The Tautological Classes of the Moduli Spaces of Stable Maps to Flag Varieties

by

Dragos Nicolae Oprea

Bachelor of Arts, Harvard University, June 2000

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2005

Author ....
Department of Mathematics
April 25, 2005

Certified by ....
Gang Tian
Simons Professor of Mathematics
Thesis Supervisor

Accepted by ....
Pavel Etingof
Chairman, Department Committee on Graduate Students

The Tautological Classes of the Moduli Spaces of Stable Maps to Flag Varieties

by

Dragos Nicolae Oprea

Submitted to the Department of Mathematics
on April 25, 2005, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

We study the tautological classes of the Kontsevich-Manin moduli spaces of genus 0 stable maps to SL flag varieties. We prove that the rational cohomology and rational Chow rings of these spaces are isomorphic and that they are generated by tautological classes.

In the case when the target is a projective space, we present a second proof of this result in the spirit of Gromov-Witten theory by making use of a suitable torus action. In addition, we explicitly describe a Bialynicki-Birula stratification of the Kontsevich-Manin spaces in terms of the Gathmann-Li spaces of relative stable morphisms.

Finally, we analyze the small codimension classes on the space of maps to arbitrary flag varieties. We obtain an explicit description of the Picard groups. We formulate a conjecture about relations between the tautological generators, which we check in low codimension.

Thesis Supervisor: Gang Tian
Title: Simons Professor of Mathematics

Acknowledgments

First of all, I would like to thank my thesis advisor, Professor Gang Tian for his formative mathematical influence. Through his mathematical pursuits and interests, he set a very powerful and motivating example. I am indebted to him for giving me many hours of guidance, for his constant encouragement, and kind patience. On a personal level, he has always been a source of tremendous moral support and good humor.

I gratefully acknowledge Professor Johan de Jong for his help, and especially for explaining a puzzling aspect of the Deligne spectral sequence whose clarification was instrumental for the computations presented in this thesis. I particularly acknowledge a discussion in September 2003, which, from my perspective, occurred at a crucial moment in the writing of this thesis.

I thank Professor Rahul Pandharipande for the interest shown in this work and for the several discussions we had last year in Princeton.

This thesis would not have been possible without Alina Marian's encouragement and advice. She helped me immensely through our numerous mathematical conversations. Most importantly, I am grateful to her for being a great friend, and for listening to me when I needed it the most.

Moreover, I thank Sergiu Vlad, Irina Mihu and Vasil Topuzov for their friendship at various stages in graduate school.

Last, but most definitely not least, I thank my sister, Mihaela Predescu for her uninterrupted support. I am sure that putting up with me all these years has not been entirely easy. It is hard to imagine how my life will be without her.

Contents

Chapter 1

Introduction

The Kontsevich-Manin moduli stacks of stable *maps* arise as generalizations of the classical Deligne-Mumford spaces of stable *curves* $\overline{\mathcal{M}}_{g,n}$. Their intersection theory has been intensively studied in the last decade in relation to enumerative geometry and string theory.

There has been a lot of interest in understanding the cohomology or the Chow groups of the Deligne-Mumford spaces, and partial results are known in low codimension or low genus. Higher genera are particularly difficult since, beginning in genus 1, non algebraic classes do exist. There is a particular collection of cycles, called "tautological," which arise from the built-in inductive structure of the Deligne-Mumford spaces. Nonetheless, the tautological classes do not generate the algebraic cohomology; explicit constructions are known starting in genus $g = 2$ [33].

Very little is known even about the tautological rings, but there is a conjectural description proposed by Faber [21]. The original conjectures concern the open moduli space $\mathcal{M}_g$ of unmarked smooth curves of genus $g > 1$, giving generators and relations for the tautological rings. There are partial extensions of the Faber conjectures for the compactified moduli spaces $\overline{\mathcal{M}}_{g,n}$; these are explained in [53].

By contrast, the moduli space of genus 0 marked curves is well understood. Keel proved that the cohomology is tautological, in fact, that it is additively generated by the boundary classes of curves with fixed dual graph [38]. An easy Hodge theoretic proof of Keel's result is outlined in [28]. All relations between the tautological generators have been found and interpreted in terms of cross ratio. This result has implications, for instance, in the study of the tree-level cohomological field theories [47].

As the moduli spaces of stable curves are examples of Kontsevich-Manin spaces, it was suggested in [53] that it may be useful to push the investigation of the tautological rings in the context of Gromov-Witten theory. A second reason for pursuing such a study comes from the fact that some understanding of the tautological rings of $\overline{\mathcal{M}}_{g,n}$ has been obtained by considering tautological cohomology classes on the Kontsevich-Manin spaces. The Getzler-Ionel vanishing [37], and the socle proof for the tautological rings [34] are such examples.

In this thesis, we prove a generalization of Keel's theorem to the moduli space of genus zero stable morphisms to flag varieties X. By analogy with the case of stable curves, there are natural cohomology classes defined on the moduli spaces of stable maps; their intersection numbers are the Gromov-Witten invariants of X. We show that these natural classes generate the rational cohomology. This implies that the Gromov-Witten invariants essentially capture the *entire* intersection theory of the Kontsevich-Manin moduli spaces.

On a rather negative note, an informal restatement of our result is that no new Gromov-Witten invariants can be defined.

Note that several types of Gromov-Witten invariants have already been studied: descendant invariants, ancestor invariants, invariants twisted by arbitrary characteristic classes. *Explicit* reconstruction results of these invariants are in fact available [13], [43], [5], [42].

We have arrived at the third reason for studying the tautological rings of the stable map spaces. Relations between the tautological classes can be used in the computation of certain intersection numbers. We conjecture that "non-trivial" relations between the genus 0 tautological classes are essentially consequences of Keel's relations. We discuss this speculation in the fourth chapter for the classes of low codimension. Our point of view fits naturally with the aforementioned reconstruction theorems.

We present two proofs of our main result in chapters 2 and 3. One of these proofs allows for partial analogues of Faber conjectures for the relevant open moduli spaces. The other one only works for $\mathbb{P}^r$, but has the advantage of giving an explicit description of the torus stratification of the Kontsevich-Manin spaces in terms of the moduli spaces of relative stable morphisms as defined by Gathmann and Li.

In the remainder of this chapter, we introduce the relevant definitions, state the results, and briefly mention the ideas involved in their proofs.

1.1 The tautological systems

To set the stage, we let X be a convex complex projective manifold, typically, a flag variety. The moduli stacks $\overline{\mathcal{M}}_{0,S}(X,\beta)$ parametrize S-pointed genus zero stable maps to X in the homology class $\beta \in H_2(X,\mathbb{Z})$. We use the notation $\overline{\mathcal{M}}_{0,n}(X,\beta)$ when the marking set is $S=\{1,2,\ldots,n\}$.

Among the natural cohomology classes, we single out the boundary classes of maps whose domain curve has a fixed dual graph. We may impose additional constraints making the marked points or nodes map to certain Schubert subvarieties of X, and requiring that the image of the map intersect various Schubert subvarieties. These classes are shown diagrammatically in figure 1.1.

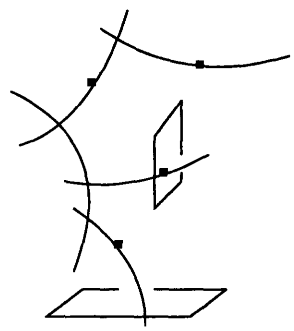


Figure 1-1: The tautological generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$.

The geometric Schubert-type classes we just described are typical elements of the tautological rings which we now define. To this end, we observe that the stable map spaces are connected by a complicated system of natural morphisms, which we enumerate below:

- forgetful morphisms: $\pi : \overline{\mathcal{M}}_{0,S}(X,\beta) \to \overline{\mathcal{M}}_{0,T}(X,\beta)$ defined whenever $T \subset S$;

- gluing morphisms which produce maps with nodal domains,

$$gl : \overline{\mathcal{M}}_{0,S_1\cup\{\bullet\}}(X,\beta_1)\times_X\overline{\mathcal{M}}_{0,\{\bullet\}\cup S_2}(X,\beta_2) \to \overline{\mathcal{M}}_{0,S_1\cup S_2}(X,\beta_1+\beta_2);$$

- evaluation morphisms to the target space, $ev_i : \overline{\mathcal{M}}_{0,S}(X,\beta) \to X$ for all $i \in S$.

The classes pulled back from the target serve as the seed data for the tautological systems. We get more classes making use of the forgetful and gluing morphisms. It is then necessary to consider all the moduli spaces $\overline{\mathcal{M}}_{0,n}(X,\beta)$ together.

Definition 1.1.1. The genus 0 tautological rings $R^\star(\overline{\mathcal{M}}_{0,n}(X,\beta))$ are the smallest system of *subrings* of the rational cohomology $H^\star(\overline{\mathcal{M}}_{0,n}(X,\beta))$ such that:

- The system is closed under pushforwards by the gluing and forgetful morphisms.
- The evaluation classes $ev_i^\star\alpha$ where $\alpha \in H^\star(X)$ are in the system.

We refer the reader to the appendix for a slightly different (though, in the relevant cases, entirely equivalent) definition of the tautological *systems* in higher genus and non-convex targets. In this thesis, the higher genus systems will be of secondary interest.

Remark 1.1.1. (i) Most stacks considered in this thesis will be smooth and of Deligne Mumford type over $\mathbb{C}$. This allows us to speak about their Chow *rings* as defined in [59] and of their cohomology as defined in [2]. Here, *rational coefficients* will always be understood.

(ii) Additionally, one can consider the rational cohomology or rational Chow rings of the coarse moduli schemes. Pullback and pushforward by $p : \overline{\mathcal{M}}_{0,n}(X,\beta) \to \overline{M}_{0,n}(X,\beta)$ induce inverse isomorphisms in cohomology [59], [2]. It is useful to observe that a generic stable map has no automorphisms. This follows from the arguments of lemma 2.1.1 in [51]. There are two exceptions when $X = \mathbb{P}^1$ or $X = \mathbb{P}^2$, $n = 0$, and the degree is 2. We will ignore these cases here. The main theorem stated below still holds in these cases by virtue of remark 2.5 in [3].

(iii) It is customary to include the ψ classes in the definition of the tautological rings of the Deligne Mumford spaces. However, it is not necessary to include the ψ classes in the genus 0 tautological system defined above. Indeed, lemma 2.2.2 in [51] expresses the ψ's in terms of evaluation classes, boundaries and the κ classes below. An alternative explanation can be found in the introduction of [24]. Here, $\psi_i = c_1(\mathcal{L}_i)$, where the fiber of the line bundle $\mathcal{L}_i \to \overline{\mathcal{M}}_{0,n}(X,\beta)$ over a stable map with domain C and markings $x_1, \dots x_n$ is the cotangent line $T^\star_{x_i}C$.

We will consider a certain collection of tautological classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ indexed by *weighted graphs.*

Definition 1.1.2. A weighted graph $(\Gamma, \mathfrak{w}, \mathfrak{f})$ consists of:

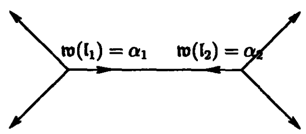
- a stable modular graph Γ of total degree β, genus 0, and with n labeled legs. This graph determines the boundary type.
- the weights $\mathfrak{w}$ are an assignment of cohomology classes on X to the half-edges and legs of Γ keeping track of the incidence conditions with certain cycles in X. The weight of an edge is then defined as the product of weights of its half-edges.

- the "forgetting" data $\mathfrak{f}$ is a subset of the legs of Γ which remembers if the incidence points with the fixed cycles determined by the weights come from markings of the domain or not.

The tautological cohomology class $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ is a class on the moduli space of maps of degree β and markings in $\mathrm{Legs}(\Gamma) \setminus \mathfrak{f}$. It is defined by equations (A.1), (A.3) and (A.4) in the appendix using the forgetful pushforward π induced by $\mathfrak{f}$, and the gluing map ζ_Γ induced by the boundary graph Γ:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = \pi_\star \zeta_\Gamma \left(\prod_{v \text{ vertex}} \left(\prod_{\text{flags } f \text{ adjacent to } v} ev_f^\star \, \mathfrak{w}(f) \cap [\overline{\mathcal{M}}_{0,n_v}(X, \beta_v)] \right) \right). \tag{1.1}$$

It is a consequence of the tautological relations discussed below that the generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ depend on the total weight of the legs and edges, not on the weights of the half-edges. For instance, the generator below is dependent on the product cohomology class $\mathfrak{w}(e) = \mathfrak{w}(\mathfrak{l}_1) \cdot \mathfrak{w}(\mathfrak{l}_2) = \alpha_1 \cdot \alpha_2$.



The classes corresponding to the graphs below are (via gluing) the building blocks of our system of tautological generators. They will appear frequently in our computations. Here Γ is a one vertex graph with $n+p$ legs such that p of them, carrying weights $\alpha_1, \alpha_2, \dots, \alpha_p \in H^\star(X)$, form the forgetting data $\mathfrak{f}$; there are no other non-trivial weights.

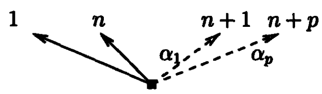


Figure 1-2: The graph corresponding to the κ classes.

We write $\kappa_n(\alpha_1, \dots, \alpha_p)$ for the corresponding class:

$$\kappa_n(\alpha_1, \dots, \alpha_p) = \pi_\star(ev_{n+1}^\star \alpha_1 \cdot \ldots \cdot ev_{n+p}^\star \alpha_p \cap [\overline{\mathcal{M}}_{0,n}(X, \beta)]). \tag{1.2}$$

1.2 Main result

As already announced above, the main result of this thesis is the following extension of Keel's theorem from the case of rational marked curves to the case of rational stable maps to SL flag varieties.

Theorem 1. *Let X be any SL flag variety over the complex numbers. Then all rational cohomology classes on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ are tautological.*

Corollary 1.2.1. *The classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ additively generate the cohomology of the spaces $\overline{\mathcal{M}}_{0,n}(X, \beta)$.*

Our proof of theorem 1 is given in chapter 2. As in [28], it relies on the Deligne spectral sequence. Most commonly, this spectral sequence is used to define the mixed Hodge structure on the cohomology of smooth open varieties. We will use the reverse procedure: we will identify the lowest weight Hodge piece in the cohomology of the open stratum and use it to derive information about the cohomology of the compactified moduli spaces. We do need to rely on Hodge theoretic considerations in order to deal with subtleties coming from the boundary cycles. Our argument is inductive, making use of the fact that the open stratum is compactified by adding normal crossing divisors which are essentially lower dimensional moduli spaces of stable maps.

Two observations are necessary in order to identify the relevant piece of the Hodge structure. First, the morphism spaces are hard to understand cohomologically. For this reason, we will appeal, as in [6], to the different compactification provided by the Hyper-Quot scheme. It turns out that replacing morphisms by quotient sheaves is more suited for understanding the cohomology of the relevant moduli spaces. Secondly, we use a technique which goes back to Atiyah and Bott: when the number of markings is small, we express the open stratum as a global quotient and carry out the computation in equivariant cohomology. A similar program was partially pursued in [52] for unpointed maps to $\mathbb{P}^r$. The general case of a flag variety is more involved, essentially because the morphism spaces to flag varieties are more difficult to describe. When X is a Grassmannian, we rely on Stromme's description of the *Quot* scheme as the base of a principal bundle sitting in an affine space [57]. Combined with the Atiyah-Bott technique we obtain enough information about the Hodge structure of the open stratum to prove our main result. When X is any SL flag variety, the proof above does not immediately carry over. We will make use of the already proved results for Grassmannians combined with a general statement about the cohomology of the Hyper-Quot scheme $HQuot$, which we obtain using a well known trick due to Ellingsrud-Stromme-Beauville [20]. This will turn out to be sufficient to finish the proof.

Our result holds either in cohomology or in the Chow groups. We will write most of the arguments in cohomology. Moreover, we will show in chapter 3 that as a consequence of localization, *rational cohomology and rational Chow groups are isomorphic.* For the reader interested in the Chow groups of the *open strata*, we indicate the necessary changes in our proofs at the appropriate places. The *only* part of the argument which we do not carry out in the Chow groups is the analysis of the boundary classes. It is conceivable that one can write down an entirely algebraic proof of our main result, which would then work over any field, possibly exploiting the torus action.

In addition, when $X = \mathbb{P}^r$, a rewriting of the above argument gives a better understanding of the Chow groups of the open stratum. Just as in Pandharipande's result, we observe that the final result is degree independent. We claim no originality for the ideas used to prove the result stated below. We will briefly review the argument in chapter 2, simply because it agrees with the philosophy of Faber's conjectures and because it allows for a proof of the first item in theorem 3. For now, we would like to remark that our computation is indeed consistent with the recent announcement [31]. We believe that the statement below should hold for a larger class of varieties (such as toric varieties and flag manifolds).

Proposition 1.2.1. *The Chow rings of $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ can be described explicitly in terms of the tautological classes in definition* 1.1.1. *They Chow rings behave like the cohomology of certain degree independent projective manifolds of lower dimension.*

1.3 Related work

All previous work on the topology of the stable map spaces considered the case when the target is $X = \mathbb{P}^r$. For general flags a generation result for the cohomology, although not surprising, is only available via the results presented in this thesis.

The computation of the Betti numbers, controlling the size of the cohomology, has been achieved only recently by Getzler and Pandharipande [31] by a clever summation trick. The answer is highly recursive, so extracting the individual numbers is slightly involved. One should be able to extend the formal part of their computation to arbitrary flags. Our main theorem should be viewed as complementary to these results, as it provides generators for the cohomology.

For projective spaces and in complex codimension 1, earlier work of Pandharipande [51] implies the statement of the main theorem. We will repeatedly use, and reprove, some of his results in this thesis. Of course, the degree 0 case had been solved by Keel, while the degree 1 case follows from work of Fulton and MacPherson. Behrend and O'Halloran [3] have a method of computing the cohomology ring in degrees 2 and 3 for maps to $\mathbb{P}^r$ without markings, essentially using localization techniques - their result implies ours in degrees 2 and 3. These authors also observe that the cohomology stabilizes as $r \to \infty$, and propose a study of the stabilized rings. After we proved our results, Mustata and Mustata found a presentation of the cohomology ring for one marking and arbitrary degree maps to $\mathbb{P}^r$ [50]. Their result implies ours in the cases where both apply. However, it is not entirely clear how to extend their approach to arbitrary flags or/and no markings. We also believe our approach is more natural for the study of the tautological classes in the context of Gromov-Witten theory, since it fits in better with the known results for $\overline{\mathcal{M}}_{0,n}$.

We should mention that the topologists have a good understanding of the spaces of holomorphic maps to flag varieties $\mathrm{Map}_\beta(\mathbb{P}^1, X)$; work in this direction was initiated by Segal. However, our results are of entirely different nature - by contrast with the spaces studied in topology, the stable map spaces have only algebraic cohomology.

1.4 Localization

As it often happens, one attempts to apply localization techniques to understand the classes on the Kontsevich-Manin moduli spaces. Such an approach is tempting in our case as well, especially because the fixed point loci are known to be products of moduli spaces of rational marked curves whose cohomology groups are indeed tautological. However, the author could not obtain a full proof of the above theorem following this line of reasoning unless the target is $\mathbb{P}^r$, as we will explain shortly. Nonetheless, Rahul Pandharipande argued that the localization theorem in [19] can be used to show that the equivariant Chow rings of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are tautological after *inverting* the torus characters. However, such an argument is not entirely straightforward and does not prove our main theorem.

Localization is a popular theme in Gromov-Witten theory, used extensively since the early papers on the subject. In chapter 3 we show that this powerful technique can be used to give a second proof of our main theorem for maps to $\mathbb{P}^r$. Our approach is novel in two ways. First, we make use of a non-generic torus action on $\mathbb{P}^r$ which fixes one point p and a hyperplane H:

$$t \cdot [z_0 : z_1 : \ldots : z_r] = [z_0 : tz_1 : \ldots : tz_r].$$

Secondly, we completely determine the Bialynicki-Birula *plus* decomposition of the *stack* of stable maps which describes the flow of maps under this action. In addition, we show that the decomposition is *filterable.* As a consequence, we build up the stack of stable maps by adding cells in a *well determined order.* This is the algebraic analogue of the Morse stratification, whose cells can be ordered by the levels of the critical sets. A filterable decomposition also gives a way of computing the Poincare polynomials of the moduli spaces of stable maps from those of the fixed loci. This method works quite well in low codimension, as we will demonstrate in chapter 4.

Note that the Bialynicki-Birula decomposition has not been established for *general* smooth Deligne-Mumford *stacks* with a torus action. Our approach for constructing the plus cells applies whenever we have an equivariant etale affine atlas. In the present case, we succeed to explicitly write it down in the context of Gathmann's stacks [25]. These stacks compactify the locus of marked maps with contact orders $\alpha_1, \ldots, \alpha_n$ with the hyperplane H. They can be obtained as blowdowns of more general spaces of relative stable morphisms defined by Jun Li.

Decorated graphs Γ will be used to bookkeep the fixed loci, henceforth denoted $\mathcal{F}_\Gamma$. Their vertices correspond to components or points of the domain mapped entirely to p or H, and carry numbered legs for each of the markings, and degree labels. The edges, also decorated by degrees, correspond to the remaining components. We repackage the datum of a decorated graph Γ into an explicit fibered product $\overline{\mathcal{Y}}_\Gamma$ of Kontsevich-Manin and Gathmann-Li spaces in equation (3.11). We summarize the properties of the plus decomposition:

Theorem 2. *The stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ can be decomposed into disjoint locally closed substacks $\mathcal{F}_\Gamma^+$ (the "plus" cells of maps "flowing" into $\mathcal{F}_\Gamma$) such that:*

(1) The fixed loci $\mathcal{F}_\Gamma$ are substacks of $\mathcal{F}_\Gamma^+$. There are projection morphisms $\mathcal{F}_\Gamma^+ \to \mathcal{F}_\Gamma$. On the level of coarse moduli schemes, we obtain the plus Bialynicki-Birula decomposition of the coarse moduli scheme of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

(2) The decomposition is filterable. That is, there is a partial ordering of the graphs Γ such that
$$\overline{\mathcal{F}_\Gamma^+} \subset \bigcup_{\Gamma' \leq \Gamma} \mathcal{F}_{\Gamma'}^+.$$

(3) The closures of $\mathcal{F}_\Gamma^+$ are images of the fibered products $\overline{\mathcal{Y}}_\Gamma$ of Kontsevich-Manin and Gathmann-Li spaces under the tautological morphisms.

(4) The codimension of $\mathcal{F}_\Gamma^+$ can be explicitly computed from the graph Γ. If $\mathfrak{u}$ is the number of H-labeled vertices of degree 0, with no legs and $\mathfrak{s}$ is the number of H-labeled vertices which have positive degree or total valency at least 3, then the codimension is $d+\mathfrak{s}-\mathfrak{u}$.

(5) The rational cohomology and rational Chow groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are isomorphic.

(6) (There exists a collection of substacks ξ which span the rational Chow groups of $\mathcal{F}_\Gamma$ and) there exist closed substacks $\overline{\xi^+}$ supported in $\overline{\mathcal{F}_\Gamma^+}$ (compactifying the locus of maps flowing into ξ), which span the rational Chow groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. The stacks $\overline{\xi^+}$ are images of fibered products of Gathmann-Li spaces and tautological substacks of the Kontsevich-Manin spaces to H.

(7) The cycles constructed in (6) are tautological.

1.5 Examples

In the case of general SL flags, we carefully analyze the low codimension strata by the techniques afforded by theorems 1 and 2. First, we extend Pandharipande's computation [51] to determine the divisors on the stable map spaces to general flags. We prove:

Proposition 1.5.1. *Let X be any SL flag whose Betti numbers in dimension 2 and 4 are h^2 and h^4. Let*

$$V \otimes \mathcal{O}_X = \mathcal{Q}_0 \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_l \to 0$$

be the tautological quotients and let β be a class with $d_i = \beta \cdot c_1(\mathcal{Q}_i) > 0$. The dimension of the rational Picard group of $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is

$$\left[2^{n-1}(d_1+1)\ldots(d_l+1) + \frac{1}{2}\right] - 1 - \binom{n}{2} + h^4 - \binom{h^2}{2}.$$

The generators of the Picard group are

- *the boundary divisors,*
- *the classes $\kappa(\alpha)$ where α is either $c_1(\mathcal{Q}_i)^2$ for $1 \leq i \leq l$ or the nonzero classes $c_2(\mathcal{K}_j)$ for $0 \leq j \leq l$. Here $\mathcal{K}_j$ is the kernel of $\mathcal{Q}_j \to \mathcal{Q}_{j+1}$.*
- *when $n = 1$ or $n = 2$, we add any one of the evaluation classes $ev_i^\star c_1(\mathcal{Q}_j)$.*

The class

$$\sum_i \kappa(c_2(\mathcal{K}_i)) + \sum_i \left(\frac{d_{i-1}+d_{i+1}}{2d_i} - 1\right)\kappa(c_1(\mathcal{Q}_i)^2)$$

is supported on the boundary. All other relations between the boundary divisors come from $\overline{\mathcal{M}}_{0,n}$ by pullback.

Unfortunately, Pandharipande's argument does not carry over because, as already observed above, the space of morphisms to arbitrary flags is harder to understand than the morphism spaces to $\mathbb{P}^r$. Instead, we use our theorem 1 to find generators. Localization is used to determine the dimension of the Picard group. This requires a computation of the symmetric group invariants on the cohomology of the moduli space of rational marked curves similar to Getzler's [28]. To reconcile the count of generators with the dimension, relations between the κ classes are exhibited. The precise statements will be given at the appropriate places.

1.6 Tautological relations

We now indicate how relations between the tautological classes are obtained via a very simple algorithmic procedure. We start by succinctly reviewing what is known in degree 0 by work of Keel. Relations between the additive generators $[\overline{\mathcal{M}}(\Gamma)]$ of $H^\star(\overline{\mathcal{M}}_{0,n})$ are obtained by cross ratio. For example, in codimension 1, we fix markings i, j, k, l and distribute them on the branches of a nodal stable curve in two ways: $(ij)(kl)$ and $(ik)(jl)$. The remaining markings are distributed arbitrarily on the branches. This yields two sums of boundary divisors which are linearly equivalent. We pictorially show this relation below. Moreover, it is easy to see how this generalizes to arbitrary codimension strata, by adding

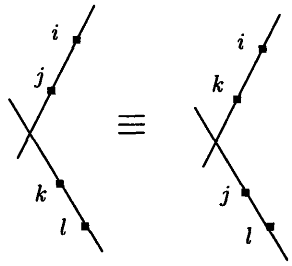


Figure 1-3: Keel's relations on $\overline{\mathcal{M}}_{0,n}$.

more branches. It is shown in [47] that these are all possible relations between the additive generators $[\overline{\mathcal{M}}(\Gamma)]$.

We propose a description of the additive structure of the tautological systems of the stable map spaces. The only difference from the case studied by Keel is the presence of the incidence conditions. This introduces additional complications in our analysis, mostly coming from degree 0, evaluation at divisor classes, and from relations in the cohomology of the target.

We regard the tautological systems as a sequence of abelian groups connected by forgetful morphisms:

$$\ldots \xrightarrow{\pi_\star} R^\star(\overline{\mathcal{M}}_{0,n}(X,\beta)) \xrightarrow{\pi_\star} R^\star(\overline{\mathcal{M}}_{0,n-1}(X,\beta)) \xrightarrow{\pi_\star} \ldots .$$

The additive generators of these groups are symbols $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ indexed by weighted graphs such that $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ lives on the moduli space of maps to X of degree the total degree of Γ and whose markings are Legs $(\Gamma) \setminus \mathfrak{f}$. We require:

- (automorphisms) The class $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ only depends on the isomorphism class of the triple $[\Gamma, \mathfrak{w}, \mathfrak{f}]$.

The forgetting maps $\pi_\star$ are defined as follows:

- (coherence) Let Γ be a stable graph with cohomology weights $\mathfrak{w}$ and forgetting data $\mathfrak{f}$, and let $\mathfrak{l} \notin \mathfrak{f}$ be a leg of Γ inducing a forgetful morphism π. Let $\tilde{\mathfrak{f}} = \mathfrak{f} \cup \{\mathfrak{l}\}$ be new forgetting data for Γ. Then:

$$\pi_\star [\Gamma, \mathfrak{w}, \mathfrak{f}] = \left[\Gamma, \mathfrak{w}, \tilde{\mathfrak{f}}\right].$$

  Note that for now we do not worry about forgetting a destabilizing marking, we will take care of this issue shortly. In the figure below, the legs which are forgotten are indicated by dashed lines.

In addition, we have the following relations between our generators:

- (no incidences) Assume $\mathfrak{w}(\mathfrak{l}) = 1 \in H^0(X)$ for a leg $\mathfrak{l}$ of Γ which is part of the forgetting data $\mathfrak{f}$, and which does not destabilize Γ by forgetting. Then:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = 0.$$

- (divisor) Let $\mathfrak{l}$ be a leg attached to a vertex of degree β, which is part of the forgetting data $\mathfrak{f}$. Assume $\mathfrak{w}(\mathfrak{l}) = \mathfrak{d}$ is a divisor with $\beta \cdot \mathfrak{d} \neq 0$. We let $\tilde{\Gamma}$ be the graph obtained by

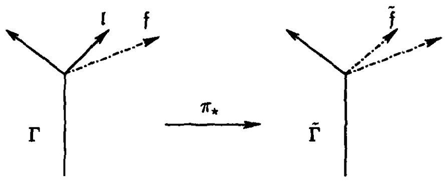


Figure 1-4: The forgetting maps

forgetting the leg $\mathfrak{l}$, with weights $\tilde{\mathfrak{w}} = \mathfrak{w} \setminus \{\mathfrak{d}\}$, and *forgetting data* $\tilde{\mathfrak{f}} = \mathfrak{f} \setminus \{\mathfrak{l}\}$. Then:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = (\beta \cdot \mathfrak{d}) \left[\tilde{\Gamma}, \tilde{\mathfrak{w}}, \tilde{\mathfrak{f}}\right]$$

- (forgetting destabilizing legs) Assume $\mathfrak{w}(\mathfrak{l}) = 1$ for a leg $\mathfrak{l}$ which is part of the forgetting data $\mathfrak{f}$ and which does destabilize Γ after forgetting. Let $\tilde{\Gamma}$ be the graph obtained by forgetting $\mathfrak{l}$ and stabilizing: the graph $\tilde{\Gamma}$ has one less vertex, and one less leg. Let $\tilde{\mathfrak{w}}$ be the induced weights (the weights of the new flags are defined by multiplying the weights of the collapsed flags) and let $\tilde{\mathfrak{f}}$ be forgetting data. Then:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = \left[\tilde{\Gamma}, \tilde{\mathfrak{w}}, \tilde{\mathfrak{f}}\right].$$

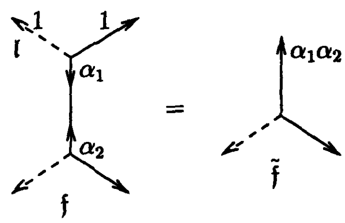


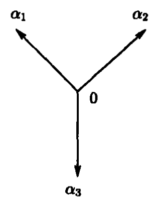
Figure 1-5: Forgetting destabilizing legs

- (mapping to a "point") Let $\pi = (\pi_1, \pi_2) : X \longrightarrow Y_1 \times Y_2$ be the embedding of X into a product of two flag varieties which is obtained by remembering complementary steps in the flags of X (for example Y_2 can be a point). Assume Γ contains two legs $\mathfrak{l}_1$ and $\mathfrak{l}_2$ adjacent to a vertex $\mathfrak{v}$ whose degree $\beta_{\mathfrak{v}} \in \pi_2^\star H^\star(Y_2)$. Let $\mathfrak{w}_1$ and $\mathfrak{w}_2$ be weights of Γ which differ only in the assignment of the cohomology class $\pi_1^\star \alpha$, $\alpha \in H^\star(Y_1)$ to the legs $\mathfrak{l}_1$ and $\mathfrak{l}_2$ respectively, otherwise being identical. Then:

$$[\Gamma, \mathfrak{w}_1, \mathfrak{f}] = [\Gamma, \mathfrak{w}_2, \mathfrak{f}].$$

  It is a consequence of this relation and gluing that for each degree 0 vertex $\mathfrak{v}$ in Γ, the corresponding generator only depends on the product of weights incident to $\mathfrak{v}$. For instance, in the figure below, the corresponding generator depends on the product $\alpha_1 \cdot \alpha_2 \cdot \alpha_3$.

- (pullbacks from X) Let $\mathfrak{w}, \mathfrak{w}_1, \mathfrak{w}_2$ be weights of the graph Γ which agree everywhere



except for a leg $\mathfrak{l}$ for which we have:

$$\mathfrak{w}(\mathfrak{l}) = \mathfrak{w}_1(\mathfrak{l}) + \mathfrak{w}_2(\mathfrak{l}).$$

Then:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = [\Gamma, \mathfrak{w}_1, \mathfrak{f}] + [\Gamma, \mathfrak{w}_2, \mathfrak{f}] .$$

In addition, Keel's theorem gives another way of obtaining non-trivial relations. We first fix a Keel relation on $\overline{\mathcal{M}}_{0,n}$, pull it back to the space of stable maps $\overline{\mathcal{M}}_{0,n}(X,\beta)$, intersect it with a monomial in evaluation classes and use the forgetful pushforward. We chose to split this procedure in several steps: at first we ignore the evaluation monomials, which we can add in later by gluing degree 0 tripods and forgetting (see section 2.3.4). For the same reason, we only need codimension 1 Keel relations:

- (Keel relations/WDVV) The pullback of a Keel relation under the stabilization map $st : \overline{\mathcal{M}}_{0,n}(X,\beta) \to \overline{\mathcal{M}}_{0,n}$ is a relation between the classes $[\Gamma, \emptyset, \emptyset]$. Here, we agree that for each graph stable graph Γ of degree 0, we have

$$st^{\star}\left[\overline{\mathcal{M}}(\Gamma)\right] = \sum_{\tilde{\Gamma}} \left[\tilde{\Gamma}, \emptyset, \emptyset\right]$$

the sum being taken over all possible ways of decorating Γ by degrees summing up to β.

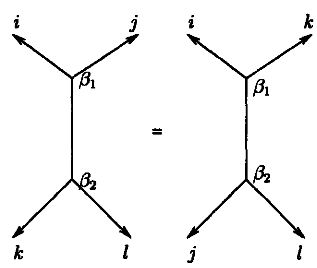


Figure 1-6: The pullbacks of Keel relations.

We can get even more relations by gluing. To this end, we define:

- (gluing generators) Let $(\Gamma_1, \mathfrak{w}_1, \mathfrak{f}_1)$ and $(\Gamma_2, \mathfrak{w}_2, \mathfrak{f}_2)$ be weighted graphs and let $\mathfrak{l}_1 \in \mathrm{Legs}(\Gamma_1) \setminus \mathfrak{f}_1$ and $\mathfrak{l}_2 \in \mathrm{Legs}(\Gamma_2) \setminus \mathfrak{f}_2$ be two unforgotten legs. The glued graph:

$$(\Gamma_1, \mathfrak{w}_1, \mathfrak{f}_1) \star (\Gamma_2, \mathfrak{w}_2, \mathfrak{f}_2)$$

is obtained by joining Γ_1 and Γ_2 along an edge whose half edges are $\mathfrak{l}_1$ and $\mathfrak{l}_2$. The weights and forgetting data are obtained by collecting the weights and forgetting data of Γ_1 and Γ_2.

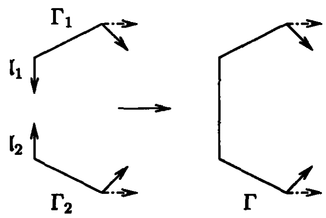


Figure 1-7: Gluing graphs.

We can now define gluing of relations as follows:

- (gluing relations) Let $(\Gamma_i, \mathfrak{w}_i, \mathfrak{f}_i)$ be data indexing generators on the same moduli space; that is, the total degree of Γ_i is independent of i, and the marking sets Legs $(\Gamma_i) \setminus \mathfrak{f}_i$ are independent of i. Let $\mathfrak{l} \in$ Legs $(\Gamma_i) \setminus \mathfrak{f}_i$. Let $(\Gamma, \mathfrak{w}, \mathfrak{f})$ be another weighted graph and let $\widehat{\mathfrak{l}} \in$ Legs $(\Gamma) \setminus \mathfrak{f}$. Assume we have a relation:

$$\sum_i [\Gamma_i, \mathfrak{w}_i, \mathfrak{f}_i] = 0.$$

Then, a new relation is obtained by gluing along $\mathfrak{l}$ and $\widehat{\mathfrak{l}}$:

$$\sum_i [\Gamma, \mathfrak{w}, \mathfrak{f}] \star [\Gamma_i, \mathfrak{w}_i, \mathfrak{f}_i] = 0.$$

We can now define the tautological relations.

Definition 1.6.1. The tautological relations are the smallest system of relations between the generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ with the following properties:

- All relations listed above are already in the system.
- The system of relations is closed under the morphisms $\pi_\star$.
- The system of relations is closed under gluing.

We believe the tautological systems are *generally* insensitive to the geometry of the target space. More formally, we pose the following:

Question 1.6.1. *Is it true that all relations between the additive generators* $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ *are obtained from the tautological relations above?*

It is remarkable that all relations we could find in the literature are of this nature [51], [43], [5], [16], [14]. However, these relations are highly non-trivial since they are incarnations, via pushforward, of relations on different moduli spaces and in higher codimension!!! We have a similar universal way of producing relations involving the cotangent lines; these will be briefly mentioned in the last chapter.

We already made several checks of the above statement in low codimensions and low degrees. We verified the cases when the target is $X = \mathbb{P}^r$, $d = 1$ and n arbitrary, $d \leq 3$ and $n \leq 1$ in complex codimension up to 4, also $X = \mathbb{P}^1$, and all $d \leq 5$ and $n \leq 1$, codimension 2 for any flag X, $n \leq 3$. We conjecture the answer to question 1.6.1 is affirmative in general, but we will investigate this question elsewhere.

We offer some more evidence in chapter 4 of this thesis:

Theorem 3. *(1) All relations between the tautological generators of the open moduli spaces $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ are restrictions of the tautological relations.*

(2) The statement in question 1.6.1 *is true for all SL flags manifolds in codimension* 1*. It also holds in codimension* 2 *for projective spaces.*

Chapter 2

Stable maps to flag varieties

In this chapter we indicate a Hodge theoretic proof of theorem 1. We organized this chapter as follows. The following section contains generalities about the spaces of stable maps. We collect there known results and we fix the notations. We discuss the stratification with respect to the dual graphs and the associated Deligne spectral sequence. We will indicate the proof of theorem 1 in the second section. An outline of the ideas involved in the proof can be found in the introductory chapter. Finally, we will discuss in some detail the case of maps to projective spaces, obtaining some improvements of earlier results of Pandharipande [52]. As a consequence we obtain the first part of theorem 3.

2.1 Preliminaries

To begin, we let $X = G/P$ be a projective algebraic homogeneous space, where G is a semisimple algebraic group and P is a parabolic subgroup. We also fix $\beta \in H_2(X, \mathbb{Z})$ a homology class.

We let $\overline{M}_{0,n}(X,\beta)$ be the coarse moduli scheme of n pointed stable maps to X in the class β. A construction of the moduli scheme in algebraic geometry was achieved by Fulton and Pandharipande in [22]. It is shown there that the moduli scheme is a normal projective variety with finite quotient singularities - an orbifold if we work in the analytic category. In this section, we will discuss the coarse moduli schemes, but it should be clear how to extend our conclusions to the moduli stacks.

2.1.1 The stratification by dual graphs.

To each stable map we associate the dual tree which carries degree labels and legs. We agree that a vertex v has degree β_v and $n(v)$ incident flags (i.e. edges and legs). The moduli space $\overline{M}_{0,n}(X,\beta)$ is a union of strata consisting in maps with fixed dual graph Γ. The closure $\overline{M}(\Gamma)$ of this stratum can be described as the image of a ramified covering of degree $Aut(\Gamma)$. The relevant morphism is:

$$\zeta_\Gamma : \left(\prod_{v \in V(\Gamma)} \overline{M}_{0,n(v)}(X, \beta_v) \right)^{\mathrm{Edge}(\Gamma)} \to \overline{M}_{0,n}(X,\beta).$$

The left hand side is a fibered product along evaluation maps at the markings on the moduli spaces determined by the edges of Γ.

There is a dense open stratum of maps with irreducible domains. Its complement is a union of divisors with normal crossings (up to a finite group action). However, this does not mean that the components of the boundary divisors do not self-intersect; in fact they always do for large degrees. We emphasize this point for later reference when we write down the Deligne spectral sequence.

The boundary strata are indexed by stable trees with one edge and two vertices. Each vertex has legs labeled by two sets A, B with $A \cup B = \{1, \dots, n\}$ and degrees β_A, β_B adding up to β. Of course, stability means that if $\beta_A = 0$ then $|A| \geq 2$, and similarly if $\beta_B = 0$ then $|B| \geq 2$. The corresponding boundary stratum:

$$\iota : \overline{D}(\beta_A, \beta_B, A, B) \hookrightarrow \overline{M}_{0,n}(X, \beta)$$

is the image of a gluing map:

$$\zeta : \overline{M}(A, B, \beta_A, \beta_B) = \overline{M}_{0,A\cup\{\bullet\}}(X, \beta_A) \times_X \overline{M}_{0,\{\star\}\cup B}(X, \beta_B) \to \overline{M}_{0,n}(X, \beta),$$

where $\bullet$ and $\star$ correspond to the double point of the domain curve. The following two cases may occur:

- If the tree has no automorphisms, then ζ is the normalization map of the boundary stratum $\overline{D}(A, B, \beta_A, \beta_B)$. If both $A, B \neq \emptyset$ then ζ is an isomorphism. Observe that if $A = \emptyset$ (or if $B = \emptyset$), then the corresponding divisor may self-intersect in a codimension two stratum of maps with three components glued at two nodes, the middle component containing all the marked points, the two external components having the same degree.

- If A and B are both empty and $\beta_A = \beta_B$, then the corresponding dual tree has a non-trivial automorphism. We need to factor out the $\mathbb{Z}/2\mathbb{Z}$ symmetry to get the normalization map of the corresponding boundary stratum.

2.1.2 The Deligne spectral sequence

In this subsection, we review the main ingredients of Deligne's spectral sequence. Then, we apply the general theory to the case of the moduli space of stable maps.

To get started, we let Y be a smooth complex projective variety (or a projective orbifold later), D be a divisor with normal crossings in Y; self-intersecting components are allowed. Let U denote the complement of the divisor D in Y, and let j be the inclusion $U \hookrightarrow Y$. We denote by D^p the subspace of Y consisting of points of multiplicity at least p, and we let $\tilde{D}_p$ be its normalization. Locally, D is union of smooth divisors, and D^p collects the points belonging to intersections of p of them. We agree that $D^0 = Y$. We will use cohomology with coefficients in the "orientation" local system ϵ_p. This local system is defined on $\tilde{D}^p$ as follows. For points y belonging to p local components of D, we set ϵ_p to be the determinant of the space of the p local components (see [15] for a complete discussion). In the case when D is union of smooth irreducible components without self-intersections, ϵ_p can be trivialized by choosing an ordering of the components. However, the case we will consider will involve self-intersecting components.

