Frobenius Transfers and p-local Finite Groups

by

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Abstract

In this thesis we explore the possibility of defining the p-local finite groups of Broto, Levi and Oliver in terms of their classifying spaces. More precisely, we consider the question posed by Haynes Miller, whether an equivalent theory can be recovered by studying maps $f\colon BS \to X$ from the classifying space of a finite p-group S to a p-complete space X equipped with a stable retract t satisfying a form of Frobenius reciprocity. In the case where S is elementary abelian, we answer this question in the affirmative, by showing that under some finiteness conditions such a triple (f, t, X) does indeed induce a p-local finite group over S. We also discuss the converse in some detail for general S.

Thesis Supervisor: Haynes R. Miller
Title: Professor of Mathematics

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Chapter 1

Introduction

Defined by Broto, Levi and Oliver [7], p-local finite groups are the culmination of a program initiated by Puig [24, 25] to find a formal framework for the p-local structure of a finite group. To a finite group G, one associates a *fusion system* (at a prime p) consisting of all p-subgroups of G and the homomorphims between them induced by conjugation in G. Puig formalised fusion systems and identified an important subclass of fusion systems, which we now call *saturated fusion systems.* Fusion systems of finite groups are contained in this class, but saturated fusion systems also arise in other important contexts, most notably in modular representation theory through Brauer subpairs of blocks of group algebras and more recently as Chevalley groups of p-compact groups [8].

The fusion system of a group G can be considered as an algebraic interpretation of the p-local structure of the group. One can also take a topological approach, and think of the p-local structure of G as being the p-completed classifying space $BG^{\wedge}_p$. By Bob Oliver's solution of the Martino-Priddy conjecture, these approaches are the same. That is, two groups induce the same fusion system if and only if their p-completed classifying spaces are homotopy equivalent. This suggests that more generally each saturated fusion system may have a unique classifying space. A *p-local finite group* consists of a saturated fusion system and an associated centric linking system, a category which offers just enough information to construct a classifying space associated to the fusion system. Thus one can think of a p-local finite group as a saturated fusion system with a chosen classifying space.

The definition of p-local finite groups is rather complicated and has the drawback that there is no straightforward concept of morphisms between p-local finite groups, so they have not yet been made to form a category in any sensible way. In this thesis, we adopt the approach used by Dwyer and Wilkerson for p-compact groups [13] and try to develop the theory of p-local finite groups in terms of classifying spaces. Specifically, we consider maps $f\colon BS \to X$ from the classifying space of a finite p-group S to a p-complete space X, which satisfy some finiteness conditions and are endowed with a transfer satisfying a form of Frobenius reciprocity. We will refer to such a triple (f, t, X) as a *Frobenius transfer triple over S* (def. 3.1.4). For a Frobenius transfer triple (f, t, X), we ask whether X is the classifying space of a p-local finite group. This question will be addressed in chapter 3, where we answer the question in the affirmative in the case where S is elementary abelian. Conversely we ask whether a p-local finite group gives rise to a Frobenius transfer triple. This question will be addressed in chapter 4.

In this chapter we give a brief overview of the theory of p-local finite groups. Most of this material is found in [7]. In this chapter and throughout the thesis, p is a fixed prime.

1.1 Some definitions and terminology

We begin by recalling some terminology regarding p-local finite groups.

Definition 1.1.1. *A fusion system $\mathcal{F}$ over a finite p-group S is a category, whose objects are the subgroups of S, and whose morphism sets $Hom_{\mathcal{F}}(P,Q)$ satisfy the following conditions:*

(a) $Hom_S(P,Q) \subseteq Hom_{\mathcal{F}}(P,Q) \subseteq Inj(P,Q)$ *for all* $P, Q \leq S$.

(b) *Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.*

Here $Hom_S(P,Q)$ is the set of group homomorphims induced by conjugation by elements in S.

Before stating the next definition, we need to introduce some additional terminology and notation. We say that two subgroups $P, P' \leq S$ are *$\mathcal{F}$-conjugate* if they are isomorphic in $\mathcal{F}$. A subgroup $P \leq S$ is *fully centralised in $\mathcal{F}$* if $|C_S(P)| \geq |C_S(P')|$ for every $P' \leq S$ which is $\mathcal{F}$-conjugate to P. Similarly P is *fully normalised in $\mathcal{F}$* if $|N_S(P)| \geq |N_S(P')|$ for every $P' \leq S$ which is $\mathcal{F}$-conjugate to P. Finally, for any finite group G, we write $Syl_p(G)$ for the set of Sylow p-subgroups of G.

Definition 1.1.2. *A fusion system $\mathcal{F}$ over a p-group S is saturated if the following two conditions hold:*

(I) *If $P \leq S$ is fully normalised in $\mathcal{F}$, then P is also fully centralised and* $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$.

(II) *If $P \leq S$ and $\varphi \in Hom_{\mathcal{F}}(P,S)$ are such that φP is fully centralised, then φ extends to $\bar{\varphi} \in Hom_{\mathcal{F}}(N_\varphi, S)$, where*

$$N_\varphi = \{g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in Aut_S(\varphi P)\}.$$

There is a class of subgroups of S of special interest to us, defined as follows.

Definition 1.1.3. *Let $\mathcal{F}$ be a fusion system over a p-group S. A subgroup $P \leq S$ is $\mathcal{F}$-centric if $C_S(P') \leq P'$ for every P', that is $\mathcal{F}$-conjugate to P. Let $\mathcal{F}^c$ denote the full subcategory of $\mathcal{F}$, whose objects are the $\mathcal{F}$-centric subgroups of S.*

Remark 1.1.4. The condition $C_S(P') \leq P'$ in the previous definition is equivalent to the condition $C_S(P') = Z(P')$.

Definition 1.1.5. *Let $\mathcal{F}$ be a fusion system over the p-group S. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$, whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor*

$$\pi : \mathcal{L} \to \mathcal{F}^c,$$

and distinguished monomorphisms $P \xrightarrow{\delta_P} Aut_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

(A) *The functor π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, the centre $Z(P)$ acts freely on $Mor_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq Aut_{\mathcal{L}}(P)$), and π induces a bijection*

$$Mor_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} Hom_{\mathcal{F}}(P,Q).$$

(B) *For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in Aut_{\mathcal{L}}(P)$ to $c_g \in Aut_{\mathcal{F}}(P)$.*

(C) *For each $f \in Mor_{\mathcal{L}}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:*

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle \delta_P(g)} & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q. \end{array}$$

We can now finally define our objects of study.

Definition 1.1.6. *A p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over a finite p-group S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$.*

The classifying space of the p-local finite group is the p-completed geometric realisation $|\mathcal{L}|^\wedge_p$.

A p-local finite group comes equipped with a natural inclusion

$$\theta\colon BS \longrightarrow |\mathcal{L}|^\wedge_p.$$

1.2 The fusion systems of groups

In this section we will discuss the fusion system arising from a Sylow subgroup inclusion $S \leq G$. This section serves as motivation for the discussion in the previous section as well as being of independent interest.

Definition 1.2.1. *Let G be a finite group. The fusion system of G is the category $\mathcal{F}(G)$, whose objects are the p-subgroups of G and whose morphism sets are given by*

$$Hom_{\mathcal{F}(G)}(P,Q) = Hom_G(P,Q)$$

for all p-subgroups $P, Q \leq G$.

For a p-subgroup $S \leq G$, the fusion system of G over S is the full subcategory $\mathcal{F}_S(G) \subseteq \mathcal{F}(G)$, whose objects are the subgroups of S.

If S is a Sylow subgroup of G, then the inclusion of $\mathcal{F}_S(G)$ in $\mathcal{F}(G)$ is an equivalence of categories, since every p-subgroup of G is conjugate to a subgroup of S.

Proposition 1.2.2. *[7, Prop. 1.3] Let G be a finite group and let S be a p-subgroup. Then the fusion system $\mathcal{F}_S(G)$ of G over S is saturated if and only if S is a Sylow subgroup.*

The centric linking system of a finite group was initially introduced in [6] as a powerful tool to study homotopy equivalences between p-completed classifying spaces of finite groups. The p-centric subgroups of a finite group G are the p-subgroups $P \leq G$ whose centre $Z(P)$ is a p-Sylow subgroup of the centraliser $C_G(P)$. This notion of centricity is equivalent to the one introduced in 1.1.3 in the sense that if $S \leq G$ is a Sylow subgroup, then a subgroup $P \leq S$ is p-centric if and only if it is $\mathcal{F}_S(G)$-centric.

For the following definition, we recall that if a group $P \leq G$ is p-centric, then one can write

$$C_G(P) = Z(P) \times C'_G(P),$$

where $C'_G(P) \leq G$ has order prime to p. The notation $C'_G(P)$ will be used in the definition. In addition, for subgroups $P, Q \leq G$, we will let $N_G(P, Q)$ denote the transporter

$$N_G(P, Q) = \{g \in G | gPg^{-1} \leq Q\}.$$

Definition 1.2.3. *Let G be a finite group. The centric linking system of G is the category $\mathcal{L}(G)$, whose objects are the p-centric subgroups of G and whose morphism sets are given by*

$$Mor_{\mathcal{L}(G)}(P, Q) = N_G(P, Q)/C'_G(P)$$

for all p-subgroups $P, Q \leq G$.

For a p-subgroup $S \leq G$, the centric linking system of G over S is the full subcategory $\mathcal{L}_S(G) \subseteq \mathcal{L}(G)$ whose objects are the subgroups of S, that are p-centric in G.

In the case of a Sylow inclusion $S \leq G$, the centric linking system $\mathcal{L}_S(G)$ is a centric linking system associated to the saturated fusion system $\mathcal{F}_S(G)$ and we have the following proposition, which serves as a motivating example for the definition of a p-local finite group.

Proposition 1.2.4. *Let S be a Sylow subgroup of a finite group G. Then the triple $(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$ is a p-local finite group over S. Furthermore, the natural map*

$$\theta \colon BS \to |\mathcal{L}_S(G)|^\wedge_p$$

is equivalent to the p-completed inclusion

$$BS \to BG^\wedge_p$$

as a space under BS.

1.3 Homotopy theoretic constructions of fusion systems

In this section we will recall how a map $f\colon BS \to X$, from the classifying space of a finite p-group S to a space X induces a fusion system $\mathcal{F}_{S,f}(X)$ over S. In general this fusion system will not be saturated.

The following definition is motivated by the fact that two group homomorphisms $\varphi, \psi\colon G \to H$ between finite groups are H-conjugate if and only if the induced maps of classifying spaces are freely homotopic.

Definition 1.3.1. *[7, Def. 7.1] For any space X, any p-group S, and any map $f\colon BS \to X$, define $\mathcal{F}_{S,f}(X)$ to be the category whose objects are the subgroups of S, and whose morphisms are given by*

$$Hom_{\mathcal{F}_{S,f}}(P,Q) = \{\varphi \in Inj(P,Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi\}$$

for each $P, Q \leq S$.

It is easy to see that $\mathcal{F}_{S,f}$ is indeed a fusion system, although it need not be saturated. In the case where $\mathcal{F}_{S,f}$ is saturated however, one can get a candidate $\mathcal{L}^c_{S,\theta}(|\mathcal{L}|^\wedge_p)$ for an associated centric linking system by retaining information about the homotopies giving the equivalence $f|_{BP} \simeq f|_{BQ} \circ B\varphi$ in the definition above.

