Some Analytical Results for the Dynamic Weapon-Target Allocation Problem*

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Abstract

We consider the dynamic version of the Weapon-Target Allocation problem. This problem is, in general, NP-Complete, so our aim is to provide insight into the problem and its solution. We will provide analytical solutions for simple cases of the problem as well as asymptotic results as the number of targets goes to infinity.

The battle scenario being modeled is as follows. The offense launches a number of weapons (the targets) which are aimed at assets of the defense. The defense has a number of defensive weapons each of which can engage at most one target. The outcome of such an engagement is stochastic. In the *static* scenario all weapons are fired simultaneously. In the *dynamic* scenario some weapons are assigned and fired and the outcomes of these engagements are observed before further assignments are made. Values are assigned to the targets and the objective is to assign weapons to targets so as to minimize the total expected value of the surviving targets after all weapons have been fired. Generally, under suitable assumptions, we show that dynamic strategies can approximately double the defense effectiveness as compared to their static counterparts.

1 Introduction

The Weapon-Target Allocation (WTA) problem is used to model the defense of assets in a military conflict. The offense (the enemy) launches a number of offensive weapons which are aimed at valuable assets of the defense. Since these weapons will be the targets of the defense's weapons, henceforth we will call them targets. The defense has a number of defensive weapons with which to engage these incoming targets. The engagement of a target by a weapon will be modeled as a stochastic event. A probability, called a kill probability, will be assigned to each weapon-target pair. This will be the probability that the weapon destroys the target if it is assigned to it. We will assume that the engagement of a weapon-target pair is independent of all other weapons and

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targets. Note that a particular target may be engaged by more than one weapon (Salvo attacks).[1] Values are assigned to the incoming targets and the objective is to assign defensive weapons to these targets so as to minimize the expected total value of the targets which survive after all engagements. This corresponds to what is known as a weighted subtractive defense.

In the static version of the problem we will assume that all weapons are assigned and fired simultaneously. Damage assessment is made after all weapon-target engagements. This is the determination of the set of surviving targets. In the dynamic version, weapons are allocated in stages with the assumption that the outcomes (i.e. survival or destruction of each target) of the weapon-target engagements of the previous stage are observed (perfectly) before assignments for the present stage are made. We will assume that each weapon can be used only once.

The efficient solution of the WTA problem is of great interest to the military. The reason for this is that, in an engagement with the enemy, the problem must be solved in real time. The enormous combinatorial complexity of the problem implies that, even with the supercomputers available today, optimal solutions cannot be obtained in real-time. One must therefore develop good heuristics for solving the problem. To provide good heuristics one must have a thorough understanding of the properties of the problem and its solution.

Some important properties of the dynamic WTA problem are that it is (a) *NP-Complete* (i.e. one must essentially resort to complete enumeration to find the optimal solution), (b) *Discrete* (fractional weapon assignments are not allowed), (c) *Dynamic* (the results of previous engagements are observed before making present assignments), (d) *Nonlinear* (the objective function is convex), (e) *Stochastic* (weapon-target engagements are modeled as stochastic events) and (f) *Large-Scale* (the number of weapons and targets is large, making enumeration techniques impractical). These properties of the problem rule out any hope of obtaining efficient optimal algorithms. In this paper we will provide properties of the solution of this problem which will be useful in providing good heuristics. Wherever it is possible, we will provide rigorous arguments. Wherever it is appropriate we will provide simple examples and computational results. We will next give a brief summary of the research literature on this problem.

In [4], denBroeder et al. consider the special case of the WTA problem in which the kill

[1] The Weapon-Target allocation problem is but one of the many problems that are addressed in the field of Command and Control (C^2) theory. The perspectives paper by Athans [1] presents some of the other basic problems.

probability of a weapon-target pair is independent of the weapon (i.e a single class of weapons). They present an optimal algorithm for solving this version of the problem. Kattar implemented this algorithm and presents some numerical results in [7].

Matlin [9] provides a review of the literature on weapon-target allocation problems. Several references are given and are classified by the model under consideration. Eckler and Burr [5] also give a review of the material on weapons allocation problems. Besides giving references, they summarize different mathematical models and provide some analysis. However, in these studies, very little emphasis is given to the dynamic allocation of weapons which is the main focus of our research.

A major result, obtained by Lloyd and Witsenhausen [8], is that the Static WTA problem is NP-Complete. What this means is that the computation time of any optimal algorithm for the problem will grow exponentially with the size of the problem. Since the static problem is a special case of the dynamic one, we can conclude that the latter is also NP-Complete.

A group at Alphatech Inc., under the leadership of Dr. D. A. Castañon, has examined the WTA problem in the context of the Strategic Defense System. Their recent reports, although unclassified, are restricted and the first author did not have access to these documents. On the other hand, personal communication with Dr. Castañon [3] ensured that no significant duplication of effort and results (unclassified and/or unrestricted) occurred.

In conclusion, we have found that the open literature on the dynamic version of the WTA problem is scant. Furthermore, the literature which addresses the dynamic problem contains few analytical results because of the difficulty of the problem. In section 2 we will present the Static WTA problem together with solution methods for simple cases. In section 3 we will present the dynamic version of the problem together with analytical results for special cases and asymptotic results as the numbers of weapons and targets go to infinity.

2 The Static Weapon-Target Allocation Problem

In this section we will present the static version of the problem. This problem has been well studied in the literature. It has been shown by Lloyd and Witsenhausen [8] to be an NP-Complete problem in general. Therefore, only sub-optimal algorithms have been proposed for its solution. In the case of a single class of weapons an optimal algorithm has been proposed by denBroeder et al. [4].

2.1 Problem Definition

In this version of the problem, the offense launches its weapons (the targets of the defense) at the defense's assets. The defense assigns values to these targets based on the predicted target type, the value of the predicted point of impact and other relevant factors. The defense has weapons which can be used to engage these targets before they impact. A one-to-one kill probability is assigned to each weapon-target pair. This is the probability that the weapon destroys the target if it is assigned to it and reflects such factors as the weapon type, the time and geometry of intercept, the characteristics of the engagement of the specific weapon-target pair and other relevant factors. Therefore, in general, the kill probability of a particular weapon-target pair will be different to the kill probabilities of all other weapon-target pairs. The objective of the defense is to assign its weapons to the targets so as to minimize the expected total value of the surviving targets. Note that in the optimal assignment some high valued targets may be engaged by more than one weapons while others (with low values) may not be engaged by any weapons.

In this version of the problem all weapons are assigned and fired simultaneously. We will also assume that the state of the targets (survived or destroyed) is observed after all weapons have been fired. In other words there is no feedback of information. This assumption will be valid in cases in which the defense has only a single opportunity to engage the targets. This would occur in conflicts in which the flight duration of the targets is short.

We will also assume that the engagement of a weapon-target pair is independent of all other weapons and targets. In practice this assumption may not hold for all engagements because targets near a weapon-target interception will be affected by the debris of the explosion. However, the problem is very difficult without this assumption because one must then include the geometry of the problem.

The following notation will be used in the mathematical definition of the Static WTA problem.

$N \stackrel{\text{def}}{=}$ the number of targets (offense weapons),

$M \stackrel{\text{def}}{=}$ the number of defense weapons,

$V_i \stackrel{\text{def}}{=}$ the value of target i, $\quad i = 1, 2, \ldots, N$,

$p_{ij} \stackrel{\text{def}}{=}$ the probability that weapon j destroys target i if it is assigned to it,

$\qquad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, M.$

The decision variables will be denoted by:

$$x_{ij} = \begin{cases} 1 & \text{if weapon } j \text{ is assigned to target } i, \\ 0 & \text{otherwise.} \end{cases}$$

The probability that target i is not destroyed by weapon j is given by $(1 - p_{ij})^{x_{ij}}$. Therefore, since it was assumed that the engagement of a target by a weapon is independent of all other targets and weapons, the probability that target i survives after all weapons have been fired is given by $\prod_{j=1}^{M}(1 - p_{ij})^{x_{ij}}$. The problem is therefore given as follows.

Problem 2.1 *The Static Weapon-Target Allocation problem (SWTA) can be stated as:*

$$\min_{\{x_{ij}\in\{0,1\}\}} F = \sum_{i=1}^{N} V_i \prod_{j=1}^{M}(1 - p_{ij})^{x_{ij}},$$

$$\text{subject to} \qquad \sum_{i=1}^{N} x_{ij} = 1, \quad j = 1, 2, \ldots, M.$$

The objective function, $F : \{0,1\}^{NM} \longrightarrow \Re$, is the total expected value of the surviving targets. We will show that this function is convex.[2] The constraint is due to the fact that each weapon must be assigned to exactly one target.

