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BIBO Stability Robustness in the Presence of Coprime Factor Perturbations*

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Abstract

In this paper, a necessary and sufficient condition for robustly stabilizing a family of plants described by perturbations of fixed coprime factors of a plant is given. The computation of the largest stability margin is discussed via solving a nonsquare ℓ^1 optimal control problem. A new algorithm for obtaining lower approximaions of the minimum value of the optimaization problem, μ_0, is proposed. This, together with the standard algorithm which provides upper approximations, μ_0 can be computed within any degree of accuracy.

1 Introduction

For unstable plants, the natural way for representing plant uncertainty is by perturbing the graph of the plant as it operates on a specific space. The choice of the space decides the admissible class of plant perturbations [10]. In [11], the problem of robustly stabilizing a family of plants generated by perturbing the graph of a nominal plant over ℓ^2 was analyzed and a Necessary and Sufficient condition was derived. Exact computation of this condition was analyzed in [6] in which the structure of the problem with normalized coprime factors was exploited. In [5], it was shown that this class of plants is equivalent to a class of plants perturbed through the gap metric. In this paper, the problem of stabilizing a family of plants characterized by perturbations of the graph of a LTI plant (for some fixed coprime factors) over ℓ^∞ is considered. This is equivalent to perturbing the coprime factors of the plant by bounded but arbitrary operators on ℓ^∞. It is shown that a similar Necessary and Sufficient condition can be derived in terms of the ℓ^1 norm. The computation of the smallest possible value of this condition is discussed, and a new iterative scheme is proposed. Although this scheme is explained for this particular ℓ^1 problem, it generalizes in a straightforward way to arbitrary nonsquare ℓ^1 problems.

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2 Mathematical Preliminaries

First, some notation regarding standard concepts for input/output systems. For more details, consult [4,8] and references therein.

ℓ_e^∞ denotes the extended space of sequences in R^N, $f = \{f_0, f_1, f_2, \ldots\}$. ℓ^∞ denotes the set of all $f \in \ell_e^\infty$ such that

$$\|f\|_{\ell^\infty} \stackrel{\text{def}}{=} \sup_i |f_i|_\infty < \infty$$

where $|f_i|_\infty$ is the standard ℓ^∞ norm on vectors. $\ell_e^\infty \backslash \ell^\infty$ denotes the set $\{f : f \in \ell_e^\infty \text{ and } f \notin \ell^\infty\}$. $\ell^p, p \in [1, \infty)$, denotes the set of all sequences, $f = \{f_0, f_1, f_2, \ldots\}$ in R^N such that

$$\|f\|_{\ell^p} \stackrel{\text{def}}{=} \left(\sum_i |f_i|_p^p \right)^{1/p} < \infty.$$

c_0 denotes the subspace of ℓ^∞ in which every function x satisfies

$$\lim_{k \to \infty} x(k) = 0.$$

S denotes the standard shift operator.

P_k denotes the k^{th}-truncation operator on ℓ_e^∞:

$$P_k : \{f_0, f_1, f_2, \ldots\} \longrightarrow \{f_0, \ldots, f_k, 0, \ldots\}$$

Let $H : \ell_e^\infty \longrightarrow \ell_e^\infty$ be a nonlinear operator. H is called *causal* if

$$P_k H f = P_k H P_k f, \quad \forall k = 0, 1, 2, \ldots,$$

H is called *strictly causal* if

$$P_k H f = P_k H P_{k-1} f, \quad \forall k = 0, 1, 2, \ldots$$

H is called *time-invariant* if it commutes with the shift operator:

$$HS = SH.$$

Finally, H is called *ℓ^p stable* if

$$\|H\| \stackrel{\text{def}}{=} \sup_k \sup_{\substack{f \in \ell_e^p \\ P_k f \neq 0}} \frac{\|P_k H f\|_{\ell^p}}{\|P_k f\|_{\ell^p}} < \infty.$$

The quantity $\|H\|$ is called the *induced operator norm over ℓ^p*.

