Chern-Simons Theory and Its Physical Applications

by

Qiang Liu

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Author ....
Department of Physics
May 18, 1995

Certified by ....
Roman Jackiw
Professor
Thesis Supervisor

Accepted by ....
George F. Koster
Chairman, Departmental Committee on Graduate Students

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Abstract

In this thesis, a brief review of Chern-Simons theory is given. Two physical applications of Chern-Simons theory are studied. First, the dynamics of non-relativistic Chern-Simons solitons are investigated. We show that in absence of background density these solitons move freely; however, when a background density is present, these solitons feel a Magnus force. In the second application, we study QCD in the high temperature limit. In this limit, the dominant effect is due to so called "hard thermal loop", which is shown to be an angular average of a Chern-Simons eikonal. In our investigation, we first show that this effect does not support soliton configurations and how the hard thermal loop arises from the stationary requirement of the composite effective action. Then, starting from classical transport theory, we demonstrate that the hard thermal loop is a classical effect. In addition, various problems related to classical transport theory, such as phase space and gauge structure of the theory, are addressed.

Thesis Supervisor: Roman Jackiw
Title: Professor

Dedicated to My Family

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Finally, I want to dedicate this thesis to my family, especially to my wife Qin Wang and to my parents Dechun Liu, Xianhuan Ye. Without their patience and constant encouragement, I would not be able to endure such a long journey.

Contents

Chapter 1

Introduction

Chern-Simons theory was introduced into physics over ten years ago as a possible cure for the infra-red problem of high-temperature gauge theory [1]. During the subsequent years, it has been widely studied by the theoretical physics community. It was discovered that the Chern-Simons action can induce spin transmutation, converting spin-zero bosons into particles carrying fractional spin, *i.e.* anyons [2] and this phenomenon is closely related to the quantum Hall effect and high-temperature superconductivity. Chern-Simons theory can be used to construct knot polynomials [3] as well as provides a unifying view point for two-dimensional conformal field theory [4].

More recently, two new applications of Chern-Simons theory were discovered. First, in a series of seminal papers, Jackiw *et al.* discovered that by coupling to ordinary scalar fields, both relativistic field and non-relativistic field, and choosing appropriate potentials, the Chern-Simons term can support soliton solutions [5, 6]. Later, the existence of soliton solutions in various generalizations of the original models was further investigated [7]. In particular, by adding background density to the original non-relativistic model, Barashenkov *et al.* found soliton solutions which exhibit distinct boundary behavior from their original counterparts. The second field in which Chern-Simons theory plays a role is high-temperature QCD. In a study of perturbative high-temperature QCD, Pisarski *et al.* observed that in order to have a consistent perturbation series, a resummation involving a subset of one-loop diagrams, the so-called "Hard Thermal Loops", is necessary [8]. Soon after that[9], Efraty and Nair showed that these Hard Thermal Loop effects can be elegantly summarized by a Chern-Simons eikonal. This discovery provides new insight on the gauge structure and dynamical nature of Hard Thermal Loops [9].

In this thesis, we will explore the physics of these two applications. More specifically, we will discuss the dynamics of two types of non-relativistic Chern-Simons solitons, as well as the physical nature and dynamical implication of Hard Thermal Loops. The thesis is structured as follows. Section 1.1 is a brief review of Chern-Simons theory. In Section 1.2, we give an overview of the two types of non-relativistic Chern-Simons solitons studied in this thesis. Section 1.3 presents the background materials of Hard Thermal Loops and its connection to the Chern-Simons eikonal. Chapters 2-4 are devoted to the study of these two topics. In Chapter 2, we study the dynamics of these Chern-Simons solitons. Chapter 3 is a discussion of the dynamical implication of hard thermal loops. In Chapter 4, we show that Hard Thermal Loops are a classical transport phenomenon and address some problems related to the classical formulation of Hard Thermal Loops. Finally, we end with conclusions in Chapter 5.

1.1 Review of Chern-Simons Theory

In this section, we shall give a brief review of Chern-Simons gauge theory [10]. Consider in $2+1$ dimensions, the Lagrange density for the Abelian Chern-Simons theory [1, 10]

$$\begin{aligned}\mathcal{L}_{cs} &= \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}\,, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\,, \end{aligned} \tag{1.1}$$

here κ is a constant, ∂_μ denotes the derivative with respect to x^μ and A_μ is a gauge field.

We start by reviewing the symmetry properties of this theory. First, we note that the Chern-Simons action $I_{cs} = \int d^3x \mathcal{L}_{cs}$ is a topological invariant [11]. Hence it is invariant against all coordinate transformations. This can be checked by computing the change of I_{cs} under an arbitrary coordinate transformation

$$\begin{aligned} &x_\mu \to X_\mu \,, \\ &A_\mu \to \tilde{A}_\mu \,, \\ &\tilde{A}_\mu(x) = A_\nu(X)\frac{\partial X^\nu}{\partial x^\mu} \,. \end{aligned} \tag{1.2}$$

Secondly, we consider the transformation of the Chern-Simons Lagrange density $\mathcal{L}_{cs}$ under a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \theta \,. \tag{1.3}$$

A straightforward calculation yields

$$\mathcal{L}_{cs} \to \mathcal{L}_{cs} + \frac{\kappa}{2}\partial_\mu(\epsilon^{\mu\alpha\beta}F_{\alpha\beta}) \,. \tag{1.4}$$

Thus I_{cs} changes by a total derivative under a gauge transformation. This guarantees that the equation of motion obtained from varying I_{cs} is gauge invariant.

In addition to these nice symmetry properties possessed by the Chern-Simons term (1.1), what makes it more interesting is the observation that it can be used as a gauge-invariant mass term for gauge fields [1]. Consider the following Lagrangian, composed of both the Chern-Simons term and the ordinary Maxwell term

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,. \tag{1.5}$$

Dimensional argument shows that κ has a dimension of mass.

Varying the above Lagrangian (1.5) with respect to the gauge field A_ν, we obtain the equation of motion

$$\partial_\mu F^{\mu\nu} + \frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \,. \tag{1.6}$$

Define the dual field $\mathcal{F}^\mu$

$$\begin{aligned}\mathcal{F}^\mu &= \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}\,,\\ F^{\alpha\beta} &= \epsilon^{\alpha\beta\mu}\mathcal{F}_\mu\,.\end{aligned} \tag{1.7}$$

Note that the dual field $\mathcal{F}^\mu$ is identically conserved

$$\partial_\mu\mathcal{F}^\mu = 0\,, \tag{1.8}$$

which is the consequence of the Bianchi identity

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0\,. \tag{1.9}$$

Using the dual field definition (1.7), we can rewrite the equation of motion (1.6) in terms of the dual field

$$(\kappa g^{\mu\alpha} + \epsilon^{\mu\alpha\beta}\partial_\beta)\mathcal{F}_\alpha = 0\,. \tag{1.10}$$

Multiplying this with the differential operator $(\kappa g^{\nu\mu} - \epsilon^{\nu\mu\gamma}\partial_\gamma)$, we have

$$(\Box + \kappa^2)\mathcal{F}^\nu = 0\,. \tag{1.11}$$

We see clearly that the gauge field excitation is massive and the Chern-Simons term plays the role of a mass term.

We can also formulate this theory in Hamiltonian formalism [2]. To do that, we first need to fix the gauge. For convenience, we choose to work in the Weyl gauge, *i.e.* $A_0 = 0$. The canonical variables in this gauge are $\mathbf{A}$. The canonically conjugate momenta $\mathbf{\Pi}$ can be determined by taking the functional derivatives of the Lagrangian with respect to $\mathbf{A}$

$$\Pi^i = -\dot{A}^i - \frac{\kappa}{2}\epsilon^{ij}A_j\,. \tag{1.12}$$

Then the Hamiltonian H can be obtained by using $H = \int d^2\mathbf{x}(\mathbf{\Pi} \cdot \dot{\mathbf{A}} - \mathcal{L})$

$$H = \frac{1}{2}\int d^2\mathbf{x}(\mathbf{E}^2 + B^2)\,, \tag{1.13}$$

where the electric fields $\mathbf{E}$ and the magnetic fields B are defined as

$$\begin{aligned} \mathbf{E} &= -\dot{\mathbf{A}} - \nabla A_0\,, \\ B &= \nabla \times \mathbf{A} = \epsilon^{ij}\partial_i A_j\,. \end{aligned} \tag{1.14}$$

Note that the Chern-Simons term makes no contribution to the energy (the Hamiltonian). This can be understood as a consequence of the topological nature of the Chern-Simons term [1, 10]: the Chern-Simons term (1.1) does not depend on the metric tensor $g_{\mu\nu}$; as a result, when we vary the action with respect to $g_{\mu\nu}$ to obtain the energy-momentum tensor, we see no contribution from $\mathcal{L}_{cs}$.

The Hamiltonian equation that follow from (1.13) must be supplemented by the Gauss's Law [2], which, in terms of $\mathbf{E}$ and B, can be written as

$$\nabla \cdot \mathbf{E} - \kappa B = 0\,. \tag{1.15}$$

When there is an external charge density ρ, the Gauss's Law is modified

$$\nabla \cdot \mathbf{E} - \kappa B = \rho\,. \tag{1.16}$$

Integrating (1.16) over the whole space yields

$$-\kappa \int d^2\mathbf{x}B = \int d^2\mathbf{x}\rho = Q\,. \tag{1.17}$$

The contribution from the electric fields $\mathbf{E}$ vanishes since $\mathbf{E}$, being massive, decrease exponentially at large distances. We see that the magnetic flux in the two-dimensional space is proportional to the external charge Q. This further implies that the magnetic fields $\mathbf{A}$ becomes long range when there is external charge, *i.e.*

$$\mathbf{A} \rightarrow -\nabla \frac{Q}{2\pi\kappa} \tan^{-1}\frac{x_2}{x_1} \ . \tag{1.18}$$

This is very similar to the soliton configuration in the Higgs model [13].

The Chern-Simons term (1.1) can be easily generalized to the non-Abelian case. The Lagrange density in that case is [10]

$$\mathcal{L}_{cs} = -\frac{\kappa}{2g^2}\epsilon^{\mu\nu\alpha}\mathrm{tr}(F_{\mu\nu}A_\alpha - \frac{2}{3}A_\mu A_\nu A_\alpha) \ . \tag{1.19}$$

Here we use a matrix notation

$$\begin{aligned} A_\mu &= gT^a A^a_\mu \, , \\ F_{\mu\nu} &= gT^a F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu \, , A_\nu] \, , \end{aligned} \tag{1.20}$$

where T^a is an anti-hermitian representation of some gauge group which satisfies the commutation relation $[T^a \, , T^b] = f^{abc}T^c$, g is the coupling constant and $\frac{\kappa}{g^2}$ is dimensionless.

Most of the results we obtained for the Abelian Chern-Simons theory can be straightforwardly generalized to the non-Abelian case. However, there is one novel feature of the non-Abelian theory. The non-Abelian Chern-Simons action $I_{cs} = \int d^3x\mathcal{L}_{cs}$ is not invariant against a finite gauge transformation

$$A_\mu \rightarrow U^{-1}A_\mu U + U^{-1}\partial_\mu U \, , \tag{1.21}$$

here U is a matrix and we shall only consider those U's satisfying $U(x) \stackrel{x\rightarrow\infty}{\longrightarrow} I$.

Indeed, a detailed calculation shows that under (1.21), the non-Abelian Chern-Simons action transforms as

$$\int d^3x\mathcal{L}_{cs} \rightarrow \int d^3x\mathcal{L}_{cs} + \frac{8\pi^2\kappa}{g^2}w(U) \, , \tag{1.22}$$

where $w(U)$ is the winding number of U

$$w(U) = \frac{1}{24\pi^2} \int d^3x \epsilon^{\alpha\beta\gamma} \mathrm{tr}[(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U)(U^{-1}\partial_\gamma U)] , \tag{1.23}$$

and only takes an integer value corresponding to the homotopy class of U [12]. While this property of the non-Abelian Chern-Simons action does not affect the classical equation of motion, it does impose severe restriction on the parameter κ of the theory upon quantization [1, 10]. This is because in quantum theory, what really matters is the exponential of the Chern-Simons action I_{cs}. Therefore the requirement of gauge invariance at quantum level translates to the condition that I_{cs} can only change by a multiple of 2π under gauge transformations. This, together with equation (1.22), leads to a quantization condition of the coefficients of the non-Abelian Chern-Simons term

$$4\pi \frac{\kappa}{g^2} = n, \qquad n = \pm 1, 2, \ldots , \tag{1.24}$$

which has no Abelian analog.

We close this review of Chern-Simons theory with a brief discussion of the Chern-Simons eikonal [14]. Consider the quantization of the Chern-Simons term (1.19) in the Schrödinger picture. First we rewrite the Lagrangian as

$$L_{cs} = \frac{\kappa}{4\pi g^2} \int d^2\mathbf{x} (A_2^a \dot{A}_1^a + A_0^a F_{12}^a) , \tag{1.25}$$

here we have indicated time differentiation by an over dot. Examining the Lagrangian, we see that A_1^a and A_2^a are a conjugated pair while A_0^a plays a role of Lagrangian multiplier, which forces F_{12}^a to vanish (the Gauss's Law).

Next, we postulate equal-time commutation relations

$$[A_1^a(\mathbf{r}, t), A_2^a(\mathbf{r}', t)] = i\delta^{ab}\delta(\mathbf{r} - \mathbf{r}') . \tag{1.26}$$

In addition, in order to satisfy the constraint enforced by A_0^a, we demand that the physical states be annihilated by F_{12}^a, *i.e.*

$$F_{12}^a \,|\!> = 0 \ . \tag{1.27}$$

To solve this equation, we choose our representation so that the state to be a functional of A_1^a. The operators A_1^a and A_2^a are realized by multiplications and functional differentiations respectively

$$\begin{aligned} |\!> &\sim \psi(A_1^a) \,, \\ A_1^a \,|\!> &\sim A_1^a \psi(A_1^a) \,, \\ A_2^a \,|\!> &\sim \frac{1}{i} \frac{\delta}{\delta A_1^a} \psi(A_1^a) \ . \end{aligned} \tag{1.28}$$

Note that this choice is consistent with the commutation relations (1.26).

In this representation, the Gauss's Law (1.27) becomes

$$\left(\partial_1 \frac{1}{i} \frac{\delta}{\delta A_1^a} - \partial_2 A_1^a + f_{abc} A_1^b \frac{1}{i} \frac{\delta}{\delta A_1^c}\right) \psi(A_1^a) = 0 \ . \tag{1.29}$$

If we define S by $\psi = e^{iS}$, we can rewrite (1.29) as

$$\partial_1 \frac{\delta}{\delta A_1^a} S - \partial_2 A_1^a + f_{abc} A_1^b \frac{\delta}{\delta A_1^c} S = 0 \ . \tag{1.30}$$

Recall that in ordinary quantum mechanics the wave function $\psi \sim e^{i\int p dx}$, we see that S is just the field theory analog of the ordinary quantum mechanical eikonal $\int p dx$. Thus we call S the Chern-Simons eikonal. An analytic form for S can then be obtained by solving equation (1.30) [14]. In Section 1.3, We will see that it has unexpected use in high-temperature QCD.

1.2 Non-relativistic Chern-Simons Solitons

Since 1990, a lot of work has gone into Chern-Simons solitons [15]. Soliton solutions have been found in both relativistic and non-relativistic models [5, 6]. In this thesis, we will mainly study two kinds of Chern-Simons solitons, both of which are non-relativistic.

1.2.1 Chern-Simons Solitons in Absence of Background Density

The first kind of Chern-Simons solitons was discovered by Jackiw and Pi [6]. In a seminal paper, Jackiw and Pi studied a theory described by the following Lagrangian

$$L = \int d^2\mathbf{x}(\frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + i\psi^* D_t\psi - \frac{1}{2}|\mathbf{D}\psi|^2 + \frac{1}{2}g(\psi^*\psi)^2) \,, \tag{1.31}$$

where $D_t = \partial_t + iA_0$, $\mathbf{D} = \nabla - i\mathbf{A}$. The Hamiltonian can be derived in the standard way

$$H = \frac{1}{2}\int d^2\mathbf{x}(|\mathbf{D}\psi|^2 - g(\psi^*\psi)^2) \,. \tag{1.32}$$

By varying the action $I = \int dtL$, we obtain the equations of motion

$$\begin{aligned} i\partial_t\psi &= -\frac{1}{2}(\mathbf{D}^2 + A^0 - g\psi^*\psi)\psi \,, \\ \frac{\kappa}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} &= J^\mu \,, \end{aligned} \tag{1.33}$$

here $J^\mu = (\rho, \mathbf{J})$ is defined as

$$\begin{aligned} \rho &= \psi^*\psi \,, \\ \mathbf{J} &= \mathrm{Im}(\psi^*\mathbf{D}\psi) \,. \end{aligned} \tag{1.34}$$

It is worth mentioning that this theory can also be formulated solely in terms of the matter field ψ [6]. In that case, the Lagrangian L' is

$$L' = \int d^2\mathbf{x} i\psi^*\partial_t\psi - H \,, \tag{1.35}$$

where H is given by (1.32) except that the vector potentials A_μ are functionals of ψ, *i.e.*

$$
\begin{aligned}
A_0(\mathbf{x},t) &= \frac{1}{\kappa}\int d^2\mathbf{x}'\mathbf{G}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}',t)\,, \\
\mathbf{A}(\mathbf{x},t) &= \frac{1}{\kappa}\int d^2\mathbf{x}'\mathbf{G}(\mathbf{x}-\mathbf{x}')\psi^*(\mathbf{x}',t)\psi(\mathbf{x}',t)\,,
\end{aligned}
\tag{1.36}
$$

with $\mathbf{G}$ satisfying

$$
\nabla\times\mathbf{G}(\mathbf{x}) = -\delta^2(\mathbf{x})\ . \tag{1.37}
$$

Before presenting the solitons solutions of this model, we would like to record some useful transformations that can be performed on this Lagrangian [6, 15]. We shall only give the transformation properties of ψ since those for the vector potentials can then be derived from equations (1.36). First, it is easy to see that the Jackiw-Pi model possesses the conventional translation and rotation symmetries, *i.e.*

$$
\begin{aligned}
\mathbf{x} &\to \mathbf{x}+\mathbf{a}, \quad t\to t+t_0, \quad \psi(\mathbf{x},t)\to\psi(\mathbf{x}+\mathbf{a},t+t_0)\,, \\
\mathbf{x} &\to \mathcal{R}\mathbf{x}, \quad\;\; t\to t, \qquad\;\; \psi(\mathbf{x},t)\to\psi(\mathcal{R}\mathbf{x},t)\,,
\end{aligned}
\tag{1.38}
$$

where $\mathcal{R}$ is a two-dimensional matrix implementing a rotation by angle φ: $\mathcal{R}(\varphi) = \delta^{ij}\cos(\varphi) - \epsilon^{ij}\sin(\varphi)$. Moreover, there exists the Galileo symmetry commonly found in non-relativistic models

$$
\mathbf{x}\to\mathbf{x}-\mathbf{v}t, \quad t\to t, \quad \psi(\mathbf{x},t)\to e^{i(\mathbf{x}\cdot\mathbf{v}-\frac{\mathbf{v}^2}{2}t)}\psi(\mathbf{x}-\mathbf{v}t,t)\ . \tag{1.39}
$$

Besides all these conventional symmetries, the Jackiw-Pi theory is also invariant under some special coordinate transformations. Consider the following transformation

$$
\begin{aligned}
& t\to T = T(t)\,, \\
& \mathbf{x}\to\mathbf{X} = \sqrt{\dot{T}(t)}\mathbf{x}\,, \\
& \psi\to\tilde{\psi}, \qquad \tilde{\psi}(\mathbf{x},t) = \sqrt{\dot{T}(t)}e^{-\frac{\mathbf{x}^2\ddot{T}}{4\dot{T}}}\psi(\mathbf{X},\mathbf{T})\ .
\end{aligned}
\tag{1.40}
$$

It is straightforward to show that the Jackiw-Pi action I is invariant when T is either

$T(t) = at$ or $T(t) = \frac{t}{1-at}$.

