Morse Inequalities, a Probabilistic Approach

by

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Abstract

In this thesis we give a probabilistic proof of the Morse inequalities in the nondegenerate and degenerate case. For the nondegenerate case the kernel associated with the Witten Laplacian has an expression via the Malliavin calculus. The first step is the analysis of this heat kernel at a point away the critical set. Using Markov property, an iteration procedure and estimates on exit times from balls, everything is reduced to the estimation of a solution to a parabolic initial-boundary problem on a ball in the Euclidean space. We achieve that by constructing a supersolution. For the case the point is close to the critical set, we use an integration by parts in the Malliavin calculus and split the analysis for paths staying inside a given distance from the critical point or exiting the corresponding ball. For the paths exiting, again an iterative Markov property argument reduces the problem to a parabolic initial-boundary value problem that can be handled by the construction of the supersolution mentioned above. For the quantity involving the paths staying inside a given ball around the critical point, we can reverse the argument, this time with the Euclidean space playing the role of the original manifold and reduce the problem to one in the Euclidean settings. This turns out to be an elementary harmonic oscillator problem that finishes the argument.

The case of the degenerate Bott-Morse function requires a bit more work due to the fact that the geometry near the critical submanifolds is in general not trivial. After some standard constructions, we have two choices of the connection around critical submanifolds. One is the Levi-Civita and the other is Bismut's connection. The main step in this analysis is to prove that the heat kernels of certain operators with respect to Levi-Civita connection and the Bismut connection stay bounded when the parameters involved become large. This is achieved by a fiberwise version of the argument given in the nondegenerate case. Using the boundedness, one can prove the basic comparison. Finally, the rest is just a fiberwise harmonic oscillator problem.

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Contents

Chapter 1

Introduction

Let M be a d-dimensional compact manifold and $h : M \to \mathbb{R}$ a Morse function with isolated critical points. By the Morse Lemma one can find local coordinates around the critical points such that the function h is quadratic in each of these. For such neighborhoods we transfer the Euclidean metric and complete with an arbitrary metric on the rest of the manifold. For α a positive number we consider on $\bigwedge(M)$ the operators

$$d^{\alpha h} = e^{-\alpha h} d e^{\alpha h}, \ \delta^{\alpha h} = e^{\alpha h} \delta e^{-\alpha h}, \tag{1.0.1}$$

and the Witten Laplacian

$$\Box^{\alpha} = d^{\alpha h} \delta^{\alpha h} + \delta^{\alpha h} d^{\alpha h}. \tag{1.0.2}$$

We consider $p_k^{\alpha}(t, x, y)$ the kernel of the operator $e^{-t\Box^{\alpha}/2}$ acting on k-forms and take

$$Q_k^{\alpha}(t) = \int_M \operatorname{Tr} p_k^{\alpha}(t, x, x) dx,$$

where Tr stands for the trace on $\bigwedge^k(M)$ and dx is the volume measure on M.

The starting point in proving Morse inequalities is the inequality

$$Q_k^{\alpha}(t) - Q_{k-1}^{\alpha}(t) + \cdots + (-1)^k Q_0^{\alpha}(t) \geq B_k - B_{k-1} + \cdots + (-1)^k B_0$$

which is true for any $t > 0, \alpha > 0$, where B_k stands for the k^{th} Betti number of the manifold. From this, the goal is to show that, for any fixed $t > 0$ the limit when α tends to infinity of the quantity $Q_k^{\alpha}(t)$ is exactly m_k, the number of critical points of index k. Then it follows that

Theorem (Non-degenerate Morse Inequalities).

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq B_k - B_{k-1} + \cdots + (-1)^k B_0, \tag{1.0.3}$$

with equality for $k = d$.

In order to achieve this program, we follow a probabilistic route. We interpret the heat kernel $p_k^{\alpha}(t, x, y)$ via the Malliavin calculus and we analyze separately the cases of $p_k^\alpha(t,x,x)$ for x away the critical points and for x close to the critical points. What we achieve by doing this is that as α tends to infinity, $Q_k^\alpha(t)$ is computed only as the integral of $p_k^\alpha(t,x,x)$ with x close to the critical points and that $p_k^\alpha(t,x,x)$ is an integral over the paths starting inside some small neighborhood of the critical points and staying inside there up to time t. Now, we are left with a local computation, and more than that, one within an Euclidean ball.

The last step in achieving the Morse inequalities is to reverse the argument given for the manifold M in reducing the computation of $Q_k^\alpha(t)$ to a local one, and to extend the computation from one in an Euclidean ball to a certain computation on the whole Euclidean space. And this last thing comes down to nothing but a harmonic oscillator calculation.

We now describe the degenerate case. Let h be a Bott-Morse function on the compact manifold M with critical connected sub-manifolds M_1, M_2, ..., M_l. The degenerate Morse Lemma says that, there are disjoint tubular neighborhoods E_1, E_2, ..., E_l of $M_1, M_2, \ldots, M_l$, open sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_l$ such that $M_i \subset \mathcal{V}_i \subset E_i$, $i = 1, \ldots, l$, and Euclidean bundles $E_i^\pm$ of dimensions $\nu_i^\pm$, such that $E_i = E_i^+ \oplus E_i^-$. Using this we can extend the metric to E_i by declaring the sub-bundles $E_i^\pm$ orthogonal to each other. In these data the function h restricted to $\mathcal{V}_i$, has the expression

$$h(z_i) = h|_{M_i} + \frac{1}{2}\left(|y_i^+|^2 - |y_i^-|^2\right) \tag{1.0.4}$$

with $y_i^\pm$ the $E_i^\pm$ components of z_i seen as a vector in $(E_i)_{\rho_i(z_i)}$, for $\rho_i : E_i \to M_i$ the canonical projection.

Now, the bundles $E_i^\pm$ can be endowed with compatible vertical connections, and thus each E_i is also endowed with a vertical connection ∇^V. Using this connection one can lift $T_{\rho_i(z_i)}(M_i)$ to a subspace $T_{z_i}^H(E_i)$ of $T_{z_i}(E_i)$. Naturally identifying $(E_i)_{\rho(z_i)}$ with a subspace $T_{z_i}^V(E_i)$ of $T_{z_i}(E_i)$, we get the decomposition $T_{z_i}(E_i) = T_{z_i}^H(E_i) \oplus T_{z_i}^V(E_i)$. Now, if we choose a metric on $T(M)$, then we can lift it to $T^H(E_i)$. Transferring the metric from E_i to $T^V(E_i)$ and declaring the spaces $T^H(E_i)$, $T^V(E_i)$ orthogonal to each other we get a metric on $T(E_i)$.

Given this metric on E_i and the Levi-Civita connection on M_i one can construct a connection ∇^B on $T(E_i)$ (sometimes called Bismut connection), by defining the associated parallel transportation. One can do this by first taking a curve α in $T(E_i)$ and β its projection on M_i. For a vector at $\alpha(0)$, consider its corresponding vertical and horizontal parts in $T_{\beta(0)}M_i$, $(E_i)_{\beta(0)}$, take their parallel transportation along β and lift these last vectors at the end point of α. Then declare the vector with these components to be the parallel transportation of the original one.

The main advantage of this connection is that it preserves the horizontal and vertical parts in E_i, but the main unpleasant feature comes from the torsion of it, which in general is non-trivial. On the other hand, compared with the Levi-Civita connection on $T(E_i)$, this new connection preserves many important properties, as for example, the Laplacians with respect to both connections are the same on functions, the Hessian of the function h on $\mathcal{V}_i$ is the same in both connections and, another important one, they coincide on $T_z(E_i)$ for any $z \in M_i$.

The main idea in this context is the following. Take an operator L on $\bigwedge(M)$ of the form

$$L^\alpha = -\Delta^\nabla + \alpha^2|\text{grad}h|^2 - \alpha\Delta h + 2\alpha(\text{hess}h) + \sum_{j=1}^{d} B(E_j)\nabla_{E_j} + C \tag{1.0.5}$$

where the data is subject to the following:

1. The connection ∇ satisfies
   (a) compatibility with the metric on M;
   (b) ∇-Laplacian on functions is the same as the standard Laplacian;
   (c) the Hessian of the function h is the same as the Hessian with respect to Levi-Civita connection.
2. B, C are extensions of tensors with $B(X)$ a skew-symmetric map.

Under these conditions, if $p^{L^\alpha}(t,x,y)$ is the heat kernel associated to the operator L, then there is a constant $K(t,B,C))$ depending on t and the data B, C in the operator L^α, such that for large α

$$\int_M \|p^{L^\alpha}(t,z,z)\|_{\bigwedge(M)} dz \leq K(t,B,C). \tag{1.0.6}$$

This is the key estimate in our approach. In fact, the proof is exactly on the same line of ideas as in the non-degenerate case. The same reasoning as in there takes the cases, z close or away the critical set. The single point one has to make is that there is a representation of the Brownian motion near critical set as an independent composition of the vertical and horizontal ones. This boils down the computation to one on fibers, which is precisely one in Euclidean space. From here, we just use the results gotten in the non-degenerate case.

For a given connection ∇ set

$$\begin{gathered} d_z^\nabla = \sum_{j=1}^{d} (E_j^*)_z \wedge \nabla_{(E_j)_z}, \\ d^{\nabla,\alpha h} = e^{-\alpha h} d^\nabla e^{\alpha h}, \quad \delta^{\nabla,\alpha h} = e^{\alpha h}\delta^\nabla e^{-\alpha h} \end{gathered} \tag{1.0.7}$$

where δ^∇ is the dual of d^∇. Define then

$$\Box^{\nabla,\alpha} = d^{\nabla,\alpha h}\delta^{\nabla,\alpha h} + \delta^{\nabla,\alpha h} d^{\nabla,\alpha h} \tag{1.0.8}$$

With these at hand, for a given $r > 0$, choose a function φ_r that is 1 on each $\{z_i \in E_i; |y| \leq r/2\}$ and 0 outside $\{z_i \in E_i; |y| \geq r\}$. Then consider the connection $\nabla_r = \varphi_r \nabla^B + (1-\varphi_r)\nabla^{LC}$ where ∇^{LC} stands for the Levi-Civita connection. Set

$\diamond_r^\alpha = \Box^{\nabla_r,\alpha}$. Using this connection together with a min-max characterization of the eigenvalues and the boundedness, one can prove that

$$\left| \int_M \left(p^{\Box^\alpha}(t,z,z) - p^{\diamond_r^\alpha}(t,z,z)\right) dz \right| \leq C(t)r \tag{1.0.9}$$

for large α. With some work one can show

$$\left| \int_M \left(p^{\diamond_r^\alpha}(t,z,z) - p^{\Box_r^\alpha}(t,z,z)\right) dz \right| \leq C(t)r \tag{1.0.10}$$

for large α with the notation

$$\Box_r^\alpha = -\Delta^{\nabla_r} + \alpha^2|\text{grad}h|^2 - \alpha\Delta h + 2\alpha(\text{hess}h) + \varphi_r^2 D^* R^{\nabla^B} + (1-\varphi_r)^2 R^{\nabla^{LC}}. \tag{1.0.11}$$

Note here that, in fact, near the critical submanifolds this operator is given entirely in terms of the Bismut connection, thus the computations can be reduced to one on fibers.

Using (1.0.9), (1.0.10), and simple analysis, as it was done in the case of the non-degenerate case, we arrive at

$$\limsup_{\alpha\to\infty} \left| \int_M \operatorname{Tr} p_k^{\Box_r^\alpha}(t,z,z)dz - \sum_{i=1}^{l} \int_{M_i} p_{k-\nu_i^-}^{i,-}(t,x,x)dx \right| =$$

$$\limsup_{\alpha\to\infty} \left| \int_M \operatorname{Tr} p_k^{L^{\nabla,\alpha}}(t,z,z)dz - \sum_{i=1}^{l} \operatorname{Tr} e^{-t\Box_{k-\nu_i^-}^{i,-}} \right| \leq C(t)r$$

where in here the operator $\Box_k^{i,-} = (d_{M_i}^- + \delta_{M_i}^-)^2$, $d_{M_i}^-$ is the differential operator acting on $\bigwedge^k(T^*(M_i) \otimes o(E_i^-))$, $o(E_i^-)$ the orientation bundle of E_i^- and $p_k^{i,-}(t,x,y)$ stands for the heat kernel of the operator $\Box_k^{i,-}$. Since the quantities under the limit are independent of r, we just obtained

$$\lim_{\alpha\to\infty} \int_M \operatorname{Tr} p_k^{\Box_r^\alpha}(t,z,z)dz = \sum_{i=1}^{l} \int_{M_i} \operatorname{Tr} p_{k-\nu_i^-}^{i,-}(t,x,x)dx = \sum_{i=1}^{l} \operatorname{Tr} e^{-t\Box_{k-\nu_i^-}^{i,-}}. \tag{1.0.12}$$

From here, if we let t tend to infinity, we get

Theorem (Degenerate Morse Inequalities).

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq B_k - B_{k-1} + \cdots (-1)^k B_0 \tag{1.0.13}$$

where

$$m_k = \sum_{i=1}^{l} \dim H^{k-\nu_i^-}(M_i; o(E_i^-)),$$

with $H^k(M_i; o(E_i^-))$ standing for the dimension of the cohomology group of M_i twisted by the orientation bundle of E_i^-. This inequality becomes equality for $k = d$.

Chapter 2

Non-degenerate Morse Inequalities

Given a compact manifold M endowed with a Morse function $h : M \to \mathbb{R}$, we give a proof of the Morse inequalities using the Witten deformation of the De Rham complex. Our approach is based on the analysis of the heat kernel associated with the Witten Laplacian acting on forms.

2.1 The Basic Inequality and The Witten Laplacian

In this section we discuss the operator $\Box^\alpha$ and we write it in an appropriate way for a probabilistic interpretation. We also give here the basic inequality we mentioned in the introduction.

2.1.1 The Basic Inequality

We start with a d dimensional, compact Riemannian manifold M and an arbitrary C^∞ function $h : M \to \mathbb{R}$. Then, consider the operators from definitions (1.0.1), (1.0.2) and $\Box_k^\alpha$ the operator (1.0.2) acting on k forms. We denote by H_k^2 the domain of this operator considered as a self-adjoint operator in $L^2(M, \bigwedge^k(M))$ the space of square integrable sections in the bundle $\bigwedge^k(M)$. Take $p_k^\alpha(t,x,y)$ the C^∞ kernel of the operator $e^{-t\Box_k^\alpha/2}$ acting on L^2 sections in $\bigwedge^k(M)$ and denote

$$Q_k^\alpha(t) = \int_M \operatorname{Tr} p_k^\alpha(t,x,x)dx. \tag{2.1.1}$$

where the trace is the standard trace on $\bigwedge^k(M)$. By well known results [8, chapter 2] (or Appendix B), such a kernel always exists. Moreover if λ_i is an increasing enumeration of the spectrum of $\Box_k^\alpha$ as a self-adjoint operator on $L^2(M, \bigwedge^k(M))$ with all the multiplicities, then one can pick up $\{\varphi^i\}_{i=1,\infty}$ a smooth orthonormal basis of $L^2(M, \bigwedge^k(M))$ with $\Box_k^\alpha \varphi^i = \lambda_i \varphi^i$. For such a base one can write the formula for

$p_k^\alpha(t,x,y) : \bigwedge_y^k(M) \to \bigwedge_x^k(M)$

$$p_k^\alpha(t,x,y)\psi_y = \sum_{i=1}^{\infty} e^{-t\lambda_i/2}\langle\psi_y, \varphi_y^i\rangle_y \varphi_x^i,$$

for any $\psi_y \in \bigwedge_y^k(M)$. From this, we immediately deduce that

$$\operatorname{Tr} p_k^\alpha(t,x,x) = \sum_{i=1}^{\infty} e^{-t\lambda_i/2}|\varphi_x^i|_x^2,$$

and that

$$\int_M \operatorname{Tr}_k p_k^\alpha(t,x,x)dx = \sum_{i=1}^{\infty} e^{-t\lambda_i/2} \tag{2.1.2}$$

Now we state the following Theorem due to Bismut [1, Theorem 1.3] which is the starting point for our analysis.

Theorem 2.1.3. *For any $\alpha > 0$, $t > 0$ we have*

$$Q_k^\alpha(t) - Q_{k-1}^\alpha(t) + \cdots + (-1)^k Q_0^\alpha(t) \geq B_k - B_{k-1} + \cdots + (-1)^k B_0$$

with equality for $k = d$, where B_k stands for the k^{th} Betti number of the manifold M.

Proof. For $\lambda \geq 0$ define $\mathcal{E}_k^\lambda = \{\varphi \in H_k^2, \square_k^\alpha\varphi = \lambda\varphi\}$. Then each such space is finite dimensional and for $\lambda > 0$ we have the following exact sequence

$$0 \to \mathcal{E}_0^\lambda \xrightarrow[d_0^{\alpha h}]{} \mathcal{E}_1^\lambda \cdots \xrightarrow[d_{n-1}^{\alpha h}]{} \mathcal{E}_n^\lambda \xrightarrow[d_n^{\alpha h}]{} \cdots \tag{*}$$

In the first place, this is a sequence because the operator $d_k^{\alpha h}$ sends space $\mathcal{E}_k^\lambda$ to $\mathcal{E}_{k+1}^\lambda$ and obviously the composition $d_k^{\alpha h}d_{k-1}^{\alpha h}$ is 0 for any k. To see that this is indeed an exact sequence, one only has to check that, if $\psi \in \mathcal{E}_k^\lambda$ and $d_k^{\alpha h}\psi = 0$ then there exits $\psi' \in \mathcal{E}_{k-1}^\lambda$ such that $\psi = d_{k-1}^{\alpha h}\psi'$. This can be simply done by taking $\psi' = \frac{1}{\lambda}\delta^{\alpha h}\psi$.