$\mathcal{L}_{TV}$ denotes the set of all *linear* causal ℓ^∞ stable operators, $T : \ell_e^\infty \longrightarrow \ell_e^\infty$. $\mathcal{L}_{TI}$ denotes the set of all $T \in \mathcal{L}_{TV}$ which are *time-invariant*. It is well known that $\mathcal{L}_{TI}$ is isomorphic to ℓ^1.

Finally, it is well known that $c_0^* = \ell^1$. Given $X \in c_0$ and $Y \in \ell^1$, then

$$< X, Y >= \sum_{i,j} < x_{ij}, y_{ij} >$$

and the induced norm on x is given by:

$$\|x\|_{c_0} = \sum_i \max_j \|x_{ij}\|_\infty .$$

3 Robustness in the Presence of Stable Coprime Factor Perturbations

Let that P_0 be a linear time invariant, finite dimensional plant, with a doubly coprime factorization given by:

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

with $P_0 = NM^{-1}$. The graph of P_0 over the space ℓ^q is given by [10]:

$$G^q(P_0) = G_{P_0}\ell^q \;\; where \;\; G_{P_0} = \begin{bmatrix} M \\ N \end{bmatrix}$$

Define the following class of plants:

$$\Omega_q = \{P | G_P = \begin{bmatrix} M + \Delta_1 \\ N + \Delta_2 \end{bmatrix} \;\; and \;\; \left\| \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \right\| \leq 1\}$$

where Δ is an ℓ^q bounded linear operator. It is well known that all controllers stabilizing P_0 are parametrized in the form [10]

$$C = (U - MQ)(V - NQ)^{-1} = (\tilde{V} - Q\tilde{N})^{-1}(-\tilde{U} + Q\tilde{M})$$

The next theorem gives a necessary and sufficient condition for a controller that stabilizes P_0 to stabilize all $P \in \Omega_\infty$. A similar result in the case of $P \in \Omega_2$ was proved in [11]. It is evident that Ω_∞ contains time varying plants which will be essential for the proof of the next theorem. For this, the following definition is introduced [9].

Definition 3.1 *Two time varying operators G_1, G_2 posses unstable cancellation if there exists an $e \in \ell_e^\infty \backslash \ell^\infty$ such that both $G_i e \in \ell^\infty$ for $i = 1, 2$.*

Theorem 3.1 *If C stabilizes P_0, then C stabilizes all $P \in \Omega_\infty$ if and only if*

$$\left\| \begin{bmatrix} \tilde{V} - Q\tilde{N} & -\tilde{U} + Q\tilde{M} \end{bmatrix} \right\|_{\mathcal{L}_{TI}} \leq 1 \qquad (1).$$

Proof To prove Sufficiency, assume (1) is satisfied. let $P = (N+\Delta_2)(M+\Delta_1)^{-1}$ for some fixed Δ_1, Δ_2. The closed loop system is stable if the operator $G_C^{\top} G_P$ is invertible inside $\mathcal{L}_{TV}$, where

$$G_C^{\top} = \begin{bmatrix} \tilde{V} - Q\tilde{N} & -\tilde{U} + Q\tilde{M} \end{bmatrix}$$

However, $G_C^{\top} G_P = I + G_C^{\top}\Delta$ and hence sufficiency follows immediately by the small gain theorem. Notice that in this case the plant cannot have any unstable cancellations since its factors satisfy

$$G_C^{\top} \begin{bmatrix} M+\Delta_1 \\ N+\Delta_2 \end{bmatrix} = unit$$

which clearly prevents any unstable cancellations.

To prove Necessity, assume that $\left\|G_C^{\top}\right\| > 1$. Then from [1,8], there exists a time varying Δ, strictly proper, with $||\Delta|| < 1$ such that $G_C^{\top} G_P$ is unstable. This will certainly imply that the closed loop system is unstable, however the instability may be a consequence of unstable cancellation in the factors of the constructed plant. This situation is undesirable since no controller can stabilize a class of plants that has hidden unstable modes. The proof will be completed if it is shown that the lack of invertibility of $G_C^{\top} G_P$ is not a consequence of cancellations in the factors of P.