For an arbitrary T, transformation (1.40) is not a symmetry of the Jackiw-Pi theory. Rather it maps the model to a model with an external harmonic force with a time-dependent frequency ω given by

$$\omega^2(t) = -\sqrt{\dot{T}(t)} \frac{d^2}{dt^2} \frac{1}{\sqrt{\dot{T}(t)}} \ . \tag{1.41}$$

One can generalize this idea and construct transformations relating the Jackiw-Pi model to models with external electric and magnetic fields [16].

Now let us discuss the static solutions of the Jackiw-Pi model. Instead of solving the equations of motion (1.33) in the static case, we take an indirect approach. Using the identity

$$|\mathbf{D}\psi|^2 = |(D_1 - i\eta(\kappa)D_2)\psi|^2 - \eta(\kappa)(B\rho + \nabla \times \mathbf{J}), \quad \eta(\kappa) = sign(\kappa) \, , \tag{1.42}$$

we rewrite the Hamiltonian (1.32) as

$$H = \frac{1}{2} \int d^2\mathbf{x} [|(D_1 - i\eta(\kappa)D_2)\psi|^2 - (g - \frac{1}{|\kappa|})\rho^2] \, , \tag{1.43}$$

here we have dropped $\int d^2\mathbf{x} \nabla \times \mathbf{J}$ by assuming $\mathbf{J}$ is well behaved at large distances. From (1.43), we notice that when $g = \frac{1}{|\kappa|}$, the Hamiltonian is non-negative and attains its minimum, *i.e.* zero, when ψ satisfies the self-dual condition

$$(D_1 - i\eta(\kappa)D_2)\psi = 0 \ . \tag{1.44}$$

Henceforth we choose $g = \frac{1}{|\kappa|}$ and without loss of generality, take κ to be positive. In this case, equation (1.44) becomes

$$(D_1 - iD_2)\psi = 0 \ . \tag{1.45}$$

Eq. (1.45), together with the Gauss's Law

$$\nabla \times \mathbf{A} = -\frac{1}{\kappa}\rho\,, \tag{1.46}$$

is equivalent to the equations of motion (1.33), as can be confirmed by explicit computation.

To solve equations (1.45), we write $\psi = \rho^{\frac{1}{2}} e^{i\omega}$. Then by setting the real part and the imaginary part of (1.45) to zero separately, we have

$$\mathbf{A} = -\frac{1}{2}\nabla \times \ln\rho + \nabla\omega\ . \tag{1.47}$$

Inserting equation (1.47) into the Gauss's Law (1.46), leads to a constraint for ρ

$$\nabla^2 \ln\rho = -\frac{2}{\kappa}\rho\ . \tag{1.48}$$

This is just the Liouville equation [6]. However, we have no equation to determine the phase ω. Rather we fix it by requiring that $\mathbf{A}$ be regular everywhere on the plane.

Solving the equation (1.48) and imposing the regularity requirement as described above, we finally obtain the N-soliton solution

$$\psi = \rho^{\frac{1}{2}} e^{i\omega} \tag{1.49}$$

with

$$\rho = \frac{4\kappa|f'|^2}{(1+|f|^2)^2}, \qquad \omega = \mathrm{Arg}(f'V^2)\,, \tag{1.50}$$

where

$$f(z) = \sum_{m=1}^{N} \frac{c^m}{z - a^m}, \qquad V(z) = \prod_{m=1}^{N}(z - a^m)\,, \tag{1.51}$$

here $z = x_1 + ix_2$, $f' = \frac{df}{dz}$, a^m is the location of the mth soliton and c^m describes its phase and scale.

1.2.2 Chern-Simons Solitons in the Presence of Background Density

Now we turn our discussion to a model studied by Barashenkov and Harin [7]. The Lagrangian is

$$L = \int d^2\mathbf{x}(\frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + A_0\rho_0 + i\psi^* D_t\psi - \frac{1}{2}|\mathbf{D}\psi|^2 - \frac{1}{2\kappa}(\psi^*\psi - \rho_0)^2) . \quad (1.52)$$

It differs from the Jackiw-Pi model in two aspects: (1) there is a background density ρ_0 in this model; (2) the potential here is repulsive while the potential in the Jackiw-Pi model is attractive. The equations of motion can be obtained in the usual way

$$\begin{aligned} i\partial_t\psi &= -\frac{1}{2}[\mathbf{D}^2 + A^0 + \frac{1}{\kappa}(\psi^*\psi - \rho_0)]\psi , \\ \frac{\kappa}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} &= J^\mu , \end{aligned} \quad (1.53)$$

where J^μ is the same as (1.34) except that ρ is replaced by $\rho - \rho_0$. The vacuum solution: $\psi = \sqrt{\rho_0}e^{in\theta}$,$A_0 = 0$ and $\mathbf{A} = \frac{n\mathbf{e}_\theta}{r}$ with $\tan\theta = \frac{x_2}{x_1}$ and $r^2 = x_1^2 + x_2^2$, which will serve as asymptotes for our soliton solutions at infinity, exhibits spontaneous symmetry breaking [7]. This implies the magnetic flux of the soliton solution is quantized, *i.e.*

$$\int d^2\mathbf{x}B = 2\pi n . \quad (1.54)$$

As in the Jackiw-Pi case, we also write down the Hamiltonian for this model

$$H = \frac{1}{2}\int d^2\mathbf{x}[|\mathbf{D}\psi|^2 + \frac{1}{\kappa}(\rho - \rho_0)^2] . \quad (1.55)$$

By virtue of the identity (1.42), (1.55) can be rewritten as

$$H = \frac{1}{2}\int d^2\mathbf{x}[|(D_1 + iD_2)\psi|^2 + B\rho + \frac{1}{\kappa}(\rho - \rho_0)^2] . \quad (1.56)$$

With the help of the Gauss's Law $B = -\frac{1}{\kappa}(\rho - \rho_0)$ and the quantization condition

(1.54), we have

$$H = \frac{1}{2}\int d^2\mathbf{x}|(D_1 + iD_2)\psi|^2 + 2\pi\rho_0 n \ , \tag{1.57}$$

here an infinite constant has been dropped. The lower bound of energy, $2\pi\rho_0 n$, is saturated when $\psi = \sqrt{\rho}e^{in\theta}$ satisfies the following self-dual conditions

$$\begin{aligned} (D_1 + iD_2)\psi &= 0 \, , \\ B &= -\frac{1}{\kappa}(\rho - \rho_0) \ . \end{aligned} \tag{1.58}$$

Following an approach similar to the one used in solving the Jackiw-Pi model, we finally obtain

$$\nabla^2 \ln\rho = -\frac{2}{\kappa}(\rho - \rho_0) \ . \tag{1.59}$$

Equation (1.59) has been studied numerically [17] and is known to possess topological soliton solutions with asymptotic behavior: $\rho \to \rho_0$ as $r \to \infty$ and $\rho(r) \sim r^{2n}$ near the origin.

It is interesting to compare the boundary behavior of the soliton solution in this model with that of the Jackiw-Pi soliton. As can be seen from (1.49), (1.50), (1.51), the Jackiw-Pi soliton vanishes at infinity; while the soliton in this model approaches a non-zero value ρ_0 when $r \to \infty$. In Chapter two, we shall demonstrate that this difference in boundary behavior leads to drastically different dynamical behavior of the solitons.

1.3 Hard Thermal Loops and Chern-Simons Theory

In the present section, we review recent works on Hard Thermal Loops (HTLs) in high-temperature QCD and its relevance to Chern-Simons theory [18, 19, 20, 21].

The motivation which leads to the discovery of HTL was the observation that physical quantities (such as the damping rate of the quark-gluon plasma) in high-temperature QCD is gauge-dependent when computed using the usual loop expansion

[22]. The solution to this problem was first proposed by Pisarski [8]. He realized that in order to obtain consistent results in high-temperature QCD computations, it is necessary to perform a resummation procedure before doing perturbative expansion. The necessity of such a procedure can be understood from the following example: Consider a one-loop amplitude $\Pi_1(p)$

$$\Pi_1(p) = \int dk I_1(p,k) \,, \tag{1.60}$$

given by the graph

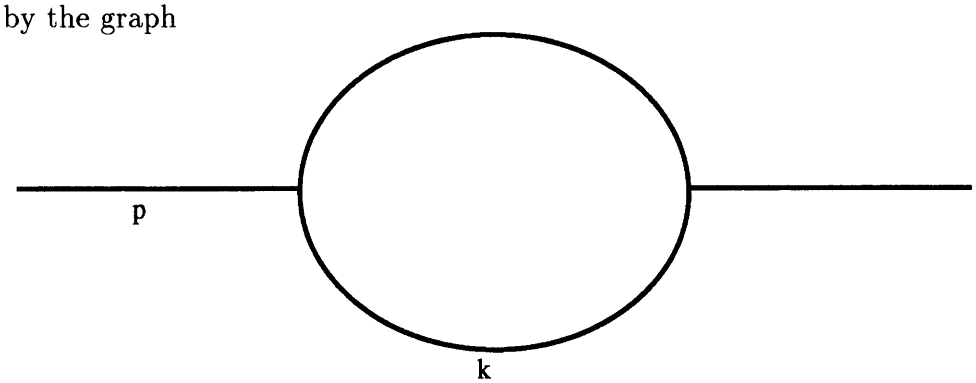

Fig. 1-1. One-loop amplitude $\Pi_1(p)$.

Compare $\Pi_1(p)$ to a two-loop amplitude $\Pi_2(p)$

$$\Pi_2(p) = \int dk I_2(p,k) \ , \tag{1.61}$$

given by

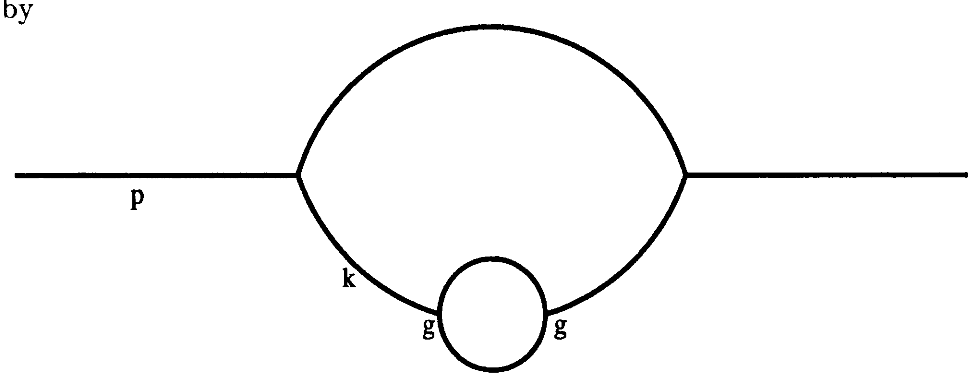

Fig. 1-2. Two-loop amplitude $\Pi_2(p)$.

Following Pisarski [8], we can estimate the relative importance of Π_2 to Π_1 by the ratio of their integrands

$$\frac{\Pi_2}{\Pi_1} \sim \frac{I_2}{I_1} = g^2 \frac{\Pi_1(k)}{k^2} \ . \tag{1.62}$$

Here g is the coupling constant and we assume that the mass of the particles are negligible. For small k and large T, we find that $\Pi_1(k) \sim T^2$, thus $\frac{\Pi_2}{\Pi_1} \sim \frac{g^2 T^2}{k^2}$. Therefore when k is $O(gT)$ or smaller, the two-loop amplitude Π_2 is not negligible compared to Π_1. It needs to be included in the computation.

In the subsequent development, Pisarski *et al.* identified the graphs needed to be resumed in SU(N) gauge theory at high-temperature limit. Their results can be summarized as follows [18]:

(i) In the leading order of g, only a subset of one-loop graphs with "soft" external momenta [$O(gT)$ or smaller] and "hard" internal momenta [$\sim T$] need to be resumed. And these so called "Hard Thermal Loops" are graphs with either none or two fermionic legs.

(ii) The HTLs are proportional to T^2 in the high temperature limit. The contribution from HTLs with fermionic legs and that from HTLs without fermionic legs are gauge-invariant separately.

In addition, generating functionals can be written down for HTLs with only gluonic legs and HTLs with fermionic legs separately [23]. The generating functional for HTLs with purely gluonic legs Γ_{HTL} exhibits remarkable simplicity. To explain this, we define two light-like vectors $Q^\mu_\pm$

$$Q^\mu_\pm = \frac{1}{\sqrt{2}}(1, \pm\hat{q}) \, , \tag{1.63}$$

where $\hat{q}$ is a unit 3-vector, pointing in an arbitrary direction. Using $Q^\mu_\pm$, we can project coordinates and gauge fields accordingly

$$x_\pm = Q^\mu_\pm x_\mu, \quad \partial_\pm = Q^\mu_\pm \frac{\partial}{\partial x^\mu}, \quad A_\pm = Q^\mu_\pm A_\mu \ . \tag{1.64}$$

The additional simplicity related to Γ_{HTL} is that: (1) Γ_{HTL} is a sum of an ultra-local contribution and an average over the angles of $\hat{q}$ of a functional W that depends only on A_+; (2) this functional W is non-local only on the two-dimensional $x_\pm$ plane and ultra-local in the remaining directions. Explicitly

$$\Gamma_{HTL} = (N + \frac{1}{2}N_F)\frac{g^2T^2}{12\pi}\left[2\pi \int d^4x A_0^a(x)A_0^a(x) + \int d\Omega_{\hat{q}} W(A_+)\right], \tag{1.65}$$

where N_F is the number of fermion species, a is the color index and $d\Omega_{\hat{q}}$ denotes an integration over the solid angle of $\hat{q}$.

Knowing (1.65), we can present the condition of gauge invariance of Γ_{HTL} as [23, 9]

$$\int d\Omega_{\hat{q}} \delta W = 4\pi \int d^4x A_0^a \omega^a \ , \tag{1.66}$$

here $\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega]$ where ω^a parametrizes an infinitesimal gauge transformation. Equation (1.66) is realized by

$$\delta W = \int d^4x \dot{A}_0^a \omega^a \ . \tag{1.67}$$

Indeed, the analysis of HTL graphs shows this is the way equation (1.66) is satisfied. To relate Γ_{HTL} to Chern-Simons theory, we define a new quantity S

$$S(A_+) \equiv W(A_+) + \frac{1}{2}\int d^4x A_+^a(x)A_+^a(x) \ . \tag{1.68}$$

In terms of S, equation (1.67) can be rewritten as

$$\partial_+ \frac{\delta}{\delta A_+^a} S - \partial_- A_+^a + f^{abc} A_+^b \frac{\delta}{\delta A_+^c} S = 0 \ . \tag{1.69}$$

Identifying $+, -$ with 1, 2, we immediately recognize that equation (1.69) is nothing but the constraint (1.30) satisfied by the Chern-Simons eikonal. Thus the generating functional for HTLs is essentially given by the Chern-Simons eikonal [9].

As a final note, we would like to mention an interesting observation by Frenkel and Taylor. They discovered that HTLs can be derived from an eikonal approximation of one-loop Feynman diagrams [24].

Chapter 2

Dynamics of Chern-Simons Vortices

2.1 Dynamics of Chern-Simons Vortices in Absence of Background Density

Recently, there has been a lot of study on Chern-Simons solitons [15]. Topological and non-topological solitons are found in a relativistic theory [5], and non-topological solitons exist in a non-relativistic model [6]. Since Chern-Simons dynamics is closely related to the quantum Hall effect and perhaps to high-T_c superconductivity, it is very interesting to study these solitons [25, 26]. Here, we analyze the dynamics of non-relativistic Chern-Simons solitons in modular parameter space.

Manton originally proposed a method for studying soliton dynamics with specific application to monopole scattering [27]. His idea can be summarized as follows: if there are no forces between static solitons, then at low energies, the dynamics of the full field theory can be described approximately on a finite dimensional space, where the degrees of freedom are the modular parameters of the general static solution. This method has been widely applied to many other systems [25, 28, 29, 30, 31]. In a recent application, the statistical interaction among non-relativistic Chern-Simons solitons has been obtained [26].

However, since only first-order time derivatives appear in non-relativistic Chern-Simons theory, it is not clear that Manton's method is directly applicable. Here this question is analyzed and we find that in order to find the correct dynamical behavior of non-relativistic Chern-Simons solitons, a phase, related to the 1- cocycle of the Galileo group, must be introduced when applying the collective coordinate method. Moreover, we use this modified method to study interactions both among well-separated solitons and between these solitons and external fields.

Let us define the notation: we shall use superscripts $m, n = 1, ..., N$ as soliton indices, and subscripts $i, j = 1, 2$ as space indices, for which a summation convention is employed.