Theorem 1.3.2. *[7, Thms. 7.4, 7.5] Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then $\mathcal{F}_{S,\theta}(|\mathcal{L}|^\wedge_p)$ is saturated and $\mathcal{L}_{S,\theta}(|\mathcal{L}|^\wedge_p)$ is a centric linking system associated to $\mathcal{F}_{S,\theta}(|\mathcal{L}|^\wedge_p)$. Furthermore, the p-local finite groups $(S, \mathcal{F}, \mathcal{L})$ and $(S, \mathcal{F}_{S,\theta}(|\mathcal{L}|^\wedge_p), \mathcal{L}^c_{S,\theta}(|\mathcal{L}|^\wedge_p))$ are isomorphic.*

1.4 Obstruction theory for centric linking systems

The main questions in the theory of p-local finite groups are the following: Given a saturated fusion system, does there exist an associated centric linking system? If so, is it unique? Broto, Levi and Oliver have developed an obstruction theory to answer these questions [7, Sec. 3]. We present their results in this section.

The problem of finding a centric linking system $\mathcal{L}$ associated to a saturated fusion system $\mathcal{F}$ can roughly be thought of as the problem of finding a compatible system of expressions of $Hom_{\mathcal{F}}(P,Q)$ as the orbit set of free $Z(P)$ actions on $Mor_{\mathcal{L}}(P,Q)$, as P and Q run through the centric subgroups of S. Compatibility here means satisfying the axioms of 1.1.5. In the special case when $P = Q$, we get the more familiar problem of finding a group $Aut_{\mathcal{L}}(P)$ along with morphisms δ_P and π fitting into the following diagram (with exact rows)

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Z(P) & \longrightarrow & P & \longrightarrow & Aut_{\mathcal{F}}(P) & \longrightarrow & Out_{\mathcal{F}}(P) & \longrightarrow & 1 \\
\downarrow & & \downarrow{=} & & \downarrow{\delta_P} & & \downarrow{=} & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z(P) & \xrightarrow{\delta_P} & Aut_{\mathcal{L}}(P) & \xrightarrow{\pi} & Aut_{\mathcal{F}}(P) & \longrightarrow & 1 & \longrightarrow & 1,
\end{array}$$

where $Out_{\mathcal{F}}(P) := Inn(P)\backslash Aut_P(\mathcal{F})$. The obstruction to the existence of the bottom row is the element $u \in H^3(Out_{\mathcal{F}}(P); Z(P))$, (where $Out_{\mathcal{F}}(P)$ acts on $Z(P)$ in the obvious way,) defined by the crossed module extension in the top row [9]. In the case when this obstruction vanishes, the group $H^2(Out_{\mathcal{F}}(P); Z(P))$ acts freely and transitively on the set of isomorphism classes of bottom rows fitting into the diagram. The obstructions to existence and uniqueness of a centric linking system associated to a given saturated fusion system $\mathcal{F}$ similarly lies in a higher limit of a "centre functor" over a certain quotient category of $\mathcal{F}$. This will be made precise below.

Definition 1.4.1. *The orbit category of a fusion system $\mathcal{F}$ over a p-group S is the category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S and whose morphisms are defined by*

$$Mor_{\mathcal{O}(\mathcal{F})}(P,Q) := Inn(Q)\backslash Hom_{\mathcal{F}}(P,Q).$$

We let $\mathcal{O}^c(\mathcal{F})$ denote the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of S.

Now consider the functor

$$\mathcal{Z} = \mathcal{Z}_{\mathcal{F}} : \mathcal{O}^c(\mathcal{F})^{op} \longrightarrow \mathtt{Ab},$$

defined for any fusion system $\mathcal{F}$ by setting $\mathcal{Z}_{\mathcal{F}}(P) = Z(P)$ and

$$\mathcal{Z}_{\mathcal{F}}(P \xrightarrow{\varphi} Q) = (Z(Q) \xrightarrow{incl} Z(\varphi(P)) \xrightarrow{\varphi^{-1}} Z(P)).$$

To make sense of this definition, the reader should keep in mind that since P and Q are both centric, one has

$$Z(Q) = C_S(Q) \leq C_S(\varphi(P)) = Z(\varphi(P))$$

and that although a morphism $\varphi \in Mor_{\mathcal{O}(\mathcal{F})}(P,Q)$ only defines a homomorphism $\tilde{\varphi} : P \to Q$ (and its image $\tilde{\varphi}(P)$) up to Q-conjugacy, the isomorphism $\varphi : Z(P) \xrightarrow{\cong} Z(\varphi(P))$ of centres is uniquely determined by φ.

The importance of this functor is evident in the following proposition.

Proposition 1.4.2. *[7, Prop. 3.1] Fix a saturated fusion system $\mathcal{F}$ over the p-group S. Then there is an element $\eta(\mathcal{F}) \in \varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z})$ such that $\mathcal{F}$ has an associated centric linking system if and only if $\eta(\mathcal{F}) = 0$. Also, if there are any centric linking systems associated to $\mathcal{F}$, then the group $\varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z})$ acts freely and transitively on the set of isomorphism classes of centric linking systems associated to $\mathcal{F}$.*

Chapter 2

A classification of p-local finite groups over abelian groups

In this chapter we classify the p-local finite groups over an abelian finite p-group S. The resulting classification shows that the isomorphism classes of p-local finite groups over S are in a bijective correspondence with the subgroups $W \leq Aut(S)$, under the assignment

$$W \mapsto (S, \mathcal{F}_S(W \ltimes S), \mathcal{L}_S(W \ltimes S)),$$

where $W \ltimes S$ is the semi-direct product. In particular, there are no exotic p-local finite groups over abelian p-groups.

We will prove this straight from the definitions given in section 1.1. The reader should note however, that this is also a direct consequence of deeper results, namely Alperin's theorem for saturated fusion systems [25] and very recent work of Broto-Castellana-Grodal-Levi-Oliver [5].

2.1 Saturated fusion systems over abelian p-groups

In the case where S is an abelian p-group, there are significant simplifications to be made in the conditions for saturation of a fusion system over S (definition 1.1.2). We note the most immediate ones:

(i) Every $P \leq S$ is both fully centralised and fully normalised, since $C_S(P) = N_S(P) = S$.

(ii) $Aut_S(P) = \{id\}$ for all $P \leq S$. More generally, for subgroups $P, Q \leq S$, the morphism set $Hom_S(P, Q)$ is either empty or consists solely of the inclusion $P \hookrightarrow Q$, depending on whether $P \leq Q$ or not.

(iii) For $P \leq S$ and $\varphi \in Hom_{\mathcal{F}}(P, S)$, we have

$$\begin{aligned} N_\varphi &= \{g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in Aut_S(\varphi P)\} \\ &= \{g \in S \mid \varphi \circ id \circ \varphi^{-1} \in \{id\}\} \\ &= S. \end{aligned}$$

These simplifications allow us to show that a saturated fusion system $\mathcal{F}$ over an abelian p-group S is completely determined by S and the automorphism group $Aut_{\mathcal{F}}(S)$. To make this more precise, we will need some notation. Given a subgroup $W \leq Aut(S)$, we let $\mathcal{F}_W$ be the category whose objects are the subgroups of S and whose morphisms $Hom_{\mathcal{F}_W}(P,Q)$ are restrictions of morphisms in W mapping P to Q. In other words,

$$Hom_{\mathcal{F}_W}(P,Q) = \{\varphi|_P \mid \varphi \in W, \varphi(P) \leq Q\}.$$

It is easy to see that $\mathcal{F}_W$ is equal to the fusion system $\mathcal{F}_S(W \ltimes S)$ of the semi-direct product $W \ltimes S$. In particular, $\mathcal{F}_W$ is a fusion system.

Lemma 2.1.1. *Let S be a finite abelian p-group and $\mathcal{F}$ a saturated fusion system over S. Put $W := Aut_{\mathcal{F}}(S)$. Then $\mathcal{F} = \mathcal{F}_W$.*

Proof: Since the categories $\mathcal{F}$ and $\mathcal{F}_W$ have the same objects, it suffices to check that $Hom_{\mathcal{F}}(P,Q) = Hom_{\mathcal{F}_W}(P,Q)$ for all $P, Q \leq S$.

Suppose we are given $\varphi \in Hom_{\mathcal{F}_W}(P,Q)$. Then $\varphi = \bar{\varphi}|_P$, for some $\bar{\varphi} \in W = Aut_{\mathcal{F}}(S)$ such that $\bar{\varphi}(P) \subseteq Q$. By condition (b) of definition 1.1.1, $\bar{\varphi}|_P$ factors as an isomorphism $\varphi' : P \xrightarrow{\cong} \bar{\varphi}(P)$ in $\mathcal{F}$, followed by the inclusion $\bar{\varphi}(P) \hookrightarrow S$. Since the inclusion $\bar{\varphi}(P) \hookrightarrow Q$ is also in $\mathcal{F}$, by condition (a) of definition 1.1.1, we see that the composition $\varphi : P \xrightarrow{\cong} \bar{\varphi}(P) \hookrightarrow Q$ is in $Hom_{\mathcal{F}}(P,Q)$.

Conversely, suppose we are given $\varphi \in Hom_{\mathcal{F}}(P,Q)$. Composing with the inclusion $Q \hookrightarrow S$, we get a morphism in $Hom_{\mathcal{F}}(P,S)$, which we also denote by φ. By (i) above, φQ is fully centralised and so by axiom (II) for a saturated fusion system (definition 1.1.2), φ extends to a map $\bar{\varphi} \in Hom_{\mathcal{F}}(N_\varphi, S)$. However, $N_\varphi = S$ by (iii) above, so $\bar{\varphi} \in Hom_{\mathcal{F}}(S,S) = W$. We have shown that φ is the restriction of an element $\bar{\varphi} \in W$ and therefore $\varphi \in Hom_{\mathcal{F}_W}(P,Q)$. □

The question now arises, for which subgroups $W \leq S$ the fusion system $\mathcal{F}_W$ is saturated. This question has a comprehensive answer.

Lemma 2.1.2. *Let $W \leq Aut(S)$. The fusion system $\mathcal{F}_W$ is saturated if and only if W has order prime to p.*

Proof: By 1.2.2, the fusion system is saturated if and only if $S \leq W \ltimes S$ is a Sylow inclusion, which in turn is true if and only if W has order prime to p. □

Combining the lemmas, we get the following classification of saturated fusion systems over abelian groups.

Proposition 2.1.3. *If S is an abelian finite p-group, then the assignment $W \mapsto \mathcal{F}_S(W \ltimes S)$ gives a bijective correspondence between subgroups $W \leq Aut(S)$ of order prime to p and saturated fusion system over S.*

Our work in this section also produces the following lemma, which describes precisely how the conditions in definition 1.1.2 are simplified under the assumption that S is abelian. We record it for later use.

Lemma 2.1.4. *Let $\mathcal{F}$ be a fusion system over a finite abelian p-group S. Then $\mathcal{F}$ is saturated if and only if the following two conditions are satisfied:*

($\mathbf{I}_{ab}$) $|Aut_{\mathcal{F}}(S)|$ *is prime to p.*

($\mathbf{II}_{ab}$) *Every $\varphi \in Hom_{\mathcal{F}}(P, Q)$ is the restriction of some $\tilde{\varphi} \in Aut_{\mathcal{F}}(S)$.*

2.2 Centric linking systems

Having classified the saturated fusion systems over a finite abelian p-group S, we now turn our attention towards the associated centric linking systems. Our work is again simplified by the abelian assumption. The main observation here is that since $C_S(P) = S$ for every $P \leq S$, the only $\mathcal{F}$-centric subgroup is S itself. Therefore $\mathcal{O}^c(\mathcal{F})$ is a category with only one object S and morphism set

$$Mor_{\mathcal{O}(\mathcal{F})}(S, S) = Inn(S) \backslash Aut_{\mathcal{F}}(S) = Aut_{\mathcal{F}}(S) =: W$$

and the obstruction theory of section 1.4 is thus simplified to ordinary group cohomology

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^{*}(\mathcal{Z}) \cong H^*(W; S).$$

Assume now that the fusion system $\mathcal{F}$ is saturated. By the results of the previous section, we know that $\mathcal{F}$ is the fusion system $\mathcal{F}_S(W \ltimes S)$ of the semi-direct product $W \ltimes S$. This fusion system has a canonical associated centric linking system $\mathcal{L}^c_S(W \ltimes S)$. We will show that there are no other centric linking systems associated to $\mathcal{F}$.