Now, to distinguish the action of $d^{\alpha h}$ in the sequence (*) we denote by $d_k^{\alpha h}|_\lambda$ the corresponding operator appearing there. Because of the exactness we have the isomorphism $\mathcal{E}_k^\lambda / \ker d_k^{\alpha h}|_\lambda \simeq \operatorname{Im}(d_k^{\alpha h}|_\lambda) = \ker d_{k+1}^{\alpha h}|_\lambda$ which implies the dimensionality equation,

$$\dim(\mathcal{E}_k^\lambda) = \dim(\ker d_k^{\alpha h}|_\lambda) + \dim(\ker d_{k+1}^{\alpha h}|_\lambda) \tag{**}$$

Now we can restate (2.1.2) in this new frame:

$$Q_k^\alpha(t) = \sum_{\lambda\geq 0} \dim(\mathcal{E}_k^\lambda)e^{-t\lambda/2}.$$

Making the alternated sum and using formula (**) we eventually get,

$$\begin{aligned}\sum_{i=0}^{k}(-1)^{k-i}Q_i^\alpha(t) &= \sum_{i=0}^{k}(-1)^{k-i}\dim(\ker(\Box_i^\alpha)) \\ &\quad + \sum_{\lambda>0}\sum_{i=0}^{k}(-1)^{k-i}(\dim(\ker d_i^{\alpha h}|_\lambda) + \dim(\ker d_{i+1}^{\alpha h}|_\lambda)e^{-t\lambda/2} \\ &= \sum_{i=0}^{k}(-1)^{k-i}\dim(\ker(\Box_i^\alpha)) + \sum_{\lambda>0}\dim(\ker d_{k+1}^{\alpha h}|_\lambda)e^{-t\lambda/2} \\ &\geq \sum_{i=0}^{k}(-1)^{k-i}\dim(\ker(\Box_i^\alpha))\end{aligned}$$

The above inequality becomes equality for $k = d$ because in this case there is no nonzero $\bigwedge^{k+1}(M)$.

Now, standard Hodge theory shows that $\dim(\ker(\Box_k^\alpha))$ is the same as the dimension of the cohomology associated to the twisted De Rham complex, $(\bigwedge^k(M), d^{\alpha h})$. But this last complex is conjugated to the usual De Rham complex, thus both cohomologies are isomorphic, in particular, their dimensions, since are finite, as vector spaces are the same. □

2.1.2 The Witten Laplacian

We now move to expressing $\Box^\alpha$ in terms of known quantities. First of all we mention that $d^{\alpha h} = d + \alpha dh\wedge$ and taking duals of this we also get that $\delta^{\alpha h} = \delta + \alpha i_{\mathrm{grad}h}$ where i_X is the contraction operator by the vector field X. Now we can perform the first steps in our computation as follows

$$\begin{aligned}\Box^\alpha\omega &= (d + \alpha dh\wedge)(\delta\omega + \alpha i_{\mathrm{grad}h}\omega) + (\delta + \alpha i_{\mathrm{grad}h})(d\omega + \alpha dh\wedge\omega) \\ &= d\delta\omega + \alpha d i_{\mathrm{grad}h}\omega + \alpha dh\wedge\delta\omega + \alpha^2 dh\wedge i_{\mathrm{grad}h}\omega \\ &\quad + \delta d\omega + \alpha\delta dh\wedge\omega + \alpha i_{\mathrm{grad}h}d\omega + \alpha^2 i_{\mathrm{grad}h}dh\wedge\omega \\ &= \Box\omega + \alpha^2 dh\wedge i_{\mathrm{grad}h}\omega + \alpha^2 i_{\mathrm{grad}h}dh\wedge\omega \\ &\quad + \alpha(d i_{\mathrm{grad}h} + i_{\mathrm{grad}h}d)\omega + \alpha(\delta dh\wedge + dh\wedge\delta)\omega\end{aligned} \tag{2.1.4}$$

for any form ω. Further

$$dh\wedge i_{\mathrm{grad}h}\omega + i_{\mathrm{grad}h}dh\wedge\omega = dh\wedge i_{\mathrm{grad}h}\omega + i_{\mathrm{grad}h}(dh)\wedge\omega - dh\wedge i_{\mathrm{grad}h}\omega = |\mathrm{grad}h|^2\omega.$$

Now, denote $L_{\mathrm{grad}h} = d i_{\mathrm{grad}h} + i_{\mathrm{grad}h}d$ and its dual by $L^*_{\mathrm{grad}h} = \delta dh\wedge + dh\wedge\delta$. To identify the quantities, we give here the following definition.

Definition 2.1.5.

1. *If ∇ is a connection on M, define the action of ∇_X, for X a vector field on M, to $\wedge(M)$.*

(a) *if f is a function, then $\nabla_X f = Xf$;*

(b) *if ω is a 1-form then we set $\nabla_X\omega$ the 1-form given by:*

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y);$$

for any vector field Y.

(c) *the extension to all forms is given as a derivation, namely*

$$\nabla_X(\omega_1 \wedge \omega_2) = (\nabla_X\omega_1) \wedge \omega_2 + \omega_1 \wedge (\nabla_X\omega_2),$$

for any two forms ω_1, ω_2.

2. *A k-tensor is an assignment*

$$x \in M \to S_x \in L(\underbrace{T_x(M) \times T_x(M) \times \cdots T_x(M)}_{k \text{ times}}, \mathbb{R})$$

such that for any $X_1, X_2, \cdots, X_k \in T(M)$ the map

$$x \to S_x((X_1)_x, (X_2)_x, \cdots, (X_k)_x)$$

is a smooth map.

3. *If S is a $2k$-tensor, we define its action on $\bigwedge(M)$, denoted D^*S such that for any point $x \in M$, D^*S_x is*

$$\sum_{j_1,\cdots,j_{2k}=1}^{d} S_x(E_{j_1}, E_{j_2}, \cdots, E_{j_{2k}})(E^*_{j_1} \wedge i_{E_{j_2}}) \circ \cdots \circ (E^*_{j_{2k-1}} \wedge i_{E_{j_{2k}}}),$$

where $(E_j)_{j=1,d}$ is any orthonormal basis of $T_x(M)$.

4. *If S is a $2k+1$-tensor, we set for $x \in M$, $(D^*S)_x(X_x)$ to be the sum*

$$\sum_{j_1,\cdots,j_{2k}=1}^{d} S_x(X_x, E_{j_1}, E_{j_2}, \cdots, E_{j_{2k}})(E^*_{j_1} \wedge i_{E_{j_2}}) \circ \cdots \circ (E^*_{j_{2k-1}} \wedge i_{E_{j_{2k}}})$$

where $(E_j)_{j=1,d}$ is any orthonormal basis of $T_x(M)$.

Note here that, in fact, the definitions just given do not depend on the basis chosen. Also, the smoothness of the maps $x \to D^*S_x$ and $x \to D^*S(X_x)_x$, if $X \in T(M)$, follows because one can choose a local orthonormal basis around a point.

A 0-tensor is nothing but a function defined on M. This way, its extension to forms is the multiplication by the function.

If S is a 2-tensor, one can show that the extension D^*S is a derivation.

If $h : M \to \mathbb{R}$ is a function, we can define its Hessian by

$$\text{hess}_x h(X_x, Y_x) = \nabla_{X_x}\text{grad}h,$$

for any $X_x, Y_x \in T_x(M)$, where ∇ is the Levi-Civita connection associated with the Riemannian structure on M. For this bilinear map we define its derivation on $\wedge(M)$ and we will call $D^*\text{hess}h$ simply by $\text{hess}h$.

Another example of tensor extension is the extension of the curvature tensor. Namely, if R is the curvature 4-tensor we will denote by D^*R its extension as given above.

Proposition 2.1.6. *Within the notations above we have*

$$L_{\text{grad}h} + L^*_{\text{grad}h} = -\Delta h + 2\text{hess}h. \tag{2.1.7}$$

Proof. If X is a vector field on M then we will denote $L_X = di_X + i_X d$ and L^*_X its adjoint. Then by a well known formula we can express the operator d in terms of the Levi-Civita connection on the manifold as:

$$d_x = \sum_{j=1}^{d} (E_j)^*_x \wedge \nabla_{E_{j_x}},$$

for any orthonormal basis $((E_j)_x)$ at $T_x(M)$. We take such an orthonormal basis locally around a point and we do computations there. Then, using this expression, we can write

$$\begin{aligned}
L_X &= \sum_{j=1}^{d} (E_j^* \wedge \nabla_{E_j} i_X + i_X E_j^* \wedge \nabla_{E_j}) \\
&= \sum_{j=1}^{d} (E_j^* \wedge \nabla_{E_j} i_X + \langle X, E_j\rangle \nabla_{E_j} - i_X E_j^* \wedge \nabla_{E_j}) \\
&= \sum_{j=1}^{d} (E_j^* \wedge i_{\nabla_{E_j} X} + \langle X, E_j\rangle \nabla_{E_j}) \\
&= \nabla_X + \sum_{j,k=1}^{d} \langle \nabla_{E_j} X, E_k\rangle E_j^* \wedge i_{E_k}
\end{aligned}$$

where we have used the fact that $\nabla_X i_Y - i_Y \nabla_X = i_{\nabla_X Y}$ for any two vector fields on M. Thus, taking adjoint and reminding that $\nabla^*_X = -\text{div}(X) - \nabla_X$, we get to

$$\begin{aligned}
L^*_X &= \nabla^*_X + \sum_{j,k=1}^{d} \langle \nabla_{E_j} X, E_k\rangle E_k^* \wedge i_{E_j} \\
&= -\text{div}(X) - \nabla_X + \sum_{j,k=1}^{d} \langle \nabla_{E_j} X, E_k\rangle E_k^* \wedge i_{E_j}
\end{aligned}$$

Adding up what we got, we arrive at

$$L_X + L_X^* = -\mathrm{div}(X) + \sum_{j,k=1}^{d} (\langle \nabla_{E_j} X, E_k \rangle + \langle \nabla_{E_k} X, E_j \rangle) E_j^* \wedge i_{E_k}.$$

Finally one has to replace X by grad$h$ and use the symmetry of the Hessians together with the definition of the Laplacian to get the required statement. $\square$

The above proposition fills in the gap left in the computation (2.1.4). Thus we have the following decomposition of $\square^\alpha$,

$$\square^\alpha = \square + \alpha^2 |\mathrm{grad} h|^2 - \alpha \Delta h + 2\alpha \mathrm{hess} h \tag{2.1.8}$$

2.2 Heat Kernels and Brownian Motion

In this section we show how one gets an expression for the heat kernel in terms of the Brownian motion on M, in fact in terms of the Malliavin calculus on the path space.

2.2.1 Orthonormal Frame Bundle and Laplacians

We refer to [7, Chapter 8] for more notations and elementary things about the orthonormal frame bundle $\mathcal{O}(M)$ over the manifold M.

We begin by reminding here a couple of notations. In the first place

$$\mathcal{O}(M) = \{(x, e(x)), e(x) = (e_1, \cdots, e_d) \text{ orthonormal basis of } T_x M\}.$$

We will identify such a pair $(x, e(x))$ with an isometry $\mathfrak{f}$ from $\mathbb{R}^d$ to $T_x M$. The object $\mathcal{O}(M)$ becomes a smooth bundle over M with the structural group $O(d)$, the group of orthogonal matrices in $\mathbb{R}^d$ and the projection $\pi : \mathcal{O}(M) \to M$ that takes $(x, e(x))$ into x.

The vertical subspace $\mathcal{V}_\mathfrak{f}\mathcal{O}(M)$ of $T_\mathfrak{f}\mathcal{O}(M)$ at $\mathfrak{f}$ consists of $\mathfrak{X} \in T_\mathfrak{f}\mathcal{O}(M)$ with $\pi_* \mathfrak{X} = 0$. The horizontal space, $\mathcal{H}_\mathfrak{f}\mathcal{O}(M)$ at $\mathfrak{f}$ is constructed as follows. First, one defines a horizontal lifting of a curve in M to a curve in $\mathcal{O}(M)$. Having a curve p in M we can define its horizontal lift $\mathfrak{p}(t) = (p(t), (e_1(t), \cdots, e_d(t))$ starting at $\mathfrak{f} = (p(0), (f_1, \cdots f_d))$ by requiring that $e_k(t) = \tau_{p\restriction[0,t]} f_k$ where $\tau_{p\restriction[0,t]}$ denotes the parallel transport along the curve p from $p(0)$ to $p(t)$. Using this notion we can define now the horizontal lift of a vector $X_x \in T_x(M)$. To this end, take any curve p with $p(0) = x$, $\dot{p}(0) = X_x$ and consider its lift $\mathfrak{p}$ starting from $\mathfrak{f}$. Then by definition set $\mathfrak{H}_\mathfrak{f}(X_x) = \dot{\mathfrak{p}}(0)$. According to [7, Lemma 8.6] this is indeed a well defined notion and if X is a smooth vector field defined on an open set U around the point x, then the map $\pi^{-1}(U) \ni \mathfrak{f} \to \mathfrak{H}_\mathfrak{f}(X_{\pi\mathfrak{f}}) \in T_\mathfrak{f}\mathcal{O}(M)$ is also a smooth map. We denote the collection of all the horizontal lifts at the point $\mathfrak{f}$ by $\mathcal{H}_\mathfrak{f}\mathcal{O}(M)$. Each of the distributions $\mathcal{HO}(M)$, $\mathcal{VO}(M)$ is smooth and for any $\mathfrak{f}$ there is the natural splitting

$$T_\mathfrak{f}\mathcal{O}(M) = \mathcal{H}_\mathfrak{f}\mathcal{O}(M) \oplus \mathcal{V}_\mathfrak{f}\mathcal{O}(M).$$

We introduce now the canonical vector fields. Given ξ, a vector in $\mathbb{R}^d$, we define

$$\mathfrak{E}(\xi)_{\mathfrak{f}} = \mathfrak{H}_{\mathfrak{f}}(\mathfrak{f}\xi) \tag{2.2.1}$$

the horizontal lift at $\mathfrak{f}$ of $\mathfrak{f}\xi$. From this definition we see that $\pi_*\mathfrak{E}(\xi)_{\mathfrak{f}} = \mathfrak{f}\xi$ for any ξ, $\mathfrak{f}$.

One of the main properties these canonical vector fields have is that they transform covariant derivatives of forms into just plain differentiation at the frame bundle level. To be more precise, we start by defining the lift of a form $\omega_{\pi\mathfrak{f}} \in \bigwedge^k_{\pi\mathfrak{f}}(M)$ to a form $\tilde{\omega}_{\mathfrak{f}} \in \bigwedge^k(\mathbb{R}^d)$. The recipe of $\tilde{\omega}$ in terms of ω is

$$\tilde{\omega}_{\mathfrak{f}}(\xi_1, \cdots, \xi_k) = \omega_{\pi\mathfrak{f}}(\mathfrak{f}\xi_1, \cdots, \mathfrak{f}\xi_k) \tag{2.2.2}$$

We mention here another way of looking at this expression. In the first place $\mathfrak{f}^{-1}$ is an isometry from $T_{\pi\mathfrak{f}}(M)$ into $\mathbb{R}^d$ that can be extended to an isometry from $\bigwedge^k_{\pi\mathfrak{f}}(M)$ into $\bigwedge^k(\mathbb{R}^d)$ using the natural extension

$$\mathfrak{f}^{-1}(v_1^* \wedge \cdots \wedge v_k^*) = (\mathfrak{f}^{-1}v_1)^* \wedge \cdots \wedge (\mathfrak{f}^{-1}v_k)^*$$

for any $v_1, \cdots, v_k \in T_{\pi\mathfrak{f}}(M)$. Within these notations one can rephrase the lifting definition as

$$\tilde{\omega}_{\mathfrak{f}} = \mathfrak{f}^{-1}\omega_{\pi\mathfrak{f}} \tag{2.2.2'}$$

Using this lifting of forms we are able to see the covariant derivatives of forms in a simple way if we lift things to $\mathcal{O}(M)$. We state that for any smooth k-form ω in M defined around $\pi\mathfrak{f}$ we have

$$\widetilde{\nabla_{\mathfrak{f}\xi}\omega} = \mathfrak{E}(\xi)_{\mathfrak{f}}\tilde{\omega} \tag{2.2.3}$$

or equivalently,

$$\widetilde{\nabla_{X_{\pi\mathfrak{f}}}\omega} = \mathfrak{H}(X_{\pi\mathfrak{f}})_{\mathfrak{f}}\tilde{\omega} \tag{2.2.3'}$$

for any $X_{\pi\mathfrak{f}} \in T_{\pi\mathfrak{f}}(M)$.

Before going into the proof of this we define temporarily another extension of the covariant derivatives on forms. Let ∇' be the action on forms given through the following:

1. $\nabla'_{X_x} f = X_x f$ for any function defined around the point x and any $X_x \in T_x(M)$;
2. if $X_x \in T_x(M)$, γ is a curve with $\gamma(0) = x$, $\dot{\gamma}(0) = X_x$ and ω is a k-form locally defined around x, then

$$(\nabla'_{X_x}\omega)((Y_1)_x, \cdots, (Y_k)_x) = \frac{d}{dt}\omega_{\gamma(t)}(\tau_{\gamma\restriction[0,t]}(Y_1)_x, \cdots, \tau_{\gamma\restriction[0,t]}(Y_k)_x)|_{t=0} \tag{2.2.4}$$

   for any $(Y_1)_x, \cdots, (Y_k)_x \in T_x(M)$.