We first introduce the Jackiw-Pi Lagragian [15],

$$L = \int d^2\mathbf{r}\left(\frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + i\psi^* D_t\psi - \frac{1}{2}|\mathbf{D}\psi|^2 + \frac{1}{2\kappa}(\psi^*\psi)^2\right) , \tag{2.1}$$

where $D_t = \partial_t + iA_0$, $\mathbf{D} = \nabla - i\mathbf{A}$.

The static solution for N solitons is

$$\psi_s = \rho^{\frac{1}{2}} e^{i\omega} \tag{2.2}$$

with

$$\rho = \frac{4\kappa|f'|^2}{(1+|f|^2)^2}, \qquad \omega = \mathrm{Arg}(f'V^2) , \tag{2.3}$$

where

$$f(z) = \sum_{m=1}^{N} \frac{c^m}{z - a^m}, \qquad V(z) = \prod_{m=1}^{N} (z - a^m) , \tag{2.4}$$

here $z = x + iy$, $f' = \frac{df}{dz}$ and ω is defined in (2.3) to make the solution nonsingular. Physically, a^m is the position of the mth soliton, and c^m parameterizes its size and phase. Note, the static solution ψ_s satisfies the self-dual equation [15],

$$(D_1 - iD_2)\psi_s = 0 \tag{2.5}$$

and the action $I = \int dtL$ can also be written as [15]

$$I = \int dtd^2\mathbf{r}(i\psi^*\partial_t\psi - \frac{1}{2}|(D_1 - iD_2)\psi|^2) \ . \tag{2.6}$$

Now we discuss the dynamics of these solitons. For sake of simplicity, we hold the c^m's time-independent, and let the a^m's be time-dependent. First, we notice: because of the Galileo invariance of our model, the static one-soliton solution acquires a phase $\Theta = \mathbf{v}\cdot\mathbf{r} - \frac{1}{2}v^2t$ when boosted with a constant velocity $\mathbf{v}$ [15]. Motivated by this fact, we consider the following function,

$$\psi = \psi_s e^{i\Theta} \ , \tag{2.7}$$

where ψ_s is the self-dual solution (2.2) and Θ is a function of $\dot{a}^m$, a^m, t and $\mathbf{r}$.

We assume that the time-evolution of well-separated Chern-Simons solitons at low energies is approximately described by the effective Lagrangian for the a^m's, which is obtained by substituting (2.7) into the original action I (2.6). Notice that ψ_s continues to satisfy equation (2.5) even with time-dependent parameters, hence we obtain

$$I_{eff} = \int dtd^2\mathbf{r}(-\rho\partial_t\Theta - \rho\partial_t\omega + \frac{i}{2}\partial_t\rho - \frac{1}{2}\rho\partial_i\Theta\partial_i\Theta) \ . \tag{2.8}$$

Since $\frac{d}{dt}\int d^2\mathbf{r}\rho = 0$,

$$I_{eff} = \int dtd^2\mathbf{r}(-\rho\partial_t\Theta - \rho\partial_t\omega - \frac{1}{2}\rho\partial_i\Theta\partial_i\Theta) \ . \tag{2.9}$$

In order to determine Θ, we shall require that near the center of each soliton ψ satisfy the equation of motion of the original Lagrangian to order $\dot{\mathbf{a}}$; we also assume that $\ddot{\mathbf{a}}$ is much smaller than $\dot{\mathbf{a}}$. This leads to

$$\Theta = \sum_m(\dot{\mathbf{a}}^m(t)G^m(\mathbf{r}))\cdot\mathbf{r} \tag{2.10}$$

and, $G^m(\mathbf{r}) \longrightarrow 1$, when $\mathbf{r}$ is near $\mathbf{a}^m$; while $G^m(\mathbf{r}) \longrightarrow 0$, when $\mathbf{r}$ is far away from $\mathbf{a}^m$. Also the derivatives of G^m are order $(\dot{\mathbf{a}})^2$, hence can be set to zero. Thus, the effective action becomes,

$$I_{eff} = \int dt d^2\mathbf{r}(-\sum_m \rho\partial_t(\dot{\mathbf{a}}^m \cdot \mathbf{r}G^m) - \rho\partial_t\omega - \frac{1}{2}\sum_m \rho(\dot{\mathbf{a}}^m G^m)\cdot(\dot{\mathbf{a}}^m G^m)) \ , \qquad (2.11)$$

or after an integration by parts,

$$I_{eff} = \int dt d^2\mathbf{r}(\sum_m \partial_t\rho(\dot{\mathbf{a}}^m G^m)\cdot\mathbf{r} - \rho\partial_t\omega - \frac{1}{2}\sum_m \rho(\dot{\mathbf{a}}^m G^m)\cdot(\dot{\mathbf{a}}^m G^m)) \ . \qquad (2.12)$$

Here an end point contribution has been dropped. Thus, our effective Lagrangian is,

$$L_{eff} = \int d^2\mathbf{r}(\sum_m \partial_t\rho(\dot{\mathbf{a}}^m G^m)\cdot\mathbf{r} - \rho\partial_t\omega - \frac{1}{2}\sum_m \rho(\dot{\mathbf{a}}^m G^m)\cdot(\dot{\mathbf{a}}^m G^m)) \ . \qquad (2.13)$$

We divide L_{eff} into two parts, L_1 and L_2, in which L_1 is the part induced by the phase Θ and L_2 is the part obtained by direct application of Manton's prescription.

$$\begin{aligned} L_{eff} &= L_1 + L_2 \\ L_1 &= \sum_m \int d^2\mathbf{r}(\partial_t\rho(\dot{\mathbf{a}}^m G^m)\cdot\mathbf{r} - \frac{1}{2}\rho(\dot{\mathbf{a}}^m G^m)\cdot(\dot{\mathbf{a}}^m G^m)) \\ L_2 &= -\int d^2\mathbf{r}\rho\partial_t\omega \ . \end{aligned} \qquad (2.14)$$

We first evaluate L_1. Using the above described properties of G^m, we obtain,

$$\begin{aligned} L_1 &= \sum_m \dot{\mathbf{a}}^m \cdot \int d^2\mathbf{r}\partial_t(\rho G^m\mathbf{r}) - \frac{1}{2}\dot{\mathbf{a}}^m\cdot\dot{\mathbf{a}}^m \int d^2\mathbf{r}\rho^m \\ &= \sum_m \dot{\mathbf{a}}^m \cdot \frac{d}{dt}(\int d^2\mathbf{r}\rho^m\mathbf{r}) - \frac{1}{2}\dot{\mathbf{a}}^m\cdot\dot{\mathbf{a}}^m \int d^2\mathbf{r}\rho^m \ , \end{aligned} \qquad (2.15)$$

where ρ^m is the spherically symmetric one-soliton density for the mth soliton. Finally, using $\int d^2\mathbf{r}\rho^m\mathbf{r} = 4\pi\kappa\mathbf{a}^m$ and $\int d^2\mathbf{r}\rho^m = 4\pi\kappa$, leaves

$$L_1 = 2\pi\kappa\sum_m \dot{\mathbf{a}}^m\cdot\dot{\mathbf{a}}^m \ . \qquad (2.16)$$

Not surprisingly, the familiar kinetic energy term for non-relativistic particles is recovered and the mass $4\pi\kappa$ is exactly what we expect from a consideration of the single-soliton momentum [15].

Now we evaluate L_2. Notice that $f'V^2$ can always be written as,

$$f'V^2 = -(\sum_{m=1}^{N} c^m) \prod_{n=1}^{2N-2} (b^n - z) , \tag{2.17}$$

where each b^n solves the following equation,

$$\sum_{m=1}^{N} (c^m \prod_{n\neq m, n=1}^{N} (z - a^n)^2) = 0 . \tag{2.18}$$

Thus, we have,

$$\omega = \mathrm{Arg}(f'V^2) = \sum_{n=1}^{2N-2} \mathrm{Arg}(b^n - z) + \mathrm{const} . \tag{2.19}$$

Using the correspondence between a complex number z and a real 2-dimensional vector $\mathbf{r}$ as well as the formula $\mathrm{Arg}(z) = \theta(\mathbf{r}) \equiv \tan^{-1}(\frac{r_2}{r_1})$, we have,

$$\begin{aligned} L_2 &= -\sum_{n=1}^{2N-2} \int d^2\mathbf{r} \rho \partial_t \theta(\mathbf{b}^n - \mathbf{r}) \\ &= -\sum_{n=1}^{2N-2} \dot{\mathbf{b}}^n \cdot \int d^2\mathbf{r} \frac{\partial}{\partial \mathbf{b}^n} \theta(\mathbf{b}^n - \mathbf{r}) \rho . \end{aligned} \tag{2.20}$$

Recall that in the original theory the Chern-Simons vector potential is given by [15],

$$\mathbf{A}(\mathbf{r}, t) = -\frac{1}{2\pi\kappa} \int d^2\mathbf{r}' \nabla \theta(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}', t) \tag{2.21}$$

and we see that (2.20) becomes,

$$L_2 = 2\pi\kappa \sum_{n=1}^{2N-2} \dot{\mathbf{b}}^n \cdot \mathbf{A}(\mathbf{b}^n, t) . \tag{2.22}$$

The interaction among the non-relativistic Chern-Simons solitons is mediated by an effective Chern-Simons vector potential induced by these solitons. A similar result was obtained in the relativistic Chern-Simons model by Kim and Min [25].

We can further simplify (2.22). As shown in Ref. [15], for the self-dual solution,

$$\mathbf{A} = -\frac{1}{2}\nabla \times \ln\rho + \nabla\omega \ . \tag{2.23}$$

Defining $\Phi(\mathbf{r}) = (|V|^2 + |fV|^2)$, we get,

$$L_2 = 2\pi\kappa \sum_{n=1}^{2N-2} \dot{\mathbf{b}}^n \cdot (\nabla \times \ln\Phi(\mathbf{r}))|_{\mathbf{r}=\mathbf{b}^n} \ . \tag{2.24}$$

Combining (2.16) with (2.24), we have,

$$L_{eff} = 2\pi\kappa \sum_m \dot{\mathbf{a}}^m \cdot \dot{\mathbf{a}}^m + 2\pi\kappa \sum_{n=1}^{2N-2} \dot{\mathbf{b}}^n \cdot (\nabla \times \ln\Phi(\mathbf{r}))|_{\mathbf{r}=\mathbf{b}^n} \ , \tag{2.25}$$

where $\mathbf{b}^n$ is determined by solving (2.18).

In principle, the interaction term L_2 for a fixed N can be simplified with the aid of (2.18). As an example, we shall make the simplification for two solitons in their center of mass frame, $\mathbf{a}^1 = -\mathbf{a}^2 = \mathbf{a}$, and arbitrary constants c^1 and c^2. In this case, equation (2.18) becomes,

$$(c^1 + c^2)(z^2 + 2dza + a^2) = 0 \ , \tag{2.26}$$

where $d = \frac{c^1 - c^2}{c^1 + c^2}$.

Equation (2.26) has two roots $b^{1,2} = a(-d \pm \sqrt{d^2 - 1})$ while $\dot{b}^{1,2} = \dot{a}(-d \pm \sqrt{d^2 - 1})$. Substituting these into the effective Lagrangian and using the explicit form of Φ for two solitons, we obtain, in complex notation,

$$\begin{aligned} L_2 &= -4\pi\kappa \sum_{n=1}^{2} \mathrm{Im}(\dot{b}^n \partial_z ln\Phi(z))|_{z=b^n} \\ &= -4\pi\kappa \sum_{n=1}^{2} \mathrm{Im}(\frac{\dot{a}^n a^*}{|a|^2}(\frac{(-1)^n d(\sqrt{d^2-1})^*}{|\sqrt{d^2-1}|^2} + 1)) \\ &= -8\pi\kappa \mathrm{Im}(\frac{\dot{a}^n a^*}{|a|^2}) \\ &= -8\pi\kappa \frac{d}{dt}\mathrm{Arg}(a) = -8\pi\kappa \frac{d}{dt}\theta(\mathbf{a}) \ . \end{aligned} \tag{2.27}$$

Recalling that $\theta(\mathbf{a})$ is the relative angle between two solitons, we see that (2.27) is the statistical interaction with spin $S = -4\pi\kappa$ [32]. This coincides with the result obtained by Hua and Chou with numerical integration [26]. Thus, classically, our Lagrangian describes two free-moving non-relativistic particles with statistical interaction.

As another interesting example, we apply our method to a single Chern-Simons soliton in the presence of external fields. To be specific, we consider one soliton either in a constant external electric field or in a constant external magnetic field or in a harmonic potential or in any combination of these three. The most general action is

$$I = \int dt d^2\mathbf{r}(i\psi^*\partial_t\psi - A_0^e\psi^*\psi - \frac{1}{2}kr^2\psi^*\psi - \frac{1}{2}|(D_1^e - iD_2^e)\psi|^2) , \tag{2.28}$$

where $\mathbf{D}^e = \nabla - i\mathbf{A} - i\mathbf{A}^e$, $A_i^e = -\frac{1}{2}\epsilon_{ij}r_jB$, $A_0^e = -\mathbf{r}\cdot\mathbf{E}$ and k is the strength of the harmonic potential .

In this case, we choose the phase Θ as follows,

$$\Theta = \dot{\mathbf{a}}\cdot\mathbf{r} - \frac{1}{8}B^2\int^t dt'\mathbf{a}(t')\cdot\mathbf{a}(t') . \tag{2.29}$$

Then we substitute the trial function (2.7) with this new Θ into (2.28), following similar procedures, we obtain,

$$L_{eff} = \frac{1}{2}\dot{\mathbf{a}}\cdot\dot{\mathbf{a}}\int d^2\mathbf{r}\rho + \int d^2\mathbf{r}\rho\mathbf{E}\cdot\mathbf{r} - \frac{1}{2}k\int d^2\mathbf{r}\rho r^2 + \int d^2\mathbf{r}\rho\dot{\mathbf{a}}\cdot\mathbf{A}^e(\mathbf{r}) . \tag{2.30}$$

By using the spherical symmetry of the one-soliton solution and the explicit form of $\mathbf{A}^e$, we have,

$$L_{eff} = 2\pi\kappa\dot{\mathbf{a}}\cdot\dot{\mathbf{a}} + 4\pi\kappa\mathbf{a}\cdot\mathbf{E} - 4\pi\kappa\dot{\mathbf{a}}\cdot\mathbf{A}^e(\mathbf{a}) - 2\pi\kappa k|\mathbf{a}|^2 . \tag{2.31}$$

Here an irrelevant constant term is dropped.

Thus, we see that the Chern-Simons soliton behaves like a non-relativistic point-like particle with charge $4\pi\kappa$ and mass $4\pi\kappa$ in these external fields. In fact, exact soliton solutions in the presence of these fields can be found by a coordinate transformation [16]. Our result agrees with the behavior of these solutions.

In summary, we comment on the phase Θ. From our work, it is clear that this phase plays a very important role in the dynamics of non-relativistic Chern-Simons solitons. However, we have not determined this phase exactly. Thus, it would be very interesting to look for some method to find this phase.

2.2 Dynamics of Chern-Simons Vortices in Presence of Background Density

In a recent letter, Barashenkov and Harin studied a non-relativistic Chern-Simons theory in the presence of a background density [7]. A similar model was also studied by Lozano [7]. As found by Barashenkov and Harin, this model admits vortex solutions with a non-vanishing field configuration at infinity, in contrast to the theory without the background density [15]. In this brief report, we study the dynamics of these non-relativistic vortices and show that the background density induces a Magnus force [33] on moving Chern-Simons vortices, a force that vanishes when the background density is absent [26, 34].

We start by reminding the reader of the conclusions of Ref. [7] that are essential to the present discussion. First, the authors of Ref. [7] have formulated a $2+1$ dimensional Chern–Simons theory in which the matter density is finite at infinity, and have found that the Euler–Lagrange equations corresponding to that theory admit vortex solutions. These equations can be derived from a Lagrangian written solely in terms of the matter field ψ. The Lagrangian can be put in the form

$$L = \int d^2\mathbf{r}(i\psi^*\partial_t\psi - \mathcal{H}(\psi)) \ , \tag{2.32}$$

where $\mathcal{H}(\psi)$ is the energy density of the system and has no explicit time dependence.

Second, the one-vortex solution centered at $\mathbf{R}$ has the following form

$$\psi_1(\mathbf{r};\mathbf{R}) = \rho^{\frac{1}{2}} e^{i\theta} \tag{2.33}$$

with $\mathbf{r} = (x, y)$ and $\mathbf{R} = (X, Y)$. Here ρ and θ are functions of $\mathbf{r} - \mathbf{R}$ where the latter is given by $\theta = \arctan\left(\frac{y-Y}{x-X}\right)$. The density ρ vanishes continuously at $\mathbf{r} = \mathbf{R}$ and approaches the background density ρ_0 as $|\mathbf{r} - \mathbf{R}| \to \infty$.