Now, W has order prime to p and S is a p-group, so a simple transfer argument shows that the cohomology groups $H^*(W; S)$ vanish for $* > 0$. By proposition 1.4.2, there is therefore a unique centric linking system associated to $\mathcal{F}$.

Taking the results of this chapter together, we get the following proposition.

Proposition 2.2.1. *If S is an abelian finite p-group, then the assignment*

$$W \mapsto (S, \mathcal{F}(W \ltimes S), \mathcal{L}^c_S(W \ltimes S)$$

gives a bijejctive correspondence between subgroups $W \leq Aut(S)$ of order prime to p and p-local finite groups over S. In particular, there are no exotic p-local finite groups over S.

Chapter 3

p-local finite groups induced by subgroup inclusions

In this chapter we study homotopy subgroup inclusions $BS \to X$ of a finite p-group S to a p-complete space X with a transfer t satisfying a form of Frobenius reciprocity. We refer to such a triple (f, t, X) as a *Frobenius transfer triple.* The goal is to prove that such an inclusion induces a p-local finite group whose classifying space is X. This goal is achieved in the case where S is elementary abelian.

In section 3.1 we explain what is meant by a subgroup inclusion and introduce Frobenius transfer triples. In section 3.2 we analyse the cohomological structure induced by a Frobenius transfer. We then restrict ourselves to the case where S is abelian and show that for (f, t, X) to induce a p-local finite group, $H^*(X)$ must be a ring of invariants in $H^*(BS)$ under the action of a group $W \leq Aut(S)$ of order prime to p. In section 3.3, we use a theorem of Adams and Wilkerson to show that in the case where S is elementary abelian, $H^*(X)$ is indeed such a ring of invariants. This turns out to be more than just a necessary condition, because in section 3.4, we apply the theorems of Miller and Lannes to conclude that $\mathcal{F}$ is saturated. This shows that the inclusion $f\colon BS \to X$ induces the same p-local finite group as the group $W \ltimes S$. Finally in section 3.5 we show that X is equivalent to the classifying space $B(W \ltimes S)^\wedge_p$ as objects under BS.

3.1 Frobenius transfers

First we make precise the setting we are working in. Cohomology will always be taken to be with $\mathbf{F}_p$-coefficients unless otherwise specified.

Definition 3.1.1. *A space F is called quasifinite at p if $Map_*(B\mathbf{Z}/p, F)$ is contractible for all choices of basepoint in F. A map $f\colon Y \to X$ is called a homotopy monomorphism at p if its homotopy fibre F (over every connected component of Y) is quasifinite at p. In the special case where $Y = BP$ is the classifying space of a finite p-group, we say that f is a p-subgroup inclusion.*

In what follows, we will speak of homotopy monomorphisms and subgroup inclusions, taking the suffix "at p" to be understood. This should cause no confusion as the prime p is fixed throughout.

The quasifiniteness condition of the fibre can be rather difficult to verify in practice. Fortunately the following proposition allows us to give a more convenient characterisation of homotopy monomorphisms between the spaces in which we are interested.

Proposition 3.1.2. *[16] Let $f: Y \to X$ be a map between two p-complete spaces, with Noetherian cohomology rings. Then f is a homotopy monomorphism if and only if the induced map in mod p-cohomology makes $H^*(Y; \mathbf{F}_p)$ a finitely generated $H^*(X; \mathbf{F}_p)$-module.*

We will demand some additional structure on our subgroup inclusions, namely, that they allow a transfer with properties similar to that of the transfer of a Sylow subgroup inclusion.

Definition 3.1.3. *Let $f: Y \to X$ be a map of spaces. A Frobenius transfer of f is a stable map $t: \Sigma^\infty_+ X \to \Sigma^\infty_+ Y$ such that $\Sigma^\infty_+ f \circ t \simeq id_{\Sigma^\infty_+ X}$ and the following diagram commutes up to homotopy*

$$\begin{array}{ccc} \Sigma^\infty_+ X & \xrightarrow{\Delta} & \Sigma^\infty_+ X \wedge \Sigma^\infty_+ X \\ \downarrow t & & \downarrow 1\wedge t \\ \Sigma^\infty_+ Y & \xrightarrow{(f\wedge 1)\circ\Delta} & \Sigma^\infty_+ X \wedge \Sigma^\infty_+ Y. \end{array} \tag{3.1}$$

The objects that will be the focus of our attention are defined as follows.

Definition 3.1.4. *A Frobenius transfer triple over a finite p-group S is a triple (f, t, X), where X is a connected, p-complete space with finite fundamental group, f is a subgroup inclusion $BS \to X$ and t is a Frobenius transfer of f.*

Since the space X in the above definition is p-complete with finite fundamental group, it follows that $\pi_1(X)$ is a finite p-group [13]. For reference, we note the following consequence.

Observation 3.1.5. *If (f, t, X) is a Frobenius transfer triple, then the space X is nilpotent.*

For a Frobenius transfer triple (f, t, X) over a finite p-group S, we ask the following questions:

- Is the fusion system $\mathcal{F}_{S,f}(X)$ saturated?
- If so, does there exist an associated centric linking system $\mathcal{L}$? Is it unique?
- If an associated centric linking system exists, then what is the relation between the classifying space $|\mathcal{L}|^\wedge_p$ and X? Are they equivalent as objects under BS?

In the course of this chapter we will answer these questions affirmatively in the case when S is an elementary abelian p-group, proving the following theorem.

Theorem 3.1.6. *Let S be a finite elementary abelian p-group. Let (f, t, X) be a Frobenius transfer triple over S and put $W := Aut_{\mathcal{F}_{S,f}(X)}(S)$. Then the following hold*

(i) *W has order prime to p.*

(ii) *$\mathcal{F}_{S,f}(X)$ is equal to the saturated fusion system $\mathcal{F}_S(W \ltimes S)$*

(iii) *$\mathcal{F}_{S,f}(X)$ has a unique associated centric linking system with classifying space $B(W \ltimes S)^{\wedge}_p$.*

(iv) *There is a natural equivalence $B(W \ltimes S)^{\wedge}_p \xrightarrow{\simeq} X$ of objects under BS.*

Thus the triple (f, t, X) induces a p-local finite group $(S, \mathcal{F}_{S,f}(X), \mathcal{L}^c_{S,f}(X^{\wedge}_p))$ over S with classifying space X.

Proof: The proof is by forward referencing. By lemma 3.4.2, W is the group of automorphisms of S acting trivially on $H^*(X)$. By proposition 3.3.10, W has order prime to p. By proposition 3.4.3, $\mathcal{F}_{S,f}(X)$ is equal to the fusion system $\mathcal{F}_S(W \ltimes S)$. This fusion system is saturated by lemma 2.1.2 and has a unique classifying space $B(W \ltimes S)^{\wedge}_p$ by section 2.2. Finally, proposition 3.5.2 states that the inclusion $f\colon BS \to X$ is equivalent to the natural inclusion $\theta\colon BS \to B(W \ltimes S)^{\wedge}_p$ as a space under BS. □

3.2 Cohomological structure of Frobenius transfer triples

In this section we discuss the cohomological structure of a Frobenius transfer triple (f, t, X) over a finite p-group S. We also discuss the cohomological structure of a p-local finite group over S, paying special attention to the case when S is abelian. This gives us a simply stated necessary condition on $H^*(X)$ for X to be the classifying space of a p-local finite group over an abelian group S. We include this discussion here, because it provides a starting point for the discussion that follows, leading to the proof of theorem 3.1.6.

Applying the cohomology functor $H^*(-; \mathbf{F}_p)$ to (3.1) we get maps

$$H^*(X) \xrightarrow{f^*} H^*(BS) \xrightarrow{t^*} H^*(X)$$

with the following properties:

CohI $t^* \circ f^* = id$.

CohII t^* is $H^*(X)$-linear by the Frobenius reciprocity property.

CohIII t^* is a morphism of unstable modules over the Steenrod algebra.

CohIV f^* is a morphism of unstable algebras over the Steenrod algebra.

Hence $H^*(X)$ is a direct summand of $H^*(BS)$ as a $H^*(X)$-module and as a module over the Steenrod algebra. CohI allows us to regard $H^*(X)$ as a subring of $H^*(BS)$ and we will often do so without further comment.

These properties are quite restrictive and the question of which unstable subalgebras $R^* \subset H^*(BS)$ over the Steenrod algebra admit a stable splitting $H^*(BS) \to R^*$ as R^*-modules and unstable modules over the Steenrod algebra is interesting in itself. However, we focus our attention on p-local finite groups.

The following finiteness properties of Frobenius transfer triples will be needed later. The first fact is a standard result, a proof of which can be found in [13, Sec. 12].

Fact 3.2.1. *If S is a finite p-group, then $H^*(BS)$ is Noetherian.*

Lemma 3.2.2. *Let S be a finite p-group and (f, t, X) be a Frobenius transfer triple over S. Then $H^*(X)$ is Noetherian and in particular X is of finite $\mathbf{F}_p$-type.*

Proof: By [13, Lemma 2.6], this follows from CohI, CohII and the fact that $H^*(BS)$ is Noetherian. □

Lemma 3.2.3. *Let S be a finite p-group and (f, t, X) be a Frobenius transfer triple over S. Then X is of $\mathbf{Z}_{(p)}$-finite type.*

Proof: By the universal coefficient theorem, it suffices to show that X is of finite $\mathbf{F}_p$-type and of finite $\mathbf{Q}$-type. The former is lemma 3.2.2 above. The latter is deduced in a similar way: By a transfer argument, BS has trivial $\mathbf{Q}$-cohomology. As in the $\mathbf{F}_p$-coefficient case, $H^*(X; \mathbf{Q})$ is a direct summand of $H^*(BS; \mathbf{Q})$. Hence X also has trivial $\mathbf{Q}$-cohomology and we are done. □

For a Frobenius transfer triple (f, t, X) over S we are interested in whether the space X is the classifying space of a p-local finite group over S. As a first test, we examine whether X has the right cohomology. In [7] it is shown that the classifying space $|\mathcal{L}|_p^\wedge$ of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ has cohomology

$$H^*(|\mathcal{L}|_p^\wedge) = H^*(\mathcal{F}) := \varprojlim_{\mathcal{O}(\mathcal{F})} H^*(B(-)).$$

We restrict our attention to the case when S is abelian. Let $\mathcal{F}$ be the fusion system $\mathcal{F}_{f,S}(X)$ induced by the inclusion $f\colon BS \to X$. By the discussion in section 2.1 we see that if $\mathcal{F}$ is saturated, then we must have

$$H^*(\mathcal{F}) = H^*(BS)^W,$$

where $W := Aut_{\mathcal{F}}(S) \leq Aut(S)$ has order prime to p. This gives us the following necessary cohomological condition.

Observation 3.2.4. *If X is the classifying space of a p-local finite group over an abelian group S, then*

$$H^*(X) = H^*(BS)^W$$

for some subgroup $W \leq Aut(S)$ of order prime to p.