If $\left\|G_C^{\top}\right\| > 1$, then there exists an $e \in \ell_e^{\infty} \backslash \ell^{\infty}$ such that

$$\frac{||P_{n-1}u||}{||P_n e||} \geq m > 1, \quad \forall n > n^*$$

and $u = G_C^{\top} e$. In [1], it was shown that there exists a strictly proper Δ, time-varying, stable with $||\Delta|| < 1$ such that

$$\Delta u = -e + P_{n^*} e$$

and

$$G_C^{\top} G_P u = (I + G_C^{\top}\Delta) u \in \ell^{\infty}$$

which implies that $G_C^{\top} G_P$ is not invertible. However, since $G_C^{\top} G_{P_0} = I$, then $G_{P_0} u = e + g_1$ where $G_C^{\top} g_1 = 0$ and $g_1 \in \ell_e^{\infty}$. Also, $G_P u = g_1 + P_{n^*} e$. If g_1 is unbounded, the proof is complete. If not, then the construction has to be adjusted as follows:
The Smith-McMillan form of $G_C^{\top}$ is given by

$$G_C^{\top} = S_1 [\Sigma \;\; 0] S_2$$

Where $S_1$and $S_2$are both invertible in $\mathcal{L}_{TI}$. Then any input of the form $g = S_2^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}$ satisfies $G_C^{\top} g = 0$. Pick g in such a way that $\tilde{e} = e + g$ satisfies

$$\frac{||P_{n-1}u||}{||P_n \tilde{e}||} \geq m_1 > 1, \quad \forall n > n^*$$

and $g \in \ell_e^\infty \backslash \ell^\infty$. It is easy to show that such a choice is possible following the construction procedure in [1,8]. The basic idea of the construction is as follows: find an input e_0 that is supported on a finite interval and captures the norm of G_C^T. The input e is constructed (roughly) by adding up translates of e_0 amplified by an increasing factor. This construction is now adjusted by adding $g = S_2^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}$ to e_0 with g having the same support as e_0 and very small norm. The details of the construction are omitted since it follows the same exact procudure as [1,8].

From here, the proof proceeds exactly as in [1], i.e a strictly proper time-varying Δ, with $||\Delta|| < 1$ is constructed such that

$$\Delta u = -\tilde{e} + P_{n^*} \tilde{e}.$$

Notice that

$$G_P u = e + g_1 - \tilde{e} + P_{n^*}\tilde{e} = g_1 + g + P_{n^*}\tilde{e} \quad \in \ell_e^\infty \backslash \ell^\infty.$$

Hence $G_C^T G_P$ is not invertible, and the lack of invertibility is not due to cancellations in the factors of G_P. This completes the proof.

Comment: Theorem 3.1 is readily generalized to the case of weighted plant perturbations. The weights will appear in the optimization problem in the obvious way. The details will be omitted.

4 Stability Margin

Let μ be defined as

$$\mu = \inf_{Q \in \mathcal{L}_{TI}} \left\| \begin{bmatrix} \tilde{V} - Q\tilde{N} & -\tilde{U} + Q\tilde{M} \end{bmatrix} \right\|$$

Then the largest stability margin defined as the maximum $||\Delta||$ such that the closed loop system remains stable is equal to $\frac{1}{\mu}$. The computation of μ was analyzed in [3] and more recently in [7]. The next theorem specializes the results in the above mentioned references. It will be assumed that the Bezout equations are all polynomials in the shift operator.

Theorem 4.1

$$\mu = \sup_{x \in c_0} \sum_i x_{ii}(0)$$

Subject to

$$||x(N^* \quad M^*)||_{c_0} \leq 1$$

with $x \in c_0$ is a square matrix, and M^* and N^* are the weak* adjoint operators associated with M, N respectively, i.e for any $x \in c_0$,

$$xN^*(t) = \sum_{k=0}^{\infty} x(t+k)N(k).$$

Before the proof is given, the following lemmas are presented.