Following the standard Manton procedure [27], we reduce the Lagrangian (2.32) to a Lagrangian of a single degree of freedom, that of the vortex' center, R. This is done by regarding R as time dependent and substituting (2.33) into (2.32). The Lagrangian thus obtained effectively describes the dynamics of the vortex center. For convenience, we consider the two terms in (2.32) separately. The second term, when integrated, gives the rest energy of the vortex and is irrelevant to our discussion of the Magnus force. On the other hand, the first term leads to

$$L^1_{eff} = \int d^2\mathbf{r} \left[\left(\frac{i}{2}\partial_t \rho(\mathbf{r} - \mathbf{R}(t)) - \rho(\mathbf{r} - \mathbf{R}(t))\partial_t \theta(\mathbf{r} - \mathbf{R}(t))\right] \ . \tag{2.34}$$

Since $\frac{d}{dt} \int d^2\mathbf{r}\rho = 0$, (2.34) can be simplified to

$$\begin{aligned} L^1_{eff} &= -\int d^2\mathbf{r}\rho(\mathbf{r} - \mathbf{R}(t))\partial_t\theta(\mathbf{r} - \mathbf{R}(t)) \\ &= \dot{\mathbf{R}} \cdot \int d^2\mathbf{r}\rho(\mathbf{r} - \mathbf{R}(t))\nabla_{\mathbf{r}}\theta(\mathbf{r} - \mathbf{R}(t)) \ . \end{aligned} \tag{2.35}$$

Here we have used the fact that θ only depends on $\mathbf{r} - \mathbf{R}$. The subscripts $\mathbf{r}$ and $\mathbf{R}$ denote the argument of the differentiation. Recalling that $\dot{\mathbf{R}}$ is the velocity of the vortex, we observe that L^1_{eff} describes an interaction of the vortex with a vector potential. The force $\mathbf{F}$ experienced by the vortex due to this interaction can then be obtained by varying (2.35) with respect to $\mathbf{R}$

$$F_i = \epsilon_{ij}\dot{R}_j B \ . \tag{2.36}$$

The field strength B is given by

$$B = \nabla_{\mathbf{R}} \times \int d^2\mathbf{r}\rho(\mathbf{r} - \mathbf{R}(t))\nabla_{\mathbf{r}}\theta(\mathbf{r} - \mathbf{R}(t)) \ . \tag{2.37}$$

The evaluation of B is straightforward. Since the integrand only depends on $\mathbf{r} - \mathbf{R}$, we write

$$B = -\int d^2\mathbf{r}\nabla_{\mathbf{r}} \times (\rho(\mathbf{r} - \mathbf{R}(t))\nabla_{\mathbf{r}}\theta(\mathbf{r} - \mathbf{R}(t))) \ . \tag{2.38}$$

Using Stoke's theorem and the asymptotic behavior of ρ, we finally have

$$B = -2\pi\rho_0 \ . \tag{2.39}$$

Thus, a force is exerted on the vortex when it moves relative to the background density ρ_0. This force is proportional and perpendicular to the vortex velocity, and proportional to the background density. It is the Magnus force. The Magnus force makes the vortex dynamics similar to that of charged particles in a magnetic field, with the role of the magnetic field played by the background density. We point out that in our derivation the essential inputs are the asymptotic behavior of the vortex at infinity and the non-relativistic nature of the theory. This suggests that the existence of the Magnus force is a universal feature for non-relativistic vortices with non-vanishing field configuration at large distances. We also note that the effective action obtained by integrating (2.35) can be obtained by a Berry phase analysis as carried out for the case of vortices in superconductors in Ref. [33]. Finally, we note that non-relativistic Chern-Simons theories can be viewed as a low-energy effective field theory of their relativistic counterparts [6]. It would then be of interest to study the relevance of our work to the dynamics of vortices in the extensively studied relativistic Chern–Simons theories [25].

Chapter 3

Hard Thermal Loops, Static Response and the Composite Effective Action

3.1 Introduction

When it was realized [9] that the gauge invariance condition [23] on the generating functional $\Gamma(A)$ for hard thermal loops in a gauge theory [18] (with or without fermions) coincides with a similar requirement on the wave functional of Chern-Simons theory, one could use the known solution for the latter, non-thermal problem [14] to give a construction of $\Gamma(A)$ relevant in the former, thermal context. The expression for $\Gamma(A)$ is non-local and not very explicit: $\Gamma(A)$ can be presented either as a power series in the gauge field A [9] [the $O(A^n)$ contribution determines the hard thermal gauge field (and fermion) loop with n external gauge field lines] or as an explicit functional of path ordered variables $P\exp\int dx^\mu A_\mu$ [14].

More accessible is the expression for the induced current $-\frac{\delta\Gamma(A)}{\delta A_\mu} \equiv -T^a\frac{\delta\Gamma(A)}{\delta A^a_\mu}$, which enters (high-temperature) response theory, in a non-Abelian generalization of

Kubo's formula (in Minkowski space-time) [20]:

$$D_\nu \, F^{\nu\mu}(x) = -\frac{\delta\Gamma(A)}{\delta A_\mu(x)} \equiv \frac{m^2}{2} \, j^\mu(x) \; . \tag{3.1}$$

T^a is an anti-hermitian representation of the Lie algebra, the gauge covariant derivative is defined as $D_\nu = \partial_\nu + g[A_\nu, \;]$, and m is the Debye mass determined by the matter content: in an $SU(N)$ gauge theory at temperature T, with fermions in the representation $\mathcal{T}^a$, and $\text{Tr}\,(\mathcal{T}^a\mathcal{T}^b) = -\frac{N_F}{2}\delta^{ab}$ where N_F counts the number of flavors, the Debye mass satisfies

$$m^2 = \frac{g^2T^2}{3}\left(N + \frac{N_F}{2}\right) \; . \tag{3.2}$$

Henceforth, through Section 3.2, we scale the gauge coupling constant to unity. The functional form of j^μ can be given as [20]

$$j^\mu(x) = \int \frac{d\hat{q}}{4\pi}\left\{Q_+^\mu\Big(a_-(x) - A_-(x)\Big) + Q_-^\mu\Big(a_+(x) - A_+(x)\Big)\right\} \; . \tag{3.3}$$

Here $Q_\pm^\mu$ are the light-like 4-vectors $\frac{1}{\sqrt{2}}(1, \pm\hat{q})$, with $\hat{q}^2 = 1$, $A_\pm$ are the light-like projections $A_\pm = Q_\pm^\mu \, A_\mu$, while $a_\pm$ are given by [14, 20]

$$a_+ = g^{-1}\,\partial_+\, g \; , \qquad a_- = h^{-1}\,\partial_-\, h \qquad (\partial_\pm \equiv Q_\pm^\mu\,\partial_\mu) \tag{3.4}$$

when $A_\pm$ are parameterized as

$$A_+ = h^{-1}\,\partial_+\, h \; , \qquad A_- = g^{-1}\,\partial_-\, g \quad . \tag{3.5}$$

In other words, $a_\pm$ satisfy the equations

$$\begin{aligned} \partial_+ a_- - \partial_- A_+ + [A_+, a_-] &= 0 \; , \\ \partial_+ A_- - \partial_- a_+ + [a_+, A_-] &= 0 \; , \end{aligned} \tag{3.6}$$

whose solution can be presented as in (3.4) when $A_\pm$ are parameterized as in (3.5) — evidently g and h involve path ordered exponential integrals of $A_\pm$. (Alternatively

$a_\pm$ may be given by a power series in $A_\mp$ [9].) Finally (3.3) requires averaging over the directions of $\hat{q}$.

It is easy to verify that (3.6) ensure covariant conservation of j^μ. Moreover, gauge invariance is maintained: for (3.1) to be gauge covariant, it is necessary that j^μ transform gauge covariantly. That the expression in (3.3) possesses this property is seen as follows. When $A_\pm$ transform by $U^{-1} A_\pm U + U^{-1}\partial_\pm U$, Eqs. (3.4) - (3.6) show that $a_\pm$ transform similarly, hence the differences $a_\pm - A_\pm$ transform covariantly. The manifest gauge covariance of (3.1) ensures that m is a gauge invariant parameter; that it also has the interpretation of an electric (Debye) mass will be evident when we consider the static limit.

It is of obvious interest to discuss solutions of (3.1). In the Abelian, electrodynamical case this is easy to do, since (3.6) can be readily solved for $a_\pm$, and the solutions of the linear problem are the well-known plasma waves [35]. The non-linear problem of finding non-Abelian plasma waves is much more formidable. Also, one inquires whether the non-linear equations support soliton solutions, and (after an appropriate continuation to imaginary time) instanton solutions. [The time-dependent equation (3.1) in Minkowski space-time must be supplemented with boundary conditions, which are determined by the physical context. For example, response theory requires retarded boundary conditions, which in fact preclude deriving (3.1) variationally [20]. Here we shall not be concerned with this issue.]

Our paper concerns the following two topics. In Section 3.2, we analyze (3.1) for static fields. It turns out that in the time-independent case (3.6) can be solved for $a_\pm$ and (3.1) is presented in closed form. We prove that the resulting equation does not possess finite-energy solutions, thereby establishing that gauge theories do not support hard thermal solitons. Also some negative conclusions about instantons are given. In Section 3.3 we present an alternative derivation of (3.1), which relies on the composite effective action [36], and makes use of approximations recently developed in an analysis of hard thermal loops based on the Schwinger-Dyson equations [21].

3.2 Static Response

When $A_\pm$ are time-independent, we seek solutions of (3.6) that are also time-independent. Acting on static fields, the derivatives $\partial_\pm$ become $\pm\frac{1}{\sqrt{2}}\hat{q}\cdot\nabla \equiv \pm\partial_\tau$, and (3.6) may be written as the equations

$$\partial_\tau \mathcal{A}_\pm \pm [A_\pm, \mathcal{A}_\pm] = 0 \tag{3.7}$$

for the unknowns $\mathcal{A}_\pm \equiv A_\pm + a_\mp$. These are solved trivially by $\mathcal{A}_\pm = 0$, that is

$$a_\mp = -A_\pm \ . \tag{3.8}$$

This solution is also the one that is deduced from the perturbative series expression for $a_\pm$, when restricted to static $A_\pm$.

[A non trivial solution can be constructed with the help of representations similar to (3.5). Upon defining in the static case

$$\begin{aligned} A_+ &= h_0^{-1}\,\partial_\tau\, h_0 \ , \\ A_- &= -g_0^{-1}\,\partial_\tau\, g_0 \end{aligned}$$

(h_0 and g_0 involve path-ordered exponentials along the path $\mathbf{r} + \hat{q}\tau$), we find

$$\begin{aligned} \mathcal{A}_+ &= h_0^{-1}\, I_+\, h_0 \ , \\ \mathcal{A}_- &= g_0^{-1}\, I_-\, g_0 \ , \end{aligned}$$

where $I_\pm$ are arbitrary Lie algebra elements, independent of τ: $\hat{q}\cdot\nabla I_\pm = 0$. Since these solutions involve the arbitrary quantities $I_\pm$, and since they do not arise in the perturbative series, we do not consider them further and remain with the trivial solution (3.8), which corresponds to $I_\pm = 0$.]

From (3.8) it follows that the current for static fields is

$$j^\mu(\mathbf{r}) \;=\; -\int \frac{d\hat{q}}{4\pi}\left(Q_+^\mu + Q_-^\mu\right)\left(A_+(\mathbf{r}) + A_-(\mathbf{r})\right)$$

$$= -\int \frac{d\hat{q}}{4\pi} \left(Q_+^\mu + Q_-^\mu\right)\left(Q_+^\nu + Q_-^\nu\right) A_\nu(\mathbf{r}) \,. \tag{3.9}$$

With $\mathbf{Q}_+ + \mathbf{Q}_- = 0$ and $Q_+^0 + Q_-^0 = \sqrt{2}$, we compute $j^\mu = -2\,\delta^{\mu 0} A_0$. The response equations (3.1) then become, in the static limit:

$$D_i E^i + m^2 A_0 = 0 \;, \tag{3.10}$$

$$\epsilon^{ijk} D_j B^k = [A_0, E^i] \;, \tag{3.11}$$

where $E^i \equiv F^{i0}$ and $F^{ij} \equiv -\epsilon^{ijk} B^k$. Eqs. (3.10), (3.11) give clear indication that m plays the role of a gauge invariant, electric mass. The fact that the static current is linear in the vector potential implies the vanishing of hard thermal loops with more that two external gauge-field lines, and zero energy — a fact which can be checked from the relevant graphs.

Unfortunately, Eqs. (3.10), (3.11) do not possess any finite energy solutions. This is established by a variant of the argument relevant to the $m^2 = 0$ case [37].

Consider the symmetric tensor

$$\theta^{ij} = 2 \,\mathrm{Tr}\, \left(E^i E^j + B^i B^j - \frac{\delta^{ij}}{2}(E^2 + B^2 + m^2 A_0^2)\right) \,. \tag{3.12}$$

Using (3.10), (3.11) one verifies that for static fields $\partial_j \theta^{ji} = 0$. Therefore

$$\int d^3r\, \theta^{ii} = \int d^3r\, \partial_j (x^i\, \theta^{ji}) = \int dS^j x^i \theta^{ji} \,. \tag{3.13}$$

Moreover, the energy of a massive gauge field (with no mass for the spatial components) can be written as

$$\mathcal{E} = \int d^3r \left\{ -\,\mathrm{Tr}\, \left(E^2 + B^2 + \frac{1}{m^2}(D_i E^i)^2\right) + \,\mathrm{Tr}\, \left(m A_0 + \frac{D_i E^i}{m}\right)^2 \right\} \,. \tag{3.14}$$

The second trace in the integrand enforces the constraint (3.10). Consequently, on the constrained surface the energy is a sum of positive terms [38]:

$$\mathcal{E} = \int d^3r \left\{ - \text{ Tr } \left(E^2 + B^2 + m^2 A_0^2 \right) \right\} \tag{3.15}$$

and $\mathbf{E}$, $\mathbf{B}$ and A_0 must decrease at large distances sufficiently rapidly so that each of them is square integrable. This in turn ensures that the surface integral at infinity in (3.13) vanishes, so that static solutions require

$$\int d^3r \theta^{ii} = 0 \ . \tag{3.16}$$

On the other hand, from (3.12), we see that θ^{ii} is a sum of positive terms

$$\theta^{ii} = - \text{ Tr } (E^2 + B^2 + 3m^2 A_0^2) \ , \tag{3.17}$$

hence (3.16),(3.17 imply the vanishing of $\mathbf{E}$, $\mathbf{B}$ and A_0.

The absence of finite energy static solutions can also be understood from the differential equations (3.10), (3.11). Eq. (3.10), (3.11) possesses solutions for A_0 that are either exponentially increasing or decreasing at infinity. Rejecting the former removes the freedom of imposing further conditions at the origin, and necessarily the exponentially damped solution evolves into one that is singular (not integrable) at the origin; see the Appendix A. (This situation can be contrasted with, *e.g.*, the magnetic dyon solution [39], where absence of the mass term allows solutions for A_0 with unconstrained large-r behavior, leaving the freedom to select the solution that is regular at the origin.)

A similar argument shows that there are no "static" instanton solutions. These would be solutions for which t is replaced by $-ix_4$, A_0 by iA_4 and presumably one would seek solutions periodic in x_4 with period $\beta = \frac{1}{T} = \frac{1}{m}\sqrt{\frac{N+N_F/2}{3}}$. An x_4-independent solution is necessarily periodic; it would satisfy (3.10), (3.11) with A_4 replacing A_0 and opposite sign in the right side of (3.11). But analysis similar to the above shows that finite-action solutions do not exist.

3.3 Hard Thermal Loops from the Composite Effective Action

In this Section, we present a derivation of the non-Abelian Kubo equation (3.1) based on the composite effective action of [36], a generalization of the usual effective action (obtained by coupling local sources to the fields) in which one additionally introduces bilocal sources. In the QCD case, the composite effective action is given by $S(A)+\Gamma_c(A,G_\phi)$, where $G_\phi(x,y)$ are (undetermined) two-point functions, and the labels $\phi = A,\psi,\zeta$ denote either gluons, or fermions-antifermions, or ghosts-antighosts, respectively (in the end, ghosts play no dynamical role, beyond maintaining gauge covariance of the final result). $S(A)$ is the pure Yang-Mills action, and

$$\begin{aligned}\Gamma_c(A,G_\phi) &= \frac{i}{2}\Big(\mathrm{Tr}\,\ln G_A^{-1} + \mathrm{Tr}\,\mathcal{D}_A^{-1}G_A\Big)\\ &\quad -i\Big(\mathrm{Tr}\,\ln G_\psi^{-1} + \mathrm{Tr}\,\mathcal{D}_\psi^{-1}G_\psi + \mathrm{Tr}\,\ln G_\zeta^{-1} + \mathrm{Tr}\,\mathcal{D}_\zeta^{-1}G_\zeta\Big) \qquad (3.18)\end{aligned}$$

when 2PI contributions are omitted (this comprises the first approximation we make). The trace is over space-time arguments as well as over Lorentz and group indices. The gauge coupling constant g, which was previously scaled to unity, is here reinserted. $\mathcal{D}_\phi^{-1}$ is computed from the usual QCD action S_{QCD} (*e.g.* in the Feynman gauge):

$$i\mathcal{D}_\phi^{-1}(x,y) = \frac{\delta^2 S_{QCD}}{\delta\phi(x)\,\delta\phi(y)}\ . \qquad (3.19)$$

The fields carry group and space-time indices, which are symbolically subsumed into the space-time labels x, y.

As indicated in [36], $S+\Gamma_c$ is stationary for physical processes. This yields the conditions

$$D_\nu F^{\nu\mu} = J^\mu\ , \qquad (3.20)$$

$$G_\phi^{-1} = \mathcal{D}_\phi^{-1}\ , \quad \phi = A,\psi,\zeta\ . \qquad (3.21)$$

Computing the local induced current $J^\mu(x) = -\frac{\delta \Gamma_c}{\delta A_\mu(x)}$ involves differentiating $\mathcal{D}_\phi^{-1}$ with respect to A_μ. Since the $\mathcal{D}_\phi^{-1}$ depend locally on A, the resulting current is the local limit of a bilocal expression constructed from the two-point functions $G_\phi(x,y)$:

$$J^\mu(x) = \lim_{y \to x} J^\mu(x,y) \ , \tag{3.22}$$

where the bilocal current $J^\mu(x,y) = T^a J_a^\mu(x,y)$ is given by

$$J^\mu(x,y) = g\left(\Gamma^\mu_{\alpha\beta\gamma}\, D_x^\alpha \mathbf{G}_A{}^{\beta\gamma}(x,y) + \partial_y^\mu \mathbf{G}_\zeta(x,y)\right) + igT^a \operatorname{tr} \gamma^\mu T^a G_\psi(x,y) \tag{3.23}$$

with $\Gamma^\mu_{\alpha\beta\gamma} \equiv 2g^\mu_\beta g_{\alpha\gamma} - g^\mu_\alpha g_{\beta\gamma} - g^\mu_\gamma g_{\beta\alpha}$. The trace "tr" is taken over Dirac spinor as well as internal symmetry indices, and we have defined $\mathbf{G}_{A,\zeta}(x,y) = [T^a, T^b] G_{A,\zeta\, ab}(x,y)$ with $D_x \mathbf{G}_A(x,y) = \partial_x \mathbf{G}_A(x,y) + g[[A(x), T^b], T^c] G_{A\, bc}(x,y)$.