Theorem 2.1 *If we relax the integrality constraint and allow $0 \leq x_{ij} \leq 1$ then the function F : $[0,1]^{NM} \longrightarrow \Re$, as defined in problem 2.1, is convex.*

Proof: See the thesis by Hosein [6]. ■

We will see in the next subsection that if we assume that the kill probabilities do not depend on the weapons (i.e. we have a single class of weapons) then the resulting problem can be solved by a polynomial time algorithm. This implies that the basic difficulty of the problem stems from the fact that there are multiple types of weapons. The problem is also difficult because of the non-linearity of the objective function.

[2]Note that convex functions are defined in convex sets. The set in which F is defined is not convex so it is incorrect to discuss the convexity of this function. In this context, what we really mean is that if we relax the integrality constraint (i.e allow $0 \leq x_{ij} \leq 1$) then the function F is convex in this set.

2.2 Special Cases of the SWTA Problem

In this subsection we will briefly describe solution methods for some special cases of the Static WTA problem.

2.2.1 A Single Class of Weapons

Two optimal algorithms exist for solving problem 2.1 under the additional assumption that the kill probabilities are independent of the weapons, i.e. $p_{ij} = p_i$. This assumption is valid if the defense has a single type of weapon and all weapons are located in the same area so that the geometry and time of intercept is the same for all of them.

denBroeder et al. [4] proposed the first algorithm for solving this special case of the problem. Their's is essentially a greedy algorithm in which weapons are assigned sequentially to the target for which the corresponding decrease in the cost is maximum. This algorithm is usually referred to as a Maximum Marginal Return algorithm. The second algorithm for solving the problem is a Local Search algorithm. The algorithm starts with any feasible solution and searches locally for a better solution until the optimal one is found. Proof of optimality of this algorithm can be found in [6].

2.2.2 Weapons with Limited Target Coverage

In the previous section we had assumed that the kill probability is independent of the weapons. In this section we will assume that for each weapon-target pair the weapon can either be assigned to the target or it cannot be assigned to the target (i.e. each weapon can only reach some of the targets). If it can be assigned to the target then we will assume that the kill probability of the pair is only dependent on the target. In other words we are assuming that the kill probability of a weapon-target pair is either 0 or some target dependent value p_i (i.e $p_{ij} \in \{0, p_i\}$). This problem can be re-formulated as a Linear Minimum Cost Network Flow problem. Any algorithm for solving such problems can then be used to find the optimal solution. Note that this problem is more general than the one presented in the previous subsection.

2.2.3 One Weapon per Target

In this subsection we will consider problem 2.1 under the additional assumption that each target can be assigned at most one weapon. This problem can be re-formulated as a Transportation problem. Any algorithm for solving Transportation problems can then be used to find the optimal solution.

3 The Dynamic Weapon-Target Allocation Problem

In this section we will consider the dynamic version of the WTA Problem. This problem consists of a number of time stages. The defense is allowed to observe the outcomes of all engagements of the previous time stage before assigning and commiting weapons for the present stage. This is called a "shoot-look-shoot-..." strategy since the defense is alternating between shooting its weapons and observing (looking) at the outcomes.

3.1 Problem Definition

In the dynamic problem the time duration of the offense's attack is divided into a number of time segments. Each segment is of sufficient length to allow the defense to fire a subset of its weapons and observe (perfectly) the outcomes of all of the engagements of the weapons. With the feedback of this information the defense can make better use of its weapons, since it will no longer engage targets which have already been destroyed.

We assume that in the initial stage the defense chooses a subset of its weapons and assigns them to targets. These weapons are then committed simultaneously. In the second stage the outcomes (i.e. the survival or destruction of each engaged target) of all of the engagements of the weapons committed in the first stage are observed. Based on this observation, the defense chooses a subset of the remaining weapons and assigns them to the targets which survived the stage 1 engagements. In the third stage the outcomes of the engagements of the weapons committed in stage two are observed. Based on this observation, a subset of the remaining weapons is chosen and assigned to the set of surviving targets. This process is repeated for all time stages. In each stage the weapons are chosen and assigned with the objective of minimizing the total expected value of the surviving targets at the end of the final stage.

Note that in each stage the problem is re-solved based on the outcomes of the previous stage.

This implies that in each stage one is interested in obtaining (a) the subset of weapons which are to be fired in that stage and (b) the optimal assignment of these weapons to targets. Note that in computing the optimal assignment for the present stage one must assume that in all subsequent stages an optimal assignment will be used. If this is not done then the expected cost for the problem could be improved by doing so. This is known as the *Principle of Optimality* in dynamic programming [2]. We will therefore implicitly assume that optimal assignments will be used in all subsequent stages.

Note that the only information required to compute the optimal assignments in a stage is the set of surviving targets, the set of remaining weapons and the number of stages left. All other information of previous stages is not relevant. Therefore at each stage the problem can be restated as one in which the present stage is the *first* stage of the restated problem. The initial set of targets for this problem is the set of surviving targets and the initial set of weapons is the set of remaining weapons. In other words the problem to be solved in each stage has the same form as the statement of the problem for stage 1. Therefore, although we will only consider the T-stage problem and solve for the optimal assignments of the first stage, the same method can be used to solve for the optimal assignments of the remaining stages.

In our notation we will index the parameters in each stage with the stage number. Therefore for a T-stage problem the parameters in stage one will have an index of 1 while those of the final stage will have an index of T. The notation, which is basically the same as for the static problem except for the stage index, is as follows:

$N \stackrel{\text{def}}{=}$ the number of targets (offense weapons),

$M \stackrel{\text{def}}{=}$ the number of defense weapons,

$T \stackrel{\text{def}}{=}$ the number of time stages,

$V_i \stackrel{\text{def}}{=}$ the value of target i, $\quad i = 1, 2, \ldots, N$,

$p_{ij}(t) \stackrel{\text{def}}{=}$ the kill probability of weapon j on target i in stage t,

$\quad i = 1, 2, \ldots, N, \qquad j = 1, 2, \ldots, M$,

$q_{ij}(t) \equiv 1 - p_{ij}(t)$, the corresponding survival probability.

The decision variables will be denoted by:

$$x_{ij} = \begin{cases} 1 & \text{if weapon } j \text{ is assigned to target } i \text{ in stage } 1 \\ 0 & \text{otherwise.} \end{cases}$$

The *target state* of the system in stage 2 will be defined as the set of targets which survive stage 1. This state will be denoted by an N-dimensional binary vector $\vec{u} \in \{0,1\}^N$ and represented by

$$u_i = \begin{cases} 1 & \text{if target } i \text{ survives stage 1,} \\ 0 & \text{if target } i \text{ is destroyed in stage 1.} \end{cases}$$

The *weapon state* of the system in stage 2 will be defined as the set of available weapons after stage 1. This state will be denoted by an M-dimensional binary vector $\vec{w} \in \{0,1\}^M$ and represented by

$$w_j = \begin{cases} 1 & \text{if weapon } j \text{ was not used in stage 1,} \\ 0 & \text{if weapon } j \text{ was used in stage 1.} \end{cases}$$

Given a first stage assignment, $\{x_{ij}\}$, the target state at the start of the second stage is an N-dimensional random vector. The probability that u_i is 1 is the probability that target i survives the first stage. The probability that u_i is 0 is the probability that target i is destroyed in the first stage. The distribution of the random variable u_i is therefore given by:

$$\Pr[u_i = k] = k \prod_{j=1}^{M} (1 - p_{ij}(1))^{x_{ij}} + [1-k] \left\{ 1 - \prod_{j=1}^{M} (1 - p_{ij}(1))^{x_{ij}} \right\}, \tag{1}$$

$$\text{for} \qquad k = 0, 1, \qquad i = 1, 2, \ldots, N.$$

Equation 1 will be called the *target state evolution* of the system.

The weapon state also evolves with time. This evolution is deterministic and depends on the assignments made in the first stage. The evolution is given by:

$$w_j = 1 - \sum_{i=1}^{N} x_{ij}, \qquad j = 1, 2, \ldots, M. \tag{2}$$

This simply says that weapon j is available in the second stage if and only if it is not used in the first stage. Equation 2 will be called the *weapon state evolution* of the system.

We will let $F_2^*(\vec{u}, \vec{w})$ denote the *optimal* cost of a $T-1$ stage problem with initial target state $\vec{u}$ and initial weapon state $\vec{w}$. Note that this problem will be defined in terms of optimal costs for $T-2$-stage problems, etc. Eventually the $(T-(T-1)$ or single stage problem will be defined in terms of optimal costs for 0-stage problems. The optimal cost of a 0-stage problem will be defined as:

$$F_{T+1}^*(\vec{u}, \vec{w}) = \sum_{i=1}^{N} V_i u_i \tag{3}$$

In other words, the cost is simply the total value of the targets which survived the final stage.