The cohomology of the open stratum $H^\star(U)$ carries a mixed Hodge structure which can be described explicitly in terms of the deRham complex of logarithmic differentials. As a consequence of this general construction, $H^n(U)$ has a weight filtration $0 \subset W^n \subset W^{n+1} \subset$

$\ldots \subset W^{2n} = H^n(U)$. We will mainly be interested in the lowest piece of the filtration which can be computed as:

$$W^n = j^\star H^n(Y).$$

Additionally, there is another filtration whose role is to give the successive quotients W^i/W^{i-1} a pure Hodge structure of weight i.

There is a spectral sequence relating all the ingredients of the above discussion. Its E_1 term is:

$$E_1^{-p,q} = H^{-2p+q}(\tilde{D}^p, \epsilon_p)$$

and the first differentials of the spectral sequence are a signed sum of Gysin inclusions. One of the main results of Hodge theory is that the higher differentials are all zero, and then the spectral sequence collapses to $E_\infty^{-p,q} = E_2^{-p,q}$ which is the piece of weight q on $H^{q-p}(U)$.

It was shown by Grothendieck [36] that the definition of the lowest Hodge piece W^n is independent of the compactification Y of U: for the purposes of defining $W^nH^n(U)$ we can pick any *smooth* compactification Y, maybe without normal crossings complement, and consider the restrictions of differential forms on Y to compute W^n. It follows immediately that if $V \subset U$ have a common *smooth* compactification then the restriction map $W^nH^n(U) \to W^nH^n(V)$ is surjective. For example, if V is Zariski open in a smooth variety U such a compactification can always be found by Nagata's theorem and resolution of singularities.

A similar remark can be proved in equivariant cohomology, if both U and V are equipped with compatible linearized actions of an algebraic group G. First, for any scheme X with a G action, the Hodge structure on the equivariant cohomology $H^\star_G(X)$ is constructed using the simplicial schemes $[X/G]_\bullet$ (see [15] for notation). The equivariant version of Grothendieck's result is obtained as follows. It is proved in [18] that for each n, there exists an open subset W of an affine G-space, equipped with a free G-action and whose complement has large codimension compared to n. The morphism of simplicial schemes $[W \times X/G]_\bullet \to [X/G]_\bullet$ induces a morphism of Hodge structures:

$$H^n_G(X) = H^n([X/G]_\bullet) \to H^n([W \times X/G]_\bullet) = H^n_G(W \times X) = H^n(W \times_G X).$$

It is shown in [18] that the map above is an isomorphism. When X is smooth, the morphism of *schemes* $X \times_G W \to W/G$ is also smooth [18]. For our applications, the base can always be chosen to be smooth, so $X \times_G W$ is also smooth. We combine the isomorphism of Hodge structures above with the non-equivariant Grothendieck remark for the schemes $W \times_G X$ when X is either U and V. We conclude that:

$$W^nH^n_G(U) \to W^nH^n_G(V) \text{ is surjective.} \tag{2.1}$$

The similar statement about the Chow groups is evident by construction [18].

We want to apply these general considerations to the space of stable maps. Although $\overline{M}_{0,n}(X,\beta)$ is not a smooth variety, its singularities are mild: they are all finite quotient singularities. There is an extension of Deligne's results to this setting which is worked out in [56]. All the above results carry over without change.

In the context of stable maps, we will mainly be concerned with the differential:

$$d_1 : E_1^{-1,k} = \oplus H^{k-2}\left(\overline{M}_{0,A\cup\{\bullet\}}(X,\beta_A)\times_X\overline{M}_{0,\{\star\}\cup B}(X,\beta_B)\right)^- \to E_1^{0,k} = H^k(\overline{M}_{0,n}(X,\beta)).$$

The cokernel of this map is the weight k piece in $H^k(M_{0,n}(X,\beta))$ which we will proceed to identify in the next section when X is a Grassmannian.

The superscript "-"on the cohomology groups of the boundary divisors comes from the orientation systems ϵ_p. If the boundary graph has no automorphisms we consider the whole cohomology group. In the case of the $\mathbb{Z}/2\mathbb{Z}$ symmetry of the boundary graph we need to take fewer classes. In general, for any graph Γ, the boundary $\overline{M}(\Gamma)$ is dominated by a product of smaller moduli spaces of stable maps. The cohomology of the product carries a representation of $Aut(\Gamma)$. Each automorphism has a sign given by its action on the one dimensional space $det(\text{Edge}(\Gamma))$. The *minus* superscript indicates that we only look at classes which are anti-invariant under the sign representation of $Aut(\Gamma)$. For the boundary divisors of nodal maps with equal degrees on the branches, these are the $\mathbb{Z}/2\mathbb{Z}$-invariant classes.

In particular, this discussion implies that the sequence:

$$\bigoplus H^{k-2}(\overline{M}(A,B,\beta_A,\beta_B)) \to H^k(\overline{M}_{0,n}(X,\beta)) \to W^k H^k(M_{0,n}(X,\beta)) \to 0 \tag{2.2}$$

is exact (see also corollary 8.2.8 in [15]). The similar statement for the Chow groups is obvious.

Once the exact sequence (2.2) is established, we can replace the coarse moduli schemes by the corresponding moduli stacks. To make sense out of the lowest piece of the Hodge structure on the smooth open stack $\mathcal{M}_{0,n}(X,\beta)$, we use the isomorphism with the cohomology of the coarse moduli scheme (remark 1.1.1). Alternatively, the construction of a functorial mixed Hodge structure on the cohomology of algebraic stacks has been outlined in [17].

2.2 Stable Maps to SL flags.

We proceed with the proof of the main result. We begin with the case when the target space is a Grassmannian. We first identify the lowest piece of the Hodge structure on the cohomology of the open stratum of irreducible maps. We start with the case of three marked points, then move down to 0, 1 and 2 markings. We conclude the argument by showing that the boundary classes are tautological. To this end, we prove a result about the cohomology of fibered products, essentially using techniques of [15] and [9]. To complete the proof for general SL flags, a discussion of the cohomology of the Hyper-Quot scheme is required.

2.2.1 Stromme's description of the Quot scheme.

Let X be the Grassmannian of r dimensional quotients of some N dimensional vector space V. We will begin with a description of the smooth scheme of degree $d \geq 1$ morphisms to X, which we denote by $\text{Mor}_d(\mathbb{P}^1, X)$. When X is the projective space, this discussion is trivial, the space in question can be described as an open subvariety in a projective space. However, for other flag varieties X such a convenient description is not as easy to come across. It was obtained by Stromme for Grassmannians [57] and generalized by [39] for arbitrary flags.

We will now explain Stromme's construction. To fix the notations, we let $\mathcal{S}$ and $\mathcal{Q}$ denote the tautological subbundle and quotient bundle on the Grassmannian, sitting in the exact sequence:

$$0 \to \mathcal{S} \to V \otimes \mathcal{O}_X \to \mathcal{Q} \to 0.$$

To give a degree d rational map $f : \mathbb{P}^1 \to X$ is the same as giving a degree d, rank r quotient vector bundle $F = f^\star \mathcal{Q}$ as follows:

$$V \otimes \mathcal{O}_{\mathbb{P}^1} \to F \to 0.$$

Allowing quotients which may not be locally free, we obtain the smooth compactification of $\mathrm{Mor}_d(\mathbb{P}^1, X)$ which is known as Grothendieck's *Quot* scheme. We will denote this by $Quot(N, r, d)$. The advantage of working with compactification will become clear below: the cohomology of the *Quot* scheme is easier to understand than that of the Kontsevich-Manin spaces. Incidentally, we note that this compactification was also used in [6] to compute the "Gromov invariants" of Grassmannians.

We consider two natural vector bundles $\mathcal{A}_{-1}$ and $\mathcal{A}_0$ on the scheme $\mathrm{Mor}_d(\mathbb{P}^1, X)$. Their fibers over a morphism f are $H^0(f^\star \mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}}(-1))$ and $H^0(f^\star \mathcal{Q})$ respectively. These vector bundles extend over the *Quot* scheme compactification. Indeed, we let $\mathcal{F}$ be the universal quotient of the trivial bundle on $\mathbb{P}^1 \times Quot$, and we let $\pi : \mathbb{P}^1 \times Quot \to Quot$ be the projection on the second factor. The extensions are, for $m \geq -1$:

$$\mathcal{A}_m = \pi_\star(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(m)).$$

The relevance of these vector bundles for our discussion comes from the following consequence of Beilinson's spectral sequence. The bundle F is the last term of the "monad":

$$0 \longrightarrow H^0(F \otimes \mathcal{O}_{\mathbb{P}}(-1)) \otimes \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow H^0(F) \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow F \longrightarrow 0.$$

If we pick bases for $H^0(F(-1))$ and $H^0(F)$, we can identify these vector spaces with $W_{-1} = \mathbb{C}^d$ and $W_0 = \mathbb{C}^{r+d}$ respectively. We rewrite the above diagram as follows:

$$\begin{tikzcd}
 & & V \otimes \mathcal{O}_{\mathbb{P}} \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & W_{-1} \otimes \mathcal{O}_{\mathbb{P}}(-1) \arrow[r] & W_0 \otimes \mathcal{O}_{\mathbb{P}} \arrow[r] & F \arrow[r] & 0.
\end{tikzcd}$$

It is evident now that the datum of the quotient $V \otimes \mathcal{O}_{\mathbb{P}} \to F$ can be encoded as an element of the affine space:

$$H^0(Hom(W_{-1} \otimes \mathcal{O}_{\mathbb{P}}(-1), W_0 \otimes \mathcal{O}_{\mathbb{P}})) \oplus \mathrm{Hom}(V, W_0) = P \otimes H^0(\mathcal{O}_{\mathbb{P}}(1)) \oplus Q$$

where $P = \mathrm{Hom}(W_{-1}, W_0)$ and $Q = \mathrm{Hom}(V, W_0)$. Of course, not every element in this affine space is allowed; we need to impose the condition that the quotient we obtain be locally free and that the map from the trivial bundle $V \otimes \mathcal{O}_{\mathbb{P}} \to F$ be surjective. In fact only an open subset of this affine space corresponds to morphisms f in $\mathrm{Mor}_d(\mathbb{P}^1, X)$. We also need to account for the $GL_d \times GL_{r+d}$ ambiguity coming from the action on the space of bases of $H^0(F(-1)) \oplus H^0(F)$.

We carried out the above discussion on the level of closed points, but in fact Stromme's construction takes care of the scheme structure as well. We obtain:

Fact 2.2.1 (Stromme, [57]). The total space $\mathcal{T}$ of the bundle of $GL_d \times GL_{r+d}$ frames of the vector bundle $\mathcal{A}_{-1} \oplus \mathcal{A}_0$ over $\mathrm{Mor}_d(\mathbb{P}^1, X)$ sits as an open subscheme in the affine space $P \otimes H^0(\mathcal{O}_{\mathbb{P}}(1)) \oplus Q$. In fact, this description extends to the *Quot* scheme compactification of $\mathrm{Mor}_d(\mathbb{P}^1, X)$.

As a consequence of equation (2.1) with trivial group, we conclude that the lowest piece of the Hodge filtration $W^\star H^\star(\mathcal{T}) = 0$.

To end the summary of Stromme's results, the following result of Grothendieck is needed. Its proof is essentially an application of the splitting principle. It suffices to consider the case when E is a line bundle. The Gysin sequence in cohomology, together with remark 2.1 to prove surjectivity, give the following:

Fact 2.2.2 (Grothendieck, [35]). Let E be any rank r vector bundle over a *smooth* compact base X. If P denotes the bundle of GL_r frames, then there is a surjective map:

$$H^\star(X) \to W^\star H^\star(P)$$

whose kernel is the ideal generated by the Chern classes $c_i(E)$. The same statement holds equivarianly for an algebraic group action, and for the Chow rings.

Applying these two facts to our setting we obtain that $H^\star(Quot)$ is generated by the Chern classes $c_i(\mathcal{A}_0)$ and $c_i(\mathcal{A}_{-1})$. Therefore, using (2.1), we obtain:

Fact 2.2.3 (Stromme, [57]). The lowest weight Hodge piece $W^\star H^\star(\mathrm{Mor}_d(\mathbb{P}^1, X))$ is spanned by the Chern classes $c_i(\mathcal{A}_0)$ and $c_i(\mathcal{A}_{-1})$. The same result holds for the Chow rings.

2.2.2 Many marked points.

Our goal is to prove the following statement about the open stratum $\mathcal{M}_{0,n}(X, d)$:

Lemma 2.2.1. *The lowest piece of the Hodge structure $W^\star H^\star(M_{0,n}(X, d))$ is spanned by restrictions of the tautological classes on $\overline{\mathcal{M}}_{0,n}(X, d)$ of definition* 1.1.1. *The same result holds for the Chow groups of $M_{0,n}(X, d)$.*

In this subsection, we will consider the case of the above lemma when $n \geq 3$. We use the results proved above about the *Quot* scheme. It will be useful to consider the sheaves on the stack $\overline{\mathcal{M}}_{0,n}(X, \beta)$ defined, for $m \geq 0$, as:

$$\mathcal{G}_m = \pi_\star ev^\star \left(\mathcal{Q} \otimes (\det \mathcal{Q})^{\otimes m}\right).$$

Here π and ev are the projection from the universal curve and the evaluation map respectively. We will be interested in the restrictions $j^\star \mathcal{G}_m$ to the open stratum $j : \mathcal{M}_{0,n}(X, d) \to \overline{\mathcal{M}}_{0,n}(X, d)$. We show that these generate the lowest Hodge piece of cohomology (or the Chow groups).

We consider the fiber diagram, where $\mathcal{C}_{0,n}(X, d)$ is the universal curve over $\mathcal{M}_{0,n}(X, d)$:

$$\begin{tikzcd}
\mathcal{C}_{0,n}(X, d) \arrow[r, "q"] \arrow[d, "\pi"] \arrow[rr, bend left, "ev"] & \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \arrow[r, "ev"] \arrow[d, "\hat{\pi}"] & X \\
\mathcal{M}_{0,n}(X, d) \arrow[r, "q"] & \mathrm{Mor}_d(\mathbb{P}^1, X) &
\end{tikzcd}$$

We let $p : \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to \mathbb{P}^1$ be the projection. Since $ev^\star \det \mathcal{Q}$ and $p^\star \mathcal{O}_{\mathbb{P}^1}(d)$ agree on the fibers of $\hat{\pi}$, there exists a line bundle $\mathcal{L}$ on $\mathrm{Mor}_d(\mathbb{P}^1, X)$ for which the following equation is satisfied:

$$ev^\star \det \mathcal{Q} = p^\star \mathcal{O}_{\mathbb{P}^1}(d) \otimes \hat{\pi}^\star \mathcal{L}.$$

We compute:

$$\begin{aligned} j^\star \mathcal{G}_m &= \pi_\star ev^\star \left(\mathcal{Q} \otimes (\det \mathcal{Q})^{\otimes m}\right) = \pi_\star \hat{q}^\star \left(ev^\star \mathcal{Q} \otimes p^\star \mathcal{O}_{\mathbb{P}^1}(dm) \otimes \hat{\pi}^\star \mathcal{L}^{\otimes m}\right) = \\ &= q^\star \hat{\pi}_\star \left(ev^\star \mathcal{Q} \otimes p^\star \mathcal{O}_{\mathbb{P}^1}(dm) \otimes \hat{\pi}^\star \mathcal{L}^{\otimes m}\right) = q^\star \mathcal{A}_{dm} \otimes q^\star \mathcal{L}^{\otimes m}. \end{aligned} \tag{2.3}$$

It is clear that the stabilization morphism and q together give an isomorphism between the moduli schemes $M_{0,n}(X, \beta) = M_{0,n} \times \mathrm{Mor}_d(\mathbb{P}^1, X)$. Kunneth decomposition can be used to understand the Hodge structure on the open stratum:

$$H^k(M_{0,n}(X, \beta)) = \bigoplus_{i+j=k} H^i(M_{0,n}) \otimes H^j(\mathrm{Mor}_d(\mathbb{P}^1, X)).$$

In [28], it is proved that the i^{th} cohomology $H^i(M_{0,n})$ carries a pure Hodge structure of weight $2i$. Since $\mathrm{Mor}_d(\mathbb{P}^1, X)$ is smooth, its j^{th} cohomology carries weights between j and $2j$. Hence to get the weight k piece $W^k H^k(M_{0,n}(X, \beta))$ we need $i = 0$, $j = k$. Thus, this weight k piece is isomorphic to the the weight k piece $W^k H^k(\mathrm{Mor}_d(\mathbb{P}^1, X))$ via the pullback:

$$q^\star : W^\star H^\star(\mathcal{M}_{0,n}(X, d)) \to W^\star H^\star(\mathrm{Mor}_d(\mathbb{P}^1, X)).$$

It follows from *fact* 3 and the exact sequence [57]:

$$0 \to \mathcal{A}_k \to \mathcal{A}_{k+1}^{\oplus 2} \to \mathcal{A}_{k+2} \to 0, \quad \text{for } k \geq 0,$$

that $W^\star H^\star(\mathrm{Mor}_d(\mathbb{P}^1, X))$ is generated by the Chern classes of $\mathcal{A}_0$ and $\mathcal{A}_d$. Better, we can pick as generators the Chern classes of $\mathcal{A}_0$ and $\mathcal{A}_d \otimes \mathcal{L}$. The isomorphism $q^\star$ above and equation (2.3) show that the Chern classes of $j^\star \mathcal{G}_0$ and $j^\star \mathcal{G}_1$ generate $W^\star H^\star(\mathcal{M}_{0,n}(X, d))$ for $n \geq 3$.

Finally, to completely prove lemma 2.2.1 when $n \geq 3$, we need to show that the Chern classes of $\mathcal{G}_m$ on $\mathcal{M}_{0,n}(X, d)$ are restrictions of the tautological classes of definition 1.1.1. We repeat the argument of lemma 3.4.1 in the next chapter. There, we explain the required Mumford-Grothendieck-Riemann-Roch computation. To conclude, we need to observe that the Chern class of the relative dualizing sheaf $c_1(\omega_\pi)$ is tautological. Here π is the forgetful morphism. This is a consequence of proposition 4.1.1 in chapter 4. One can argue differently by using the Plucker embedding to reduce the statement to the case of a projective space $\mathbb{P}^N$ in which the Grassmannian embeds. Then, we invoke [51]. It is shown there that all codimension 1 classes on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$ are tautological.

The statement for the Chow groups follows in the same fashion. We need the observation that for any scheme T, the map:

$$q^\star : A_\star(T) \to A_\star(M_{0,n} \times T)$$

is an isomorphism. This is well known. For example, it follows from comparing the exact sequences:

$$\begin{tikzcd}
\bigoplus_i A_*(\overline{D}_i) \otimes A_*(T) \arrow[r] \arrow[d] & A_*(\overline{\mathcal{M}}_{0,n}) \otimes A_*(T) \arrow[r] \arrow[d] & A_*(M_{0,n}) \otimes A_*(T) \cong A_*(T) \arrow[r] \arrow[d, "q^*"] & 0 \\
\bigoplus_i A_*(\overline{D}_i \times T) \arrow[r] & A_*(\overline{\mathcal{M}}_{0,n} \times T) \arrow[r] & A_*(M_{0,n} \times T) \arrow[r] & 0
\end{tikzcd}$$

Here $\overline{D}_i$ are the boundary divisors of $\overline{\mathcal{M}}_{0,n}$, which are products of lower dimensional moduli spaces of stable marked rational curves. The first two vertical arrows are isomorphisms as it is shown in section 2 of [38].

2.2.3 Fewer marked points.

We will now prove lemma 2.2.1 when the domain has fewer marked points. A case by case analysis depending on the number of markings is required. In the case of one or two markings, it will be satisfying to note the emergence of the ψ classes in the proof which would justify their usual inclusion in the tautological systems.

No marked points.

To begin, we consider the case of no marked points. We let SL_2 act on $\mathbb{C}^2$ in the usual way. In turn, we obtain an action on the scheme of morphisms $\mathrm{Mor}_d(\mathbb{P}^1, X)$:

$$SL_2 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \ni (g, f) \to f \circ g^{-1} \in \mathrm{Mor}_d(\mathbb{P}^1, X).$$

Since each morphism is finite onto its image, it is easy to derive that the action has finite stabilizers. It is known that the PSL_2 quotient of $\mathrm{Mor}_d(\mathbb{P}^1, X)$ equals $M_{0,0}(X, d)$ and since the center in SL_2 acts trivially, we see that the space of SL_2 orbits of $\mathrm{Mor}_d(\mathbb{P}^1, X)$ is the topological space underlying $M_{0,0}(X, d)$. A well known result, which is proved for example in [10], gives an isomorphism between the cohomology of the orbit space and the equivariant cohomology. In our case, this translates into an isomorphism:

$$H^*(M_{0,0}(X,d)) = H^*_{SL_2}(\mathrm{Mor}_d(\mathbb{P}^1, X)). \tag{2.4}$$

Even more, the right hand side can be given a Hodge structure using simplicial schemes [15], which, by functoriality is compatible with the structure on the left hand side.

We now move the discussion to the algebraic category. It is easy to see using the numerical criterion of stability that the action of SL_2 on $\mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^N) \hookrightarrow \mathbb{P}^{Nd+N+d}$ has only stable points. The same statement then holds for $\mathrm{Mor}_d(\mathbb{P}^1, X)$ using the Plucker embedding. In turn, this implies the existence of a geometric quotient $\mathrm{Mor}_d(\mathbb{P}^1, X)/SL_2$. This quotient can be identified with $M_{0,0}(X, d)$, since both schemes solve the same moduli problem (see also the argument in [52]). We will make use of the isomorphism [18]:

$$A^k(M_{0,0}(X,d)) = A^k_{SL_2}(\mathrm{Mor}_d(\mathbb{P}^1, X)) = A^k(\mathrm{Mor}_d(\mathbb{P}^1, X) \times_{SL_2} W). \tag{2.5}$$

In the topological category, we compute equivariant cohomology taking for W the contractible space ESL_2. In the algebraic case, we let $W = W_k$ be any a smooth open subvariety of an affine SL_2-space, which has large codimension compared to k and which has a free SL_2 action.

Let us write, for now, $G = SL_2$; later, we will make use of other groups as well. We will use the following standard notation. For any G scheme X, X_G will denote the equivariant Borel construction. In the algebraic setting, X_G will stand for any of the mixed *schemes* $X \times_G W$ (cf. [18]) where W is described in the previous paragraph. We observed that all equivariant models X_G can be chosen to be smooth. In fact, properties of equivariant morphisms such as smoothness, or flatness and properness, still hold for the induced morphisms between the mixed spaces [18]. Moreover, any G-linearized bundle $E \to X$ lifts to a bundle $E_G \to X_G$ whose total space is $E \times_G W$.

We will compute the right hand side of (2.4) and (2.5) using arguments similar to Stromme's. We will need to extend the SL_2 action to the *Quot* scheme and to lift it to the bundles $\mathcal{A}_{-1}, \mathcal{A}_0$. For example, the SL_2 linearization of $\mathcal{A}_{-1}$ is essentially determined by the usual SL_2 linearization of $L = \mathcal{O}_{\mathbb{P}^1}(-1)$. To be precise, we consider the following diagram:

$$\begin{tikzcd}
SL_2 \times \mathbb{P}^1 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\alpha"'] & \mathbb{P}^1 \\
SL_2 \times \mathbb{P}^1 \times Quot \arrow[u, "p"] \arrow[d, "\pi"] \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\alpha"'] & \mathbb{P}^1 \times Quot \arrow[u, "p"'] \arrow[d, "\pi"'] \\
SL_2 \times Quot \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\alpha"'] & Quot
\end{tikzcd}$$

Here, σ is the action, while the morphisms α, π and p are projections. We have an isomorphism $\eta : \sigma^\star\mathcal{O}_{\mathbb{P}^1}(-1) \to \alpha^\star\mathcal{O}_{\mathbb{P}^1}(-1)$. Writing $\mathcal{F}$ for the universal quotient sheaf on $\mathbb{P}^1 \times Quot$, the linearization of $\mathcal{A}_{-1}$ is obtained as follows:

$$\begin{aligned}
\sigma^\star\mathcal{A}_{-1} &= \sigma^\star\pi_\star(\mathcal{F} \otimes p^\star\mathcal{O}_{\mathbb{P}}(-1)) = \pi_\star\sigma^\star(\mathcal{F} \otimes p^\star\mathcal{O}_{\mathbb{P}^1}(-1) = \\
&\cong \pi_\star(\alpha^\star\mathcal{F} \otimes p^\star\sigma^\star\mathcal{O}_{\mathbb{P}^1}(-1)) \cong \pi_\star(\alpha^\star\mathcal{F} \otimes p^\star\alpha^\star\mathcal{O}_{\mathbb{P}^1}(-1)) = \\
&= \pi_\star\alpha^\star(\mathcal{F} \otimes p^\star\mathcal{O}_{\mathbb{P}^1}(-1)) = \alpha^\star\pi_\star(\mathcal{F} \otimes p^\star\mathcal{O}_{\mathbb{P}^1}(-1)) = \alpha^\star\mathcal{A}_{-1}.
\end{aligned}$$

We will be concerned with the mixed space $Quot_{SL_2}$ and the bundles $\mathcal{A}_m^{SL_2}$ which are the lifts of the equivariant bundles $\mathcal{A}_m$, for $m \in \{-1, 0\}$. A moment's thought shows that the Stromme embedding described in *fact* 1 is SL_2 equivariant. It is immediate that taking frames of a vector bundle commutes with the construction of the mixed spaces and of mixed bundles over them. Then, the bundle of (split) frames of $\mathcal{A}_{-1}^{SL_2} \oplus \mathcal{A}_0^{SL_2}$ is the mixed space $\mathcal{T}_{SL_2}$ and moreover, it can be realized as a subscheme:

$$\mathcal{T}_{SL_2} \hookrightarrow (P \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))) \oplus Q)_{SL_2}.$$

The latter space can be described explicitly. In the topological category, BGL_2 can be realized as the infinite Grassmannian $\mathbb{G}$ of 2 dimensional planes endowed with a tautological rank 2 bundle $\mathbb{S}$. Its frame bundle serves as a model for $EGL_2 = ESL_2$. In the algebraic category, we consider the truncated models of the infinite dimensional constructions. For instance, W/GL_2 will be a finite dimensional Grassmannian of 2 dimensional planes. Then, W will be the bundle of GL_2 frames of the tautological rank 2 bundle $\mathbb{S}$ over W/GL_2. W/SL_2 is the bundle of $\mathbb{C}^\star$ frames of the determinant det $\mathbb{S} \to W/GL_2$. This is a consequence of the fact that any frame of $\mathbb{S}$, say $w \in W$, gives rise to a frame $\det w$ of $\det \mathbb{S}$. We will also denote by $\mathbb{S}$ the pullback to W/SL_2 of the tautological bundle on the Grassmannian.

It is then clear that the mixed space we are trying to describe can be realized as the total space of the bundle:

$$P \otimes \mathbb{S}^\star \oplus \mathcal{O} \otimes Q \to W/SL_2.$$

Therefore, its cohomology (or Chow rings) can be computed from that of the base. In turn, this is easily seen to be (via *fact* 2):

$$W^\star H^\star_{SL_2} = H^\star_{GL_2}/c_1(\Lambda^2\mathbb{S}) = \mathbb{C}\,[c_2(\mathbb{S})]$$

Repeating the arguments of the previous subsection in the equivariant setting (making use of *fact* 2 and equation (2.1)) we obtain two surjections:

$$W^\star H^\star_{SL_2} \to W^\star H^\star_{SL_2}(\mathcal{T}),\;\; H^\star_{SL_2}(Quot) \to W^\star H^\star_{SL_2}(\mathcal{T}).$$

We have explicit generators for the kernel of the second map, namely the equivariant Chern classes of the bundles $\mathcal{A}_{-1}, \mathcal{A}_0$. We conclude the following:

Claim 2.2.1. *The cohomology $H^\star_{SL_2}(Quot)$ is generated by the equivariant classes of $\mathcal{A}_0$ and $\mathcal{A}_{-1}$ together with the pullback of the Chern class $4c_2(\mathbb{S}) = c_2(Sym^2\mathbb{S})$ from the model W/SL_2 constructed above. Therefore, the same is true about the lowest piece of the Hodge structure on $H^\star_{SL_2}(Mor_d(\mathbb{P}^1, X))$.*

We are now ready for the proof of lemma 2.2.1. We consider the following diagram of fiber squares:

$$\begin{tikzcd}
\mathbb{P}(\mathbb{S}) = \mathbb{P}^1 \times_{SL_2} W \arrow[d, "\hat{\pi}"] & (\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)) \times_{SL_2} W \arrow[l, "\hat{q}"'] \arrow[r, "\tilde{\epsilon}"] \arrow[d, "\tilde{\pi}"] & \mathcal{C} \arrow[r, "ev"] \arrow[d, "\pi"] & X \\
W/SL_2 & \mathrm{Mor}_d(\mathbb{P}^1, X) \times_{SL_2} W \arrow[l, "q"'] \arrow[r, "\epsilon"] & \mathcal{M}_{0,0}(X,d) \arrow[r, "p"] & M_{0,0}(X,d)
\end{tikzcd}$$

In the above diagram, we start with the flat algebraic family $\tilde{\pi}$ whose fibers are irreducible genus 0 curves (which are not canonically identified to $\mathbb{P}^1$ because we factored out the SL_2-action). The classifying map to the moduli stack is denoted by ϵ. In the same diagram, $\mathcal{C}$ is the universal curve, and $p : \mathcal{M}_{0,0}(X,d) \to M_{0,0}(X,d)$ is the natural morphism to the coarse moduli scheme.

We know that p and $p\epsilon$ induce isomorphisms in rational cohomology [2], [18], hence the same is true about ϵ. It suffices to show that any class of lowest Hodge weight on the smooth space $\mathrm{Mor}_d(\mathbb{P}^1, X) \times_{SL_2} W$ is the pullback from $\mathcal{M}_{0,0}(X,d)$ of restrictions of the tautological classes of definition 1.1.1.

We will use the conclusions summarized in the *claim* above. It is clear that $\mathcal{A}_0^{SL_2}$ corresponds to the bundle $\mathcal{G}_0 = \pi_\star(ev^\star\mathcal{Q})$ under the isomorphism (2.4) induced by ϵ.

We argue that the Chern classes of the bundle $\mathcal{A}_{-1}^{SL_2}$ also come as pullbacks under ϵ of tautological classes on $\mathcal{M}_{0,0}(X,d)$. Let L be the lift of the linearized bundle $p^\star\mathcal{O}_{\mathbb{P}^1}(-1)$ on $\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)$ to the equivariant mixed space $(\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)) \times_{SL_2} W$. First we note that $\mathcal{A}_{-1}^{SL_2} = \tilde{\pi}_\star(\tilde{\epsilon}^\star ev^\star\mathcal{Q} \otimes L)$. Since both sides are locally free, it suffices to check equality after pullback by the smooth morphism κ of the following fiber diagram:

$$\begin{tikzcd}
& X & \\
\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \arrow[ur, "ev"] \arrow[d, "\pi"] & \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \times W \arrow[l, "pr"'] \arrow[r, "\tilde{\kappa}"] \arrow[u, "ev"'] \arrow[d, "\underline{\pi}"] & (\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)) \times_{SL_2} W \arrow[ul, "ev"'] \arrow[d, "\tilde{\pi}"] \\
\mathrm{Mor}_d(\mathbb{P}^1, X) & \mathrm{Mor}_d(\mathbb{P}^1, X) \times W \arrow[l, "pr"'] \arrow[r, "\kappa"] & \mathrm{Mor}_d(\mathbb{P}^1, X) \times_{SL_2} W
\end{tikzcd}$$

We have:

$$\kappa^\star \mathcal{A}_{-1}^{SL_2} = pr^\star \mathcal{A}_{-1} = \underline{\pi}_\star(ev^\star \mathcal{Q} \otimes p^\star \mathcal{O}_{\mathbb{P}^1}(-1)) = \underline{\pi}_\star(\tilde{\kappa}^\star ev^\star \mathcal{Q} \otimes \tilde{\kappa}^\star L) = \kappa^\star \tilde{\pi}_\star(ev^\star \mathcal{Q} \otimes L).$$

To compute the Chern classes of $\mathcal{A}_{-1}^{SL_2}$ we need to use Grothendieck-Riemann-Roch. We need to explain that the class $c_1(L)$ is a pullback under ϵ of a tautological class on $\mathcal{M}_{0,0}(X,d)$. To this end, we replace this class by a multiple of $c_1(\omega_{\tilde{\pi}})$, which is tautological as explained in section 2.2. This is justified since after pullback by $\tilde{\kappa}$ the two line bundles $\omega_{\tilde{\pi}} = \epsilon^\star \omega_\pi$ and $L^{\otimes 2}$ agree.

Finally, we discuss $c_2(Sym^2\mathbb{S})$. We claim that on $\mathrm{Mor}_d(\mathbb{P}^1, X) \times_{SL_2} W$ the following equation holds true:

$$\epsilon^\star \pi_\star \omega_\pi^\star = q^\star Sym^2 \mathbb{S}^\star. \tag{2.6}$$

The Chern classes of $\pi_\star \omega_\pi^\star$ are tautological by the usual argument involving Grothendieck-Riemann-Roch (the explicit computation shows that the second Chern class we need is 0).

To establish the equation above, observe that we are in the particularly favorable situation when all relative dualizing sheaves involved are line bundles over smooth bases, hence taking duals causes no problems. We have $\epsilon^\star \pi_\star \omega_\pi^\star = q^\star \hat{\pi}_\star \omega_{\hat{\pi}}^\star$. The following Euler sequence on $\mathbb{P}(\mathbb{S})$:

$$0 \to \mathcal{O} \to \hat{\pi}^\star \mathbb{S} \otimes \mathcal{O}_{\mathbb{P}(\mathbb{S})}(1) \to \omega_{\hat{\pi}}^\star \to 0$$

shows:

$$\omega_{\hat{\pi}}^\star = \det(\hat{\pi}^\star \mathbb{S} \otimes \mathcal{O}_{\mathbb{P}(\mathbb{S})}(1)) = \mathcal{O}_{\mathbb{P}(\mathbb{S})}(2).$$

Here we used that W/SL_2 is the space of frames for $\Lambda^2\mathbb{S}$, so the pullback of $\Lambda^2\mathbb{S} \to W/GL_2$ is trivial. We immediately obtain $\hat{\pi}_\star \omega_{\hat{\pi}}^\star = Sym^2\mathbb{S}^\star$, thus establishing (2.6).

One marked point.

The remaining two cases are entirely similar. For the case of one marking, we will use the action of the subgroup N of SL_2 of matrices:

$$N = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^\star, b \in \mathbb{C} \right\}.$$

We carry out the equivariant arguments of the previous section replacing SL_2 with the group N.

We will identify EN and BN. To construct algebraic families, we will work with the finite dimensional approximations of the topological models. As before, W/GL_2 will be a Grassmannian of 2 dimensional planes of large, but finite, dimension. We identify W with the bundle of GL_2 frames of the tautological bundle $\mathbb{S}$. We have seen that W/SL_2 is the bundle of $\mathbb{C}^\star$ frames in det $\mathbb{S}$. Moreover, the space W/SL_2 comes equipped with the pullback bundle, which we denote by $\tilde{\mathbb{S}}$. The space W can be used to compute the N-equivariant cohomology (or Chow groups). Then, W/N will be the projective bundle $\mathbb{P}(\tilde{\mathbb{S}})$ over W/SL_2. In addition, letting $\tau : W/N \to W/GL_2$ be the projection, we have $\mathbb{P}^1 \times_N W = \mathbb{P}(\tau^\star\mathbb{S})$. The last two statements follow by observing the map:

$$\{\text{frames in a 2 dimensional vector space } S\}/N \to \mathbb{P}^1(S), \text{ frame } \{e, f\} \to \text{line spanned by } e.$$

The relevant diagram of fiber squares is:

$$\begin{tikzcd}
\mathbb{P}(\tau^\star\mathbb{S}) = \mathbb{P}^1 \times_N W \arrow[d, shift left, "\hat\pi"] & (\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)) \times_N W \arrow[l, "\hat q"'] \arrow[r, "\tilde\epsilon"] \arrow[d, shift left, "\tilde\pi"] & \mathcal{C} \arrow[r, "ev"] \arrow[d, shift left, "\pi"] & X \\
\mathbb{P}(\tilde{\mathbb{S}}) = W/N \arrow[u, shift left, "\hat z"] & \mathrm{Mor}_d(\mathbb{P}^1, X) \times_N W \arrow[l, "q"'] \arrow[r, "\epsilon"] \arrow[u, shift left, "\tilde z"] & \mathcal{M}_{0,1}(X,d) \arrow[r, "p"] \arrow[u, shift left, "z"] & M_{0,1}(X,d)
\end{tikzcd}$$

We will denote by ξ the dual hyperplane bundle on $W/N = \mathbb{P}(\tilde{\mathbb{S}})$ and similarly Ξ will be the dual hyperplane bundle on $\mathbb{P}(\tau^\star\mathbb{S})$. The morphism $\hat\pi : \mathbb{P}(\tau^\star\mathbb{S}) \to \mathbb{P}(\tilde{\mathbb{S}})$ has a canonical section $\hat z$ such that $\hat z^\star\Xi = \xi$. The flat algebraic family $\tilde\pi$ also has a section $\tilde z$ whose image in each fiber is the N-invariant basepoint $[1:0] \in \mathbb{P}^1$. We let ϵ be the classifying map to the moduli stack $\mathcal{M}_{0,1}(X,d)$.

The above description of W/N as a projective bundle over W/SL_2 shows that:

$$W^\star H_N^\star = W^\star H_{SL_2}^\star[c_1(\xi)]/(c_1(\xi)^2 + c_2(\tilde{\mathbb{S}})) = \mathbb{C}[c_1(\xi), c_2(\tilde{\mathbb{S}})]/(c_1(\xi)^2 + c_2(\tilde{\mathbb{S}})) = \mathbb{C}[c_1(\xi)].$$

To complete the proof of lemma 2.2.1, we use the arguments in the previous subsection. We only have to write the class $q^\star c_1(\xi)$ on $\mathrm{Mor}_d(\mathbb{P}^1, X) \times_N W$ as the pullback of a tautological class on $\mathcal{M}_{0,1}(X,d)$.

The Euler sequence on $\mathbb{P}(\tau^\star\mathbb{S})$ shows that:

$$0 \to \mathcal{O} \to \hat\pi^\star\tau^\star\mathbb{S} \otimes \Xi \to \omega_{\hat\pi}^\star \to 0.$$

Taking determinants and recalling that $\tau^\star \det \mathbb{S}$ is trivial, we obtain $\omega_{\hat\pi}^\star = \Xi^{\otimes 2}$. Therefore,

$$\hat z^\star c_1(\omega_{\hat\pi}) = -2c_1(\hat z^\star\Xi) = -2c_1(\xi).$$

It is then clear that:

$$-2q^\star c_1(\xi) = q^\star \hat z^\star c_1(\omega_{\hat\pi}) = \tilde z^\star c_1(\omega_{\tilde\pi}) = \epsilon^\star(z^\star c_1(\omega_\pi)) = \epsilon^\star\psi_1.$$

The proof is now complete, since we explained in the introduction (cf. remark 1.1.1) the tautology of the ψ classes.

Two marked points.

The argument is again similar to the case of one marked point. We let $\mathbb{C}^\star$ act on $\mathbb{P}^1$ as follows $t \cdot [z:w] = [t^{-1}z : tw]$. We obtain the isomorphism:

$$H^\star(\mathcal{M}_{0,2}(X,d)) = H_{\mathbb{C}^\star}^\star(\mathrm{Mor}_d(\mathbb{P}^1, X)).$$

We compute the right hand side. $W/\mathbb{C}^\star$ will be a projective space and W the bundle of $\mathbb{C}^\star$ frames of the tautological line $\mathbb{S} \to W/\mathbb{C}^\star$.

The following diagram of fiber squares will be useful in our computation:

$$\begin{tikzcd}
\mathbb{P}(\mathbb{S} \oplus \mathbb{S}^\star) = \mathbb{P}^1 \times_{\mathbb{C}^\star} W \arrow[d, shift left=2, "\hat\pi"] & (\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)) \times_{\mathbb{C}^\star} W \arrow[l, "\hat q"'] \arrow[r, "\tilde\epsilon"] \arrow[d, shift left=2, "\tilde\pi"] & \mathcal{C} \arrow[r, "ev"] \arrow[d, shift left=2, "\pi"] & X \\
W/\mathbb{C}^\star \arrow[u] \arrow[u, shift left] & \mathrm{Mor}_d(\mathbb{P}^1, X) \times_{\mathbb{C}^\star} W \arrow[l, "q"'] \arrow[r, "\epsilon"] \arrow[u] \arrow[u, shift left] & \mathcal{M}_{0,2}(X,d) \arrow[r, "p"] \arrow[u] \arrow[u, shift left] & M_{0,2}(X,d)
\end{tikzcd}$$

The family $\hat{\pi}$ has two tautological sections $\hat{z}, \hat{w}$ such that $\hat{z}^\star\xi = \mathbb{S}$ and $\hat{w}^\star\xi = \mathbb{S}^\star$; here, ξ is the dual hyperplane bundle on $\mathbb{P}(\mathbb{S} \oplus \mathbb{S}^\star)$. The family $\tilde{\pi}$ also has two sections, their images are the two $\mathbb{C}^\star$ invariant points of $\mathbb{P}^1$ in each fiber.

We obtain generators for the lowest piece of the Hodge structure on $H^\star_{\mathbb{C}^\star}(\mathrm{Mor}_d(\mathbb{P}^1, X))$. In the light of the previous discussion, we will only need to explain that the class $q^\star c_1(\mathbb{S})$ coming from $W/\mathbb{C}^\star$ is a pullback of a tautological class on $\mathcal{M}_{0,2}(X, d)$. The argument is identical to the one in the previous section. From the Euler sequence along the fibers of $\hat{\pi}$, it follows that $c_1(\omega_{\hat{\pi}}) = -2c_1(\xi)$. The computation below finishes the proof:

$$-2q^\star c_1(\mathbb{S}) = -2q^\star \hat{z}^\star c_1(\xi) = \tilde{z}^\star \hat{q}^\star c_1(\omega_{\hat{\pi}}) = \tilde{z}^\star c_1(\omega_{\tilde{\pi}}) = \epsilon^\star z^\star c_1(\omega_\pi) = \epsilon^\star \psi_1.$$

2.2.4 General SL flag varieties.

Let us now consider the case of a general SL flag variety X parameterizing l successive quotients of dimensions $r_1, \ldots, r_l$ of some N dimensional vector space V.

Pulling back the tautological sequence on X:

$$0 \to \mathcal{S}_1 \to \ldots \to \mathcal{S}_l \to V \otimes \mathcal{O}_X \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_l \to 0. \tag{2.7}$$

under a morphism $f : \mathbb{P}^1 \to X$, we obtain a sequence of locally free quotients:

$$V \otimes \mathcal{O}_{\mathbb{P}^1} \to F_1 \to \ldots \to F_l \to 0$$

of rank r_i and degree d_i. Allowing for arbitrary (not necessarily locally free) quotients, we obtain the Hyper-Quot scheme compactification $HQuot$ of the space of morphisms $\mathrm{Mor}_\beta(\mathbb{P}^1, X)$. We will use the more explicit notation $HQuot(N, \mathbf{r}, \mathbf{d})$ when necessary.