In the first place one has to notice that in fact this definition depends on the curve γ. To clarify the issue, take any smooth extension of the vectors, $(Y_1)_x, \cdots, (Y_k)_x$

around the point x, then write

$$
\begin{aligned}
(\nabla'_{X_x}\omega)((Y_1)_x,\cdots,(Y_k)_x) &= \frac{d}{dt}\omega_{\gamma(t)}(\tau_{\gamma\restriction[0,t]}(Y_1)_x,\tau_{\gamma\restriction[0,t]}(Y_2)_x,\cdots,\tau_{\gamma\restriction[0,t]}(Y_k)_x)|_{t=0}\\
&= \frac{d}{dt}\omega_{\gamma(t)}(\tau_{\gamma\restriction[0,t]}(Y_1)_x-(Y_1)_{\gamma(t)},\tau_{\gamma\restriction[0,t]}(Y_2)_x,\cdots,\tau_{\gamma\restriction[0,t]}(Y_k)_x)|_{t=0}\\
&\quad+\frac{d}{dt}\omega_{\gamma(t)}((Y_1)_{\gamma(t)},\tau_{\gamma\restriction[0,t]}(Y_2)_x-(Y_2)_{\gamma(t)},\cdots,\tau_{\gamma\restriction[0,t]}(Y_k)_x)|_{t=0}+\dots\\
&\quad+\frac{d}{dt}\omega_{\gamma(t)}((Y_1)_{\gamma(t)},(Y_2)_{\gamma(t)},\cdots,\tau_{\gamma\restriction[0,t]}(Y_k)_x-(Y_k)_{\gamma(t)})|_{t=0}\\
&\quad+\frac{d}{dt}\omega_{\gamma(t)}((Y_1)_{\gamma(t)},(Y_2)_{\gamma(t)},\cdots,(Y_k)_{\gamma(t)})|_{t=0}\\
&= -\omega_x(\nabla_{X_x}Y_1,(Y_2)_x,\cdots,(Y_k)_x)-\omega_x((Y_1)_x,\nabla_{X_x}Y_2,\cdots,(Y_k)_x)-\dots\\
&\quad-\omega_x((Y_1)_x,(Y_2)_x,\cdots,\nabla_{X_x}Y_k)+X_x(\omega(Y_1,Y_2,\cdots,Y_k)).
\end{aligned}
$$

On the other hand, the map

$$
\begin{aligned}
Y_1,Y_2,\cdots,Y_k \to\ & X_x(\omega((Y_1)_x,(Y_2)_x,\cdots,(Y_k)_x))-\omega_x(\nabla_{X_x}Y_1,(Y_2)_x,\cdots,(Y_k)_x)\\
&-\cdots-\omega_x((Y_1)_x,\nabla_{X_x}Y_2,\cdots,(Y_k)_x)-\omega_x((Y_1)_x,(Y_2)_x,\cdots,\nabla_{X_x}Y_k)
\end{aligned}
$$

is tensorial in $Y_1,Y_2,\cdots,Y_k$, which shows that the right hand side depends only on the values of Y's at the point x. This proves that ∇' is well-defined. Notice here that, for ω a 1-form we get

$$(\nabla'_{X_x}\omega)(Y)=X_x(\omega(Y))-\omega_x(\nabla_{X_x}Y)=(\nabla_{X_x}\omega)(Y),$$

which proves that ∇ and ∇' coincide on functions and one forms.

Turning back to the proof of (2.2.3), take $\mathfrak{p}:\mathbb{R}\to\mathcal{O}(M)$ the integral curve of $\mathfrak{E}(\xi)$ with $\mathfrak{p}(0)=\mathfrak{f}$ and then for any vectors $\eta_1,\cdots,\eta_k\in\mathbb{R}^d$,

$$
\begin{aligned}
\widetilde{\nabla'_{\mathfrak{f}\xi}\omega}(\eta_1,\cdots,\eta_k) &= \nabla'_{\mathfrak{f}\xi}\omega(\mathfrak{f}\eta_1,\cdots,\mathfrak{f}\eta_k)\\
&= \frac{d}{dt}\omega_{\pi\mathfrak{p}(t)}(\mathfrak{p}(t)\eta_1,\cdots,\mathfrak{p}(t)\eta_k)|_{t=0}\\
&= \frac{d}{dt}\tilde{\omega}_{\mathfrak{p}(t)}(\eta_1,\cdots,\eta_k)\\
&= \mathfrak{E}(\xi)_{\mathfrak{f}}\tilde{\omega}(\eta_1,\cdots,\eta_k).
\end{aligned}
$$

This proves at first that ∇' is a derivation and because it coincides with ∇ we have (2.2.3).

Next, we give the definitions of the Laplacian on M acting on $C^\infty(M,\bigwedge^k(M))$ and of the Bochner Laplacian acting on $C^\infty(\mathcal{O}(M),\bigwedge^k(\mathbb{R}^d))$. For the sake of simplicity we will drop any superscript referring to k. We only mention here that both operators preserve the degree of a form.

Definition 2.2.5. *1. Given $x \in M$ and a smooth ω, the Laplacian is*

$$(\Delta\omega)(x) = \sum_{j=1}^{d}(\nabla_{(E_j)_x}\nabla_{E_j}\omega - \nabla_{\nabla_{(E_j)_x}E_j}\omega),$$

where $(E_j)_{j=1,d}$ is an arbitrary local orthonormal basis around x and ω is a local form defined around x.

2. The Bochner Laplacian Δ_B is

$$\Delta_B = \sum_{j=1}^{d}\mathfrak{E}(e_j)^2,$$

for any orthonormal basis $(e_j)_{j=1,d}$ in $\mathbb{R}^d$.

First of all one has to check that the definition of Δ does not depend on the local orthonormal basis around x. In fact one can easily check that the map

$$(X, Y) \to \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$$

is a tensor, thus for a fixed point x it depends only on the values of X, Y at the point. Then, it is easy to see that ultimately it does not depend on the choice of $((E_j)_x)_{j=1,d}$.

We record here an extension of the basic property from the scalar case that relates the operators Δ and Δ_B.

Proposition 2.2.6. *For any $\mathfrak{f}$ and ω locally defined around $\pi\mathfrak{f}$,*

$$\widetilde{\Delta\omega}(\pi\mathfrak{f}) = \Delta_B\tilde{\omega}(\mathfrak{f})$$

Proof. As we mentioned above, $\nabla_{X_{\pi\mathfrak{f}}}\nabla_Y - \nabla_{\nabla_{X_{\pi\mathfrak{f}}}Y}$ depends only on the values of X, Y at $\pi\mathfrak{f}$. Consequently, in computing $\nabla_{X_{\pi\mathfrak{f}}}\nabla_Y\omega - \nabla_{\nabla_{X_{\pi\mathfrak{f}}}Y}\omega$, one can take any smooth extensions of $X_{\pi\mathfrak{f}}, Y_{\pi\mathfrak{f}}$. Thus, choose a curve p so that $p(0) = \pi\mathfrak{f}$, $\dot{p}(0) = X_{\pi\mathfrak{f}}$, and select Y to be a smooth vector field around $\pi\mathfrak{f}$ such that along p it is given by parallel transportation of $Y_{\pi\mathfrak{f}}$ along p. That is, $Y_{p(t)} = \tau_{p\restriction[0,t]}Y_{\pi\mathfrak{f}}$. Since this gives $\nabla_{X_{\pi\mathfrak{f}}}Y = 0$, we are left with $\nabla_{X_{\pi\mathfrak{f}}}\nabla_Y\omega$. This can be computed using formula (2.2.3') as

$$\widetilde{\nabla_{X_{\pi\mathfrak{f}}}\nabla_Y\omega} = \mathfrak{E}(\mathfrak{f}^{-1}X_{\pi\mathfrak{f}})_{\mathfrak{f}}\widetilde{\nabla_Y\omega}.$$

Take $\mathfrak{p}$, the horizontal lift of p, and use again (2.2.3') to arrive at

$$\widetilde{\nabla_{Y_{p(t)}}\omega} = \mathfrak{E}(\mathfrak{p}(t)^{-1}Y_{p(t)})_{\mathfrak{p}(t)}\tilde{\omega}.$$

The next step is

$$\mathfrak{E}(\mathfrak{f}^{-1}X_{\pi\mathfrak{f}})_{\mathfrak{f}}\widetilde{\nabla_Y\omega}(\eta_1,\cdots,\eta_k) = \frac{d}{dt}\widetilde{\nabla_{Y_{p(t)}}\omega}(\eta_1,\cdots,\eta_k)|_{t=0}$$
$$= \frac{d}{dt}[\mathfrak{E}(\mathfrak{p}(t)^{-1}Y_{p(t)})_{\mathfrak{p}(t)}\tilde{\omega}](\eta_1,\cdots,\eta_k)|_{t=0}.$$

Translating the parallel transportation as, $\mathfrak{p}(t)^{-1}Y_{p(t)} = \mathfrak{p}(0)^{-1}Y_{p(0)} = \mathfrak{f}^{-1}Y_{\pi\mathfrak{f}}$, we finally conclude that

$$\mathfrak{E}(\mathfrak{f}^{-1}X_{\pi\mathfrak{f}})_{\mathfrak{f}}\widetilde{\nabla_Y\omega} = \frac{d}{dt}[\mathfrak{E}(\mathfrak{f}^{-1}Y_{\pi\mathfrak{f}})_{\mathfrak{p}(t)}\tilde{\omega}]|_{t=0}$$
$$= \mathfrak{E}(\mathfrak{f}^{-1}X_{\pi\mathfrak{f}})_{\mathfrak{f}}\mathfrak{E}(\mathfrak{f}^{-1}Y_{\pi\mathfrak{f}})\tilde{\omega},$$

and then

$$\widetilde{\nabla_{X_{\pi\mathfrak{f}}}\nabla_Y\omega} = \mathfrak{E}(\mathfrak{f}^{-1}X_{\pi\mathfrak{f}})_{\mathfrak{f}}\mathfrak{E}(\mathfrak{f}^{-1}Y_{\pi\mathfrak{f}})\tilde{\omega}$$

which ends the proof. □

Next in line, we want to say something about the lifts of tensors on M to tensors on the frame bundle with values in $\mathbb{R}^d$.

If S_x is a k-tensor on $T_x(M)$ we define its lift $\mathfrak{S}_{\mathfrak{f}}$, at $\mathfrak{f}$ with $\pi\mathfrak{f} = x$, to be the k-tensor on $\mathbb{R}^d$ given by the prescription

$$\mathfrak{S}_{\mathfrak{f}}(\xi_1,\cdots,\xi_k) = S_{\pi\mathfrak{f}}(\mathfrak{f}\xi_1,\cdots,\mathfrak{f}\xi_k) \tag{2.2.7}$$

for any $\xi_1,..,\xi_k \in \mathbb{R}^d$. If S is a k-tensor on M then we define its lift $\mathfrak{S}$ given by the above formula for each $\mathfrak{f} \in \mathcal{O}(M)$.

Now, assume we are given $\mathfrak{S} : \mathcal{O}(M) \to L(\underbrace{\mathbb{R}^d \times \mathbb{R}^d \cdots \times \mathbb{R}^d}_{k \text{ times}};\mathbb{R})$. Then we can extend this to $D^*\mathfrak{S} : \mathcal{O}(M) \to L(\bigwedge\mathbb{R}^d, \bigwedge\mathbb{R}^d)$ in the same way we did for tensors on M, namely

1. If k is even, set $D^*\mathfrak{S}_{\mathfrak{f}}$ for

$$\sum_{j_1,j_2,\cdots,j_k=1}^{d} \mathfrak{S}_{\mathfrak{f}}(e_{j_1},e_{j_2},\cdots,e_{j_k})(e^*_{j_1}\wedge i_{e_{j_2}})\circ\cdots\circ(e^*_{j_{k-1}}\wedge i_{e_{j_k}}), \tag{2.2.8}$$

2. if k is odd, set $(D^*\mathfrak{S})_{\mathfrak{f}}(\xi)$ for the expression

$$\sum_{j_1,j_2,\cdots,j_k=1}^{d} \mathfrak{S}_{\mathfrak{f}}(\xi,e_{j_1},e_{j_2},\cdots,e_{j_k})(e^*_{j_1}\wedge i_{e_{j_2}})\circ\cdots\circ(e^*_{j_{k-1}}\wedge i_{e_{j_k}}), \tag{2.2.9}$$

in any orthonormal basis $(e_i)_{i=1,d}$ of $\mathbb{R}^d$ and any $\xi \in R^d$.

The relation between the extensions to forms is given by the following. If S is a tensor on M, $\mathfrak{S}$ its lift, then

$$\begin{aligned} &\text{If } k \text{ is even,} \quad D^*\mathfrak{S}_{\mathfrak{f}}\tilde{\omega}_{\mathfrak{f}} = \widetilde{D^*S_{\pi\mathfrak{f}}\omega_{\pi\mathfrak{f}}}, \\ &\text{If } k \text{ is odd,} \quad D^*\mathfrak{S}(\xi)_{\mathfrak{f}}\tilde{\omega}_{\mathfrak{f}} = \widetilde{D^*S(\mathfrak{f}\xi)_{\pi\mathfrak{f}}\omega_{\pi\mathfrak{f}}}. \end{aligned} \tag{2.2.10}$$

for any form $\omega_{\pi\mathfrak{f}} \in T_{\pi\mathfrak{f}}(M)$ and $\xi \in \mathbb{R}^d$.

2.2.2 Semigroups and Kernels

We can now put to work all these formulae in computing the semi-group generated by various operators subordinated to the Laplacian. We can do this sort of analysis for operators of the form

$$L = \frac{1}{2}\Delta + \sum_{j=1}^{d} B(E_j)\nabla_{E_j} + C \tag{2.2.11}$$

where B and C are defined below by

$$B(X_x) = (D^*S_0)_x(X_x) + (D^*S_1)_x(X_x) + \cdots + (D^*S_k)_x(X_x)$$

$$C = D^*T_0 + D^*T_1 + \cdots + D^*T_l$$

for any $X_x \in T_x(M)$, with respect to the following data:

- $k \geq 1$, S_i, $i = 1, \cdots, k$ odd tensors on M
- $l \geq 1$, T_j, $j = 1, \cdots, l$ even tensors on M

and the corresponding meaning for D^* given in definition (2.1.5).

For such an operator we consider its lift to $\mathcal{O}(M)$ given by

$$\mathfrak{L} = \frac{1}{2}\Delta_B + \sum_{j=1}^{d} \mathfrak{B}(e_j)\mathfrak{E}(e_j) + \mathfrak{C}, \tag{2.2.12}$$

where $\mathfrak{B}$, $\mathfrak{C}$ are

$$\mathfrak{B}(\xi)_{\mathfrak{f}} = D^*\mathfrak{S}_0(\xi)_{\mathfrak{f}} + D^*\mathfrak{S}_1(\xi)_{\mathfrak{f}} + \cdots + D^*\mathfrak{S}_k(\xi)_{\mathfrak{f}}$$

$$\mathfrak{C} = D^*\mathfrak{T}_0 + D^*\mathfrak{T}_1 + \cdots + D^*\mathfrak{T}_l$$

with $\mathfrak{S}_i$ the lift of S_i, $\mathfrak{T}_j$ the lift of T_j and D^* stands for the corresponding extension to the frame bundle.

We mention here one important relationship between these quantities, namely:

$$\mathfrak{B}(\xi)_{\mathfrak{f}}\tilde{\omega}_{\mathfrak{f}} = \widetilde{B(\mathfrak{f}\xi)\omega_{\pi\mathfrak{f}}} \quad \text{and} \quad \mathfrak{C}_{\mathfrak{f}}\tilde{\omega}_{\mathfrak{f}} = \widetilde{C_{\pi\mathfrak{f}}\omega_{\pi\mathfrak{f}}},$$

which are consequences of (2.2.10). These in particular imply that $\mathfrak{L}_{\mathfrak{f}}\tilde{\omega} = \widetilde{L_{\pi\mathfrak{f}}\omega}$ for any smooth form ω defined around the point $\pi\mathfrak{f}$.

We are going to use $p(t, x, \mathbf{w})$ and $\mathfrak{p}(t, \mathfrak{f}, \mathbf{w})$ which are constructed in [7, Chapter 8]. Because the manifold is compact, there is no problem with the explosion. What is important for us here is that the distribution of $\mathbf{w} \to p(\cdot, x, \mathbf{w})$ is the solution to the martingale problem for $\frac{1}{2}\Delta$ starting at x, while the distribution of $\mathbf{w} \to \mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w})$ is the solution to the martingale problem for $\frac{1}{2}\Delta_B$ starting at $\mathfrak{f}$. Then, for any function $F \in C^\infty(\mathbb{R}_+ \times \mathcal{O}(M); \bigwedge(\mathbb{R}^d))$, Ito's formula in this context is:

$$F(t, \mathfrak{p}(t, \mathfrak{f}, \mathbf{w})) = F(0, \mathfrak{f}) + \sum_{j=1}^{d} \int_0^t \mathfrak{E}(e_j)_{\mathfrak{p}(s,\mathfrak{f},\mathbf{w})} F(s, \cdot) d\mathbf{w}_j(s) + \int_0^t (\frac{\partial F}{\partial s} + \frac{1}{2}\Delta_B F)(s, \mathfrak{p}(s, \mathfrak{f}, \mathbf{w}))ds. \quad (2.2.13)$$

Now, fix a function $\tilde{\theta} \in C^\infty(\mathcal{O}(M); \wedge(\mathbb{R}^d))$, and let $\tilde{\omega}(t, \mathfrak{f})$ be the solution to the equation:

$$\begin{cases} \frac{d\tilde{\omega}}{dt} = \mathfrak{L}\tilde{\omega} \\ \tilde{\omega}(0) = \tilde{\theta}. \end{cases} \quad (2.2.14)$$

Fix a $t > 0$ and for $0 \le s \le t$, take $\mathfrak{V}(s, \mathfrak{f}) = \tilde{\omega}(t - s, \mathfrak{f})$ and $\mathfrak{U}(s, \mathfrak{f}, \mathbf{w})$ the solution to

$$\begin{cases} d\mathfrak{U}(s, \mathfrak{f}, \mathbf{w}) = \mathfrak{U}(s, \mathfrak{f}, \mathbf{w}) \Big(\mathfrak{C}_{\mathfrak{p}(s,\mathfrak{f},\mathbf{w})} ds + \sum_{j=1}^{d} \mathfrak{B}(e_j)_{\mathfrak{p}(s,\mathfrak{f},\mathbf{w})} d\mathbf{w}_j(s) \Big) \\ \mathfrak{U}(0) = \mathrm{Id}_{\bigwedge(\mathbb{R}^d)} \end{cases} \quad (2.2.15)$$

