{cases}$$

Then, the problem (5.6) becomes

$$\inf\left\{ \int_\Omega \tilde{f}(x, v, \nabla v)\, dx + \int_{S_v} \tilde{\psi}(v^+, v^-, \nu_v)\, d\mathcal{H}_{n-1}(x) : v \in SBV(\Omega;\mathbf{R}^k),\ |\{v = 0\}| = c \right\}, \tag{5.7}$$

where

$$\tilde{f}(x, u, p) = \begin{cases} f(x, u, p) & \text{if } u \in \Gamma; \\ 0 & \text{if } u = 0; \\ +\infty & \text{if } u \notin \Gamma', \end{cases}$$

and

$$\tilde{\psi}(u, v, \nu) = \begin{cases} \psi(u, \nu) & \text{if } v = 0; \\ \psi(v, -\nu) & \text{if } u = 0; \\ +\infty & \text{otherwise.} \end{cases} \tag{5.8}$$

If $\psi \geq c > 0$, theorem 2.1 implies that the functional (5.7) is coercive in $SBV(\Omega;\mathbf{R}^k)$. On the other hand, the functional is not necessarily lower semicontinuous, because (5.8) allows only jumps between a point in Γ and 0. To deal with a lower semicontinuous functional, we must assign a finite energy to discontinuities corresponding to points where u^+ and u^- both belong to Γ. The most natural way to do it is to relax the functional in (5.6), setting

$$\overline{F}(v) = \inf\left\{ \liminf_{h\to+\infty} \int_\Omega \tilde{f}(x, v, \nabla v)\, dx + \int_{S_v} \tilde{\psi}(v^+, v^-, \nu_v)\, d\mathcal{H}_{n-1}(x) : v_h \to v \text{ in measure} \right\} \tag{5.9}$$

for every function $v \in SBV(\Omega;\mathbf{R}^k)$. We conjecture that the relaxed functional $\overline{F}$ admits the following representation

$$\overline{F}(v) = \int_\Omega \tilde{f}(x, v, \nabla v)\, dx + \int_{S_v} \overline{\psi}(v^+, v^-, \nu_v)\, d\mathcal{H}_{n-1}(x) \tag{5.10}$$

for every $v \in SBV(\Omega;\mathbf{R}^k)$, where

$$\overline{\psi}(u, v, \nu) = \chi_{\Gamma'}(u)\psi(u, \nu) + \chi_{\Gamma'}(v)\psi(v, -\nu). \tag{5.11}$$

Recalling the remarks following proposition 5.1, the functional in the right hand side of (5.10) is lower semicontinuous and admits minimum. The inequality $\geq$ in (5.10) is consequence of the lower semicontinuity of the functional in the right hand side. The opposite inequality, which is only conjectured, requires an explicit construction of the minimizing sequences in (5.9), which might be done placing along S_v a strip (whose thickness tends to 0) on which the functions v_h are 0.
It must be noted that the integrand $\overline{\psi}$ is equal to ψ for jumps between $u \in \Gamma$ and 0, and it is the greatest function with this property among biconvex integrands (recall (3.6)).
We are hopeful that such mathematical formulation of problem (5.6) could be useful to solve problems of static theory of liquid crystals. In this case, $\Omega \subset \mathbf{R}^3$, $\Gamma = \mathbf{S}^2 \subset \mathbf{R}^3$, and the function $u(x)$ in (5.6) represents

the average direction (optic axis) of the crystal. Tipically, the functions $f(x,u,p)$ contains contributions due to electric and magnetic fields, plus the Oseen-Frank energy (see [21], [24])

$$k_1(\mathrm{div}u)^2 + k_2|\langle u, \mathrm{curl}u\rangle|^2 + k_3|u\wedge \mathrm{curl}u|^2 + (k_2+k_4)\big(\mathrm{tr}(\nabla u)^2 - (\mathrm{div}u)^2\big),$$

with k_i constants depending on temperature. The integrand ψ is frequently taken as ([26], [38])

$$\psi(u,\nu) = \tau(1+\omega|\langle u,\nu\rangle|^2)$$

with $\tau > 0$ and $\omega > -1$, and represents the interface energy with an isotropic liquid.
Problem (5.6) has been studied when D and the boundary values are prescribed (the so-called strong anchoring problem, [13], [26]) or in the case when, being the constant c very small, the first term is negligible with respect to the second one ([8], [37], [41]).
The formulation above in $SBV(\Omega;\mathbf{R}^k)$ could perhaps be useful to deal with intermediate problems. The choice of $\overline{\psi}$ in (5.11) corresponds to imagine an infiltration of the isotropic liquid along the discontinuity.

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