We now use eqs. (3.20) – (3.23) to study "soft" plasma excitations. "Soft" means that both the energy and the momentum carried by a particle are of order gT, for a coupling constant $g \ll 1$, while particles with energy or momentum of order T are called hard (see *e.g.* [18]). The strategy is to solve the system of coupled equations (3.20), (3.21), in order to derive from (3.23) the expression (3.3) for the local current J^μ. We approximate eqs. (3.20), (3.21) by expanding them in powers of g. The approximation scheme we use was first proposed in [21]. Accordingly, we introduce relative and center of mass coordinates, $s = x - y$ and $X = \frac{1}{2}(x+y)$, respectively. Note that in the new variables the partial derivatives carry different dependences on g: $\partial_s \sim T$ and $\partial_X \sim gT$. This comes from the fact that ∂_s corresponds to hard loop momenta, whereas ∂_X is related to soft external momenta.

Next, we expand G_ϕ in powers of g:

$$G_\phi = G_\phi^{(0)} + gG_\phi^{(1)} + g^2 G_\phi^{(2)} + \dots \ , \tag{3.24}$$

where $G_\phi^{(0)}$ is just the free propagator at temperature T and $G_\phi^{(i)}$, $i \geq 1$ are determined by (3.21). At leading order in g (to which we restrict ourselves in the sequel), the bilocal current (3.23) depends on $G_\phi^{(0)}$ and $G_\phi^{(1)}$:

$$\begin{aligned} J_a^\mu(X,s) &= g^2 f^{abc}\Big[\Gamma^\mu_{\alpha\beta\gamma}\Big(\partial_s^\alpha G_{A\,bc}^{(1)\,\beta\gamma}(X,s) + f^{bde}A_d^\alpha(X)G_{A\,ec}^{(0)\,\beta\gamma}(X,s)\Big) - \partial_s^\mu G_{\zeta\,bc}^{(1)}(X,s)\Big] \\ &\quad + ig^2\,\mathrm{tr}\,\gamma^\mu T^a G_\psi^{(1)}(X,s) + \delta J_a^\mu(X,s)\ , \end{aligned} \tag{3.25}$$

where $G_\phi(X,s) \equiv G_\phi(X+\frac{s}{2}, X-\frac{s}{2})$ [and similarly for $J(X,s)$]. We have added the term $\delta J_a^\mu(X,s)$ in order to compensate for the loss of gauge covariance due to non-locality:

$$\delta J_a^\mu(X,s) = g^2 s\cdot A^b(X)\Big[f^{ace}f^{bcd}\Big(3\,\partial_\nu^s G_{A\,de}^{(0)\,\mu\nu}(s) + \partial_s^\mu G_\zeta^{(0)\,de}(s)\Big) + i\,\mathrm{tr}\,T^bT^a\gamma^\mu G_\psi^{(0)}(s)\Big]. \tag{3.26}$$

Note that this term vanishes in the local limit.

Now, we derive from (3.21) a condition on $G_\phi^{(1)}$. [It turns out to be convenient to expand, instead of (3.21), the equivalent equations $\mathcal{D}_\phi^{-1}G_\phi = G_\phi\mathcal{D}_\phi^{-1} = I$, in which we disregard temperature-independent contributions.] The $\mathcal{O}(g)$-condition does not fix $G_\phi^{(1)}$ uniquely; hence we need to go to $\mathcal{O}(g^2)$. The condition so obtained on $G_\phi^{(1)}$ is then used to derive a constraint on the bilocal current. The subsequent analysis is straightforward and lengthy (it is similar to the one given in [21], to which we refer the reader); momentum space is most convenient, *i.e.*

$$G_\phi(X,k) = \int d^4s\, e^{ik\cdot s}G_\phi(X,s)\ , \tag{3.27}$$

the explicit forms for the thermal parts of the free propagators being (*e.g.* in Feynman gauge):

$$\begin{aligned} G_{A\,ab}^{(0)\,\mu\nu}(k) &= -2\pi\,\delta^{ab}g^{\mu\nu}\delta(k^2)\,n_B(k_0)\ , \\ G_\psi^{(0)\,mn}(k) &= -2\pi\,\delta^{mn}\not{k}\delta(k^2)\,n_F(k_0)\ , \\ G_\zeta^{(0)\,ab}(k) &= 2\pi\,\delta^{ab}\delta(k^2)\,n_B(k_0)\ , \end{aligned} \tag{3.28}$$

where $n_{B,F}(k_0) = 1/(e^{\beta|k_0|} \mp 1)$ are the bosonic and fermionic probability distributions.

Similarly, for the bilocal current in momentum space one writes

$$J^\mu(X,k) = \int d^4s\, e^{ik\cdot s} J^\mu(X,s) \ . \tag{3.29}$$

In the limit $s \to 0$, or equivalently $y \to x$, where $X = x$,

$$J^\mu(x) = J^\mu(X) = \int \frac{d^4k}{(2\pi)^4} J^\mu(X,k) \ . \tag{3.30}$$

The resulting constraint on the bilocal current is [21]:

$$Q \cdot D_X\, J^\mu(X,k) = 4\pi g^2 Q^\mu Q^\rho k_0\, F_{\rho 0}\, \delta(k^2) \frac{d}{dk_0}[N\, n_B(k_0) + N_F\, n_F(k_0)] \ , \tag{3.31}$$

where $Q^\mu \equiv \frac{k^\mu}{k_0} = (1, \mathbf{Q})$. We integrate this equation over $|\mathbf{k}|$ and $k_0 \geq 0$. Due to the $\delta(k^2)$ on the right side, the bilocal current is non-vanishing only when $k_0 = |\mathbf{k}|$; hence $\mathbf{Q}$ can be replaced by a unit vector $\hat{q} \equiv \frac{\mathbf{k}}{|\mathbf{k}|}$. The integration thus yields:

$$Q_+ \cdot D_X\, \mathcal{J}^\mu_+(X,\hat{q}) = -2\sqrt{2}\,\pi^3 m^2 Q^\mu_+ Q^\rho_+ F_{\rho 0} \ , \tag{3.32}$$

where we have defined

$$\mathcal{J}^\mu_+(X,\hat{q}) = \int |\mathbf{k}|^2 d|\mathbf{k}| \int_0^\infty dk_0\, J^\mu(X,k) \ . \tag{3.33}$$

Similarly, upon introducing

$$\mathcal{J}^\mu_-(X,\hat{q}) = \int |\mathbf{k}|^2 d|\mathbf{k}| \int_{-\infty}^0 dk_0\, J^\mu(X,k) \ , \tag{3.34}$$

the integration of (3.31) over $|\mathbf{k}|$ and $k_0 \leq 0$ gives:

$$Q_- \cdot D_X\, \mathcal{J}^\mu_-(X,\hat{q}) = -2\sqrt{2}\,\pi^3 m^2 Q^\mu_- Q^\rho_- F_{\rho 0} \ , \tag{3.35}$$

wherefrom one sees that $\mathcal{J}^\mu_-(X,-\hat{q})$ satisfies the same equation (3.32) as $\mathcal{J}^\mu_+(X,\hat{q})$.

Now, using $\int d^4k = \int d\Omega |\mathbf{k}|^2 d|\mathbf{k}| dk_0$, we rewrite the expression (3.30) for the local current as $J^\mu(X) = \int \frac{d\hat{q}}{(2\pi)^4}[\mathcal{J}_+^\mu(X,\hat{q}) + \mathcal{J}_-^\mu(X,\hat{q})]$. Here, $\hat{q}$ can be replaced by $-\hat{q}$ in each term of the integrand separately, since $\hat{q}$ spans the whole solid angle. Therefore, we can write

$$J^\mu(X) = \int \frac{d\hat{q}}{(2\pi)^4} \mathcal{J}^\mu(X,\hat{q}) \ , \tag{3.36}$$

where $\mathcal{J}^\mu(X,\hat{q})$ is defined as

$$\mathcal{J}^\mu(X,\hat{q}) \equiv \mathcal{J}_+^\mu(X,\hat{q}) + \mathcal{J}_-^\mu(X,-\hat{q}) \ , \tag{3.37}$$

and satisfies, as a consequence of (3.32) and (3.35),

$$Q_+ \cdot D_X \, \mathcal{J}^\mu(X,\hat{q}) = -4\sqrt{2}\,\pi^3 m^2 Q_+^\mu Q_+^\rho F_{\rho 0} \ . \tag{3.38}$$

From this, after decomposing

$$\mathcal{J}^\mu(X,\hat{q}) = \tilde{\mathcal{J}}^\mu(X,\hat{q}) - 4\sqrt{2}\,\pi^3 m^2 Q_+^\mu A_0 \ , \tag{3.39}$$

we get as our final condition on the bilocal current:

$$Q_+ \cdot D_X \, \tilde{\mathcal{J}}^\mu(X,\hat{q}) = 4\sqrt{2}\,\pi^3 m^2 Q_+^\mu \partial_X^0 (Q_+ \cdot A) \ . \tag{3.40}$$

Let us now assume that $\tilde{\mathcal{J}}^\mu(X,\hat{q})$ can be obtained from a functional $W(A,\hat{q})$ as

$$\tilde{\mathcal{J}}^\mu(X,\hat{q}) = \frac{\delta W(A,\hat{q})}{\delta A_\mu(X)} \ . \tag{3.41}$$

Equation (3.40) then implies that $W(A,\hat{q})$ depends only on A_+, *i.e.* $W(A,\hat{q}) = W(A_+)$, and $\tilde{\mathcal{J}}^\mu = \frac{\delta W(A_+)}{\delta A_+} Q_+^\mu$. In turn, $W(A_+)$ satisfies, as a consequence of (3.40),

$$Q_+ \cdot D_X \, \frac{\delta W(A_+)}{\delta A_+} = 4\sqrt{2}\,\pi^3 m^2 \partial_X^0 A_+ \ . \tag{3.42}$$

By introducing new coordinates $(x_+, x_-, \mathbf{x}_\perp)$,

$$x_+ = Q_- \cdot X, \qquad x_- = Q_+ \cdot X, \qquad \mathbf{x}_\perp \cdot \hat{q} = 0 \ , \tag{3.43}$$

we can rewrite $Q_+ \cdot \partial_X$ as ∂_+ and (3.42) becomes

$$\partial_+ \frac{\delta W(A_+)}{\delta A_+} + g \left[A_+ \, , \frac{\delta W(A_+)}{\delta A_+} \right] = 4\sqrt{2}\, \pi^3 m^2 \partial_X^0 A_+ \ . \tag{3.44}$$

This equation was first derived in [23], as an expression of gauge invariance of the generating functional for hard thermal loops, and has since then been studied by several authors. Here, it is seen to be a consequence of the stationarity requirement on the composite effective action.

It has been shown in [9] that $W(A_+)$ is given by the eikonal of a Chern-Simons gauge theory. This observation is our last step towards deriving the approximate expression for the local current $J^\mu(x)$ in eq. (3.20). The subsequent development follows [20] and the result is exactly the non-Abelian Kubo equation (3.1) with the form (3.3) for the induced current.

3.4 Conclusions

The behavior of the quark-gluon plasma at high temperature is described by the non-Abelian Kubo equation (3.1) – (3.3). We have studied the static response of such a plasma and proved that there are no hard thermal solitons. The absence of "static" instantons is established by invoking a similar argument. In addition, the static non-Abelian Kubo equation indicates that the non-Abelian electric field is screened by a gauge invariant Debye mass $m = gT\sqrt{\frac{N+N_F/2}{3}}$. Furthermore, we have derived the non-Abelian Kubo equation from the composite effective action formalism. Indeed, the requirement that the composite effective action be stationary leads, within a kinematical approximation scheme taken at the leading order, to the equation obtained in [23] by imposing gauge invariance on the generating functional of hard thermal loops.

Let us mention some problems deserving further investigation. Finding non-static solutions to the non-Abelian Kubo equation is an appealing — if difficult — task, since such solutions would correspond to collective excitations of the quark-gluon plasma at high temperature. Also, it would be interesting to investigate the next-to-leading order effects in the kinematical approximation and to see if they are gauge invariant. It is clear that $\Gamma_c(A, G_\phi)$, when evaluated on the solution for G_ϕ obtained from (3.21) and (3.24), coincides with the $\Gamma(A)$ constructed from the Chern-Simons eikonal. While our derivation establishes this fact indirectly, an explicit evaluation of the relevant functional determinants in the hard thermal limit would be welcome.

NOTE ADDED

We have now seen two papers [41] wherein the static response equations (3.10), (3.11) are also obtained. Moreover, a local equation is found (and analyzed) for time-dependent, but space-independent gauge fields. The starting point of these investigations is a non-local expression for the induced current (see [41]),

$$j^{\text{ind}}_{\mu\ a}(x) = 3\,\omega_p^2 \int \frac{d\Omega}{4\pi}\, v_\mu \int_0^\infty du\, U_{ab}(x, x - vu)\, \mathbf{v} \cdot \mathbf{E}^b(x - vu) \ , \tag{3.45}$$

which appears different from our local, but coupled, form (3.3) – (3.6). Here we exhibit the steps that explicitly relate the two. We also derive the time-dependent, space-independent equations from our formalism.

Beginning with our form for the induced current, (3.3) – (3.6), we observe that, owing to the integration over the angles of $\hat{q}$, we may collapse these expressions into

$$\frac{m^2}{2}\, j^\mu(x) = m^2 \int \frac{d\hat{q}}{4\pi}\, Q_+^\mu \Big(a_-(x) - A_-(x) \Big) \ , \tag{3.46}$$

where

$$\partial_+ a_- + [A_+, a_-] = \partial_- A_+ \ . \tag{3.47}$$

Eq. (3.47) may be integrated, yielding

$$a^a_-(x) = \int_0^\infty du\, U_{ab}(x, x - Q_+u)\, \partial_- A^b_+(x - Q_+u)\ . \tag{3.48}$$

Here U_{ab} satisfies

$$\begin{aligned} &\frac{\partial}{\partial u} U_{ab}(x, x - Q_+u) = U_{ac}(x, x - Q_+u)\, f_{cbd}\, A^d_+(x - Q_+u)\ , \\ &U_{ab}(x, x) = \delta_{ab}\ . \end{aligned} \tag{3.49}$$

Also A^a_- may be presented as

$$\begin{aligned} A^a_-(x) &= -\int_0^\infty du\, \frac{d}{du} \Big\{ U_{ab}(x, x - Q_+u)\, A^b_-(x - Q_+u) \Big\} \\ &= \int_0^\infty du\, U_{ab}(x, x - Q_+u) \Big\{ \partial_+ A^b_-(x - Q_+u) \\ &\qquad - f^{bcd}\, A^c_-(x - Q_+u)\, A^d_+(x - Q_+u) \Big\}. \end{aligned} \tag{3.50}$$

[We have assumed that no contributions arise at infinity.] From (3.46), (3.48) and (3.50), it follows that the induced current can be written as

$$\frac{m^2}{2} j^\mu_a(x) = m^2 \int \frac{d\hat{q}}{4\pi}\, Q^\mu_+ \int_0^\infty du\, U_{ab}(x, x - Q_+u)\ F^b_{-+}(x - Q_+u)\ , \tag{3.51}$$

which coincides with the expression (3.45) derived in [41], after the notational replacements $m \to \sqrt{3}\,\omega_p$, $d\hat{q} \to d\Omega$, $Q^\mu_+ \to v^\mu$ and $F_{-+} \to \mathbf{v} \cdot \mathbf{E}$ are performed.

The time-dependent, space-independent equation found in [41] is easily derived in our formalism, also. When there is no space dependence, eqs. (3.6) can be written as

$$\partial_+(a_\mp - A_\pm) + [A_\pm, a_\mp - A_\pm] = 0 \tag{3.52}$$

and are solved by $a_\mp = A_\pm$. Hence:

$$\frac{m^2}{2} j^\mu = m^2 \int \frac{d\hat{q}}{2\pi}\, (Q_+ - Q_-)^\mu (Q_+ - Q_-)^\nu A_\nu\ , \tag{3.53}$$

of which only the spatial component is non-vanishing:

$$\frac{m^2}{2}\, j^i = m^2 \int \frac{d\hat{q}}{\pi}\, \hat{q}^i \hat{q}^j A_j = -\frac{4}{3}\, m^2 A^i \,. \tag{3.54}$$

This is the result obtained in [41].

Chapter 4

Classical Transport Theory and Hard Thermal Loops in the Quark-gluon Plasma

Following a recent Letter [42], we present an expanded and self-contained account of the derivation of the hard thermal loops (HTLs) of QCD from classical transport theory. In addition, we justify the use of the *ad hoc* phase-space integration measure for classical colored particles. This justification is based on the phase-space symplectic structure, and relates directly the (dependent) color charges to a set of (independent) Darboux variables. We also discuss formally the gauge invariance properties of the system of coupled non-Abelian Vlasov equations, and exploit the gauge principle to justify the approximation scheme we use. In order to show how physical information can be extracted, we analyze color polarization of the quark–gluon plasma in a plane-wave *Ansatz.*

We start by reviewing the work relevant to hard thermal loops in QCD. The motivation that led to their discovery was that physical quantities (such as damping rates) in hot QCD were *gauge-dependent* when computed using the usual loop expansion [22]. The solution to this puzzle was first proposed by Pisarski [8]. Subsequent development was carried out by Braaten and Pisarski [18], and by Frenkel and Taylor [19]. These authors realized that, in the diagrammatic approach to high-temperature QCD, a resummation procedure is necessary in order to take into account consistently all contributions at leading-order in the coupling constant. Such contributions were found to arise only from one-loop diagrams with "soft" external and "hard" internal momenta. "Soft" denotes a scale $\sim gT$ and "hard" refers to one $\sim T$, where $g \ll 1$ is the coupling constant, and T denotes the plasma temperature. Such diagrams were called "hard thermal loops" in [8, 18]. The HTL approach was successful in providing gauge-invariant results for physical quantities. Identifying the momentum scales that are relevant to the study of a hot quark–gluon plasma, as was done in [8, 18], was an essential step for all further developments on HTLs.

An effective action for HTLs was given by Taylor and Wong [23] who, after imposing gauge invariance, solved the resulting condition on the generating functional. Efraty and Nair [9] have identified this gauge invariance condition with the equation of motion for the topological Chern-Simons theory at zero temperature, thereby providing a non-thermal framework for studying hard thermal physics. Along the same line of research, the eikonal for a Chern-Simons theory has been used by Jackiw and Nair [20] to obtain a non-Abelian generalization of the Kubo formula, which governs, through the current induced by HTLs, the response of a hot quark–gluon plasma.