Problem 3.1 *The Dynamic Weapon-Target Allocation problem (DWTA) can now be stated as:*

$$\min_{\{x_{ij}\}} F_1 = \sum_{\vec{\omega}\in\{0,1\}^N} \Pr[\vec{u} = \vec{\omega}] F_2^*(\vec{\omega}, \vec{w})$$

$$\text{subject to} \quad x_{ij} \in \{0,1\}, \qquad i = 1,2,\ldots,N \quad j = 1,2,\ldots,M,$$

$$\text{with} \quad w_j = 1 - \sum_{i=1}^{N} x_{ij}.$$

The objective function is the sum over all possible stage 2 target states of the probability of occurrence of that state times the optimal cost given that state. The probability distribution of the target state was given in 1. Note that the distribution of the stage 2 target state and the stage 2 weapon state both depend on the first stage assignment. The first constraint restricts each weapon to be assigned at most once in the first stage. The second constraint is due to the weapon state evolution.

This problem is considerably more difficult than the static one. This can be illustrated by attempting to use a straightforward dynamic programming approach to the problem. Let us consider a two stage problem. The number of possible weapon subsets that can be chosen in the first stage is 2^M. If m_1 weapons are used in stage 1 the number of possible assignments that must be checked is N^{m_1}. If $\tilde{N}$ of the N targets are engaged in the first stage the number of possible outcomes is $2^{\tilde{N}}$. If $\bar{N}$ of the N targets survive stage 1 and m_2 weapons are available in stage 2 then the number of assignments that must be checked to obtain the optimal cost for this outcome is $\bar{N}^{m_2}$. These numbers show the enormous number of computations that will be required if a straightforward dynamic programming approach is used. Note that to simply evaluate the expected value of a first stage assignment requires a tremendous computational effort.

3.2 Unit Valued Targets and Stage Dependent Kill Probabilities

In this subsection we will study the effect of stage dependent kill probabilities $p(t)$ on the optimal assignment. We will assume that the targets all have a value of unity and that the kill probabilities $p(t)$ are independent of the weapons and the targets. We were not able to obtain an analytical solution to this problem even for the case of two targets. However, we were able to obtain results for the limiting case, as the number of targets goes to infinity. We will first present some properties of the optimal solution.

Theorem 3.1 *Consider the dynamic version of the WTA problem in which there are T stages, N unit-valued targets, stage dependent kill probabilities $p(t)$, and M weapons. The optimal strategy has the property that the weapons to be used at each stage are spread as evenly as possible among the surviving targets.*

Proof: See the thesis by Hosein [6]. ■

The above result simplifies the problem to be solved since we can use the number of weapons to be used at each stage, m_t, as the decision variable and optimize over this variable. Given the optimal values of m_t, the optimal assignment can be obtained by spreading these weapons evenly among the targets. In the case of $T = 2$ the resulting problem is a one dimensional optimization problem since $m_1 + m_2 = M$. Intutively we would expect the expected cost to be a unimodal function with respect to the number of weapons used in stage 1. However, this is not the case as we see in the following two-stage example.

Let us choose m_1, the number of first stage weapons, as the independent variable. We will write the expected value if m_1 weapons are used in stage 1 and $M - m_1$ weapons are used in stage 2 by $F_1(m_1)$. The optimal solution can then be obtained by minimizing $F_1(m_1)$ over the set $\{0, 1, \ldots, M\}$.

If $F_1(m_1)$ was a unimodal function of m_1 then the above minimization could be done efficiently by using a local search algorithm. Unfortunately, this is not the case as can be seen in the following example. Consider the problem in which $T = 2$, $M = 14$, $N = 3$ and $p(1) = p(2) = 0.9$. In Figure 1 we have plotted $\log F_1(m_1)$ versus m_1. We used a log scale because the variations near the global minimum are so small that, with a linear scale, the function "appears" to have a single minimum. This suggests that for all practical purposes any of the local minima will suffice. A local minimum can easily be obtained by a local search algorithm (i.e. repeatedly increase or decrease m_1, if doing so decreases the cost, until any change in m_1 results in an increase in the cost.

Our next theorem concerns the case in which the number of weapons is less than the number of targets. Our intuition tells us that a dynamic allocation should not perform any better than a static one. This is indeed the case.

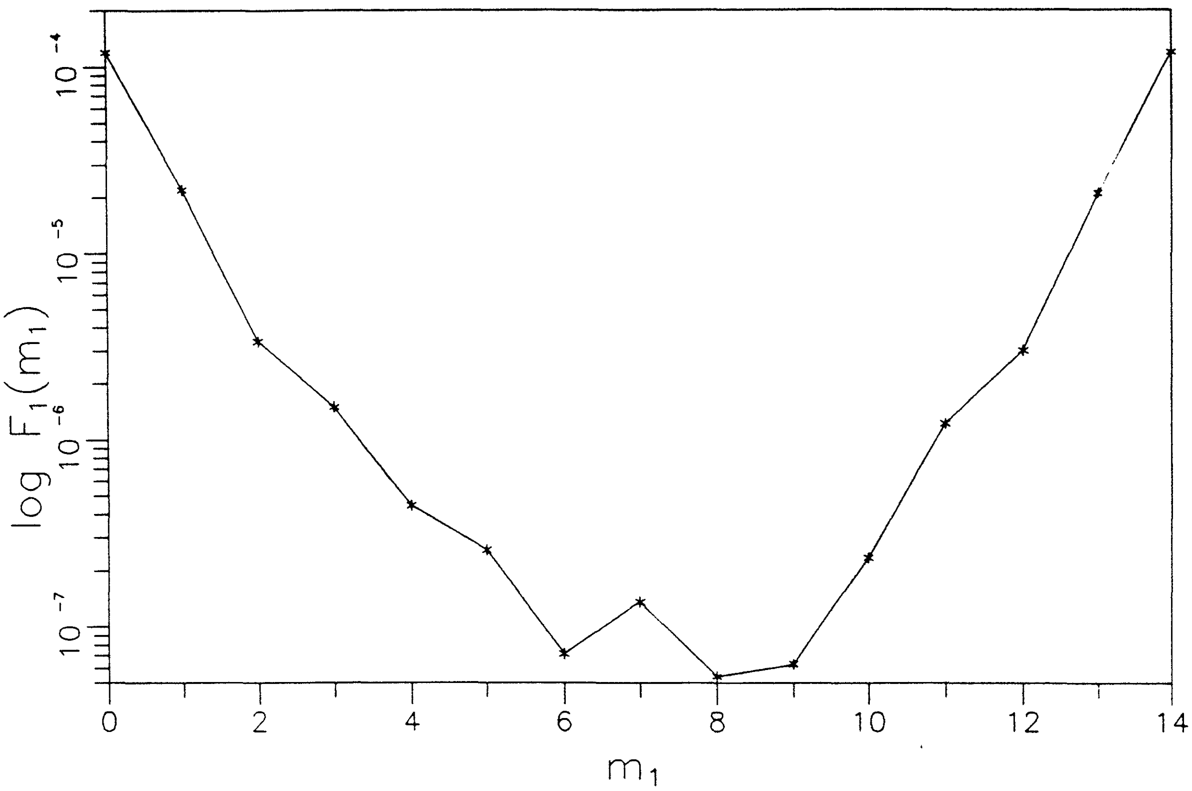

Figure 1: A two-stage example in which the expected cost as a function of the number of first stage weapons, $F_1(m_1)$, has multiple local minima.

Theorem 3.2 *If $M \leq N$, then the optimal strategy is to assign all of the weapons in the stage with the highest kill probability.*

Proof: See the thesis by Hosein [6]. ■

The above theorem is not particularly enlightening. However, it allows us to concentrate on the cases $M \geq N$. Our next result pertains to these cases. It states that, if $M \geq N$ and $p(t) \geq p(t+1)$ then the optimal assignment has the property $m_1^* \geq N$. In other words the optimal number of weapons to be used in the first stage is at least as big as the number of targets.

Theorem 3.3 *If $M \geq N$, and $p(t) \geq p(t+1)$ for $t = 1, 2, \ldots, T-1$, then $m_1^* \geq N$.*

Proof: See the thesis by Hosein [6]. ■

The next theorem concerns the case in which the number of stages is large. One would expect that if this is the case then at each stage one should assign a single weapon to each surviving target at that stage. If two weapons are assigned to a target at a stage and one of them destroys the target then the other weapon has essentially been wasted. This result is given in the following theorem.