3.3 A theorem of Adams and Wilkerson

In this section, we restrict ourselves to the case where S is an elementary abelian finite p-group. In this case, we use a theorem of Adams and Wilkerson [2] to prove that if (f, t, X) is a Frobenius transfer triple over S, then $H^*(X)$ is a ring of invariants $H^*(BS)^W$ for a subgroup $W \leq Aut(S)$. We then use a result of Lannes [14] to deduce that W must in fact have order prime to p. In addition to verifying that the necessary condition in 3.2.4 is satisfied, we will be able to draw further conclusions from this result in the following sections.

In this section we will consider only the case of an odd prime. The results still hold true for the case of an even prime and the proofs proceed in more or less the same way, but are simpler at times.

In [2], Adams and Wilkerson study the following category.

Definition 3.3.1. *Let $\mathcal{AW}$ be the category of evenly graded unstable algebras R^* over the Steenrod algebra, that are integral domains.*

They also make precise the notions of "algebraic extension" and "algebraic closure" in this setting and prove the following.

Proposition 3.3.2. *[2, Prop 1.5] Every object R^* in $\mathcal{AW}$ has an algebraic closure H^* in $\mathcal{AW}$. If R^* has finite transcendence degree, then so does H^*.*

Theorem 3.3.3. *[2, Thm 1.6] The objects H^* in $\mathcal{AW}$, that are "algebraically closed" and of finite transcendence degree are precisely the polynomial algebras $\mathbf{F}_p[x_1, \ldots, x_n]$ on generators x_i of degree 2.*

The theorem we wish to apply is the following.

Theorem 3.3.4. *[2, Thm 1.2] Let R^* be an algebra in $\mathcal{AW}$ of finite transcendence degree and let $H^* = \mathbf{F}_p[x_1, \ldots, x_n]$ be the algebraic closure in $\mathcal{AW}$. In order that R^* should admit an isomorphism*

$$R^* \cong (H^*)^W,$$

for some group W of automorphisms of H^*, the following two conditions are necessary and sufficient:

AW1 *The integral domain R^* is integrally closed in its field of fractions.*

AW2 *If $y \in R^{2dp}$ and $Q^r y = 0$ for each $r \geq 1$, then $y = x^p$ for some $x \in R^{2d}$.*

The second condition is really an "inseparably closed" condition. The operation Q^r is the Milnor primitive of dimension $2p^r - 2$ in $\mathcal{A}^*$.

We now assume that S is a finite elementary abelian group. Before applying theorem 3.3.4 we have to put ourselves in the framework of $\mathcal{AW}$. To do this, we apply some techniques of [14].

Let $\mathcal{U}$ denote the category of unstable modules over the Steenrod algebra and $\mathcal{K}$ denote the category of unstable algebras over the Steenrod algebra. In both cases morphisms are of degree zero. Let $\mathcal{U}'$ and $\mathcal{K}'$ denote the corresponding full subcategories whose objects are evenly graded. The forgetful functor $\theta\colon \mathcal{K}' \to \mathcal{K}$ has a right adjoint $\tilde{\theta}\colon \mathcal{K} \to \mathcal{K}'$.

Proposition 3.3.5. *[14, Cor. 3.5] If K_1 and K_2 are two unstable Steenrod algebras, whose images in $\mathcal{U}$ are reduced $\mathcal{U}$-injectives, then K_1 is isomorphic to K_2 in $\mathcal{K}$ if and only if $\tilde{\theta}K_1$ is isomorphic to $\tilde{\theta}K_2$ in $\mathcal{K}'$.*

When S is elementary abelian, the cohomology ring $H^*(BS)$ is a reduced $\mathcal{U}$-injective by [17]. If (f, t, X) is a Frobenius transfer triple over S, then $H^*(X)$ is a direct summand of $H^*(BS)$ and hence $H^*(X)$ is also a reduced $\mathcal{U}$-injective. The following result of Lannes, which appeared in [14], describes precisely in which cases a ring of invariants of $H^*(BS)$ is a reduced $\mathcal{U}$-injective.

Proposition 3.3.6. *[14, Prop. 4.1.1] Let S be an elementary abelian p-group and let $W \leq Aut(S)$. Then the ring of invariants $H^*(BS)^W$ is a reduced $\mathcal{U}$-injective if and only if W has order prime to p.*

Remark 3.3.7. Proposition 3.3.6 equivalently says that $H^*(BS)^W$ is a direct summand of $H^*(BS)$ as unstable modules over the Steenrod algebra if and only if W has order prime to p. By the same proof, we can deduce that $(\tilde{\theta}H^*(BS))^W$ is a direct summand of $\tilde{\theta}H^*(BS)$ if and only if W has order prime to p.

To show that $H^*(X) = H^*(BS)^W$ for some $W \leq Aut(S)$ of order prime to p, it is now enough to show that the corresponding equality holds after applying $\tilde{\theta}$. Before doing so, we prove the following lemma in order to remove the confusion caused by working with automorphisms in many different categories.

Lemma 3.3.8. *Let S be a finite elementary abelian p-group. Then there are isomorphisms*

$$Aut(S) \xrightarrow[\cong]{H^*(-)} Aut_{\mathcal{K}}(H^*(BS)) \xrightarrow[\cong]{\tilde{\theta}} Aut_{\mathcal{K}'}(\tilde{\theta}H^*(BS)).$$

Proof: Let n be the rank of S. Recall, that as an algebra

$$H^*(BS) \cong E[y_1, \ldots, y_n] \otimes \mathbf{F}_p[x_1, \ldots, x_n], \tag{3.2}$$

is a tensor product of an exterior algebra over $\mathbf{F}_p$ on n generators y_i of degree 1 and a polynomial algebra over $\mathbf{F}_p$ on n generators x_i of degree 2. The Steenrod module structure is determined by the unstable condition and that the Bockstein maps y_i to x_i. Therefore, we see that any homomorphism $H^*(BS) \to H^*(BS)$ in $\mathcal{K}$ is determined by its restriction to a linear self map of the $\mathbf{F}_p$-vector subspace $V \cong S$ of $H^2(BS)$ spanned by the elements $x_1, \ldots, x_n$. Conversely, any linear map $V \to V$ extends to a homomorphism $H^*(BS) \to H^*(BS)$ in $\mathcal{K}$ by simply applying the cohomology functor to the corresponding group homomorphism $S \to S$. In particular, this holds for automorphisms, in which case we get the desired isomorphism

$$H^*(-)\colon Aut(S) \xrightarrow{\cong} Aut_{\mathcal{K}}(H^*(BS)).$$

For the second isomorphism, we recall [17, 29] that (3.2) implies that

$$\tilde{\theta}H^*(BS) \cong \mathbf{F}_p[x_1, \ldots, x_n]. \tag{3.3}$$

By the same reasoning as above, we get an isomorphism

$$\tilde{\theta}H^*(-)\colon Aut(S) \xrightarrow{\cong} Aut_{\mathcal{K}'}(\tilde{\theta}H^*(BS)),$$

which factors through the isomorphism $H^*(-)\colon Aut(S) \xrightarrow{\cong} Aut_{\mathcal{K}}(H^*(BS))$. It follows that we have an isomorphism

$$\tilde{\theta}\colon Aut_{\mathcal{K}}(H^*(BS)) \xrightarrow{\cong} Aut_{\mathcal{K}'}(\tilde{\theta}H^*(BS)).$$

Alternatively, this isomorphism follows directly from [14, Cor. 3.3], since $H^*(BS)$ is a reduced $\mathcal{U}$-injective. □

Proposition 3.3.9. *Let (f,t,X) be a Frobenius transfer triple over an elementary p-group S. Then*

$$H^*(X) = H^*(BS)^W$$

for some subgroup $W \leq Aut(S)$ of order prime to p.

Proof: Put

$$R^* := \tilde{\theta}H^*(X)$$

and

$$H^* := \tilde{\theta}H^*(BS) = \mathbf{F}_p[x_1, \ldots, x_n],$$

where the x_i are algebraically independent elements of degree 2 (cf. proof of 3.3.8). It is clear that $H^* \in \mathcal{AW}$. Since R^* is a subobject of H^* in $\mathcal{K}'$, it follows that H^* is also in $\mathcal{AW}$.

By assumption, the map $f^*\colon H^*(X) \to H^*(BS)$ makes $H^*(BS)$ into a finitely generated $H^*(X)$ algebra and therefore $R^* \hookrightarrow H^*$ is an algebraic extension. Since H^* is algebraically closed in $\mathcal{AW}$, this means that H^* is the algebraic closure of R^* in $\mathcal{AW}$. Thus theorem 3.3.4 applies. We use the properties of the morphisms

$$R^* \xrightarrow{f^*} H^* \xrightarrow{t^*} R^*$$

discussed in section 3.2 and the fact that H^* satisfies conditions AW1 and AW2 to show that the same is true of R^*.

Proof of AW1: Let $x \in Fr(R^*)$ be in the field of fractions of R^* and suppose that x is integral over R^*. Write $x = a/b$, with $a, b \in R^*$. Now, x is also integral over H^* and since H^* is integrally closed, this implies that $x \in H^*$. We now have the equation $a = bx$ in H^*. Applying t^* and using R^*-linearity (CohII), we get:

$$a = t^*(a) = t^*(bx) = bt^*(x).$$

Since H^* is an integral domain, this implies that

$$x = a/b = t^*(x) \in R^*.$$

Proof of AW2: Let $y \in R^{2dp}$ and assume that $Q^r y = 0$ for all $r \geq 1$. Since H^* satisfies AW2, this implies that there is an $x \in H^{2d}$ such that $x^p = y$. Applying t^* to this equation and using the fact that t^* preserves Steenrod operations (CohIII), we get

$$y = t^*(y) = t^*(x^p) = t^*(P^d x) = P^d t^*(x) = (t^*x)^p.$$

Since $t^*(x) \in R^{2d}$ we are done.

We have now shown that $R^* = (H^*)^W$ for some subgroup W of $Aut(S)$. Since R^* is a direct summand of H^* as unstable modules over the Steenrod algebra, we deduce by remark 3.3.7 that W has order prime to p. The right adjoint $\tilde{\theta}$ preserves inverse limits and in particular rings of invariants. We therefore have an isomorphism

$$(H^*)^W \cong \tilde{\theta}(H^*(BS)^W)$$

compatible with the inclusion into H^*. As noted earlier, $H^*(X)$ is a reduced $\mathcal{U}$-injective. Since W has order prime to p, proposition 3.3.6 applies to show that $H^*(BS)^W$ is also a reduced $\mathcal{U}$-injective. Hence, by proposition 3.3.5, the isomorphism

$$\tilde{\theta}H^*(X) = R^* \cong (H^*)^W \cong \tilde{\theta}(H^*(BS)^W)$$

implies that

$$H^*(X) \cong H^*(BS)^W.$$

□

With very litte extra work, we can get a slightly nicer statement.

Proposition 3.3.10. *Let (f, t, X) be a Frobenius transfer triple over an elementary p-group S and let $W \leq Aut(S)$ be the subgroup of automorphisms acting trivially on $H^*(X)$. Then*

$$H^*(X) = H^*(BS)^W$$

and W has order prime to p.

Proof: The inclusion $H^*(X) \subseteq H^*(BS)^W$ is obvious. For the converse, we apply proposition 3.3.9 to get a subgroup $W' \leq Aut(S)$ such that $H^*(X) = H^*(BS)^{W'}$. Then W' acts trivially on $H^*(X)$, so $W' \leq W$. It follows that

$$H^*(BS)^W \subseteq H^*(BS)^{W'} = H^*(X).$$

This shows that $H^*(X) = H^*(BS)^W$ and theorem 3.3.6 applies to show that W has order prime to p. □

Remark 3.3.11. A posteriori, one can show that $W' = W$ in the above proof by comparing the fusion systems of $W \ltimes S$ and $W' \ltimes S$.