For the product $\mathfrak{U}(s, \mathfrak{f}, \mathbf{w})\mathfrak{V}(s, \mathfrak{p}(s, \mathfrak{f}, \mathbf{w}))$, apply integration by parts to get

$$\mathfrak{U}(s)\overline{\mathfrak{V}}(s) = \mathfrak{U}(0)\overline{\mathfrak{V}}(0) + \int_0^s \mathfrak{U}(\sigma) d\overline{\mathfrak{V}}(\sigma) + \int_0^s d\mathfrak{U}(\sigma)\overline{\mathfrak{V}}(\sigma) + \langle \mathfrak{U}(s), \overline{\mathfrak{V}}(s) \rangle_s$$

$$= \mathfrak{U}(0)\overline{\mathfrak{V}}(0) + \int_0^s \mathfrak{U}(\sigma) \sum_{j=1}^{d} \mathfrak{E}(e_j)_{\mathfrak{p}(\sigma)} \overline{\mathfrak{V}}(\sigma)) d\mathbf{w}_j(\sigma) + \int_0^s \mathfrak{U}(\sigma) \overline{\Big(\frac{\partial \mathfrak{V}}{\partial \sigma}(\sigma) + \frac{1}{2}\Delta_B \mathfrak{V} \Big)}(\sigma) d\sigma$$

$$+ \int_0^s \mathfrak{U}(\sigma)(\mathfrak{C}_{\mathfrak{p}(\sigma)} d\sigma + \sum_{j=1}^{d} \mathfrak{B}(e_j)_{\mathfrak{p}(\sigma)} d\mathbf{w}_j(\sigma))\overline{\mathfrak{V}}(\sigma) + \int_0^s \mathfrak{U}(\sigma) \sum_{j=1}^{d} \mathfrak{B}(e_j)_{\mathfrak{p}(\sigma)} \mathfrak{E}(e_j)_{\mathfrak{p}(\sigma)} \overline{\mathfrak{V}}(\sigma) d\sigma$$

$$= \mathfrak{U}(0)\overline{\mathfrak{V}}(0) + \int_0^s \mathfrak{U}(\sigma) \sum_{j=1}^{d} (\mathfrak{E}(e_j)_{\mathfrak{p}(\sigma)} + \mathfrak{B}(e_j)_{\mathfrak{p}(\sigma)}) \overline{\mathfrak{V}}(\sigma) d\mathbf{w}_j(\sigma) + \int_0^s \mathfrak{U}(\sigma) \overline{\Big(\frac{\partial \mathfrak{V}}{\partial \sigma} + \mathfrak{L}\mathfrak{V} \Big)}(\sigma) d\sigma$$

$$= \mathfrak{U}(0)\overline{\mathfrak{V}}(0) + M(s) + \int_0^s \mathfrak{U}(\sigma)\overline{\left(\frac{\partial \mathfrak{V}}{\partial \sigma} + \mathfrak{L}\mathfrak{V}\right)}(\sigma)d\sigma.$$

where, M is an integrable martingale with $M(0) = 0$ and for simplicity we denoted $\overline{F}(s)(\mathfrak{f}, \mathbf{w}) = F(s, \mathfrak{p}(s, \mathfrak{f}, \mathbf{w}))$, for a function. On the other hand, we know that $\mathfrak{V}$ satisfies $\frac{d\mathfrak{V}(s)}{ds} + \mathfrak{L}\mathfrak{V}(s) = 0$, so that we deduce $\mathfrak{U}(s)\overline{\mathfrak{V}}(s) = \mathfrak{U}(0)\overline{\mathfrak{V}}(0) + M(s)$. Thus, taking expectations we obtain

$$\mathbb{E}[\mathfrak{U}(t)\overline{\mathfrak{V}}(t)] = \mathbb{E}[\mathfrak{U}(0)\overline{\mathfrak{V}}(0)].$$

Since $\mathfrak{V}(0) = \tilde{\omega}(t)$, $\mathfrak{V}(t) = \tilde{\theta}$ and $\mathfrak{U}(0) = \mathrm{Id}_{\bigwedge(\mathbb{R}^d)}$, we finally arrive at:

$$\tilde{\omega}(t, \mathfrak{f}) = \mathbb{E}^{\mathcal{W}_d}[\mathfrak{U}(t, \mathfrak{f}, \mathbf{w})\tilde{\theta}(\mathfrak{p}(t, \mathfrak{f}, \mathbf{w}))] \tag{2.2.16}$$

or in terms of the semi-group $\mathbf{P}_t^{\mathfrak{L}}$ generated by $\mathfrak{L}$

$$(\mathbf{P}_t^{\mathfrak{L}}\tilde{\theta})(\mathfrak{f}) = \mathbb{E}^{\mathcal{W}_d}[\mathfrak{U}(t, \mathfrak{f}, \mathbf{w})\tilde{\theta}(\mathfrak{p}(t, \mathfrak{f}, \mathbf{w}))] \tag{2.2.17}$$

Our next goal is to project this down to M. Take $\theta \in \bigwedge(M)$ and consider $\omega(t, x)$ the solution to the equation:

$$\begin{cases} \frac{d\omega}{dt} = L\omega \\ \omega(0) = \theta. \end{cases}$$

Now, lift θ and ω to the frame bundle and denote their lifts by $\tilde{\theta}$ and $\tilde{\omega}(t, \mathfrak{f})$. The equation satisfied by $\tilde{\omega}$ is:

$$\begin{cases} \frac{d\tilde{\omega}}{dt} = \mathfrak{L}\tilde{\omega} \\ \tilde{\omega}(0) = \tilde{\theta}. \end{cases}$$

Now, by (2.2.16) we know that

$$\tilde{\omega}(t, \mathfrak{f}) = \mathbb{E}^{\mathcal{W}_d}[\mathfrak{U}(t, \mathfrak{f}, \mathbf{w})\tilde{\theta}(\mathfrak{p}(t, \mathfrak{f}, \mathbf{w}))] \tag{2.2.18}$$

where $\mathfrak{U}$ is the solution given in (2.2.15). The projection on M of this equality gives $\omega(t, x)$ by

$$\begin{aligned} \omega(t, x) &= \mathbb{E}^{\mathcal{W}_d}[\mathfrak{f}\mathfrak{U}(t, \mathfrak{f}, \mathbf{w})\mathfrak{p}(t, \mathfrak{f}, \mathbf{w})^{-1}\theta(p(t, x, \mathbf{w}))] \\ &= \mathbb{E}^{\mathcal{W}_d}[\mathfrak{f}\mathfrak{U}(t, \mathfrak{f}, \mathbf{w})\mathfrak{f}^{-1}\mathfrak{f}\mathfrak{p}(t, \mathfrak{f}, \mathbf{w})^{-1}\theta(p(t, x, \mathbf{w}))] \\ &= \mathbb{E}^{\mathcal{W}_d}[U(t, x, \mathbf{w})\tau_{p(\cdot, x, \mathbf{w})\restriction[t,0]}\theta(p(t, x, \mathbf{w}))], \end{aligned} \tag{2.2.19}$$

with the notations pointed in (2.2.2'), $\mathfrak{f}$ such that $\pi\mathfrak{f} = x$, $U(t, x, \mathbf{w}) = \mathfrak{f}\mathfrak{U}(t, \mathfrak{f}, \mathbf{w})\mathfrak{f}^{-1}$ and $\tau_{p(\cdot, x, \mathbf{w})\restriction[0,t]} = \mathfrak{p}(t, \mathfrak{f}, \mathbf{w})\mathfrak{f}^{-1}$, the parallel transportation along the path $p(\cdot, x, \mathbf{w})$.

The task now is to identify what is the equation satisfied by U. For this purpose, use the equation (2.2.15) to get the equation for U:

$$dU(t,x,\mathbf{w}) = \mathfrak{f} d\mathfrak{U}(t,\mathfrak{f},\mathbf{w})\mathfrak{f}^{-1} = \mathfrak{f}\mathfrak{U}(t,\mathfrak{f},\mathbf{w})\left(\mathfrak{C}_{\mathfrak{p}(t,\mathfrak{f},\mathbf{w})}dt + \sum_{j=1}^{d}\mathfrak{B}(e_j)_{\mathfrak{p}(t,\mathfrak{f},\mathbf{w})}d\mathbf{w}_j(t)\right)\mathfrak{f}^{-1}$$

$$= U(t,x,\mathbf{w})\mathfrak{f}\left(\mathfrak{C}_{\mathfrak{p}(t,\mathfrak{f},\mathbf{w})}dt + \sum_{j=1}^{d}\mathfrak{B}(e_j)_{\mathfrak{p}(t,\mathfrak{f},\mathbf{w})}d\mathbf{w}_j(t)\right)\mathfrak{f}^{-1}$$

$$= U(t,x,\mathbf{w})\tau_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\left(C_{p(t,x,\mathbf{w})}dt + \sum_{j=1}^{d}B(\mathfrak{p}(t,\mathfrak{f},\mathbf{w})e_j)_{p(t,x,\mathbf{w})}d\mathbf{w}_j(t)\right)\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]}.$$

Now we interpret $\mathfrak{p}(t,\mathfrak{f},\mathbf{w})e_j$ as $\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]}\mathfrak{f}e_j$. Also, we point out that $E_j = \mathfrak{f}e_j$ is an orthonormal basis at $T_x(M)$ that identifies it with $\mathbb{R}^d$. This way we can think of the Brownian motion $\mathbf{w}$ as a Brownian motion in $T_x(M)$ via this identification.

We further make some notations. For a path $\mathbf{w}$ in $\mathbb{R}^d$, set $C(t,x,\mathbf{w})$, $B_j(t,x,\mathbf{w})$, for $j=1,d$, the $L(\bigwedge_x(M))$-valued functions

$$\begin{aligned} C(t,x,\mathbf{w}) &= \tau_{p(\cdot,x,\mathbf{w})\restriction[t,0]}C_{p(t,x,\mathbf{w})}\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]} \\ B_j(t,x,\mathbf{w}) &= \tau_{p(\cdot,x,\mathbf{w})\restriction[t,0]}B((\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]}E_j)_{p(t,x,\mathbf{w})}\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]}. \end{aligned} \tag{2.2.20}$$

Thus, we can finally give the equation of U as the solution to the stochastic differential equation in the space $L(\bigwedge_x(M))$.

$$\begin{cases} dU(t,x,\mathbf{w}) = U(t,x,\mathbf{w})\left(C(t,x,\mathbf{w})dt + \sum_{j=1}^{d}B_j(t,x,\mathbf{w})d\mathbf{w}_j(t)\right) \\ U(0,x,\mathbf{w}) = \mathrm{Id}_{\bigwedge_x(M)} \end{cases} \tag{2.2.21}$$

Conclude this section with the final formula stated as a proposition. The proof of this is given in Appendix B in a more general situation.

Proposition 2.2.22. *The semi-group $\mathbf{P}_t^L$ generated by L is given by*

$$(\mathbf{P}_t^L\theta)(x) = \mathbb{E}^{\mathcal{W}_d}[U(t,x,\mathbf{w})\tau^{-1}_{p(\cdot,x,\mathbf{w})\restriction[0,t]}\theta(p(t,x,\mathbf{w}))]$$

and the heat kernel $p^L(t,x,y)$ corresponding to the operator L

$$p^L(t,x,y) = \mathbb{E}^{\mathcal{W}_d}[U(t,x,\mathbf{w})\tau^{-1}_{p(\cdot,x,\mathbf{w})\restriction[0,t]}\delta_y(p(t,x,\mathbf{w}))],$$

where the integral is interpreted via the Malliavin calculus.

2.2.3 The Heat Kernel of $\Box^\alpha$

Let h be a Morse function on M with isolated critical points $c_1,\cdots,c_l$. Then, by the Morse Lemma, one can find coordinates (U_i,φ_i), with $c_i\in U_i$ such that $\varphi_i(c_i)=0$ and

$$h(\varphi_i^{-1}(x_1,\cdots,x_d)) = h(c_i) - \frac{1}{2}\sum_{j=1}^{\mathrm{ind}(i)}x_j^2 + \frac{1}{2}\sum_{k=\mathrm{ind}(i)+1}^{d}x_k^2$$

where $\mathrm{ind}(i)$ is the index of c_i. Using these coordinates we pull-back the metric from $\mathbb{R}^d$ around critical points. Then, we complete with an arbitrary metric on the rest of the manifold.

Denote by $p_k^\alpha(t,x,y)$ the heat kernel of the operator $\Box^\alpha$ acting on k-forms. In order to express this heat kernel by (2.2.22) we recall here the Weitzenböck's formula. Define, using (2.1.5), the extension of the curvature tensor D^*R to an operator on forms. Then Weitzenböck's formula states that

$$\Box = -\Delta + D^*R \tag{2.2.23}$$

Then by (2.1.8) and (2.2.22) we can write

$$p_k^\alpha(t,x,y) = \mathbb{E}^{\mathcal{W}_d}\left[U_k^\alpha(t,x,\mathbf{w})\tau_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\delta_y(p(t,x,\mathbf{w}))\right]$$

where U_k^α is the solution to the ODE:

$$\begin{cases} dU_k^\alpha(t,x,\mathbf{w}) = U_k^\alpha(t,x,\mathbf{w})\Big(-\dfrac{\alpha^2}{2}|\mathrm{grad}h(p(t,x,\mathbf{w}))|^2 I\,dt + \dfrac{\alpha}{2}\Delta h(p(t,x,\mathbf{w}))I\,dt \\ \qquad\qquad + \tau_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\left(-\alpha\mathrm{hess}_{p(t,x,\mathbf{w})}h + \dfrac{1}{2}D^*R_{p(t,x,\mathbf{w})}\right)\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]}dt\Big) \\ U_k^\alpha(0,x,\mathbf{w}) = \mathrm{Id}_{\bigwedge_x^k(M)} \end{cases}$$

After simple computations we arrive at the simpler form of $p_k^\alpha(t,x,y)$ as

$$\mathbb{E}^{\mu_x^M}\left[e^{-\frac{\alpha^2}{2}\int_0^t|\mathrm{grad}h(\varphi(\sigma))|^2d\sigma + \frac{\alpha}{2}\int_0^t\Delta h(\varphi(\sigma))d\sigma}V_k^\alpha(t,x,\varphi)\tau_{\varphi\restriction[t,0]}\delta_y(\varphi(t))\right] \tag{2.2.24}$$

with the following updates:

1. μ_x^M is the Wiener measure on M starting at x. Otherwise stated, it is the distribution under μ of $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d) \to p(\cdot,x,\mathbf{w}) \in \mathcal{P}(M)$
2. V_k^α is the solution to the ODE on $L(\bigwedge_x^k(M))$

$$\begin{cases} \dot{V}_k^\alpha(t,x,\varphi) = V_k^\alpha(t,x,\varphi)\left(\tau_{\varphi\restriction[t,0]}\left(-\alpha\mathrm{hess}_{\varphi(t)}h + R_{\varphi(t)}\right)\tau_{\varphi\restriction[0,t]}\right) \\ V_k^\alpha(0,x,\varphi) = \mathrm{Id}_{\bigwedge_x^k(M)} \end{cases} \tag{2.2.25}$$

2.3 Away from the Critical Set Case

This is the first case, and in some sense, is the key to our estimates. Before starting up the machinery involved in the analysis we want to state from the beginning what is the goal of this section.

Theorem 2.3.1 (The x away case). *For small $r > 0$, set $\Omega_r = \{x \in M, \mathrm{dist}(x,c_i) > r, \forall i\}$. Then, there exist constants $\alpha_0(t,r) > 0$, $C_1(t,r) > 0$, $C_2(t,r) > 0$ depending on t and* *r such that for any $\alpha \geq \alpha_0(t,r)$,*

$$\left\|p_k^\alpha(t,x,y)\right\|_{y,x} \leq C_1(t,r)e^{-C_2(t,r)\alpha}$$

uniformly for $x \in \Omega_r$ and $y \in M$ where the norm $\|\cdot\|_{y,x}$ is the Hilbert-Schmidt norm on the space of linear maps from $\bigwedge_y(M)$ to $\bigwedge_x(M)$.

The whole section is devoted to proving this theorem, the formal proof appears right after Theorem (2.3.15).

We estimate $p_k^\alpha(t,x,y)$ by first estimating the size of V_k^α. See for instance (A.2.11) To estimate V_k^α we will choose a smooth function f_k such that

$$\begin{aligned} &f_k(x) = \text{ind}(i) \text{ for } x \text{ close to } c_i, \\ &-\text{hess}_x h \leq f_k(x) Id_{\bigwedge_x^k(M)} \text{ for any point } x \in M. \end{aligned} \tag{2.3.2}$$

Then by (2.2.25), and (2.2.24) we get the first estimate

$$\|p_k^\alpha(t,x,y)\|_{y,x} \leq C_d \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^t H_\alpha(v,\varphi)dv\right)\delta_y(\varphi(t))\right] \tag{2.3.3}$$

where C_d is a dimensional constant and

$$H_\alpha(v,\varphi) = -\frac{\alpha^2}{2}|\text{grad}h(\varphi(v))|^2 + \frac{\alpha}{2}\Delta h(\varphi(\sigma)) + \alpha f_k(\varphi(\sigma))$$

Next, we estimate this last integral.

Proposition 2.3.4. *For any positive η there exists $P_{\eta,t}(\alpha)$ a polynomial in α such that*

$$\|p_k^\alpha(t,x,y)\|_{y,x} \leq P_\eta(\alpha)\left\{\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^t H_\eta^\alpha(v,\varphi)dv\right)\right]\right\}^{\frac{1}{1+\eta}}$$

uniformly on M in x, y, where we use the notation

$$H_\eta^\alpha(v,\varphi) = (1+\eta)\left(-\frac{\alpha^2}{2}|\text{grad}h(\varphi(v))|^2 + \frac{\alpha}{2}\Delta h(\varphi(v)) + \alpha f_k(\varphi(v))\right). \tag{2.3.5}$$

Proof. (Sketch) By the Malliavin calculus (see for details Appendix A), one can make sense of the integral involving the Dirac function through a number of integration by parts. Taking derivatives of the integrand we will be left with the exponential times a couple of terms involving derivatives of the function h, and geometric quantities. We mention in here that all these derivatives are in all $L^p(\mu_x^M)$ and their norms are uniformly bounded in x, y. Thus, by applying Hölder's inequality to the integral with one term containing the exponential and the other term the polynomial in α with coefficients the derivatives obtained from the integration by parts, we get the estimates stated. In this manner we pick up the $(1+\eta)$ Hölder norm of the exponential times some polynomial in α. □

Denote

$$q_k^\alpha(t,x) = \mathbb{E}^{\mu_x^M}\left[\exp\left(\left(\int_0^t H_\eta^\alpha(v,\varphi)dv\right)\right]\right. \tag{2.3.6}$$

From the above Proposition it is clear that estimates on $p_k^\alpha(t,x,y)$ are obtained from estimates on $q_k^\alpha(t,x)$. In the next subsections we analyze the behavior of this quantity when α gets large.