Theorem 3.4 *If $T > 1 + \frac{M-N}{2}$, $M > N$ and $p(t) = p$ for $t = 1, \ldots, T$ then $m_1^* = N$.*

Proof: See the thesis by Hosein [6]. ■

3.2.1 The Limit of an Infinite Number of Targets

In this subsection we will consider what happens for very large numbers of unit-valued targets. We will keep the ratio of weapons to targets fixed and solve the problem in the limit as the number of targets goes to infinity. We will find that, in the limit, the problem can be considered as a deterministic one in which the number of targets in a stage is the expected number of targets which survive the previous stage.

Let us introduce the variable $\kappa_t \equiv \frac{m_t}{N}$. This is the number of weapons reserved for stage t per initial number of targets. We will also define the *vector* $\vec{\kappa_t} \in \Re^t$ for $1 \le t \le T$ by

$$\vec{\kappa_t} \equiv [\kappa_t, \kappa_{t+1}, \ldots, \kappa_T]^T.$$

Note that the values of κ_t may not be optimal for the problem. We will address the question of finding optimal values for κ_t in subsection 3.2.3. By theorem 3.1 we know that the weapons to be used in each stage should be spread evenly among the surviving targets. The expected cost of the T-stage problem with N targets and in which $m_t = \kappa_t N$ weapons are used in stage t will be denoted by $F_1(N, \vec{\kappa}_1)$. Let α denote the expected fraction, of the initial number of targets, which survive stage 1 i.e.

$$\alpha = [1 - (\kappa_1 - \lfloor \kappa_1 \rfloor) p(1)](1 - p(1))^{\lfloor \kappa_1 \rfloor}. \quad (4)$$

Note that α is independent of N. Consider the case of the static problem (i.e. $T = 1$). We have

$$\frac{F_1(N, \kappa_1)}{N} \equiv \alpha = [1 - (\kappa_1 - \lfloor \kappa_1 \rfloor) p(1)](1 - p(1))^{\lfloor \kappa_1 \rfloor}.$$

Taking the limit as N goes to infinity on both sides we get

$$\lim_{N\to\infty} \frac{F_1(N,\kappa_1)}{N} = [1-(\kappa_1 - \lfloor\kappa_1\rfloor)p(2)](1-p(2))^{\lfloor\kappa_1\rfloor} = \alpha. \tag{5}$$

In other words, for the static problem, if the weapon to target ratio is kept fixed then the expected fraction of targets which survive is the same for all values of N. This will also be the value in the limit as the number of targets goes to infinity. We will now show how the limit of this ratio can be obtained for more than one stages. The limit for the T-stage problem will be obtained in terms of the limit for the $T-1$ stage problem, etc. Since the limit for the case $T = 1$ (the static problem) is well defined then the limit for the two-stage problem is well defined etc. The T-stage limit is therefore well defined. The main result will now be presented.

Theorem 3.5 *Consider the T-stage problem with N unit valued targets, $M = \kappa N$ weapons and stage dependent kill probabilities $p(t)$. Assume that the number of weapons to be used in stage t is given by $m_t = \kappa_t N$, where $\kappa_t \in [0,\kappa]$ is a fixed constant which may be different for each stage. We then have that*

$$\lim_{N\to\infty} \frac{F_1(N,\vec{\kappa}_1)}{N} = \alpha \lim_{N\to\infty} \frac{F_2(N,\vec{\kappa}_2/\alpha)}{N}, \tag{6}$$

where α is given by equation 4.

Proof: Let N_2 represent the number of targets which survive stage 1. N_2 is a random variable. If κ_1 is an integer then it is a binomial random variable; otherwise, its distribution can be obtained by the convolution of two binomial distributions. The mean and variance of this distribution is given by:

$$E[N_2] \equiv \bar{N}_2 = \alpha N,$$

$$Var[N_2] \equiv \sigma^2 = \beta N,$$

where

$$\beta = q(1)^{\lfloor\kappa_1\rfloor}[(1-(\kappa_1-\lfloor\kappa_1\rfloor))(1-q(1)^{\lfloor\kappa_1\rfloor}) + (\kappa_1-\lfloor\kappa_1\rfloor)q(1)(1-q(1)^{\lfloor\kappa_1\rfloor+1})]$$

Note that β is independent of N. For any $\mu > 0$ we have

$$\begin{aligned} F_1(N,\vec{\kappa_1}) &= \Pr(|N_2-\bar{N}_2| \geq \mu N)E[F_2(N_2,\vec{\kappa}_2)||N_2-\bar{N}_2| \geq \mu N)] \\ &\quad + \Pr(|N_2-\bar{N}_2| < \mu N)E[F_2(N_2,\vec{\kappa}_2)||N_2-\bar{N}_2| < \mu N)]. \end{aligned} \tag{7}$$

By Chebyshev's Inequality we know that

$$\Pr(|N_2 - \bar{N}_2| < \mu N) \geq 1 - \frac{\sigma^2}{(\mu N)^2} = 1 - \frac{\beta}{\mu^2 N}. \tag{8}$$

Since $F_2(N_2, \vec{\kappa}_2)$ is a monotonically increasing function of N_2 then

$$E[F_2(N_2, \vec{\kappa}_2)||N_2 - \bar{N}_2| < \mu N] \leq F_2(\bar{N}_2 + \mu N, \vec{\kappa}_2), \tag{9}$$

and

$$E[F_2(N_2, \vec{\kappa}_2)||N_2 - \bar{N}_2| < \mu N] \geq F_2(\bar{N}_2 - \mu N, \vec{\kappa}_2), \tag{10}$$

and also

$$E[F_2(N_2, \vec{\kappa}_2)||N_2 - \bar{N}_2| \geq \mu N] \leq F_2(N, \vec{\kappa}_2), \tag{11}$$

Using 8 9, 10 and 11 in 7 we obtain

$$(1 - \frac{\beta}{\mu^2 N})F_2(\bar{N}_2 - \mu N, \vec{\kappa}_2) \leq F_1(N, \vec{\kappa_1}) \leq \frac{\beta}{\mu^2 N}[F_2(N, \vec{\kappa}_2)] + F_2(\bar{N}_2 + \mu N, \vec{\kappa}_2) \tag{12}$$

Dividing by N and taking the limit as N goes to infinity we obtain

$$\lim_{N \to \infty} \frac{F_2(\bar{N}_2 - \mu N, \vec{\kappa}_2)}{N} \leq \lim_{N \to \infty} \frac{F_1(N, \vec{\kappa_1})}{N} \leq \lim_{N \to \infty} \frac{F_2(\bar{N}_2 + \mu N, \vec{\kappa}_2)}{N}.$$

Using the fact that $\bar{N}_2 = \alpha N$ and taking μ arbitrarily close to 0 we obtain

$$\lim_{N \to \infty} \frac{F_1(N, \vec{\kappa_1})}{N} = \lim_{N \to \infty} \frac{F_2(\alpha N, \vec{\kappa}_2)}{N}.$$

Using a change of variables we finally obtain

$$\lim_{N \to \infty} \frac{F_1(N, \vec{\kappa_1})}{N} = \alpha \lim_{N \to \infty} \frac{F_2(N, \vec{\kappa}_2/\alpha)}{N}.$$

This completes the proof. ■

Note that the theorem gives the limit of the T-stage problem in terms of the limit for a $(T-1)$-stage problem. The latter can be expressed in terms of the limit of a $(T-2)$-stage problem etc. The limit for the case $T = 1$ is given in equation 5. This limit provides us with a lower bound for finite values of N. This result is given in the next theorem.

Theorem 3.6 *Consider the T-stage problem with N unit valued targets, $M = \kappa N$ weapons and stage dependent kill probabilities $p(t)$. Assume that the number of weapons to be used in stage t is given by $m_t = \kappa_t N$, where $\kappa_t \in [0, \kappa]$ is a fixed constant which may be different for each stage. We then have that*

$$F_1(N, \vec{\kappa}_1) \geq N \lim_{\hat{N} \to \infty} \frac{F_1(\hat{N}, \vec{\kappa}_1)}{\hat{N}}. \tag{13}$$

Proof: Let $k \in \aleph$, be any positive integer. Consider the problem with kN targets and in which $m_t = k\kappa_t N$ weapons are used in stage t. Let $F_1(kN, \vec{\kappa}_1)$ denote the optimal cost for this problem. A sub-optimal solution for this problem is the following. Split the problem into k subproblems. Each of these subproblems has N targets and uses $m_t = \kappa_t N$ weapons in each stage. The optimal cost for the problem under this restriction is given by $kF_1(N, \vec{\kappa}_1)$. Since this solution is suboptimal we have:

$$F_1(kN, \vec{\kappa}_1) \leq kF_1(N, \vec{\kappa}_1).$$

Dividing both sides by kN and taking the limit as k goes to infinity we have

$$\frac{F_1(N, \vec{\kappa}_1)}{N} \geq \lim_{k \to \infty} \frac{F_1(kN, \vec{\kappa}_1)}{kN} = \lim_{\hat{N} \to \infty} \frac{F_1(\hat{N}, \vec{\kappa}_1)}{\hat{N}}.$$

The result 13 now follows. ∎

Theorem 3.6 provides us with a lower bound on the optimal cost for the problem with finite values of N. Theorem 3.5 is more easily understood if we look at some examples.