3.4 Proving saturation of $\mathcal{F}_{f,S}(X)$

We have now determined the cohomological structure of a Frobenius transfer triple (f, t, X) over an elementary abelian p-group S. In this section we use the theorems of Miller and Lannes to draw conclusions in homotopy from the results in the previous section and prove that the fusion system $\mathcal{F}_{f,S} := \mathcal{F}_{f,S}(X)$ is saturated.

Theorem 3.4.1. *[15] Let Y be a connected space and V an elementary abelian p-group. Suppose that Y is nilpotent, that $\pi_1 Y$ is finite and that $H^*(Y)$ is of finite type. Then the natural map*

$$[BV, Y] \to Hom_{\mathcal{K}}(H^*(Y), H^*(BV))$$

is a bijection.

The case when Y is a space with cohomology of the type $H^*(Y) = U(M)$ is due to Miller [21].

Let (f, t, X) be a Frobenius transfer triple over an elementary abelian p-group S. By observation 3.1.5, X is a nilpotent space. By assumption, $\pi_1(X)$ is finite. By CohI in section 3.2, $H^*(X)$ is of finite type. Thus theorem 3.4.1 applies to X.

The two things we need to show in order to prove saturation of $\mathcal{F}_{f,S}$ are expressed in lemma 2.1.4. The first part, that $Aut_{\mathcal{F}_{f,S}}(S)$ has order prime to p, follows from proposition 3.3.10 and the following lemma.

Lemma 3.4.2. *Let (f, t, X) be a Frobenius transfer triple over a finite elementary abelian p-group S and let W be the subgroup of $Aut(S)$ acting trivially on $H^*(X)$. Then the isomorphism*

$$Hom(S, S) \to [BS, BS] \to Hom_{\mathcal{K}}(H^*(BS), H^*(BS))$$

sends $Aut_{\mathcal{F}_{f,S}(X)}(S)$ to W.

Proof: By definition, an automorphism $\varphi\colon BS \to BS$ is in $Aut_{\mathcal{F}_{f,S}}(S)$ if and only if $f \circ B\varphi \simeq f$. By theorem 3.4.1, this is true if and only if the corresponding equality $B\varphi^* \circ f^* \simeq f^*$ holds in cohomology. Since f^* is the inclusion $H^*(X) \hookrightarrow H^*(BS)$ and W is the group of automorphisms of $H^*(BS)$ fixing $H^*(X)$, the equality in cohomology holds if and only if $B\varphi^* \in W$. □

Given the fact that $Aut_{\mathcal{F}_{f,S}}(S) \simeq W$ and the classification of p-local finite groups over elementary abelian groups in chapter 2, we see that $\mathcal{F}_{f,S}$ is saturated if and only if it is isomorphic to the fusion system $\mathcal{F}_S(W \ltimes S)$. This follows from the results in the next section, but can also be proved independently in a fashion similar to the proof of the last lemma.

Proposition 3.4.3. *Let (f, t, X) be a Frobenius transfer triple over an elementary abelian group S and put $W := Aut_{\mathcal{F}_{f,S}}(S)$. Then W has order prime to p and $\mathcal{F}_{f,S}$ is equal to the fusion system $\mathcal{F}_S(W \ltimes S)$. In particular $\mathcal{F}_{f,S}$ is saturated.*

Proof: By lemma 3.4.2, W is the group of automorphisms of S acting trivially on $H^*(X)$. By proposition 3.3.10, W has order prime to p.

We will now show that $\mathcal{F}_{f,S} = \mathcal{F}_S(W \ltimes S)$. Let V and V' be two subgroups in S. Let ι_V and $\iota_{V'}$ be the inclusions $V \hookrightarrow S$ and $V' \hookrightarrow S$, respectively. By definition of $\mathcal{F}_{f,S}$, a morphism $\varphi\colon V \to V'$ is in $Hom_{\mathcal{F}_{f,S}}(V, V')$ if and only if

$$f \circ B\iota_{V'} \circ B\varphi \simeq f \circ B\iota_V.$$

By theorem 3.4.1, this is true if and only if the corresponding equality

$$B\varphi^* \circ B\iota_{V'}^* \circ f^* \simeq B\iota_V^* \circ f^* \tag{3.4}$$

holds in cohomology.

Similarly, let θ be the natural map $BS \to B(W \ltimes S)^\wedge_p$. Then a morphism $\varphi\colon V \to V'$ is in $Hom_{\mathcal{F}_S(W \ltimes S)}(V, V')$ if and only if

$$\theta \circ B\iota_{V'} \circ B\varphi \simeq \theta \circ B\iota_V.$$

Again, by theorem 3.4.1, this is true if and only if the corresponding equality

$$B\varphi^* \circ B\iota_{V'}^* \circ \theta^* \simeq B\iota_V^* \circ \theta^* \tag{3.5}$$

holds in cohomology.

By the Cartan-Eilenberg theorem [11, Thm. XII.10.1], we have

$$H^*(B(W \ltimes S)) \cong H^*(BS)^W,$$

so the two maps f^* and θ^* are both the inclusion

$$H^*(BS)^W \hookrightarrow H^*(BS).$$

Therefore the two conditions 3.4 and 3.5 in cohomology agree and

$$\varphi \in Hom_{\mathcal{F}_{f,S}}(V, V')$$

if and only if

$$\varphi \in Hom_{\mathcal{F}_S(W \ltimes S)}(V, V').$$

Since $\mathcal{F}_{f,S}$ is equal to the fusion system $\mathcal{F}_S(W \ltimes S)$ and W has order prime to p, lemma 2.1.2 applies to show that $\mathcal{F}_{f,S}$ is saturated. □

3.5 Comparing classifying spaces

So far we have proved that if (f, t, X) is a Frobenius transfer triple over an elementary abelian finite p-group S, then the induced fusion system $\mathcal{F}_{f,S} := \mathcal{F}_{f,S}(X)$ agrees with the fusion system $\mathcal{F}_S(W \ltimes S)$, where $W := Aut_{\mathcal{F}}(S)$. This fusion system has a unique associated centric linking system $\mathcal{L}_S(W \ltimes S)$, by the results in chapter 2. Thus we have shown that to the triple (f, t, X) one can associate a unique p-local finite group $(S, \mathcal{F}_{f,S}, \mathcal{L}_S(W \ltimes S))$.

In this section we will use Wojtkowiak's obstruction theory to compare the classifying space $|\mathcal{L}_S(W \ltimes S)|_p^\wedge = B(W \ltimes S)_p^\wedge$ of the p-local finite group $(S, \mathcal{F}_{f,S}, \mathcal{L}_S(W \ltimes S))$ to the space X. We will prove that $BS \xrightarrow{f} X$ is isomorphic to $BS \xrightarrow{\theta} B(W \ltimes S)_p^\wedge$ as objects under BS, thus completing the proof of theorem 3.1.6.

In [27, 28], Wojtkowiak develops an obstruction theory for the existence and uniqueness of maps from homotopy colimits to p-complete classifying spaces. This obstruction theory is quite general and it is interesting to note that, when applied to the classifying spaces of p-local finite groups, Wojtkowiak's obstructions to the existence of maps are identical to the obstructions to uniquness of classifying spaces in proposition 1.4.2. It is this phenomenon, which we will use to our advantage in this section.

The particular result we wish to apply is reproduced below. For the sake of simplicity, we state the result only in a special case, relevant to our needs.

Theorem 3.5.1. *[27, 28] Let S be a finite abelian p group and W a group of order prime to p, which acts on S. For any nilpotent p-local space X of $\mathbf{Z}_{(p)}$-finite type with trivial W-action, the natural map*

$$[B(W \ltimes S), X] \overset{-\circ Bi_S}{\longrightarrow} [BS, X]^W$$

is a bijection.

The application of the theorem allows us to prove the following.

Proposition 3.5.2. *Let (f, t, X) be a Frobenius transfer triple over an elementary abelian group S and put $W := Aut_{\mathcal{F}_{f,S}}(S)$. Let θ be the p-completed inclusion $BS \hookrightarrow B(W \ltimes S)^\wedge_p$. Then there is an isomorphism*

$$h\colon (\theta, B(W \ltimes S)^\wedge_p) \to (f, X)$$

of spaces under BS.

Proof: Since X is p-complete, it is in particular p-local. By lemma 3.2.3, X is of finite $\mathbf{Z}_{(p)}$-type. Thus theorem 3.5.1 applies to give a bijection

$$[B(W \ltimes S), X] \overset{-\circ B\iota}{\longrightarrow} [BS, X]^W,$$

where $B\iota$ is the inclusion $BS \hookrightarrow B(W \ltimes S)$ (before completion).

Now, $f \in [BS, X]^W$ by definition of W. Hence the bijection yields a map $h\colon B(W \ltimes S) \to X$ such that

$$h \circ B\iota \simeq f.$$

Upon applying the cohomology functor $H^*(-)$, we see, as in the proof of proposition 3.4.3, that the induced maps f^* and $B\iota^*$ are both the inclusion

$$H^*(BS)^W \hookrightarrow H^*(BS).$$

From $B\iota^* \circ h^* \simeq f^*$, we can therefore deduce that h induces an isomorphism in cohomology. Upon p-completion, we now get a homotopy equivalence

$$h^\wedge_p\colon B(W \ltimes S)^\wedge_p \overset{\simeq}{\to} X^\wedge_p = X$$

such that

$$h^\wedge_p \circ \theta = h^\wedge_p \circ \iota^\wedge_p \simeq f.$$

□

Chapter 4

Transfer properties of p-local finite groups

In chapter 3 we introduced the notion of a Frobenius transfer triple over a finite p-group S and showed that in the case where S is elementary abelian, such a triple induces a p-local finite group. In this chapter, we consider the reverse implication. Namely, we start with a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ over a general finite p-group S and attempt to show that the inclusion $\theta\colon BS \to |\mathcal{L}|_p^\wedge$ has a Frobenius transfer t, which makes $(\theta, t, |\mathcal{L}|_p^\wedge)$ into a Frobenius transfer triple.

To determine the cohomological structure of a saturated fusion system, Broto, Levi and Oliver [7, prop. 5.5] have constructed a stable self-map $[\Omega]\colon \Sigma_+^\infty BS \to \Sigma_+^\infty BS$, which is idempotent in cohomology. Building on their result we will produce a stable idempotent $\omega\colon \Sigma_+^\infty BS \to \Sigma_+^\infty BS$ satisfying the following Frobenius reciprocity relation

$$(\omega \wedge \omega) \circ \Delta = (\omega \wedge 1) \circ \Delta \circ \omega,$$

where $\Delta\colon \Sigma_+^\infty BS \to \Sigma_+^\infty BS \wedge \Sigma_+^\infty BS$ is the diagonal. This stable idempotent induces a splitting

$$\Sigma_+^\infty BS \xrightarrow{f} \Sigma_+^\infty |\mathcal{L}|_p^\wedge \xrightarrow{t} \Sigma_+^\infty BS.$$

The final step is now to relate f to the map $\Sigma_+^\infty BS \xrightarrow{\Sigma_+^\infty \theta} \Sigma_+^\infty |\mathcal{L}|_p^\wedge$. This part is not yet complete, but we will identify a plausible conjecture (4.3.1), which, if true implies that $(\theta, t, |\mathcal{L}|_p^\wedge)$ is a Frobenius transfer triple.