Since $HQuot$ is a fine moduli scheme, there is a universal sequence on $\mathbb{P}^1 \times HQuot$:

$$0 \to \mathcal{E}_1 \to \ldots \to \mathcal{E}_l \to V \otimes \mathcal{O} \to \mathcal{F}_1 \to \ldots \to \mathcal{F}_l \to 0.$$

We seek to show that:

Lemma 2.2.2. *The cohomology (and the Chow rings) of $HQuot(N, \mathbf{r}, \mathbf{d})$ is generated by the Kunneth components of $c_j(\mathcal{F}_i)$.*

Unfortunately, the arguments of the previous subsections do not extend to the present case. Even though a description of the Hyper-Quot scheme similar to the one in subsection 2.1 does exist [39], nonetheless we obtain an embedding of a principal bundle over $HQuot$ into a *singular* affine variety. The existence of singularities is a one (and not the only) obstacle in extending the proofs in subsection 2.2 to our new setting.

Nevertheless, the proof of the lemma stated above should be well known, but we could not find a suitable reference. To prove it, we will use a well-known trick of Beauville-Ellingsrud-Stromme [20]. Some of the details appear in the next subsection. There are difficulties in applying the same argument equivariantly. Instead, we will use a combination of the Leray spectral sequence and the already established equivariant results for the *Quot* scheme. We prove the following:

Lemma 2.2.3. *For any flag variety X and any degree β, the lowest piece of the Hodge structure $W^\star H^\star(\mathcal{M}_{0,n}(X, \beta))$ is spanned by the restrictions of the tautological classes on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ of definition 1.1.1. The same results hold for the Chow groups.*

To see this, we will assume lemma 2.2.2. We consider the following product of forgetful morphisms:

$$i = \prod_j i_j : HQuot_{\mathbb{P}^1}(N, \mathbf{r}, \mathbf{d}) \to \prod_j Quot(N, r_j, d_j).$$

We put together lemma 2.2.2, and the observation that the universal quotients on $\mathbb{P}^1 \times Quot(N, r_j, d_j)$ pull back to the universal sheaves $\mathcal{F}_j$ on $\mathbb{P}^1 \times HQuot$. We conclude that the pullback map:

$$i^\star : H^\star(\prod_j Quot(N, r_j, d_j)) \to H^\star(HQuot(N, \mathbf{r}, \mathbf{d}))$$

is surjective. The results of section 2.1 imply that in fact the cohomology of $HQuot$ is generated by the Chern classes $c_i(\mathcal{A}_{j,m})$ for $m \in \{-1, 0\}$. Here $\mathcal{A}_{j,m} = R\pi_\star \mathcal{F}_j(m)$ for $m \geq -1$, and $\pi : \mathbb{P}^1 \times HQuot \to HQuot$ is the natural projection. Of course, this could also be seen more directly.

When $n \geq 3$, the statement in lemma 2.2.3 follows by the same argument we used in section 2.2 for Grassmannians.

To deal with fewer marked points, we will explain that the map $i^\star$ is surjective in G-equivariant cohomology. Here G denotes one of the groups SL_2, N or $\mathbb{C}^\star$ which we used in section 2.3. Surjectivity is a consequence of the collapse of the Leray spectral sequence in equivariant cohomology as proved for example in [30]. Strictly speaking, the group N is not covered by the results of [30], but the argument in the algebraic category presented below takes care of this case as well.

We obtain diagram:

$$\begin{array}{ccc} E_2^{p,q} = H_G^p \otimes H^q \left(\prod_j Quot(N, r_j, d_j)\right) & \Rightarrow & H_G^{p+q}\left(\prod_j Quot(N, r_j, d_j)\right) \\ \Big\downarrow{\scriptstyle i^\star} & & \Big\downarrow{\scriptstyle i_G^\star} \\ E_2^{p,q} = H_G^p \otimes H^q(HQuot) & \Rightarrow & H_G^{p+q}(HQuot) \end{array}$$

Surjectivity of the equivariant cohomology restriction map $i_G^\star$ follows now from the non-equivariant statement. It suffices to observe that the collapse of the spectral sequence shows that the equivariant groups admit filtrations such that $i^\star$ induces surjections between their associated graded algebras.

We dedicated the previous subsection to the computation of the equivariant cohomology of the $Quot$ schemes. From the above, we obtain a generation result of the equivariant cohomology of $HQuot$ in terms of the equivariant tautological Chern classes of the bundles $\mathcal{A}_{j,m}$. Now the arguments which occupy the rest of section 2.3 can be used to finish the proof of lemma 2.2.3.

The argument in the Chow groups is slightly more involved, but we will include it here for completeness. For simplicity, let us write X and Y for $HQuot$ and $\prod_j Quot(N, r_j, d_j)$ respectively, and then $i : X \to Y$ is the forgetful morphism. We know that the pullback:

$$i^! : A_\star(Y) \cong A^\star(Y) \xrightarrow{i^\star} A^\star(X) \cong A_\star(X) \text{ is surjective.}$$

We want to derive the same statement equivariantly for the action of the groups $SL_2, N, \mathbb{C}^\star$. As before, let W be an open smooth subvariety of an affine space with a free action of G. We claim that the map $i_W^! : A_\star(Y \times W) \to A_\star(X \times W)$ is also surjective. This follows from the surjectivity of $i^!$ and of the following two maps:

$$A_\star(Y) \otimes A_\star(W) \to A_\star(Y \times W) \text{ and } A_\star(X) \otimes A_\star(W) \to A_\star(X \times W).$$

To see that the exterior product maps are surjective, we make use of the fact that both X and Y are smooth projective varieties admitting torus actions with isolated fixed points [57]. Such torus actions are obtained from a generic torus action on $\mathbb{P}^1$ and on the fibers of the sheaf $\mathcal{O}_{\mathbb{P}^1}^N$ whose quotients give the *Quot* schemes. Therefore, the theorem of Bialynicki-Birula shows that X and Y can be stratified by unions of affine spaces. For affine spaces surjectivity is clear. Our claim follows inductively, by successively building X and Y from their strata.

To finish the proof, it suffices to explain the surjectivity of the map:

$$i_G^! : A_\star^G(Y) = A_\star(Y \times_G W) \to A_\star(X \times_G W) = A_\star^G(X).$$

We have a fiber diagram:

$$\begin{array}{ccc} X \times W & \xrightarrow{i_W} & Y \times W \\ {\scriptstyle \pi_X}\downarrow & & {\scriptstyle \pi_Y}\downarrow \\ X \times_G W & \xrightarrow{i_G} & Y \times_G W \end{array}$$

where the vertical arrows are principal G bundles. Let α be any class in $A_k(X \times_G W)$. Then, our assumption and theorem 1 in [60] respectively show that there are classes $\overline{\beta}$ and β on $Y \times W$ and $Y \times_G W$ such that:

$$\pi_X^\star \alpha = i_W^! \overline{\beta} \text{ and } \overline{\beta} = \pi_Y^\star \beta.$$

Therefore,

$$\pi_X^\star \alpha = i_W^! \pi_Y^\star \beta = \pi_X^\star i_G^! \beta.$$

We first assume G is either SL_m or GL_m and use theorem 2 in [60]. Then,

$$\alpha = i_G^! \beta + \sum_i c_i^X \cap \alpha_i$$

for some classes $\alpha_i \in A_{k+i}(X \times_G W)$ and for some operational classes c_i^X on $X \times_G W$. Moreover, we find classes c_i^Y operating on $A_\star(Y \times_G W)$ with $i_G^\star c_i^Y = c_i^X$. Inductively on codimension, we know $\alpha_i = i_G^! \beta_i$. The following computation concludes the proof:

$$\alpha = i_G^! \beta + \sum_i i_G^\star c_i^Y \cap i_G^! \beta_i = i_G^! \left(\beta + \sum_i c_i^Y \cap \beta_i \right).$$

There is one remaining case needed for our arguments, namely that of the group N. Let $E \to X \times_{GL_2} W$ be the vector bundle associated to the principal GL_2 bundle:

$$X \times W \to X \times_{GL_2} W.$$

Then $X \times_{SL_2} W$ is total space of the bundle of frames in $\det E$. It is equipped with a projection $\eta : X \times_{SL_2} W \to X \times_{GL_2} W$. More importantly, $X \times_N W$ is the projective bundle $\mathbb{P}(\eta^\star E)$ over $X \times_{SL_2} W$ and comes equipped with a tautological bundle τ_X. Then

$A_\star(X \times_N W)$ is generated by the class $c_1(\tau_X)$ and $A_\star(X \times_{SL_2} W)$. The analogous discussion holds for Y. From here, the surjectivity of $i^!_N$ follows from that of $i^!_{SL_2}$.

2.2.5 The Cohomology of the Hyper-Quot scheme.

In this section we will prove lemma 2.2.2 using the diagonal trick of Beauville-Ellingsrud-Stromme [20]. We will express the class of the diagonal embedding $\Delta : HQuot \hookrightarrow HQuot \times HQuot$ as a combination of classes $\pi_1^\star\alpha \cdot \pi_2^\star\beta$ on $HQuot \times HQuot$, where α and β are among the tautological classes listed in lemma 2.2.2. Here π_1, π_2 are the two projections $HQuot \times HQuot \to HQuot$. This will establish lemma 2.2.2 completely. Since such arguments are well known, we will only sketch some of the details.

Let $\mathcal{K}$ be the kernel of the natural sheaf morphism on $\mathbb{P}^1 \times HQuot \times HQuot$:

$$\bigoplus_{i=1}^{l} \text{Hom }(\pi_1^\star\mathcal{E}_i, \pi_2^\star\mathcal{F}_i) \to \bigoplus_{i=1}^{l-1} \text{Hom }(\pi_1^\star\mathcal{E}_i, \pi_2^\star\mathcal{F}_{i+1}) \to 0.$$

Let $p : \mathbb{P}^1 \times HQuot \times HQuot \to HQuot \times HQuot$ denote the natural projection. It can be shown that $p_\star\mathcal{K}$ is a vector bundle whose rank equals the dimension of $HQuot$, essentially by showing that there are no H^1's along the fibers of p. In turn, this can be observed via the following argument borrowed from [12]. Assume we are given two geometric points of $HQuot$:

$$0 \to E_\bullet \to V \otimes \mathcal{O}_{\mathbb{P}^1} \to F_\bullet \to 0, \quad 0 \to E'_\bullet \to V \otimes \mathcal{O}_{\mathbb{P}^1} \to F'_\bullet \to 0.$$

These define a morphism $\mathbb{P}^1 \to \mathbb{P}^1 \times HQuot \times HQuot$ and we let K be the pullback of $\mathcal{K}$. We have a natural map from a trivial bundle:

$$\text{Hom }(V, V) \otimes \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{i=1}^{l} \text{Hom }(E_i, F'_i) \to \bigoplus_{i=1}^{l-1} \text{Hom }(E_i, F'_{i+1})$$

which factors through K. One easily checks that the map $\text{Hom }(V, V) \otimes \mathcal{O}_{\mathbb{P}^1} \to K$ is generically surjective. Therefore $H^1(\mathbb{P}^1, K) = 0$.

Moreover, a section of $\mathcal{K}$ is canonically obtained from the natural morphisms:

$$\pi_1^\star\mathcal{E}_i \to V \otimes \mathcal{O} \to \pi_2^\star\mathcal{F}_i.$$

We also obtain a section of the bundle $p_\star\mathcal{K}$ on $HQuot \times HQuot$. This section vanishes precisely along the diagonal. Therefore $[\Delta] = c_{top}(p_\star\mathcal{K})$. A Grothendieck Riemann Roch computation expresses the Chern character/classes of $p_\star\mathcal{K}$ as combination of classes $\pi_1^\star\alpha \cdot \pi_2^\star\beta$ where α, β are among the candidates we listed in lemma 2.2.2, as desired.

2.2.6 Cohomology of fibered products.

To finish the proof of theorem 1 we need to understand the cohomology of the boundary strata. To this end, we will need to make use of the following result about the cohomology of fibered products.

Lemma 2.2.4. *Assume there is a fiber square where p_1 and p_2 are proper morphisms of projective orbifolds with surjective orbifold differentials, and B simply connected:*

$$\begin{array}{ccc} Z = X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow p_2 \\ X & \xrightarrow{p_1} & B \end{array}$$

The cohomology of $H^\star(Z)$ is generated by the image of $H^\star(X) \otimes H^\star(Y)$.

Proof. We reformulate the statement as follows. We have a fiber diagram:

$$\begin{array}{ccc} S' & \xrightarrow{i'} & S \\ \pi' \downarrow & & \pi \downarrow \\ T' & \xrightarrow{i} & T \end{array}$$

where $T' = B, T = B \times B, S' = Z, S = X \times Y$. We observe that $i^\star : H^\star(T) \to H^\star(T')$ is surjective. We want to prove that $(i')^\star : H^\star(S) \to H^\star(S')$ is also surjective. This will follow from a more general argument.

There are two Leray sequences corresponding to the maps π and π'. Their collapsing is a well known result of Deligne. To apply it we need to know that the differentials of both maps are surjective. Strictly speaking, Deligne works with smooth projective varieties, but his result extends to projective orbifolds (the main ingredient in the proof is the Hard Lefschetz theorem, which holds for orbifolds - see [56]). There are natural morphisms between these spectral sequences:

$$\begin{array}{ccc} H^p(T, R^q\pi_\star\mathbb{Q}) & \Rightarrow & H^{p+q}(S) \\ i^\star \downarrow & & (i')^\star \downarrow \\ H^p(T', R^q\pi'_\star\mathbb{Q}) & \Rightarrow & H^{p+q}(S') \end{array}$$

We claim that the second vertical arrow $(i')^\star$ is surjective. We first observe that the first vertical arrow $i^\star$ is surjective. Indeed, $i^\star : H^\star(T) \to H^\star(T')$ is surjective. The two local systems given by the direct images $R^q\pi_\star\mathbb{Q}$ and $R^q\pi'_\star\mathbb{Q}$ on T, T' are trivial, these spaces being simply connected. Surjectivity of $i^\star$ follows. Because the two spectral sequences degenerate, there are suitable filtrations $F^\bullet$ and $F'^\bullet$ of $H^{p+q}(S)$ and $H^{p+q}(S')$, compatible with the map $(i')^\star$ such that the map $(i')^\star : Gr^\bullet_F \to Gr^\bullet_{F'}$ is surjective. It follows inductively that $(i')^\star$ restricted to the successive pieces of the filtrations is also surjective, hence $(i')^\star : H^\star(S) \to H^\star(S')$ is surjective. This completes the proof.

2.2.7 The main result.

Let X be an arbitrary SL flag variety. In this subsection we will conclude the cohomological proof of our main result, theorem 1.

We will apply the lemma proved above to the evaluation maps $ev : \overline{M}_{0,n+1}(X, \beta) \to X$. Then, the cohomology of the fibered product $\overline{M}_{0,A\cup\{\bullet\}}(X, \beta_A) \times_X \overline{M}_{0,B\cup\{\star\}}(X, \beta_B)$ is generated by classes coming from each factor. To place ourselves in the context of the lemma, we need to show that the (orbifold) differentials have maximal rank i.e. that these differentials are surjective. Recall from [22] the construction of $\overline{M}_{0,n+1}(X, \beta)$. First, one rigidifies the moduli problem. Consider an embedding of X into an r dimensional projective space $\mathbb{P}(V)$, and fix $t = (t_0, \dots, t_r)$ a basis of $V^\star$. We define a moduli space of t-rigid stable maps to $\mathbb{P}(V)$. This space parametrizes stable maps $f : C \to \mathbb{P}(V)$ of degree d with $n + 1$

markings; in addition to the $n+1$ standard markings of the domain p_i, we also fix $d(r+1)$ markings $q_{i,j}$ ($0 \le i \le r$ and $1 \le j \le d$) stabilizing the domain curve, such that we have an equality of Cartier divisors $f^\star t_i = q_{i,1} + \ldots + q_{i,d}$.

We consider the closed subscheme of the scheme of t-rigid maps to $\mathbb{P}(V)$ which factor through the inclusion of X in $\mathbb{P}(V)$. The moduli space of such t-rigid maps M_t is a smooth variety. To get to the Kontsevich-Manin moduli space, we need to quotient out the action of a finite group. It suffices to show that the map $ev : M_t \to \mathbb{P}^r$ which evaluates at the last point has maximal rank. This is essentially explained in [22] and it is quite straightforward. Let $(f, C, p_1, \ldots p_{n+1}, q_{ij})$ be a t-rigid map and let $p = f(p_{n+1})$. The differential of the evaluation sends the deformation space of the rigid stable map, Def, to T_pX. This map factors as the composition of two surjections Def $\to$ **Def** $\to T_pX$. Here $\mathbf{Def} = H^0(f^\star TX/T_C(-p_{n+1}))$ is the deformation space of the triple (f, C, p_{n+1}). The second map is simply the fiber evaluation. It is surjective because for any genus 0 stable map f, $f^\star TX$ is generated by global sections [22].

We are now ready to prove theorem 1. The statement is proved by double induction, first on the degree β, and then on the number of markings. In degree $\beta = 0$, the result follows from Keel's theorem. Next, we consider the indecomposable classes β and $n \le 1$. When $X = G(k, V)$, the result is a consequence of the description of the moduli space $\overline{\mathcal{M}}_{0,0}(X, 1)$ as the flag variety $Fl(k-1, k+1, V)$ of two step flags in V of dimensions $k-1$ and $k+1$ respectively (this is explained for example in [55] lemma 3.2). For $n = 1$, there is a similar description of the moduli space as the flag variety $Fl(k-1, k, k+1, V)$. For general flags X, the class β is Poincare dual to $c_1(\mathcal{Q}_i)$, for some i. There exists a flag variety Y (obtained by skipping the i^{th} quotient in X) and a projection morphism $\pi : X \to Y$ such that $\pi_\star\beta = 0$. All stable maps to X in the class β are entirely contained in the fibers of π which are Grassmannians. Therefore, the moduli space $\overline{\mathcal{M}}_{0,0}(X, \beta)$ maps to Y, the fibers being flag varieties as above. The main theorem follows immediately.

All other moduli spaces for higher values of n or β have nonempty boundary divisors $\overline{D}(A, B, \beta_A, \beta_B)$, where either β_A and β_B are both smaller than β, or A, B have fewer than n points. The cohomology of the fibered product scheme $\overline{M}(A, B, \beta_A, \beta_B)$ (and hence of the fibered product stack) dominating the boundary is computed by lemma 2.2.4. It is spanned by tautological classes in the light of the induction assumption. We apply the Deligne spectral sequence, specifically the exact sequence (2.2). We start with an arbitrary codimension k class α in $\overline{\mathcal{M}}_{0,n}(X, \beta)$. Its restriction $j^\star\alpha$ on $\mathcal{M}_{0,n}(X, \beta)$ has Hodge weight k. By virtue of lemma 2.2.3, we derive that $j^\star\alpha$ is equal to the restriction $j^\star\alpha'$ of some tautological class α'. Then Deligne's theorem shows that $\alpha - \alpha'$ is supported on $\overline{\mathcal{M}}(A, B, \beta_A, \beta_B)$, hence inductively it is sum of tautological classes. This proves the theorem.

2.3 Stable Maps to Projective Spaces.

In this section we revisit the computation of the Chow groups of the *open* stratum of irreducible maps of degree $d \ge 1$ to $\mathbb{P}^r$. We will seek to prove the following analogue of lemma 2.2.1. In addition, we prove a "Gorenstein" property of the tautological rings reminiscent of Faber's conjecture.

Proposition 2.3.1. *(1) The Chow rings of $M_{0,n}(\mathbb{P}^r, d)$ can be described explicitly in terms of the tautological classes in definition* 1.1.1 *- the precise description is given in lemmas* 2.3.1 *-* 2.3.3*. This description is independent of the degree* $d \ge 1$.

The tautological rings behave like the cohomology of certain projective manifolds of lower dimension (which do not depend on the degree).

(2) *All relations between the tautological generators* $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ *are tautological in the sense of definition* 1.6.1.

Our results hold equally well for Chow groups and the the lowest weight Hodge piece of the cohomology. We only include here the proofs in the Chow groups; they carry over in cohomology, almost verbatim, only replacing $A^\star$ by $W^\star H^\star$.

Remark 2.3.1. When $n = 0$, Pandharipande proved the fact below [52]. Our goal here is to obtain similar results for more markings.

Fact 2.3.1 (Pandharipande). The Chow ring of $M_{0,0}(\mathbb{P}^r, d)$ is isomorphic to the Chow ring of the Grassmannian $\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)$. The (restrictions of the) classes $\kappa(H^{i+1}, H^{j+1})$, where $0 \leq i \leq j \leq r-1$, form a set of generators.

To prove the first item in proposition 2.3.1, we follow the same line of reasoning as the original paper, *claiming no new ideas.* We include these computations because they fit quite naturally with our earlier arguments, and because they are necessary in proving the second part of the proposition. Moreover, we would like to point out consistency with the results of [31] where the Betti numbers of the relevant spaces are determined.

2.3.1 Three marked points

Let us start with the case when $n \geq 3$. As in section 2, $A^\star(M_{0,n}(\mathbb{P}^r, d))$ is isomorphic to $A^\star(\mathrm{Map}_d(\mathbb{P}^1, \mathbb{P}^r))$ via the natural projection $p : M_{0,n}(\mathbb{P}^r, d) \to \mathrm{Map}_d(\mathbb{P}^1, \mathbb{P}^r)$.

The last group can be computed from the image of any compactification of $\mathrm{Map}_d(\mathbb{P}^1, \mathbb{P}^r)$. There is an obvious candidate for a compactification, namely the projective space $\mathbb{P}(V)$ where $V = \oplus_0^r H^0(\mathbb{P}^1, \mathcal{O}(d))$. We need to identify the image of the restriction map:

$$i^\star : A^k(\mathbb{P}(V)) \to A^k(\mathrm{Map}_d(\mathbb{P}^1, \mathbb{P}^r)).$$

We claim that the image is 1 dimensional for $k \leq r-1$ and zero otherwise. If h is the hyperplane class on $\mathbb{P}(V)$, it is enough to show that $i^\star h^{r-1} \neq 0$ and $i^\star h^r = 0$.

We use the different compactification of $\mathrm{Map}_d(\mathbb{P}^1, \mathbb{P}^r)$ by the Kontsevich-Manin space of maps $\overline{M}_{0,3}(\mathbb{P}^r, d)$, letting $\tilde{D}$ be the disjoint union of the boundary divisors. We apply the exact sequence (2.2):

$$A^{k-1}(\tilde{D}) \to A^k(\overline{M}_{0,3}(\mathbb{P}^r, d)) \to \text{ Image } i^\star \to 0.$$

We define the following class on the Kontsevich-Manin space: $L_i = ev_i^\star H$ where H is the hyperplane class in $\mathbb{P}^r$. In lemma 2.3.2 we show that the class:

$$L_1^r + L_1^{r-1} L_2 + \ldots + L_1 L_2^{r-1} + L_2^r$$

is supported on the boundary. We let j denote the inclusion of $\mathrm{Map}_d(\mathbb{P}^1, \mathbb{P}^r)$ into the Kontsevich-Manin space. Since $j^\star L_1 = j^\star L_2 = i^\star h$ we see that $i^\star h^r = 0$.

It remains to show that $i^\star h^{r-1} \neq 0$, or that $i^\star$ is nonzero in degree $r-1$. By the exact sequence above, it suffices to show that the class L_1^{r-1} is not supported on the boundary.

To see this, we follow an idea of Pandharipande [52] to reduce to the case of maps of degree 1. Let ν be a self-morphism of $\mathbb{P}^r$ of degree d. Let C be the universal curve over $\overline{M}_{0,3}(\mathbb{P}^r,1)$. We have a diagram:

$$\begin{tikzcd} C \arrow[r, "ev"] \arrow[d, "\pi"] & \mathbb{P}^r \arrow[r, "\nu"] & \mathbb{P}^r \\ \overline{M}_{0,3}(\mathbb{P}^r,1) & & \end{tikzcd}$$

which induces a map $\tau : \overline{M}_{0,3}(\mathbb{P}^r,1) \to \overline{M}_{0,3}(\mathbb{P}^r,d)$. Letting $\tilde{L}_1$ be the evaluation class on the space of degree 1 maps, we have:

$$\tau^\star L_1 = \tau^\star ev_1^\star H = ev_1^\star \nu^\star H = d \cdot ev_1^\star H = d\tilde{L}_1.$$

Assuming that L_1^{r-1} is supported on the boundary of $\overline{M}_{0,3}(\mathbb{P}^r,d)$ then we conclude that the class $\tilde{L}_1^{r-1}$ is supported on the boundary of $\overline{M}_{0,3}(\mathbb{P}^r,1)$. Therefore, $j^\star\tilde{L}_1^{r-1} = 0$ on $M_{0,3}(\mathbb{P}^r,1)$. Here j denotes, just as for degree d, the inclusion $M_{0,3}(\mathbb{P}^r,1) \to \overline{M}_{0,3}(\mathbb{P}^r,1)$.

We will now derive the contradiction by looking at the space of degree 1 maps. Denote again by i the inclusion of $M_{0,3}(\mathbb{P}^r,1)$ into its "obvious" compactification $\mathbb{P}^{2r+1}$. In fact, $M_{0,3}(\mathbb{P}^r,1)$ can be described as $\mathbb{P}^{2r+1} \setminus S$, where S is the subvariety corresponding to $r+1$-tuples of polynomials of degree 1 with a common root. Then S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^r$. The restriction map $i^\star : A^{r-1}(\mathbb{P}^{2r+2}) \to A^{r-1}(M_{0,3}(\mathbb{P}^r,1))$ is an isomorphism. It is clear that $j^\star\tilde{L}_1 = i^\star h$, where h is the hyperplane class on $\mathbb{P}^{2r+1}$. We obtain the contradiction $0 = j^\star\tilde{L}_1^{r-1} = i^\star h^{r-1} \neq 0$. Our claim is now proved.

Combining the claim with the observation opening this subsection, we obtain:

Lemma 2.3.1. *When $n \geq 3$, $A^\star(M_{0,n}(\mathbb{P}^r,d))$ is isomorphic to $A^\star(\mathbb{P}^{r-1})$, and $L_1 = ev_1^\star H$ is a multiplicative generator.*

2.3.2 Two marked points

In this subsection, we will compute $A^\star(M_{0,2}(\mathbb{P}^r,d))$ using the ideas of Pandharipande [52].

We let ν be a degree d self-morphism of $\mathbb{P}^r$. As before, composition with ν induces a morphism $\tau : M_{0,2}(\mathbb{P}^r,1) \to M_{0,2}(\mathbb{P}^r,d)$.

Observe that $M_{0,2}(\mathbb{P}^r,1) = \mathbb{P}^r \times \mathbb{P}^r \setminus \Delta$ where Δ is the diagonal. Moreover, $\overline{M}_{0,2}(\mathbb{P}^r,1) = \mathrm{Bl}_\Delta(\mathbb{P}^r \times \mathbb{P}^r)$ is the blowup along the diagonal. The two hyperplane classes on the two factors of $\mathbb{P}^r \times \mathbb{P}^r$, as well as on the blow up, will be denoted by h_1 and h_2. The evaluation classes $\tilde{L}_1$ and $\tilde{L}_2$ coincide with h_1 and h_2 on $\overline{M}_{0,2}(\mathbb{P}^r,1)$. Notice that $\tau^\star L_i = d \cdot h_i$.

Letting $s_r = \sum_{i+j=r} h_1^i h_2^j$, we see that $A^\star(\mathbb{P}^r \times \mathbb{P}^r \setminus \Delta) = \mathbb{C}[h_1,h_2]/(h_1^{r+1}, h_2^{r+1}, s_r)$. It is clear that τ induces a homomorphism:

$$\tau^\star : A^\star(M_{0,2}(\mathbb{P}^r,d)) \to \mathbb{C}[h_1,h_2]/(h_1^{r+1}, h_2^{r+1}, s_r) = A^\star(M_{0,2}(\mathbb{P}^r,1)). \tag{2.8}$$

The map $\tau^\star$ is surjective since we saw h_1 and h_2 are contained in its image.

We seek to show that $\tau^\star$ is an isomorphism. To this end, we will analyze $M_{0,2}(\mathbb{P}^r,d)$ differently, by exhibiting this space as a quotient. Let $\mathbb{P}^1 = \mathbb{P}(V)$ where $V \cong \mathbb{C}^2$ is a two dimensional vector space with the natural action of the torus $T = \mathbb{C}^\star \times \mathbb{C}^\star$. Let

$$U \hookrightarrow \bigoplus_0^r H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \bigoplus_0^r Sym^d(V^\star)$$

be the open subvariety corresponding to $(r+1)$-tuples of degree d polynomials with no common vanishing.

The torus T acts with finite stabilizers on U and the geometric quotient is $M_{0,2}(\mathbb{P}^r, d)$ [52]. There is an isomorphism:

$$A^\star(M_{0,2}(\mathbb{P}^r, d)) = A^\star_T(U) = A^\star(U_T).$$

Here $U_T = U \times_T ET$ is the Borel construction. In the topological category, we take $BT = \mathbb{P} \times \mathbb{P}$, where $\mathbb{P}$ is the infinite projective space, while $ET \to BT$ is the T-bundle whose associated vector bundle is $\mathbb{S} = p_1^\star \mathcal{O}_{\mathbb{P}}(-1) \oplus p_2^\star \mathcal{O}_{\mathbb{P}}(-1)$. Here, we write p_1 and p_2 for the two projections. In the algebraic category, for the purposes of finding classes of *fixed* codimension, we will pass to projective spaces $\mathbb{P}$ of large, but finite, dimension.

Now, we observe that U_T sits naturally as a subspace of the following bundle $p : B \to \mathbb{P} \times \mathbb{P}$:

$$B = \bigoplus_0^r Sym^d V^\star \times_T ET = \bigoplus_0^r Sym^d(p_1^\star \mathcal{O}_{\mathbb{P}}(1) \oplus p_2^\star \mathcal{O}_{\mathbb{P}}(1)) = \bigoplus_0^r \bigoplus_{i+j=d} p_1^\star \mathcal{O}_{\mathbb{P}}(i) \oplus p_2^\star \mathcal{O}_{\mathbb{P}}(j).$$

We let D be the complement of U_T in this bundle. We let $\mathbf{h}_1$ and $\mathbf{h}_2$ denote the two generators of $A^\star(\mathbb{P} \times \mathbb{P})$. The pullbacks $p^\star \mathbf{h}_1$ and $p^\star \mathbf{h}_2$ are the generators for B.

There is an exact sequence:

$$A_{dim-k}(D) \xrightarrow{i} A^k(B) \xrightarrow{j^\star} A^k(U_T) \to 0 \tag{2.9}$$

Therefore, $A^\star(U_T)$ is spanned by the restrictions $j^\star p^\star \mathbf{h}_1$ and $j^\star p^\star \mathbf{h}_2$.

For the reader interested in the cohomological proofs, the first term in the sequence above should be replaced by the Borel-Moore homology. The image of the map j is $W^k H^k(U_T)$. This follows from (2.1) with trivial group, since the schemes B and U_T have the common compactification $\mathbb{P}(B \oplus \mathcal{O}_{\mathbb{P}})$.

We consider the following element in the Chow group of $\mathbb{P} \times \mathbb{P}$: $\mathbf{s}_j = \sum_{a+b=j} \mathbf{h}_1^a \mathbf{h}_2^b$. We claim that the elements $p^\star \mathbf{s}_{k+r}$ are in the image of the inclusion map i so they are also in the kernel of $j^\star$ for all $k \geq 0$.

The proof of the claim is almost identical to Pandharipande's argument. We introduce the following notation. We let π be the projection $\mathbb{P}(\mathbb{S}) \to \mathbb{P} \times \mathbb{P}$. We look at the total space Q of the bundle $\pi^\star B$ which sits over $\mathbb{P}(\mathbb{S})$, and comes equipped with the pullback of the tautological bundle $p^\star \mathcal{O}_{\mathbb{P}(\mathbb{S})}(1)$:

$$\begin{tikzcd}
Q = \pi^\star B \arrow[r, "p"] \arrow[d, "\pi"] & \mathbb{P}(\mathbb{S}) \arrow[d, "\pi"] \\
B \arrow[r, "p"] & \mathbb{P} \times \mathbb{P}.
\end{tikzcd}$$

Just as in [52] one arrives at the following diagram:

$$\begin{tikzcd}
Q \setminus \overline{D} \arrow[r, "ev"] \arrow[d, "\pi"] & \mathbb{P}^r \\
B \setminus D = U_T \arrow[r, "\epsilon"] & U/T = M_{0,2}(\mathbb{P}^r, d).
\end{tikzcd}$$

Indeed, unwinding the definitions, we obtain a natural evaluation map $ev : Q \setminus \overline{D} \to \mathbb{P}^r$ undefined over the common vanishing $\overline{D}$ of $r+1$ canonical sections of $p^\star \mathcal{O}_{\mathbb{P}(\mathbf{S})}(d)$. It is clear that $\overline{D}$ maps birationally to D via π. The classifying map ϵ to the coarse moduli scheme is the natural quotient map.

We observed that $[\overline{D}] = c_1\left(p^\star \mathcal{O}_{\mathbb{P}(\mathbf{S})}(d)^{r+1}\right)$. Hence, the class $\pi_\star\left(c_1(p^\star \mathcal{O}_{\mathbb{P}(\mathbf{S})}(d))^{r+1}\alpha\right)$ is supported on D, so it is in the image of i, for all classes α on Q. We apply this observation to the class:

$$\alpha = p^\star c_1(\mathcal{O}_{\mathbb{P}(\mathbf{S})}(d))^k.$$

The following class is contained in the image of i:

$$\pi_\star\left(p^\star c_1\left(\mathcal{O}_{\mathbb{P}(\mathbf{S})}(d)\right)^{r+1+k}\right) = d^{r+k+1} p^\star \pi_\star\left(c_1(\mathcal{O}_{\mathbb{P}(\mathbf{S})}(1))^{r+k+1}\right) = p^\star \pi_\star\left(c_1(\mathcal{O}_{\mathbb{P}(\mathbf{S})}(1))^{r+k+1}\right) =$$

$$= p^\star\left(\frac{1}{c(\mathbf{S})}\right)_{r+k} = p^\star\left(\frac{1}{(1-\mathbf{h}_1)(1-\mathbf{h}_2)}\right)_{r+k} = p^\star \mathbf{s}_{r+k}.$$

Using what we just proved together with the exact sequence (2.9), we obtain a surjection:

$$j^\star : \mathbb{C}[p^\star \mathbf{h}_1, p^\star \mathbf{h}_2] / \left(p^\star \mathbf{s}_{k+r}\right)_{k \geq 0} \to A^\star(M_{0,2}(\mathbb{P}^r, d)). \tag{2.10}$$

The reader can check that there is an obvious isomorphism between the right hand side of (2.8) and the left hand side of (2.10). Hence both (2.8) and (2.10) are isomorphisms.

We know that $h_1^i h_2^j$ $(0 \leq i, j \leq r)$ span the right hand side of (2.8) with the relation $s_r = 0$ and $h_1^i h_2^j = d^{-i-j} \tau^\star j^\star(L_1^i L_2^j)$. Therefore,

Lemma 2.3.2. *The map $\tau^\star$ induces a ring isomorphism between $A^\star(M_{0,2}(\mathbb{P}^r, d))$ and $A^\star(\mathbb{P}^r \times \mathbb{P}^r \setminus \Delta) = \mathbb{C}[h_1, h_2]/(h_1^{r+1}, h_2^{r+1}, \sum_{i+j=r} h_1^i h_2^j)$. The class*

$$\sum_{i+j=r} ev_1^\star H^i \cdot ev_2^\star H^j \tag{2.11}$$

is supported on the boundary.

2.3.3 One marked point

The discussion for one marked point is similar. We first observe that $\overline{M}_{0,0}(\mathbb{P}^r, 1) = \mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)$ and $\overline{M}_{0,1}(\mathbb{P}^r, 1) = \mathbb{P}(\mathbf{S})$ where $\mathbf{S}$ is the tautological bundle over the Grassmannian.

We fix ν a degree d self-morphism of $\mathbb{P}^r$, and as usual, we use composition with ν to get degree d maps from degree 1 maps. We obtain two morphisms:

$$\tau : \mathbf{G}(\mathbb{P}^1, \mathbb{P}^r) \to M_{0,0}(\mathbb{P}^r, d) \text{ and } \tau : \mathbb{P}(\mathbf{S}) \to M_{0,1}(\mathbb{P}^r, d).$$

We conclude that there is a diagram:

$$\begin{array}{ccc} A^\star(M_{0,0}(\mathbb{P}^r, d)) & \xrightarrow{\tau^\star} & A^\star(\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)) \\ {\scriptstyle \pi^\star}\downarrow & & {\scriptstyle \pi^\star}\downarrow \\ A^\star(M_{0,1}(\mathbb{P}^r, d)) & \xrightarrow{\tau^\star} & A^\star(\mathbb{P}(\mathbf{S})). \end{array}$$

The lower horizontal arrow:

$$\tau^\star : A^\star(M_{0,1}(\mathbb{P}^r, d)) \to A^\star(\mathbb{P}(\mathbf{S})) \tag{2.12}$$

is surjective. Indeed, the Chow ring of the projective bundle $\mathbb{P}(\mathbf{S})$ is generated by the Chow ring of the base $\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)$ together with the additional class $\lambda = c_1(\mathcal{O}_{\mathbb{P}(\mathbf{S})}(1))$. The class λ is in the image of $\tau^\star$. Indeed, pulling back under the two evaluation maps: $ev : \mathbb{P}(\mathbf{S}) = M_{0,1}(\mathbb{P}^r, 1) \to \mathbb{P}^r$ and $ev : M_{0,1}(\mathbb{P}^r, d) \to \mathbb{P}^r$ it is clear that:

$$\tau^\star L_1 = \tau^\star ev^\star H = ev^\star \nu^\star H = d \cdot ev^\star H = d\lambda.$$

We also know by fact 2.3.1 that the upper arrow $\tau^\star$ is surjective. This proves the claim.

The next step involves the computation in equivariant Chow groups. We keep the notation of the previous subsection. We observe that $M_{0,1}(\mathbb{P}^r, d) = U/N$, where N is the group of 2×2 upper triangular matrices acting on $V \cong \mathbb{C}^2$. We obtain the isomorphism:

$$A^\star(M_{0,1}(\mathbb{P}^r, d)) = A^\star_N(U) = A^\star(U_N) = A^\star(U \times_N EN).$$

We denote by $\mathbb{G}$ the infinite Grassmannian of 2 dimensional planes, and by $\mathbb{S}$ the tautological rank 2 bundle. BN can be identified with the projective bundle $\mathbb{P}(\mathbb{S})$. We let $\pi : \mathbb{P}(\mathbb{S}) \to \mathbb{G}$ denote the projection. Also $V \times_N EN = \pi^\star\mathbb{S}$. For our purposes, we will pass to finite dimensional truncations of $\mathbb{G}$ by large dimensional Grassmannians.

We can view $U \times_N EN$ as a subvariety of the bundle:

$$\bigoplus_0^r Sym^d(V^\star) \times_N EN = \bigoplus_0^r Sym^d \pi^\star \mathbb{S}^\star.$$

Let B denote this bundle and let D denote the complement of U_N in B. We obtain an exact sequence:

$$A_{dim-k}(D) \xrightarrow{i} A^k\left(\bigoplus_0^r Sym^d(\pi^\star\mathbb{S}^\star)\right) = A^k(\mathbb{P}(\mathbb{S})) \xrightarrow{j^\star} A^k(M_{0,1}(\mathbb{P}^r, d)) \to 0. \tag{2.13}$$

We denote by $p : B \to \mathbb{P}(\mathbb{S})$ and $q : \mathbb{P}(\pi^\star\mathbb{S}) \to \mathbb{P}(\mathbb{S})$ the two projections. We let Q be the total space of the bundle $q^\star B$ so that we obtain a commutative diagram:

$$\begin{tikzcd}
 & Q = q^\star B \arrow[r, "p"] \arrow[d, shift right, "q"'] & \mathbb{P}(\pi^\star\mathbb{S}) \arrow[d, shift right, "q"'] & \\
M_{0,1}(\mathbb{P}^r, d) & B \arrow[l, dotted, "\epsilon"'] \arrow[r, "p"] \arrow[u, shift right] & \mathbb{P}(\mathbb{S}) \arrow[r, "\pi"] \arrow[u, shift right] & \mathbb{G}.
\end{tikzcd}$$

Just as before, the space $\mathbb{P}(\pi^\star\mathbb{S})$ comes equipped with a bundle $\mathcal{O}(d)$, which gives by pullback the bundle $p^\star\mathcal{O}(d)$ over Q.

Unwinding the definitions, we obtain an evaluation morphism $ev : Q \backslash \overline{D} \to \mathbb{P}^r$ undefined over the common vanishing $\overline{D}$ of $r+1$ sections of the line bundle $p^\star\mathcal{O}(d)$ on Q. Hence, $[\overline{D}] = c_1(p^\star\mathcal{O}(d))^{r+1}$. Therefore, the image of i contains the classes:

$$q_\star(c_1(p^\star\mathcal{O}(d))^{r+k+1}) = p^\star q_\star(c_1(\mathcal{O}(d))^{r+k+1}) = d^{r+k+1} p^\star\pi^\star s_{r+k}(\mathbb{S}),\ k \geq 0,$$

where s_{r+k} are the Segre classes. We use the exact sequence (2.13) to obtain a surjection:

$$A^\star(\mathbb{P}(\mathbb{S}))/\left(\pi^\star s_{r+k}(\mathbb{S})\right)_{k\geq 0} \to A^\star(M_{0,1}(\mathbb{P}^r, d)).$$

We recall the surjection (2.12) and the isomorphism:

$$A^\star(\mathbb{P}(\mathbb{S}))/(\pi^\star s_{r+k}(\mathbb{S}))_{k\geq 0} \to A^\star(\mathbb{P}(\mathbf{S})).$$

This follows from the usual description of the Chow rings of projective bundles and the fact that over the base we have the analogous isomorphism:

$$A^\star(\mathbb{G})/(s_{r+k}(\mathbb{S}))_{k\geq 0} \to A^\star(\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)).$$

This is enough to infer that the map $\tau^\star$ is an isomorphism. However, we push the analysis further. We see that $A^\star(\mathbb{P}(\mathbf{S}))$ is generated by $A^\star(\mathbf{G})$ together with the class $\lambda = c_1(\mathcal{O}_{\mathbb{P}(\mathbf{S})}(1))$ which satisfies the relation $\lambda^2 + c_1(\mathbf{S})\lambda + c_2(\mathbf{S}) = \mathbf{0}$. We have:

$$\lambda = \frac{1}{d}\tau^\star L_1,\ \frac{1}{d^2}\tau^\star\kappa(H^2) = -c_1(\mathbf{S}),\ \frac{1}{d^4}\tau^\star\kappa(H^2)^2 - \frac{1}{d^3}\tau^\star\kappa(H^3) = c_2(\mathbf{S}).$$

The last two follow from a fact explained in [52], namely that the Chern classes of the quotient bundles $c_1(Q)$ and $c_2(Q)$ are the pullbacks of $\frac{1}{d^2}\kappa(H^2)$ and $\frac{1}{d^3}\kappa(H^3)$ under τ. We derive:

Lemma 2.3.3. *$A^\star(M_{0,1}(\mathbb{P}^r, d))$ is generated multiplicatively by $L_1 = ev_1^\star H$ and the pullback of $A^\star(M_{0,0}(\mathbb{P}^r, d))$ as determined in 2.3.1. $\tau^\star$ establishes an isomorphism with $A^\star(\mathbb{P}(\mathbf{S}))$, where $\mathbf{S}$ is the tautological bundle over the Grassmannian $\mathbf{G}(\mathbb{P}^1, \mathbb{P}^r)$. The codimension 2 class:*

$$ev_1^\star H^2 - \frac{1}{d}ev_1^\star H \cdot \kappa(H^2) + \frac{1}{d^2}\kappa(H^2)^2 - \frac{1}{d}\kappa(H^3) \tag{2.14}$$

is supported on the boundary.