2.3.1 About Stopping Times

In this little section we want to recall a few things about stopping times. We give them in a more general situation, when the manifold M is not necessary compact.

We start by stating a part of [7, Theorem 8.62]

Theorem 2.3.7. *Let M be a complete Riemannian manifold with a fixed point o such that*

$$Ric_x X_x \geq C(1+\text{dist}(x,o)^2)|X_x|$$

for any $x \in M$, $X_x \in T_x(M)$. Then, for a compact set $K \subset M$ and $t > 0$, there exist two constants $C_1 > 0$, $C_2 > 0$ depending on K, t, such that for $r > 0$, $t \geq s > 0$

$$\sup_{x\in K} \mu_x^M(\tau_r \leq s) \leq C_1 e^{-\frac{C_2 r^2}{s}} \tag{2.3.8}$$

where τ_r is the exit time from the ball $B(x,r)$.

Proof. Because K is compact, there is a constant C such that

$$Ric_x X_x \geq C(1+\text{dist}(x,y)^2)|X_x|$$

for any $x \in M$, $X_x \in T_x(M)$, $y \in K$. Therefore, from the Theorem invoked above we get two constants $K_1 > 0$, $K_2 > 0$, such that

$$\sup_{x\in K} \mu_x^M(\tau_r \leq t) \leq K_1 e^{-\frac{r^2}{te^{K_2 t}}}$$

for any $r, t > 0$. From this, the statement above follows at once. □

There are two important corollaries we need in our estimates.

Corollary 2.3.9. *Within the assumption of the above Theorem, for $K \subset M$, a compact set and $t > 0$, there exist two constants $C_1 > 0$, $C_2 > 0$, depending on K and t, such that for $t \geq s > 0$, $r > 0$ and $\beta > 0$*

$$\sup_{x\in K} \mathbb{E}^{\mu_x^M}\left[e^{-\beta(s\wedge\tau_r)}\right] \leq e^{-\beta s} + C_1(t)e^{-C_2(t)r\sqrt{\beta}}. \tag{2.3.10}$$

In particular, there exist two constants $C_1 > 0$, $C_2 > 0$ such that for any $\beta > 0$, $t \geq s > 0$,

$$\sup_{x\in K} \mathbb{E}^{\mu_x^M}\left[e^{-\beta(s\wedge\tau_r)}\right] \leq C_1(t,r)e^{-C_2(t,r)\sqrt{\beta}}. \tag{2.3.11}$$

Proof. First, split and estimate the integral above as:

$$\begin{aligned}\mathbb{E}^{\mu_x^M}\left[e^{-\beta(s\wedge\tau_r)}\right] &= \mathbb{E}^{\mu_x^M}\left[e^{-\beta(s\wedge\tau_r)}, \tau_r > s\right] + \mathbb{E}^{\mu_x^M}\left[e^{-\beta(s\wedge\tau_r)}, \tau_r \leq s\right] \\ &= e^{-\beta s}\mu_x^M(\tau_r > s) + \mathbb{E}^{\mu_x^M}\left[e^{-\beta(s\wedge\tau_r)}, \tau_r \leq s\right].\end{aligned}$$

We have in general

$$\mathbb{E}^{\mu_x^M}\left[f(\tau_r), \tau_r \leq s\right] = f(s)\mu_x^M(\tau_r \leq s) - \int_0^s f'(s)\mu_x^M((\tau_r \leq \sigma)d\sigma$$

for any smooth function f. For $f(\sigma) = e^{-\beta\sigma}$ and the Theorem above we have

$$\begin{aligned}\mathbb{E}^{\mu_x^M}\left[e^{-\beta(t\wedge\tau_r)}, \tau_r \leq s\right] &= \mathbb{E}^{\mu_x^M}\left[e^{-\beta\tau_r}, \tau_r \leq s\right] \\ &= e^{-\beta s}\mu_x^M(\tau_r \leq s) + \beta\int_0^s e^{-\beta\sigma}\mu_x^M((\tau_r \leq \sigma)d\sigma \\ &\leq e^{-\beta s}\mu_x^M(\tau_r \leq s) + \beta C_1\int_0^t e^{-\beta\sigma - \frac{r^2C_2}{\sigma}}d\sigma \\ &\leq e^{-\beta s}\mu_x^M(\tau_r \leq s) + \beta C_1\int_0^\infty e^{-\beta\sigma - \frac{r^2C_2}{\sigma}}d\sigma.\end{aligned}$$

Now, for given $a, b > 0$ using the change of variable $\xi = a\sigma^{1/2} - b\sigma^{-1/2}$ we have,

$$\begin{aligned}\int_0^\infty e^{-a^2\sigma - \frac{b^2}{\sigma}}d\sigma &= \frac{e^{ab}}{2a^2}\int_{-\infty}^\infty\left(\sqrt{\xi^2 + 4ab} + \frac{\xi^2}{\sqrt{\xi^2 + 4ab}}\right)e^{-\xi^2}d\xi \\ &\leq \frac{e^{ab}}{a^2}\int_{-\infty}^\infty\left(\sqrt{\xi^2 + 4ab}\right)e^{-\xi^2}d\xi \\ &\leq \frac{e^{ab}}{a^2}\int_{-\infty}^\infty\left(|\xi| + 2\sqrt{ab}\right)e^{-\xi^2}d\xi \\ &\leq K(1 + 2\sqrt{ab})\frac{e^{ab}}{a^2},\end{aligned}$$

from which with $a = \sqrt{\beta}$, $b = r\sqrt{C_2}$, we get

$$\mathbb{E}^{\mu_x^M}\left[e^{-\beta(t\wedge\tau_r)}, \tau_r \leq t\right] \leq e^{-\beta t}\mu_x^M(\tau_r \leq t) + C_3(t)(1 + r\sqrt{\beta})e^{-r\sqrt{C_2\beta}}$$

and then the desired result follows. □

Corollary 2.3.12. *Continuing with the assumptions above, let Ω_1, Ω_2 be two open sets in M, such that $\overline{\Omega}_2 \subset \Omega_1$, $\text{dist}(\Omega_1^c, \Omega_2) > 0$ and $\partial\Omega_1$ is compact. Define σ to be the first exit time from Ω_1 and τ the first hitting time of $\overline{\Omega}_2$. Set $\sigma_1 = \sigma$ and define for $n \geq 1$ the sequence of stopping times*

$$\begin{aligned}\tau_n(\varphi) &= \inf\{t \geq \sigma_n(\varphi) \mid \varphi(t) \in \overline{\Omega}_2\} \\ \sigma_{n+1}(\varphi) &= \inf\{t \geq \tau_n(\varphi) \mid \varphi(t) \in \Omega_1^c\}.\end{aligned} \qquad (2.3.13)$$

If $\sigma_\infty = \lim_{n\to\infty}\sigma_n$*, then* $\mu_x^M(\sigma_\infty < \infty) = 0$.

Proof. Fix a $r < \mathrm{dist}(\Omega_1^c, \overline{\Omega}_2)$ and define the sequence of stopping times ζ_n with $\zeta_0 = 0$ and for $n \geq 1$

$$\zeta_n = \inf\{t \geq \sigma_n(\varphi) \mid \mathrm{dist}(\varphi(\sigma_n), \varphi(t) \geq r\},$$

$$\zeta_\infty = \lim_{n\to\infty} \zeta_n.$$

Then, $\mu_x^M(\sigma_\infty < \infty) \leq \mu_x^M(\zeta_\infty < \infty)$, so, it suffices to prove that $\mu_x^M(\zeta_\infty < \infty) = 0$.

Now we have

$$\mu_x^M(\zeta_n \leq t) \leq e^{\beta t}\mathbb{E}^{\mu_x^M}\left[e^{-\beta\zeta_n}\right] \leq e^{\beta t}\left(K_1 e^{-r\sqrt{K_2\beta}}\right)^n$$

where we used repeatedly Markov property combined with the result from the previous Corollary applied for $K = \partial\Omega_1$. Taking β so that $K_1 e^{-r\sqrt{K_2\beta}} \leq e^{-1}$ we get that $\mu_x^M(\zeta_n \leq t) \leq e^{\beta t - n}$, which ends the proof. □

2.3.2 The Iteration

The main idea in estimating (2.3.6) is to use the Markov property to break the integral into integrals on paths which either stay inside some neighborhoods of the critical points or stay a certain distance from the critical set.

Let's start up with a small $r > 0$ such that the balls $B(c_i, 4r)$, $i = 1, \cdots, l$ are all disjoint and the metric in each of them is flat. Then take the sets

$$\Omega_1 = \{x \in M, \mathrm{dist}(x, c_i) > r/2, \forall i\}, \quad \Omega_2 = \{x \in M, \mathrm{dist}(x, c_i) > r, \forall i\}$$

and let τ_n and σ_n be given by (2.3.13). Using (2.3.12) we can justify that

$$q_k^\alpha(t,x) = \lim_{n\to\infty} \mathbb{E}^{\mu_x^M}\left[\exp\Big(\int_0^{t\wedge\sigma_n} H_\eta^\alpha(v,\varphi)dv\Big)\right] \tag{2.3.14}$$

We record here the following theorem which is in fact the main idea of the estimation.

Theorem 2.3.15. *There exist* $\eta_0(t,r) > 0$ *and* $\alpha_0(t,r) > 0$ *such that, for any* $\alpha \geq \alpha_0(t,r)$, $\eta \leq \eta_0(t,r)$, $n \geq 1$ *and* $x \in \overline{\Omega}_2$

$$\mathbb{E}^{\mu_x^M}\left[\exp\Big(\int_0^{t\wedge\sigma_{n+1}} H_\eta^\alpha(v,\varphi)dv\Big)\right] \leq \mathbb{E}^{\mu_x^M}\left[\exp\Big(\int_0^{t\wedge\sigma_n} H_\eta^\alpha(v,\varphi)dv\Big)\right] \tag{2.3.16}$$

The proof of this will be given in a sequence of steps and it will be the content of the rest of this section. But, as we mentioned at the beginning of this section, given Theorem (2.3.15) we have enough to prove Theorem (2.3.1).
Here is the proof.

Proof of Theorem (2.3.1) Theorem (2.3.1) is just a corollary of (2.3.4), (2.3.14), (2.3.16) and (2.3.10). To see this, one only has to apply (2.3.10) to (2.3.16). This can be seen by pointing out that, for any $x \in \overline{\Omega}_2$ any path φ starting at x stays inside Ω_1 up to time $\sigma(\varphi)$. Now, because there is a constant $c_1(r) > 0$ such that $|\text{grad}_y h| \geq c_1(r)$ for $y \in \Omega_1$, one gets that

$$\begin{aligned}&\text{if } \varphi(0) \in \Omega_1, \text{ then}\\ &H_\eta^\alpha(v,\varphi) \leq -c_1(r)\alpha^2 + c_2\alpha \leq -c_1(r)\alpha^2/2 \\ &\text{for large enough } \alpha \text{ and any } v \leq \sigma(\varphi),\end{aligned} \tag{2.3.17}$$

where c_2 is a constant depending only on the bounds of the Hessian and the Laplacian of h on M. Thus, by (2.3.14) and Theorem (2.3.15) we get $q_k^\alpha(t,x)$ bounded above by

$$\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\sigma} H_\eta^\alpha(v,\varphi)dv\right)\right] \leq \mathbb{E}^{\mu_x^M}\left[\exp\left(-\frac{c_1(r)\alpha^2(t\wedge\sigma)}{2}\right)\right].$$

From here, Corollary (2.3.10) finishes the proof. □

Now we return to the business of proving Theorem (2.3.15).

Step 1. In this step we prove that

$$\mathbb{E}^{\mu_x^M}\left[e^{\int_0^{t\wedge\sigma_{n+1}} H_\eta^\alpha(v,\varphi)dv}\right] \leq \mathbb{E}^{\mu_x^M}\left[e^{\int_0^{t\wedge\tau_n} H_\eta^\alpha(v,\varphi)dv}\right]. \tag{2.3.18}$$

In order to see this, first apply the Markov property to justify that

$$\mathbb{E}^{\mu_x^M}\left[e^{\int_0^{t\wedge\sigma_{n+1}} H_\eta^\alpha(v,\varphi)dv}\right] = \\ \iint e^{\int_0^{\sigma(\psi)\wedge(t-t\wedge\tau_n(\varphi))} H_\eta^\alpha(v,\psi)dv}\mu^M_{\varphi(\tau_n)}(d\psi)e^{\int_0^{t\wedge\tau_n(\varphi)} H_\eta^\alpha(v,\varphi)dv}\mu_x^M(d\varphi).$$

Now we remark that the path ψ starts at $\varphi(\tau_n)$, hence in $\overline{\Omega}_2$, and runs till $\sigma(\psi) \wedge (t - t\wedge\tau_n(\varphi))$. Then by (2.3.17) we get that for α large, $H_\eta^\alpha(v,\psi) \leq 0$ for $v \leq \sigma(\psi) \wedge (t - t\wedge\tau_n(\varphi))$. This brings us to

$$\iint e^{\int_0^{\sigma(\psi)\wedge(t-t\wedge\tau_n(\varphi))} H_\eta^\alpha(v,\psi)dv}\mu^M_{\varphi(\tau_n)}(d\psi)e^{\int_0^{t\wedge\tau_n(\varphi)} H_\eta^\alpha(v,\varphi)dv}\mu_x^M(d\varphi) \leq \mathbb{E}^{\mu_x^M}\left[e^{\int_0^{t\wedge\tau_n} H_\eta^\alpha(v,\varphi)dv}\right],$$

from which we get (2.3.19).

Step 2. In this step we show that

$$\mathbb{E}^{\mu_x^M}\left[e^{\int_0^{t\wedge\tau_n} H_\eta^\alpha(v,\varphi)dv}\right] \leq \mathbb{E}^{\mu_x^M}\left[e^{\int_0^{t\wedge\sigma_n} H_\eta^\alpha(v,\varphi)dv}\right]. \tag{2.3.19}$$

To see this, start again with the Markov property to justify that

$$\mathbb{E}^{\mu_x^M}\left[e^{\int_0^{t\wedge\tau_n} H_\eta^\alpha(v,\varphi)dv}\right] =$$

$$\iint e^{\int_0^{\tau(\psi)\wedge(t-t\wedge\sigma_n(\varphi))} H_\eta^\alpha(v,\psi)dv}\mu^M_{\varphi(\sigma_n)}(d\psi)e^{\int_0^{t\wedge\sigma_n(\varphi)} H_\eta^\alpha(v,\varphi)dv}\mu_x^M(d\varphi).$$

Notice now that the point $\varphi(\sigma_n)$ is on the sphere $S(c_i, r/2)$ for some i. Also notice that each of the balls $B(c_i, r)$ is just an Euclidean ball with the corresponding Euclidean metric on it. So, let's fix a critical point and call it c. Define the function $u : \mathbb{R}_+ \times B(c,r) \to \mathbb{R}$

$$u_\eta^\alpha(s,y) = \int \exp\left(\int_0^{\tau(\psi)\wedge s} H_\eta^\alpha(v,\psi)dv\right)\mu_y^M(d\psi) \tag{2.3.20}$$

Because the integral inside runs up to the exit time from the ball, combined with the invariance of the Brownian motion in the Euclidean space, we can make the assumption that $c = 0$.

At this stage, what we need is the following estimate.

Theorem 2.3.21. *There exist $\alpha(r,t) > 0$ and $\eta(r,t) > 0$ such that*

$$\sup_{s\in[0,t] r/2\le|y|\le r} u_\eta^\alpha(s,y) \le 1$$

for $\alpha \ge \alpha(r,t)$, $\eta \le \eta(r,t)$.

Using this theorem we complete the Step 2 and the proof of Theorem (2.3.15). The proof of (2.3.21) is given in the next subsection.

2.3.3 The proof of Theorem (2.3.21)

In this section we work entirely in local coordinates. Moreover because the metric we choose around the critical points is flat, we are in fact working in the Euclidean balls in $\mathbb{R}^d$. Thus, everything in this section is done on an Euclidean ball centered at 0 in $\mathbb{R}^d$.

For convenience we drop the dependence on α and η of $u_\eta^\alpha(s,y)$. We start by identifying the quantity $u(s,y)$ as a solution to a certain PDE.

Proposition 2.3.22. *u is the solution to the initial-boundary problem on $[0,\infty) \times B(0,r)$*

$$\begin{cases} \partial_s u(s,x) = \frac{1}{2}\Delta u(s,x) + \frac{1}{2}((1+\eta)\alpha d - (1+\eta)\alpha^2|x|^2)u(s,x) \\ u(0,x) = 1 \; if \; x \in B(0,r), \\ u(s,y) = 1 \; if \; s > 0, \; y \in \partial B(0,r). \end{cases} \tag{2.3.23}$$

Proof. First, recall the choice of the function f_k in (2.3.2) and then point out that for a path φ that starts inside $B(c,r)$ we have

$$H(v,\varphi) = \frac{1}{2}((1+\eta)\alpha d - (1+\eta)\alpha^2|\varphi(v)|^2)$$

for $v \le \tau(\varphi)$.