4.1 The double Burnside ring and the Segal conjecture

In this section we give a brief discourse about the double Burnside ring and its connection to the ring of stable self-maps of the classifying space of a finite p-group.

For finite groups G and H, let $\mathcal{M}or(G, H)$ be the set of isomorphism classes of finite sets with a left G-action and a free right H-action. The disjoint union operation makes $\mathcal{M}or(G, H)$ into a commutative monoid. We denote the Grothendieck group completion by $A(G,H)$. The group structure of $A(G,H)$ is easy to describe. It is a free abelian group with one basis element $G \times_{(G',\varphi)} H$ for each conjugacy class of pairs (G',φ), where $G' \leq G$ and φ is a homomorphism $\varphi\colon G' \to H$. Here $G \times_{(G',\varphi)} H$ denotes the biset

$$G \times_{(G',\varphi)} H = (G \times H)/\sim,$$

with the obvious left G-action and right H-action, where

$$(xg, y) \sim (x, \varphi(g)y)$$

for $x, y \in S,\ g \in G'$.

Given three finite groups G, H, and K, we get a morphism of monoids

$$\circ\colon \mathcal{M}or(H,K) \times \mathcal{M}or(G,H) \to \mathcal{M}or(G,K)$$

by

$$(\Omega, \Lambda) \mapsto \Omega \circ \Lambda := \Lambda \times_H \Omega,$$

which extends to a bilinear map

$$A(H,K) \times A(G,H) \to A(G,K). \tag{4.1}$$

In the case where $G = H = K$, this defines a ring structure on $A(G,G)$.

Definition 4.1.1. *Let G be a finite group. The double Burnside ring of G is the ring $A(G,G)$ described above.*

The ring structure of the double Burnside ring can be described in terms of the basis elements using the double coset formula. See for example [3, sec. 2].

For a (G,H)-biset Ω, a subgroup $G' \leq G$ and a homomorphism $\varphi\colon G' \to G$, we let $\Omega|_{(\varphi,H)}$ denote the restriction of Ω to a (G',H)-biset, where the left G'-action is induced by φ. In the special case where φ is the inclusion $G' \leq G$, we will denote the restriction by $\Omega|_{(G',H)}$. We note that $\Omega|_{(\varphi,H)}$ can also be constructed by the pairing

$$\circ\colon \mathcal{M}or(G,H) \times \mathcal{M}or(G',G) \to \mathcal{M}or(G',G)$$

as

$$\Omega|_{(\varphi,H)} = \Omega \circ (G' \times_{(G',\varphi)} G).$$

For spaces X and Y, let $\{X_+, Y_+\}$ be the group of homotopy classes of stable maps $\Sigma^\infty_+ X \to \Sigma^\infty_+ Y$. Given a finite (G,H)-biset $\Omega \in \mathcal{M}or(G,H)$, we get a stable map $[\Omega] \in \{BG_+, BH_+\}$ as follows. Let $\Lambda := \Omega/H$. Since the right H-action on Ω was free, we get a principal fibre sequence

$$H \to EG \times_G \Omega \to EG \times_G \Lambda.$$

Let

$$\xi_\Omega\colon EG \times_G \Lambda \to BH$$

be the classifying map of this fibration. Furthermore, the projection map

$$EG \times_G \Lambda \to BG$$

is a finite covering. Let $\tau\colon \Sigma^\infty_+ BG \to \Sigma^\infty_+ EG \times_G \Lambda$ be the associated transfer map. The map $[\Omega]$ is now defined by

$$[\Omega] := \Sigma^\infty_+ \xi_\Omega \circ \tau.$$

This assignment extends to a homomorphism

$$\alpha\colon A(G,H) \to \{BG_+, BH_+\}$$

of abelian groups. Although it may not be immediate from the definition, the map α sends the pairing of 4.1 to the composition pairing for stable maps:

$$\alpha(\Omega \circ \Lambda) = \alpha(\Omega) \circ \alpha(\Lambda).$$

Thus α is a ring homomorphism when $G = H$. For basis elements, one can check that

$$[G \times_{(G',\varphi)} H] = \Sigma^\infty_+ B\varphi \circ tr_{G'},$$

where $tr_{G'}$ denotes the transfer of the inclusion $G' \leq G$.

The homomorphism α gives a way to relate $A(G,H)$ to the group of stable maps $\{BG_+, BH_+\}$. Lewis, May and McClure have made this relationship precise [18]. As a consequence of the Segal conjecture (proved by Carlsson in [10]), they show that this map is a certain completion. In the case where G is a p-group, this completion takes a simple form, which we will describe below.

Consider the augmentation homomorphism given by

$$\epsilon\colon A(G,H) \to A(G,1), \;\; \Omega \mapsto \Omega/H.$$

This sends a basis element $G \times_{(G',\varphi)} H$ to G/G'. Let $\tilde{A}(G,H)$ be the kernel of ϵ. Then $\tilde{A}(G,H)$ is free abelian on elements $G \times_{(G',\varphi)} H - (G/G' \times H)$ as (G',φ) runs through H-conjugacy classes of subgroups $G' \leq G$ and nontrivial homomorphisms $\varphi\colon G' \to H$. Moreover, α induces a homomorphism

$$\tilde{\alpha}\colon \tilde{A}(G,H) \to \{BG_+, BH_+\},$$

$$G \times_{(G',\varphi)} H - (G/G' \times H) \mapsto \Sigma^\infty_+ \varphi \circ tr_{G'}.$$

When $G = H$, this is a ring homomorphism.

Theorem 4.1.2 (Segal conjecture). *[10, 18] If G is a p-group, then $\tilde{\alpha}$ induces an isomorphism*

$$\tilde{A}(G,H)^\wedge_p \xrightarrow{\cong} \{BG_+, BH_+\},$$

where $(-)^\wedge_p = (-) \otimes \mathbf{Z}^\wedge_p$ is p-adic completion. If in addition $H = G$, then this is an isomorphism of rings.

Remark 4.1.3. We note that there is a surjection

$$A(G,H) \twoheadrightarrow \tilde{A}(G,H), \Omega \mapsto \Omega - (\Omega/H \times H)$$

and that α factors through this surjection:

$$\alpha\colon A(G,H) \twoheadrightarrow \tilde{A}(G,H) \xrightarrow{\tilde{\alpha}} \{BG_+, BH_+\}.$$

Hence there is a surjection

$$\alpha_p^\wedge\colon A(G,H)_p^\wedge \twoheadrightarrow \tilde{A}(G,H)_p^\wedge \xrightarrow{\cong} \{BG_+, BH_+\}.$$

4.2 Stable idempotents induced by p-local finite groups

In this section we turn our attention back to fusion systems. For a saturated fusion system $\mathcal{F}$ over a finite p-group S, we will prove the existence of an idempotent $\omega \in \{BS_+, BS_+\}$ making the following diagram commute:

$$\begin{array}{ccc} \Sigma_+^\infty BS & \xrightarrow{\Delta} & \Sigma_+^\infty BS \wedge \Sigma_+^\infty BS \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega\wedge\omega} \\ \Sigma_+^\infty BS & \xrightarrow{(\omega\wedge id)\circ\Delta} & \Sigma_+^\infty BS \wedge \Sigma_+^\infty BS. \end{array} \tag{4.2}$$

In [7, sec. 5] Broto, Levi and Oliver determine the cohomological structure of a p-local finite group $(S,\mathcal{F},\mathcal{L})$. In short, they prove that there is a natural isomorphism

$$H^*(|\mathcal{L}|_p^\wedge) \xrightarrow{\cong} H^*(\mathcal{F}),$$

where

$$H^*(\mathcal{F}) := \varprojlim_{\mathcal{O}(\mathcal{F})} H^*(B(-))$$

is the "ring of stable elements for $\mathcal{F}$", regarded as a subring of $H^*(BS)$, and that the map θ^* makes $H^*(BS)$ a finitely generated $H^*(|\mathcal{L}|_p^\wedge)$-module.

One of the key ingredients in their proof is the construction of a biset $\Omega \in \mathcal{M}or(S,S)$, for which the induced map in cohomology, $[\Omega]^*$, is idempotent and satisfies the analogue of (4.2). We will take advantage of their result (included below as proposition 4.2.2) and produce our idempotent by showing the convergence of a judiciously chosen subsequence of the sequence

$$[\Omega], [\Omega]^2, [\Omega]^3, \ldots$$

The author would like to thank Bob Oliver and Ran Levi for their patient and helpful suggestions for the work in this section.

For a fusion system $\mathcal{F}$ over a finite p-group S, let $\mathcal{M}or_{\mathcal{F}}(S,S) \subseteq \mathcal{M}or(S,S)$ be the subset of $\mathcal{M}or(S,S)$ whose irreducible components are of the form $S \times_{(P,\varphi)} S$, for $P \leq S$ and $\varphi \in Hom_{\mathcal{F}}(P,S)$. Then $\mathcal{M}or_{\mathcal{F}}(S,S)$ is clearly a submonoid of $\mathcal{M}or(S,S)$. We let $A_{\mathcal{F}}(S,S)$ denote the corresponding subgroup of $A(S,S)$, i.e. $A_{\mathcal{F}}(S,S) \leq A(S,S)$ is the subgroup of $A(S,S)$ generated by the basis elements $S \times_{(P,\varphi)} S$, for $P \leq S$ and $\varphi \in Hom_{\mathcal{F}}(P,S)$.

Lemma 4.2.1. *The submonoid $\mathcal{M}or_{\mathcal{F}}(S,S) \leq \mathcal{M}or(S,S)$ is closed under the product operation*

$$\mathcal{M}or(S,S) \times \mathcal{M}or(S,S) \to \mathcal{M}or(S,S),\ (\Omega, \Lambda) \mapsto \Omega \circ \Lambda.$$

Consequently $A_{\mathcal{F}}(S,S)$ is a subring of $A(S,S)$.

Proof: By the double coset formula, the product of two basis elements $S \times_{(P,\varphi)} S$ and $S \times_{(P',\varphi')} S$ can be written as a sum of basis elements $S \times_{(Q,\psi)} S$, where each ψ is obtained from φ and φ' by composition, restriction and conjugation in S. By the axioms of a fusion system, it follows that $\psi \in Hom_{\mathcal{F}}(Q,S)$. □

After p-completion, we get a subring $A_{\mathcal{F}}(S,S)^{\wedge}_p$ of $A(S,S)^{\wedge}_p$. We let the subring $\{BS_+, BS_+\}_{\mathcal{F}}$ of $\{BS_+, BS_+\}$ denote the image of $A_{\mathcal{F}}(S,S)^{\wedge}_p$ under the surjection $\alpha^{\wedge}_p$ of remark 4.1.3. Then $\{BS_+, BS_+\}_{\mathcal{F}}$ is a free $\mathbf{Z}^{\wedge}_p$-module on the basis elements $\Sigma^{\infty}_{+} B\varphi \circ tr_P$ as (P,φ) runs through conjugacy classes of $P \leq S$ and $\varphi \in Hom_{\mathcal{F}}(P,S)$.

The properties of the (S,S)-biset Ω constructed by Broto, Levi and Oliver and its induced map in cohomology are described in the proposition that follows.