2.3.4 The tautological relations

In the remainder of this chapter we will show that all relations between the tautological classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ on $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ are tautological in the sense of definition 1.6.1. We will only briefly indicate the ideas involved.

We make two preliminary observations. First, the Keel relations stated in the introduction did not involve assignining weights to the legs. However, this can be achieved via the gluing and forgetting equations.

Indeed, assigning a class α to the leg $\mathfrak{l}$ is equivalent to gluing in a tripod of degree 0 along the leg $\mathfrak{l}$, with weights $(1, 1, \alpha)$, and then forgetting one of the markings of the newly added vertex (and stabilizing).

These Keel relations with weights assigned to the legs are therefore tautological. The same argument shows that:

- multiplication of tautological relations by $ev_{\mathfrak{l}}^\star\alpha$ still gives a tautological relation.

Secondly,

- the system of tautological relations between $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ is closed under pullback by the forgetful morphisms π.

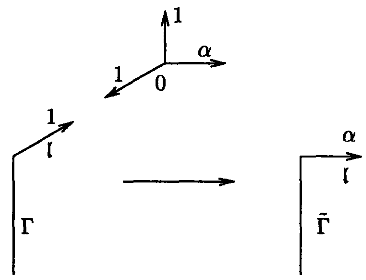


Figure 2-1: Adding in weights via gluing and forgetting.

We agree that:

$$\pi^\star\left[\Gamma,\mathfrak{w},\mathfrak{f}\right]=\sum_{\tilde{\Gamma}}\left[\tilde{\Gamma},\mathfrak{w},\mathfrak{f}\right]$$

the sum being taken over all possible graphs $\tilde{\Gamma}$ obtained from Γ by attaching a leg at any of its vertices. This is again a consequence of gluing a tripod of degree 0 and forgetting a destabilizing leg.

As a consequence, we can generalize Keel's relations accounting for $n+4$ attached legs with arbitrary weights, the last n of which are distributed arbitrarily among two vertices, and the first 4 being distributed among the vertices as $(ij)(kl)$ and $(ik)(jl)$ (see the figure).

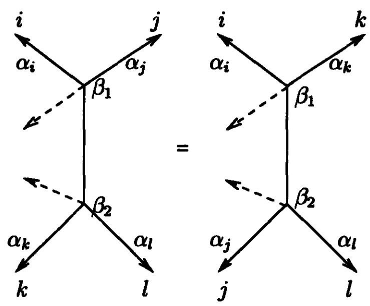


Figure 2-2: Generalized Keel relations.

We now prove proposition 2.3.1. We start our analysis starts with the case $n=0$. All tautological classes $[\Gamma,\mathfrak{w},\mathfrak{f}]$ on $\mathcal{M}_{0,0}(\mathbb{P}^r,d)$ are restrictions of the classes $\kappa(H^{i_1},\ldots,H^{i_m})$. According to fact 2.3.1, the two point classes $\kappa(H^{l_1},H^{l_2})$ are additive generators without relations. It suffices to express $\kappa(H^{i_1+1},\ldots,H^{i_l+1})$ in terms of the two point classes by means of the tautological relations. This can be thought of as an instance of Kontsevich-Manin reconstruction, and it can be proved in the same manner.

We use the Keel relations on $\overline{\mathcal{M}}_{0,n+4}(\mathbb{P}^r,d)$ with cohomology weights $H^{i_1},\ldots,H^{i_4}$, $H^{j_1},\ldots H^{j_n}$ assigned to the legs of the graph. Using invariance under the forgetful morphisms, we conclude that the following equation on $\mathcal{M}_{0,0}(\mathbb{P}^r,d)$:

$$\kappa(H^{i_1+i_4},H^{i_2},H^{i_3},H^{j_1},\ldots,H^{j_n})+\kappa(H^{i_1},H^{i_4},H^{i_2+i_3},H^{j_1},\ldots,H^{j_n})=$$

$$=\kappa(H^{i_1+i_3},H^{i_2},H^{i_4},H^{j_1},\ldots,H^{j_n})+\kappa(H^{i_1},H^{i_3},H^{i_2+i_4},H^{j_1},\ldots,H^{j_n})\qquad(2.15)$$

is tautological. We have made use of the mapping to a point and forgetting destabilizing legs relations. Setting $i_4 = 1$, and using the divisor equation, we express

$$\kappa(H^{i_1+1}, H^{i_2}, H^{i_3}, H^{j_1}, \ldots, H^{j_n}) - \kappa(H^{i_1}, H^{i_2+1}, H^{i_3}, H^{j_1}, \ldots, H^{j_n})$$

in terms of classes with fewer insertions. This works as long as we have at least at least 3 insertions. This system of equations together determine uniquely the κ classes with several insertions in terms of the two point classes $\kappa(H^{l_1}, H^{l_2})$.

When $n = 1$, the same reasoning applies. We only need to write down tautological equations expressing the classes $L_1^e \cdot \kappa(H^{i_1}, \ldots, H^{i_l})$ in terms of the generators with $e \leq 1$ and $l \leq 2$. We can multiply (2.15) by $ev_1^\star H^e$ to get tautological relations which reduce us to the case $l \leq 2$.

The last step of the reduction consists in proving that the equation (2.14) is tautological. In fact, we claim the following tautological equation on $\mathcal{M}_{0,1}(\mathbb{P}^r, d)$:

$$ev_1^\star \alpha - \frac{1}{d} ev_1^\star H \cdot \kappa(\alpha) + \frac{1}{d^2}\kappa(\alpha, H^2) - \frac{1}{d}\kappa(\alpha H) = 0. \tag{2.16}$$

Indeed, consider the Keel relation on $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^r, d)$ with distribution of the markings (12)(34) and (13)(24) among two vertices, such that the weights of the legs are $1, \alpha, H, H$ respectively. We then forget the the last three markings via the morphism $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, d)$, and restrict to the open part $\mathcal{M}_{0,1}(\mathbb{P}^r, d)$. The statement then follows using the divisor relation, and the contracting unstable tripods relations.

When $n = 2$, we need to prove that the generators

$$ev_1^\star H^{i_1} \cdot ev_2^\star H^{i_2} \cdot \kappa(H^{j_1}, \ldots, H^{j_l})$$

can be expressed in terms of the classes $ev_1^\star H^{i_1} \cdot ev_2^\star H^{i_2}$ via tautological equations. This is a consequence of the above discussion and of equation (2.17). To include more insertions in the κ classes we multiply (2.17) below by a monomial in the evaluation classes and apply the forgetful morphisms. The identity (2.17) below also shows that equation (2.11) is tautological; we specialize to $k = 0$ and $l = r$, also using the pullback from the target relations.

We claim that the following equation on $\mathcal{M}_{0,2}(\mathbb{P}^r, d)$:

$$\sum_{i=0}^{l} ev_1^\star H^{k+l-i} \cdot ev_2^\star H^{k+i} = \frac{1}{d} ev_1^\star H^k \cdot ev_2^\star H^k \cdot \kappa(H^{l+1}) \tag{2.17}$$

is tautological. It is clear that the case $l = 0$ is just the divisor equation. The case $l = 1$ is a tautological equation since it is obtained by multiplication by evaluation classes of the tautological equation:

$$ev_1^\star H + ev_2^\star H = \frac{1}{d}\kappa(H^2). \tag{2.18}$$

In turn, this is obtained from the Keel equation on $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^r, d)$, splitting the legs in the two configurations (12)(34) and (13)(24) among two vertices. The weights on the legs are $(1, 1, H, H)$. We then pushforward the relation by the forgetful morphism $\pi : \overline{\mathcal{M}}_{0,4}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,2}(\mathbb{P}^r, d)$, and use the divisor equation to obtain (2.18). Similarly, one proves that the equation:

$$ev_1^\star H \cdot ev_2^\star H = \frac{1}{d^2}\kappa(H^2, H^2) - \frac{1}{d}\kappa(H^3) \tag{2.19}$$

is a tautological identity on $\mathcal{M}_{0,2}(\mathbb{P}^r, d)$.

As a corollary of (2.16) and (2.18) the following more general tautological equation on $\mathcal{M}_{0,2}(\mathbb{P}^r, d)$ holds true:

$$ev_1^\star H^l + ev_2^\star H^l = \frac{2}{d}\kappa(H^{l+1}) - \frac{1}{d^2}\kappa(H^2, H^l) \tag{2.20}$$

Finally, equation (2.17) for $(k+1, l-2)$ and (2.19) imply the statement for (k, l) if one observes that:

$$\kappa(H^{l+1}) - \frac{1}{d}\kappa(H^2, H^l) + \frac{1}{d^2}\kappa(H^2, H^2, H^{l-1}) - \frac{1}{d}\kappa(H^3, H^{l-1}) = 0.$$

This again is a tautological equation obtained from Keel's relation (2.15) with $i_1 = 1, i_2 = l-1, l_3 = 1, i_4 = 2$ and the divisor equation.

Finally, let $n \geq 3$. The system of equations (2.20) and (2.17) for all pairs of indices (i, j) now becomes solvable. We obtain the following tautological equation on $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$:

$$ev_1^\star H^l = \ldots = ev_n^\star H^l = \frac{1}{d(l+1)}\kappa(H^{l+1}). \tag{2.21}$$

For $k = 0, l = r$, we obtain that

$$ev_1^\star H^r = 0$$

is a tautological equation on the open part.

We have seen in the case $n = 2$ that all classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ can be expressed via tautological equations in terms of the evaluation classes at the first 2 markings. In turn, these can be expressed via the tautological relation (2.21) in terms of evaluation classes with one marking. In the light of lemma (2.3.1), the proof of proposition 2.3.1 is complete.

Chapter 3

The Bialynicki-Birula stratification

In this chapter we give another proof of our main theorem when the target is a projective space, using a different point of view. In fact, we will prove the stronger version, theorem 2 stated in the introduction. This result explicitly describes the Bialynicki-Birula stratification on the space of stable maps in terms of the Gathmann-Li spaces. The main ingredient of the proof is an explicit description of the torus flow on the Kontsevich-Manin spaces. To obtain the tautology of the Chow classes, we exploit the inductive structure of the Gathmann and Kontsevich-Manin spaces.

This chapter is organized as follows. The first section contains preliminary observations about localization on the moduli spaces of stable maps and about the Gathmann stacks. Then, we construct the Bialynicki-Birula cells on a general smooth Deligne Mumford stack with an equivariant atlas. We establish the "homology basis theorem" under a general filterability assumption. The third section contains the main part of the argument. There, we identify explicitly the torus decomposition for the stacks $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$, and show its filterability. Filterability essentially entails to defining an ordering on the fixed loci which allows us to build the moduli space by *successively* adding cells. Finally, the last section proves the main results.

3.1 Preliminaries.

In this section we collect several useful facts about the fixed loci of the torus action on the moduli spaces of stable maps. We also discuss the Gathmann compactification of the stack of maps with prescribed contact orders to a fixed hyperplane.

3.1.1 Localization on the moduli spaces of stable maps.

The main theme of this chapter is a description of the flow of stable maps under the torus action on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. This flow is obtained by translation of maps under the action on the target $\mathbb{P}^r$. Traditionally, actions with isolated fixed points have been used. As it will become manifest in the next sections, it is better to consider the following action which in homogeneous coordinates is given by:

$$t \cdot [z_0 : z_1 : \ldots : z_r] = [z_0 : tz_1 : \ldots : tz_r], \text{ for } t \in \mathbb{C}^\star. \tag{3.1}$$

There are two fixed sets: one of them is the isolated point $p = [1 : 0 : \ldots : 0]$ and the other one is the hyperplane H given by the equation $z_0 = 0$. We observe that

$$\text{if } z \in \mathbb{P}^r - H \text{ then } \lim_{t \to 0} t \cdot z = p. \tag{3.2}$$

The fixed stable maps $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^r$ are obtained as follows. The image of f is an invariant curve in $\mathbb{P}^r$, while the images of the marked points, contracted components, nodes and ramification points are invariant i.e. they map to p or to H. The non-contracted components are either entirely contained in H, or otherwise they map to invariant curves in $\mathbb{P}^r$ joining p to a point q_H in H. The restriction of the map f to these latter components is totally ramified over p and q_H. This requirement determines the map uniquely. To each fixed stable map we associate a tree Γ such that:

- The edges correspond to the non-contracted components which are not contained in H. These edges are decorated with degrees.
- The vertices of the tree correspond to the connected components of the set $f^{-1}(p) \cup f^{-1}(H)$. These vertices come with labels p and H such that adjacent vertices have distinct labels. Moreover, the vertices labeled H also come with degree labels, corresponding to the degree of the stable map on the component mapped to H (which is 0 if these components are isolated points).
- Γ has n numbered legs coming from the marked points.

We introduce the following notation for the graph Γ.

- Typically, v *stands for a vertex labeled* p and we let $n(v)$ be its total valency (total number of incident flags i.e. legs and edges).
- Typically, w *stands for a vertex labeled* H and we let $n(w)$ be its total valency. The corresponding degree is d_w.
- The set of vertices is denoted $V(\Gamma)$. We write V and W for the number of vertices labeled p and H respectively.
- The set of edges is denoted $E(\Gamma)$, and the degree of the edge e is d_e. We write E for the total number number of edges.
- For each vertex v, we write α_v for the *ordered* collection of degrees of the incoming flags. We agree that the degrees of the legs are 0. We use the notation $d_v = |\alpha_v|$ for the sum of the incoming degrees.
- A vertex w labeled H of degree $d_w = 0$ is called *unstable* if $n(w) \leq 2$ and *very unstable* if $n(w) = 1$. The unstable vertices have the following interpretation:
  - the very unstable vertices come from unmarked smooth points of the domain mapping to H;
  - the unstable vertices with one leg come from marked points of the domain mapping to H;
  - the unstable vertices with two incoming edges come from nodes of the domain mapping to H.

The vertices w labeled H of positive degree or with $n(w) \geq 3$ are *stable*. Let $\mathfrak{s}$ be the number of stable vertices labeled H, and $\mathfrak{u}$ be the number of very unstable vertices.

The fixed locus corresponding to the decorated graph Γ will be denoted by $\mathcal{F}_\Gamma$. It can be described as the image of a finite morphism:

$$\zeta_\Gamma : \prod_{v \text{ labeled } p} \overline{\mathcal{M}}_{0,n(v)} \times \prod_{w \text{ labeled } H} \overline{\mathcal{M}}_{0,n(w)}(H, d_w) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \tag{3.3}$$

To get the fixed locus we need to factor out the action of a finite group A_Γ of automorphisms, which is determined by the exact sequence below whose last term is the automorphism group of the decorated graph Γ:

$$1 \to \prod_{e \in E(\Gamma)} \mathbb{Z}/d_e\mathbb{Z} \to A_\Gamma \to \mathrm{Aut}_\Gamma \to 1$$

The map ζ_Γ can be described as follows.

- For each vertex v labeled p pick a genus 0, $n(v)$-marked stable curve C_v .
- For each vertex w labeled H pick a genus 0 stable map f_w to H of degree d_w with $n(w)$ markings on the domain C_w.
- When necessary, we need to interpret C_v or C_w as points.

A fixed stable map f with n markings to $\mathbb{P}^r$ is obtained as follows.

- The component C_v will be mapped to p. The components C_w will be mapped to H with degree d_w under the map f_w.
- We join any two curves C_v and C_w by a rational curve C_e whenever there is an edge e of the graph Γ joining v and w. We map C_e to $\mathbb{P}^r$ with degree d_e such that the map is totally ramified over the special points.
- Finally, the marked points correspond to the legs of the graph Γ.

3.1.2 Gathmann's moduli spaces.

Gathmann's moduli spaces are an important ingredient of our localization proof. We briefly describe them below, referring the reader to [25] for the results quoted in this section.

We let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a n tuple of non-negative integers. We will usually assume that:

$$|\alpha| = \sum \alpha_i = d.$$

The substack $\overline{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ parametrizes stable maps $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^r$ such that:

- $f(x_i) \in H$ for all i such that $\alpha_i > 0$;
- $f^\star H - \sum_i \alpha_i x_i$ is effective.

Gathmann shows that this is an irreducible, reduced, proper substack of the expected codimension $|\alpha| = \sum_i \alpha_i$ of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

There are more general versions of the Gathmann stacks which were introduced by Jun Li [44]. Jun Li's construction holds in any genus and for arbitrary target. It allows, via a careful study of the virtual fundamental classes involved, for a proof a degeneration formula of the Gromov-Witten invariants. The discussion in the next few paragraphs will not be essential for our proofs. However, since Jun Li's relative stable morphism spaces have become standard in Gromov-Witten theory, we will like to briefly explain their relationship to Gathmann's stacks.

To define the relative stable morphisms, we need to consider higher degenerations $\mathbb{P}^r[k]$ of X. The k^{th} degeneration $\mathbb{P}^r[k]$ of $\mathbb{P}^r$ is obtained by gluing in k additional "levels" isomorphic to the projective bundle $P = \mathbb{P}(\mathcal{O}_H(-H) \oplus \mathcal{O}_H) \to H$ to the initial level isomorphic to $\mathbb{P}^r$. The projective bundle P has two sections H_0 and H_∞ corresponding to each of the two summands. The new levels are attached identifying the section H_∞ of the i^{th} level with H_0 in the $(i+1)^{st}$ level. The singular locus of $\mathbb{P}^r[k]$ consists in the k copies of H where the gluing occurs. The last copy H_∞ of H in the kth level of $\mathbb{P}^r[k]$ will be of special importance to us, as we will measure multiplicities with respect to this hypersurface. Also, observe the fiberwise collapsing map $\pi : \mathbb{P}^r[k] \to \mathbb{P}^r$.

A relative genus 0 stable morphism to $\mathbb{P}^r$ of degree d with multiplicities $(\alpha_1, \ldots, \alpha_n)$ along H consists in the following data:

- A usual stable map $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^r[k]$ of genus 0 to some degeneration $\mathbb{P}^r[k]$ of $\mathbb{P}^r$ satisfying the properties below;
- $\pi_\star f_\star[C] = d$;
- $f^{-1}H_\infty = \sum \alpha_i x_i$;
- no irreducible component maps to the singular locus of $\mathbb{P}^r[k]$;
- every point mapping to the singular locus is a node. Local branches around the node map to H with same contact order on both levels;
- (stability) the restriction of the morphism f to the level i copy of P is not a union of rational tails which are totally ramified over H_0 and H_∞ and which have no special points away from H_0 and H_∞.

The automorphism group of $\mathbb{P}^r[k]$ is, by definition, the torus $(\mathbb{C}^\star)^k$ acting on the fibers of each of the levels P. Note that we do not take into account the automorphisms of the initial level isomorphic to $\mathbb{P}^r$. Morphisms of relative stable maps have the usual meaning in terms of commuting diagrams, allowing the above automorphisms act on $\mathbb{P}^r[k]$.

It can be shown that the relative stable morphisms are parametrized by a separated and proper Deligne Mumford stack $\mathfrak{M}^H_{0,\alpha}(\mathbb{P}^r, d)$ of the same dimension as Gathamann's. Moreover, the projection morphisms $\pi : \mathbb{P}^r[k] \to \mathbb{P}^r$ give rise to a natural morphism:

$$\mathfrak{M}^H_{0,\alpha}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d).$$

It can be shown that the image of this morphism is the Gathmann stack $\overline{\mathcal{M}}^H_{0,n}(\mathbb{P}^r, d)$.

This particular description of the Gathmann stacks is useful is one pursues the study of their boundary, as we will below. Informally, morphisms in (the boundary of) the Gathmann spaces which have components contained entirely in H, arise as collapsing maps to some degeneration of the pair $(\mathbb{P}^r, H)$ in Jun Li's construction. For example, figure 3.1.2 shows a map in the boundary of Gathmann space with only one marking mapping to H with multiplicity α (which should equal the degree d). This map can be seen as the collapse of the relative stable morphism depicted in figure below to the level 1 degeneration of $\mathbb{P}^r$. The "internal" component C_0 mapped to H is the collapsed image of the relative morphism restricted to the first level. The "teeth" are the components of the relative stable morphism restricted on the initial level. The arrow along the level 1 part of the degeneration indicates the presence of automorphisms.

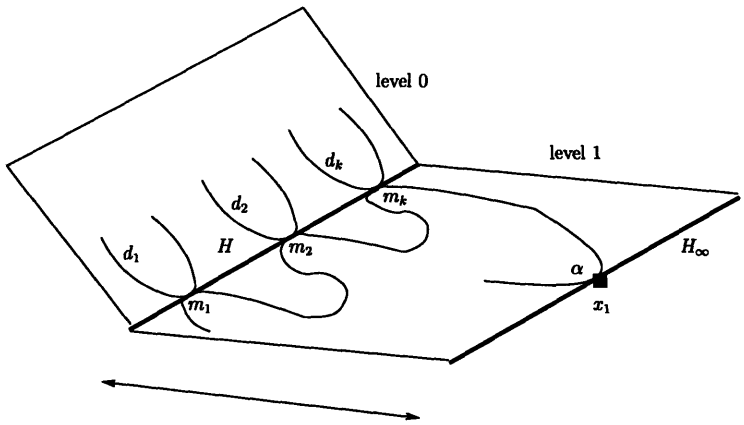


Figure 3-1: A level 1 relative stable morphism.

We will show later that the Gathmann stacks define tautological classes on the moduli spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. To this end we will make use of the recursive structure of the Gathmann stacks explained in the equations (3.4) and (3.5) below. We describe what happens if we increase the multiplicities. We let e_j be the elementary n-tuple with 1 in the j^{th} position and 0 otherwise. Then, we have the following relation in $A_*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$:

$$[\overline{\mathcal{M}}^H_{\alpha+e_j}(\mathbb{P}^r, d)] = -(\alpha_j \psi_j + ev_j^\star H) \cdot [\overline{\mathcal{M}}^H_{\alpha}(\mathbb{P}^r, d)] + [\mathcal{D}_{\alpha,j}(\mathbb{P}^r, d)] \tag{3.4}$$

The correction terms $\mathcal{D}_{\alpha,j}(\mathbb{P}^r, d)$ come from the boundary of the Gathmann stacks. These boundary terms account for the stable maps f with one "internal" component C_0 mapped to H with some degree d_0 and with some multiplicity conditions α^0 at the marked points points of f lying on C_0. Moreover, we require that the point x_j lie on C_0. There are r (union of) components attached to the internal component at r points. On each of these r components C_i the map has degree d_i and sends the intersection point with the internal component to H with multiplicity m^i. In addition, there are multiplicity conditions α^i at the marked points of f lying on C_i. We require that the d_i's sum up to d and that the α^i's form a partition of the n-tuple α.

The boundary terms we described are fibered products of lower dimensional Kontsevich-Manin and Gathmann stacks:

$$\overline{\mathcal{M}}_{0,r+|\alpha^0|}(H, d_0) \times_{H^r} \prod_{i=1}^{r} \overline{\mathcal{M}}^H_{\alpha^i \cup m^i}(\mathbb{P}^r, d_i).$$

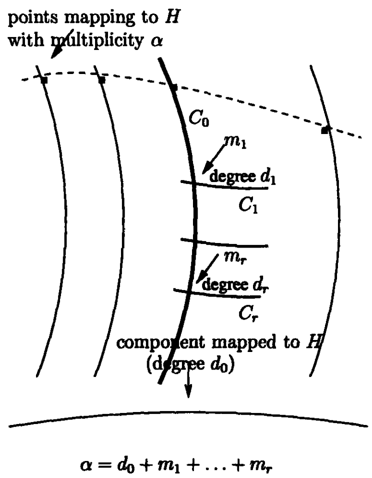


Figure 3-2: A map in the boundary of the Gathmann compactification.

Their multiplicities are found from the equation:

$$[\mathcal{D}_{\alpha,j}(\mathbb{P}^r,d)] = \sum \frac{m^1 \ldots m^r}{r!} \left[\overline{\mathcal{M}}_{0,r+|\alpha^0|}(H,d_0) \times_{H^r} \prod_{i=1}^{r} \overline{\mathcal{M}}^H_{\alpha^i \cup m^i}(\mathbb{P}^r,d_i) \right] \tag{3.5}$$

3.2 The decomposition on smooth stacks with a torus action.

In this section we will construct the Bialynicki-Birula cells of a smooth Deligne-Mumford stack with a torus actions under the additional assumption that there exists an equivariant affine etale atlas. We show that the plus decomposition on the atlas descends to the stack. The existence of such an atlas should be a general fact, which we do not attempt to prove here since in the case of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ it can be constructed explicitly by hand. Finally, in lemma 3.2.3 we prove a "homology basis theorem" for such stacks. Most (but not all) results presented in this section can in fact be proved from the corresponding statements for the coarse moduli schemes.

3.2.1 The equivariant etale affine atlas.

In this subsection will construct an equivariant affine atlas for the moduli stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$. Fix an arbitrary $\mathbb{T}$-action on $\mathbb{P}^r$ inducing an action by translation on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$.

We start with an identification $\mathbb{T} = \mathbb{C}^*$. We may need to change this identification later. For a scheme/stack X with a torus action, we denote the fixed locus by $\mathcal{X}^{\mathbf{T}}$.

Lemma 3.2.1. *Possibly after lifting the action, there exists a smooth etale affine $\mathbb{C}^*$-equivariant surjective atlas $S \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$.*

As a first step, we will find for any invariant stable map f, an equivariant etale atlas $S_f \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$. The construction in [22] shows that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is a global quotient $[J/PGL(W)]$, thus giving a smooth surjective morphism $\pi : J \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$. Here, J is a quasiprojective scheme which is smooth since π is smooth and $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is smooth. In fact, J can be explicitly constructed as a locally closed subscheme of a product of Hilbert schemes on $\mathbb{P}(W) \times \mathbb{P}^r$ for some vector space W. The starting point of the construction is an embedding of the stable map domain in $\mathbb{P}(W) \times \mathbb{P}^r$. It is clear that the $\mathbb{T}$-action on the second factor equips J with a $\mathbb{T}$-action such that the morphism $\pi : J \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is equivariant. Moreover, from the explicit construction, it follows that $\pi^{\mathbb{T}} : J^{\mathbb{T}} \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)^{\mathbb{T}}$ is surjective.

For any invariant f, there exists a $\mathbb{T}$-invariant point j_f of J whose image is f. It follows from [56] that there exists an equivariant affine neighborhood J_f of $j = j_f$ in J. The map on tangent spaces $d\pi : T_j J_f \to T_f\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is equivariantly surjective. We can pick an equivariant subspace $V_f \hookrightarrow T_j J_f$ which maps isomorphically to $T_f\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. By theorem 2.1 in [7], we can construct a smooth equivariant affine subvariety S_f of J_f containing j such that $T_j S_f = V_f$. The map $\pi_f : S_f \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is etale at j. Replacing S_f to an equivariant open subset, we may assume π_f is etale everywhere. Shrinking further, we can assume S_f is equivariant smooth affine [56].

We consider the case of non-invariant maps f. We let $\alpha : \mathbb{C}^\star \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ be the equivariant *nonconstant* translation morphism:

$$\mathbb{C}^\star \ni t \to f^t \in \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d).$$

Proposition 6 in [22] or corollary 3.3.1 below show that, after possibly a base-change $\mathbb{C}^\star \to \mathbb{C}^\star$, we can extend this morphism across 0. The image of $0 \in \mathbb{C}$ under α is a $\mathbb{T}$-invariant map F so we can utilize the atlas S_F constructed above. We claim that the image of the atlas $\pi_F : S_F \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ contains f. Indeed, we consider the equivariant fiber product $C = \mathbb{C} \times_{\overline{\mathcal{M}}} S$. Since the morphism $C \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is non-constant, the image of some closed point j in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is of the form f^t for $t \neq 0$. Then, f is the image of the closed point $t^{-1}j$.

We obtained equivariant smooth affine atlases $S_f \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ whose images cover $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Only finitely many of them are necessary to cover $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$, and their disjoint union gives an affine smooth etale surjective atlas $S \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

Corollary 3.2.1. *Let X be any convex smooth projective variety with a $\mathbb{T}$-action. There exists an equivariant smooth etale affine surjective atlas $S \to \overline{\mathcal{M}}_{0,n}(X, \beta)$ as in lemma* 3.2.1.

We embed $i : X \hookrightarrow \mathbb{P}^r$ equivariantly, and base change the atlas S constructed in the lemma under the closed immersion $i : \overline{\mathcal{M}}_{0,n}(X, \beta) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Convexity of X is used to conclude that since $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is smooth, the etale atlas S is also smooth.

3.2.2 The Bialynicki-Birula cells.

In this section we construct the Bialynicki-Birula cells for a smooth Deligne-Mumford stack $\mathcal{M}$ with a $\mathbb{T}$ action which admits an equivariant atlas as in proposition 3.2.1. This presupposes the identification $\mathbb{T} = \mathbb{C}^\star$ obtained in the lemma. We first establish:

Lemma 3.2.2. *Let $f : X \to Y$ be an equivariant etale surjective morphism of smooth schemes (stacks) with torus actions. Let Z be any component of the fixed locus of Y. Then $f^{-1}(Z)$ is union of components of $X^{\mathbb{T}}$ all mapping onto Z.*

It suffices to show that the torus action on $f^{-1}(Z)$ is trivial. The $\mathbb{T}$-orbits in $f^{-1}(Z)$ need to be contracted by f since Z has a trivial action. Since the differential $df : TX \to TY$ is an isomorphism, it follows that all orbits are 0 dimensional. They must be trivial since they are also reduced and irreducible.

Proposition 3.2.1. *Let $\mathcal{M}$ be any smooth Deligne Mumford stack with a $\mathbb{C}^\star$-action and assume a $\mathbb{C}^\star$-equivariant affine etale surjective atlas $\pi : S \to \mathcal{M}$ has been constructed. Let $\mathcal{F}$ be the fixed substack and $\mathcal{F}_i$ be its components. Then $\mathcal{M}$ can be covered by locally closed disjoint substacks $\mathcal{F}_i^+$ which are vector bundles over $\mathcal{F}_i$.*

Let $R = S \times_{\mathcal{M}} S$. The two etale surjective morphisms $s, t : R \to S$ together define a morphism $j : R \to S \times S$. It is clear that R has a torus action such that s, t are both equivariant. Moreover, since $\mathcal{M}$ is Deligne-Mumford, j is quasi-finite, hence a composition of an open immersion and an affine morphism. Since S is affine, it follows that R is quasi-affine. As s is etale, we obtain that R is also smooth.

If $F = S \times_{\mathcal{M}} \mathcal{F}$ then $F \hookrightarrow S$ is a closed immersion. Since $S \to \mathcal{M}$ is etale and equivariant, by the above lemma, F coincides with $S^{\mathbb{T}}$. Similarly $s^{-1}(F)$ and $t^{-1}(F)$ coincide with $R^{\mathbb{T}}$. Fixing i, we let $F_i = S \times_{\mathcal{M}} \mathcal{F}_i$. Then F_i is union of components F_{ij} of $S^{\mathbb{T}}$. Similarly, $s^{-1}(F_i) = t^{-1}(F_i)$ is a union of components R_{ik} of $R^{\mathbb{T}}$. We will construct the substack $\mathcal{F}_i^+$ of $\mathcal{M}$ and the vector bundle projection $\alpha_i : \mathcal{F}_i^+ \to \mathcal{F}_i$ on the atlas S. We will make use of the results of [7], where a *plus* decomposition for quasi-affine schemes with a $\mathbb{C}^\star$-action was constructed. For each component F_{ij}, we consider its *plus* scheme F_{ij}^+; similarly for the R_{ik}'s we look at the cells R_{ik}^+. We claim that:

$$s^{-1}(\cup_j F_{ij}^+) = t^{-1}(\cup_j F_{ij}^+) = \cup_k R_{ik}^+$$

and we let $\mathcal{F}_i^+$ be the stack which $F_i^+ = \cup_j F_{ij}^+ \hookrightarrow S$ defines in $\mathcal{M}$. It suffices to show that if R_{ik} is mapped to F_{ij} under s, then R_{ik}^+ is a component of $s^{-1}(F_{ij}^+)$. Let $r \in R_{ik}$. Then, using that s is etale and the construction in [7], we have the following equality of tangent spaces:

$$T_r s^{-1}(F_{ij}^+) = ds^{-1}\left(T_{s(r)} F_{ij}^+\right) = ds^{-1}\left((T_{s(r)} S)^{\geq 0}\right) = (T_r R)^{\geq 0} = T_r R_{ik}^+.$$

Here $V^{\geq 0}$ denotes the subspace of the equivariant vector space V where the $\mathbb{C}^\star$-action has non-negative weights. The uniqueness result in corollary to theorem 2.2 in [7] finishes the proof. Note that the argument here shows that the codimension of $\mathcal{F}_i^+$ in $\mathcal{M}$ is given by the number of negative weights on the tangent bundle $T_r\mathcal{M}$ at a fixed point r.

To check that $\mathcal{F}_i^+ \to \mathcal{F}_i$ is a vector bundle, we start with the observation that $F_{ij}^+ \to F_{ij}$ are vector bundles. We also need to check that the pullback *bundles* under s and t are isomorphic:

$$s^\star\left(\oplus_j F_{ij}^+\right) \simeq t^\star\left(\oplus_j F_{ij}^+\right) \simeq \oplus_k R_{ik}^+$$

The argument is identical to the one above, except that one needs to invoke corollary of proposition 3.1 in [7] to identify the bundle structure.

3.2.3 The homology basis theorem.

In this subsection we will establish the "homology basis theorem" (lemma 3.2.3) extending a result which is well known for smooth projective schemes [11], at least in the case of isolated fixed points. The proof does not contain any new ingredients, but we include it below, for completeness. We agree on the following conventions.

Let us consider a smooth Deligne Mumford stack $\mathcal{M}$ with a torus action whose fixed loci $\mathcal{F}_i$ are indexed by a finite set I, and whose Bialynicki-Birula cells $\mathcal{F}_i^+$ were defined above. We furthermore assume that the decomposition is filterable. That is, there is a partial (reflexive, transitive and anti-symmetric) ordering of the indices such that:

(a) We have $\overline{\mathcal{F}_i^+} \subset \bigcup_{j \le i} \mathcal{F}_j$;

(b) There is a unique maximal index $\mathfrak{m} \in I$.

Filterability of the Bialynicki-Birula decomposition was shown in [8] for projective schemes. For the stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$, filterability follows from the similar statement on the coarse moduli scheme. However, to prove the tautology of the Chow classes, we need the *stronger filterability condition* (*c*), which we will demonstrate in the next section, and which does not follow from the known arguments:

(c) There is a family Ξ of cycles supported on the fixed loci such that:

- The cycles in Ξ span the rational Chow groups of the fixed loci.
- For all $\xi \in \Xi$ supported on $\mathcal{F}_i$, there is a *plus* substack ξ^+ (flowing into ξ) supported on $\mathcal{F}_i^+$. We assume that ξ^+ is contained in a closed substack $\widehat{\xi^+}$ supported on $\overline{\mathcal{F}_i^+}$ (usually, but not necessarily, its closure) with the property:

$$\widehat{\xi^+} \setminus \xi^+ \subset \bigcup_{j<i} \mathcal{F}_j^+.$$

Lemma 3.2.3. *Assume that $\mathcal{M}$ is a smooth Deligne Mumford stack which satisfies the assumptions (a) and (b) above.*

(i) The Betti numbers $h^m(\mathcal{M})$ of $\mathcal{M}$ can be computed as:

$$h^m(\mathcal{M}) = \sum_i h^{m-2n_i^-}(\mathcal{F}_i).$$

Here n_i^- is the codimension of $\mathcal{F}_i^+$ which equals the number of negative weights on the tangent bundle of $\mathcal{M}$ at a fixed point in $\mathcal{F}_i$.

(ii) If the rational Chow rings and the rational cohomology of each fixed stack $\mathcal{F}_i$ are isomorphic, then the same is true about $\mathcal{M}$.

(iii) Additionally, if assumption (c) is satisfied, the cycles $\widehat{\xi^+}$ for $\xi \in \Xi$ span the rational Chow groups of $\mathcal{M}$.

Thanks to item (*b*), we can define an integer valued function $L(i)$ as the length of the shortest descending path from $\mathfrak{m}$ to i. Observe that $i < j$ implies $L(i) > L(j)$. Because of (*a*), we observe that

$$\mathcal{Z}_k = \bigcup_{L(i)>k} \mathcal{F}_i^+ = \bigcup_{L(i)>k} \overline{\mathcal{F}_i^+}$$

is a closed substack of $\mathcal{M}$. Letting $\mathcal{U}_k$ denote its complement, we conclude that

$$\mathcal{U}_{k-1} \hookrightarrow \mathcal{U}_k \text{ and } \mathcal{U}_k \setminus \mathcal{U}_{k-1} \text{ is union of cells } \bigcup_{L(i)=k} \mathcal{F}_i^+.$$

The Gysin sequence associated to the pair $(\mathcal{U}_k, \mathcal{U}_{k-1})$ is:

$$\ldots \to \bigoplus_{L(i)=k} H^{m-2n_i^-}(\mathcal{F}_i^+) = \bigoplus_{L(i)=k} H^{m-2n_i^-}(\mathcal{F}_i) \to H^m(\mathcal{U}_k) \to H^m(\mathcal{U}_{k-1}) \to \ldots$$

One imitates the usual argument for smooth schemes in [1] to prove that the long exact sequence splits. Item (i) follows by estimating the dimensions.

To prove (ii), we compare all short exact Gysin sequences to the Chow exact sequences (for m even) and use the five lemma:

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{L(i)=k} H^{m-2n_i^-}(\mathcal{F}_i) \arrow[r] \arrow[d] & H^m(\mathcal{U}_k) \arrow[r] \arrow[d] & H^m(\mathcal{U}_{k-1}) \arrow[r] \arrow[d] & 0 \\ & \bigoplus_{L(i)=k} A^{m/2-n_i^-}(\mathcal{F}_i) \arrow[r] & A^{m/2}(\mathcal{U}_k) \arrow[r] & A^{m/2}(\mathcal{U}_{k-1}) \arrow[r] & 0 \end{tikzcd} \tag{3.6}$$

Finally, for (iii), we use (3.6) to prove inductively that

the cycles $\widehat{\xi^+} \cap \mathcal{U}_k$ for $\xi \in \Xi$ supported on $\mathcal{F}_i$ with $L(i) \leq k$ span $A^\star(\mathcal{U}_k)$.

Condition (c) is used to prove that the image of ξ in $\mathcal{U}_k$ is among the claimed generators:

$$\xi^+ = \widehat{\xi^+} \cap \mathcal{U}_k.$$

3.3 The Bialynicki-Birula decomposition on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

In the previous section we constructed the Bialynicki-Birula plus cells on the stack of stable maps $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. In this section, we identify the decomposition explicitly. We start by analyzing the $\mathbb{C}^\star$-flow of individual stable maps. We will relate the decomposition to Gathmann's stacks in the next subsection. Finally, we will prove the filterability condition (c) needed to apply lemma 3.2.3.

3.3.1 The flow of individual maps.

To fix the notation, we let $f : (C, x_1, \ldots, x_n) \to \mathbb{P}^r$ be a degree d stable map to $\mathbb{P}^r$. We look at the sequence of translated maps:

$$f^t : (C, x_1, \ldots, x_n) \to \mathbb{P}^r, \;\; f^t(z) = tf(z).$$

By the "compactness theorem," this sequence will have a stable limit. We want to understand this limit $F = \lim_{t \to 0} f^t$.

To construct F explicitly we need to lift the torus action $t \to t^D$ where $D = d!$. Henceforth, we will work with the lifted action:

$$t \cdot [z_0 : z_1 : \ldots : z_r] = [z_0 : t^D z_1 : \ldots : t^D z_r].$$

We seek to construct a family of stable maps $G : \mathcal{X} \to \mathbb{P}^r$ over $\mathbb{C}$, whose fiber over $t \neq 0$ is f^t and whose central fiber $F : \mathcal{C} \to \mathbb{P}^r$ will be explicitly described below.

$$\begin{tikzcd} C \arrow[r] \arrow[rr, bend left, "F"] \arrow[d] & \mathcal{X} \arrow[r, "G"] \arrow[d, "\pi"] & \mathbb{P}^r \\ 0 \arrow[r] \arrow[u, bend left, "x_i"] & \mathbb{C} \arrow[u, bend left, "x_i"] & \end{tikzcd} \tag{3.7}$$

First we assume that the domain C is an irreducible curve. In case f is mapped entirely to H, the family (3.7) is trivial and $F = f$.

Otherwise, f intersects the hyperplane H at isolated points, some of them possibly being among the marked points. We make a further simplifying assumption: we may assume that all points in $f^{-1}(H)$ are marked points of the domain. If this is not the case, we mark the remaining points in $f^{-1}(H)$ thus getting a new stable map $\overline{f}$ living in a moduli space with more markings $\overline{\mathcal{M}}_{0,n+k}(\mathbb{P}^r, d)$. We will have constructed a family (3.7) whose central fiber is $\overline{F} = \lim_{t\to 0} \overline{f}^t$. A new family having f^t as the t-fiber is obtained by forgetting the markings. We use a multiple of the line bundle

$$\omega_\pi(\sum_i x_i) \otimes G^\star \mathcal{O}_{\mathbb{P}^r}(3)$$

to contract the unstable components of the central fiber. Thus, we obtain the limit F from $\overline{F}$ by forgetting the markings we added and stabilizing.

Henceforth we assume that all points in $f^{-1}(H)$ are among the markings of f, and f is not a map to H. Let $s_1, \ldots, s_k$ be the markings which map to H, say with multiplicities $n_1, \ldots, n_k$ such that $\sum n_i = d$. We let $t_1, \ldots, t_l$ be the rest of the markings. We let $q_i = f(s_i)$. The following lemma will be of crucial importance to us. The method of proof is an explicit stable reduction, and it is similar to that of proposition 2 in [40].

Lemma 3.3.1. *Let F be the following stable map with reducible domain:*

- *The domain has one component of degree 0 mapped to p. This component contains markings $T_1, \ldots, T_l$.*
- *Additionally, there are k components $C_1, \ldots, C_k$ attached to the degree 0 component. The restriction of F to C_i has degree n_i, its image is the line joining p to $q_i = f(s_i)$ and the map is totally ramified over p and q_i.*
- *Moreover, if we let $S_i = F^{-1}(q_i)$, then $S_1, \ldots, S_k, T_1, \ldots, T_l$ are the marked points of the domain of F.*

Then, the stabilization of F is the limit $\lim_{t\to 0} f_t$.

It suffices to exhibit a family as in (3.7). We let $f^0, \ldots, f^r$ be the homogeneous components of the map f. We let C be the domain curve with coordinates $[z : w]$. The assumption about the contact orders of f with H shows that f^0 vanishes at $s_1, \ldots, s_k$ of orders $n_1, \ldots, n_k$ with $\sum_i n_i = d$.