Consider $V(x) = \frac{1}{2}((1+\eta)\alpha d - (1+\eta)\alpha^2|x|^2)$. Then, by Ito's formula, for any smooth function f on $\mathbb{R}_+ \times \mathbb{R}^d$

$$f(s,\varphi(s))e^{\int_0^s V(\varphi(v))dv} - \int_0^s (\partial_s f + \frac{1}{2}\Delta f + Vf)(v,\varphi(v))e^{\int_0^v V(\varphi(u))du}dv$$

is a local martingale. In particular,

$$f(s\wedge\tau,\varphi(s\wedge\tau))e^{\int_0^{s\wedge\tau} V(\varphi(v))dv} - \int_0^{s\wedge\tau} (\partial_s f + \frac{1}{2}\Delta f + Vf)(v,\varphi(v))e^{\int_0^v V(\varphi(u))du}dv$$

is also a local martingale. Then by an easy extension we get that for any smooth function f in $\mathbb{R}_+ \times B(0,r)$ and continuous up to the boundary with $\partial_s f + \frac{1}{2}\Delta f + Vf$ also bounded in $[0,t]\times B(0,r)$ for any time t,

$$f(s\wedge\tau,\varphi(s\wedge\tau))e^{\int_0^{s\wedge\tau} V(\varphi(v))dv} - \int_0^{s\wedge\tau} (\partial_s f + \frac{1}{2}\Delta f + Vf)(v,\varphi(v))e^{\int_0^v V(\varphi(u))du}dv$$

remains a martingale. Take now $\tilde{u}$ the solution to the PDE (2.3.23) and apply the martingale property with $f(s,x) = \tilde{u}(t-s,x)$ on $[0,t]$ to get that

$$\tilde{u}(t-s\wedge\tau,\varphi(s\wedge\tau))e^{\int_0^{s\wedge\tau} V(\varphi(v))dv}$$

is a martingale. After taking the expectations, the function obtained will be independent of s and this at $s=0$ and $s=t$ gives that $\tilde{u}(t,x) = u(t,x)$. □

Next we want to get estimates on the solution to the equation (2.3.23). We will do this by constructing a supersolution. To compare the solution with a supersolution we are going to use a comparison principle for parabolic equations which is an extension of the classical comparison for the heat equation. To do this we start with the following proposition.

Proposition 2.3.24. *Suppose* $V \in C(B(0,r))$ *is bounded above,* $v \in C\big(([0,t]\times B(0,r)) \cup ((0,t]\times\partial B(0,r));\mathbb{R}\big)$ *and assume there exists a closed subset* $A \subset B(0,2r)$ *with the following two properties:*

1. *we have*

$$v \in C^{1,2}((0,t]\times(B(0,r)\backslash A));\mathbb{R}), \tag{i}$$

2. *and for any* $a \in A$ *there exists a unitary vector* w_a *such that for any fixed* $s \in (0,t]$ *the function* $\psi(\sigma) = v(s, a+\sigma w_a)$ *defined around* 0 *has left and right*

derivatives and

$$\psi_l'(0) > \psi_r'(0). \tag{ii}$$

If v satisfies

$$\begin{cases} \partial_s v(s,x) \geq \frac{1}{2}\Delta v(s,x) + V(x)v(s,x), \ \textit{for } (s,x) \in (0,t] \times B(0,r)\backslash A, \\ v(0,x) \geq 0 \quad \textit{for } x \in B(0,r), \\ v(s,y) \geq 0 \quad \textit{for } y \in \partial B(0,r), \end{cases} \tag{2.3.25}$$

then $v(s,x) \geq 0$ for all $(s,x) \in [0,t] \times B(0,r)$.

Proof. After taking $\overline{v}(s,x) = e^{Cs}v(s,x)$ with $C > \sup_{x \in B(0,r)} V(x)$, we can assume without loss of generality that $\sup_{x\in B(0,r)} V(x) < 0$. Now we prove that $v \leq 0$ on $[0,t] \times B(0, r-\epsilon)$ for small ϵ. The idea is the same as the proof of classical minimum principle given in [3, Theorem 9, Chapter 7].

Replacing $v(s,x)$ by $v(s,x) + \delta s$, we may assume that

$$\partial_s v(s,x) > \frac{1}{2}\Delta v(s,x) + V(x)v(s,x) \tag{*}$$

for all $(s,x) \in [0,t] \times B(0,r)\backslash A$.

Take then the point (s_ϵ, x_ϵ) to be the minimum point of v on the set $[0,t] \times \overline{B(0, r-\epsilon)}$. We claim that

$$v(s_\epsilon, x_\epsilon) \geq \min_{([0,t]\times\partial B(0,r-\epsilon))\cup(0\times B(0,r-\epsilon))} (v \wedge 0). \tag{**}$$

If this is not the case, then the point (s_ϵ, x_ϵ) is not in $([0,t] \times \partial B(0, r-\epsilon)) \cup (0 \times B(0, r-\epsilon))$. We distinguish three cases now

1. $(s_\epsilon, x_\epsilon) \in (0,t) \times B(0, r-\epsilon)\backslash A$. In this case $\partial_s v(s_\epsilon, x_\epsilon) = 0$, $\Delta v(s_\epsilon, x_\epsilon) \geq 0$ and if we plug in (*) we get a contradiction.

2. $(s_\epsilon, x_\epsilon) \in \{t\} \times B(0, r-\epsilon)\backslash A$. This is treated similarly, the only difference is that we only get $\partial_s v(s_\epsilon, x_\epsilon) \leq 0$. This also leads to a contradiction with (*).

3. If $x_\epsilon \in A \cap B(0, r-\epsilon)$ then certainly the point (s_ϵ, x_ϵ) can not be a local minimum because, if it were, then the function ψ associated with (s_ϵ, x_ϵ) would have a local minimum at 0 and then

$$\psi_r' \geq 0 \geq \psi_l'$$

in contradiction with the assumption made.

Thus, having proved (**), we let ϵ to 0 and get that $v \geq 0$ in $[0,t] \times B(0,r)$. □

We give here the corollary of this proposition, which will be the backbone of our construction of the supersolution.

Corollary 2.3.26 (The comparison principle). *Let $V \in C(B(0,r))$ be a continuous bounded above function and suppose we are given the following data:*

1. *a function u in the set*

$$C^{1,2}\left((0,t] \times B(0,r)\right) \cap C\left(([0,t] \times B(0,r)) \cup ((0,t] \times \partial B(0,r))\right)$$

2. *a subset A of $B(0,r)$ such that the function u belongs to*

$$C^{1,2}\left((0,t] \times B(0,r)\backslash A\right) \cap C\left(([0,t] \times B(0,r)) \cup ((0,t] \times \partial B(0,r))\right)$$

with the following property

3. *for any $a \in A$ there exists a unit vector w_a such that for any s in$(0,t]$ the function $\psi(\sigma) = v(s, a + \sigma w_a)$ defined around 0 has left and right derivatives and*

$$\psi_l'(0) > \psi_r'(0). \tag{*}$$

Further assume that,

i. *on $(0,t] \times B(0,r)\backslash A$,*

$$\partial_s u = \frac{1}{2}\Delta u + Vu, \quad \partial_s v \geq \frac{1}{2}\Delta v + Vv, \tag{**}$$

ii. *for any $x \in B(0,r)$, $(s,y) \in (0,t] \times B(0,r)$*

$$u(0,x) \leq v(0,x), \quad u(s,y) \leq v(s,y). \tag{***}$$

Then we have $u \leq v$ on $[0,t] \times B(0,r)$.

Proof. Just take the difference $v - u$ and apply Proposition (2.3.24). □

We are now prepared to begin the proof of Theorem (2.3.21).

Remark first that it suffices to deal with the case when $r = 1$. Indeed, if $\overline{u}(s,x) = u(r^2 s, rx)$, then $\overline{u}$ satisfies

$$\partial_s \overline{u}(s,x) = \frac{1}{2}\Delta\overline{u}(s,x) + \frac{1}{2}\left(\overline{\alpha} d\sqrt{1+\eta} - \overline{\alpha}^2 |x|^2\right)\overline{u}(s,x)$$

with $\overline{u}$ equal to 1 on the parabolic boundary with $\overline{\alpha} = r^2\alpha\sqrt{1+\eta}$. Thus we will assume that $r = 1$ and the final constants $\eta(t,r)$, $\alpha(t,r)$ will be easily related to those obtained from the case $r = 1$.

We split the proof of (2.3.21) in two separate lemmas.

Lemma 2.3.27. *If u is the solution in $(0,\infty)\times B(0,1)$ to*

$$\begin{cases} \partial_s u(s,x) = \frac{1}{2}\Delta u(s,x) + \frac{1}{2}\left(d\,(1+\epsilon)\,\alpha - \alpha^2|x|^2\right) u(s,x) \\ u(0,x) = 1 - \exp\left(\frac{\beta\left(|x|^2-1\right)}{2}\right) \ \text{for } |x|<1, \\ u(s,y) = 0 \ \text{for } s>0, |y|=1, \end{cases} \tag{2.3.28}$$

then, for any $K>0$, there exist constants $\epsilon_0(t)$, $\alpha_0(t)$ such that for $\beta=\alpha-K$ and any $\epsilon\le\epsilon_0(t)$, $\alpha\ge\alpha_0(t)$,

$$u(s,x) \le 1 - \exp\left(\frac{\beta\left(|x|^2-1\right)}{2}\right) \tag{2.3.29}$$

when $s\in[0,t]$, $\frac{1}{2}\le|x|\le 1$.

Proof. Set

$$k(x) = 1 - \exp\left(\frac{\beta\left(|x|^2-1\right)}{2}\right)$$

and

$$w(s,x) = \left(\frac{2e^{\epsilon\alpha s}}{1+e^{-2\alpha s}}\right)^{\frac{d}{2}} \exp\left(-\frac{\alpha\tanh\alpha s}{2}|x|^2\right).$$

A simple computation shows that w is the solution on $\mathbb{R}^d$ of the equation

$$\partial_s w(s,x) = \frac{1}{2}\Delta w(s,x) + \frac{1}{2}\left(d\,(1+\epsilon)\,\alpha - \alpha^2|x|^2\right) w(s,x) \tag{*}$$

with the initial condition $w(0,x)=1$.

Let t_α be the unique solution of $\tanh\alpha t = \frac{d+\beta}{2\alpha}$. The idea of the proof is to show that for the time interval $[0,t_\alpha]$, the solution u is bounded above by kw and for the time interval $[t_\alpha,t]$ the solution u is bounded above by w itself. We do this in two separate claims.

Claim 1. For $s\in[0,t_\alpha]$, $x\in B(0,1)$

$$u(s,x) \le k(x)w(s,x). \tag{i}$$

Proof. We check that the right hand side of (i) is a supersolution on $[0,t_\alpha]\times B(0,1)$. This comes down to checking

$$\partial_s(kw)(s,x) \ge \frac{1}{2}\Delta(kw)(s,x) + \frac{1}{2}\left(d\,(1+\epsilon)\,\alpha - \alpha^2|x|^2\right) k(x)w(s,x)$$

which is equivalent to

$$k(x)\partial_s w(s,x) \geq \frac{1}{2}(\Delta k)(x)w(s,x) + \langle \nabla k(x), \nabla w(s,x)\rangle + \frac{1}{2}k(x)\Delta w(s,x) \\ + \frac{1}{2}\left(d\left(1+\epsilon\right)\alpha - \alpha^2|x|^2\right)k(x)w(s,x)$$

and further, because of (*) comes to

$$(\Delta k)(x)w(s,x) + 2\langle \nabla k(x), \nabla w(s,x)\rangle \leq 0 \Leftrightarrow \\ |x|^2(2\alpha \tanh \alpha s - \beta) \leq d,$$

which is true for all x with $|x| \leq 1$ and $0 \leq s \leq t_\alpha$. Since both u and kw have the same initial-boundary values we get the claim by a simple application of (2.3.26) with the set A the empty set.

Claim 2. For $s \in [t_\alpha, t]$, $x \in B(0,1)$ we have

$$u(s,x) \leq w(s,x) \tag{ii}$$

Proof. Because w satisfies (*) and on the parabolic boundary it dominates u this is again a simple application of (2.3.26) with the set A the empty set.

With these two claims at hand, we want to check (2.3.29).

Case ($s \in [0, t_\alpha]$). By claim 1, we need to verify that $k(x)w(s,x) \leq k(x)$ for $\frac{1}{2} \leq |x| \leq 1$, which is the same with $w(s,x) \leq 1$ for that range of x. Using the expression of w, we are down to checking under what conditions

$$\left(\frac{2e^{\epsilon\alpha s}}{1+e^{-2\alpha s}}\right)^{\frac{d}{2}} \exp\left(-\frac{\alpha \tanh \alpha s}{8}\right) \leq 1 \tag{**}$$

for $s \in [0, t_\alpha]$.

To deal with this, first remark that from $\tanh \alpha t_\alpha = \frac{d+\beta}{2\alpha}$ we obtain for large enough α, $\tanh \alpha s \leq \frac{3}{4}$ if $s \in [0, t_\alpha]$. In this case case we claim that for $\epsilon \leq 1$ and α large (**) is true. To see this, take $\sigma = \tanh \alpha s$. Then $e^{-2\alpha s} = \frac{1-\sigma}{1+\sigma}$ and the expression on the left hand side in (**) is

$$\psi(\sigma) = \frac{(1+\sigma)^{\frac{d(\epsilon+2)}{4}}}{(1-\sigma)^{\frac{d\epsilon}{4}}} e^{-\frac{\alpha\sigma}{8}}$$

Now,

$$\psi'(\sigma) = \left(-\frac{\alpha}{8} + \frac{d(\epsilon+2)}{4(1+\sigma)} + \frac{d\epsilon}{4(1-\sigma)}\right)\psi(\sigma)$$

and obviously on the interval $[0, \frac{3}{4}]$ this is negative for large enough α and ϵ less than 1. So, $\psi(\sigma) \leq \psi(0) = 1$ which in turn proves (**).

Case ($s \in [t_\alpha, t]$). By claim 2 one has the bounds on u in terms of bounds on w. Thus, we need to show that for ϵ small and α large we have $w(s,x) \le k(x)$ for $t_\alpha \le s \le t$, $\frac{1}{2} \le |x| \le 1$. Indeed, for $\frac{1}{2} \le |x| \le 1$,

$$w(s,x) \le \left(\frac{2e^{\epsilon\alpha s}}{1+e^{-2\alpha s}}\right)^{\frac{d}{2}} \exp\left(-\frac{\alpha \tanh \alpha s}{8}\right) \le 2^{\frac{d}{2}} \exp\left(\frac{d\alpha\epsilon t}{2} - \frac{d+\beta}{4}\right)$$

and

$$1 - e^{-\frac{3\beta}{8}} \le k(x).$$

From these, if $\epsilon_0(t) \le \frac{1}{4dt}$ and $\alpha_0(t)$ large enough we have $w(s,x) \le k(x)$ for $(s,x) \in [t_\alpha, t] \times \{x, \frac{1}{2} \le |x| \le 1\}$ for $\epsilon \le \epsilon_0(t)$ and $\alpha \ge \alpha_0(t)$, which ends the proof of this case. □

Lemma 2.3.30. *If u is the solution in $(0,\infty) \times B(0,1)$ to*

$$\begin{cases} \partial_s u(s,x) = \frac{1}{2}\Delta u(s,x) + \frac{1}{2}\left(d\,(1+\epsilon)\,\alpha - \alpha^2|x|^2\right) u(s,x) \\ u(0,x) = \exp\left(\frac{\beta\left(|x|^2-1\right)}{2}\right) \quad \textit{for } |x| < 1, \\ u(s,y) = 1 \quad \textit{for} \ s > 0, |y| = 1, \end{cases} \tag{2.3.31}$$

then, there exist dimensional constants $C, K > 0$ and constants, $\epsilon_0(t)$ and $\alpha_0(t)$ such that for $\beta = \alpha - K$ and any $\epsilon \le \epsilon_0(t)$, $\alpha \ge \alpha_0(t)$,

$$u(s,x) \le \exp\left(\frac{\beta\left(|x|^2-1\right)}{2}\right) \tag{2.3.32}$$

for $(s,x) \in [0,t] \times \{x, \frac{1}{2} \le |x| \le 1\}$, and

$$u(s,x) \le e^{-C\alpha} \tag{2.3.33}$$

for $(s,x) \in [0,t] \times \{x, 0 \le |x| \le \frac{1}{2}\}$.

Proof. The strategy of proving this lemma is to make use of Corollary (2.3.26) and to construct our supersolution on pieces of the ball $B(0,1)$.

Take first $0 < \delta$ a small number and fix $m_1 \le \frac{1}{2}$ and denote $m_2 = \frac{m_1}{3}$. The set A in Corollary (2.3.26) is chosen to be $S(0,m_1) \cup S(0,m_2)$.

Region 1: $\mathbf{\{x : m_1 \le |x| \le 1\}}$. In this region we take

$$u_1(s,x) = \exp\left(\frac{(\beta - 2\delta)\left(|x|^2-1\right)}{2}\right).$$

1. Checking

$$\partial_s u_1(s,x) \ge \frac{1}{2}\Delta u_1(s,x) + \frac{1}{2}\left(d\,(1+\epsilon)\,\alpha - \alpha^2|x|^2\right) u_1(s,x)$$

is equivalent to checking

$$0 \geq \left[d(\beta - 2\delta) + |x|^2(\beta - 2\delta)^2\right] + \left[d(1+\epsilon)\alpha - \alpha^2|x|^2\right],$$

for $m_1 \leq |x| \leq 1$, which is further equivalent to

$$(\alpha - \beta + 2\delta)(\alpha + \beta - 2\delta)|x|^2 \geq d((1+\epsilon)\alpha + \beta - 2\delta).$$

This is true for any $m_1 \leq |x| \leq 1$ iff

$$m_1^2 \geq \frac{d(2+\epsilon)\alpha - d(K+2\delta)}{2(K+2\delta)\alpha - (K+2\delta)^2},$$

which is true for $K > \frac{3d}{m_1^2}$, $0 \leq \epsilon < 1$, and α large.