Example 1

Suppose that $\kappa = 2, \kappa_1 = 0.5, \kappa_2 = 1.5$ and $p = 0.6$. In other words the defense has $2N$ weapons, $N/2$ weapons are used in stage 1 and the remainder are used in stage 2. The expected fraction of targets which survive stage 1 is given by

$$\alpha = \frac{1}{2}[(1-p)+1] = 0.7.$$

Therefore the expected value in stage 2 given that the expected number of targets survive stage 1 is given by:

$$F_2(\alpha, \kappa_2) = F_2(0.7, 1.5) = [(1-p)^3 + 6(1-p)^2]/10 = 0.1024$$

Note that we had to scale the number of weapons and the number of targets by a factor of 10 so that there are an integral number of each. If we now use the theorem we obtain:

$$\lim_{N\to\infty} \frac{F_1(N,[.5,1.5])}{N} = 0.1024.$$

In words this says the following. For very large N, if 25% of the weapons are used in stage 1 then approximately 10% of the targets will survive both stages. For comparison, if a static strategy is used then 16% of the targets will survive. If we consider the case of two targets, $N = 2$, then 13.12% of the targets will survive both stages. Let us now consider the case of a three-stage problem.

Example 2

Consider the 3 stage problem with $\kappa = 3, \kappa_1 = \kappa_2 = \kappa_3 = 1$, and $p = 0.5$. In the limit the expected fraction of targets which survive stage 1 is $\frac{1}{2}$. The expected fraction which survive stage 2 is $\frac{1}{8}$ and the expected fraction which survives stage 3 is $\frac{1}{2^{11}}$. Therefore,

$$\lim_{N\to\infty} \frac{F_1(N,[1,1,1])}{N} = 2^{-11}.$$

Let us now consider a case with stage dependent kill probabilities.

Example 3

Suppose that $\vec{\kappa_1} = [1.5, 1, .5]$ and that $p(1) = .6, p(2) = .5, p(3) = .4$. The expected fraction of targets which survive stage 1 is given by

$$\alpha = 0.5[(1-p(1)) + (1-p(1))^2] = 0.28.$$

The expected fraction which survives stage 2 is the solution to a static problem with 0.28 targets and 1 weapon. To find the limit for this problem we find the cost for the case of 7 targets and 25 weapons (i.e multiply by 25) and divide the cost by 25. We obtain

$$\alpha = [4(1-p(1))^4) + 3(1-p(1))^3]/25 = 0.025.$$

The expected fraction which survives the final stage is the solution to a static problem with 0.025 targets and .5 weapons. Multiplying the parameters by 40 etc. we obtain

$$\alpha = (1-p(2))^{20}/40 = 9.1 \times 10^{-7}.$$

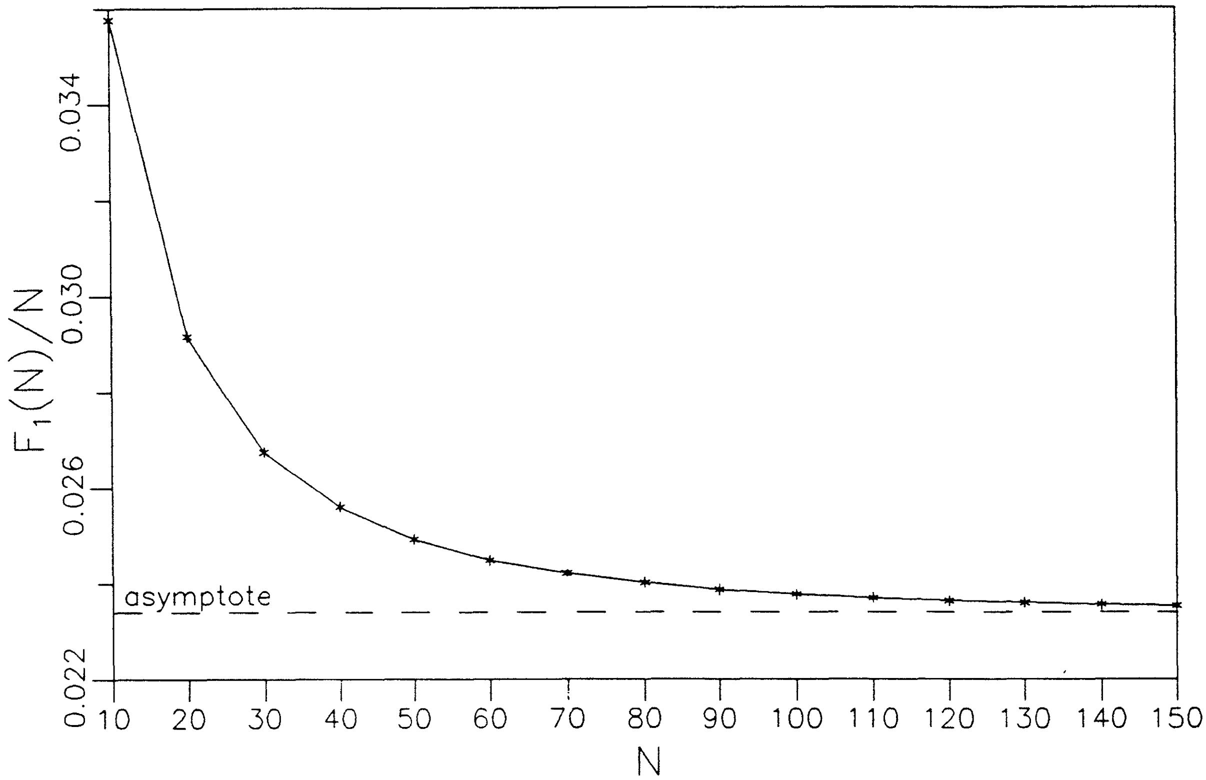

Figure 2: The ratio of the expected two-stage cost and the initial number of targets N vs. N for $p(1) = 0.6, p(2) = 0.7$; N weapons are used in each stage.

this case we used a second stage kill probability of $p(2) = 0.7$ and a first stage kill probability of $p(1) = 0.6$. Additional examples can be found in [6].

3.2.2 Optimal Number of First-Stage Weapons for a Two-Stage problem with a Large Number of Targets

Note that in the discussion in the previous subsection the number of weapons to be used in each stage was fixed. In this section we will find optimal values for $\vec{\kappa}$ as N goes to infinity. This will give us a good approximation to the optimal solution for large values of N.

We will only consider the two-stage case, $T = 2$. The optimization could also be attempted for $T > 2$, but it is doubtful whether one can find an analytical solution for such cases. For the case $T = 2$ we know that $\kappa_2 = \kappa - \kappa_1$ since all remaining weapons are used in the second stage. We therefore have a one dimensional optimization problem. We will let κ_1 be the free variable. The

Therefore in the limit as the number of targets goes to infinity, the expected fraction of the initial number of targets which survives all stages is 9.1×10^{-7}.

Theorem 3.5 is important because it allows us to compute approximate costs for the case of large N. This approximation is typically good for values of N greater than 100. Theorem 3.6 says that this limit provides a lower bound on the cost for finite values of N.

In words theorem 3.5 says the following. Let us suppose that the number of weapons reserved for a stage is linearly dependent on the initial number of targets N. Therefore, as we increase the number of targets, the number of weapons in each stage will increase at the same rate. As we increase the number of targets, the expected number of targets which survive the final stage will also increase. Let us instead consider the ratio of the expected number of surviving targets and the initial number of targets. The theorem says that we can compute this ratio in the limit of an infinite number of targets N by solving a related deterministic problem. This deterministic problem is obtained as follows. Let us suppose that at each stage the number of surviving targets is equal to the *expected* number of surviving targets. Pick the initial number of targets N so that the exepcted number of surviving targets at each stage is integral. Using this value of N we evaluate the expected surviving number of targets at the end of the final stage of the deterministic problem in which, at each stage the expected number of surviving targets survive the previous stage. The ratio of the expected number of surviving targets for this problem and the initial number of targets N is the same as the ratio, in the limit as N goes to infinity, of the expected number of surviving targets and the initial number of targets. Note that the former ratio is obtained by solving a deterministic problem while the latter ratio must be obtained by solving a *stochastic* problem for an infinite number of targets. This limit provides a lower bound for the ratio for finite values of N. Furthermore, it provides an approximate answer for large values of N. An interesting question is how large does N have to be for the approximation to be good.