Another description of hard thermal loops in QCD has been proposed by Blaizot and Iancu [21]. It is based on a truncation of the Schwinger-Dyson hierarchy and yields quantum kinetic equations for the QCD induced color current. These kinetic equations, as well as the generating functional for HTLs, were obtained in [21] by performing a consistent expansion in the coupling constant, which amounts to taking into account the coupling constant dependence carried by the space-time derivatives. This dependence is extracted by going to a coordinate system which separates long-wavelength, collective excitations carrying soft momenta from the typical hard energies of plasma particles.

Alternatively, Jackiw, Liu and Lucchesi [43] have shown how HTLs can be derived from the Cornwall-Jackiw-Tomboulis composite effective action [36] by requiring its stationarity, and by using the approximation scheme developed in [21].

The resummation prescription of Braaten-Pisarski and Frenkel-Taylor, as well as the consistent expansion in the coupling constant developed by Blaizot-Iancu, although remarkably insightful, are technically very involved and necessitate lengthy computations. Furthermore, they are puzzling with respect to the very nature of hard thermal loops. One wonders if a quantum field theoretical description of hard thermal loops (involving gauge-fixing and ghost fields) is required. Indeed, we are faced with the following situation: in the resummation approach, HTLs emerge from loop diagrams, and in Blaizot and Iancu's work, they arise from the Schwinger-Dyson equations. However, hard thermal effects are UV-finite since they are due exclusively to *thermal* fluctuations. One might therefore be able to describe such effects within a *classical*, more transparent, context.

This motivated us to develop a classical formalism [42] for hard thermal loops in QCD, the natural starting point being the classical transport theory of plasmas (see for instance [35]). Our effort was encouraged by the fact that for an Abelian plasma of electrons and ions, the dielectric tensor computed [35] from classical transport theory is the same as that extracted from the hard thermal corrections to the vacuum polarization tensor [18, 19, 20]. Moreover, the same situation is encountered for non-Abelian plasmas [40, 44].

The classical transport theory for non-Abelian plasmas has been established by Elze and Heinz [40], before hard thermal effects were an issue. The HTLs of QCD were not uncovered in these early works, mainly due to the lack of a motivation to do so and because the transport equations had been linearized, thereby neglecting non-Abelian contributions. There has not been, to the best of our knowledge, any attempt to derive the complete set of HTLs for QCD from classical transport theory. The aim of the present paper is to give a detailed account of this derivation, the results of which have already appeared recently in a Letter [42].

In our approach, the generating functional of HTLs (with an arbitrary number of soft external bosonic legs) arises as a leading-order effect in the coupling constant. We start by reviewing the Wong equations for classical colored particles. Following [40], these are substituted into the transport equation, which governs the time evolution of the one-particle distribution function, thereby yielding the so-called Boltzmann equation. The latter, augmented with the Yang-Mills equation relating the field strength to the color current, form a consistent set of coupled, gauge-invariant differential equations known as non-Abelian Vlasov equations.

Expanding the distribution function in powers of the coupling constant and considering the lowest-order effects, we obtain a constraint on the color current. The latter constraint is equivalent to the condition found in [43], and previously in [21], on the induced current. The constraint on the color current leads to the generating functional of hard thermal loops.

This work is structured as follows. Section 4.1 describes classical transport theory for the quark–gluon plasma. The latter is reviewed in Subsection 4.1.1. In Subsection 4.1.2, we discuss and justify the *ad hoc* phase-space integration measure, using Darboux variables. Subsection 4.1.3 presents an analysis of the gauge invariance of the system of non-Abelian Vlasov equations. Section 4.2 contains the derivation of the hard thermal loops of QCD. As a consequence of constraints satisfied by the induced current (which are derived in Subsection 4.2.1), we obtain the generating functional of hard thermal loops (Subsection 4.2.2). In Section 4.3, we compute the polarization tensor from classical transport theory (at leading-order in g) and extract the expression for Landau damping. The consistency of our result with previous ones is discussed. Section 4.4 states our conclusions. In particular, we discuss there the validity of our approximations. In Appendix B, we check the validity of the Boltzmann equation. Appendix C presents a proof of the covariant conservation of the color current.

4.1 Classical Transport Theory for a Non-Abelian Plasma

4.1.1 Classical motion and non-Abelian Vlasov equations

The classical transport theory for the QCD plasma was developed in [40], which we follow in this subsection. Consider a particle bearing a non-Abelian $SU(N)$ color charge Q^a, $a = 1, ..., N^2 - 1$, traversing a worldline $x^\alpha(\tau)$, where τ denotes the proper time. The dynamical effects of the spin of the particles shall be ignored, as they are typically small. The Wong equations [45] describe the dynamical evolution of the variables[1] x^μ, p^μ and Q^a:

$$m\,\frac{dx^\mu}{d\tau} = p^\mu\ , \tag{4.1}$$

$$m\,\frac{dp^\mu}{d\tau} = g\,Q^a F_a^{\mu\nu} p_\nu\ , \tag{4.2}$$

$$m\,\frac{dQ^a}{d\tau} = -g\,f^{abc} p^\mu A_\mu^b Q^c\ . \tag{4.3}$$

The f^{abc} are the structure constants of the group, $F_a^{\mu\nu}$ denotes the field strength, g is the coupling constant, and we set $c = \hbar = k_B = 1$ henceforth. Equation (4.2) is the non-Abelian generalization of the Lorentz force law, and (4.3) describes the precession in color space of the charge in an external color field A_μ^a. It is noteworthy that the color charge Q^a is itself subject to dynamical evolution, a feature which distinguishes the non-Abelian theory from electromagnetism.

The usual (x, p) phase-space is now enlarged to (x, p, Q) by including into it color degrees of freedom for colored particles. Physical constraints are enforced by inserting delta-functions in the phase-space volume element $dx\,dP\,dQ$. The momentum measure

$$dP = \frac{d^4p}{(2\pi)^3}\,2\,\theta(p_0)\ \delta(p^2 - m^2) \tag{4.4}$$

guarantees positivity of the energy and on-shell evolution. The color charge measure enforces the conservation of the group invariants, *e.g.*, for $SU(3)$,

$$dQ = d^8Q\ \delta(Q_aQ^a - q_2)\ \delta(d_{abc}Q^aQ^bQ^c - q_3)\ , \tag{4.5}$$

where the constants q_2 and q_3 fix the values of the Casimirs and d_{abc} are the totally symmetric group constants. The color charges which now span the phase-space are

[1] Note that we are using the kinetic momentum, rather than the canonical one. A formulation in terms of canonical variables would be equivalent [35].

dependent variables. These can be formally related to a set of independent phase-space Darboux variables. This derivation is presented in Subsection 4.1.2 below.

The one-particle distribution function $f(x,p,Q)$ denotes the probability for finding the particle in the state (x,p,Q). It evolves in time via a transport equation,

$$m\,\frac{df(x,p,Q)}{d\tau} = C[f](x,p,Q) \ , \tag{4.6}$$

where $C[f](x,p,Q)$ denotes the collision integral, which we henceforth set to zero. Using the equations of motion (4.1), (4.2), (4.3), (4.6) becomes, in the collisionless case, the Boltzmann equation:

$$p^\mu \left[\frac{\partial}{\partial x^\mu} - g\,Q_a F^a_{\mu\nu}\frac{\partial}{\partial p_\nu} - g\,f_{abc}A^b_\mu Q^c \frac{\partial}{\partial Q_a}\right] f(x,p,Q) = 0 \ . \tag{4.7}$$

In Appendix B, an explicit microscopic distribution function is presented and used to check the validity of (4.7).

A complete, self-consistent set of non-Abelian Vlasov equations for the distribution function and the mean color field is obtained by augmenting the Boltzmann equation with the Yang-Mills equations:

$$[D_\nu F^{\nu\mu}]^a(x) = J^{\mu\,a}(x) \ . \tag{4.8}$$

The covariant derivative is defined as $D^{ac}_\mu = \partial_\mu \delta^{ac} + g\,f^{abc}A^b_\mu$. The total color current $J^{\mu\,a}(x)$ is given by the sum of all contributions from particle species and helicities,

$$J^{\mu\,a}(x) = \sum_{\text{species}}\ \sum_{\text{helicities}} j^{\mu\,a}(x) \ . \tag{4.9}$$

Each $j^{\mu\,a}(x)$ (species and spin indices are implicit) is computed from the corresponding distribution function as

$$j^{\mu\,a}(x) = g \int dPdQ\ p^\mu Q^a f(x,p,Q) \tag{4.10}$$

and it is covariantly conserved,

$$(D_\mu j^\mu)^a(x) = 0 \ , \tag{4.11}$$

as can be checked by using the Boltzmann equation (a detailed proof is presented in Appendix C). For later convenience, we define the total and individual current momentum-densities:

$$J^{\mu\, a}(x,p) = \sum_{\text{species}} \sum_{\text{helicities}} j^{\mu\, a}(x,p) \ , \qquad j^{\mu\, a}(x,p) = g \int dQ \ p^\mu Q^a f(x,p,Q) \ . \tag{4.12}$$

Note that a solution to the set of Vlasov equations (4.7)–(4.8) is specified by giving the forms for the gauge potential $A_\mu(x)$ and for the distribution function $f(x,p,Q)$.

4.1.2 Phase-space for colored particles

In order to carry out the transport theory analysis for classical colored particles, it has been necessary to extend phase-space by the addition of the color charges. In (4.5), the charges are constrained to remain within the group manifold by means of delta-functions which fix the values of the (representation-dependent) group Casimirs. In fact, at an operational level, this is the approach adopted in the rest of the present paper. In this subsection, we formally justify this approach by analysis of the symplectic structure of the group manifold [47, 46]. We work out explicitly the $SU(2)$ and $SU(3)$ cases.

The group $SU(2)$, is generated by three charges, (Q_1, Q_2, Q_3), and has one Casimir, $Q^a Q_a$. The structure constants are $f_{abc} = \epsilon_{abc}$, while $d_{abc} = 0$. From the point of view adopted throughout the rest of this paper the phase-space color measure is

$$dQ = dQ_1\, dQ_2\, dQ_3\, \delta(Q^a Q_a - q_2) \ , \tag{4.13}$$

where q_2 denotes the value of the quadratic Casimir.

New coordinates (ϕ, π, J) may be introduced by the following transformation [46]:

$$Q_1 = \cos\phi\,\sqrt{J^2 - \pi^2}\ , \qquad Q_2 = \sin\phi\,\sqrt{J^2 - \pi^2}\ , \qquad Q_3 = \pi\ . \tag{4.14}$$

Note that π is bounded, $-J \leq \pi \leq J$. That the group manifold has spherical geometry is readily apparent if one chooses $\pi = J\cos\theta$. The variables ϕ and π form a canonically conjugate pair; the Poisson bracket may be formed in the conventional manner:

$$\{A, B\}_{\mathrm{PB}} \equiv \frac{\partial A}{\partial \phi}\frac{\partial B}{\partial \pi} - \frac{\partial A}{\partial \pi}\frac{\partial B}{\partial \phi}\ . \tag{4.15}$$

It is easily verified that the charges as given by (4.14) form a representation of $SU(2)$ under the Poisson bracket, *i.e.*,

$$\{Q_a, Q_b\}_{\mathrm{PB}} = \epsilon_{abc}\, Q_c\ . \tag{4.16}$$

The above Poisson bracket structure allows one to identify ϕ and π as Darboux variables (see for instance [47]). The Jacobian of the transformation from (Q_1, Q_2, Q_3) to (ϕ, π, J) takes the value

$$\left|\frac{\partial(Q_1, Q_2, Q_3)}{\partial(\phi, \pi, J)}\right| = J\ . \tag{4.17}$$

Performing the change of variables (4.14) in (4.13) and substituting the value of the quadratic Casimir, $Q^a Q_a = J^2$, the color measure reads

$$dQ = d\phi\, d\pi\, dJ\, J\, \delta(J^2 - q_2)\ , \tag{4.18}$$

which, upon integration over the constrained variable J, is just the proper, canonical volume element $d\phi\, d\pi$, up to an irrelevant constant.

The group $SU(3)$ has eight charges, $(Q_1, \ldots, Q_8)$ and two conserved quantities, the quadratic and the cubic Casimirs, $Q^a Q_a$ and $d_{abc} Q^a Q^b Q^c$, respectively. The phase-space color measure is quoted above in (4.5).

As in the $SU(2)$ case, new coordinates $(\phi_1, \phi_2, \phi_3, \pi_1, \pi_2, \pi_3, J_1, J_2)$ may be introduced by means of the following transformations [46]:

$$
\begin{aligned}
Q_1 &= \cos\phi_1\,\pi_+\,\pi_- \ , & Q_2 &= \sin\phi_1\,\pi_+\,\pi_- \ , \\
Q_3 &= \pi_1 \ , \\
Q_4 &= C_{++}\,\pi_+\,A + C_{+-}\,\pi_-\,B \ , & Q_5 &= S_{++}\,\pi_+\,A + S_{+-}\,\pi_-\,B \ , \\
Q_6 &= C_{-+}\,\pi_-\,A - C_{--}\,\pi_+\,B \ , & Q_7 &= S_{-+}\,\pi_-\,A - S_{--}\,\pi_+\,B \ , \\
Q_8 &= \pi_2 \ ,
\end{aligned}
\tag{4.19}
$$

in which we have used the definitions:

$$
\begin{aligned}
\pi_+ &= \sqrt{\pi_3+\pi_1} \ , & \pi_- &= \sqrt{\pi_3-\pi_1} \ , \\
C_{\pm\pm} &= \cos\left[\tfrac{1}{2}(\pm\phi_1+\sqrt{3}\phi_2\pm\phi_3)\right] \ , & S_{\pm\pm} &= \sin\left[\tfrac{1}{2}(\pm\phi_1+\sqrt{3}\phi_2\pm\phi_3)\right] \ ,
\end{aligned}
\tag{4.20}
$$

and A, B are given by

$$
\begin{aligned}
A &= \frac{1}{2\pi_3}\sqrt{\left(\frac{J_1-J_2}{3}+\pi_3+\frac{\pi_2}{\sqrt{3}}\right)\left(\frac{J_1+2J_2}{3}+\pi_3+\frac{\pi_2}{\sqrt{3}}\right)\left(\frac{2J_1+J_2}{3}-\pi_3-\frac{\pi_2}{\sqrt{3}}\right)} \ , \\
B &= \frac{1}{2\pi_3}\sqrt{\left(\frac{J_2-J_1}{3}+\pi_3-\frac{\pi_2}{\sqrt{3}}\right)\left(\frac{J_1+2J_2}{3}-\pi_3+\frac{\pi_2}{\sqrt{3}}\right)\left(\frac{2J_1+J_2}{3}+\pi_3-\frac{\pi_2}{\sqrt{3}}\right)} \ .
\end{aligned}
\tag{4.21}
$$

Note that in this representation, the set (Q_1, Q_2, Q_3) forms an $SU(2)$ subgroup with quadratic Casimir $Q_1^2+Q_2^2+Q_3^2=\pi_3^2$. It can be verified that the expressions above for $Q_1,\ldots,Q_8$ form a representation of the group $SU(3)$:

$$
\{Q_a, Q_b\}_{\text{PB}} = f_{abc}Q_c \ , \tag{4.22}
$$

under the Poisson bracket

$$
\{A, B\}_{\text{PB}} \equiv \sum_{i=1}^{3}\left(\frac{\partial A}{\partial\phi_i}\frac{\partial B}{\partial\pi_i}-\frac{\partial A}{\partial\pi_i}\frac{\partial B}{\partial\phi_i}\right) \ , \tag{4.23}
$$

where the canonical pairs are $\{\phi_i, \pi_i\}_{i=1,2,3}$.

As is implicit in the above, the two Casimirs depend only on J_1 and J_2. They can be computed, using the values given in the table below, as:

$$\begin{aligned} Q^a Q_a &= \frac{1}{3}(J_1^2 + J_1 J_2 + J_2^2) \ , \\ d_{abc} Q^a Q^b Q^c &= \frac{1}{18}(J_1 - J_2)(J_1 + 2J_2)(2J_1 + J_2) \ . \end{aligned} \tag{4.24}$$

d_{abc}	d_{118}	d_{146}	d_{157}	d_{228}	d_{247}	d_{256}	d_{338}	d_{344}	d_{355}	d_{366}	d_{377}	d_{448}	d_{558}	d_{668}
Value	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$

Table: Values of the (non-zero) $SU(3)$ totally symmetric constants.

The phase-space color measure for $SU(3)$, given in (4.5), may be transformed to the new coordinates through use of (4.24) and evaluation of the Jacobian

$$\left| \frac{\partial(Q_1, Q_2, \ldots, Q_8)}{\partial(\phi_1, \phi_2, \phi_3, \pi_1, \pi_2, \pi_3, J_1, J_2)} \right| = \frac{\sqrt{3}}{48} \, J_1 \, J_2 \, (J_1 + J_2) \ . \tag{4.25}$$

The measure reads:

$$\begin{aligned} dQ &= d\phi_1 \, d\phi_2 \, d\phi_3 \, d\pi_1 \, d\pi_2 \, d\pi_3 \, dJ_1 \, dJ_2 \frac{\sqrt{3}}{48} \, J_1 \, J_2 \, (J_1 + J_2) \, \delta\Big(\frac{1}{3}(J_1^2 + J_1 J_2 + J_2^2) - q_2\Big) \ \times \\ & \quad \delta\Big(\frac{1}{18}(J_1 - J_2)(J_1 + 2J_2)(2J_1 + J_2) - q_3\Big) \ . \end{aligned} \tag{4.26}$$

Since the two Casimirs are linearly independent, the delta-functions uniquely fix both J_1 and J_2 to be representation-dependent constants. Upon integrating over J_1 and J_2, (4.26) reduces to a constant times the proper canonical volume element $\prod_{i=1}^{3} d\phi_i \, d\pi_i$.

The construction of the canonical phase-space measure for the general $SU(N)$ case is a departure from our purposes and will not be undertaken here. Nevertheless, based on the examples we treated explicitly, it is apparent that no difficulties will arise for $N > 3$ [47].

In principle, the classical transport theory analysis should be carried out using canonical, independent integration variables and the phase-space volume element should be taken to be the proper canonical volume element. In this subsection, we have shown the equivalence of the *ad hoc* phase-space color measure and the proper canonical volume element. Hence, the use of the color charges as phase-space coordinates is justified.