2. The initial-boundary condition is satisfied for $s = 0$, $m_1 \leq |x| \leq 1$ and also for any $s > 0$, $|x| = 1$.

Region 2: $\{\mathbf{x} : \mathbf{m_2} \leq |\mathbf{x}| \leq \mathbf{m_1}\}$. In this region we take

$$u_2(s,x) = \exp\left((\beta - \delta)|x|\left(|x| - m_1\right)\right) \exp\left(\frac{(\beta - 2\delta)\left(m_1^2 - 1\right)}{2}\right).$$

1. Checking

$$\partial_s u_2(s,x) \geq \frac{1}{2}\Delta u_2(s,x) + \frac{1}{2}\left(d\left(1+\epsilon\right)\alpha - \alpha^2|x|^2\right)u_2(s,x)$$

is equivalent to

$$0 \geq (\beta - \delta)\left(2d - \frac{(d-1)m_1}{|x|}\right) + (\beta - \delta)^2(2|x| - m_1)^2 + d\alpha(1+\epsilon) - \alpha^2|x|^2$$

which will be fulfilled if

$$0 \geq 2d(\beta - \delta) + (\beta - \delta)^2(2|x| - m_1)^2 + d\alpha(1+\epsilon) - \alpha^2|x|^2$$

for $m_2 \leq |x| \leq m_1$. To check this out, observe that the right hand side of this is a quadratic in $|x|$ with dominant coefficient $4(\beta - \delta)^2 - \alpha^2$. Thus for α large, this will be positive, and then, to check the above inequality for $m_2 \leq |x| \leq m_1$ it suffices to do this for $|x|$ at the end points, namely at m_1, m_2. This comes down to verifying the following inequalities

- for $|x| = m_1$, $0 \geq 2d(\beta - \delta) + (\beta - \delta)^2 m_1^2 + d\alpha(1+\epsilon) - \alpha^2 m_1^2$ which is equivalent to:
$$m_1^2 \geq \frac{d(3+\epsilon)\alpha - d(2K+2\delta)}{2(K+\delta)\alpha - (K+\delta)^2},$$
and this is true for $K > \frac{4d}{m_1^2}$, $0 \leq \epsilon < 1$ and α large.

- for $|x| = m_2 = \frac{m_1}{3}$, $0 \geq 2d(\beta - \delta) + (\beta - \delta)^2(\frac{m_1}{3})^2 + d\alpha(1+\epsilon) - \alpha^2(\frac{m_1}{3})^2$ which is equivalent to:

$$\left(\frac{m_1}{3}\right)^2 \geq \frac{d(3+\epsilon)\alpha - d(2K+2\delta)}{2(K+\delta)\alpha - (K+\delta)^2}$$

  and again this is true for $K > \frac{4d}{m_1^2}$, $0 \leq \epsilon < 1$ and α large.

2. The boundary condition is reduced here to checking that

$$u_2(0,x) \geq \exp\left(\frac{\beta(|x|^2-1)}{2}\right)$$

   for $m_2 \leq |x| \leq m_1$. This comes down to $2(\beta-\delta)|x|(|x|-m_1)+(\beta-2\delta)(m_1^2-1) \geq \beta(|x|^2-1)$ which can be written as $\beta(|x|-m_1)^2 \geq 2\delta(|x|(|x|-m_1)+(m_1-1))$, certainly true for any $\beta > 0$, $\delta \geq 0$, $|x| \leq m_1$.

3. Next in line is to check the conditions required in Corollary (2.3.26). First we have to start pointing that $u_2(s,x) = u_1(s,x)$ for any s and x with $|x| = m_1$. Now for a given $a \in S(0, m_1)$ we choose $w_a = \frac{a}{|a|}$ and

$$\psi(\sigma) = \begin{cases} u_1(s, a+\sigma w_a) & if\ \sigma \geq 0 \\ u_2(s, a+\sigma w_a) & if\ \sigma < 0. \end{cases}$$

   Then

$$\psi_l'(0) = \langle \nabla u_2(s,a), w_a\rangle = (\beta-\delta)m_1 u_2(s,a)$$
$$\psi_r'(0) = \langle \nabla u_1(s,a), w_a\rangle = (\beta-2\delta)m_1 u_1(s,a)$$

   which shows that $\psi_l'(0) > \psi_r'(0)$.

Region 3: $\{\mathbf{x} : \mathbf{0} \leq |\mathbf{x}| \leq \mathbf{m_2}\}$**.** In this region we take the function

$$u_3(s,x) = \exp\left(\frac{\alpha}{2}\left(1 - \exp(-\frac{2ds}{m_2^2})\right)(m_2^2 - |x|^2)\right) u_2(s, m_2)$$

1. Checking

$$\partial_s u_3(s,x) \geq \frac{1}{2}\Delta u_3(s,x) + \frac{1}{2}\left(d(1+\epsilon)\alpha - \alpha^2|x|^2\right) u_3(s,x)$$

   is equivalent to

$$\frac{2d\alpha}{m_2^2}(m_2^2 - |x|^2)\exp\left(-\frac{2ds}{m_2^2}\right) \geq \left(d(1+\epsilon)\alpha - \alpha^2|x|^2\right)$$
$$+\left[-d\alpha\left(1-\exp\left(-\frac{2ds}{m_2^2}\right)\right) + \alpha^2\left(1-\exp\left(-\frac{2ds}{m_2^2}\right)\right)^2|x|^2\right]$$

or a bit more explicitly

$$d\exp\left(-\frac{2ds}{m_2^2}\right) - \frac{2d|x|^2}{m_2^2}\exp\left(-\frac{2ds}{m_2^2}\right)$$
$$\geq d\epsilon - 2\alpha|x|^2\exp\left(-\frac{2ds}{m_2^2}\right) + \alpha|x|^2\exp\left(-\frac{4ds}{m_2^2}\right)$$

Here is the last change for $\epsilon_0(t)$, small enough, less than $\exp\left(-\frac{2dt}{m_2^2}\right)$. Then the inequality above is satisfied if the following is true:

$$2\alpha - \alpha\exp\left(-\frac{2ds}{m_2^2}\right) - \frac{2d}{m_2^2} \geq 0$$

for all $0 \leq s \leq t$, certainly true if α is large enough.

2. The initial-boundary comparison in this case is

$$u_3(0,x) \geq \exp\left(\frac{\beta(|x|^2-1)}{2}\right)$$

for $0 \leq |x| \leq m_2$, true in this case, because

$$u_3(0,x) = u_2(0,m_2) \geq \exp\left(\frac{\beta\left(m_2^2-1\right)}{2}\right) \geq \exp\left(\frac{\beta\left(|x|^2-1\right)}{2}\right).$$

3. We have to verify the conditions of Corollary (2.3.26). Observe that $u_3(s,x) = u_2(s,x)$ for any $s \in [0,t]$ and x with $|x| = m_2$. Now for a given $a \in S(0,m_2)$ we choose $w_a = \frac{a}{|a|}$ and the function

$$\psi(\sigma) = \begin{cases} u_2(s,a+\sigma w_a) & if\ \sigma \geq 0 \\ u_3(s,a+\sigma w_a) & if\ \sigma < 0. \end{cases}$$

Then

$$\psi_l'(0) = \langle\nabla u_3(s,a), w_a\rangle = -m_2\alpha\left(1-\exp\left(-\frac{2ds}{m_2^2}\right)\right)u_2(s,a)$$
$$\psi_r'(0) = \langle\nabla u_2(s,a), w_a\rangle = -m_2(\beta-\delta)u_2(s,a)$$

Thus $\psi_l'(0) > \psi_r'(0)$ iff $\alpha\left((1-\exp\left(-\frac{2ds}{m_2^2}\right)\right) < \beta - \delta$ for all $s \in [0,t]$. This last one is true iff $\alpha > (1+K)\exp(\frac{2dt}{m_2^2})$, certainly the case for α large.

Define now

$$v_\delta(s,x) = \begin{cases} u_1(s,x) & \text{if } m_1 \leq |x|, \\ u_2(s,x) & \text{if } m_2 \leq |x| \leq m_1, \\ u_3(s,x) & \text{if } |x| \leq m_2. \end{cases}$$

Then v_δ satisfies all the requirements in Corollary (2.3.26), and this gives an upper bound on u. Because this is true for all small enough δ we finally get

$$u(s,x) \le \begin{cases} \exp\left(\frac{\beta(|x|^2-1)}{2}\right) & m_1 \le |x| \le 1, \\ \exp\left(\frac{\beta}{2}\left(2|x|^2 - 2|x|m_1 + m_1^2 - 1\right)\right) & m_2 \le |x| \le m_1, \\ \exp\left(\frac{\alpha}{2}\left(1-\exp(-\frac{2ds}{m_2^2})\right)(m_2^2 - |x|^2) + \frac{\beta}{2}\left(5m_2^2-1\right)\right) & 0 \le |x| \le m_2. \end{cases} \tag{2.3.34}$$

From this we get (2.3.32).

To obtain (2.3.33) we need to analyze each case separately. If $m_1 \le |x| \le \frac{1}{2}$, $s \in [0,t]$ we have

$$u(s,x) \le \exp\left(-\frac{3\beta}{8}\right).$$

If $m_2 \le |x| \le m_1$, $s \in [0,t]$, then

$$u(s,x) \le \exp\left(\frac{\beta}{2}\left(2|x|(|x|-m_1) + m_1^2 - 1\right)\right) \le \exp\left(-\frac{4\beta}{9}\right).$$

At last for $0 \le |x| \le m_2$, $s \in [0,t]$ we get

$$u(s,x) \le \exp\left(\frac{\alpha}{2}m_2^2 + \frac{\beta}{2}(5m_2^2 - 1)\right) \le \exp\left(-\frac{31\beta}{72} + \frac{\alpha}{36}\right) \le \exp\left(-\frac{28\beta}{72}\right)$$

for large β. From all these cases and the fact that $\beta = \alpha - K$ with K a dimensional constant, we see that it is possible to choose a dimensional constant $C > 0$ as required in (2.3.33). □

2.4 Near the Critical Set Case

In this section we analyze the heat kernel $p_k^\alpha(t,x,x)$ for x close to the critical set. The analysis is based on the same line of ideas as in the away case, so we will point out the basic ideas and differences.

In the away case we started by estimating the quantity V_k^α and then we estimated a scalar expression, namelly $q_k^\alpha(t,x,y)$ instead the original heat kernel $p_k^\alpha(t,x,x)$. In this case we can not do this because it will be too rough and we can not get too much.

Here in the first place one uses integration by parts in the Malliavin calculus to eliminate the Dirac function from the expression of the heat kernel. Thus, $p_k^\alpha(t,x,x)$ equals

$$\mathbb{E}^{\mu_x^M}\left[T_x^\alpha(t,\varphi)\exp\left(-\frac{\alpha^2}{2}\int_0^t |\text{grad}h(\varphi(v))|^2 dv + \frac{\alpha}{2}\int_0^t \Delta h(\varphi(v))dv\right)\right] \tag{2.4.1}$$

where T is a $\text{End}(\bigwedge_x M)$-valued Wiener functional that is in all $L^p(\mu_x^M)$, $p > 1$.

Moreover there exists a polynomial $P_p(t,\alpha)$ such that

$$\|T_x^\alpha(t,\varphi)\|_{H.S} \leq R_x^\alpha(t,\varphi)\exp\left(\alpha\int_0^t f_k(\varphi(v))dv\right) \tag{2.4.2}$$

with the function f given by (2.3.2) and

$$\|R_x^\alpha(t,\varphi)\|_{L^p(\mu_x^M)} \leq P_p(t,\alpha) \tag{2.4.3}$$

uniformly on $x \in M$.

The integral on the right hand side of (2.4.1) can be written as a sum of two terms, $I_x^{int}(\alpha)$ the integral taken over the paths staying inside the set $x \in \{x : |x - c_i| < 2r \text{ for some } i\}$ up to time t and $I_x^{ext}(\alpha)$ the integral over paths exiting the set $x \in \{x : |x - c_i| \leq 2r \text{ for some } i\}$ before time t.

Proposition 2.4.4. *For the integral, $I_x^{ext}(\alpha)$ there exist two constants $C_1(t,r) > 0$, $C_2(t,r) > 0$ such that for large α*

$$I_x^{ext}(\alpha) \leq C_1(t,r)e^{-C_2(t,r)\alpha} \tag{*}$$

uniformly for $x \in \{x : |x - c_i| < r$ for some $i\}$.

Proof. To see this we follow the same route as the one in proving Theorem (2.3.1) in the away case. First, from (2.4.2) and (2.4.3), a simple application of Hölder's inequality reduces the estimate to the estimate of the following quantity

$$q_{ext}^\alpha(t,x) = \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{\tau(\varphi)} H_\eta^\alpha(v,\varphi)dv\right), \tau(\varphi) < t\right], \tag{2.4.5}$$

with the notation (3.4.10) and τ the exit time from the set $x \in \{x : |x - c_i| < 2r \text{ for some } i\}$. To estimate this, let's fix x in the ball $B(c,r)$ for a critical point c and then start the procedure we used in (2.3.15) for the away case. Theorem (2.3.15) can be applied here. The last integral in the iteration is

$$\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{\tau(\varphi)} H_\eta^\alpha(v,\varphi)dv\right), \tau(\varphi) < t\right].$$

The key point is to replace the estimate on the exit time we used in the away case with the estimate that is described now. Using local coordinates we identify $B(c,2r)$ with the ball $B(0,2r)$ in R^d. Then, since $H_\eta^\alpha(v,\varphi) = (1+\eta)\alpha d - (1+\eta)\alpha^2|\varphi(v)|^2$ for $v \leq \tau(\varphi)$, we can describe, as in (2.3.22),

$$u(s,x) = \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{\tau(\varphi)} H_\eta^\alpha(v,\varphi)dv\right), \tau(\varphi) < s\right]$$

as the solution to the initial-boundary problem in $(0,\infty)\times B(0,2r)$:

$$\begin{cases} \partial_s u(s,x) = \frac{1}{2}\Delta u(s,x) + \frac{1}{2}(1+\eta)(\alpha d - \alpha^2|x|^2)u(s,x) \\ u(0,x) = 0 \quad \text{for } \; x \in B(0,2r), \\ u(s,y) = 1 \quad \text{for } \; s>0, \; y \in \partial B(0,2r). \end{cases} \tag{2.4.6}$$

Now, as we did before for the away case, we can reduce the problem to the problem in $(0,\infty)\times B(0,1)$

$$\begin{cases} \partial_s v(s,x) = \frac{1}{2}\Delta v(s,x) + \frac{1}{2}\Big(d\big(1+\epsilon\big)\alpha - \alpha^2|x|^2\Big)v(s,x) \\ v(0,x) = 0 \;\; \text{for } \; |x|<1, \\ v(s,y) = 1 \;\; \text{for } \; s>0, |y|=1, \end{cases} \tag{2.4.7}$$

and then (2.3.33), gives the bound

$$u(s,x) \le K_1(r,t)e^{-K_2(r,t)\alpha}$$

for $(s,x)\in[0,t]\times B(0,r)$ which ends the proof of (2.4.4). □

We now turn to the integral $I_x^{int}(\alpha)$, with $x\in B(c_i,r)$ for some i. Remark that this is an integral over paths staying up to time t in $B(c_i,2r)$ and that we are allowed to think in local coordinates which are identified with Euclidean coordinates in the ball $B(0,2r)$ in $\mathbb{R}^d$. We think now of the function h as the Morse function in $\mathbb{R}^d$ given by

$$h_i(x) = -\frac{1}{2}\sum_{j=1}^{\operatorname{ind}(i)} x_j^2 + \frac{1}{2}\sum_{k=\operatorname{ind}(i)+1}^{d} x_k^2.$$

Now we will reverse the roles. Namely we take the starting manifold to be R^d and the Morse function by the above expression. Then we can construct the operator $\Box_i^\alpha$ and the heat kernel for this. The main important ingredient in the estimation of the heat kernel on the compact manifold was boundedness of the curvature of the manifold. We have that in this context as well and then the whole machinery can be applied for the heat kernel analysis. Thus we can approximate the heat kernel $\overline{p}_{i,k}^\alpha(t,x,y)$ of $\Box_i^\alpha$ on $\mathbb{R}^d$ up to an exponentially decaying term by $I_x^{int}(\alpha)$. Consequently,

Theorem 2.4.8. *With the notations just given, there are constants $C_1>0$, $C_2>0$, depending on t, r, such that for large α*

$$\left\|p_k^\alpha(t,x,x) - \overline{p}_{i,k}^\alpha(t,x,x)\right\| \le C_1\exp(-\alpha C_2) \tag{2.4.9}$$

for all $x\in\cup_{i=1}^l B(c_i,r)$ and, for some other constants C_1, C_2,

$$\left|\int_M \operatorname{Tr} p_k^\alpha(t,x,x)dx - \sum_{i=1}^{l}\int_{B(c_i,r)} \operatorname{Tr}\overline{p}_{i,k}^\alpha(t,x,x)dx\right| \le C_1\exp(-\alpha C_2) \tag{2.4.10}$$

or otherwise

$$\left| Q_k^\alpha(t) - \sum_{i=1}^{l} \int_{B(c_i,r)} \operatorname{Tr} \overline{p}_{i,k}^\alpha(t,x,x)dx \right| \leq C_1 \exp(-\alpha C_2). \tag{2.4.11}$$

2.5 The Proof of the Non-degenerate Morse Inequalities

We are now ready to give the proof of Morse Inequalities.