There is a well defined map $G_0 : \mathbb{C}^\star \times C \to \mathbb{P}^r$ given by:

$$(t, [z : w]) \mapsto \left[f^0(z : w) : t^D f^1(z : w) : \ldots : t^D f^r(z : w)\right].$$

The projection map $\pi : \mathbb{C}^\star \times C \to \mathbb{C}^\star$ has constant sections $s_1, \ldots, s_k, t_1, \ldots, t_l$. It is clear

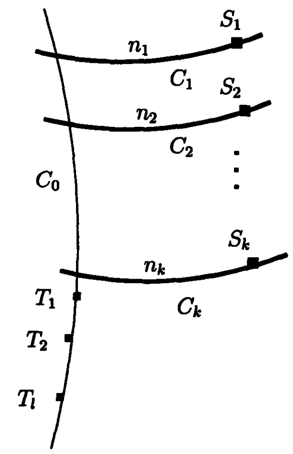


Figure 3-3: The limit in lemma 3.3.1.

that G_0 can be extended to a map

$$G_0 : \mathbb{C} \times C \setminus \bigcup_i (\{0\} \times \{s_i\}) \to \mathbb{P}^r.$$

A suitable sequence of blowups of $\mathbb{C} \times C$ at the points $\{0\} \times \{s_i\}$ will give a family of stable maps $G : \mathcal{X} \to \mathbb{P}^r$ as in (3.7).

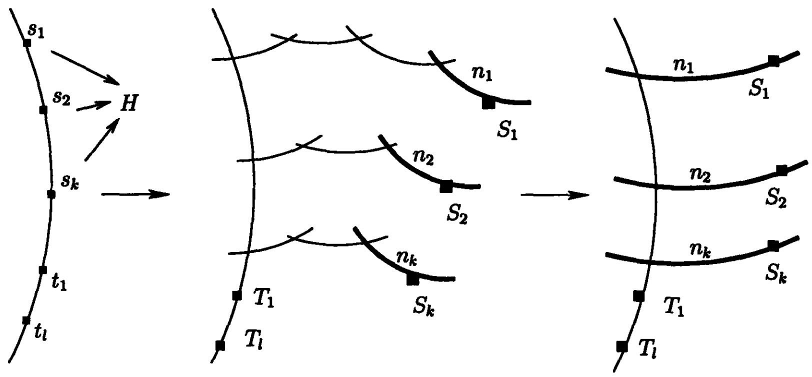


Figure 3-4: Obtaining the stable limit.

It is useful to understand these blowups individually. It suffices to work locally in quasi-affine patches U_i near s_i and then glue. An affine change of coordinates will ensure $s_i = 0$. For $n = n_i$, we write $f^0 = z^n h$. We may assume that on U_i, h does not vanish and that $f_1, \dots, f_r$ do not all vanish. Let $D = n \cdot e$. We will perform e blowups to resolve the map $G_0 : U_i \times \mathbb{P}^1 \setminus \{(0, [0:1])\} \to \mathbb{P}^r$:

$$(t, [z:w]) \to [z^n h(z:w) : t^{ne} f_1(z:w) : \dots : t^{ne} f_r(z:w)] .$$

The blowup at $(0, [0:1])$ gives a map:

$$G_1 : \mathcal{X}_1 \dashrightarrow \mathbb{P}^r .$$

In coordinates,

$$\mathcal{X}_1 = \{(t, [z:w], [A_1, B_1]) \text{ such that } A_1 z = B_1 t w\}$$

and

$$G_1 = \left[B_1^n h(tB_1 : A_1) : t^{ne-n} f_1(B_1 t : A_1) : \ldots : t^{ne-n} f_r(B_1 t : A_1)\right].$$

The map is still undefined at $t = 0$ and $B_1 = 0$ so we will need to blow up again. After the k^{th} blowup, we will have obtained a map:

$$G_k : \mathcal{X}_k \dashrightarrow \mathbb{P}^r,$$

which in coordinates becomes:

$$\mathcal{X}_k = \{(t, [z:w], [A_1, B_1], \ldots, [A_k : B_k]) \mid A_1 z = B_1 t w,\ A_{i+1} B_i = t A_i B_{i+1},\ 1 \le i \le k-1\}$$

$$G_k = \left[B_k^n h(t^k B_k : A_k) : t^{ne-nk} f_1(t^k B_k : A_k) : \ldots : t^{ne-nk} f_r(t^k B_k : A_k)\right].$$

After the e^{th} blow up we obtain a well defined map. This map is constant on the first $e-1$ exceptional divisors (hence they are unstable). On the e^{th} exceptional divisor the map is given by:

$$G_e = [B_e^n : A_e^n f_1(0:1) : \ldots : A_e^n f_r(0:1)].$$

There, the map is totally ramified over two points in its image. It is easy to check that the sections $s_1, \ldots, s_k, t_1, \ldots, t_l$ extend over $t = 0$ as claimed in the lemma.

We obtain a family $G : \mathcal{X} \to \mathbb{P}^r$ of maps parametrized by $\mathbb{C}$ as in (3.7). The profile of the central fiber is the middle shape in figure $3-4$. There are unstable components coming from the exceptional divisors which need to be contracted successively to obtain the final limit we announced. This completes the proof.

We consider the case when the domain curve is not irreducible. Assume that the stable map f is obtained by gluing maps f_1 and f_2 with fewer irreducible components at markings $\star$ and $\bullet$ on their domains with $f_1(\star) = f_2(\bullet)$. Inductively, we will have constructed families (3.7) of stable maps over $\mathbb{C}$ whose fibers over $t \neq 0$ are f_1^t and f_2^t. We glue the two families together at the sections $\star$ and $\bullet$ thus obtaining a family whose fiber over t is f^t. The argument above proves that the limit for reducible maps can be obtained by taking the limits of each irreducible component and gluing the limits together along the corresponding sections.

Example 3.3.1. Figure $3-5$ shows the limit in the case of a node x mapping to H with contact orders a_1 and a_2 on the two components C_1 and C_2 transversal to H. The node is replaced by two rational components of degrees a_1 and a_2 joined at node. These components are joined to the rest of the domain $C_1 \cup C_2$ at nodes mapping to p.

We obtain the following algorithm for computing the limit F:

(i) We consider each irreducible component of the domain individually. We mark the nodes on each such component.

(ii) The map F leaves unaltered the irreducible components mapping to H.

(iii) The components which are transversal to H are replaced in the limit by reducible maps. The reducible map has one back-bone component mapped to p. This component contains all markings which are not mapped to H. Moreover, rational tails are

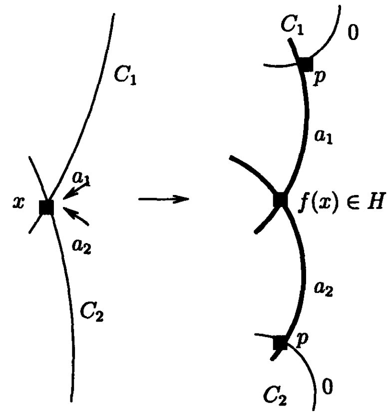


Figure 3-5: The limit of the $\mathbb{C}^*$ flow when a node maps to H.

glued to the back-bone component at the points which map to H according to the item below. The markings which map to H are replaced by markings on the rational tails.

(iv) In the limit, each isolated point x of the domain curve which maps to $f(x) \in H$ with multiplicity n is replaced by a rational tail glued at a node to the rest of the domain. The node is mapped to p. The image under F of the rational tail is a curve in $\mathbb{P}^r$ joining p to $f(x) \in H$. The map F is totally ramified over these two points with order n. If the point x happens to be a section, we mark the point $F^{-1}(f(x))$ on the rational tail.

(v) The map F is obtained by gluing all maps in (*ii*) and (*iii*) along the markings we added in (*i*) and then stabilizing.

Corollary 3.3.1. *For each stable map f, there is a family of stable maps* (3.7) *over $\mathbb{C}$, whose fiber over $t \neq 0$ is the translated map f^t and whose central fiber F is obtained by the algorithm above.*

3.3.2 Relation to the Gathmann stacks

We will proceed to identify the Bialynicki Birula cells of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Recall that the fixed loci for the torus action on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are indexed by decorated graphs Γ. We will identify the closed stacks $\overline{\mathcal{F}_\Gamma^+}$ in terms of images of fibered products of Kontsevich-Manin and Gathmann stacks under the tautological morphisms.

In this chapter, we will need the following versions of Gathmann's construction.

(i) The substacks $\widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$ parametrize maps with the additional condition that the components of f are transversal to H. The maps in the open Gathmann stack $\mathcal{M}_\alpha^H(\mathbb{P}^r, d)$ satisfy this condition by definition [25], hence:

$$\mathcal{M}_\alpha^H(\mathbb{P}^r, d) \hookrightarrow \widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d).$$

(ii-1) For each map f in $\widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$, the dual graph Δ is obtained as follows:

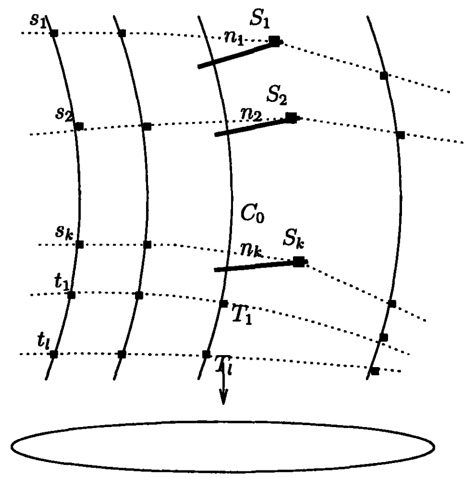


Figure 3-6: A family as in corollary 3.3.1.

- Vertices labeled by degrees correspond to the irreducible components of f. Vertices of degree 0 satisfy the usual stability condition.
- The edges correspond to the nodes of f.
- Numbered legs correspond to the markings. The multiplicities α are distributed among the legs of Δ.
- We write α_v for the *ordered* collection of multiplicities of the legs incoming to the vertex v to which we adjoin 0's for all incoming edges (corresponding to the fact that the nodes of a map in $\widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)$ cannot be sent to H). The assignment of the multiplicities to the incoming flags is part of the datum of α_v.
- The degree d_v of the vertex v is computed from the multiplicities: $|\alpha_v| = d_v$.

For each graph Δ as above, we consider the stratum in $\widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)$ of maps whose dual graph is precisely Δ. This is the image of the fibered product $\mathcal{M}^{\Delta,H}_\alpha(\mathbb{P}^r, d)$ of open Gathmann spaces under the gluing maps:

$$\mathcal{M}^{\Delta,H}_\alpha(\mathbb{P}^r, d) = \left(\prod_{v\in V(\Delta)} \mathcal{M}^H_{\alpha_v}(\mathbb{P}^r, d_v)\right)^{E(\Gamma)} \to \widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)$$

The fibered product is computed along the evaluation maps on the corresponding moduli spaces at the markings determined by the edges of Δ.

(ii-2) One also defines the stack $\widetilde{\mathcal{M}}^{\Delta,H}_\alpha(\mathbb{P}^r, d)$ by taking the analogous fiber product of spaces $\widetilde{\mathcal{M}}^H_{\alpha_v}(\mathbb{P}^r, d_v)$. Its image in $\widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d)$ are the maps transversal to H with domain type at least Δ. We write $\overline{\mathcal{M}}^\Delta_{0,n}$ for the closure of the stratum of marked stable curves whose dual graph is the unlabeled graph underlying Δ. It follows that:

$$\widetilde{\mathcal{M}}^{\Delta,H}_\alpha(\mathbb{P}^r, d) = \overline{\mathcal{M}}^\Delta_{0,n} \times_{\overline{\mathcal{M}}_{0,n}} \widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d). \tag{3.8}$$

(ii-3) The similar fibered product of closed Gathmann spaces is defined as:

$$\overline{\mathcal{M}}_{\alpha}^{\Delta,H}(\mathbb{P}^r,d) = \left(\prod_{v\in V(\Delta)} \overline{\mathcal{M}}_{\alpha_v}^{H}(\mathbb{P}^r,d_v)\right)^{E(\Gamma)}. \tag{3.9}$$

We will see later that $\overline{\mathcal{M}}_{\alpha}^{\Delta,H}(\mathbb{P}^r,d)$ is truly a compactification of $\mathcal{M}_{\alpha}^{\Delta,H}(\mathbb{P}^r,d)$.

(iii) We will need to deal with unmarked smooth points of the domain mapping to H. This requires manipulations of a stack obtained from Gathmann's via the forgetful morphisms. We fix a collection of non-negative integers $\beta = (\beta_1,\ldots,\beta_n)$ and a collection of positive integers $\delta = (\delta_1,\ldots,\delta_m)$, satisfying the requirement $d = |\beta| + |\delta|$. The write $\mathcal{M}_{\beta,\delta}^{H}(\mathbb{P}^r,d)$ for the image of the open Gathmann stack via the forgetful morphism:

$$\mathcal{M}_{\beta\cup\delta}^{H}(\mathbb{P}^r,d) \hookrightarrow \overline{\mathcal{M}}_{n+m}(\mathbb{P}^r,d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d).$$

The open stack $\mathcal{M}_{\beta,\delta}^{H}(\mathbb{P}^r,d)$ parametrizes irreducible stable maps $f : (C,x_1\ldots,x_n) \to \mathbb{P}^r$ such that:

$$f^\star H = \sum \beta_i x_i + \sum \delta_j y_j,$$

for some distinct unmarked points of the domain y_j.

With these preliminaries under our belt, we repackage the datum carried by each of the graphs Γ indexing a fixed locus, into a fibered product $\mathcal{X}_\Gamma$ of Kontsevich-Manin and Gathmann spaces. Precisely, we define:

$$\mathcal{X}_\Gamma = \left(\prod_v \mathcal{M}_{\beta_v,\delta_v}^{H}(\mathbb{P}^r,d_v) \times_H \prod_w \overline{\mathcal{M}}_{0,n(w)}(H,d_w)\right)^{E(\Gamma)}$$

The set of integers α_v defined in section 1.1 is partitioned into two subsets $\beta_v \cup \delta_v$. δ_v collects the degrees of the incoming edges whose endpoint labeled w is very unstable. The fibered product above is obtained as usual along the evaluation maps on the moduli spaces determined by the edges of Γ.

A general point of $\mathcal{X}_\Gamma$ is obtained as follows.

- For each vertex w labeled (H,d_w) we construct a stable map f_w of degree d_w, with $n(w)$ markings and rational domain curve C_w. For the *unstable* vertices w this construction should be interpreted as points mapping to H.
- For each vertex v labeled p, we construct a stable map f_v with smooth domain C_v and $n(v)$ marked points.
- We join the domain curves C_v and C_w at a node each time there is an edge incident to both v and w. Each edge e which contains unstable w's gives a special point of the domain. The special point should be a node mapping to H if w has two incoming edges, or a marking if w has one incoming edge and an attached leg, or an unmarked point mapping to H when w is very unstable.
- For each v labeled p, the map f_v has degree $d_v = |\beta_v| + |\delta_v|$ on the component C_v. Moreover, each incident edge e corresponds to a point on C_v which maps to H and we require that the contact order of the map with H at that point be d_e.

It is clear that by corollary 3.3.1, the limit of the flow of the above map has dual graph Γ.

Even though the fibered product above is, modulo automorphisms, the Bialynicki-Birula cell, we will carry out our discussion so that it only involves the stacks in (i) and (ii). To this end, we mark all the smooth points of the domain mapping to H. Combinatorially, this corresponds to eliminating the very unstable vertices in Γ. We let γ be the graph obtained from Γ by attaching legs to each terminal very unstable vertex w. Then $\mathcal{X}_\Gamma$ is the image of the fibered product:

$$(\mathcal{X}_\gamma =)\, \mathcal{Y}_\Gamma = \left(\prod_v \mathcal{M}^H_{\alpha_v}(\mathbb{P}^r, d_v) \times_H \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w)\right)^{E(\Gamma)}$$

under the morphism:

$$\overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$$

which forgets the markings corresponding to the $\mathfrak{u}$ newly added legs of $\gamma \to \Gamma$. We analogously define the companion stacks $\widetilde{\mathcal{Y}}_\Gamma$ and $\overline{\mathcal{Y}}_\Gamma$ (and their images $\widetilde{\mathcal{X}}_\Gamma$ and $\overline{\mathcal{X}}_\Gamma$):

$$\widetilde{\mathcal{Y}}_\Gamma = \left(\prod_v \widetilde{\mathcal{M}}^H_{\alpha_v}(\mathbb{P}^r, d_v) \times_H \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w)\right)^{E(\Gamma)} \tag{3.10}$$

$$\overline{\mathcal{Y}}_\Gamma = \left(\prod_v \overline{\mathcal{M}}^H_{\alpha_v}(\mathbb{P}^r, d_v) \times_H \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w)\right)^{E(\Gamma)}. \tag{3.11}$$

There is a morphism $\overline{\mathcal{Y}}_\Gamma \to \overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ obtained as compositions of:

- gluing morphisms;
- forgetful morphisms;
- inclusions of Gathmann stacks $\overline{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_{0,m}(\mathbb{P}^r, d)$;
- inclusions of Kontsevich-Manin stacks $\overline{\mathcal{M}}_{0,m}(H, d) \hookrightarrow \overline{\mathcal{M}}_{0,m}(\mathbb{P}^r, d)$.

Lemma 3.3.2. *The stack $\widetilde{\mathcal{Y}}_\Gamma$ is smooth. Its image in $\overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d)$ has codimension $d + \mathfrak{s} - \mathfrak{u}$.*

We observe that for any collection of weights α, the evaluation morphism:

$$ev_1 : \widetilde{\mathcal{M}}^H_\alpha(\mathbb{P}^r, d) \to H \tag{3.12}$$

is smooth. First, the source is smooth. This is proved in [25] for $\mathcal{M}^H_\alpha(\mathbb{P}^r, d)$. To pass to the nodal locus, an argument identical to that of lemma 10 in [22] is required. As a consequence, there is a non-empty open set of the base over which the morphism is smooth. As $PGL(H)$ acts transitively on H, the claim follows.

To prove the lemma, we follow an idea of [40]. We will induct on the number of vertices of the tree Γ, the case of one vertex being clear. We will look at the terminal vertices of Γ with only one incident edge.

Pick a terminal stable vertex $\mathfrak{w}$ labeled $(H, d_{\mathfrak{w}})$, if it exists. A new graph Γ' is obtained by relabeling $\mathfrak{w}$ by $(H, 0)$ and removing all its legs. Inductively, $\widetilde{\mathcal{Y}}_{\Gamma'}$ is smooth. It remains to observe that the morphism:

$$\widetilde{\mathcal{Y}}_\Gamma \to \widetilde{\mathcal{Y}}_{\Gamma'}$$

is smooth, as it is obtained by base change from the smooth morphism:

$$ev : \overline{\mathcal{M}}_{0,n(\mathfrak{w})}(H, d_{\mathfrak{w}}) \to H.$$

We can now assume all terminal vertices are either labeled p or labeled H but unstable. Removing all terminal H labeled vertices from Γ, we obtain a new tree whose terminal vertices are all labeled p. Pick a terminal vertex $\mathfrak{v}$ in the new tree. It is connected to (at most) one vertex $\mathfrak{w}$. Assume $\mathfrak{v}$ was connected to the terminal vertices $\mathfrak{w}_1, \ldots, \mathfrak{w}_k$ in the old tree Γ. A new graph Γ' is obtained from Γ by removing all flags incident to $\mathfrak{v}, \mathfrak{w}_1, \ldots, \mathfrak{w}_k$ and replacing them by a leg attached at $\mathfrak{w}$. The same argument as before applies. We base change $\widetilde{\mathcal{Y}}_{\Gamma'}$ by the smooth morphism (3.12). In our case, $\alpha = \alpha_{\mathfrak{v}}$ is the collection of degrees of the flags incoming to $\mathfrak{v}$. The evaluation is taken along the marking corresponding to the edge joining $\mathfrak{v}$ and $\mathfrak{w}$.

To compute the dimension of $\widetilde{\mathcal{Y}}_\Gamma$, we look at the contribution of each vertex w labeled (H, d_w), of each vertex v labeled p, and we subtract the contribution of each edge e. Assuming all H labeled vertices are stable, we obtain the following formula for the dimension of $\widetilde{\mathcal{Y}}_\Gamma$:

$$\begin{aligned}
&\sum_w (rd_w + (r-1) + n(w) - 3) + \sum_v ((r+1)d_v + r + n(v) - 3 - |\alpha_v|) - \sum_e (r-1) = \\
= \;& r\left(\sum_w d_w + \sum_v d_v\right) + \left(\sum_w n(w) + \sum_v n(v)\right) + ((r-4)W + (r-3)V - (r-1)E) = \\
= \;& rd + (n + 2E) + (-2E - W + r - 3) = ((r+1)d + r + n - 3) - d - W.
\end{aligned}$$

Thus the codimension of $\widetilde{\mathcal{Y}}_\Gamma$ in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ equals $d + W$. The formula needs to be appended accordingly for the unstable vertices. The final answer for the codimension of $\widetilde{\mathcal{Y}}_\Gamma$ in $\overline{\mathcal{M}}_{0,n+\mathfrak{u}}(\mathbb{P}^r, d)$ becomes $d + \mathfrak{s} - \mathfrak{u}$.

We constructed open immersions $\mathcal{Y}_\Gamma \hookrightarrow \widetilde{\mathcal{Y}}_\Gamma \hookrightarrow \overline{\mathcal{Y}}_\Gamma$. The following lemma clarifies the relationship between these spaces.

Lemma 3.3.3. *The image of $\mathcal{Y}_\Gamma$ is dense in $\overline{\mathcal{Y}}_\Gamma$. The stack $\overline{\mathcal{Y}}_\Gamma$ is reduced and irreducible.*

Using the above discussion, the only thing we need to show is that $\overline{\mathcal{Y}}_\Gamma$ is irreducible. We observe that the smooth stack $\mathcal{Y}_\Gamma$ is irreducible. Indeed, we can prove $\mathcal{Y}_\Gamma$ is connected by analyzing the $\mathbb{C}^\star$ action. Using corollary 3.3.1 all maps in $\mathcal{Y}_\Gamma$ flow to one connected fixed locus which is the image of the connected stack:

$$\prod_v \mathcal{M}_{0,n(v)} \times \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w).$$

To prove the irreducibility of $\overline{\mathcal{Y}}_\Gamma$ one uses the same arguments as in lemma 1.13 in [25]. By the previous paragraph, it is enough to show that any map f in $\overline{\mathcal{Y}}_\Gamma$ can be deformed to a map with fewer nodes. Picking a map in $\overline{\mathcal{Y}}_\Gamma$ is tantamount to picking maps f_v and f_w in $\overline{\mathcal{M}}^H_{\alpha_v}(\mathbb{P}^r, d_v)$ and in $\overline{\mathcal{M}}_{0,n(w)}(\mathbb{P}^r, d_w)$ with compatible gluing data. For each vertex v, Gathmann constructed a deformation of f_v over a smooth base curve such that the generic fiber has fewer nodes. We attach the rest of f to the aforementioned deformation. To glue in the remaining components, we match the images of the markings by acting with automorphisms of $\mathbb{P}^r$ which preserve H. The details are identical to those in [25].

Lemma 3.3.4. *There are $d + \mathfrak{s} - \mathfrak{u}$ negative weights on the normal bundle of $\mathcal{F}_\Gamma$.*

The arguments used to prove this lemma are well known (see for example [32] for a similar computation). In the computation below, we will repeatedly use the fact that the tangent space $T_x\mathbb{P}^r$ has $\mathbb{C}^\star$ weights $D, \ldots, D$ for $x = p$ and weights $0, 0, \ldots, -D$ if $x \in H$.

Recall the description of the stable maps in $\mathcal{F}_\Gamma$ which was given in the discussion following equation (3.3). We let $(f, C, x_1, \ldots, x_n)$ be a generic stable map in $\mathcal{F}_\Gamma$ such that C_v and C_w are irreducible. We will compute the weights on the normal bundle at this generic point. These are the non-zero weights of the term $\mathcal{T}_f$ of the following exact sequence:

$$0 \to Ext^0(\Omega_C(\sum_i x_i), \mathcal{O}_C) \to H^0(C, f^\star T\mathbb{P}^r) \to \mathcal{T}_f \to Ext^1(\Omega_C(\sum_i x_i), \mathcal{O}_C) \to 0 \quad (3.13)$$

We will count the negative weights on the first, second and fourth term above.

The first term gives the infinitesimal deformations of the marked domain. All contributions come from deformation of the components of type C_e. An explicit computation shows that the deformation space of such rational components with two special points, which need to be fixed by the deformation, is one dimensional with trivial weight. There is one exception in case the special points are not marked or nodes. This exceptional case corresponds to very unstable vertices. We obtain one negative weight for each such vertex, a total number of $\mathfrak{u}$.

Similarly, the fourth term corresponds to deformations of the marked domains. We are interested in the smoothings of nodes x lying on two components D_1 and D_2. The deformation space is $T_xD_1 \otimes T_xD_2$. The nodes lying on C_e and C_v give positive contributions. We obtain negative weights for nodes joining components C_e and C_w for stable w, and also for nodes lying on two components C_{e_1} and C_{e_2}, which correspond to unstable w's with two incoming edges. The number of such weights equals the number F of edges whose vertex labeled w is stable plus the number of unstable w's with two incoming edges.

The weights on the second term will be computed from the exact sequence

$$0 \to H^0(C, f^\star T\mathbb{P}^r) \to \bigoplus_v H^0 \quad (C_v, f_v^\star T\mathbb{P}^r) \bigoplus_w H^0(C_w, f_w^\star T\mathbb{P}^r) \bigoplus_e H^0(C_e, f_e^\star T\mathbb{P}^r) \to$$
$$\to \bigoplus_{\mathfrak{f}_v} T_{\mathfrak{f}_v}\mathbb{P}^r \bigoplus_{\mathfrak{f}_w} T_{\mathfrak{f}_w}\mathbb{P}^r \to 0$$

Here $\mathfrak{f}_v$, $\mathfrak{f}_w$ are flags of Γ labeled by their initial vertices v and w. They correspond to nodes of the domain mapping to p and H, hence the terms $T_{\mathfrak{f}_v}\mathbb{P}^r$ and $T_{\mathfrak{f}_w}\mathbb{P}^r$ in the exact sequence above. The last term of the exact sequence above receives one negative contribution for each of the flags $\mathfrak{f}_w$. We obtain the following contributions to the negative weights of $H^0(C, f^\star T\mathbb{P}^r)$ coming from the middle term. There are no negative contribution to $H^0(C_v, f_v^\star T\mathbb{P}^r) = T_p\mathbb{P}^r$. The Euler sequence:

$$0 \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathbb{C}^{r+1} \to T\mathbb{P}^r \to 0$$

and the arguments of [32] can be used to deal with the remaining two middle terms. Stable vertices labeled w will contribute $d_w + 1$ negative weights on $H^0(C_w, f_w^\star T\mathbb{P}^r)$. Similarly there will be d_e negative weights on $H^0(C_e, f_e^\star T\mathbb{P}^r)$. We find that the number of negative weights of $H^0(C, f^\star T\mathbb{P}^r)$ equals:

$$\sum_w (d_w + 1) + \sum_e d_e - F = d + \mathfrak{s} - F.$$

Thus the combined contributions of the terms in (3.13) is $d + \mathfrak{s} - u$.

Proposition 3.3.1. *The closed cell $\overline{\mathcal{F}_\Gamma^+}$ is the stack theoretic image of the fibered product $\overline{\mathcal{Y}}_\Gamma$ of closed Gathmann and Konstevich-Manin spaces to H under the tautological morphisms. Alternatively, it is the generically finite image of the stack $\overline{\mathcal{X}}_\Gamma$.*

It is enough to show, by taking closures and using lemma 3.3.3, that the stack theoretic image of $\widetilde{\mathcal{Y}}_\Gamma \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is dense in $\mathcal{F}_\Gamma^+$. We observe that the geometric points of the image of $\widetilde{\mathcal{Y}}_\Gamma$ are contained in $\overline{\mathcal{F}_\Gamma^+}$ because of corollary 3.3.1. Moreover the dimensions match by lemmas 3.3.2 and 3.3.4. $\overline{\mathcal{F}_\Gamma^+}$ is reduced and irreducible because $\mathcal{F}_\Gamma$ clearly is, thanks to equation (3.3). Same is true about $\widetilde{\mathcal{Y}}_\Gamma$. These observations give our claim. The proof of proposition 3.3.2 shows that maps in $\overline{\mathcal{Y}}_\Gamma \setminus \widetilde{\mathcal{Y}}_\Gamma$ cannot flow to a map whose dual graph is Γ. As an afterthought, we obtain that the stack theoretic image of $\widetilde{\mathcal{Y}}_\Gamma$ equals $\mathcal{F}_\Gamma^+$.

3.3.3 Filterability of the decomposition

We will now establish the filterability condition (c) of lemma 3.2.3 which will allow us to prove the tautology of all Chow classes. In this subsection we define the partial ordering on the set of graphs indexing the fixed loci.

For any two decorated graphs Γ and Γ' indexing the fixed loci, we decree that $\Gamma \geq \Gamma'$ if there is a sequence of combinatorial surgeries called splits, joins and transfers changing the graph Γ into Γ'. Each one of these moves is shown in figure $3-7$. Figure $3-8$ explains the intuition behind this ordering; we exhibit families of maps in a given Bialynicki-Birula cell degenerating to a boundary map which belongs to a different cell. The new cell should rank lower in our ordering. In figure $3-8$, the non-negative integers a are degrees, and the positive integers m are multiplicity orders with H. Components mapping to H are represented by thick lines.

Explicitly,

- The split move takes an edge of degree m and cuts it into two (or several) edges with positive degrees m_1 and m_2. The vertex labeled (H, a) is relabeled $(H, a+d_0)$ for some $d_0 \geq 0$, while the vertex labeled p is replaced by two vertices labeled p. The incoming edges and legs to the vertex p are distributed between the newly created vertices. We require that $m = d_0 + m_1 + m_2$. The split move is obtained by degenerating a sequence of maps containing a point mapping to H with multiplicity m. The central fiber is a stable map in the boundary of the Gathmann space. There is an "internal" component mapped to H of degree d_0, to which other components are attached, having multiplicities $m_1, \ldots, m_k$ with H at the nodes. The figure also shows an additional component mapped to H with degree a which is attached to the family.

- The join move takes two edges of degrees m_1 and m_2 meeting in a vertex labeled p and replaces them by a single edge whose degree is m_1+m_2, also collecting the two vertices labeled H, their degrees and all their incoming flags to a single vertex. Locally, the join move corresponds to a family of maps having two domain points mapping to H with multiplicities m_1 and m_2 (there may be additional components mapping to H

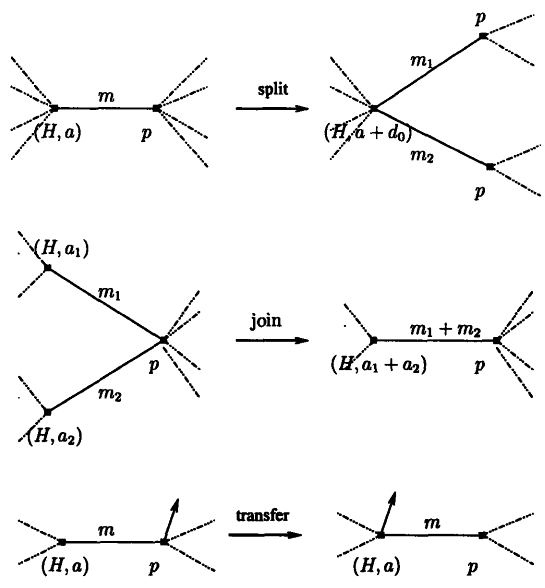


Figure 3-7: Split, joins and transfers

with degrees a_1 and a_2 attached at these points). Letting the two points collapse, we obtain a boundary map with a point mapping to H with multiplicity $m_1 + m_2$.

- The transfer move can be applied to edges whose vertex labeled p has an attached leg. We move the leg to the other end of the edge, labeled H. This move can be realized by a family of maps with one marking, and with domain points which map to H with multiplicity m. In the limit, the marking and the point mapping to H collapse.

To check that we have indeed defined a partial ordering we introduce the following length function:

$$\begin{aligned} l(\Gamma) &= \sum_e (e-1) \cdot \#\{\text{vertices labeled}(H,e)\} + \#\{\text{labeled vertices labeled } p\} + \\ &+ \#\{\text{legs incident to } H \text{ labeled vertices}\}. \end{aligned}$$

The binary relation " $\geq$ " is indeed anti-symmetric since if

$$\Gamma > \Gamma' \text{ then } l(\Gamma) < l(\Gamma').$$

Moreover, it is clear that condition (b) is satisfied; the unique maximal graph is shown in figure $3-9$.

3.3.4 The spanning cycles.

We will construct a family of cycles Ξ satisfying the filterability condition (c) of lemma 3.2.3.

To begin with, we compare the cohomology and the Chow groups of the fixed loci, assuming that the filterability condition is satisfied.

Lemma 3.3.5. *The rational cohomology and rational Chow groups of $\mathcal{F}_\Gamma$ are isomorphic. The rational cohomology and rational Chow groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are isomorphic.*

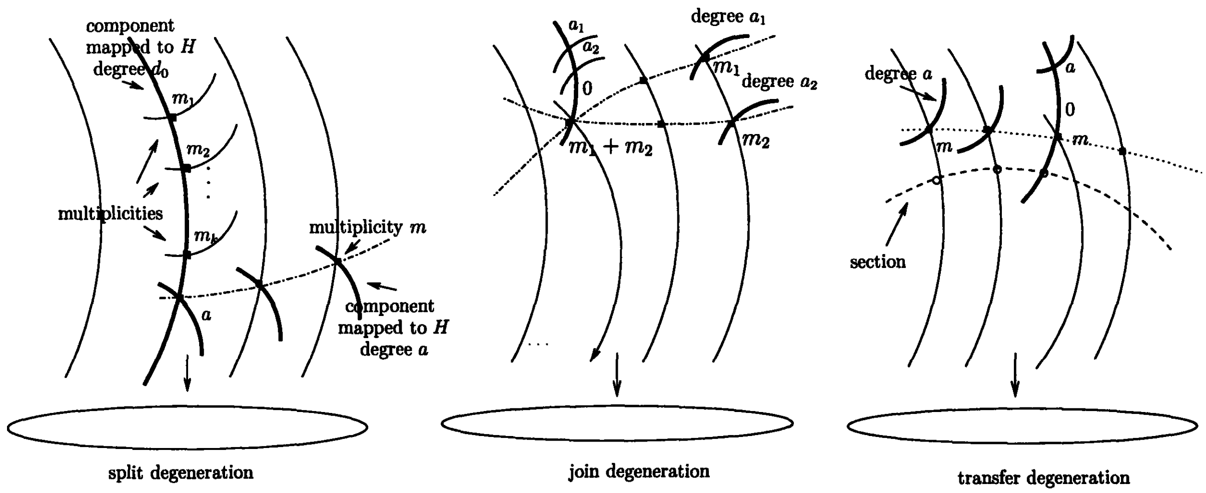


Figure 3-8: Models for the combinatorial moves

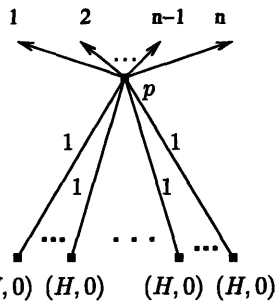


Figure 3-9: The maximal graph.

We will use induction on r. There are two statements to be proved, we call them A_r and B_r respectively. It is proved in [38] that B_0 is true. Lemma 3.2.3 shows that $A_r \implies B_r$. We conclude the proof by showing $B_{r-1} \implies A_r$. Indeed, the cohomology of $\mathcal{F}_\Gamma$ can be computed using equation (3.3):

$$H^\star(\prod_v \overline{\mathcal{M}}_{0,n(v)} \times \prod_w \overline{\mathcal{M}}_{0,n(w)}(H, d_w))^{A_\Gamma} = \left(\otimes_v H^\star(\overline{\mathcal{M}}_{0,n(v)}) \otimes_w H^\star(\overline{\mathcal{M}}_{0,n(w)}(H, d_w))\right)^{A_\Gamma}.$$

It is remarkable that the same formula holds for the Chow groups. This follows from theorem 2 in [38]. Our claim is established.

Corollary 3.3.2. *Let $X = G/P$ be any homogeneous space where G is semisimple algebraic group and P is a parabolic subgroup and $\beta \in A_1(X)$. Then parts (i) and (ii) of lemma* 3.2.3 *are true for $\overline{\mathcal{M}}_{0,n}(X, \beta)$.*

We use a $\mathbb{T}$-action on X with isolated fixed points. The fixed loci of the induced action on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ are, up to a finite group action, products of the Deligne Mumford spaces $\overline{\mathcal{M}}_{0,n}$. The rational cohomology and Chow groups of the fixed loci are isomorphic. Using corollary 3.2.1 and proposition 3.2.1 we obtain a Bialynicki-Birula decomposition on $\overline{\mathcal{M}}_{0,n}(X, \beta)$. We need to verify the conditions (a) and (b) of lemma 3.2.3. We can check them on the closed points, hence we can pass to the coarse moduli schemes (considered in the sense of Vistoli [59]). The two conditions are satisfied on the projective irreducible [40] coarse moduli scheme $\overline{M}_{0,n}(X,\beta)$ of $\overline{\mathcal{M}}_{0,n}(X,\beta)$ as shown in [8]. We conclude observing that the image of $\mathcal{F}_i^+$ in $\overline{M}_{0,n}(X,\beta)$ is the corresponding Bialynicki-Birula cell F_i^+. In fact, one can show that F_i^+ is a coarse moduli scheme for $\mathcal{F}_i^+$. This is because $\mathcal{F}_i^+$ is reduced or equivalently that $\mathcal{F}_i$ is reduced.

The proof of lemma 3.3.5 also suggests the family Ξ. For each graph Γ, we will perform the following construction:

- For each vertex v, pick a cycle class σ_v on $\overline{\mathcal{M}}_{0,n(v)}$.
- For each vertex w, pick a cycle class σ_w on $\overline{\mathcal{M}}_{0,n(w)}(H, d_w)$.
- Assume that our choices define an A_Γ invariant collection of classes.

Henceforth, we will use explicit representatives for the above classes. Since $A^\star(\overline{M}_{0,n(v)})$ is generated by the boundary classes, we may assume:

- σ_v is the closed cycle $\overline{\mathcal{M}}_{0,n(v)}^{\Delta_v}$ of curves with dual graph (at least) Δ_v. Here Δ_v is a stable graph with $n(v)$ labeled legs.

In the following, ξ will be any one of the cycles:

$$\left[\prod_v \overline{\mathcal{M}}_{0,n(v)}^{\Delta_v} \times \prod_w \sigma_w / A_\Gamma\right]. \tag{3.14}$$

Proposition 3.3.2. *The filterability condition (c) is satisfied for the cycles ξ defined above.*

By construction, it is clear that the cycles ξ span the Chow groups of the fixed loci.

We describe the maps in ξ^+ informally. The dual graphs Δ_v determine the type of the domain curves. We consider maps with such domains which are transversal to H; points mapping to H (with multiplicities determined by the edge degrees d_e in Γ) are distributed on the irreducible components. Then ξ^+ will be a fibered product of smaller Gathmann spaces and the cycles σ_w.

Formally, we begin by adding one leg at each very unstable vertex of Γ, thus obtaining a graph γ without very unstable vertices. Geometrically, this corresponds to marking *all* the smooth points on the domain which map to H. There is a forgetful morphism:

$$\overline{\mathcal{M}}_{0,n+u}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \tag{3.15}$$

corresponding to the collapsing map $\gamma \to \Gamma$.

A priori, the only decorations Δ_v carries are the labeled legs. The legs are in one-to-one correspondence to the incoming flags to v in the graph Γ. However, we have seen in section 1.1 that all flags of Γ incident to v carry the degrees α_v, which are the degrees of the incident edges or 0 for the incident legs. In this manner, we enrich the decorations of Δ_v using these degrees as "multiplicities" associated to the legs. We denote by $\alpha(v)$ the datum of the collection of multiplicities α_v together with their distribution along the legs of Δ_v. We can then form the fibered product $\widetilde{\mathcal{M}}_{\alpha(v)}^{\Delta_v,H}(\mathbb{P}^r, d_v)$ as in section 3.3.2 (ii-2).

We let $\xi^+ = \xi \times_{\mathcal{F}_\Gamma} \mathcal{F}_\Gamma^+$. We showed in proposition 3.3.1 that there is a surjective morphism $\widetilde{\mathcal{Y}}_\Gamma/\mathrm{Aut}_\Gamma \to \mathcal{F}_\Gamma^+$, inducing a surjective morphism $\xi \times_{\mathcal{F}_\Gamma} \left[\widetilde{\mathcal{Y}}_\Gamma/\mathrm{Aut}_\Gamma\right] \to \xi^+$. By equations (3.14), (3.3), (3.10) and (3.8), we derive that ξ^+ is the image of:

$$\chi^+ = \left[\left(\prod_v \widetilde{\mathcal{M}}_{\alpha(v)}^{\Delta_v,H}(\mathbb{P}^r, d_v) \times_H \prod_w \sigma_w\right)^{E(\Gamma)} /\mathrm{Aut}_\Gamma\right].$$

This is clear on the level of geometric points. An argument is required to match the stack structures, and such an argument can be made. However, since we work in the Chow groups, we get around by endowing the above stacks with their reduced structures.

We similarly define:

$$\overline{\chi^+} = \left[\left(\prod_v \overline{\mathcal{M}}_{\alpha(v)}^{\Delta_v,H}(\mathbb{P}^r, d_v) \times_H \prod_w \sigma_w\right)^{E(\Gamma)} /\mathrm{Aut}_\Gamma\right]. \tag{3.16}$$

We let $\overline{\xi^+}$ be the its image under the forgetful map (3.15). We obtain morphisms:

$$\chi^+ \hookrightarrow \overline{\chi^+} \to \overline{\mathcal{Y}}_\Gamma/\mathrm{Aut}_\Gamma \to \overline{\mathcal{F}_\Gamma^+}.$$

The first one is an open immersion and by flatness of (3.15) the same is true about the first inclusion below:

$$\xi^+ \hookrightarrow \overline{\xi^+} \to \overline{\mathcal{F}_\Gamma^+}.$$

We do not know that $\overline{\xi^+}$ is the closure of ξ^+ (we do not know $\overline{\xi^+}$ is irreducible). However, when formulating lemma 3.2.3 we were careful not to include this as a requirement in condition (c).

Finally, we show that a map f contained in the boundary $\overline{\xi^+} \setminus \xi^+$ flows to a fixed locus indexed by a graph Γ' with $\Gamma' < \Gamma$. We first make a few reductions. Replacing Γ by γ and ξ^+ by χ^+, we may assume Γ has no very unstable vertices. We want to show that the graph of $F = \lim_{t\to 0} f^t$ is obtained from Γ by a sequence of the combinatorial moves which we called joins, splits and transfers.