4.1.3 Gauge invariance of the non-Abelian Vlasov equations

Before addressing the question of the gauge invariance of the system of Vlasov equations, we consider the Wong equations (4.1), (4.2),(4.3). They are invariant under the finite gauge transformations[2]:

$$\begin{aligned} \bar{x}^\mu &= x^\mu \, , \\ \bar{p}^\mu &= p^\mu \, , \\ \bar{Q} &= U\, Q\, U^{-1} \, , \\ \bar{A}_\mu &= U\, A_\mu\, U^{-1} - \frac{1}{g}\, U \frac{\partial}{\partial x_\mu} U^{-1} \, , \end{aligned} \tag{4.27}$$

where $U(x) = \exp[-g\, \varepsilon^a(x)\, t^a]$ is a group element.

Accordingly, the derivatives appearing in the Boltzmann equation (4.7) transform as:

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &= \frac{\partial}{\partial \bar{x}^\mu} - 2\ \mathrm{Tr}\left([\, (\frac{\partial}{\partial \bar{x}^\mu} U) U^{-1} \, , \, \bar{Q}\,] \frac{\partial}{\partial \bar{Q}} \right) , \\ \frac{\partial}{\partial p^\mu} &= \frac{\partial}{\partial \bar{p}^\mu} \, , \\ \frac{\partial}{\partial Q} &= U^{-1} \frac{\partial}{\partial \bar{Q}} U \, . \end{aligned} \tag{4.28}$$

Consequently, the Boltzmann equation (rewritten here in terms of traces):

$$\left[p^\mu \frac{\partial}{\partial x^\mu} + 2\, g\, p^\mu\ \mathrm{Tr}\ (Q F_{\mu\nu}) \frac{\partial}{\partial p_\nu} + 2\, g\, p^\mu\ \mathrm{Tr}\ \left([\, A_\mu \, , \, Q\,] \frac{\partial}{\partial Q} \right) \right] f(x,p,Q) = 0 \tag{4.29}$$

[2] We use here matrix notation, *e.g.* $Q = Q_a t^a$, $\frac{\partial}{\partial Q} = \frac{\partial}{\partial Q^a}\, t^a$, where the generators are represented by antihermitian matrices t^a in the fundamental representation, $[t^a, t^b] = f^{abc} t^c$, and they are normalized as $\mathrm{Tr}\ (t^a t^b) = -\frac{1}{2}\, \delta^{ab}$.

becomes, in the new coordinates:

$$
\Big[\quad \bar{p}^\mu \frac{\partial}{\partial \bar{x}^\mu} - 2\,g\,\bar{p}^\mu \text{ Tr } \Big([\, (\frac{\partial}{\partial \bar{x}^\mu} U) U^{-1} \;,\; \bar{Q}\,] \frac{\partial}{\partial \bar{Q}} \Big) + 2\,g\,\bar{p}^\mu \text{ Tr } (\bar{Q} \bar{F}_{\mu\nu}) \frac{\partial}{\partial \bar{p}_\nu}
$$

$$
+ 2\,g\,\bar{p}^\mu \text{ Tr } \Big([\, \bar{A}_\mu + \frac{1}{g} (\frac{\partial}{\partial \bar{x}^\mu} U) U^{-1} \;,\; \bar{Q}\,] \frac{\partial}{\partial \bar{Q}} \Big) \Big] \; \bar{f}(\bar{x}, \bar{p}, \bar{Q}) = 0 \;, \tag{4.30}
$$

where we have defined

$$
\bar{f}(\bar{x}, \bar{p}, \bar{Q}) = f\Big(x(\bar{x}, \bar{p}, \bar{Q}), p(\bar{x}, \bar{p}, \bar{Q}), Q(\bar{x}, \bar{p}, \bar{Q}) \Big) \;. \tag{4.31}
$$

Simplifying (4.30), we obtain:

$$
\Big[\bar{p}^\mu \frac{\partial}{\partial \bar{x}^\mu} + 2\,g\,\bar{p}^\mu \text{ Tr } (\bar{Q} \bar{F}_{\mu\nu}) \frac{\partial}{\partial \bar{p}_\nu} + 2\,g\,\bar{p}^\mu \text{ Tr } \Big([\, \bar{A}_\mu \;,\; \bar{Q}\,] \frac{\partial}{\partial \bar{Q}} \Big) \Big] \; \bar{f}(\bar{x}, \bar{p}, \bar{Q}) = 0 \;. \tag{4.32}
$$

This proves that the Boltzmann equation is invariant under gauge transformations. On the other hand, the Yang-Mills equation (4.8) is gauge-covariant. Indeed, the color current (4.10) transforms under (4.27) as a gauge covariant vector: $j^\mu(x) \rightarrow \bar{j}^\mu(\bar{x}) = \int d\bar{P}\, d\bar{Q}\, \bar{p}^\mu\, \bar{Q}\, \bar{f}(\bar{x}, \bar{p}, \bar{Q})$. Due to the gauge-invariance of the phase-space measure, to the transformation property of f (4.31), and to (4.27), $\bar{j}^\mu(x)$ may be rewritten as:

$$
\bar{j}^\mu(\bar{x}) = \int dP\, dQ\, p^\mu\, U\, Q\, U^{-1}\, f(x, p, Q) = U\, j^\mu(x)\, U^{-1} \;. \tag{4.33}
$$

Hence, the system of non-Abelian Vlasov equations is gauge-covariant, with the distribution function $f(x, p, Q)$ transforming as a scalar. Note that the gauge symmetry also implies that the gauge transform $\{\bar{A}_\mu(x), \bar{f}(x, p, Q)\}$ of a set of solutions $\{A_\mu(x), f(x, p, Q)\}$ to the Vlasov equations:

$$
\begin{aligned}
\bar{A}_\mu(x) &= U\, A_\mu(x)\, U^{-1} - \frac{1}{g} U \frac{\partial}{\partial x^\mu} U^{-1} \;, \\
\bar{f}(x, p, Q) &= f(x, p, U\, Q\, U^{-1})
\end{aligned} \tag{4.34}
$$

is still a solution.

4.2 Emergence of Hard Thermal Loops

4.2.1 Constraint on the color current

Classical transport theory is now employed to study soft excitations in a hot, color-neutral quark–gluon plasma. In the high-temperature limit, the masses of the particles can be neglected and shall henceforth be assumed to vanish. The wavelength of a soft excitation is of order $\frac{1}{g\,|A|}$ and the coupling constant g is assumed to be small. We then expand the distribution function $f(x,p,Q)$ in powers of g:

$$f = f^{(0)} + g f^{(1)} + g^2 f^{(2)} + \dots \,, \tag{4.35}$$

where $f^{(0)}$ is the equilibrium distribution function in the absence of a net color field, and is given by:

$$f^{(0)}(p_0) = C\; n_{B,F}(p_0)\;. \tag{4.36}$$

Here C is a normalization constant and $n_{B,F}(p_0) = 1/(e^{\beta|p_0|} \mp 1)$ is the bosonic, resp. fermionic, probability distribution.

At leading-order in g, the color current (4.12) is

$$j^{\mu\, a}(x,p) = g^2 \int dQ\; p^\mu Q^a f^{(1)}(x,p,Q)\;, \tag{4.37}$$

while the Boltzmann equation (4.7) reduces to

$$p^\mu \left(\frac{\partial}{\partial x^\mu} - g\, f^{abc} A^b_\mu Q_c \frac{\partial}{\partial Q^a} \right) f^{(1)}(x,p,Q) = p^\mu Q_a F^a_{\mu\nu} \frac{\partial}{\partial p_\nu} f^{(0)}(p_0)\;. \tag{4.38}$$

Due to the softness of the excitation, the $\frac{\partial}{\partial x^\mu}$ in the above equation is of order $g\,|A|$, so we are taking into account consistently all contributions of order g. The approximation we use guarantees that the non-Abelian gauge symmetry of the exact Boltzmann equation (4.7) is preserved in the approximate equation (4.38). As a consequence $f^{(0)}$ and $f^{(1)}$, like f, transform separately as gauge-invariant scalars. Other approximations, which have been carried out in the past [40], have discarded the non-Abelian contributions, thereby breaking the non-Abelian gauge symmetry of the Boltzmann equation.

The equations (4.37) and (4.38) yield the following constraint on the color current:

$$[p\cdot D\ j^{\mu}(x,p)]^{a} = g^{2}\, p^{\mu}p^{\nu}F^{b}_{\nu\rho}\frac{\partial}{\partial p_{\rho}}\left(\int dQ\ Q^{a}Q_{b}f^{(0)}(p_{0})\right)\ , \tag{4.39}$$

where, from color symmetry, we have $\int dQ\ Q^{a}Q_{b}f^{(0)}(p_{0}) = C_{B,F}\ n_{B,F}(p_{0})\,\delta^{a}_{b}$ with $C_{B}=N$, $C_{F}=\frac{1}{2}$ for gluons, resp. fermions. Thus, upon summation over all species (N_F quarks, N_F antiquarks and one $[(N^{2}-1)$-plet$]$ gluon) and helicities (2 for quarks-antiquarks and for the massless gluon), (4.39) yields,

$$[p\cdot D\ J^{\mu}(x,p)]^{a} = 2\,g^{2}\,p^{\mu}p^{\nu}F^{a}_{\nu 0}\frac{d}{dp_{0}}[N\,n_{B}(p_{0}) + N_{F}\,n_{F}(p_{0})]\ . \tag{4.40}$$

Similar results have been obtained in [21, 43], in a quantum field theoretic setting.

4.2.2 Derivation of hard thermal loops

Subsequent steps which lead to the generating functional of HTLs have been described in [43], the results of which were used straightforwardly in [42], for the sake of brevity. Here, we present a simpler derivation of HTLs by exploiting fully the structure of the momentum integration measure (4.4).

We first integrate equation (4.40) over $|\mathbf{p}|$ and p_0 using the massless limit of the momentum measure dP (4.4). Therefore, the (massless) mass-shell constraint enforces $|\mathbf{p}| = p_0$, and we thus introduce the unit vector $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$. Introducing also $v \equiv (1, \hat{\mathbf{p}})$, the integration of (4.40) yields (group indices are henceforth omitted):

$$v\cdot D\ \mathcal{J}^{\mu}(x,v) = -2\,\pi^{2}\,m_{D}^{2}\,v^{\mu}\,v^{\rho}\,F_{\rho 0}(x)\ , \tag{4.41}$$

where m_D is the Debye screening mass

$$m_{D} = gT\sqrt{\frac{N + N_{F}/2}{3}}\ , \tag{4.42}$$

and we have defined

$$\mathcal{J}^\mu(x,v) = \int |\mathbf{p}|^2\, d|\mathbf{p}|\, dp_0\, 2\,\theta(p_0)\,\delta(p^2)\; J^\mu(x,p)\ . \tag{4.43}$$

Notice (for later use) that, using $\int dP = \int \frac{d\Omega}{(2\pi)^3}|\mathbf{p}|^2 d|\mathbf{p}| dp_0\, 2\theta(p_0)\,\delta(p^2)$, where $d\Omega$ denotes integration over all angular directions of the unit vector $\hat{\mathbf{p}}$, we can rewrite the expression $J^\mu(x) = \int dP J^\mu(x,p)$ for the color current as

$$J^\mu(x) = \int \frac{d\Omega}{(2\pi)^3}\, \mathcal{J}^\mu(x,v)\ . \tag{4.44}$$

After decomposing $\mathcal{J}^\mu(x,v)$ as

$$\mathcal{J}^\mu(x,v) = \tilde{\mathcal{J}}^\mu(x,v) - 2\,\pi^2\, m_D^2\, v^\mu A_0(x)\ , \tag{4.45}$$

we get as our final condition on the color current:

$$v\cdot D\; \tilde{\mathcal{J}}^\mu(x,v) = 2\,\pi^2\ m_D^2\ v^\mu \frac{\partial}{\partial x^0}\Big(v\cdot A(x)\Big)\ . \tag{4.46}$$

It has been shown that solutions to (4.46) can be obtained from a functional $W(A,v)$ as [23]

$$\tilde{\mathcal{J}}^\mu(x,v) = \frac{\delta W(A,v)}{\delta A_\mu(x)}\ . \tag{4.47}$$

Equation (4.46) then implies that $W(A,v)$ depends only on $A_+ \equiv v\cdot A$, *i.e.* $W(A,v) = W(A_+)$, and $\tilde{\mathcal{J}}^\mu = \frac{\delta W(A_+)}{\delta A_+}\, v^\mu$. In turn, $W(A_+)$ satisfies, as a consequence of (4.46),

$$v\cdot D\ \frac{\delta W(A_+)}{\delta A_+} = 2\,\pi^2\ m_D^2\ \frac{\partial}{\partial x^0}\, A_+\ . \tag{4.48}$$

By introducing new coordinates $(x_+, x_-, \mathbf{x}_\perp)$,

$$x_+ = \bar{v}\cdot x\ , \qquad x_- = v\cdot x\ , \qquad \mathbf{x}_\perp = \mathbf{x} - (\hat{\mathbf{p}}\cdot\mathbf{x})\hat{\mathbf{p}}\ , \tag{4.49}$$

with $\bar{v} \equiv (1, -\hat{\mathbf{p}})$ and $\mathbf{x}_\perp \cdot \hat{\mathbf{p}} = 0$, we can rewrite $v \cdot \frac{\partial}{\partial x}$ as ∂_+ and (4.48) becomes:

$$\partial_+ \frac{\delta W(A_+)}{\delta A_+} + g \left[A_+, \frac{\delta W(A_+)}{\delta A_+} \right] = 2\,\pi^2\, m_D^2 \,\frac{\partial}{\partial x^0} A_+ \ . \tag{4.50}$$

Now using (4.44), (4.45) and (4.47), we define an effective action Γ that generates the color current, *i.e.*, $J^\mu(x) = -\frac{\delta \Gamma[A(x)]}{\delta A_\mu(x)}$, where Γ takes the form:

$$\Gamma = \frac{m_D^2}{2} \int d^4x\, A_0^a(x) A_0^a(x) - \int \frac{d\Omega}{(2\pi)^3}\, W(A_+) \ . \tag{4.51}$$

This is the expression for the effective action generating hard thermal loops [18, 19], while equation (4.50) represents the condition of gauge invariance [23] for this generating functional. By solving (4.50), Taylor and Wong [23], as well as Efraty and Nair [9], have given an explicit form for the functional $W(A_+)$ in the second term of (4.51). The first term is a mass term for $A_0^a(x)$ and describes Debye screening.

This concludes our derivation of the hard thermal loops of QCD from classical transport theory.

4.3 Application: Color Polarization

As an application of the classical transport formalism presented above, we solve the approximate Boltzmann equation (4.38) for plane-wave excitations in a collisionless isotropic plasma of quarks and gluons. Recall that in the case of a collisionless plasma of electrons and ions the Abelian version of equation (4.38) has been solved exactly for an electromagnetic plane-wave [35], making it possible to study the response of an Abelian plasma to a weak field. We shall proceed analogously in the non-Abelian case. We consider a plane-wave *Ansatz* in which the vector gauge fields only depend on x^μ through the combination $x \cdot k$, where $k^\mu = (\omega, \mathbf{k})$ is the wave vector, *i.e.*, $A_\mu^a(x) \equiv A_\mu^a(k \cdot x)$. With this *Ansatz* (which has been used in [41] to study the non-Abelian Kubo equation) the solution of (4.38) is

$$f^{(1)}(x,p,Q) = Q_a \left(A_0^a(x) - \omega \frac{p \cdot A^a(x)}{p \cdot k} \right) \frac{d}{dp_0} f^{(0)}(p_0) \ . \tag{4.52}$$

Hence, the color current is given by

$$j_a^\mu(x) = g^2 \int dP\, dQ\, p^\mu Q_a Q_b \left(A_0^b(x) - \omega \frac{p \cdot A^b(x)}{p \cdot k} \right) \frac{d}{dp_0} f^{(0)}(p_0) \ . \tag{4.53}$$

The integration over color charges can be done by using $\int dQ\ Q^a Q_b = C_{B,F}\, \delta_b^a$ with $C_B = N$, $C_F = \frac{1}{2}$ for gluons, resp. fermions. The integration over p_0 and $|\mathbf{p}|$ is straightforward as well. Upon summation over all species and helicities (see Section III for notations and conventions), we get the following expression for the total color current:

$$J_a^\mu(x) = m_D^2 \int \frac{d\Omega}{4\pi} v^\mu \left(\omega \frac{v \cdot A_a(x)}{v \cdot k} - A_a^0(x) \right) \ . \tag{4.54}$$

The polarization tensor $\Pi_{ab}^{\mu\nu}$ can be computed from (4.54) by using the relation

$$J_a^\mu(x) = \int d^4y\, \Pi_{ab}^{\mu\nu}(x-y)\, A_\nu^b(y) \ . \tag{4.55}$$

It reads:

$$\Pi_{ab}^{\mu\nu}(k) = m_D^2 \left(-g^{\mu 0} g^{\nu 0} + \omega\, I^{\mu\nu}(\omega, \mathbf{k}) \right) \ , \tag{4.56}$$

where $I^{\mu\nu}$ is defined as

$$I^{\mu\nu}(\omega, \mathbf{k}) = \int \frac{d\Omega}{4\pi} \frac{v^\mu v^\nu}{\omega - \mathbf{k} \cdot \mathbf{v}} \ . \tag{4.57}$$

To avoid the poles in the above integrand, we impose retarded boundary conditions, *i.e.*, we replace ω by $\omega + i\epsilon$. Using the identity

$$\frac{1}{z + i\epsilon} = \mathcal{P} \frac{1}{z} - i\pi\, \delta(z) \ , \tag{4.58}$$

where $\mathcal{P}$ stands for the principal value, the real and imaginary parts of the polarization tensor are

$$\begin{aligned}
\mathrm{Re}\,\Pi^{\mu\nu}_{ab}(\omega,\mathbf{k}) &= -\delta_{ab}\, m_D^2 \left(-g^{\mu 0}g^{\nu 0} + \omega\,\mathcal{P}\int \frac{d\Omega}{4\pi}\,\frac{v^\mu\, v^\nu}{\omega - \mathbf{k}\cdot\mathbf{v}}\right) , \\
\mathrm{Im}\,\Pi^{\mu\nu}_{ab}(\omega,\mathbf{k}) &= -\delta_{ab}\, m_D^2\,\pi\,\omega \int \frac{d\Omega}{4\pi}\, v^\mu\, v^\nu\, \delta(\omega - \mathbf{k}\cdot\mathbf{v}) . \qquad (4.59)
\end{aligned}$$

The imaginary part of the polarization tensor (4.59) describes Landau damping in the quark–gluon plasma. Explicitly:

$$\begin{aligned}
\mathrm{Im}\,\Pi^{00}_{ab}(\omega,\mathbf{k}) &= -\delta_{ab}\, m_D^2\,\pi\, \frac{\omega}{2|\mathbf{k}|}\,\theta(|\mathbf{k}|^2-\omega^2) , \\
\mathrm{Im}\,\Pi^{0i}_{ab}(\omega,\mathbf{k}) &= -\delta_{ab}\, m_D^2\,\pi\, \frac{\omega^2}{2|\mathbf{k}|^2}\frac{k^i}{|\mathbf{k}|}\,\theta(|\mathbf{k}|^2-\omega^2) , \\
\mathrm{Im}\,\Pi^{ij}_{ab}(\omega,\mathbf{k}) &= -\delta_{ab}\, m_D^2\,\pi \left[\frac{\omega^2}{4|\mathbf{k}|^2}\left(\frac{|\mathbf{k}|}{\omega}-\frac{\omega}{|\mathbf{k}|}\right)\left(\delta^{ij}-\frac{k^i k^j}{|\mathbf{k}|^2}\right)+\frac{\omega^3}{2|\mathbf{k}|^3}\frac{k^i k^j}{|\mathbf{k}|^2}\right]\theta(|\mathbf{k}|^2-\omega^2) .
\end{aligned} \qquad (4.60)$$

From the above θ-functions it is apparent that Landau damping only occurs for color fields with space-like wave vectors. This is also true for an Abelian plasma [35].