Lemma 4.1 *Let*

$$S = \{K \in \ell^1 | (K_1 \quad K_2) = Q(\tilde{N} \quad -\tilde{M}), Q \in \ell^1\}$$

Then $K \in S$ if and only if $K \in \ell^1$ and

$$(K_1 \quad K_2)\begin{pmatrix} M \\ N \end{pmatrix} = 0 \qquad (2).$$

Proof Necessity is straightforward. For Sufficiency, assume $K \in \ell^1$ satisfies (2). Then Q is uniquely defined as $Q = -K_2\tilde{M}^{-1}$. The proof is completed if it is shown that Q lies in ℓ^1. From the Bezout identity, $\tilde{N}, \tilde{M}$ satisfy

$$-\tilde{N}U + \tilde{M}V = I$$

or equivalently

$$-\tilde{M}^{-1}\tilde{N}U + V = \tilde{M}^{-1}.$$

Multiplying both sides by K_2:

$$-K_2\tilde{M}^{-1}\tilde{N}U + K_2V = K_2\tilde{M}^{-1}.$$

From (2),

$$K_1 = -K_2NM^{-1} = -K_2\tilde{M}^{-1}\tilde{N}$$

The above equations then imply that

$$Q = -K_1U - K_2V$$

which is clearly stable. This completes the proof.

Lemma 4.2

$$^{\perp}S = \{y \in c_0 | y = x(N^* \quad M^*), x \in c_0\}$$

where $^{\perp}S$ denotes the left annihilator subspace of S.

Proof Follows immediately from [3,7].

Comment: The first lemma shows that no extra interpolation conditions are needed to guarantee the stability of Q. The combination of both lemmas above make the final solution quite elegant.

Proof of Theorem 4.1 From the above lemmas, the problem is given by

$$\mu = \inf_{K \in S} \left\| \left[\tilde{V} \quad -\tilde{U}\right] - [K_1 \quad K_2] \right\|$$

The dual problem is given by

$$\mu = \sup_{y \in ^{\perp}S, \|y\|_{c_0} \leq 1} < y, (\tilde{V} \quad -\tilde{U}) >$$

Finally by noting that

$$(\tilde{V} \quad -\tilde{U}) \begin{pmatrix} M \\ N \end{pmatrix} = I.$$

the result follows.

In [3], it was shown that if the factoriztion is obtained over the space of polynomial matrices, then the above problem is readily a semi-infinite linear programming problem. An iterative procedure was proposed and convergence was proved. This procedure corresponds to the following problem:

$$\overline{\mu}_n = \sup_{x \in c_0} \sum_i x_{ii}(0)$$

Subject to

$$\|P_n(x(N^* \quad M^*))\|_{c_0} \leq 1$$

This problem is a finite linear program, and $\overline{\mu}_n \geq \mu$. In the limit, $\overline{\mu}_n$ converges to μ.

On the other hand, it is desirable to know at each iteration how far $\overline{\mu}_n$ is from μ. For this purpose, we propose another method that approximates μ from below, and has guaranteed convergence in the limit. Consider the problem:

$$\underline{\mu}_n = \sup_{x \in c_0} \sum_i x_{ii}(0)$$

Subject to

$$\|(P_n x)(N^* \quad M^*)\|_{c_0} \leq 1$$

This problem is also a finite linear program, and $\underline{\mu}_n \leq \mu$. Also, $\underline{\mu}_n$ will converge to μ in the limit since the space c_0 can be approximated arbitrarily closely by $P_n c_0$ for n large enough. Notice that the above two problems are different. Basically, the first corresponds to truncating the constraints (the output) after n steps in the semi-infinite problem, and the second corresponds to truncating the input x. It is interesting to note that this procedure is valid for approximating μ from below for the general l^1 problem, and hence it provides a consistent way of computing the minimum performance to any desired accuracy.

5 Conclusions

In this paper, the problem of robustly stabilizing a class of plants characterized by coprime factors perturbations is analyzed and a necessary and sufficient condition is derived. The computation of the largest stability margin is discussed and an alternative algorithm that provides lower estimates is furnished. By having both upper and lower estimates, it is possible to get arbitrarily close to the minimum solution.

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