The datum of a map f is tantamount to giving maps f_v and f_w in the Gathmann spaces $\overline{\mathcal{M}}_{\alpha(v)}^{\Delta_v,H}(\mathbb{P}^r, d_v)$ and the cycles σ_w with compatible gluing conditions. As usual, the unstable vertices w require special care as they only give points on the domain not actual maps. We have seen that the limit F of f^t is obtained from gluing the individual limits F_v and $F_w = f_w$ (for stable w's) of f_v^t and f_w^t. The dual graphs are also obtained by gluing. Since to compute the limit we consider each vertex at a time, we may further assume that Γ consists in one vertex labeled v to which we attach legs and unstable vertices w. Thus, we may take Γ to be the graph in figure $3-10$.

Now recall that Δ_v encodes the domain type of the nodal map f_v. The markings of f_v are distributed on the components of the domain and come with multiplicities encoded by the flags of Δ_v. As we can treat the components individually, we may assume Δ_v has only one vertex. Moreover, the map f_v has to be in the boundary of the Gathmann space

$$\overline{\mathcal{M}}_{\alpha(v)}^{H}(\mathbb{P}^r, d_v) \setminus \widetilde{\mathcal{M}}_{\alpha(v)}^{H}(\mathbb{P}^r, d_v).$$

Changing notation slightly, we prove the following. We consider the multindex $\alpha =$

$(\alpha_1, \dots, \alpha_n, 0, \dots, 0)$ with $|\alpha| = d$ and $\alpha_i > 0$. We will consider marked maps in the Gathmann space $\overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$. For such a map $(f, C, x_1, \dots, x_n, y_1, \dots, y_m)$ we have $f^! H = \sum_i \alpha_i x_i$. If f were an element in $\widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$, then its limit F would have the dual graph Γ shown in figure $3-10$. This graph has one vertex v labeled p, n edges labeled $\alpha_1, \dots, \alpha_n$ joining v to unstable vertices w labeled $(H, 0)$.

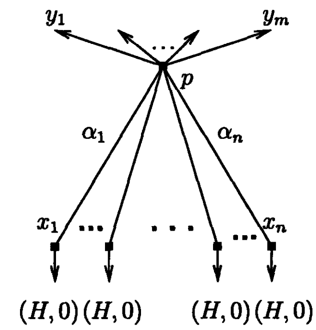


Figure 3-10: The graph Γ of a limit for a generic map in the Gathmann space.

Lemma 3.3.6. *Let f be a map in the boundary of the Gathmann stack $\overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d) \setminus \widetilde{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)$. Let F be the limit of the flow of f. Then the dual graph Γ_F of F can be obtained from Γ by splits, joins and transfers.*

The map f will have components which are not transversal to H and which are responsible for the different dual graph. Let C_0 be a nontrivial connected component of $f^{-1}(H)$ on which the map has total degree d_0, and let $C_1, \dots, C_k$ be the irreducible components joined to C_0, having multiplicities $m_1, \dots, m_k$ with H at the nodes. Figure 3.1.2 shows an example of such a map. In any case, C_0 will contain some of the markings mapping to H, say x_i for $i \in I$, and some of the remaining markings y_j for $j \in J$. The contribution of the components $C_0 \cup C_1 \dots \cup C_k$ to the dual graph Γ_F, as computed by corollary 3.3.1, is shown in figure $3-11$. The figure also shows the moves we need to apply to this portion of Γ_F to obtain its corresponding contribution to Γ. The rest of the graph Γ_F is attached to the portion shown there and is carried along when performing the combinatorial moves. Observe that existence of the join move is guaranteed by the equation $d_0 + \sum m_i = \sum_{i \in I} \alpha_i$ which follows by considering intersection multiplicities with H. This completes the proof.

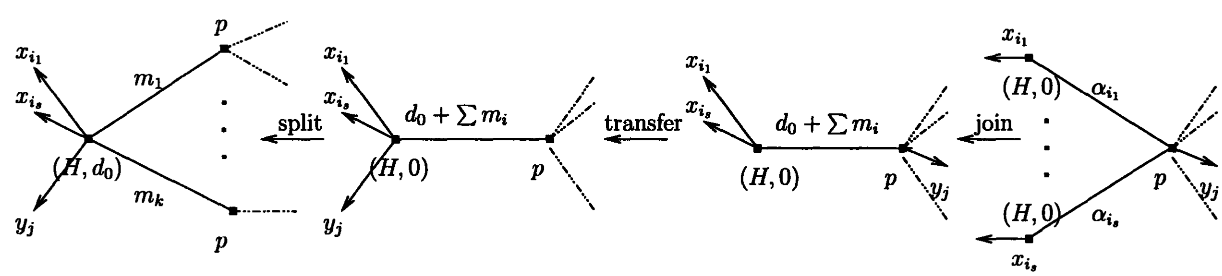


Figure 3-11: The combinatorial moves comparing Γ_F to Γ.

3.4 The tautology of the Chow classes

In this section we tie the loose ends and prove the main result, theorem 2. Items (1)-(5) are contained in proposition 3.2.1, proposition 3.3.2, proposition 3.3.1, lemma 3.3.4 and lemma 3.3.5 respectively. Item (6) is a consequence of the proof of proposition 3.3.2 and equations (3.9) and (3.16). The last item (7) follows from (6) combined with the following result.

Lemma 3.4.1. *(i) Let $i : H \to \mathbb{P}^r$ be a hyperplane and let $i : \overline{\mathcal{M}}_{0,n}(H, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ denote the induced map. The pushforward map*

$$i_\star : A_\star(\overline{\mathcal{M}}_{0,n}(H, d)) \to A_\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$$

preserves the tautological classes.

(ii) For each n-multindex α, the class of the Gathmann space $\left[\overline{\mathcal{M}}_\alpha^H(\mathbb{P}^r, d)\right]$ is tautological.

Consider the bundle $\mathcal{B} = R\pi_\star ev^\star \mathcal{O}_{\mathbb{P}^r}(1)$ where ev and π are the universal evaluation and projection morphisms. This is a rank $d+1$ vector bundle on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. As usual, the equation of H gives a section of $\mathcal{B}$ which vanishes precisely on $\overline{\mathcal{M}}_{0,n}(H, d)$.

We claim that

$$R^\star(\overline{\mathcal{M}}_{0,n}(H, d)) \subset i^\star R^\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)).$$

We need to check $i^\star R^\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$ satisfies the two conditions of definition 1. For the first one, invariance under pullbacks is obvious, while invariance under pushforwards follows from standard manipulations of the projection formula. The second condition is immediate, as all classes α_H on H are obtained by restrictions of classes $\alpha_{\mathbb{P}^r}$ on $\mathbb{P}^r$ and

$$ev^\star \alpha_H = ev^\star i^\star \alpha_{\mathbb{P}^r} = i^\star ev^\star \alpha_{\mathbb{P}^r}.$$

Any tautological class α on $\overline{\mathcal{M}}_{0,n}(H, d)$ is the restriction of a tautological class β on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Therefore,

$$i_\star \alpha = i_\star i^\star \beta = \beta \cdot c_{d+1}(\mathcal{B}).$$

It suffices to prove that $c_{d+1}(\mathcal{B})$ is tautological. A computation identical to Mumford's [49] using Grothendieck-Riemann-Roch shows:

$$ch(\mathcal{B}) = \pi_\star \left(e^{ev^\star H} \cdot \left(\frac{c_1(\omega_\pi)}{e^{c_1(\omega_\pi)} - 1} + i_\star P(\psi_\star, \psi_\bullet) \right) \right). \tag{3.17}$$

Here P is a universal polynomial whose coefficients can be explicitly written down in terms of the Bernoulli numbers. The morphism i is the codimension 2 inclusion of the nodes of the fibers of the universal curve $\pi : \overline{\mathcal{M}}_{0,n\cup\{\diamond\}}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. Under the standard identifications, this can be expressed as union of images of fibered products :

$$i : \overline{\mathcal{M}}_{0,S_1\cup\{\star\}}(\mathbb{P}^r, d_1) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,\{\star,\diamond,\bullet\}}(\mathbb{P}^r, 0) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,\{\bullet\}\cup S_2}(\mathbb{P}^r, d_2) \to \overline{\mathcal{M}}_{0,S_1\cup S_2\cup\{\diamond\}}(\mathbb{P}^r, d)$$

for all partitions $S_1 \cup S_2 = \{1, \ldots, n\}$ and $d_1 + d_2 = d$. The classes $\psi_\star$ and $\psi_\bullet$ of equation (3.17) are the cotangent lines at the markings $\star$ and $\bullet$ which are joined at a node.

To prove out claim, we need to argue that $c_1(\omega_\pi)$ and ψ are tautological. This follows from the results of [51], where it is shown that all codimension 1 classes are tautological.

An argument may be required to justify the application of the Grothendieck-Riemann-Roch theorem in our stacky context. There are several ways to go about this. For example, one can argue on the coarse moduli schemes using [59]. Alternatively, lemma 2.1.1 in [51] shows that if r is large enough the locus of maps with automorphisms has codimension at least $d+2$. Its complement is a fine moduli scheme $\mathcal{M}^\star$ and we can apply GRR for the universal morphism over $\mathcal{M}^\star$. Since we are only interested in $c_{d+1}(\mathcal{B}) \in A^{d+1}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) = A^{d+1}(\mathcal{M}^\star)$, the formula (3.17) holds up to codimension $d+1$. To deal with the small values of r, we pick N large enough and use the inclusion $j : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{r+N}, d)$. The class in question can be expressed as a pullback of a class we already know to be tautological:

$$\left[\overline{\mathcal{M}}_{0,n}(H, d)\right] = j^\star \left[\overline{\mathcal{M}}_{0,n}(\mathcal{H}, d)\right] \in j^\star R^{d+1}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{r+N}, d)) = R^{d+1}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)).$$

Here $\mathcal{H} \hookrightarrow \mathbb{P}^{r+N}$ is a hyperplane which intersects $\mathbb{P}^r \hookrightarrow \mathbb{P}^{r+N}$ along H. This proves the claim.

Part (ii) is a consequence of equation (3.4), using induction on the multindex α. The correction terms are pushforwards of classes on the boundary strata. These classes are either lower dimensional Gathmann spaces or Kontsevich-Manin spaces to H which are tautological by induction and by part (i) of the lemma respectively.

Chapter 4

The low codimension classes

In this chapter we will consider some examples and applications of our main results. We will focus on the low codimension tautological classes. We will determine the Picard group of the moduli space of stable maps to flag varieties, generalizing results of Pandharipande [51]. We will also prove that the relations between the low codimension tautological classes are tautological in the sense of definition 1.6.1.

4.1 Generators for the codimension one Chow group.

To begin with, we review the relevant facts about homogeneous spaces and their Schubert stratification which we will use. To set the stage, let X be the algebraic homogeneous space G/P where G is a semisimple group and P is a parabolic subgroup. We pick T a maximal torus, B a Borel subgroup such that $T \subset B \subset P \subset G$. Our convention is that the Lie algebra $\mathfrak{b}$ contains all the negative roots with respect to some choice of a Weyl chamber. We let U^+ denote the unipotent subgroup of G whose Lie algebra is the sum of all positive root spaces. We let W be the Weyl group and $W^{\mathfrak{p}}$ be the Hesse diagram of $\mathfrak{p}$.

We consider the decomposition of X coming from the maximal torus action on X. This action has isolated fixed points indexed by the elements of the Hasse diagram $W^{\mathfrak{p}}$. The corresponding *plus* Bialynicki-Birula stratification coincides with the more familiar Schubert decomposition:

$$G/P = \cup_{w \in W^{\mathfrak{p}}} U^+ \cdot (wP).$$

We let X_w be the orbit $U^+ \cdot (wP)$ and we let Y_w be its closure. Y_w is a subvariety of X whose codimension equals the length $l(w)$ of w. Y_w can be written as union of lower dimensional strata (where $\geq$ refers to the Bruhat ordering):

$$Y_w = \cup_{w' \in W^{\mathfrak{p}}, w' \geq w} X_{w'}.$$

The codimension 1 cells Y_w are important to us. They are in one to one correspondence with the *simple* roots α of $\mathfrak{g}$ not contained in $\mathfrak{p}$, the corresponding w being the reflection s_α across the wall α. The cycle X_α corresponds to the points of X which flow to $q_\alpha = s_\alpha P$ as $t \to 0$. The Y_α's can also be described by the zeros of holomorphic sections of some very ample line bundles L_α. We can write any class $\beta \in H^2(X, \mathbb{Z})$ in the form $\beta = \sum_\alpha d_\alpha \beta_\alpha$, where β_α is the codimension 1 class $[Y_\alpha] = c_1(L_\alpha)$ for each simple root α not in $\mathfrak{p}$.

In addition, each simple root α determines a rational curve in X joining P to $q_\alpha = s_\alpha P$. The class of this rational curve is dual to the class of Y_α. The rational curve can be parametrized as:

$$t \to exp(tv)P, \text{ where } v \text{ is a vector in the root space of } \alpha.$$

Similarly, such T invariant curves in X can be generated for all positive (not necessarily simple) roots α which are not in $\mathfrak{p}$; all T invariant curves passing through P are obtained this way. More generally, a T-invariant rational curve joining two general fixed points wP and $w'P$ exists provided $w' = ws_\alpha$ for some root α not in $\mathfrak{p}$.

After these preliminaries, we proceed to prove the following result about the generators of the rational codimension 1 Chow group of $\overline{M}_{0,n}(X,\beta)$. We obtain the same results on the moduli stack, essentially because the locus of maps with non trivial automorphisms has large codimension by an argument of [51]. As a corollary of localization and the rationality of the moduli space of stable maps we easily see that the rational Picard group, the rational codimension 1 Chow group and the complex codimension 1 rational cohomology all coincide. This was explained for cohomology and Chow groups in the previous chapter. The equality with the Picard group follows from the exponential sequence and the rationality of the moduli spaces involved [40]. Henceforth, we will use Chow groups/cohomology interchangeably, but in the proof of this proposition it is more convenient to make use of the Chow group of $\overline{M} = \overline{M}_{0,n}(X,\beta)$.

Proposition 4.1.1. *The codimension* 1 *Chow group with rational coefficients of* $\overline{M}_{0,n}(X,\beta)$ *(or* $\overline{\mathcal{M}}_{0,n}(X,\beta)$*) is generated by the following classes:*

- *boundary divisors of nodal maps, the degrees and marked points being distributed arbitrarily on the two components.*
- *evaluation classes* $ev_i^\star c_1(L_\alpha)$ *for each* $1 \leq i \leq n$ *and each simple root* α *which is not contained in* $\mathfrak{p}$,
- *kappa classes* $\kappa(Y_w)$ *where* $l(w) = 2$ *(so that* Y_w *has complex codimension* 2 *in* X*).*

We let Y be the complement in X of all classes Y_w with $l(w) \geq 2$. We also let U be the open dense cell of the Schubert decomposition. It can be defined as follows:

$$U = \{x \in X \text{ such that } \lim_{t \to 0} t \cdot x \to P\}.$$

We consider the following subschemes of $\overline{M}$:

(a) The codimension 1 boundary divisors.

(b) The subscheme of maps intersecting Y_w for all w with $l(w) = 2$.

(c) The subscheme of maps with markings in Y_α for all simple roots α.

(d) The subscheme of maps which cut Y_α with multiplicity higher than 1.

It is clear that the complement of all these subschemes is the locus $\mathcal{X}$ of maps $f : (C, x_1, \ldots, x_n) \to X$ with the following properties:

- The domain curve is irreducible,
- the image of the map is contained in Y,

- the markings of f map to U,
- f intersects Y_α transversally.

We claim that the Chow group $A^1(\mathcal{X}) = 0$. It follows then that the four types of classes $(a) - (d)$ span the space of divisors. To complete the proof it remains to show that the classes in item (d) are indeed among the generators we enumerated in the proposition.

The subscheme in item (d) can be described as the image of the cycle $\overline{M}^{Y_\alpha}_{(0,\dots,0,2)}(X,\beta)$ on $\overline{M}_{0,n+1}(X,\beta)$ under the map π forgetting the last marking. Here, $\overline{M}^{Y_\alpha}_{(0,\dots,0,2)}(X,\beta)$ is the Gathmann space of stable maps with contact order at least 2 with the very ample hypersurface Y_α (see [25] for the relevant definitions). Due to the fact that Y_α is a very ample, there is an embedding $\phi : (X, Y_\alpha) \to (\mathbb{P}^N, \mathbb{P}^{N-1})$. The following equation:

$$\pi_\star\left[\overline{M}^{Y_\alpha}_{\bullet}(X,\beta)\right] = \phi^\star\pi_\star\left[\overline{M}^{\mathbb{P}^{N-1}}_{\bullet}(\mathbb{P}^N,\phi_\star\beta)\right]$$

holds in the Chow group of $\overline{M}_{0,n}(\mathbb{P}^N,\phi_\star\beta)$ (see [25], theorem 2.6). Here $\overline{M}^{\mathbb{P}^{N-1}}_{\bullet}(\mathbb{P}^N,\phi_\star\beta)$ denotes the corresponding Gathmann space of maps to $\mathbb{P}^N$ with contact order 2 at the hyperplane $\mathbb{P}^{N-1}$. Therefore, it suffices to show that on $\overline{M}_{0,n}(\mathbb{P}^N,\phi_\star\beta)$ the pushforward class $\pi_\star\left[\overline{M}^{\mathbb{P}^{N-1}}_{\bullet}(\mathbb{P}^N,\phi_\star\beta)\right]$ is in the span of the corresponding classes we claimed as generators. That is, we need to show this class is a sum boundary divisors, κ classes and evaluation classes $ev^\star\mathcal{O}_{\mathbb{P}^N}(1)$, and then observe that these classes pullback to similar classes under ϕ. However, this statement is already proved by Pandharipande [51] who enumerated all divisor classes for $\overline{M}_{0,n}(\mathbb{P}^N,\phi_\star\beta)$.

To prove the vanishing of $A^1(\mathcal{X})$ we first consider the case when $n+\sum_\alpha d_\alpha \geq 4$. We let $\mathfrak{S} = \times_\alpha S_{d_\alpha}$ and

$$\mathcal{F} = M_{0,n+\sum_\alpha d_\alpha}/\mathfrak{S},\ \overline{\mathcal{F}} = \overline{M}_{0,n+\sum_\alpha d_\alpha}/\mathfrak{S}.$$

In fact $\overline{\mathcal{F}}$ is the big fixed locus for the T action on $\overline{M}$. The embedding $\overline{j} : \overline{\mathcal{F}} \to \overline{M}$ (and similarly $j : \mathcal{F} \to \overline{M}$) is obtained as follows:

- We consider a stable curve with $n+\sum_\alpha d_\alpha$ marked points. This will be a contracted component of the stable map whose image is the origin P of X. We make the first n marked points of the stable curve be the marked points of the stable map to X.
- At the d_α marked points we add $\mathbb{P}^1$'s of degree 1 mapping to the rational curve joining P to q_α constructed in the beginning of this section.

We let

$$\mathcal{E} = \{f \text{ stable map in } \overline{M} \text{ such that } t\cdot f \to F \in \mathcal{F} \text{ as } t \to 0\}.$$

Proposition 2 of [40] shows that $\mathcal{X}$ is an open subvariety of $\mathcal{E}$. It is enough to show $A^1(\mathcal{E}) = 0$.

Let $\pi : \widehat{M} \to \overline{M}$ be a T-equivariant resolution of singularities for $\overline{M}$. The image of restricted map $j : \mathcal{F} \to \overline{M}$ lies in the smooth (automorphism-free) locus of $\overline{M}$. Since π is an isomorphism over the smooth locus, we obtain an inclusion $\hat{j} : \mathcal{F} \to \widehat{M}$.

Let $\widehat{\mathcal{E}}$ be the subset of $\widehat{M}$ of points flowing to $\mathcal{F}$. Since, π is an isomorphism on $\mathcal{F}$, we have $\pi^{-1}\mathcal{E} = \widehat{\mathcal{E}}$. Now, $\pi : \widehat{\mathcal{E}} \to \mathcal{E}$ can be chosen to be a composition of blowups, so we conclude that $A^1(\widehat{\mathcal{E}}) \to A^1(\mathcal{E})$ is surjective. It is enough to show $A^1(\widehat{\mathcal{E}}) = 0$. This follows easily, since $\widehat{M}$ is smooth and for smooth varieties, it is well known that $\widehat{\mathcal{E}}$ is a bundle over the fixed set $\mathcal{F}$. Thus the claimed vanishing of $A^1(\widehat{\mathcal{E}})$ follows from the vanishing of

$A^1(\mathcal{F}) = A^1(M_{0,n+\sum_\alpha d_\alpha})^{\mathfrak{S}}$. This is well known, it can be derived, for example from Keel's result that the boundary classes generate all codimension one classes on $\overline{M}_{0,n+\sum_\alpha d_\alpha}$.

To finish the proof we have to analyze each of the remaining cases when $n + \sum_\alpha d_\alpha \leq 3$ individually. Then $\mathcal{F}$ is to be interpreted as a point, and as long as this point has no automorphisms in the moduli space $\overline{M}$ we are done by the same arguments as before. There are four cases to consider. We will briefly show the argument for $n = 0, \beta = 3\beta_\alpha$. The remaining cases can be obtained as in theorem 3 in [40], by adding more marked points to place ourselves in the case we already discussed.

We let $\mathcal{Y}$ be the open subscheme of $M_{0,3}(X, 3\beta_\alpha)$ consisting in maps with image in Y such that all 3 markings map to Y_α with multiplicity 1. It follows that $\mathcal{Y}/S_3 = \mathcal{X}$. It is therefore enough to show $A^1(\mathcal{Y}) = 0$. Now, let $\widehat{\mathcal{Y}}$ be an equivariant resolution of singularities for the closure $\overline{\mathcal{Y}}$ of $\mathcal{Y}$ in $\overline{M}_{0,3}(X, 3\beta_\alpha)$. Since $\mathcal{Y}$ is smooth we can view it as a subscheme of $\widehat{\mathcal{Y}}$. The arguments of proposition 2 in [40] show that all $f \in \mathcal{Y}$ flow to a unique map $\mu \in \overline{\mathcal{Y}}$. This map μ has an internal component of degree 0 mapping to P to which we attach external components of degree β_α, each of them having a marked point mapping to q_α. Therefore μ sits in the smooth part of $\overline{\mathcal{Y}}$ and we can therefore regard it as an element in $\widehat{\mathcal{Y}}$. Let us look at the subvariety $\mathcal{V}$ of points of $\widehat{\mathcal{Y}}$ flowing to μ under the $\mathbb{C}^\star$ action. $\mathcal{Y}$ is contained in $\mathcal{V}$ by the above discussion. Now since $\widehat{\mathcal{Y}}$ is smooth, $\mathcal{V}$ is an affine space. Therefore $A^1(\mathcal{Y}) = 0$, as desired. This completes the proof of the proposition.

4.1.1 Divisors on the space of maps to Grassmannians.

This subsection will contain another way of finding divisor classes on the space of maps to the Grassmannian X parameterizing k dimensional projective subspaces of $\mathbb{P}^r$. The method is quite ad-hoc and it involves decomposing the space of stable maps into pieces we understand better. The author could not make this procedure work for other flag manifolds. We seek to reprove the generation result of the previous subsection.

Let $\mathcal{S}$ and $\mathcal{Q}$ denote the tautological and quotient bundles on X. We let W be a copy of $\mathbb{P}^{r-k-2}$ (given by the vanishing of the first $k+2$ homogeneous coordinates). Then the subvariety

$$\mathcal{V} = \{\Lambda \in X \text{ such that } \Lambda \cap W \neq \emptyset\}$$

has codimension 4. It is easy to observe that $c_1(\mathcal{Q})^2$ and $[\mathcal{V}]$ generate the complex codimension 2 classes on X.

The complement of $\mathcal{V}$ parametrizes subspaces Λ in $\mathbb{P}^r \setminus \mathbb{P}^{r-k-2}$. The key observation is that $\mathbb{P}^r \setminus \mathbb{P}^{r-k-2}$ can be understood as the total space of the bundle

$$\pi : \mathcal{O}_{\mathbb{P}^{k+1}}(1)^{\oplus(r-k-1)} \to \mathbb{P}^{k+1}.$$

Since the fibers of π are affine spaces, the k dimensional subspace Λ contained in $X \setminus \mathcal{V}$ projects to a k dimensional subspace $L = \pi(\Lambda)$ in $\mathbb{P}^{k+1}$ i.e. L gives an element of the projective space $\mathbb{P}^{k+1}$ of k dimensional subspaces of $\mathbb{P}^{k+1}$. We conclude that $X \setminus \mathcal{V}$ can be described as a bundle $\mathcal{E}$ over $\mathbb{P}^{k+1}$ whose fiber over a subspace L is $H^0(L, \mathcal{O}_L(1))^{\oplus(r-k-1)}$.

An easy argument, involving the Euler sequence identifies this bundle with $T\mathbb{P}^{k+1}(-1)$. The long exact sequence in cohomology induced by the Euler sequence shows that this bundle is convex. That is, for any morphism $g : \mathbb{P}^1 \to \mathbb{P}^{k+1}$ we have $H^1(\mathbb{P}^1, g^\star\mathcal{E}) = 0$.

We define $\mathcal{X}$ to be the subscheme of $\overline{M}_{0,n}(X, d)$ parameterizing stable maps f whose images intersect $\mathcal{V}$. The open set $\overline{M}_{0,n}(X, d) \setminus \mathcal{X}$ consists in maps whose images are contained in the total space $X \setminus \mathcal{V}$ of the bundle $\mathcal{E} \to \mathbb{P}^{k+1}$. Proposition 2.1 in [3] and the convexity of the bundle $\mathcal{E}$ imply that $\overline{M}_{0,n}(X,d) \setminus \mathcal{X}$ is in fact a bundle over $\overline{M}_{0,n}(\mathbb{P}^{k+1},d)$. Therefore, $A^1(\overline{M}_{0,n}(X,d) \setminus \mathcal{X}) = A^1(\overline{M}_{0,n}(\mathbb{P}^{k+1},d))$. This last group is well known, it has been computed by Pandharipande in [51]. For example, in the case when $n \geq 3$ or $n = 0$, the generators are the boundary divisors and the class $\pi_\star(ev_{n+1}^\star c_1(\mathcal{O}_{\mathbb{P}^{k+1}}(1))^2)$. This implies that the boundary divisors and the class $\pi_\star ev_{n+1}^\star c_1(\mathcal{Q})^2$ on $\overline{M}_{0,n}(X,\beta)$ restrict to generators for $A^1(\overline{M}_{0,n}(X,d) \setminus \mathcal{X})$. To get all divisors classes on $A^1(\overline{M}_{0,n}(X,\beta))$ we need to add the class $[\mathcal{X}] = \pi_\star ev_{n+1}^\star [\mathcal{V}]$. A similar argument works for $n = 1$ or $n = 2$. We recover the statement of proposition 4.1.1.

4.2 The classes on the moduli spaces of maps to SL flags.

In this section we will focus explicitly on the case of divisor classes on the space of maps to SL flag varieties. We will start by restating the results of the previous section for SL flags, then describe relations between the generators we found and finally prove their independence by a dimension computation. We will show first:

Proposition 4.2.1. *Let X be any SL flag whose Betti numbers in dimension 2 and 4 are h^2 and h^4. Let*

$$V \otimes \mathcal{O}_X = \mathcal{Q}_0 \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_l \to 0$$

be the tautological quotients and let β be a class with $d_i = \beta \cdot c_1(\mathcal{Q}_i) > 0$. The dimension of the rational Picard group of $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is

$$\left[2^{n-1}(d_1+1)\ldots(d_l+1)+\frac{1}{2}\right]-1-\binom{n}{2}+h^4-\binom{h^2}{2}.$$

The generators of the Picard group are

- *the boundary divisors,*
- *the classes $\kappa(\alpha)$ where α is either $c_1(\mathcal{Q}_i)^2$ for $1 \leq i \leq l$ or the nonzero classes $c_2(\mathcal{K}_j)$ for $0 \leq j \leq l$. Here $\mathcal{K}_j$ is the kernel of $\mathcal{Q}_j \to \mathcal{Q}_{j+1}$.*
- *when $n = 1$ or $n = 2$, we add any one of the evaluation classes $ev_i^\star c_1(\mathcal{Q}_j)$.*

The class

$$\sum_i \kappa(c_2(\mathcal{K}_i)) + \sum_i \left(\frac{d_{i-1}+d_{i+1}}{2d_i}-1\right)\kappa(c_1(\mathcal{Q}_i)^2)$$

is supported on the boundary. All other relations between these generators come from $\overline{\mathcal{M}}_{0,n}$ by pullback.

4.2.1 Divisors on the moduli spaces of maps to SL_n flags.

To set the stage, we let X be the flag variety parameterizing quotients of a vector space V of fixed dimensions $n_1, \ldots, n_l$, or equivalently of subspaces of V of dimensions $m_1, \ldots, m_l$:

$$0 \to S_1 \to \ldots \to S_l \to V \to Q_1 \to \ldots \to Q_l \to 0.$$

There is a tautological sequence on X given by

$$0 \to \mathcal{S}_1 \to \ldots \to \mathcal{S}_l \to V \otimes \mathcal{O}_X \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_l \to 0. \tag{4.1}$$

We let $\mathcal{K}_j$ denote the kernel of the map $\mathcal{Q}_j \to \mathcal{Q}_{j+1}$ for all $0 \leq j \leq l$, where by convention $\mathcal{Q}_0 = V \otimes \mathcal{O}_X$ and $\mathcal{Q}_{l+1} = 0$.

It is well known that the Chern classes $c_1(\mathcal{Q}_j)$ form a basis for $H^2(X, \mathbb{Z})$ for $1 \leq j \leq l$. We let $h^2(X)$ denote the dimension of this vector space. Each stable map to X will have a multi-degree $(d_1, \ldots, d_l)$ determined by the above generators of $H^2(X)$. Similarly, $H^4(X, \mathbb{Z})$ is generated by the classes $c_1(\mathcal{Q}_i)c_1(\mathcal{Q}_j)$ together with the *nonzero* Chern classes $c_2(\mathcal{K}_j)$. There is only one relation between these generators:

$$\sum_i c_2(\mathcal{K}_i) - \sum_i c_1(\mathcal{Q}_i)^2 + \sum_i c_1(\mathcal{Q}_i)c_1(\mathcal{Q}_{i+1}) = 0. \tag{4.2}$$

We enumerate the generators we obtained in proposition 4.1.1:

(a) boundary classes. We have $[2^{n-1}(d_1+1)\ldots(d_l+1)]^+ - 1 - n$ such boundaries.

(b) κ classes $\kappa(c_1(\mathcal{Q}_i) \cdot c_1(\mathcal{Q}_j))$ and all classes $\kappa(c_2(\mathcal{K}_i))$ when $\mathcal{K}_i$ has rank at least 2.

(c) evaluation classes $ev_i^\star c_1(\mathcal{Q}_j)$, for each $1 \leq i \leq n$ and $1 \leq j \leq l$.

Now, these generators turn out not to be independent. We exhibit relations between them. We start with the boundary classes. The obvious way of getting relations is to pull back relations from $\overline{\mathcal{M}}_{0,n}$ under the forgetful map:

$$\overline{M}_{0,n}(X, \beta) \to \overline{\mathcal{M}}_{0,n}.$$

There are $2^{n-1} - 1 - n$ boundary classes on $\overline{\mathcal{M}}_{0,n}$ but there are $\frac{n(n-3)}{2}$ independent relations between them. This cuts down the number of independent classes in (a) to at most $[2^{n-1}(d_1+1)\ldots(d_l+1)]^+ - 1 - \binom{n}{2}$, with equality when all relations come from $\overline{\mathcal{M}}_{0,n}$. All these relations are tautological as defined in the introduction.

Next, we consider the classes of type (c). When $n \geq 1$, we pick H an ample generator and pick m large enough such that the bundles $\mathbf{Q_j} = \det \mathcal{Q}_j(mH)$ are all very ample. We will replace the bundles $\mathcal{Q}_j$ in (c) by their very ample counterparts $\mathbf{Q_j}$. Note that the span of the classes in (b) and (c) will be not be affected by this change.

When $n \geq 3$ and when $X = \mathbb{P}^r$ all divisor classes, including the corresponding evaluation classes of type (c), are spanned by boundaries and the κ class $\kappa(c_1(\mathcal{O}_{\mathbb{P}^r}(1))^2)$. This is the contents of lemma 1.1.1 in [51]. Using the linear system $|\mathbf{Q_j}|$ we get an *embedding* of X into a projective space. Pulling back under this map, we can therefore conclude that the class $ev_i^\star c_1(\mathbf{Q_j})$ is in the span of boundaries and of κ classes, which will necessarily be on the list (b). When $n \geq 3$ we will henceforth dispense with the classes (c).

However, the same conclusion can be reached in a slightly different way. An analogue of equation (2.20), proved in the same way, is the following tautological equation:

$$ev_i^\star c_1(\mathcal{Q}_k) + ev_j^\star c_1(\mathcal{Q}_k) = \frac{1}{c_1(\mathcal{Q}_k) \cdot \beta}\kappa(c_1(\mathcal{Q}_k)^2) + \text{ boundaries }. \tag{4.3}$$

When $n \geq 3$, this system can be solved to express the evaluation classes in terms of the κ's via the tautological relations.

When $n = 1$ a different discussion is needed. We use Lemma 2.2.2 in [51]. Pull back the relation provided by the lemma for $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, d)$ under the embedding given by the linear system $|\mathbf{Q_j}|$. Then, modulo boundary classes the following relation holds for some constants

$D_j = c_1(\mathbf{Q_j}) \cdot \beta$:

$$\psi_1 = \frac{1}{D_j^2}\kappa(c_1(\mathbf{Q_j})^2) - \frac{2}{D_j} ev_1^\star c_1(\mathbf{Q_j}) \tag{4.4}$$

It follows that the span of the evaluation classes $ev^\star c_1(\mathbf{Q_j})$ is exactly 1 dimensional modulo boundaries and the κ classes in (b). For example ψ_1 is a generator of the one dimensional span. Intersecting with suitable curves or restricting to a copy of $\mathbb{P}^1 \hookrightarrow X$ and invoking the results of section 2.3, it can be shown this class is independent from the boundaries.

We remark that equation (4.4) can also be interpreted geometrically via the topological recursion relations. The following equation on $\overline{\mathcal{M}}_{0,3}(X,\beta)$ is obtained by pullback from $\overline{\mathcal{M}}_{0,3}$:

$$\pi^\star\psi_1 + \Delta^{0,\beta}_{(123),\emptyset} + \Delta^{0,\beta}_{(12),(3)} + \Delta^{0,\beta}_{(13),(2)} = \sum_{\beta_1+\beta_2=\beta} \Delta^{\beta_1,\beta_2}_{(1),(23)} \; (=\psi_1).$$

Here $\Delta^{\beta_1,\beta_2}_{S,T}$ is the boundary divisor with markings labeled S and T distributed on two branches of degree β_1, β_2, and $\pi : \overline{\mathcal{M}}_{0,3}(X,\beta) \to \overline{\mathcal{M}}_{0,1}(X,\beta)$ is the map forgetting the last two points. We intersect the above equation with the class $ev_2^\star c_1(\mathbf{Q_j}) \cdot ev_3^\star c_1(\mathbf{Q_j})$ and then use the pushforward by π to obtain (4.4).

For $n = 2$, a similar discussion as above shows that the span of the evaluation classes in (c) is at most 2 dimensional modulo boundaries and the κ classes in (b). The classes ψ_1 and ψ_2 can be chosen as generators for the two dimensional span (alternatively we can pick pairs of evaluation classes). However, corollary 1 in [43] rewritten as equation

$$\psi_1 + \psi_2 = \Delta(\{1\},\{2\})$$

shows that the sum of these two classes is also in the span of the boundaries $\Delta(\{1\},\{2\})$ where the markings are on two distinct branches. It follows that the span of the evaluation classes (c) is exactly 1 dimensional modulo boundaries and κ classes. We immediately conclude that for all values of n, the combined contribution of the classes in (a) and (c) is at most $[2^{n-1}(d_1+1)\dots(d_l+1)]^+ - 1 - \binom{n}{2}$.

Finally, the classes of type (b) are also connected by relations. Lemma 4.2.1 of the next subsection shows that $\kappa(c_1(\mathcal{Q}_i)c_1(\mathcal{Q}_j))$ can be expressed in terms of $\kappa(c_1(\mathcal{Q}_i)^2)$ and $\kappa(c_1(\mathcal{Q}_j)^2)$ modulo boundaries. In fact, applying this lemma to each pair $(\mathcal{Q}_i, \mathcal{Q}_{i+1})$ we derive that the following equation is true modulo boundaries:

$$\kappa(c_1(\mathcal{Q}_i)c_1(\mathcal{Q}_{i+1})) = \frac{d_{i+1}}{2d_i}\kappa(c_1(\mathcal{Q}_i)^2) + \frac{d_i}{2d_{i+1}}\kappa(c_1(\mathcal{Q}_{i+1})^2). \tag{4.5}$$

Using (4.2), we easily arrive at the following relation:

$$\sum_i \kappa(c_2(\mathcal{K}_i)) + \sum_i \left(\frac{d_{i-1}+d_{i+1}}{2d_i} - 1\right)\kappa(c_1(\mathcal{Q}_i)^2) = 0 \text{ modulo boundaries.} \tag{4.6}$$

The coefficients of the boundary terms can be written down explicitly, but they

To summarize, we obtain the following upper bound for the dimension of the space of divisors on $\overline{M}_{0,n}(X,\beta)$:

$$[2^{n-1}(d_1+1)\dots(d_l+1)]^+ - 1 - \binom{n}{2} + h^4(X) - \binom{h^2(X)}{2}. \tag{4.7}$$

In section 2.4, we will use localization to give a lower bound for the dimension of the space of divisors. In fact, we will prove that the bound obtained above is sharp. This will give the proof of theorem 1 stated in the introduction.

4.2.2 Relations between the κ classes.

We will now indicate the statement and proof of the lemma invoked in the previous subsection to find relations between the κ classes. We hope this lemma could also be of use to understand divisors on product spaces. Similar results were obtained in [16] using completely different methods. We will indicate a third method later, which we discovered after we originally wrote this chapter.

Lemma 4.2.1. *Let L, M be two line bundles on a projective variety X. Then the following κ class on $\overline{M}_{0,0}(X,\beta)$:*

$$\kappa\left(\left(\frac{c_1(L)}{\int_\beta c_1(L)}-\frac{c_1(M)}{\int_\beta c_1(M)}\right)^2\right)$$

is in the span of the boundary divisors.

We claim that it is enough to prove the statement for L and M very ample. Indeed, assuming we proved the statement in this case, we pick a very ample divisor H on X. For n large enough, $L+nH, M+nH$ will both be very ample. We obtain that

$$\kappa\left(\left(\frac{c_1(L)+nH}{\int_\beta c_1(L)+nH\cdot\beta}-\frac{c_1(M)+nH}{\int_\beta c_1(M)+nH\cdot\beta}\right)^2\right)$$

is in the span of boundaries. Clearing denominators, and then looking at the term independent of n, we derive that the κ class in the statement of the lemma is also in the span of boundaries.

Assume now L and M are very ample. We consider the embedding $i : X \to \mathbb{P}^n \times \mathbb{P}^m$ determined by the linear systems $|L|$ and $|M|$. We let $d = \int_\beta c_1(L)$ and $e = \int_\beta c_1(M)$. We let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the two hyperplane bundles on the projective spaces $\mathbb{P}^n$ and $\mathbb{P}^m$. Let $D_{i,j}$ denote the boundary divisor of maps with nodal target such that the bidgree of the map on one of the components is (i, j); then the bidgree on the other component is $(d-i, e-j)$.

The lemma will follow pulling back under i the following relation on $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e))$:

$$\kappa\left(\left(\frac{c_1(\mathcal{H}_1)}{d}-\frac{c_1(\mathcal{H}_2)}{e}\right)^2\right)=\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{e}D_{i,j}\left(\frac{i}{d}-\frac{j}{e}\right)^2 \tag{4.8}$$

Claim 4.2.1. *As a first step in establishing* (4.8)*, we show that the codimension* 1 *classes on $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e))$ are in the span of the boundaries and of the two kappa classes $\kappa(c_1(\mathcal{H}_1)^2)$ and $\kappa(c_2(\mathcal{H}_2)^2)$.*

The boundary in $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e))$ is a divisor with normal crossings. It follows from the Deligne spectral sequence that the cokernel of the Gysin map

$$\oplus H^0(\text{boundaries}) \to H^2(\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e)))$$

can be identified with the weight 2 piece of the Hodge structure on the cohomology of the open stratum $W^2H^2(M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e)))$. We will show this is at most two dimensional.

We will write the open stratum $M = M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e))$ as a global quotient and make the computation in equivariant cohomology. Let V be a two dimensional space so that $\mathbb{P}^1 = \mathbb{P}(V)$ comes with the obvious $PGL(V)$ action. The space of maps Map $=$ $\text{Map}_{(d,e)}(\mathbb{P}^1, \mathbb{P}^n \times \mathbb{P}^m)$ of bidgree (d,e) is an open set in the product of two projective spaces

$$\mathbb{P}\left(\bigoplus_{i=0}^{n} Sym^d V^\star\right) \times \mathbb{P}\left(\bigoplus_{i=0}^{m} Sym^e V^\star\right).$$

We need to factor out the action of $PGL(V)$ on the two factors to obtain $M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e))$. Equivalently, we can think of $M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e))$ as sitting in a quotient of the affine space

$$\mathbb{A} = \bigoplus_{i=0}^{n} Sym^d V^\star \oplus \bigoplus_{i=0}^{m} Sym^e V^\star$$

by the action of the group $GL(V) \times \mathbb{C}^\star$. The action of $GL(V)$ is the usual one on the two factors, while $\mathbb{C}^\star$ acts in the usual way only on the second factor, and trivially on the first. This action is easily seen to have finite stabilizers. It is well known that in such cases we have an isomorphism between the cohomology of the orbit space and equivariant cohomology:

$$H^\star(M_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e))) = H^\star(\text{Map} \times_{GL_2 \times \mathbb{C}^\star} (EGL_2 \times E\mathbb{C}^\star)).$$

Both sides have Hodge structures (for equivariant cohomology, we need to use finite dimensional approximations of the equivariant models) compatible with the above isomorphism. Moreover, Map $\times_{GL_2 \times \mathbb{C}^\star} (EGL_2 \times E\mathbb{C}^\star)$ sits inside the space

$$\left(\bigoplus_{i=0}^{n} Sym^d V^\star \oplus \bigoplus_{i=0}^{m} Sym^e V^\star\right) \times_{GL_2 \times \mathbb{C}^\star} (EGL_2 \times E\mathbb{C}^\star).$$

This space is the total space of a bundle over the product of two classifying spaces $BGL_2 \times B\mathbb{C}^\star$. The restriction map

$$H^2(BGL_2 \times B\mathbb{C}^\star) = H^2_{GL_2 \times \mathbb{C}^\star}(\mathbb{A}) \to W^2H^2_{GL_2 \times \mathbb{C}^\star}(\text{Map}) = W^2H^2(M)$$

is surjective. Therefore $W^2H^2(M)$ is at most 2 dimensional. The surjectivity of the map can be explained by the usual arguments in the second chapter, using (Grothendieck's) remark 2.1.

Our claim follows if we show that the two classes $\kappa(c_1(\mathcal{H}_1)^2)$ and $\kappa(c_2(\mathcal{H}_2)^2)$ are not in the linear span of the boundary divisors. This is done in [51], Lemma 1.2.1(i) by intersecting with curves in the moduli space in the case of $\mathbb{P}^r$, but the argument goes through without change for $\mathbb{P}^n \times \mathbb{P}^m$.