Proposition 4.2.2. *[7, Prop. 5.5] For any saturated fusion system $\mathcal{F}$ over a p-group S, there is an (S,S)-biset Ω with the following properties:*

(a) $\Omega \in \mathcal{M}or_{\mathcal{F}}(S,S)$

(b) *For each $P \leq S$ and each $\varphi \in Hom_{\mathcal{F}}(P,S)$, the restrictions $\Omega|_{(P,S)}$ and $\Omega|_{(\varphi,S)}$ are isomorphic as (P,S)-bisets.*

(c) $|\Omega|/|S| \equiv 1$ *(mod p).*

Furthermore, for any biset Ω which satisfies these properties, the induced map $[\Omega]^*$ in cohomology is a $H^*(\mathcal{F})$-linear idempotent in $End_{\mathcal{U}}(H^*(BS))$ with

$$Im[H^*(BS) \xrightarrow{[\Omega]^*} H^*(BS)] = H^*(\mathcal{F}).$$

We now proceed by a sequence of lemmas about (S,S)-bisets.

Lemma 4.2.3. *Let Ω and Λ be two (S,S)-bisets satisfying properties (a),(b) and (c) in proposition 4.2.2. Then $\Omega \circ \Lambda$ also satisfies the properties. In particular, any power of Ω satisfies these properties.*

Proof: That $\Omega \circ \Lambda$ satisfies property (a) is lemma 4.2.1. To see that $\Omega \circ \Lambda$ satisfies property (b), we note that

$$\begin{aligned}(\Omega \circ \Lambda)|_{(\varphi,S)} &= (\Omega \circ \Lambda) \circ (P \times_{(\varphi,P)} S) \\ &= \Omega \circ (\Lambda \circ (P \times_{(\varphi,P)} S)) \\ &= \Omega \circ (\Lambda|_{(\varphi,S)}) \\ &= \Omega \circ (\Lambda|_{(P,S)}) \\ &= \Omega \circ (\Lambda \circ (P \times_{(\iota_P,P)} S)) \\ &= (\Omega \circ \Lambda) \circ (P \times_{(\iota_P,P)} S) \\ &= (\Omega \circ \Lambda)|_{(P,S)}.\end{aligned}$$

Finally,

$$|\Omega \circ \Lambda| = |\Omega \times_S \Lambda| = |\Omega| \cdot |\Lambda|/|S|,$$

so

$$|\Omega \circ \Lambda|/|S| = (|\Omega|/|S|)(|\Lambda|/|S|) \equiv 1 \cdot 1 \equiv 1 \ (mod\ p),$$

Proving that $\Omega \circ \Lambda$ satisfies (c). The final statement now follows by induction. □

Lemma 4.2.4. *Let $\Omega \in A(S,S)$. Then there exists an $M > 0$ such that Ω^M is idempotent mod p.*

Proof: Let $\bar{\Omega}$ denote the image of Ω under the projection $A(S,S) \to A(S,S)/pA(S,S)$. It is equivalent to show that $\bar{\Omega}^M$ is idempotent for some $M > 0$. Now, $A(S,S)$ is a finitely generated **Z**-module and hence $A(S,S)/pA(S,S)$ is finite. Consider the sequence

$$\bar{\Omega}, \bar{\Omega}^2, \bar{\Omega}^3, \ldots$$

in $A(S,S)/pA(S,S)$. By the pigeonhole principle, there must be numbers, $N, t > 0$ such that $\bar{\Omega}^N = \bar{\Omega}^{N+t}$. It follows that

$$\bar{\Omega}^n = \bar{\Omega}^{n+t}$$

for all $n \geq N$. Now take $m \geq 0$ such that $mt > N$ and put $M := mt$. Then

$$\bar{\Omega}^{2M} = \bar{\Omega}^{M+mt} = \bar{\Omega}^{M+(m-1)t} = \cdots = \bar{\Omega}^{M+t} = \bar{\Omega}^M.$$

□

The following two lemmas were suggested to the author by Bob Oliver through private correspondence. Although they hold for any p-torsion-free ring, we will state them only for $A(S,S)$.

Lemma 4.2.5. *If $\Omega \in A(S,S)$ is an idempotent mod p^k, where $k > 0$, then Ω^p is an idempotent mod p^{k+1}.*

Proof: For notational convenience, put $q := p^k$. By assumption we can write

$$\Omega^2 = \Omega + q\Lambda \tag{4.3}$$

for some $\Lambda \in A(S,S)$. It follows that

$$\Omega^2 + q\Omega\Lambda = \Omega(\Omega + q\Lambda) = \Omega^3 = (\Omega + q\Lambda)\Omega = \Omega^2 + q\Lambda\Omega,$$

so

$$q\Omega\Lambda = q\Lambda\Omega.$$

Since $A(S,S)$ is torsion-free as a **Z**-module, we deduce that Ω and Λ commute. This allows us to apply the binomial formula to (4.3) and get

$$\Omega^{2p} = \Omega^p + \binom{p}{1}\Omega^{p-1}q\Lambda + \binom{p}{2}\Omega^{p-2}q^2\Lambda^2 + \cdots + \binom{p}{p-1}\Omega q^{p-1}\Lambda^{p-1} + q^p\Lambda^p.$$

A brief inspection of the coefficients occurring on the right hand side, taking into account that p divides q since $k > 0$, shows that we can therefore write

$$\Omega^{2p} = \Omega^p + pq\Lambda'$$

for some $\Lambda' \in A(S,S)$. Since $pq = p^{k+1}$ we finally deduce that Ω^p is idempotent *mod* p^{k+1}. □

Lemma 4.2.6. *If $\Omega \in A(S,S)$ is idempotent mod p, then the sequence*

$$\Omega, \Omega^p, \Omega^{p^2}, \ldots$$

converges in $A(S,S)^\wedge_p$. Furthermore the limit is idempotent.

Proof: By lemma 4.2.5 and induction, Ω^{p^k} is idempotent *mod* p^{k+1} for each $k \geq 0$. That is to say, that

$$\Omega^{2p^k} - \Omega^{p^k} \in p^{k+1}A(S,S), \tag{4.4}$$

for $k \geq 0$. By induction it follows that

$$\Omega^{np^k} - \Omega^{p^k} \in p^{k+1}A(S,S),$$

for $k \geq 0$, $n > 0$. In particular

$$\Omega^{p^l} - \Omega^{p^k} \in p^{k+1}A(S,S),$$

when $l \geq k > 0$, so

$$\Omega, \Omega^p, \Omega^{p^2}, \ldots$$

is a Cauchy sequence in the p-adic topology of $A(S,S)$. Hence, it converges to a unique element $\hat{\Omega} \in A(S,S)^\wedge_p$. Since the multiplication in $A(S,S)$ is continuous with respect to the p-adic topology, $\hat{\Omega}^2$ is the limit of the sequence

$$\Omega^2, \Omega^{2p}, \Omega^{2p^2}, \ldots$$

Idempotence of $\hat{\Omega}$ now follows from taking the limit of (4.4) over k. □

We can now prove the main result of this section.

Proposition 4.2.7. *For any saturated fusion system $\mathcal{F}$ over a p-group S, there is a stable self-map $\omega \in \{BS_+, BS_+\}$ with the following properties:*

(a) $\omega \in \{BS_+, BS_+\}_{\mathcal{F}}$

(b) *For each $P \leq S$ and each $\varphi \in Hom_{\mathcal{F}}(P, S)$, the restrictions $\omega|_{\Sigma^\infty_+ BP}$ and $\omega \circ \Sigma^\infty_+ B\varphi$ are homotopic as maps $\Sigma^\infty_+ BP \to \Sigma^\infty_+ BS$.*

(c) *ω is idempotent.*

Proof: Let Ω be an (S, S)-biset given by proposition 4.2.2. By lemmas 4.2.4 and 4.2.3, we may assume that Ω is an idempotent *mod p*. By lemma 4.2.6, the sequence

$$\Omega, \Omega^p, \Omega^{p^2}, \ldots$$

converges to an idempotent $\hat{\Omega} \in A(S, S)^\wedge_p$. Let ω be the image of $\hat{\Omega}$ under the surjection

$$A(S, S)^\wedge_p \twoheadrightarrow \{BS_+, BS_+\}.$$

Then ω is idempotent, proving (c). It is not hard to see that $A_{\mathcal{F}}(S, S)$ is a closed subspace of $A(S, S)$ in the p-adic topology and hence that $A_{\mathcal{F}}(S, S)^\wedge_p$ is a closed subspace of $A(S, S)^\wedge_p$. Since each Ω^n is in $A_{\mathcal{F}}(S, S)$ by lemma 4.2.1, it follows that the limit $\hat{\Omega}$ is in $A_{\mathcal{F}}(S, S)^\wedge_p$ and hence that $\omega \in \{S_+, S_+\}_{\mathcal{F}}$, proving (a).

By property (b) of proposition 4.2.2, we have

$$\Omega \circ (P \times_{(\varphi, P)} S) = \Omega \circ (P \times_{(\iota_P, P)} S)$$

and consequently

$$\Omega^{p^k} \circ (P \times_{(\varphi, P)} S) = \Omega^{p^k} \circ (P \times_{(\iota_P, P)} S),$$

for all $k \geq 0$. Since the pairing

$$\circ \colon A(S, S) \times A(P, S) \to A(P, S)$$

is continuous in the p-adic topology on the relevant $\mathbf{Z}$-modules, we can take limits to get

$$\hat{\Omega} \circ (P \times_{(\varphi, P)} S) = \hat{\Omega} \circ (P \times_{(\iota_P, P)} S).$$

Applying α, we now get

$$\omega \circ \Sigma^\infty_+ B\iota_P \simeq \omega \circ \Sigma^\infty_+ \varphi,$$

proving (b). □

The following lemma is a homotopy version of the $H^*(\mathcal{F})$-linearity in proposition 4.2.2. The argument in the proof given here is the same as in the proof in [7], but formally lifted to the stable homotopy category.

Lemma 4.2.8. *Let ω be a stable selfmap $\Sigma^\infty_+ BS \to \Sigma^\infty_+ BS$ satisfying properties (a),(b) and (c) in proposition 4.2.7. Then ω satisfies the Frobenius reciprocity relation*

$$(\omega \wedge \omega) \circ \Delta = (\omega \wedge 1) \circ \Delta \circ \omega,$$

where $\Delta \colon \Sigma^\infty_+ BS \to \Sigma^\infty_+ BS \wedge \Sigma^\infty_+ BS$ is the image of the diagonal of BS under the infinite suspension functor Σ^∞_+.

Proof: For the sake of notational convenience, we will let $B\varphi$ denote the stable map $\Sigma_+^\infty B\varphi$ in this proof.