Let us take the following example. Consider the problem of two stages $T = 2$ with $M = 2N$ weapons. N weapons are used in each of the stages (i.e. $\vec{\kappa} = [1,1]$). We computed the exact value of the ratio $\frac{F_1(N,\vec{\kappa}_1)}{N}$ for $N = 10, 20, \ldots, 150$, and also in the limit as N goes to infinity. In figure 2 we have plotted this ratio for finite values of N as well as the ratio in the limit of infinite N. In optimization problem can be stated as:

$$\min_{\kappa_1} F_2(\alpha, \kappa - \kappa_1) \tag{14}$$

$$\text{subject to} \qquad \kappa_1 \in [0, \kappa]$$

where

$$\alpha = [1 - (\kappa_1 - \lfloor \kappa_1 \rfloor) p(1)](1 - p(1))^{\lfloor \kappa_1 \rfloor}.$$

The function $F_2(\alpha, \kappa_2)$ is given by:

$$F_2(\alpha, \kappa_2) = [\alpha - p(2)(\kappa_2 - \alpha \lfloor \frac{\kappa_2}{\alpha} \rfloor) q(2)^{\lfloor \frac{\kappa_2}{\alpha} \rfloor}.$$

This expression is difficult to optimize. However, if the integrality constraint is relaxed, then the expected cost is given by $\alpha q(2)^{\frac{\kappa_2}{\alpha}}$. Since this is a lower bound for the non-relaxed problem, then

$$F_2(\alpha, \kappa_2) \geq \alpha q(2)^{\frac{\kappa_2}{\alpha}}. \tag{15}$$

This states that the solution obtained by allowing fractional assignments in the second stage is a lower bound to the solution in which only integral assignments are allowed. Note that if $\frac{\kappa_2}{\alpha}$ is a non-negative integer then equality holds in expression 15. Therefore, if the solution to the problem using the lower bound as the objective function is a multiple of α then it is optimal for the true problem.

The optimization problem using the lower bound in 15 as the objective function can be stated as:

$$\min_{\kappa_1} \alpha q(2)^{\frac{\kappa - \kappa_1}{\alpha}}. \tag{16}$$

$$\text{subject to} \qquad \kappa_1 \in [0, \kappa]$$

where

$$\alpha = [1 - (\kappa_1 - \lfloor \kappa_1 \rfloor) p(1)](1 - p(1))^{\lfloor \kappa_1 \rfloor}.$$

Let us first consider the case $\kappa = 1$. Note that theorem 3.2 has already provided us with a solution for this case. The solution is simply that all weapons should be assigned in the stage with the higher kill probability. Therefore,

$$\kappa_1^* = 0 \quad \text{for} \quad p(1) < p(2) \tag{17}$$

$$\kappa_1^* = 1 \quad \text{for} \quad p(1) \geq p(2) \tag{18}$$

Let us now consider the case in which $\kappa = 2$, i.e. a 2:1 weapon to target ratio. Using straightforward calculus one can show that the optimal values of κ_1 are given by

$$\kappa_1^* = 0 \quad \text{for} \quad \frac{2p(1)-1}{p(1)} \leq \frac{1}{\log(1-p(2))} \tag{19}$$

$$\kappa_1^* = 1 \quad \text{for} \quad \frac{2p(1)-1}{p(1)[1-p(1)]} \geq \frac{1}{\log(1-p(2))} \geq \frac{-1}{p(1)} \tag{20}$$

$$\kappa_1^* = 2 \quad \text{for} \quad \frac{-1}{p(1)[1-p(1)]} \geq \frac{1}{\log(1-p(2))} \tag{21}$$

Note that if $\frac{1}{1-p(1)}$ is a positive integer then equality holds in 15. If this is the case then κ_1^* is optimal for problem 14. Otherwise κ_1^* is approximately optimal.

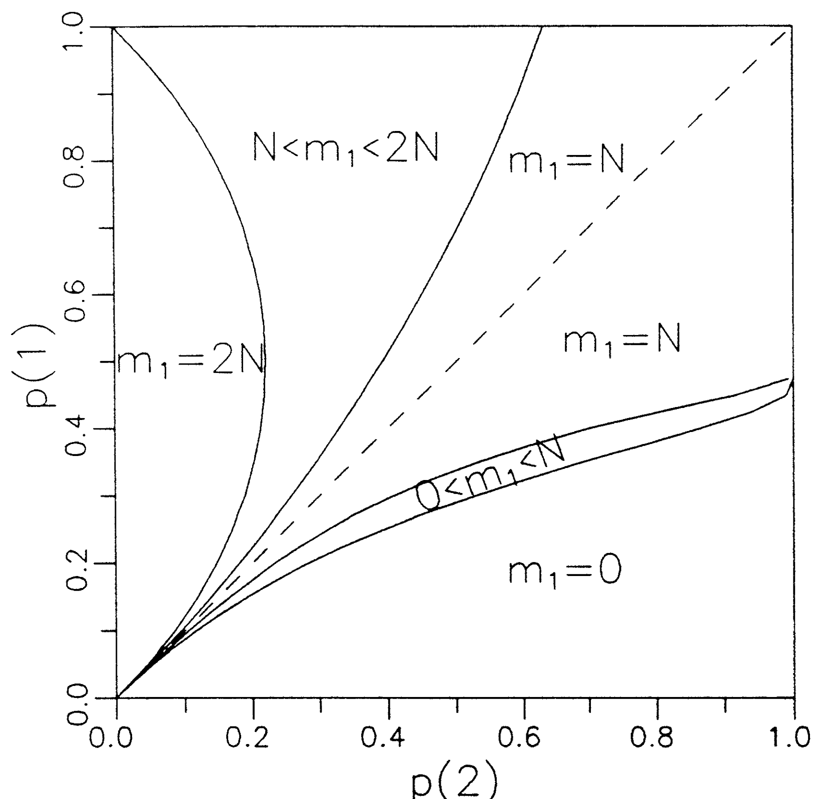

Figure 3: Optimal number of first-stage weapons, m_1, for various kill probabilities with $M = 2N$ weapons, in the limit of an infinite number of targets, N.

In the plot in figure 3 the vertical axis represents the kill probability in stage 1 while the horizontal axis represents the kill probability in stage 2. In each region we have indicated the optimal value of m_1, the number of weapons allocated in the first stage (recall that $m_1^* = \kappa_1^* N$) for the kill probabilities in that region. For example, consider the case $p(1) = 0.8$. If $0 < p(2) < 0.15$ then it is optimal to use all weapons in stage 1. If $0.15 < p(2) < 0.55$ then the optimal number of weapons to be used in stage 1 lies between N and $2N$. If $p(2) > 0.55$ it is optimal to use half of the weapons in stage 1.

Note that for $0.6 \leq p(1) \leq 0.9$ and $0.6 \leq p(2) \leq 0.9$ it is optimal to use half of the weapons in stage 1. This implies that for the problems of interest to us (i.e large-scale problems with kill probabilities greater than 0.6) it is optimal to use half of the weapons in stage 1, even if the kill probabilities are different in each stage. This insensitivity of the optimal strategy to the kill probabilities is very interesting. We should stress that this result is valid for large numbers of unit-valued targets and weapons

3.3 The Case of Two Targets

In the previous section we considered the problem in the limit of an infinite number of targets. In this section we will consider the case of two targets.

3.3.1 Two Different-Valued Targets and a Uniform Kill Probability

In this subsection we will assume that the two targets have different values and that the kill probability is the same for all weapon-target pairs in all stages. In the following theorem, we will show that, in the optimal strategy, the weapons are spread as evenly as possible among the stages left. Once the number of weapons to be used in the first stage is known the optimal assignment of these weapons to targets must be computed. We will see that this assignment can be obtained by solving a static problem.