The claim we just proved suffices to establish the results needed here. However for the sake of completeness, we will also prove the precise relation (4.8). It follows from what we proved above that a linear combination of the three classes:

$$\kappa(c_1(\mathcal{H}_1)c_1(\mathcal{H}_2)) + A \cdot \kappa(c_1(\mathcal{H}_1)^2) + B \cdot \kappa(c_1(\mathcal{H}_2)^2) = \text{sum of boundary classes} \tag{4.9}$$

To identify the coefficients of this relation it is enough to intersect (4.9) with curves in $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^m, (d,e))$. A moment's thought shows that it is enough to check (4.8) for all curves of $\overline{M}$ transversal to the boundary. Indeed, assuming this is the case, we show that (4.9) and an appropriately scaled version of (4.8) coincide. Subtracting the two equations, we get an expression involving only $\kappa(c_1(\mathcal{H}_1)^2)$, $\kappa(c_1(\mathcal{H}_2)^2)$ and boundary classes. This expression vanishes on each curve in $\overline{M}$ transversal to the boundary. We have seen already in the proof of the claim that this implies that the coefficients of the κ's must vanish. It is not any harder to conclude the same about the coefficients of the boundary classes. [1]

It remains to show (4.8) holds after intersecting with the smooth curves intersecting the boundary divisors transversally. Let us now consider such a curve. This is the same as a family of stable maps to $\mathbb{P}^n \times \mathbb{P}^m$ parametrized by a one dimensional base B:

$$\begin{tikzcd} S \arrow[r, "F=(f,g)"] \arrow[d, "\pi"'] & \mathbb{P}^n \times \mathbb{P}^m \\ B & \end{tikzcd}$$

It can be proved that S is the blow up of a projective bundle $P = \mathbb{P}(V)$ at the points $x_1, \ldots, x_s$ where B meets the boundary divisors.

We let E_i be the exceptional divisors of the blowups and we let $h = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$. We assume that the map F has bidegree (d_i, e_i) on each exceptional divisor E_i. It is then clear that for some line bundles $\mathcal{J}_1$ and $\mathcal{J}_2$ on B we have:

$$F^\star\mathcal{H}_1 = \pi^\star\mathcal{J}_1 \otimes \mathcal{O}_{\mathbb{P}(V)}(d) \otimes \mathcal{O}(-\sum_i d_i E_i) \tag{4.10}$$

$$F^\star\mathcal{H}_2 = \pi^\star\mathcal{J}_2 \otimes \mathcal{O}_{\mathbb{P}(V)}(e) \otimes \mathcal{O}(-\sum_i e_i E_i). \tag{4.11}$$

It is also obvious that $B \cdot D_{i,j} = n(i,j) + n(d-i, e-j)$ where $n(u,v)$ is the number of points among $x_1, \ldots, x_s$ such that the bidegree of the map F on the corresponding exceptional divisor is (u,v).

We now intersect both sides of the equation (4.8) with the curve B. We will need to show that

$$\begin{aligned} \pi_\star\left(\left(\frac{1}{d}c_1(F^\star\mathcal{H}_1) - \frac{1}{e}c_1(F^\star\mathcal{H}_2)\right)^2\right) &= \frac{1}{2}\sum_{i,j}(n(i,j) + n(d-i,e-j))\left(\frac{i}{d} - \frac{j}{e}\right)^2 \\ &= \sum_{i,j} n(i,j)\left(\frac{i}{d} - \frac{j}{e}\right)^2 = \sum_i \left(\frac{d_i}{d} - \frac{e_i}{e}\right)^2 \end{aligned}$$

The proof of this equality involves equations (4.10) and (4.11). Indeed, the right hand

[1]Recall that as a consequence of localization, numerical and rational equivalence coincide, essentially because the same is true for each of the fixed loci. One can perhaps conceive an argument which would establish (4.8) without appealing to the **claim** above, simply by intersecting with the smooth curves of $\overline{M}$. We actually do this in the proof below for curves transversal to the boundary. The general argument should not be more complicated.

side of the expression above equals

$$\pi_\star\left(\left(\frac{1}{d}(\pi^\star c_1(\mathcal{J}_1)+dh-\sum_i d_iE_i)-\frac{1}{e}(\pi^\star c_1(\mathcal{J}_2)+eh-\sum_i e_iE_i)\right)^2\right)$$

After a few cancellations, we finally arrive at the desired result. The proof of the lemma is complete.

4.2.3 The computation of the symmetric group invariants.

Our next digression will be useful in the dimension computation needed to finish the proof of theorem 1.5.1. We will prove a preliminary result about the S_n action on the cohomology of the moduli space of rational pointed curves $\overline{\mathcal{M}}_{0,n}$.

To fix notation, for each permutation $\sigma \in S_n$ we write $n_j(\sigma)$ for the number of cycles of length j. We denote by $c(\sigma)$ the total number of cycles of σ.

Lemma 4.2.2. *For each $\sigma \in S_n$, the trace of σ on $H^2(\overline{\mathcal{M}}_{0,n})$ is given by*

$$2^{c(\sigma)-1}-1-n_2(\sigma)-\binom{n_1(\sigma)}{2}+\delta(\sigma)$$

where

$$\delta(\sigma)=\begin{cases}2^{c(\sigma)-1} & \text{if } \sigma \text{ has only cycles of even length}\\ 0 & \text{otherwise}\end{cases}$$

The proof of this lemma makes use of the ideas of Getzler's paper ([28]). Getzler works out the Deligne spectral sequence of the mixed Hodge structure on the open manifold $M_{0,n}$. He shows that there is an exact sequence:

$$0 \to H^1(M_{0,n}) \to \oplus_\Gamma H^0(D_\Gamma) \to H^2(\overline{\mathcal{M}}_{0,n}) \to 0. \tag{4.12}$$

Here D_Γ are the boundary divisors of $\overline{\mathcal{M}}_{0,n}$. They correspond to unordered partitions of the n marked points into two subsets A and B such that $|A|,|B| \geq 2$.

It is clear that the the middle term of the exact sequence (4.12) is a sum of one dimensional spaces, one for each unordered partition of $\{1,\dots,n\}$ into 2 subsets as above. If n is odd, the trace of σ on the middle term of the exact sequence (4.12) equals the number of partitions $\{A,B\}$ such that

$$\sigma(A)=A,\ \sigma(B)=B,\ |A|,|B|\geq 2$$

This number is easily seen to be $2^{c(\sigma)-1}-1-n_1(\sigma)$. Indeed, both A and B have to be unions of full cycles of σ. The last two terms are the corrections corresponding to the non-stable cases when A or B have 0 or 1 elements. In the case when n is even, we also need to consider the partitions $\{A,B\}$ such that

$$\sigma(A)=B,\sigma(B)=A,n\geq 4$$

Such partitions exist only if σ has all cycles of even length and their number equals $2^{c(\sigma)-1}$.

The proof will be complete using the exact sequence (4.12), the remarks above and if we also show that:

$$\mathrm{Tr}_\sigma H^1(M_{0,n}) = n_2(\sigma) + \frac{n_1(\sigma)^2 - 3n_1(\sigma)}{2} \tag{4.13}$$

To establish (4.13), we will use the following facts collected from [28].

(a) First, Getzler shows that as a consequence of the Serre spectral sequence, we have an isomorphism $H^\bullet(M_{0,n+1}) = H^\bullet(M_{0,n} \times \mathbb{C} \setminus \{1, 2 \ldots, n\})$. It is then clear that

$$\mathrm{Tr}_\sigma H^1(M_{0,n+1}) = \mathrm{Tr}_\sigma H^1(M_{0,n}) + \mathrm{Tr}_\sigma H^1(\mathbb{C}\setminus\{1,2\ldots,n\}) = \mathrm{Tr}_\sigma H^1(M_{0,n}) + n_1(\sigma) - 1.$$

(b) Getzler also shows that if F_n denotes the configuration space of n pairwise distinct points in $\mathbb{C}$, then $H^\bullet(F_n) = H^\bullet(M_{0,n+1} \times S^1)$. Therefore,

$$\mathrm{Tr}_\sigma H^1(M_{0,n+1}) = \mathrm{Tr}_\sigma H^1(F_n) - 1.$$

(c) Finally, it is a consequence of a formula of Lehrer and Solomon, also discussed in [29] that $\mathrm{Tr}_\sigma H^1(F_n) = n_2(\sigma) + \binom{n_1(\sigma)}{2}$. These three items together prove equation (4.13), thus completing the proof of the lemma.

Lemma 4.2.3. *Let $n, a_1, \ldots a_l$ be positive integers. Consider the obvious action of $S_{a_1} \times \ldots \times S_{a_l}$ on $H^2(\overline{M}_{n+a_1+\ldots+a_l})$. The dimension of the invariant subspace is computed by the formula:*

$$dim\ H^2(\overline{M}_{n+a_1+\ldots+a_l})^{S_{a_1}\times\ldots\times S_{a_l}} = \left[2^{n-1}(a_1+1)\ldots(a_l+1)\right]^+ - 1 - \binom{n}{2} - ln - \binom{l+1}{2} + \mathfrak{a}.$$

Here $\mathfrak{a}$ denotes the number of indices i such that $a_i = 1$. We also write $[x]^+ = x$ if x is an integer and $[x]^+ = x + \frac{1}{2}$ if x is a half integer.

To prove this statement, we will average out the trace of each permutation $\sigma \in S_{a_1} \times \ldots \times S_{a_l}$ on $H^2(\overline{M}_{n+a_1+\ldots+a_l})$. For this combinatorial computation we will need the following identities which can be proved by induction on k.

$$\sum_{\sigma\in S_k} 2^{c(\sigma)} = (k+1)! \tag{4.14}$$

$$\sum_{\sigma\in S_k} \delta(\sigma) = \begin{cases} \frac{k!}{2} & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

$$\sum_{\sigma\in S_k} n_1(\sigma) = k! \tag{4.16}$$

$$\sum_{\sigma\in S_k} \left(n_2(\sigma) + \binom{n_1(\sigma)}{2}\right) = k! \text{ for } k \geq 2. \tag{4.17}$$

Another induction, this time on l, making use of equation (4.16) gives the following two identities:

$$\sum_{\sigma\in S_{a_1}\times\ldots\times S_{a_l}} n_1(\sigma) = la_1!\ldots a_l! \tag{4.18}$$

$$\sum_{1\le i<j\le l}\sum_{\sigma_i\in S_{a_i}}\sum_{\sigma_j\in S_{a_j}} n_1(\sigma_i)n_1(\sigma_j) = \binom{l}{2} a_1!\dots a_l! \tag{4.19}$$

We can now compute the dimension of the invariant subspace. A permutation $\sigma \in S_{a_1} \times \ldots \times S_{a_l}$ is tantamount to l permutations $\sigma_i \in S_{a_i}$. Lemma 4.2.2 shows that

$$\mathrm{Tr}_\sigma H^2(\overline{M}_{n+a_1+\ldots+a_l}) = 2^{\sum_i c(\sigma_i)+n-1} - 1 - \sum_{i=1}^{l} n_2(\sigma_i) - \binom{n+n_1(\sigma_1)+\ldots+n_1(\sigma_l)}{2}$$
$$+\begin{cases} 2^{\sum_i c(\sigma_i)-1} & \text{if } n=0 \text{ and the } \sigma_i\text{'s have only even length cycles} \\ 0 & \text{otherwise} \end{cases}.$$

We average out these traces to compute the dimension of the invariant subspace. First, we assume $n \neq 0$.

$$\dim H^2(\overline{M}_{n+a_1+\ldots+a_l})^{S_{a_1}\times\ldots\times S_{a_l}} = \frac{1}{a_1!\dots a_l!}\sum_{\sigma\in S_{a_1}\times\ldots\times S_{a_l}} \mathrm{Tr}_\sigma H^2(\overline{M}_{n+a_1+\ldots+a_l}) = \tag{4.20}$$

$$= \frac{1}{a_1!\dots a_l!}\left(\sum_{i=1}^{l}\sum_{\sigma_i\in S_{a_i}} 2^{\sum_i c(\sigma_i)+n-1} - 1 - \sum_{i=1}^{l} n_2(\sigma_i) - \binom{n+n_1(\sigma_1)+\ldots+n_1(\sigma_l)}{2}\right)$$
$$= 2^{n-1}\prod_{i=1}^{l}\left(\sum_{\sigma_i\in S_{a_i}}\frac{2^{c(\sigma_i)}}{a_i!}\right) - 1 - \sum_{i=1}^{l}\frac{1}{a_i!}\left(n_2(\sigma_i)+\binom{n_1(\sigma_i)}{2}\right) - \binom{n}{2} -$$
$$-\frac{n}{a_1!\dots a_l!}\sum_{i=1}^{l}\sum_{\sigma_i\in S_{a_i}} n_1(\sigma_i) - \frac{1}{a_1!\dots a_l!}\sum_{1\le i<j\le l} n_1(\sigma_i)n_1(\sigma_j) =$$
$$= 2^{n-1}(a_1+1)\dots(a_l+1) - 1 - (l-\mathfrak{a}) - \binom{n}{2} - nl - \binom{l}{2}.$$

where in the last line we used equations (4.14), (4.17), (4.18) and (4.19) respectively.

In the case when $n = 0$, we have extra-terms corresponding to the case when all permutations σ_i have even cycles. The contribution of these terms is

$$\frac{1}{a_1!\dots a_l!}\sum_{\sigma_i\in S_{a_i}\text{with only even cycles}} 2^{c(\sigma_1)+\ldots+c(\sigma_l)-1} = \frac{1}{2}\prod_i\left(\sum_{\sigma_i\in S_{a_i}}\frac{2}{a_i!}\delta(\sigma_i)\right) = \frac{1}{2},$$

by virtue of equation (4.15). This completes the proof of the lemma.

4.2.4 The fixed loci contributions.

We will exhibit particular fixed point loci for the $\mathbb{C}^\star$ action on $\overline{M}_{0,n}(X,\beta)$ which comes from a torus action on X. We will argue that the normal bundles of the fixed loci we exhibit have at most 1 negative weight. We will employ the "homology basis theorem" to give a lower bound for the dimension of the space of codimension 1 classes (according to lemma 3.2.3).

The lower bound will match the upper bound (4.7), completing the proof of theorem 1.5.1. Therefore, the argument will also give **all** fixed loci which contribute to the computation of $H^2(\overline{M}_{0,n}(X,\beta))$.

We consider an action of $\mathbb{C}^\star$ on V with weights $\lambda_1 < \ldots < \lambda_N$ and weight vectors $e_1, \ldots, e_N$. In fact we will need assume that the weight λ_i is much bigger than λ_{i-1}. This assumption will be needed when evaluating the negative weights for the normal bundles to the fixed loci below, especially when dealing with the vertices of valency 2. We write our flag variety as a quotient $X = SL(V)/P$ where P is the parabolic of upper triangular block matrices of size m_1, $m_2 - m_1$, ..., $m_l - m_{l-1}$, $N - m_l$. We will frequently use the notation $W(i)$ for the vector subspace of V spanned by $e_1, \ldots, e_i$. We enumerate the following fixed points of the $\mathbb{C}^\star$ action on X:

(a) the origin P. The tangent space T_PX has no negative $\mathbb{C}^\star$ weights. Explicitly, this fixed point is represented by the flag:

$$W(m_1) \subset \ldots \subset W(m_l) \subset V.$$

(b) There are $h^2(X)$ fixed points $q_1, \ldots, q_l$ corresponding to the simple roots not in the parabolic subalgebra. There is only one negative weight on the tangent spaces $T_{q_i}X$. Each q_i can be joined to P by a rational curve R_i whose Poincare dual is the generator β_i of $H^2(X, \mathbb{Z})$. Explicitly q_i is represented by a flag which differs from the one in the previous item only at the i th step:

$$W(m_1) \subset \ldots \subset W(m_i - 1) \oplus \text{ span } \langle e_{m_i+1} \rangle \subset W(m_{i+1}) \subset \ldots \subset W(m_l) \subset V.$$

Additionally, the curve R_i can be parametrized as:

$$[t : s] \mapsto \text{ the flag } W(m_1) \subset \ldots \subset W(m_i - 1) \oplus \langle te_{m_i} + se_{m_i+1} \rangle \subset W(m_{i+1}) \subset \ldots \subset W(m_l) \tag{4.21}$$

(c) There are $h^4(X)$ fixed points with 2 negative roots on their tangent spaces. There are three types of such points which we now describe. Let $m_0 = 0$ and $m_{l+1} = n$.

- for each pair $1 \leq i < j \leq l$ such that $m_j - m_i \geq 2$ we have a fixed point denoted q_{ij}. It is obtained from the reference flag in (a) by modifying its ith and jth steps:

  $$W(m_1) \subset \ldots \subset W(m_i - 1) \oplus \langle e_{m_i+1} \rangle \subset \ldots \subset W(m_j - 1) \oplus \langle e_{m_j+1} \rangle \subset \ldots \subset W(m_l) \subset V$$

  It is clear that q_{ij} can be joined to both q_i and q_j by rational curves in the cohomology classes dual to β_j and β_i.

- For each $1 \leq i \leq l$ such that $m_{i+1} - m_i \geq 2$ we obtain a fixed point r_i which has the property that it can be joined to P by a rational curve in the cohomology class dual to β_i. The r_i's can be obtained from the reference flag in (a) by modifying its ith step :

  $$W(m_1) \subset \ldots \subset W(m_i - 1) \oplus \langle e_{m_i+2} \rangle \subset W(m_{i+1}) \subset \ldots \subset W(m_l) \subset V.$$

  In addition, for all $1 \leq i \leq l$ such that $m_i - m_{i-1} \geq 2$, we get the fixed points r_i' which can be joined to the origin by a curve in the cohomology class dual to β_i. They can be defined modifying the ith step of the reference flag in (a):

  $$W(m_1) \subset \ldots \subset W(m_i - 2) \oplus \langle e_{m_i}, e_{m_i+1} \rangle \subset W(m_{i+1}) \subset \ldots W(m_l) \subset V.$$

- For each $m_{i+1} - m_i = 1$, $1 \leq i < l$, we have a fixed point s_i which can be joined to q_{i+1} by a rational curve in the cohomology class dual to β_i. Explicitly it is given by changing the ith and $(i+1)$st steps of the standard flag:

$$W(m_1) \subset \ldots \subset W(m_i - 1) \oplus \langle e_{m_i+2} \rangle \subset W(m_i) \oplus \langle e_{m_i+2} \rangle \subset \ldots W(m_l) \subset V.$$

  Similarly, for all $1 < l \leq l$ such that $m_i - m_{i-1} = 1$, we obtain the fixed point s'_i which can be joined to q_{i-1} by a rational curve in the cohomology class dual to β_i. Explicitly, this point is obtained by modifying the $(i-1)$st and ith steps of the reference flag

$$W(m_1) \subset \ldots \subset W(m_i - 2) \oplus \langle e_{m_i} \rangle \subset W(e_{m_i-2}) \oplus \langle e_{m_i}, e_{m_i+1} \rangle \subset \ldots \subset W(m_l) \subset V.$$

Turning to the fixed loci on $\overline{M}_{0,n}(X, \beta)$, we will employ the usual method of bookkeeping the fixed loci by means of decorated graphs Γ. The vertices of Γ are in one to one correspondence with the components of $f^{-1}(\mathbb{C}^\star$ fixed points on $X)$. The vertices come with labels corresponding to the $\mathbb{C}^\star$ fixed points on X. The edges of the graph correspond to non-contracted components of the map f and are decorated with the degree on that component. The graph Γ has legs attached to its vertices. A flag $\mathfrak{f}$ determines an edge $e(\mathfrak{f})$ and thus a non contracted component C_e of the stable map. We let R_e be the image of this component. The flag $\mathfrak{f}$ also determines a unique vertex $v(\mathfrak{f})$ which gives a point on the curve C_e mapping to a fixed point of the $\mathbb{C}^\star$ action on X. We let $\omega_\mathfrak{f}$ denote the $\mathbb{C}^\star$ weight on the fiber of the bundle $f^\star TR_e$ at the point $v(\mathfrak{f})$.

The weights on the normal bundles of each fixed locus were computed by Kontsevich-Graber-Pandharipande (see [32]). The list of weights on the normal bundle of the fixed locus indexed by Γ is generated by the following algorithm:

- flag contributions: for each flag $\mathfrak{f}$ whose vertex $v(\mathfrak{f})$ has total valency ≥ 3 we include the weight $\omega_\mathfrak{f}$ on our list of weights.

- vertex contributions: for each vertex v corresponding to a fixed point q of the $\mathbb{C}^\star$ action on X, we include the $\mathbb{C}^\star$ weights on T_qX.

- vertex contributions: the vertices v with valency 2 and no legs have two incident flags $\mathfrak{f}_1$ and $\mathfrak{f}_2$. We include the weight $\omega_{\mathfrak{f}_1} + \omega_{\mathfrak{f}_2}$.

- edge contributions: for each edge e, we include the weights of the $\mathbb{C}^\star$ action on $H^0(C_e, f^\star TX)$.

- flag contributions: for each flag $\mathfrak{f}$, the vertex $v(\mathfrak{f})$ maps to a fixed point q of the $\mathbb{C}^\star$ action on X. We remove the weights on T_qX from the list.

- vertex contributions: for each vertex v with valency 1 and no legs, i.e. those vertices contained in only one flag $\mathfrak{f}$, we remove the weight $\omega_\mathfrak{f}$ from the list.

The graph in figure 4.2.4 corresponds to a fixed locus with no negative weights on its normal bundle. Its *plus* cell is the big locus of the $\mathbb{C}^\star$ flow on $\overline{M}_{0,n}(X, \beta)$. The corresponding fixed locus is isomorphic to the quotient $\overline{M}_{0,n+\sum_i d_i}/S_{d_1} \times \ldots \times S_{d_l}$. Its contribution to $H^2(\overline{M}_{0,n}(X, \beta))$, as determined by the "homology basis theorem" (lemma 3.2.3) in the previous chapter, equals the second Betti number. By lemma 4.2.3 this contribution is:

$$[2^{n-1}(d_1+1)\ldots(d_l+1)]^+ - 1 - ln - \binom{l+1}{2} - \binom{n}{2} + \mathfrak{a},$$

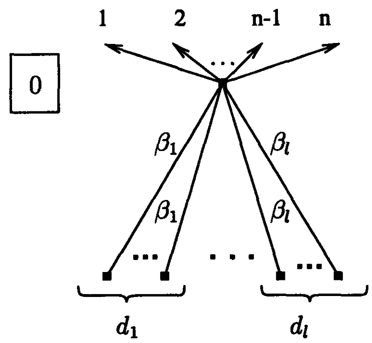


Figure 4-1: The big locus of the $\mathbb{C}^\star$ flow on $\overline{M}_{0,n}(X,\beta)$

where $\mathfrak{a}$ is the number of indices such that $d_j = 1$.

There are (at least) five types of graphs which correspond to fixed loci with only one negative weight on their tangent bundle. To compute their contribution to $H^2(\overline{M}_{0,n}(X,\beta))$, we only have to count all such graphs.

We will analyze each of the five types one by one. In drawing these graphs, we used continuous lines for the edges representing the rational curves R_i in the cohomology class dual to β_i, which join the origin P to the fixed points q_i. We also indicated just below the graph the number of such curves that we use.

The graphs of type A have one leg labeled i attached to a vertex labeled by q_j. There are $n \cdot l$ such graphs. The graphs of type B have one thicker edge labeled $2\beta_j$. This edge corresponds to components mapping to R_j with degree 2 with ramification only over P and q_j. These graphs only exist if $d_j \geq 2$, and their number equals $l - \mathfrak{a}$. For the graphs of type C, one of the edges corresponding to the rational curve R_j is replaced by a rational curve, still in the homology class β_j which joins the origin to r_j or r'_j. This new edge is represented by a dotted line. For each $1 \leq i < j \leq l$ we obtain a graph of type D. An edge representing the curve R_j has been replaced by a rational curves with cohomology class β_j joining q_i to q_{ij}. The new edge is attached to one of the edges representing the curve R_i. We used dotted lines for the two rational curves in question. Notice the apparent asymmetry between i, j. Indeed, the corresponding graph obtained by switching i and j has one more negative weight. This comes from the contribution of the vertex of valency 2. In our case, that vertex contributes with positive weight as one immediately checks remembering that the weight λ_{j+1} is the dominant one among the weights which appear on the rational curves coming into the valency 2 vertex in question. Finally, the graphs of type E are described in the same manner. We have replaced two rational curves in the cohomology classes β_i and β_{i+1} by two rational curves with the same cohomology classes, which are graphically represented by dotted lines. There are $h^4(X)$ such graphs of type C, D and E.

Adding up all these contributions from the graphs above, we find a lower bound for the dimension of $H^2(\overline{M}_{0,n}(X,\beta))$. As promised, this coincides with the number given by equation (4.7). This completes the proof of theorem 1.5.1.

It remains to explain why the five types of fixed loci listed above have exactly one negative weight on their normal bundles. Needless to say, the computation involves the Konsevich-Graber-Pandharipande recipe for computing the weights. This is a straightforward argument for the most part. There are two ingredients which are important to the count of the negative weights.

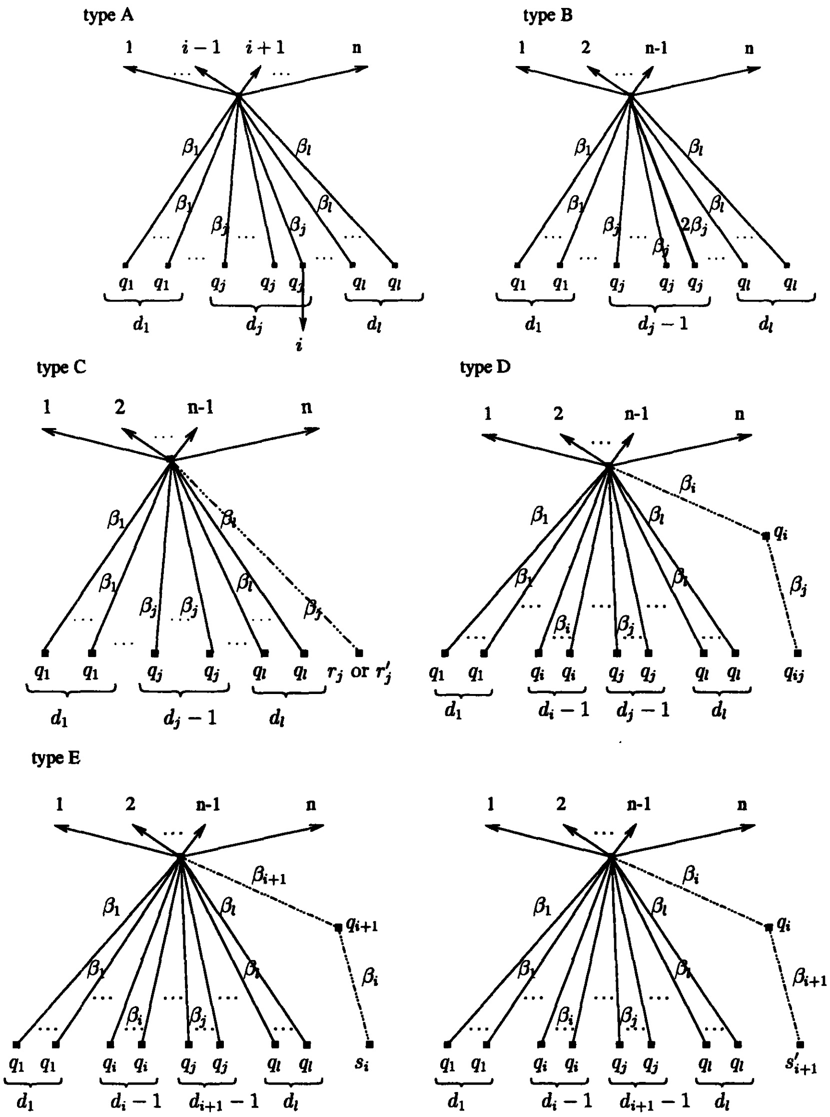


Figure 4-2: Fixed loci with one negative weight on the normal bundle.

(a) For each curve $R = R_i$, the tangent space $T_P R$ at the fixed point P has one positive weight, as all weights on $T_P X$ are positive. Therefore the tangent space $T_{q_i} R$ has the opposite/negative weight. When R is the curve joining q_i to q_{ij}, the tangent space at q_i has one positive weight: there should be only one negative weight on $T_{q_i} X$ which we've seen occurs along the curve R_i. Therefore the tangent space of R at q_{ij} has the opposite/negative weight. Moreover, an explicit computation of the weights shows that each vertex of valency 2 in the graphs of type D, E contributes with the positive weight $\omega_{\mathfrak{f}_1} + \omega_{\mathfrak{f}_2}$.

(b) For each rational curve $R = R_j$ joining P to q_j and each map $f : \mathbb{P}^1 \to R$ of degree d which is totally ramified over P and q_j, the number of negative weights on $H^0(\mathbb{P}^1, f^\star TX)$ equals d. The case in hand can be checked rather easily recalling the following "Euler" sequence on X:

$$0 \to TX \to \bigoplus_i Hom(\mathcal{S}_i, \mathcal{Q}_i) \to \bigoplus_i Hom(\mathcal{S}_i, \mathcal{Q}_{i+1}) \to 0. \tag{4.22}$$

It is enough to compute the weights on the virtual $\mathbb{C}^\star$ representation

$$\bigoplus_i H^0(f^\star(\mathcal{S}_i^\star \otimes \mathcal{Q}_i)) - \bigoplus_i H^0(f^\star(\mathcal{S}_i^\star \otimes \mathcal{Q}_{i+1})).$$

We claim that when $i \neq j$, there are no negative weights on $H^0(f^\star(\mathcal{S}_i^\star \otimes \mathcal{Q}_i))$. Indeed, we observe that for $i \neq j$, we have the equivariant isomorphism

$$f^\star \mathcal{S}_i = W(m_i) \otimes \mathcal{O}_{\mathbb{P}}$$

so the weights on $H^0(f^\star(\mathcal{S}_i^\star \otimes \mathcal{Q}_i))$ are the positive numbers $\lambda_u - \lambda_v$ for $v \leq m_i < u$.

We claim d negative weights on $H^0(f^\star(\mathcal{S}_j^\star \otimes \mathcal{Q}_j))$. Equation (4.21) shows that we have an equivariant identification:

$$\mathcal{S}_j|_{R_j} = W(m_j - 1) \otimes \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-1). \tag{4.23}$$

In the above, the $\mathbb{C}^\star$ action on $\mathcal{O}_{\mathbb{P}}(-1)$ has the weight λ_{m_j} at $[1:0]$ and the weight λ_{m_j+1} at $[0:1]$.

The exact sequence (4.1) tensored with $f^\star\mathcal{S}_j^\star$ shows that $H^0(f^\star\mathcal{S}_j^\star \otimes f^\star\mathcal{Q}_j)$ is equivalent to the $\mathbb{C}^\star$ virtual representation

$$V \otimes H^0(f^\star\mathcal{S}_j^\star) - H^0(f^\star\mathcal{S}_j^\star \otimes f^\star\mathcal{S}_j) + H^1(f^\star\mathcal{S}_j^\star \otimes f^\star\mathcal{S}_j). \tag{4.24}$$

We can now consider each of the three terms in (4.24) individually. Using (4.23), we rewrite the first term as

$$V \otimes W(m_j - 1)^\star \oplus V \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(d)).$$

The first summand has weights $\lambda_u - \lambda_v$ for $v \leq m_j - 1$ and all u. In addition, we also have the weights $\lambda_u - \frac{1}{d}(a\lambda_{m_j} + b\lambda_{m_j+1})$ for all nonnegative a, b such that $a + b = d$. The second term in (4.24) can be rewritten as

$$W(m_j-1)^\star \otimes W(m_j - 1) \oplus H^0(\mathcal{O}_{\mathbb{P}}(d)) \otimes W(m_j - 1) \oplus \mathbb{C}$$

with a trivial action on the last term. This has exactly the same non-zero weights as the first term in (4.24), except that we need to require that $u \leq m_j - 1$. Finally, the third term in (4.24) equals

$$W(m_j - 1)^\star \otimes H^1(\mathcal{O}_{\mathbb{P}}(-d))$$

with positive weights $-\lambda_u + \frac{a\lambda_{m_j} + b\lambda_{m_j+1}}{d}$ for all $u \leq m_j - 1$ and a, b are positive integers summing up to d. Summarizing, we find that the only negative weights in the list above are the d values $\frac{a}{d}(\lambda_{m_j} - \lambda_{m_j+1})$ for $1 \leq a \leq d$.

In the same way we verify that there are no negative weights on $H^0(f^\star(\mathcal{S}_i^\star \otimes \mathcal{Q}_{i+1}))$. We leave the details to the reader since they are similar to the computations above.

Our initial claim is now proved.

Similarly, one shows that the number of negative weights on $H^0(R, TX)$ is exactly 2 whenever R is a rational curve in the cohomology class dual to β_j of one of the following types:

- the rational curve joining P to one of the points r_j or r'_j.
- the rational curve joining q_i to q_{ij}.
- the rational curve joining q_{j+1} to s_j or the curve joining q_{j-1} to s'_j.

Remark 4.2.1. The author believes that the arguments of the last few subsections can be repeated for $X = G/P$. The proofs should not be any more difficult. The count of negative $\mathbb{C}^\star$ weights on various cohomology groups can be done in terms of the roots of $\mathfrak{p}$. Moreover, the count of the negative weights on the various tangent spaces follows from standard computations. We leave these arguments to the interested reader.

Proposition 4.2.2. *All relations between the codimension* 1 *tautological classes* $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ *on* $\overline{\mathcal{M}}_{0,n}(X, \beta)$ *are tautological.*

The discussion in section 4.2.1 essentially established this claim. We have seen that all relations between the boundary divisors are Keel relations. It suffices to show all relations between the κ's and evaluation classes are tautological. When $n \geq 3$, it was observed already in the proof of proposition 4.2.1 that the evaluation classes can be expressed in terms of the $\kappa's$ via the tautological equations (see (4.3)). Similarly, (4.6) is tautological being a linear combination of a pullback relation and (4.5), which we will show to be tautological. It remains to make the following two observations.

- When $n = 1$ and $n = 2$, the fact that the span of $ev_i^\star \mathcal{Q}_j$ is one dimensional modulo boundaries and κ's also follows from the tautological equations. This is not entirely clear from the discussion above since in equation (4.4) we made use of the ψ classes. We need to prove that the tautological equations express any evaluation class $ev_i^\star H'$ in terms of a fixed one, say $ev_i^\star H$. Here H, H' are two divisors, such as the Chern classes of the quotient bundles $\mathcal{Q}_i$ and $\mathcal{Q}_j$ on X. The tautological equation (4.3) reduces our analysis to one marking $i = 1$.

  We produce a tautological equation which expressed $ev_1^\star H'$ in terms of κ classes and the evaluation $ev_1^\star H$. Such an equation is not so easy to come by, and in fact, we observed this fact only after proving lemma 4.2.1. The reason is the fact that our relation is an incarnation of a Keel relation in codimension 5 on the space $\overline{\mathcal{M}}_{0,5}(X, \beta)$. The legs are split as (15)(23) and (12)(53) among two vertices, and the weights assigned to the legs are $(1, H', H, H', H)$ respectively. After forgetting the first 4 markings, using the divisor equation, the no incidence equation, the unstable tripod equation, we obtain the following tautological relation:

$$\frac{1}{\beta \cdot H} ev_1^\star H - \frac{1}{\beta \cdot H'} ev_1^\star H' = \frac{1}{(\beta \cdot H)^2} \kappa(H^2) - \frac{1}{(\beta \cdot H)(\beta \cdot H')} \kappa(HH') + \text{boundaries}.$$

- Equation (4.5) is tautological. This can be seen by adding the above equations when H, H' are interchanged. As a consequence, we obtain the desired equation (4.5):

$$\frac{1}{(\beta \cdot H)^2} \kappa(H^2) + \frac{1}{(\beta \cdot H')^2} \kappa(H'^2) = 2 \frac{1}{(\beta \cdot H)(\beta \cdot H')} \kappa(HH') + \text{ boundaries }.$$

This completes the proof of the proposition.

4.3 The codimension 2 classes.

In this section, we will prove the remaining part of theorem 3. We seek to show:

Proposition 4.3.1. *All relations between the codimension* 2 *tautological generators* $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ *on* $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ *are tautological.*

To begin with, we consider the case of maps without markings $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. There are several cases to consider: $r = 1$, $r = 2$ and $r \geq 3$. We will consider $X = \mathbb{P}^1$ first. In the previous chapters, we saw that the whole cohomology of the moduli space of stable maps to $\mathbb{P}^1$ is generated by the tautological classes. The search for the codimension 2 tautological classes gives the following results:

1. boundary classes of nodal maps with three irreducible components,
2. classes of nodal stable maps whose node maps to a fixed point in $\mathbb{P}^1$.

It turns out the classes in (1) and (2) are in fact linearly independent. To see this it is enough to match their number with the dimension of $H^4(\overline{M}_{0,0}(\mathbb{P}^1, d))$. This dimension count can be done in several ways - either using Deligne's spectral sequence or localization.

Let us briefly indicate how the localization argument works, then turning to the computation by Deligne's spectral sequence. We will use a $\mathbb{C}^\star$ action on $\mathbb{P}^1$ with two fixed points which we call $0, 1$ such that the tangent bundle at 0 has a positive weight, while the tangent bundle at 1 has a negative weight. As before, we index the fixed point loci on $\overline{M}_{0,0}(\mathbb{P}^1, d)$ by decorated graphs Γ.

We count the negative weighs on the normal bundles of the fixed loci. More precisely, we let $\mathfrak{s}$ denote the number of vertices of Γ labeled by 1 with at least three incident flags and let $\mathfrak{u}$ denote the number of vertices of Γ labeled by 1 with one incident flag. The number of negative weights on the normal bundle of the fixed locus indexed by Γ is $d - \mathfrak{u} + \mathfrak{s}$. An easy argument shows that there are 4 types of graphs indexing fixed loci with at most 2 negative weights on their normal bundle. These graphs are shown in the figure 4.3. The number of negative weights is indicated in a box to the left of each graph and the degrees of the edges are written below the graphs.

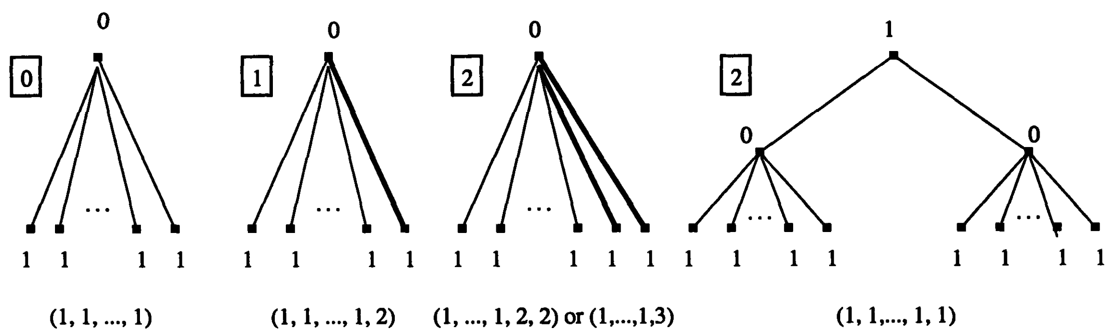


Figure 4-3: The fixed loci for the $\mathbb{C}^\star$ action on $\overline{M}_{0,0}(\mathbb{P}^1, d)$.

The first graph gives the fixed locus $\overline{M}_{0,d}/S_d$ with no negative weights on the normal bundle, while the second graph gives the fixed locus $\overline{M}_{0,d-1}/S_{d-2}$ with one negative weight. There are $2+[d/2]$ graphs with 2 negative weights. We find that the dimension of H^4 equals

$$h^4(\overline{M}_{0,d})^{S_d} + h^2(\overline{M}_{0,d-1})^{S_{d-2}} + 2 + [d/2] = h^4(\overline{M}_{0,d})^{S_d} + d - 2 + [d/2]$$

where we also used lemma 4.2.3 to evaluate the second term of the sum.

It remains to compute the first term. We already know that the generators of $H^4(\overline{M}_{0,d})^{S_d}$ are the boundary classes of curves with at least 3 irreducible components. These boundary classes $\mathcal{B}_{ijl}$ are indexed by triples (i,j,l) such that $i+j+l=d,\ j\geq 1,\ 2\leq i\leq l$, these integers corresponding to the number of marked points on each component.

Claim 4.3.1. *The classes $\mathcal{B}_{ijl}$ form a basis of $H^4(\overline{M}_{0,d})^{S_d}$.*

We show these classes are linearly independent by matching their number with the actual dimension of $H^4(\overline{M}_{0,d})^{S_d}$. The dimension computation is identical to that in lemma 4.2.3 by averaging out the traces of all $\sigma\in S_d$ on $H^4(\overline{M}_{0,d})$. We will omit the details. An alternate proof will be established by the arguments below. This proof has the advantage of extending to arbitrary markings.

We will now redo the same computation making use of Deligne's spectral sequence. There are two cases to consider depending on the parity of d. Let us show the details for $d=2k$. An identical argument also works for odd d's - the numerical details are slightly different. We make use of the fact that the boundary divisors in the space of stable maps have normal crossings. However, these boundary divisors have self-intersections and writing down the Deligne spectral sequence with the right system of coefficients is delicate, as we have to account for automorphisms and self-intersections. This is essentially done in chapter 2 in a similar context.

In our case, the k boundary divisors $D_1,\ldots D_k$ correspond to nodal maps whose degrees on the components are $(1,2k-1),\ \ldots(k,k)$. The codimension two strata are denoted by D_{ijl} for $1\leq i\leq l$ and $1\leq j$ with $i+j+l=d$; they correspond to stable maps with three components, the degree on the middle component being j. For $1\leq i<k$, the stratum $D_{i,2k-2i,i}$ is the self intersection of $D_{i,2k-i}$; there are no anti invariant classes in its zeroth cohomology group because of the $\mathbb{Z}/2\mathbb{Z}$ symmetry which switches the edges. Therefore, this term does not appear in the Deligne spectral sequence. We obtain the following complex:

$$\bigoplus_{1\leq i<l,1\leq j,i+j+l=d} H^0(D_{ijl}) \to \bigoplus_{i=1}^{k-1} H^2(D_i)\oplus H^2(D_k)^- \to H^4(\overline{M}_{0,0}(\mathbb{P}^1,d)). \tag{4.25}$$

Here the minus superscript on $H^2(D_k)^-$ stands for the anti invariant part of the cohomology under the sign representation of $Aut(\Gamma)$. Recall that each automorphism has a sign, given by the action it induces on the determinant det(Edge (Γ)). The minus sign refers to cohomology classes which are invariant under this sign representation. Since the edge of the dual graph indexing D_k is preserved under the $\mathbb{Z}/2\mathbb{Z}$ symmetry of the graph, $H^2(D_k)^-$ is the same as the $\mathbb{Z}/2\mathbb{Z}$ invariant part of $H^2(\overline{M}_{0,1}(\mathbb{P}^1,k)\times_{\mathbb{P}^1}\overline{M}_{0,1}(\mathbb{P}^1,k))$. We use the computation of lemma 4.2.3 to conclude that the dimension of the middle term of the complex (4.25) is $2k^2-2k+1$. The first term is easily seen to be k^2-2k+1 dimensional.