By condition (a), we can write ω as a linear combination with $\mathbf{Z}_p^\wedge$-coefficients of maps $[S \times_{(P,\varphi)} S] \in \{BS_+, BS_+\}$, where $\varphi \in Hom_{\mathcal{F}}(P,S)$. For such a map $[S \times_{(P,\varphi)} S]$, we have by condition (b):

$$\omega \circ B\varphi = \omega \circ B\iota_P, \tag{4.5}$$

where ι_P is the inclusion $P \leq S$. We will take advantage of this and the fact [1] that the transfer tr_P of the inclusion ι_P satisfies the Frobenius relation

$$(1 \wedge tr_P) \circ \Delta_S = (B\iota_P \wedge 1) \circ \Delta_P \circ tr_P, \tag{4.6}$$

where Δ_P and Δ_S are the diagonals of $\Sigma_+^\infty BP$ and $\Sigma_+^\infty BS$ respectively. We will also use the fact that since $B\varphi$ has a desuspension, it commutes with the diagonals as follows

$$\Delta_S \circ B\varphi = (B\varphi \wedge B\varphi) \circ \Delta_P. \tag{4.7}$$

Now,

$$\begin{aligned}
(\omega \wedge 1) \circ \Delta_S \circ [S \times_{(P,\varphi)} S] &= (\omega \wedge 1) \circ \Delta_S \circ B\varphi \circ tr_P \\
&\overset{(4.7)}{=} (\omega \wedge 1) \circ (B\varphi \wedge B\varphi) \circ \Delta_P \circ tr_P \\
&= ((\omega \circ B\varphi) \wedge B\varphi) \circ \Delta_P \circ tr_P \\
&\overset{(4.5)}{=} ((\omega \circ B\iota_P) \wedge B\varphi) \circ \Delta_P \circ tr_P \\
&= (\omega \wedge B\varphi) \circ (B\iota_P \wedge 1) \circ \Delta_P \circ tr_P \\
&\overset{(4.6)}{=} (\omega \wedge B\varphi) \circ (1 \wedge tr_P) \circ \Delta_S \\
&= (\omega \wedge (B\varphi \circ tr_P)) \circ \Delta_S \\
&= (\omega \wedge [S \times_{(P,\varphi)} S] \circ \Delta_S.
\end{aligned}$$

and summing over the different $[S \times_{(P,\varphi)} S]$, we get the desired result. □

Remark 4.2.9. By analogy with proposition 4.2.2, for any stable selfmap $\omega \in \{BS_+, BS_+\}$, that satisfies the properties in proposition 4.2.7, the induced map in cohomology ω^* is an idempotent in $End(H^*(BS))$, is $H^*(\mathcal{F})$-linear and a homomorphism of modules over the Steenrod algebra; and

$$Im[H^*(BS) \xrightarrow{\omega^*} H^*(BS)] = H^*(\mathcal{F}).$$

Only the last statement is non-obvious at this point. The proof proceeds along the same lines as in the proof of proposition 4.2.2 in [7].

4.3 Frobenius transfers induced by p-local finite groups

In this section we discuss the stable splitting of BS induced by the stable idempotent ω of the previous section. More precisely, we will apply mapping telescope techniques

[12] to produce a factorisation

$$\omega\colon \Sigma^\infty_+ BS \xrightarrow{f} \Sigma^\infty_+ |\mathcal{L}|^\wedge_p \xrightarrow{t} \Sigma^\infty_+ BS$$

and discuss the problems involved with showing that this splitting yields a Frobenius transfer triple over S. For the sake of notational convenience, we will denote the suspension spectrum functor $\Sigma^\infty_+(-)$ by $\widetilde{(-)}$ in this section.

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S and let ω be an idempotent in $\{BS_+, BS_+\}$ as given by proposition 4.2.7.

Let M be the infinite mapping telescope of the idempotent ω. In other words,

$$M := \underrightarrow{\text{Holim}} \left(\widetilde{BS} \xrightarrow{\omega} \widetilde{BS} \xrightarrow{\omega} \widetilde{BS} \xrightarrow{\omega} \dots \right),$$

We denote the structure map of the homotopy colimit by $\sigma\colon \widetilde{BS} \to M$. Since ω is idempotent, we get a factorisation of ω through the homotopy colimit

$$\omega\colon \widetilde{BS} \xrightarrow{\sigma} M \xrightarrow{t'} \widetilde{BS},$$

such that $\sigma \circ t' \simeq id_M$. Thus M is a retract of the p-complete spectrum $\widetilde{BS}$ and is therefore p-complete. Furthermore, the induced map of σ in cohomology is the inclusion

$$Im[H^*(BS) \xrightarrow{\omega^*} H^*(BS)] \hookrightarrow H^*(BS),$$

which by remark 4.2.9 is the inclusion

$$H^*(\mathcal{F}) \hookrightarrow H^*(BS).$$

Consider next the case, in which the fusion system $\mathcal{F}$ belongs to a p-local finite group $(S, \mathcal{F}, \mathcal{L})$. The natural map $\theta\colon BS \to |\mathcal{L}|^\wedge_p$ also induces the inclusion $H^*(\mathcal{F}) \hookrightarrow H^*(BS)$ in cohomology. Hence the composite $\widetilde{\theta} \circ t'\colon M \to \widetilde{|\mathcal{L}|^\wedge_p}$ is an isomorphism in cohomology and since the spectra involved are p-complete, it follows that we have a homotopy equivalence

$$h := \widetilde{\theta} \circ t'\colon M \xrightarrow{\simeq} \widetilde{|\mathcal{L}|^\wedge_p}.$$

Putting $f := h \circ f'$ and $t := t' \circ h^{-1}$, we now get a splitting

$$\omega\colon \widetilde{BS} \xrightarrow{f} \widetilde{|\mathcal{L}|^\wedge_p} \xrightarrow{t} \widetilde{BS},$$

where $f \circ t \simeq id_{\widetilde{|\mathcal{L}|^\wedge_p}}$.

To show that the p-local finite group induces a Frobenius triple it now remains to show that f desuspends to a map $BS \to |\mathcal{L}|^\wedge_p$ and that our splitting satisfies the Frobenius reciprocity relation

$$(1 \wedge t) \circ \Delta_{|\mathcal{L}|^\wedge_p} \simeq (f \wedge 1) \circ \Delta_S \circ t. \tag{4.8}$$

In fact, we will show below that if f does desuspend, then (4.8) follows from lemma 4.2.8. For our purposes, we are not happy with just any desuspension γ of f to get a Frobenius transfer triple $(\gamma, t, |\mathcal{L}|_p^\wedge)$ over S associated to $(S, \mathcal{F}, |\mathcal{L}|_p^\wedge)$. We demand that f should desuspend to the natural inclusion $\theta\colon BS \to |\mathcal{L}|_p^\wedge$ of BS into the classifying space of $(S, \mathcal{F}, \mathcal{L})$. The problem with relating $\widetilde{\theta}$ to f is that the equivalence $h\colon M \xrightarrow{\cong} \widetilde{|\mathcal{L}|_p^\wedge}$ is not natural. The missing ingredient to achieve the desired level of naturality is the truth of the following conjecture.

Conjecture 4.3.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then the stable self-map $\omega \in \{BS_+, BS_+\}$ of proposition 4.2.7 can be chosen so that $\widetilde{\theta} \circ \omega \simeq \widetilde{\theta}$.*

This conjecture is plausible, since θ has the property that

$$\theta \circ B\varphi \simeq \theta \circ B\iota_P$$

for all $\varphi \in Hom_{\mathcal{F}}(P, S)$ and $\omega \in \{BS_+, BS_+\}_{\mathcal{F}}$. These properties in themselves do not imply the conjecture. As an example, the set Ω of proposition 4.2.2 has the analogous properties (a) and (b) of the proposition, yet it does not follow that Ω is idempotent. However, the failure of $\widetilde{\theta} \circ \omega \simeq \widetilde{\theta}$ to hold is similar in nature to the failure of idempotence of Ω. Therefore it is entirely possible that the idempotent ω, as constructed in proposition 4.2.7 to get an idempotent version of Ω, does satisfy the conjecture.

In order to prove the conjecture, one may be able amend lemma 5.4 of [7] by carefully studying the inductive step to be able to produce an Ω in proposition 4.2.2 satisfying $\widetilde{\theta} \circ [\Omega] \simeq \widetilde{\theta}$. However, this Ω will certainly have to be an element of $A(S, S)$ or even $A(S, S)_p^\wedge$ rather than an actual (S, S)-biset.

Proposition 4.3.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. If conjecture 4.3.1 is true, then there is a transfer map $t\colon \Sigma_+^\infty |\mathcal{L}|_p^\wedge \to \Sigma_+^\infty BS$ such that $(\theta, t, |\mathcal{L}|_p^\wedge)$ is a Frobenius transfer triple.*

Proof: By [7, Prop. 5.2], the map θ makes $H^*(BS)$ into a finitely generated $H^*(\mathcal{F})$-module and is therefore a homotopy monomorphism by lemma 3.1.2. By [7, Prop. 1.12] there is a surjection $S \twoheadrightarrow \pi_1(|\mathcal{L}|_p^\wedge)$. In particular, $|\mathcal{L}|_p^\wedge$ has finite fundamental group.

Let ω be a stable self-map of BS as prescribed by proposition 4.2.7, such that $\widetilde{\theta} \circ \omega \simeq \widetilde{\theta}$. Let M be the mapping telescope of ω and let

$$\omega\colon \widetilde{BS} \xrightarrow{\sigma} M \xrightarrow{t'} \widetilde{BS},$$

be the corresponding stable splitting. Since $\widetilde{\theta} \circ \omega \simeq \widetilde{\theta}$, the universal mapping property of M gives us a map

$$u\colon M \to \widetilde{|\mathcal{L}|_p^\wedge}$$

such that $u \circ \sigma \simeq \tilde{\theta}$. Since the maps σ and θ both induce the inclusion $H^*(\mathcal{F}) \hookrightarrow H^*(BS)$ in cohomology, we see that u is an isomorphism in cohomology. Since the spectra involved are p-complete, it follows that u is a homotopy equivalence. Putting $t := t' \circ u^{-1}$, we now get a splitting

$$\omega\colon \widetilde{BS} \xrightarrow{\widetilde{\theta}} \widetilde{|\mathcal{L}|_p^\wedge} \xrightarrow{t} \widetilde{BS},$$

with $\widetilde{\theta} \circ t \simeq id_{\widetilde{|\mathcal{L}|^{\wedge}_p}}$.

Since the stable map $\widetilde{\theta}$ has a desuspension θ, it commutes with the diagonals Δ_S and $\Delta_{|\mathcal{L}|^{\wedge}_p}$ of $\widetilde{BS}$ and $\widetilde{|\mathcal{L}|^{\wedge}_p}$, respectively, in the following sense:

$$\Delta_{|\mathcal{L}|^{\wedge}_p} \circ \widetilde{\theta} \simeq (\widetilde{\theta} \wedge \widetilde{\theta}) \circ \Delta_S.$$

This allows us to deduce

$$(1 \wedge t) \circ \Delta_{|\mathcal{L}|^{\wedge}_p} \simeq (\tilde{\theta} \wedge 1) \circ \Delta_S \circ t$$

from

$$(\omega \wedge \omega) \circ \Delta_S \simeq (\omega \wedge 1) \circ \Delta_S \circ \omega \tag{4.9}$$

as follows. Applying $(\widetilde{\theta} \wedge 1) \circ - \circ t$ to the left hand side of (4.9) and rewriting, we get

$$\begin{aligned}
(\widetilde{\theta} \wedge 1) \circ (\omega \wedge \omega) \circ \Delta_S \circ t &\simeq ((\widetilde{\theta} \circ t) \wedge t) \circ (\widetilde{\theta} \wedge \widetilde{\theta}) \circ \Delta_S \circ t \\
&\simeq (1_{\widetilde{|\mathcal{L}|^{\wedge}_p}} \wedge t) \circ \Delta_{|\mathcal{L}|^{\wedge}_p} \circ \widetilde{\theta} \circ t \\
&\simeq (1_{\widetilde{|\mathcal{L}|^{\wedge}_p}} \wedge t) \circ \Delta_{|\mathcal{L}|^{\wedge}_p}
\end{aligned}$$

Doing the same with the right hand side yields

$$\begin{aligned}
(\widetilde{\theta} \wedge 1) \circ (\omega \wedge 1) \circ \Delta_S \circ \omega \circ t &\simeq ((\widetilde{\theta} \circ t \circ \widetilde{\theta}) \wedge 1) \circ \Delta_S \circ (t \circ \widetilde{\theta} \circ t) \\
&\simeq (\widetilde{\theta} \wedge 1) \circ \Delta_S \circ t.
\end{aligned}$$

By (4.9) we can take the last two equations together to form a new equation

$$(1 \wedge t) \circ \Delta_{|\mathcal{L}|^{\wedge}_p} \simeq (\tilde{\theta} \wedge 1) \circ \Delta_S \circ t.$$

□

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