Theorem 3.7 *An optimal strategy for the special case of the Dynamic WTA problem in which $N = 2$ and $p_i(t) = p$ is as follows. Let $\bar{x}_1$ and $\bar{x}_2$ denote the optimal assignment of the two-target* **static** *problem with the same target values and kill probabilities as the dynamic problem but with $\lfloor \frac{M}{T} \rfloor$ weapons. The optimal decision variables for the dynamic problem is given by $x_1^* = \bar{x}_1, x_2^* = \bar{x}_2, m_1^* = \lfloor \frac{M}{T} \rfloor$.*

Proof: See the thesis by Hosein [6]. ■

Theorem 3.7 is an interesting result because we find that the weapons are spread evenly among the stages. We will now compute the cost of the optimal strategy. Define the following variables:

$$m_l \stackrel{\text{def}}{=} \lfloor \frac{M}{T} \rfloor,$$

$$
\begin{aligned}
m_u &\stackrel{\text{def}}{=} \lceil \tfrac{M}{T} \rceil, \\
x_{ll} &\stackrel{\text{def}}{=} \lfloor \tfrac{m_l}{T} \rfloor, \\
x_{lu} &\stackrel{\text{def}}{=} \lfloor \tfrac{m_u}{T} \rfloor, \\
x_{ul} &\stackrel{\text{def}}{=} \lceil \tfrac{m_l}{T} \rceil, \\
x_{uu} &\stackrel{\text{def}}{=} \lceil \tfrac{m_u}{T} \rceil
\end{aligned}
$$

Using the results from theorem 3.7 it can be shown that the optimal cost $F_1^*(M)$ for the case $V_1 = V_2 = 1$ is given by:

$$F_1^*(M) = (M - Tm_l)[q^{x_{lu}} + q^{x_{uu}}] + (Tm_u - M)[q^{x_{ll}} + q^{x_{ul}}] - 2(T-1)q^M. \tag{22}$$

In the special case in which $M = 2kT$ for some positive integer k (i.e. if the two targets survive all stages then, in each stage, k weapons will be assigned to each of them), the optimal cost can be simplified to

$$F_1^*(M) = 2[Tq^{M-k} - (T-1)q^M]. \tag{23}$$

Note that if the number of stages is large then $F_1^*(M) \approx 2q^M$. The optimal cost for the static problem with M weapons is $2q^{\frac{M}{2}}$. This implies that roughly half as many weapons are required for the dynamic case to produce the same optimal cost as for the static one.

3.3.2 Two Different-Valued Targets with Stage and Target Dependent Kill Probabilities

In this subsection we will consider the case of two different valued targets with stage and target dependent kill probabilities. Since the kill probabilities are weapon independent then one can show that the decision variables are the number of weapons to be used in stage one and the assignment of these weapons to the targets. The problem can therefore be solved as follows. We first solve the problem in which the number of weapons to be used in stage one is fixed. This will be called the assignment sub-problem. We can then find the solution to the orginal problem by optimizing over the number of weapons to be used in stage one.

We have already shown that the problem of obtaining the optimal number of weapons to be used in stage one is a difficult one since multiple minima may exist (recall figure 1). One must essentially do a global search to obtain the global optimum.

Let us now consider the assignment sub-problem. In this case the number of weapons to be used in stage one is fixed and we need to find the optimal assignment of weapons to targets. We can show that this assignment can be found by using a greedy algorithm *with modified target values.* Details of this algorithm can be found in [6].

3.4 Numerical Results

In this section we will consider the case of N equally valued targets and a uniform kill probability. In the previous subsections we considered the case of two targets as well as the case of an infinite number of targets. For general N there does not appear to be an analytic solution to the problem. One must therefore compute solutions numerically. In this section we will compute the solutions for some simple cases and use the results of the previous sections to provide bounds.

Theorem 3.1 states that, in the optimal strategy of this problem, the weapons to be used in each stage should be spread evenly among the surviving targets. The decision variable will therefore be the number of weapons to be used in the first stage, m_1. The remaining weapons are used in the remaining stages. Given the optimal values of m_1 the optimal assignment can be obtained by spreading these weapons evenly among the targets. The expected cost for the T stage problem in which m_1 weapons are used in the first stage will be denoted by $F_1(m_1)$. We computed optimal solutions for a two stage problem with N unit-valued targets, M weapons and a single kill probability p

Figure 4 is a plot of the ratio of the optimal dynamic two-stage cost to the optimal static cost versus the kill probability p with a 2:1 weapon to target ratio (i.e $M = 2N$). We have plotted the cases N=2,4,6,8 and 10. We have also plotted the ratio in the limit as N goes to infinity. Note that this provides a lower bound for the case of finite N. Here we see that, as the sizes of both offensive and defensive stockpiles increase, the cost advantage of the dynamic strategy increases. This implies that, for large-scale problems, the dynamic shoot-look-shoot strategy will have a significant cost advantage over the static one.

Figure 5 contains a plot of the ratio of the optimal two-stage cost to the optimal static cost versus the number of weapons M with a kill probability of $p = 0.5$. We have plotted the cases N=2,4,6,8 and 10. Note that the cost advantage of the dynamic strategy increases roughly exponentially with the number of weapons. This implies that the dynamic strategy is significantly better even for

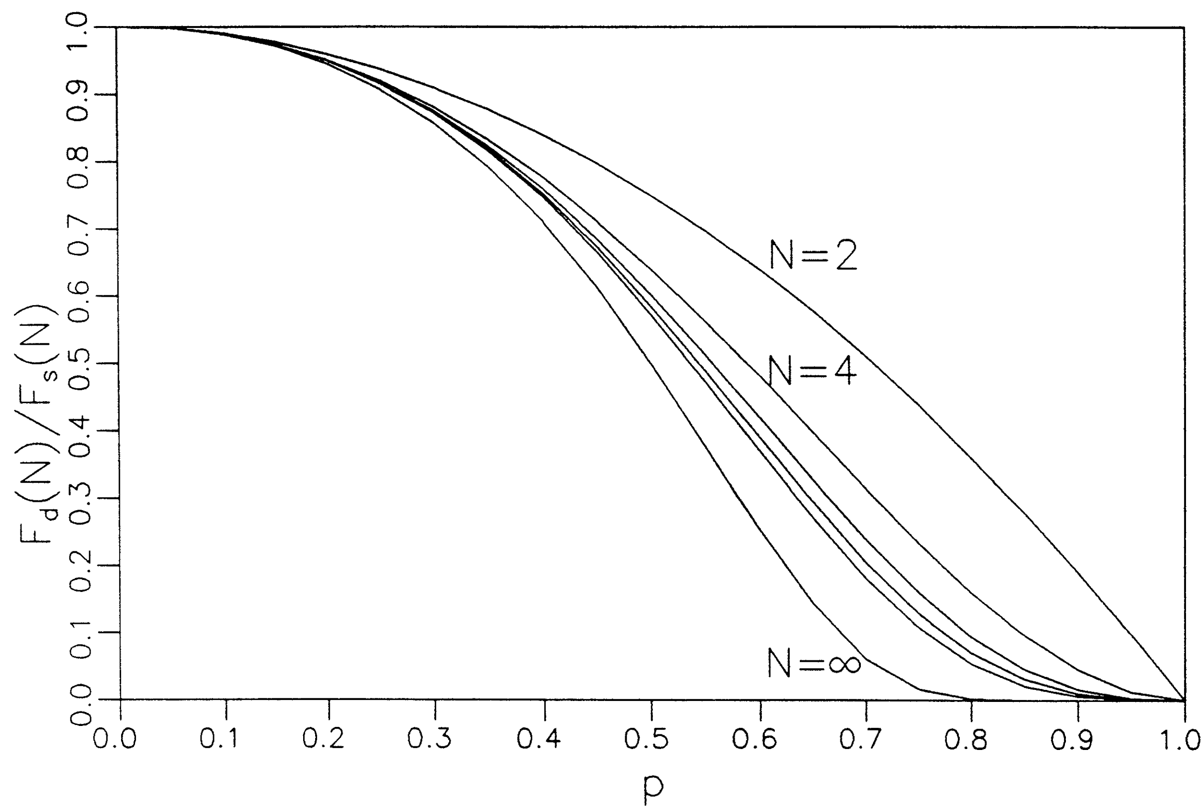

Figure 4: Plot of the ratio of the optimal dynamic (two-stage) and static costs vs the kill probability for a 2:1 weapon-target ratio, ($N = 2, 4, 6, 8, 10, \infty$).

relatively small weapon to target ratios.

Figure 6 contains a plot of the ratio of the optimal dynamic and static costs versus the number of stages T. We used a 2:1 weapon to target ratio and $p = 0.5$. The cases $N = 2, 3$ and 4 were plotted as well as the limiting case as N goes to infinity. The latter plot provides a lower bound for all cases of finite N. Using theorem 3.5 we can show that in the limit as N goes to infinity the ratio of the T stage cost to the static cost is equal to 2^{1-T}.