Evaluation of the real part of the polarization tensor (4.59) yields

$$\begin{aligned}
\mathrm{Re}\,\Pi^{00}_{ab}(\omega,\mathbf{k}) &= \delta_{ab}\,\Pi_l(\omega,\mathbf{k}) , \\
\mathrm{Re}\,\Pi^{0i}_{ab}(\omega,\mathbf{k}) &= \delta_{ab}\,\omega\,\frac{k^i}{|\mathbf{k}|^2}\,\Pi_l(\omega,\mathbf{k}) , \\
\mathrm{Re}\,\Pi^{ij}_{ab}(\omega,\mathbf{k}) &= \delta_{ab}\left[\left(\delta^{ij}-\frac{k^i k^j}{|\mathbf{k}|^2}\right)\Pi_t(\omega,\mathbf{k}) + \frac{k^i k^j}{|\mathbf{k}|^2}\,\frac{\omega^2}{|\mathbf{k}|^2}\,\Pi_l(\omega,\mathbf{k})\right] , \qquad (4.61)
\end{aligned}$$

where

$$\begin{aligned}
\Pi_l(\omega,\mathbf{k}) &= m_D^2\left(\frac{\omega}{2|\mathbf{k}|}\ln\left|\frac{\omega+|\mathbf{k}|}{\omega-|\mathbf{k}|}\right| - 1\right) , \\
\Pi_t(\omega,\mathbf{k}) &= -m_D^2\,\frac{\omega^2}{2|\mathbf{k}|^2}\left[1+\frac{1}{2}\left(\frac{|\mathbf{k}|}{\omega}-\frac{\omega}{|\mathbf{k}|}\right)\ln\left|\frac{\omega+|\mathbf{k}|}{\omega-|\mathbf{k}|}\right|\right] . \qquad (4.62)
\end{aligned}$$

Equations (4.61)–(4.62) characterize Debye screening, as well as longitudinal and transverse plasma waves.

Our results for the HTLs of the polarization tensor agree with those obtained in the high temperature limit using quantum field theoretic techniques[3] [44, 18, 19, 20, 21, 41]. We emphasize that the above results are gauge-independent, and obey the Ward identity

$$k_\mu \Pi^{\mu\nu}_{ab} = 0 \ , \tag{4.63}$$

as should be expected from the gauge invariance of our formalism.

Previous applications of classical transport theory to QCD have utilized an Abelian-dominance approximation to compute the polarization tensor [40]. It is noteworthy that there one recovers the same values of the polarization tensor that we found here. The reason for this agreement is that the leading-order contribution to the color current is made linear in the gauge field by the plane-wave *Ansatz* [41], exactly as happens in the Abelian-dominance approximation. However, the Abelian-dominance approximation cannot give a proper account of the *whole* set of HTLs, such as thermal corrections to n-point functions, $n \geq 3$.

4.4 Conclusions

In this paper, we have shown how classical transport theory can be used to derive the hard thermal loops of QCD. This formalism, we believe, is more direct and transparent than previous approaches based on perturbative quantum field theory. Indeed, hard thermal loops represent UV-finite thermal corrections to propagators and vertices. They arise from thermal scattering within a hot assembly of particles, and one may reasonably expect them to be describable in terms of classical physics.

The fact that we are modeling the high-temperature, deconfined, phase of QCD allows us to treat color classically, and enables the colored constituents of the plasma to be identified as quarks and gluons. Employing classical transport theory to study colored particles requires incorporating the color degrees of freedom into phase-space. A consistent measure must be defined over the new color coordinates. Furthermore,

[3]The connection between the retarded polarization tensor computed here and the time-ordered polarization tensor that is commonly used in quantum field theory has been studied in [20].

conservation of the group Casimirs under the dynamical evolution must be ensured. One means to accomplish these goals is to include delta-function constraints into the phase-space volume element. We have formally justified this *ad hoc* procedure by relating the dependent color degrees of freedom to a set of independent Darboux canonical variables, and by proving that the corresponding volume elements are equivalent.

A system of non-Abelian Vlasov equations describes transport phenomena in the QCD plasma. This system governs the evolution of both the single-particle phase-space distribution functions and the mean color fields. It would be a formidable task to solve the transport equation in the most general case, hence suitable approximations must be made.

First, we specialize to a collisionless plasma, in which there is no direct scattering between particles. This situation is not devoid of interest since collective mean-field effects can, and indeed do, arise.

Second, we employ a perturbative approximation scheme. We assume that the plasma is near equilibrium and expand the phase-space distribution function in powers of g, the gauge coupling constant. At the high temperature which must prevail for the formation of a quark–gluon plasma, the thermal energies of the particles are sufficiently large that the effects of the interactions with the gauge fields are comparatively small, and we expect perturbation theory to be valid.

Taking the high temperature limit constitutes our third approximation. In this limit, the masses of the particles can be neglected. Furthermore, the plasma is in a highly degenerate state, so that the equilibrium distribution functions are determined by the spin–statistics theorem.

We demand that gauge invariance be preserved by our perturbative expansion. In its lowest, non-trivial, order this expansion leads to the generating functional of HTLs. That gauge invariance is the appropriate guiding principle in uncovering hard thermal loops is not surprising. To apply this principle to the quantum field theoretic calculation of HTLs involves gauge fixing, ghosts, and resummation of classes of Feynman diagrams. In contrast, the route that one has to follow in order to adhere to the gauge principle is straightforward within classical physics. Therefore, the gauge invariance property of hard thermal loops is self-evident in our formalism.

Chapter 5

Conclusions

In this last chapter, we would like to summarize the results we obtained in the previous chapters:

1) In the first part of the thesis, we applied Manton's procedure to study the dynamics of non-relativistic Chern-Simons solitons. We found that by using the unmodified soliton solution as the trial function, this procedure only yields information linear in v, where v is the velocity of the solitons. In this order, the dynamics of solitons depends crucially on the existence of a background density: when there is a background density present, the solitons can interact with the background density and feel the Magnus force; while in absence of such a density, the solitons only exhibit statistical interaction between themselves.

2) We showed in the first part how to improve Manton's procedure to obtain information about the soliton dynamics at higher orders of v. The basic idea is to modify the soliton solution by using equations of motion and use the modified solution as the trial function. We successfully applied this modified Manton's procedure to the Chern-Simons solitons in absence of background densities, *i.e.* the Jackiw-Pi solitons. We found that at order v^2, the soliton solution is modified by a phase, which is related to the 1-cocyle of the Galileo group. This modification gives the correct dynamical behavior of these solitons, *e.g.* the mass of the soliton.

3) In the second part of the thesis, we studied Hard Thermal Loops, a phenomena closely related to the Chern-Simons eikonal. We first studied the static response of

the gauge fields in presence of HTLs and proved that HTLs do not support solitons or instantons. Then, we investigated the physical origin of HTLs. By using classical transport theory, we successfully rederived HTLs. This shows that HTLs are classical effects. Furthermore, we clarified several problems related to classical transport theory, such as phase space and gauge invariance. In particular, we showed that gauge invariance is the guiding principle in uncovering the HTLs.

Appendix A

Numerical Solutions of (3.10) and (3.11)

In this Appendix we analyze in greater detail and integrate numerically the radially symmetric version of the static response equations (3.10), (3.11), in the $SU(2)$ case. Radially symmetric $SU(2)$ gauge potentials take the forms:

$$
\begin{aligned}
A_i^a &= (\delta^{ai} - \hat{r}^a \hat{r}^i)\,\frac{\phi_2(r)}{r} + \varepsilon^{aij}\,\hat{r}^j\,\frac{1-\phi_1(r)}{r}\,, \\
A_0^a &= \hat{r}^a\,\frac{g(r)}{r}\,,
\end{aligned}
\qquad \text{(A.1)}
$$

where a residual gauge freedom has been used to eliminate a term proportional to $\hat{r}^a \hat{r}^i$.

We substitute the *Ansatz* (A.1) into (3.10), (3.11). The resulting equations give us the freedom to set one of the two ϕ_i's to zero; we obtain,

$$
\begin{aligned}
x^2 \frac{d^2}{dx^2} J &= (x^2 + 2K^2)\,J\,, \\
x^2 \frac{d^2}{dx^2} K &= (K^2 - J^2 - 1)\,K\,,
\end{aligned}
\qquad \text{(A.2)}
$$

where we have set ϕ_2 to zero, rescaled $x = mr$ and defined $J(x) = g(r)$, $K(x) = \phi_1(r)$.

We now investigate this system of coupled second-order differential equations.

First, we see that they possess the following two exact solutions:

$$J = 0,\ K = \pm 1\ , \tag{A.3}$$

$$J = J_0\, e^{-x},\ K = 0\ . \tag{A.4}$$

Eq. (A.3) corresponds to the Yang-Mills vacuum, while (A.4) is the celebrated Wu-Yang monopole plus a screened electric field.

In the asymptotic region $x \to \infty$, the regular solution of the system (A.2) tends to (A.3), with J approaching its asymptote exponentially. (Of course there is also the solution with J growing exponentially, which we do not consider.)

Near the origin, J and K behave either like the vacuum (A.3) or approach the monopole solution (A.4) as follows,

$$\begin{aligned} J(x) &\to J_0 + \dots\ , \\ K(x) &\to K_0\sqrt{x}\, cos\Big(\frac{2\pi}{\tau} ln\frac{x}{x_0}\Big) + \dots\ , \end{aligned} \tag{A.5}$$

where τ is correlated with J_0 as

$$\tau = \frac{4\pi}{\sqrt{4J_0^2 + 3}}\ . \tag{A.6}$$

Only the vacuum alternative at the origin leads to finite energy. However, since we must choose one of two possible solutions at infinity (obviously we pick the regular one), the behavior at the origin is determined and can be exhibited explicitly by integrating the equations (A.2) numerically. Starting with regular boundary conditions at infinity, we find the profiles presented in Figure A-1.

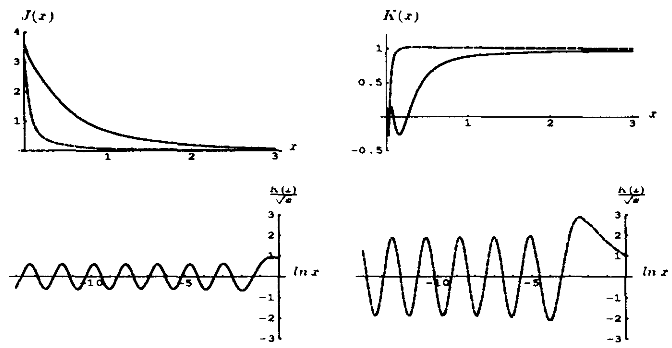

Fig. A-1. Profiles for eqs. (A.2). The plain and dashed lines represent two different rates of approach to the asymptotes (A.3) at $x = \infty$ (with $K = 1$).

They show that the monopole solution (A.4) is reached at the origin, with K vanishing as in (A.5) - (A.6), a result consistent with our analytic proof that there are no finite energy static solutions in hard thermal gauge theories.

Appendix B

Microscopic description

In a microscopic description, the particle's trajectory in phase-space is known exactly. With this knowledge, we can construct a distribution function $f(x,p,Q)$ (without loss of generality, we shall consider only one particle):

$$f(x,p,Q) = \int \frac{d\tau}{m}\, \delta^{(4)}\big(x - x(\tau)\big)\, \delta^{(4)}\big(p - p(\tau)\big)\, \delta^{(N^2-1)}\big(Q - Q(\tau)\big) \quad , \tag{B.1}$$

where $x(\tau)$, $p(\tau)$ and $Q(\tau)$ obey the Wong equations (4.1),(4.2),(4.3), *i.e.* they naturally fulfill the mass-shell and Casimir constraints. For convenience, those constraints are here subsumed into the distribution function instead of being contained in the phase-space volume element. Had we not done this, (B.1) would have to be written in terms of both a 3-dimensional δ-function in momentum space, and an $N(N-1)$-dimensional δ-function in color space.

We now prove that the expression (B.1) for f satisfies the collisionless Boltzmann equation (4.7). The first term in (4.7) can be rewritten, by using the properties of the δ-function, as:

$$p^\mu \frac{\partial}{\partial x^\mu} f = -\int \frac{d\tau}{m}\, p^\mu(\tau) \left[\frac{\partial}{\partial x^\mu(\tau)}\, \delta^{(4)}\big(x - x(\tau)\big)\right] \delta^{(4)}\big(p - p(\tau)\big)\, \delta^{(N^2-1)}\big(Q - Q(\tau)\big)\ . \tag{B.2}$$

From this, after using the Wong equation for the variable $x^\mu(\tau)$ and applying the chain rule, we get:

$$p^\mu \frac{\partial}{\partial x^\mu} f = - \int d\tau \left[\frac{d}{d\tau} \, \delta^{(4)}\Big(x - x(\tau)\Big) \right] \, \delta^{(4)}\Big(p - p(\tau)\Big) \, \delta^{(N^2-1)}\Big(Q - Q(\tau)\Big) \, . \qquad \text{(B.3)}$$

Similar arguments yield, for the second term in the Boltzmann equation:

$$- \, g \, p^\mu Q_a F^a_{\mu\nu} \frac{\partial}{\partial p_\nu} f = - \int d\tau \, \delta^{(4)}\Big(x - x(\tau)\Big) \left[\frac{d}{d\tau} \delta^{(4)}\Big(p - p(\tau)\Big) \right] \delta^{(N^2-1)}\Big(Q - Q(\tau)\Big) \, , \qquad \text{(B.4)}$$

and for the third term:

$$-g \, p^\mu f_{abc} A^b_\mu Q^c \frac{\partial}{\partial Q_a} f = - \int d\tau \, \delta^{(4)}\Big(x - x(\tau)\Big) \, \delta^{(4)}\Big(p - p(\tau)\Big) \, \left[\frac{d}{d\tau} \, \delta^{(N^2-1)}\Big(Q - Q(\tau)\Big) \right] \, . \qquad \text{(B.5)}$$

Adding together the equations (B.3), (B.4) and (B.5), we observe that the left hand side – the τ-integral of a total τ-derivative – vanishes, thereby yielding the collisionless Boltzmann equation (4.7).

Appendix C

Conservation of the Color Current

Let us verify that the color current (4.10) is covariantly conserved. Using the collisionless transport equation, one can compute

$$\begin{aligned}
\partial_\mu j^{\mu\,a}(x) &= g\int dP\,dQ\,p^\mu Q^a \partial_\mu f(x,p,Q) \\
&= g^2\int dP\,dQ\,p^\mu Q^a\left(Q_b F^b_{\mu\nu}(x)\frac{\partial}{\partial p_\nu} + f^{dbc}A_{\mu\,b}(x)Q_c\frac{\partial}{\partial Q^d}\right)f(x,p,Q)\ ,
\end{aligned} \tag{C.1}$$

where the color measure is $dQ = d^{(N^2-1)}Q\ C(Q)$, and $C(Q)$ specifies the color constraints in phase-space. Integrating by parts and discarding surface terms, one gets

$$\begin{aligned}
\partial_\mu j^{\mu\,a}(x) &= -g^2\int dP\Big[\int dQ\big(Q^a Q_b g^\mu_\nu F^b_{\mu\nu}(x) + p^\mu \delta_{cd} f^{dbc}A_{\mu\,b}(x)Q^a + p^\mu f^{dbc}A_{\mu\,b}(x)\delta^a_d Q_c\big) \\
&\quad + \int d^{(N^2-1)}Q\ p^\mu A_{\mu\,b}(x) f^{dbc} Q_c \frac{\partial}{\partial Q^d}C(Q)\Big] f(x,p,Q)\ .
\end{aligned} \tag{C.2}$$

Among the four terms in the right side, only the third one survives. The first two terms cancel due to antisymmetry of $F^a_{\mu\nu}$ and f^{dbc}, respectively. The last term also cancels, since the constraints $C(Q)$ are gauge-invariant, *i.e.*,

$$0 = \delta Q^a\,\frac{\delta C(Q)}{\delta Q^a} = -g f^{abc}\epsilon_b(x)Q_c\,\frac{\partial C(Q)}{\partial Q^a}\ , \tag{C.3}$$

where δQ^a denotes an infinitesimal gauge transformation with arbitrary parameter $\epsilon_b(x)$. (For $SU(3)$ this last property can be explicitly checked by using the Jacobi-like identity $f_{abc}d_{dec} + f_{adc}d_{ebc} + f_{aec}d_{bdc} = 0$.) Finally, one obtains the expression for the covariant conservation of the color current: $\partial_\mu j^{\mu\,a}(x) + g f^{abc} A_{\mu\,b}(x) j^\mu_c(x) = 0$.

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