We claim the dimension of H^4 is k^2, which also turns out to be the number of generators we have exhibited for H^4. It suffices to show that the alternating sum of dimensions of the terms in the complex (4.25) is 0. This alternating sum equals the coefficient of $q^{2\dim -4}$ in the virtual Poincare polynomial of $M = M_{0,0}(\mathbb{P}^1, d)$. By definition, the Poincare polynomial can be computed from the associated graded of the Hodge weight filtration:

$$P(M) = \sum_{i,j} (-1)^{i+j} \dim Gr_j^W(H_c^i(M)) q^j \tag{4.26}$$

In [46], it is proved that $P(M) = q^{4d-4}$. The claim follows from these observations.

The argument above is no more complicated when we deal with general projective spaces. We will pick a $\mathbb{C}^*$ action with weights $\lambda_0, \ldots, \ldots, \lambda_r$ such that λ_i is much bigger than λ_{i-1}. We denote by $0, 1, \ldots, r$ the isolated fixed points of this action. To find the dimension of H^4 we use localization. In addition to the fixed loci listed above for $\mathbb{P}^1$ we have 4 more types of graphs when $r = 2$ and one additional graph for $r = 3$. There are no new graphs added for $r > 3$. This is an aspect of the "stabilization of cohomology" theorem proved in [3]. Our computation shows that when $r = 2$ we gain d dimensions (we will need to invoke lemma 4.2.3 again to compute the contribution of the first graph). For $r \geq 3$ we gain $d + 1$ dimensions.

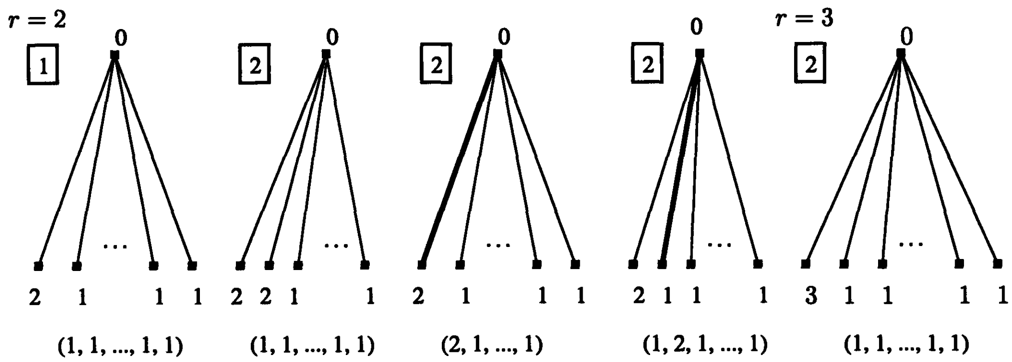


Figure 4-4: The fixed loci for the $\mathbb{C}^*$ action on $\overline{M}_{0,0}(\mathbb{P}^r, d)$.

We find generators for H^4 by enumerating all tautological classes and matching their number with the dimension we just computed. We obtain the following classes:

(A.1) the boundary classes of maps whose domain has at least three components. These correspond to graphs with three vertices and no legs, no weights, and no forgetting data.

(A.2.1) the nodal classes of maps whose node is mapped to a codimension 1 subspace; these classes correspond to graphs with two vertices, and the weight of the edge is H.

(A.2.2) (when $r \geq 2$) classes of nodal maps, one component passing through a fixed codimension 2 subspace; these correspond to graphs with two vertices, and a forgotten leg with weight H^2.

(A.3.1) (when $r \geq 2$) classes of maps whose images pass through two general codimension two subspaces; these correspond to graphs with one vertex with two forgotten legs with weight H^2.

(A.3.2) (when $r \geq 3$) the class of maps intersecting a codimension 3 subspace; these correspond to graphs with one vertex and one forgotten leg with weight H^3.

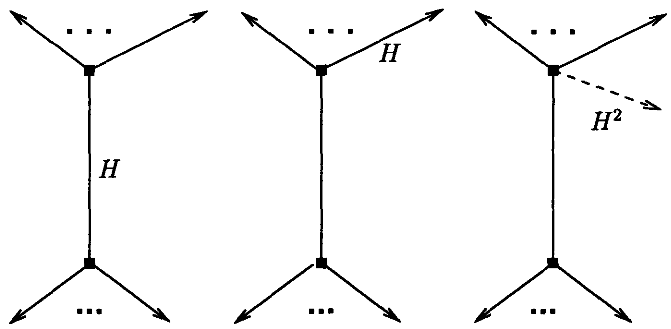


Figure 4-5: The codimension 2 tautological generators for graphs with two vertices.

Lemma 4.3.1. *The collections of codimension* 2 *classes on* $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ *form a basis for the codimension* 2 *Chow group.*

We will omit the case when $n = 1$ and $n = 2$. The essential part of the proof of proposition 4.3.1 consists in checking the case $n \geq 3$. We will do so only in the case $r \geq 3$.

(i) First, the proposition 2.3.1 expresses all generators $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ where Γ has only one vertex of degree d as a sum of $ev_1^\star H^2$ with boundary classes via the tautological equations. Therefore, we retain only one generator of this form.

(ii) The codimension 2 classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ for which Γ has two vertices are obtained when:

   (a) an edge of Γ is decorated with the weight H, there are no other weights or forgotten legs;

   (b) a leg of Γ is decorated with the weight H, there are no other weights or forgotten legs;

   (c) a forgotten leg of Γ is decorated with the weight H^2, there are no additional weights or forgotten legs.

   Moreover, the tautological equations (2.18) and (2.21) show (via gluing) that we only need to consider the classes in (a) and the classes in (c) when the forgotten leg is incident to a vertex with at most 2 unforgotten legs. A simple count count gives

$$2^n + (d-1)(2^{n-1} + n + 1) \tag{4.27}$$

   generators as above. In addition, there are

$$\frac{n(n-3)}{2} \tag{4.28}$$

   independent Keel relations between them.

(iii) Finally, we consider the classes for which Γ has two vertices. We will first discuss the case $d = 1$. An easy count gives

$$\frac{3^{n+1}+3}{2} + \frac{n(n+3)}{2} - 2^n(n+3) \tag{4.29}$$

boundary terms. We will exhibit

$$2^{n-2}(n^2-n-8)+2-\frac{n(3n^3-10n^2+21n-86)}{24} \tag{4.30}$$

independent relations between them.

Indeed, we think of the graph Γ as a vertex v with $k+1$ legs (one of them is distinguished) glued to a graph with 2 vertices and $n-k+1$ legs (one of them is distinguished). Once v and its incident flags are fixed, we obtain

$$\frac{(n-k+1)^2-3(n-k+1)}{2}$$

Keel relations coming from the $n-k+1$ legs distributed between the remaining two vertices. This is however not entirely correct since the resulting relations may not be independent. We count the relations differently: first, there are relations obtained when v has degree 1. Secondly, there are relations obtained when v has degree 0, but these may not be independent from those exhibited before. Instead, a second set of relations are pulled back from $\overline{\mathcal{M}}_{0,n}$. These can be expressed in terms of Keel's relations. Moreover, they are independent from the relations of the first kind. Indeed, linear combinations of the first set of relations do not involve graphs whose middle vertex has degree 1, while any non-trivial combination of an independent set of relations of the second kind would. We count relations of the first type. We must have $k \leq n-3$, because the remaining two degree 0 vertices are stable. We obtain

$$\sum_{k\leq n-3}\binom{n}{k}\cdot\frac{(n-k+1)^2-3(n-k+1)}{2} \tag{4.31}$$

independent relations. The relations coming from $\overline{\mathcal{M}}_{0,n}$ are counted next. There are

$$\frac{3^n+1}{2}-2^{n-1}(n+3)+\frac{(n+1)(n+2)}{2}$$

codimension 2 boundaries in $\overline{\mathcal{M}}_{0,n}$. The recursions in [47] give

$$h^4(\overline{\mathcal{M}}_{0,n})=\frac{3^n+1}{2}-2^{n-3}(n^2+3n+4)+\frac{n(n-1)(3n^2-7n+26)}{24}.$$

Therefore, we obtain

$$2^{n-3}(n^2-n-8)+\frac{(n+1)(n+2)}{2}-\frac{n(n-1)(3n^2-7n+26)}{24} \tag{4.32}$$

independent relations between these boundaries. Equation (4.30) follows from (4.31) and (4.32).

Finally, we match the number of generators modulo the number of relations obtained in items $(i)-(iii)$ above to the actual dimension of $H^4(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,1))$. This proves that there are no other relations we need to account for.

The Betti number in question is computed via localization. We use a torus action as in chapter 3 fixing a point p and a hyperplane H. The number of negative weights on the normal bundle is obtained from equation (3.3.4). We sum the following three contributions of the following fixed loci shown in figure 4.3:

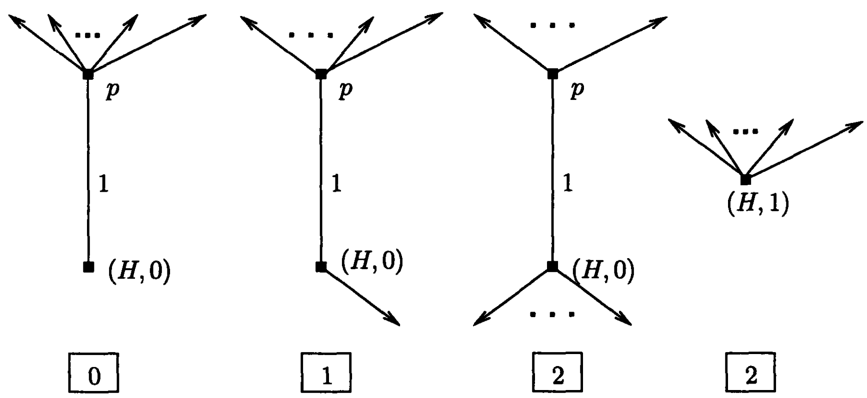


Figure 4-6: The fixed loci on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1)$.

- The fixed locus $\overline{\mathcal{M}}_{0,n+1} \times H$ with no negative weights on the normal bundle. It corresponds to the graph with one edge labeled 1, all the n legs being attached to the vertex labeled p. The contribution of this locus to the Betti number is

$$h^4(\overline{\mathcal{M}}_{0,n+1} \times H) = h^4(\overline{\mathcal{M}}_{0,n+1}) + h^2(\overline{\mathcal{M}}_{0,n+1}) + 1 =$$
$$= \frac{3^{n+1}+1}{2} - 2^{n-2}(n^2+5n+4) + \frac{n(n+1)(3n^2-n+10)}{24}. \tag{4.33}$$

- There are n loci corresponding to graphs with one edge, the vertex labeled $(H,0)$ supporting an attached leg. These fixed loci have 1 negative weight on their normal bundle and are isomorphic to $\overline{\mathcal{M}}_{0,n} \times H$. Their contribution to the Betti number is

$$n \cdot h^2(\overline{\mathcal{M}}_{0,n} \times H) = n\left(2^{n-1} - \frac{n(n-1)}{2}\right). \tag{4.34}$$

- There are $2^n - 1 - n$ fixed loci corresponding to graphs with an edge of degree 1 and the vertex labeled $(H,0)$ supporting at least two legs. There is an additional fixed locus corresponding to the graph with one vertex labeled $(H,1)$ with n legs attached. All these fixed loci have 2 negative weights on their normal bundle. Their total contribution to the Betti number is

$$2^n - n. \tag{4.35}$$

Now, the computation for d arbitrary is similar, and we will not reproduce it here entirely. The count obtained in items $(i) - (iii)$ needs to be modified in two places. First, the total number of boundary graphs with two edges is:

$$\frac{(d+1)(d+2)}{2}\cdot\frac{3^n+1}{2} - 2^{n-1}(n+3)(d+1) + \frac{(n+1)(n+2)}{2} - \left\lceil\frac{d+1}{2}\right\rceil\cdot\left\lceil\frac{d+2}{2}\right\rceil \tag{4.36}$$

Secondly, there are

$$(d-1)\cdot\left(\sum_{k\leq n-3}\binom{n}{k}\cdot\frac{(n-k+1)^2-3(n-k+1)}{2}\right) \tag{4.37}$$

independent relations between these classes *in addition* to (4.30).

The count of generators modulo relations is then matched to the Betti number. The dimension of the cohomology $H^4(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))$ is computed via localization, or better via the Deligne spectral sequence. We obtain complexes:

$$0\to\bigoplus_{\alpha}H^0(D^{(2)}_{\alpha})\to\bigoplus_{\alpha}H^2(D^{(1)}_{\alpha})\to H^4(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))\to 0. \tag{4.38}$$

Here, $D^{(2)}_{\alpha}$ are the codimension 2 boundary strata corresponding to dual graphs without automorphisms. Their number is obtained from formula (4.36) subtracting $\left[\frac{d}{2}\right]$. The middle terms are isomorphic to fibered products

$$\overline{\mathcal{M}}_{0,A\cup\{\star\}}(\mathbb{P}^r,d_A)\times_{\mathbb{P}^r}\overline{\mathcal{M}}_{0,B\cup\{\star\}}(\mathbb{P}^r,d_B)$$

for all possible splittings of the markings and degrees such that the corresponding graphs are stable. The dimensions of the middle terms are computed as using proposition 1.5.1. The alternating sums of the dimensions of the terms in (4.38) can be read off from the virtual Poincare polynomial $P(\mathcal{M}_{0,n}(\mathbb{P}^r,d))$ defined in (4.26). The expression in [31] proves that the alternating sums of dimensions is 1. We conclude that

$$h^4(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))=\frac{(d-1)(d+4)}{2}\cdot\frac{3^n-1}{2}-2^{n-3}(d-1)(n^2+6n+9)+$$
$$+\left[\frac{d+1}{2}\right]\cdot\left[\frac{d+2}{2}\right]+\left[\frac{d}{2}\right]-1\;+\;(4.33)\;+\;(4.34)\;+\;(4.35).$$

Putting everything together, we find the claimed result.

Remark 4.3.1. The method of computing the Betti numbers via localization shown here works very well in degrees 0 and 1, recovering the results of [47] precisely. In higher degrees a complete computation has been recently achieved in [31], but the answer is still highly recursive. We plan to investigate the extension of our method to arbitrary degrees elsewhere.

Chapter 5

Further discussion

In this short chapter we discuss several conjectures and open questions. First, the results discussed in this thesis should hold for general flag varieties over any field. We conjecture:

Conjecture 5.0.1. *If $X = G/P$ is a general flag variety, where G is a semisimple algebraic group and P a parabolic subgroup, then all rational Chow classes of $\overline{\mathcal{M}}_{0,n}(X,\beta)$ are tautological. What are other examples where this holds (e.g. toric varieties, cellular varieties)?*

The results of section 2.3, certain localization computations, and also a combination of the results of [46] and [31] lead us to the following statement:

Conjecture 5.0.2. *If X is any flag variety, (the dimensions of) the Chow rings of the open part $\mathcal{M}_{0,n}(X,\beta)$ are independent on the degree β, provided that β is a linear combination of the indecomposable classes with positive coefficients.*

We believe a similar statement could be true for toric varieties. When X is a convex toric variety, the scheme of rational morphisms into X is a rational smooth variety which admits a description similar to the one for $\mathbb{P}^r$: it is obtained as a base of a torus bundle whose total space sits in an affine space [54]; similar arguments as in chapter 2 can be applied. We mention this fact also because it is known that SL flags admit degenerations into toric, though singular, varieties. We will leave this investigation for a future project. A positive answer will be useful, via vanishing of high codimension classes on the open part, to obtain reconstruction theorems for a certain class of varieties.

Question 5.0.1. *If X is any SL flag, is it true that the cohomology/tautological classes on the open moduli spaces satisfy Poincare duality? Does this hold for other classes of varieties (e.g. toric)?*

On a similar note, is would be interesting to investigate if the tautological systems of the compactified spaced (in higher genus, for non-convex targets, including the ψ classes) have any special structure e.g. if they satisfy Poincare duality.

Since the number of tautological classes is huge compared to the dimension, it would be interesting to find a universal way of producing relations. For instance, relations can be obtained using Mumford's method: one exploits the vanishing of higher Chern classes of tautological bundles which can otherwise be computed by Grothendieck-Riemann-Roch. It is not clear that this method produces new relations other than the ones proposed in the introduction. We believe it does not, and we checked this in a number of examples in low degrees; in arbitrary degrees, we checked this fact up to the boundary terms.

Question 5.0.2. *Is it true that all relations between the tautological classes (or better, stabilized cohomology classes in the sense of [3]) are tautological as in definition* 1.6.1*?*

In chapter 4 we showed that in codimension 1 and 2, all relations are *essentially consequences* of Keel's relations. We also observed in chapter 2 that the same is true on the open part of the moduli space in any codimension when $X = \mathbb{P}^r$. This question can perhaps be approached when the target is $\mathbb{P}^1$ by a naive count of generators which should then be compared to the computation of the Betti numbers in [31]. We will consider this investigation in a future project. We have already checked this in particular cases.

Of course, the presentation of the ring announced in [50] would in principle solve this question in the case when the target is $\mathbb{P}^r$ and one marking; there is yet no written account for more markings, and no claims have been made for zero markings. Nonetheless, even in this special case, the geometric interpretation of the proposed relations is not entirely clear. Even better, it would be more interesting if we could find a concrete description of the moduli spaces of stable maps similar to Keel's blowup construction.

Question 5.0.3. *Find a universal way of obtaining relations in the tautological rings. Find a concrete "geometric" description of the cohomology rings for any target.*

Our point of view ties in with certain reconstruction theorems of the Gromov-Witten invariants [42], [43], [5]: all recursions between the Gromov-Witten invariants considered by these authors are consequences of a general procedure of obtaining relations between the tautological generators which include the cotangent insertions. This procedure can be formalized in the language presented in the introduction. It replicates all equations contained in [51]. We will explain this in more detail elsewhere.

In [5] and [43], the authors prove reconstruction of descendant Gromov-Witten invariants of $\mathbb{P}^r$ (or of any smooth manifold whose cohomology is generated by divisor classes) from the J function encoding the 1 point invariants; this function can be explicitly computed in several examples. It is defined as:

$$J_d^X = ev_\star \left(\frac{\left[\overline{\mathcal{M}}_{0,1}(X,d)\right]^{vir}}{1-\psi_1} \right) \in H^\star(X).$$

The recursions in [5] are obtained in a complicated fashion using virtual localization on the graph space. However these recursions can be immediately obtained via the equation below which can be used to reduce the number of markings when evaluating Gromov-Witten invariants:

$$ev_2^\star H = ev_1^\star H + d\psi_1 + \sum \text{boundary divisors} . \tag{5.1}$$

This is also proved in [43] by intersecting both sides with curves. We rederive this equation in a more geometric fashion. A similar procedure was applied in chapter 4. The following topological recursion relation on $\overline{\mathcal{M}}_{0,3}(X,\beta)$ follows by pullback from $\overline{\mathcal{M}}_{0,3}$, and was essentially known to Witten:

$$\pi^\star\psi_1 + \Delta^{(0,d)}_{(13),(2)} = \sum_{a,b} \Delta^{a,b}_{(1),(23)}. \tag{5.2}$$

Here $\pi : \overline{\mathcal{M}}_{0,3}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,2}(\mathbb{P}^r, d)$ forgets the third point, and $\Delta^{a,b}_{S,T}$ denotes the divisor of nodal maps with markings labeled S and T distributed on the two branches of degrees a, b. Intersecting (5.2) with $ev_3^\star H$ and pushing forward via π we obtain (5.1).

Finally, we observe that virtual fundamental classes $[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir}$ for all projective manifolds $i : X \hookrightarrow \mathbb{P}^r$ define classes in $H^\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$, which are tautological by our main result. Even more generally, the virtual classes of all relative stable morphism spaces $[\mathfrak{M}^Y_{0,\alpha}(X,\beta)]^{vir}$ are tautological. For computational purposes, it would be interesting to find an efficient way or expressing such classes.

For example, when $n = 1$, and X is a Fano hypersurface of degree $l < r$, then the virtual class in question can be expressed as $c_{dl+1}(\pi_\star ev_2^\star \mathcal{O}_{\mathbb{P}^r}(l))$. Grothendieck Riemann Roch combined with topological relations as above show that *on the open part of the moduli space* we have:

$$ch(\pi_\star ev_2^\star \mathcal{O}_{\mathbb{P}^r}(l)) = \sum_{k=0}^{dl} e^{l \cdot ev_1^\star H + k\psi_1}.$$

If no boundary contributions existed, then this computation would show that:

$$[\overline{\mathcal{M}}_{0,1}(X,d)]^{vir} = \prod_{k=0}^{dl} (l \cdot ev_1^\star H + k\psi_1).$$

In turn, this would give the following formula for the J function of X:

$$i_\star J_d^X = \prod_{k=0}^{dl} (lH + k) J_d^{\mathbb{P}^r}.$$

Specializing to $l = 1$, we inductively determine the J function of the projective space:

$$J_d^{\mathbb{P}^r} = \frac{1}{\prod_{k=1}^{d} (H+k)^{r+1}}.$$

It can be shown that all boundary contributions vanish in the Fano case by a dimension counting argument [26], so the above computation is indeed correct.

It is then of interest to obtain complete formulas of the virtual classes $[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir}$, including the boundary contributions, in terms of tautological classes. For example, it would be interesting to study the case when X is a cut out by the zero locus of a section of an indecomposable bundle.

As far as the higher genera are concerned, there is by now a large body of work aiming to understand the tautological rings of the stable curve spaces by making use of versions of Kontesevich-Manin spaces. In a different direction, one can define the higher genus tautological Gromov-Witten systems, as we will in the appendix. It is beyond the scope of this thesis to discuss this case, but its study in the context of Gromov-Witten theory will be of interest. For example, it would be interesting to have results about the structure of these systems or, say, about their dimensions.

As a first check that our definition is the correct one, the following question needs to be answered. An affirmative result is easily obtained in the presence of a torus action via virtual localization. A similar question was studied in [24]: the authors prove that the moduli space of Hurwitz covers yield tautological classes.

Question 5.0.4. *Is it true that the pushforwards of the tautological classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ under the stabilization map $st : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$ are tautological? Is the same statement true for the tautological classes on the space of relative stable morphisms $\mathfrak{M}^Y_{g,\alpha}(X,\beta)$ for* *any pair* (X, Y)*?*

Moreover, it would be interesting to have a way of obtaining relations between the tautological classes. Already in degree 0, that is for moduli spaces of curves, the higher genus tautological systems seem to be very complicated. Ionel's vanishing (generalizing Getzler's vanishing low genus) provides examples of nontrivial relations between the tautological classes in degree 0 [37]. These relations can be pulled back to any moduli space of stable maps to obtain relations between the tautological classes in any degree. Unfortunately the number of terms in such relations grows very fast and a concrete study is difficult even in low genus.

It is not yet clear if these degree 0 ideas can be extended to Gromov-Witten theory. One possible application concerns reconstruction of Gromov-Witten invariants with the aid of the possible relations between the tautological classes. A genus 0 example was explained above, but higher genus examples are undoubtedly more interesting.

In higher genus all work that has been done started from the degree 0 relations. The reason behind is the belief that, at least in the case of targets with semisimple quantum cohomology such as $\mathbb{P}^r$, the degree 0 universal equations determine all higher genus invariants. This has been checked for $g \leq 2$ [45]. For example, in genus 0, all that is needed are the WDVV equations. A related observation of Gathmann shows the Virasoro constraints together with one additional degree 0 equation determine all genus g invariants of $\mathbb{P}^r$ [27]. It would perhaps be interesting to obtain relations between the tautological classes in higher degrees.

Appendix A

The tautological systems in higher genus.

To define the tautological systems $\mathcal{R}$ for non-convex targets or higher genera we need to make use of the virtual fundamental classes. As the moduli spaces involved are not necessarily smooth, we will ignore any possible *ring* structure $\mathcal{R}$ may have.

We let $\mathcal{R} \subset A_\star(\overline{\mathcal{M}}_{0,n}(X,\beta))$ be the minimal system satisfying the requirements:

- $ev_1^\star\alpha_1 \cdot \ldots \cdot ev_n^\star\alpha_n \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cap \left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{vir} \in \mathcal{R}$, for all $\alpha_i \in A^\star(X)$, $a_i \geq 0$;
- The system is closed under pushforward by the forgetful morphisms;
- The system is closed under the gluing maps ζ_Γ.

To define the gluing maps ζ_Γ, we fix a stable dual graph Γ of genus g, degree β, with n legs. For each vertex v, we write g_v, β_v, n_v for the corresponding genus, degree and total valency (half edges and legs). The boundary stratum of maps with fixed dual graph Γ is obtained from the fibered diagram below, where $E(\Gamma)$ and $H(\Gamma)$ stand for the set of edges and half edges of Γ:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n}(X,\beta) & \overline{\mathcal{M}}(\Gamma) \arrow[l, "gl_\Gamma"'] \arrow[r] \arrow[d] & \prod_v \overline{\mathcal{M}}_{g_v,n_v}(X,\beta_v) \arrow[d] \\
& X^{E(\Gamma)} \arrow[r, "\Delta_\Gamma"] & X^{H(\Gamma)}
\end{tikzcd}$$

The gluing map ζ_Γ is obtained as composition of the gluing pushforward, the Gysin morphism and the exterior product (which we will omit from the notation):

$$\zeta_\Gamma : \bigotimes_v A_\star\left(\overline{\mathcal{M}}_{g_v,n_v}(X,\beta_v)\right) \to A_\star\left(\overline{\mathcal{M}}_{g,n}(X,\beta)\right), \;\; \zeta_\Gamma = (gl_\Gamma)_\star \Delta_\Gamma^! . \tag{A.1}$$

Lemma A.0.2. *In genus* 0 *and for convex targets, we recover the definition proposed in the introduction.*

To begin, we define the κ classes using the forgetful pushforward:

$$\kappa_n(\alpha_{n+1},\ldots,\alpha_{n+p}) = \pi_\star(ev_{n+1}^\star\alpha_{n+1} \cdot \ldots \cdot ev_{n+p}^\star\alpha_{n+p} \cap \left[\overline{\mathcal{M}}_{0,n+p}(X,\beta)\right]^{vir}).$$

More generally, we can define the operational classes $\tilde{\kappa}_n(\alpha_{n+1},\dots,\alpha_{n+p})$ as the operational proper flat pushforward [49] of monomials in the evaluation classes. Then:

$$\tilde{\kappa}_n(\alpha_{n+1},\dots,\alpha_{n+p}) \cap [\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir} = \kappa_n(\alpha_{n+1},\dots,\alpha_{n+p}).$$

We let $\mathcal{S}^d = \mathcal{S}^d_{\beta,n} \subset A_\star(\overline{\mathcal{M}}_{0,n}(X,\beta))$ be the following collection of descendant classes:

$$\theta = ev_1^\star\alpha_1 \cdot \ldots \cdot ev_n^\star\alpha_n \cdot \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} \cap \kappa_n(\alpha_{n+1},\dots,\alpha_{n+p}) \in A_\star(\overline{\mathcal{M}}_{0,n}(X,\beta)). \qquad \text{(A.2)}$$

We let $\tilde{\mathcal{S}}^d$ be the collection of classes:

$$\zeta_\Gamma(\theta_\Gamma), \text{ where } \theta_\Gamma = \prod_v \theta_v, \text{ and } \theta_v \in \mathcal{S}^d_{\beta_v,n_v},$$

for all stable dual graphs Γ. We define the similar collections of primary classes $\mathcal{S}^p$ and $\tilde{\mathcal{S}}^p$ only allowing $a_1 = \ldots = a_n = 0$ in equation (A.2).

The lemma will follow from the facts below:

(1) For all targets X, $\tilde{\mathcal{S}}^p$ is preserved by the natural pushforwards.

(2) For convex targets X, $\tilde{\mathcal{S}}^d$ is closed under the multiplication in the Chow ring.

(3) For all targets X, $\tilde{\mathcal{S}}^d = \tilde{\mathcal{S}}^p$; consequently, both collections give additive generators for the tautological systems.

To prove (1), we first observe that closure of $\tilde{\mathcal{S}}^p$ under the gluing pushforwards is obvious. We check closure under the forgetful morphism $\pi : \overline{\mathcal{M}}_{0,n}(X,\beta) \to \overline{\mathcal{M}}_{0,n-1}(X,\beta)$. Letting Γ' be the graph obtained from Γ by forgetting the n^{th} leg, we obtain:

$$\pi_\star\zeta_\Gamma(\theta_\Gamma) = \pi_\star(gl_\Gamma)_\star\Delta^!_\Gamma\theta_\Gamma = (gl_{\Gamma'})_\star\pi_\star\Delta^!_\Gamma\theta_\Gamma = (gl_{\Gamma'})_\star\Delta^!_{\Gamma'}(\pi_\star\theta_\Gamma) = \zeta_{\Gamma'}(\pi_\star\theta_\Gamma).$$

Special care must be taken when the graph Γ' is unstable. At any rate, we reduce our check to classes in $\mathcal{S}^p$. Then, let θ be a class as in (A.2), with $a_1 = \ldots = a_n = 0$. The projection formula shows:

$$\pi_\star\theta = ev_1^\star\alpha_1 \cdot \ldots \cdot ev_{n-1}^\star\alpha_{n-1} \cap \kappa_{n-1}(\alpha_n,\dots,\alpha_{n+p}) \in \mathcal{S}^p.$$

To check (2), we follow an idea of [33]. We fix two classes $\zeta_\Gamma(\theta_\Gamma)$ and $\zeta_{\Gamma'}(\theta_{\Gamma'})$ supported on $\overline{\mathcal{M}}(\Gamma)$ and $\overline{\mathcal{M}}(\Gamma')$. Their product is computed by the excess intersection formula. The excess bundle will be distributed over the components $\overline{\mathcal{M}}(\Gamma'')$ of the stack theoretic intersection of $\overline{\mathcal{M}}(\Gamma)$ and $\overline{\mathcal{M}}(\Gamma')$. These are indexed by dual graphs which are given additional structure. The graph Γ'' is endowed with two collapsing maps $\Gamma'' \to \Gamma$ and $\Gamma'' \to \Gamma'$ which replace whole subgraphs of Γ'' with vertices of either Γ or Γ', also collecting the incident legs and the degree labels. Moreover, we require that each half-edge of Γ'' correspond to a half edge in either Γ or in Γ'. Just as in equation (11) in [33], we derive that the excess normal bundle splits as sum of line bundles which are expressed in terms of the cotangent lines. The top Chern class of the excess bundle equals:

$$\prod_e (-\psi_v - \psi_w)$$

where v, w are vertices lying on an edge e which "comes" from both Γ and Γ'. Therefore, the excess intersection formula shows that:

$$\zeta_\Gamma(\theta_\Gamma) \cdot \zeta_{\Gamma'}(\theta_{\Gamma'}) = \sum_{\Gamma''} \zeta_{\Gamma''}(\theta_{\Gamma''}).$$

We argue that $\theta_{\Gamma''}$ is an exterior product of classes in $\mathcal{S}^d$. To this end, we observe that:

(i) Pullback under the gluing morphisms gl_Γ preserves the evaluation classes $ev^\star\alpha$ and the ψ classes.

(ii) The pullback of a κ class is sum of κ classes. Product of κ classes is a κ class.

Finally, for the last item on our list, it suffices to check that $\mathcal{S}^d \subset \tilde{\mathcal{S}}^p$, since $\tilde{\mathcal{S}}^p$ is invariant under the gluing morphisms. This will follow if we show that:

$$\psi_1 \cap \_ : \tilde{\mathcal{S}}^p \to \tilde{\mathcal{S}}^p.$$

Using the projection formula for the boundary maps, observation (i) and the compatibility of Chern classes with the Gysin morphisms, it suffices to prove that:

$$\psi_1 : \mathcal{S}^p \to \tilde{\mathcal{S}}^p.$$

Let θ be given by (A.2). The projection formula for the morphism $\pi : \overline{\mathcal{M}}_{0,n+p}(X,\beta) \to \overline{\mathcal{M}}_{0,n}(X,\beta)$ shows that $\psi_1 \cap \theta$ is the pushforward of the class:

$$\pi^\star\psi_1 \cdot ev_1^\star\alpha_1 \cdot \ldots \cdot ev_{n+p}^\star\alpha_{n+p} \cap \left[\overline{\mathcal{M}}_{0,n+p}(X,\beta)\right]^{vir}.$$

Using the invariance of $\tilde{\mathcal{S}}^p$ under the forgetful pushforward by π established above, it suffices to prove that this class belongs to $\tilde{\mathcal{S}}^p$. It is an immediate consequence of the commutation between Gysin morphisms and flat pullbacks that:

$$ev_i^\star\alpha_i \cap \_ : \tilde{\mathcal{S}}^p \to \tilde{\mathcal{S}}^p.$$

It remains to prove that:

$$\pi^\star\psi_1 \cap \left[\overline{\mathcal{M}}_{0,n+p}(X,\beta)\right]^{vir} \in \tilde{\mathcal{S}}^p.$$

When $X = \mathbb{P}^r$, this is a consequence of the equation VI 6.17 in [49] and lemma 2.2.2 in [51]. The general case follows pulling back under the closed embedding $i : \overline{\mathcal{M}}_{0,n+p}(X,\beta) \to \overline{\mathcal{M}}_{0,n+p}(\mathbb{P}^r, i_\star\beta)$ induced by a projective embedding of X. We only need to observe that the operational $\tilde{\kappa}$ classes on the space of maps to $\mathbb{P}^r$ pullback to the similar classes for X. Then, to finish, we cap with the virtual fundamental class.

As a corollary of the proof we obtain a system of generators for the genus 0 tautological systems. Recall the definition of weighted graphs given in the introduction. For each weighted graph $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ and each vertex v of Γ we obtain a monomial in evaluation classes:

$$\theta_v = \prod_f ev_f^\star\, \mathfrak{w}(f) \cap \left[\overline{\mathcal{M}}_{0,n_v}(X,\beta_v)\right]^{vir}, \tag{A.3}$$

where the product is taken over flags f incident to v. The "forgetting" data $\mathfrak{f}$ determines a morphism π by discarding the legs in $\mathfrak{f}$. Therefore, each triple $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ determines a cohomology class:

$$[\Gamma, \mathfrak{w}, \mathfrak{f}] = \pi_\star \zeta_\Gamma \left(\prod_{v \in V(\Gamma)} \theta_v \right), \quad \theta_v \in \mathcal{E}_{\beta_v, n_v}. \tag{A.4}$$

Corollary A.0.1. *For all targets X, the classes $[\Gamma, \mathfrak{w}, \mathfrak{f}]$ form a system of additive generators for the genus 0 tautological systems.*

Bibliography

[1] M. F. Atiyah, R. Bott, *The Yang Mills equations over Riemann surfaces*, Philos. Trans. Royal Soc. London, 308(1982), 523-615.

[2] K. Behrend, *Cohomology of stacks.* Lectures at MSRI and ICTP. Available at http://www.msri.org/publications/video/ and http://www.math.ubc.ca/behrend/preprints.html

[3] K. Behrend, A. O'Halloran, *On the cohomology of stable map spaces*, Invent. Math. 154 (2003), no. 2, 385-450, AG/0202288.

[4] K. Behrend, Y. Manin, *Stacks of Stable Maps and Gromov-Witten Invariants*, Duke Math. J. 85 (1996), no. 1, 1-60, AG/9506023.

[5] A. Bertram, H. Kley, *New Recursions for genus-zero Gromov-Witten invariants*, Topology 44 (2005), 1-24.

[6] A. Bertram, G. Daskalopoulos, R. Wentworth, *Gromov Invariants for Holomorphic Maps from Riemann Surfaces to Grassmannians*, J. Amer. Math. Soc. 9 (1996), no. 2, 529 - 571, AG/9306005.

[7] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math, (2) 98 (1973), 480-497.

[8] A. Bialynicki-Birula, *Some properties of the decompositions of algebraic varieties determined by actions of a torus*, Bull. Acad. Polon. Sci., 24 (1976), no. 9, 667-674.

[9] G. Bini, C. Fontanari, *On the cohomology of $\overline{M}_{0,n}(\mathbb{P}^1, d)$*, Commun. Contemp. Math. 4 (2002), no. 4, 751–761.

[10] M. Brion, *Equivariant cohomology and equivariant intersection theory*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci, Representation theories and algebraic geometry, 1-37, Kluwer Acad. Publ., Dordrecht, 1998.

[11] J. B. Carrell, *Torus actions and cohomology*, Encyclopaedia Math. Sci., 131, Springer, Berlin, 2002.

[12] I. Ciocan-Fontanine, *The quantum cohomology ring of flag varieties*, Transactions of the AMS, 351 (7) (1999), 2695-2729.

[13] T. Coates, A. Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, AG/0110142.

[14] J. Cox, *An additive basis for the Chow ring of $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^r, 2)$*, AG/0501322.

[15] P. Deligne, *Teorie de Hodge II, III*, Publ. Math. I.H.E.S, 40 (1972), 44(1974), 5-58 and 5-77.

[16] J. deJong, J. Starr *Divisor classes and the virtual canonical bundle for genus* 0 *maps*, preprint.

[17] A. Dhillon, *On the cohomology of moduli of vector bundles*, AG/0310299.

[18] D. Edidin, W. Graham, *Equivariant intersection theory*, Invent. Math. 131 (1998), no. 3, 595–634.

[19] D. Edidin, W. Graham, *Localization in equivariant intersection theory and the Bott residue formula*, Amer. J. Math. 120 (1998), no. 3, 619-636, AG/9508001.

[20] G. Ellingsrud, S. A. Stromme, *Towards the Chow ring of the Hilbert scheme of* P^2, J. Reine Angew. Math. 441 (1993), 33–44.

[21] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, Moduli of curves and abelian varieties, 109 - 129, Aspects Math., E33, Vieweg, Braunschweig, 1999.

[22] W. Fulton, R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry, Santa Cruz 1995, 45-96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997, AG/9608011.

[23] W. Fulton, R. MacPherson, *A compactification of configuration spaces*, Annals of Math, 139 (1994), 183-225.

[24] C. Faber, R. Pandharipande, *Relative maps and tautological classes*, AG/0304485.

[25] A. Gathmann, *Absolute and relative Gromov-Witten invariants of very ample hypersurfaces*, Duke Math. J. 115 (2002), no. 2, 171-203, AG/9908054.

[26] A. Gathmann, *Relative Gromov-Witten invariants and the mirror formula*, Math. Ann. 325 (2003), no. 2, 393-412.

[27] A. Gathmann, *Topological recursion relations and Gromov-Witten invariants in higher genus*, AG/0305361.

[28] E. Getzler, *Operads and moduli spaces of genus* 0 *Riemann surfaces*, The moduli space of curves, 199-230, Progr. Math., 129, Birkhuser Boston, Boston, MA, 1995, AG/9411004.

[29] E. Getzler, *Mixed Hodge structures on configuration spaces*, AG/9510018.

[30] V. Ginzburg, *Equivariant cohomology and Kahler geometry*, Funktsional. Anal. i Prilozhen. 21 (1987), no. 4, 19–34, 96.

[31] E. Getzler, R. Pandharipande, *The Betti numbers of* $\overline{M}_{0,n}(r,d)$, AG/0502525.

[32] T. Graber, R. Pandharipande, *Localization of virtual classes*, Invent. Math. 135 (1999), no. 2, 487-518, AG/9708001.

[33] T. Graber, R. Pandharipande, *Construction of non-tautological classes on the moduli spaces of curves*, Michigan Math J, 51 (2003), 93-109.

[34] T. Graber, R. Vakil, *Relative virtual localization, and vanishing of tautological classes on moduli spaces of curves"*, Duke Math. J. - to appear.

[35] A. Grothendieck, *Quelques proprietes fondamentales en theorie des intersections*, Seminaire Chevalley *Anneaux de Chow et applications*, 1959.

[36] A. Grothendieck, *Le groupe de Brauer II. Dix exposes sur la cohomologie de schemas*, North Holland, 1968.

[37] E. Ionel, *Topological recursive relations in* $H^{2g}(\mathcal{M}_{g,n})$, Innvent. Math. 148 (2002),627-658.

[38] S. Keel, *Intersection theory of the moduli space of stable n pointed curves of genus zero*, Trans. Amer. Math. Soc, 330(1992), no 2, 545-574.

[39] B. Kim, *Quot schemes for flags and Gromov invariants for flag varieties*, AG/9512003

[40] B. Kim, R. Pandharipande, *The connectedness of the moduli space of maps to homogeneous spaces*, Symplectic geometry and mirror symmetry (Seoul, 2000), 187-201, World Sci. Publishing, River Edge, NJ, 2001, AG/0003168.

[41] M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves, 335-368, Progr. Math., 129, Birkhuser Boston, Boston, MA, 1995, hep-th/9405035.

[42] M. Kontsevich, Y. Manin, *Gromov-Witten classes, quantum cohomology and enumerative geometry*, Comm. Math. Phys, 163 (1994), 525-562.

[43] Y. P. Lee, R. Pandharipande, *A reconstruction theorem in quantum cohomology and quantum K theory*, AG/0104084.

[44] J. Li, *A degeneration formula of GW invariants*, J. Diff. Geom, 60 (2002), no. 2, 199-293.

[45] X. Liu, *Genus* 2 *Gromov-Witten invariants for manifolds with semisimple quantum cohomology*, DG/0310410.

[46] Y. Manin, *Stable maps of genus zero to flag spaces*, Topol. Methods Nonlinear Anal. 11 (1998), no. 2, 207–217.

[47] Y. Manin, *Frobenius Manifolds, Quantum cohomology and Moduli Spaces*, AMS Colloquim Publications, v.47, 1999.

[48] A. Marian, *Intersection theory on the moduli spaces of stable bundles via morphism spaces*, Harvard University Thesis.

[49] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in *Arithmetic and Geometry*, Part II, Birkhauser, 1983, 271-328.

[50] D. Mustata, A. Mustata, *Intermediate Moduli Spaces of Stable Maps*, AG/0409569.

[51] R. Pandharipande, *Intersection of Q-divisors on Kontsevich's Moduli Space* $\overline{M}_{0,n}(\mathbb{P}^r, d)$ *and enumerative geometry*, Trans. Amer. Math. Soc. 351 (1999), no. 4, 1481-1505, AG/9504004.

[52] R. Pandharipande, *The Chow Ring of the nonlinear Grassmanian*, AG/9604022.

[53] R. Pandharipande, *Three questions in Gromov-Witten theory*, AG/0302077.

[54] B. Siebert, *An update on the (small) quantum cohomology*, Proceedings of the conference on Geometry and Physics (D.H. Phong, L. Vinet, S.T. Yau eds.), Montreal 1995, International Press 1998.

[55] B. Siebert, G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, AG/9403010

[56] J. Steenbrinck, *Mixed Hodge Structure on the Vanishing Cohomology*, Nordic Summer School - Symposium in Mathematics Olso, 1976, 525- 563.

[57] S. Stromme, *On parametrized rational curves in Grassmann varieties*, Space curves (Rocca di Papa, 1985), 251-272, Lecture Notes in Math, 1266, Berlin- New York, 1987.

[58] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ., 14 (1974), 1-28.

[59] A. Vistoli, *Intersection theory on algebraic stacks and their moduli spaces*, Invent. Math. 97 (1989), 613-670.

[60] A. Vistoli, *Chow groups of quotient varieties*, J. Algebra 107 (1987), 410-424.