Note that the advantage of the dynamic strategy increases with the number of stages. For finite values of N the advantage increases up to a finite number of stages. Beyond this point the advantage remains constant because there are not enough weapons to make use of the additional stages. Note that, for large values of N, most of the improvement is obtained for a small number of stages (approximately 5 for this example). For a kill probability of 0.8 most of the improvement will be obtained for three stages. Recall that the computational complexity of the problem increases exponentially with the number of stages. This suggests that the defense should use a small number of stages in its strategy (roughly 3) since this provides a significant increase in performance over

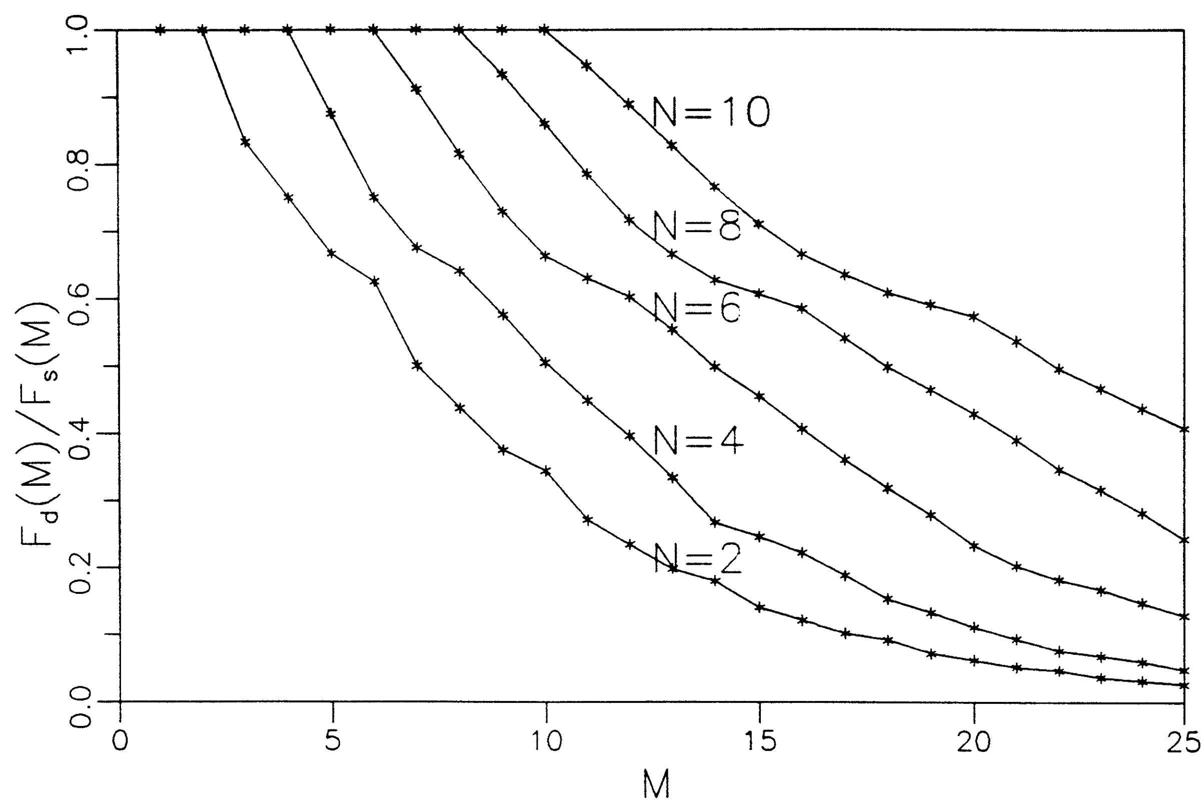

Figure 5: Plot of the ratio of the optimal dynamic (two-stage) and static costs vs. the number of weapons M, for different numbers of targets N=2,4,6,8,10, with $p = 0.5$.

the static strategy and the computational complexity is not too great.

4 Conclusions

The following conclusions about the dynamic WTA problem can be drawn. We have seen that an optimal solution cannot be obtained for the general problem 3.1 (in practice) because of the computational complexity of the problem.

Even under the assumption of weapon independent kill probabilities, the problem is still computationally difficult because multiple minima may exist (proven by example). However, we have also found that, if this is the case then, the difference in cost between any two local minima is small compared to the cost of either of them. This suggests that each of these local minima corresponds to a near-optimal solution to the problem.

If we assume weapon independent kill probabilities and assume that the number of weapons to be used in each stage is fixed, the problem is still difficult. The difficulty is due to the fact that the cost-to-go function is not separable with respect to the assignment variables. We can show that for

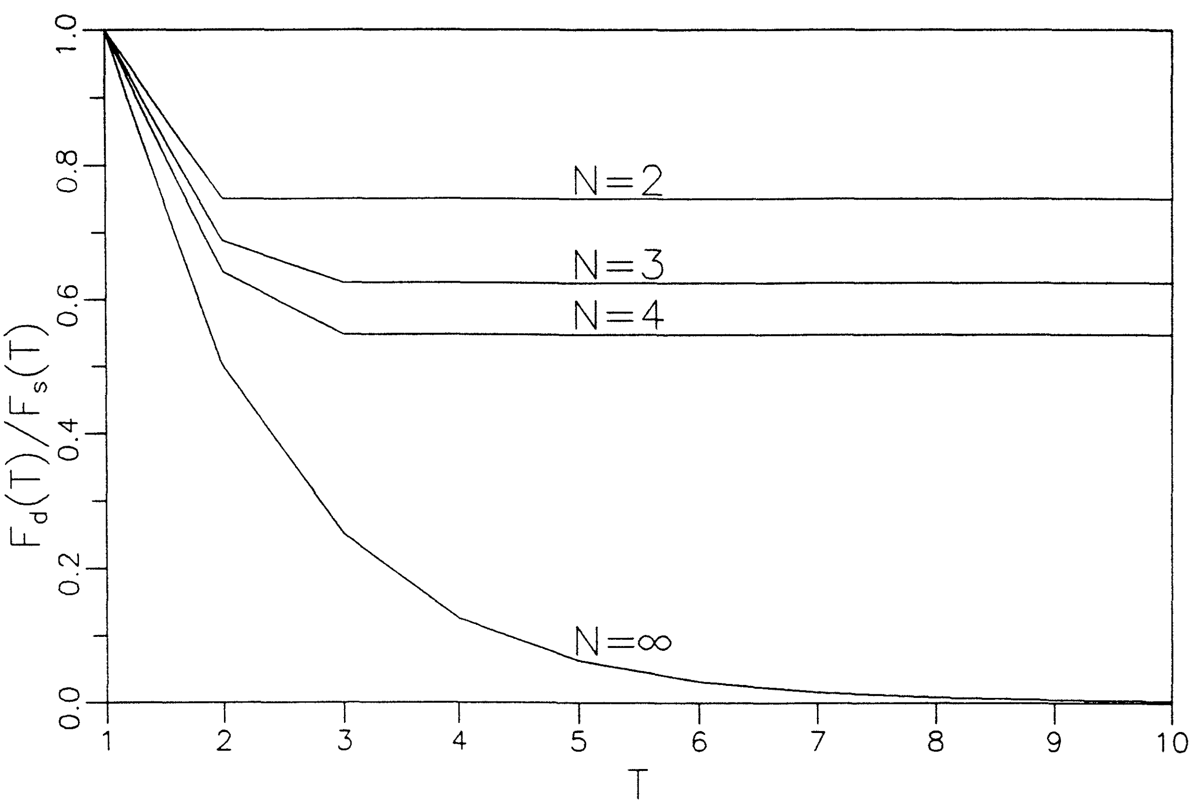

Figure 6: Plot of the ratio of the optimal dynamic and static costs vs the number of stages, T, available; $M/N = 2$ and $p = 0.5$, $(N = 2, 3, 4, \infty)$.

the case of two targets a greedy algorithm is optimal. We conjecture that such an algorithm will produce a near optimal solution for more than two targets.

For the case of unit valued targets, a single kill probability and many stages we have found that roughly half as many weapons are required for the dynamic strategy to obtain the same performance as the static one. This result was shown for the case of two targets. We can also show that it holds approximately for large numbers of targets. Our results also show that most of the efficiency of the dynamic problem is obtained by having 3-5 stages.

In the case of the two-stage problem with a large number of unit-valued targets, stage dependent kill probabilities in the range $0.6 \leq p(1), p(2) \leq 0.9$, and a 2:1 weapon target ratio, it is optimal to use half of the weapons in stage 1. This suggests that, for the more general problem, if the dependency of the kill probabilities on the stage number is small then a good approximate solution can be obtained by assuming stage independent kill probabilities.

In conclusion, the dynamic version of the WTA problem is significantly more difficult than the static version. However, based on our results, we believe that by using good heuristics one can reduce the computational complexity of the problem while maintaining its cost advantage. This suggests that further research should be concentrated on the dynamic rather than the static problem.

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