The kernel $\overline{p}_{i,k}^\alpha(t,x,y)$ in (2.4.8) is the heat kernel of the operator $\frac{1}{2}\Box_i^\alpha$ where

$$\Box_c^\alpha = -\Delta + \alpha^2|x|^2 - \alpha(d - 2\text{ind}(i)) + 2\text{hess}h_i$$

acting on k-forms in $\mathbb{R}^d$. Because

$$\bigwedge^k(\mathbb{R}^d) = \bigoplus_{\substack{k_1+k_2=k \\ 0\leq k_1\leq d-\text{ind}(i), 0\leq k_2\leq \text{ind}(i)}} \bigwedge^{k_1}(\mathbb{R}^{d-\text{ind}(i)}) \wedge \bigwedge^{k_2}(\mathbb{R}^{\text{ind}(i)}), \tag{2.5.1}$$

and the fact that the Hessians of h_i is diagonal we can use the representation given in (2.2.24) to arrive at the writing of $\operatorname{Tr}\overline{p}_{i,k}^\alpha(t,x,x)$ as the sum

$$\sum_{\substack{k_1+k_2=k \\ 0\leq k_1\leq d-\text{ind}(i) \\ 0\leq k_2\leq \text{ind}(i)}} e^{t\alpha(-k_1+k_2-\text{ind}(i))}\mathbb{E}_x^{\mathcal{W}_d}\left[\exp\left(-\frac{\alpha^2}{2}\int_0^t |\varphi(\sigma)|^2 d\sigma\right)\delta_x(\varphi(t))\right]$$

The integral above can be identified with the heat kernel of the operator $\frac{1}{2}\Delta - \frac{\alpha^2}{2}|x|^2$ which by Mehler's formula makes the expression above equal to

$$\sum_{\substack{k_1+k_2=k \\ 0\leq k_1\leq d-\text{ind}(i) \\ 0\leq k_2\leq \text{ind}(i)}} \left(\frac{\alpha}{\pi(1-e^{-2t\alpha})}\right)^{\frac{d}{2}} e^{-\alpha\tanh(t\alpha/2)|x|^2 + t\alpha(-k_1+k_2-\text{ind}(i))}.$$

Taking the integral over $B(0,r)$ and changing $x \to \frac{1}{\sqrt{\alpha\tanh(t\alpha/2)}}x$ we get that

$$\int_{B(0,r)} \operatorname{Tr}\overline{p}_{i,k}^\alpha(t,x,x)dx = \sum_{\substack{k_1+k_2=k \\ 0\leq k_1\leq d-\text{ind}(i) \\ 0\leq k_2\leq \text{ind}(i)}} \exp(-\alpha t k_1 - \alpha t(\text{ind}(i) - k_2))A(\alpha)$$

where $A(\alpha) = \left(\frac{1}{\pi(1+e^{-t\alpha})^2}\right)^{\frac{d}{2}} \int_{B(0,r\sqrt{\alpha\tanh(t\alpha/2)})} e^{-|x|^2}dx$. Taking into account that $\lim_{\alpha\to\infty} A(\alpha) = 1$, the integral above tends either to 0 or 1. It does tend to 1 only in the case $k_1 = 0$ and $k_2 = \text{ind}(i)$ which is equivalent to $k = \text{ind}(i)$. Taking the sum over all the critical points we arrive at the following theorem.

Theorem 2.5.2. *For $t > 0$,*

$$\lim_{\alpha\to\infty} Q_k^\alpha(t) = m_k.$$

From here and Theorem (2.1.3) we get to

Theorem 2.5.3 (Non-degenerate Morse Inequalities).

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq B_k - B_{k-1} + \cdots + (-1)^k B_0, \tag{2.5.4}$$

with equality for $k = d$.

2.6 Refinements and Remarks

In this section we want to make some extensions of the Morse inequalities. The case we would like to deal with is the case the critical points are isolated but the Hessian may be degenerate. Even if we were not able to prove the Morse inequalities in this general setting, we want to point out some direct consequences of the techniques developed here.

The functions we want to deal with here are functions $h : M \to \mathbb{R}$ with the set of critical points, $Crit$, isolated, and with the property that for any $c \in Crit$, there exists a local coordinate (U_c, φ_c) in which $\varphi(c_i) = 0$ and the function h has the form

$$h(\varphi_c^{-1}(x)) = h(c) + \frac{1}{2}\sum_{i=1}^{d} \epsilon_i x_i^{q_i} \tag{2.6.1}$$

for any $x \in \varphi_c(U_c)$, with $\epsilon_i = \pm 1$ and $q_i \geq 2$ integer numbers for $i = 1, \cdots, d$. Without loss of generality we may assume that $q_i = 2$ for $i = 1, \cdots, \nu(c)$ and $q_i \geq 3$ for $i = \nu(c), \cdots, d$. Also denote here $\nu^-(c)$ the number of $\epsilon_i = -1$ for $i = 1, \cdots, \nu(c)$, and $\nu^+(c) = \nu(c) - \nu^-(c)$. We then say that the critical point c is *an index point* if all the powers q_i in the above representation are even. If the critical point is an index point then we define, using (2.6.1)

$$index(c) = \text{the number of } i = 1, \cdots, d \text{ such that } \epsilon_i = -1 \tag{2.6.2}$$

Within these notations we set m_k the number of index critical points of index k. Then, we have the following refinements of Morse inequalities for the type of functions we considered above.

Theorem 2.6.3 (Morse Inequalities). *We have the following inequalities*

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq B_k - B_{k-1} + \cdots + (-1)^k B_0, \tag{2.6.4}$$

for any $k = 1, \cdots, d$ with equality for $k = d$.

Proof. We assume that the Hessian at every critical point c has rank at least 1. At the end of the proof we show how one can reduce the problem to this case.

The idea of proving this is the same as the proof of the Morse inequalities. We will run through the steps of the proof and we will point out the main difference and the necessary tuning to this situation.

In estimating the heat kernel we start with the same basic inequality as in (2.1.3), and up to the section (2.3) nothing is changed, except the representation given by the Morse Lemma which is replaced by (2.6.1). To state the corresponding of Theorem (2.3.1), we need to adjust the choice of the balls around the critical points. Because hess$h(c)$ has the rank at least 1 for any critical point c, this means that $\nu(c) \geq 1$ at any critical point c. Since the metric around the critical points is flat, one can choose instead of the balls we worked with in (2.3.1), a product $B(c,r) \times B(c,\rho(t,r))$ with $r, \rho(t,r) < 1$, where $B(c,r)$ is a ball in the variables $x_1, \cdots, x_{\nu(c)}$ and $B(c,\rho(r))$ stands for the ball in the variables $x_{\nu(c)+1}, \cdots, x_d$ with the radius $\rho(t,r)$ to be chosen appropriately below.

We state now

Theorem 2.6.5. *For small $r > 0$, there exist constants $\rho(t,r) > 0$ $\alpha_0(t,r) > 0$, $C_1(t,r) > 0$, $C_2(t,r) > 0$ depending on t and r such that for any $\alpha \geq \alpha_0(t,r)$,*

$$\|p_k^\alpha(t,x,y)\|_{y,x} \leq C_1(t,r)e^{-C_2(t,r)\alpha}$$

uniformly for x outside the sets $B(c,r) \times B(c,\rho(t,r))$ and $y \in M$.

Proof. We give here the outline of the proof. We only mark the important changes to this situation of the proof of Theorem (2.3.1).

We use Lemma (A.2.11) to estimate the associated V_k^α. Here we remark that the properties of the function f_k in (2.3.2) are replaced by

$$\begin{aligned} &f_k(x) \leq \nu^-(c) + D(c)\rho(t,r) \text{ for } x \in B(c,r) \times B(c,\rho(t,r)), \\ &-\text{hess}_x h \leq f_k(x) Id_{\bigwedge_x^k(M)} \text{ for any point } x \in M, \end{aligned} \tag{2.6.6}$$

where $D(c)$ is a constant depending only on the degrees q_i, $i = \nu(c), \ldots, d$. This choice comes naturally since the Hessian of the function h in $B(c,r) \times B(c,\rho(t,r))$ is diagonal with $\nu(c)$ elements on the diagonal equal to 1 and the rest bounded in terms of $\rho(t,r)$. From here we get the estimate

$$\|p_k^\alpha(t,x,y)\|_{y,x} \leq C_d \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^t H_\alpha(v,\varphi)dv\right)\delta_y(\varphi(t))\right] \tag{2.6.7}$$

where C_d is a dimensional constant and

$$H_\alpha(v,\varphi) = -\frac{\alpha^2}{2}|\text{grad}h(\varphi(v))|^2 + \frac{\alpha}{2}\Delta h(\varphi(\sigma)) + \alpha f_k(\varphi(\sigma)).$$

Further we use (2.3.4) to get rid of the delta function. Then, denote

$$q_k^\alpha(t,x) = \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^t H_\eta^\alpha(v,\varphi)dv\right)\right]. \tag{2.6.8}$$

The iteration is basically the same with a few adjustments. First, the set Ω_1 is the complement of the set $\bigcup_{c\in Crit} B(c,r/2)\times B(c,\rho(t,r)/2)$ and Ω_2 the complement of the set $\bigcup_{c\in Crit} B(c,r)\times B(c,\rho(t,r))$. The statement of the Theorem (2.3.15) is the same. Then the proof of Theorem (2.3.1) for this situation is the same. The same is also the proof of Step 1 of Theorem (2.3.15). For Step 2 of the proof of Theorem (2.3.15) we adapt a bit the situation. The main thing we do is we further estimate

$$\begin{aligned}
&\int \exp\left(\int_0^{\tau(\psi)\wedge s} H_\eta^\alpha(v,\psi)dv\right)\mu_y^M(d\psi)\\
&= \iint \exp\left(\int_0^{\tau(\psi')\wedge\tau(\psi'')\wedge s} H_\eta^\alpha(v,\psi',\psi'')dv\right)\mu_{y'}^{\mathbb{R}^{\nu(c)}}(d\psi')\mu_{y''}^{\mathbb{R}^{d-\nu(c)}}(d\psi'') \qquad (*)\\
&\le \iint \exp\left(\int_0^{\tau(\psi')\wedge\tau(\psi'')\wedge s} \overline{H}_\eta^\alpha(v,\psi')dv\right)\mu_{y'}^{\mathbb{R}^{\nu(c)}}(d\psi')\mu_{y''}^{\mathbb{R}^{d-\nu(c)}}(d\psi'')
\end{aligned}$$

where we used $'$ and $''$ to denote the dependence on the first $\nu(c)$ variables respectively for the last $d-\nu(c)$ ones. Also we use the fact that up to the first exit time from a product of two Euclidean balls, the Brownian motion is just the product of the two corresponding Brownian motions. The last inequality in (*) is carried out by

$$\begin{aligned}
H_\eta^\alpha(v,\psi',\psi'') &\le -\frac{\alpha^2(1+\eta)}{2}\left(|\psi'(v)|^2 + \sum_{i=\nu(c)+1}^{d} |q_i\psi_i''|^{2(q_i-1)}\right)\\
&\quad + \frac{\alpha(1+\eta)}{2}\left(\nu^+(c)-\nu^-(c) + \sum_{i=\nu(c)+1}^{d} q_i(q_i-1)\,(\psi_i'')^{q_i-2}\right)\\
&\quad + \frac{\alpha(1+\eta)}{2}\left(2\nu^-(c)+2D(c)\rho(t,r)\right)\\
&\le -\frac{\alpha^2(1+\eta)}{2}|\psi'(v)|^2 + \frac{\alpha(1+\eta)}{2}\left(\nu(c)+O(\rho(t,r))\right) = \overline{H}_\eta^\alpha(v,\psi')
\end{aligned}$$

for $v\in[0,\tau(\psi')\wedge\tau(\psi'')\wedge s]$. Now we take

$$u_\eta^\alpha(s,y) = \int \exp\left(\int_0^{\tau(\psi')\wedge s} \overline{H}_\eta^\alpha(v,\psi')dv\right)\mu_y^{\mathbb{R}^{\nu(c)}}(d\psi').$$

Then as in the section for the proof of Theorem (2.3.21) we identify u as the solution of the initial-boundary problem in $(0,\infty)\times B(0,r)$,

$$\begin{cases}\partial_s u(s,x)=\frac{1}{2}\Delta u(s,x)+\frac{(1+\eta)}{2}(\alpha(1+O(\rho(t,r)))d-\alpha^2|x|^2)u(s,x)\\ u(0,x)=1\ if\ x\in B(0,r),\\ u(s,y)=1\ if\ s>0,\ y\in\partial B(0,r).\end{cases}\tag{2.6.9}$$

Using theorem (2.3.21) we can now choose $\rho(t,r)$ small enough and $\alpha(r,t)>0$, $\eta(r,t)>0$ such that

$$\sup_{\substack{s\in[0,t]\\ r\leq|y|\leq 2r}} u_\eta^\alpha(s,y)\leq 1\tag{2.6.10}$$

for $\alpha\geq\alpha(r,t)$, $\eta\leq\eta(r,t)$. This suffices to end the proof of the away case. □

For the second case, where we analyze the heat kernel for a point close to $Crit$, we follow the same strategy as in section (2.4).

The set of points we work on now is the set $B(c,r)\times B(c,\rho(t,r))$, with $\rho(t,r)$ chosen in such a way that solution to (2.6.9) satisfies the estimates given by Lemmas (2.3.27), (2.3.30). Let's fix a critical point $c\in Crit$ and focus on the heat kernel on the set $B(c,r)\times B(c,\rho(t,r))$. We perform integration by parts and split the resulting integral in two integrals, $I^\alpha_{int}(x)$ and $I^\alpha_{ext}(x)$, according to whether or not the path has left the set $B(c,2r)\times B(c,2\rho(t,r))$.

The integral $I^\alpha_{ext}(x)$ has an exponential decay in α. To proof goes exactly as in the proof of (2.4.4) with the adjustments given above in the proof of the away case for this situation. For the integral $I^\alpha_{int}(x)$, we can prove by the same procedure as in the standard case that the heat kernel is exponentially approximated by the heat kernel of the operator $\square^\alpha_c$ on $\mathbb{R}^d$ associated to the function

$$h_c(x)=\frac{1}{2}\sum_{i=1}^{d}\epsilon_i x_i^{q_i}$$

on $\mathbb{R}^d$. We denote by $\overline{p}^\alpha_{c,k}(t,x,y)$ the heat kernel of the operator $\frac{1}{2}\square^\alpha_c$ with

$$\square^\alpha_c=-\Delta+\alpha^2|\text{grad}h_c|^2-\alpha\Delta h_c+2\text{hess}h_c.$$

Theorem 2.6.11. *With the notations just given, there are constants ρ, $C_1>0$, $C_2>0$, depending on t, r, such that for large α*

$$\left|\int_M \operatorname{Tr} p^\alpha_k(t,x,x)dx-\sum_{c\in Crit}\int_{B(c,r)\times B(c,\rho)}\operatorname{Tr}\overline{p}^\alpha_{c,k}(t,x,x)dx\right|\leq C_1e^{-\alpha C_2},\tag{2.6.12}$$

or in other words

$$\left|Q^\alpha_k(t)-\sum_{c\in Crit}\int_{B(c,r)\times B(c,\rho)}\operatorname{Tr}\overline{p}^\alpha_{i,k}(t,x,x)dx\right|\leq C_1e^{-\alpha C_2}.\tag{2.6.13}$$

With this result at hand we have to do computations on $\mathbb{R}^d$. First note that

$$\bigwedge^k \mathbb{R}^d = \bigoplus_{\Lambda} \mathbb{R} e_\Lambda.$$

where the sum is taken over all subsets Λ of k elements of $\{1, \cdots, d\}$, e_i is the standard basis of $\mathbb{R}^d$ and $e_\Lambda = e_{i_1} \wedge \cdots \wedge e_{i_k}$ if $\Lambda = \{i_1, \cdots, i_k\}$. Also, write $h(x_1, \cdots, x_d) = \sum_{i=1}^d h_i(x_i)$. Then the Hessian of the function h is diagonal and $\text{hess}_x h(e_\Lambda) = \sum_{i \in \Lambda} h_i(x_i) e_\Lambda$. The operator $\Box_c^\alpha$ restricted to $\mathbb{R}e_\Lambda$ is given by

$$\Box_c^\alpha f e_\Lambda = \left(-\Delta + \alpha^2 |\text{grad} h_c|^2 - \alpha \Delta h_c + 2 \sum_{i \in \Lambda} h_{i_j} \right) f e_\Lambda$$

for any function f. From this one, we deduce

$$\text{Tr}\, \overline{p}_{c,k}^\alpha(t, x, x) = \sum_{\Lambda} \overline{p}_{c,\Lambda}^\alpha(t, x, x).$$

with $\overline{p}_{c,\Lambda}^\alpha(t, x, y)$ the heat kernel of the operator $-\frac{1}{2}\Delta + \frac{\alpha^2}{2}|\text{grad} h_c|^2 - \frac{\alpha}{2}\Delta h_c + \sum_{i \in \Lambda} h_i$. This implies

$$\overline{p}_{c,\Lambda}^\alpha(t, x, x) = \prod_{j=1}^d \overline{p}_{c,\Lambda,j}^\alpha(t, x_j, x_j)$$

where $\overline{p}_{c,\Lambda,j}^\alpha(t, x, y)$ is the heat kernel of the one dimensional operator

$$\begin{cases} -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\alpha^2}{2}(h_i')^2 - \frac{\alpha}{2}h_i'' & if\ j \notin \Lambda \\ -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\alpha^2}{2}(h_i')^2 + \frac{\alpha}{2}h_i'' & if\ j \in \Lambda \end{cases}$$

If $h_i(x) = x^{q_i}$ then a simple scaling argument shows that

$$\overline{p}_{c,\Lambda,j}^\alpha(t, x, y) = \alpha^{\frac{1}{q_i}} \overline{p}_{c,\Lambda,j}(\alpha^{\frac{2}{q_i}} t, \alpha^{\frac{1}{q_i}} x, \alpha^{\frac{1}{q_i}} y)$$

with $\overline{p}_{c,\Lambda,j}(t, x, y)$ the heat kernel of the operator

$$L_{c,\Lambda,j} = \begin{cases} -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}(h_i')^2 - \frac{1}{2}h_i'' & if\ j \notin \Lambda \\ -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}(h_i')^2 + \frac{1}{2}h_i'' & if\ j \in \Lambda. \end{cases} \tag{2.6.14}$$

From here, a standard argument shows that

$$\lim_{\alpha \to \infty} \int_{[-r_i, r_i]} \overline{p}_{c,\Lambda,j}^\alpha(t, x, x) dx = \lim_{\alpha \to \infty} \int_{\left[-\alpha^{\frac{1}{q_i}} r_i, \alpha^{\frac{1}{q_i}} r_i\right]} \overline{p}_{c,\Lambda,j}(\alpha^{\frac{2}{q_i}} t, x, x) dx = \dim(\ker(L_{c,\Lambda,j}))$$

and that

$$\lim_{\alpha\to\infty}\int_{B(0,r)\times B(0,\rho(t,r))} \overline{p}^{\alpha}_{c,\Lambda}(t,x,x)dx = \prod_{j=1}^{d}\dim(\ker(L_{c,\Lambda,j})). \tag{2.6.15}$$

To compute the dimension of the kernel of the operator $L = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}(g')^2 - \frac{1}{2}g''$ for a given function $g:\mathbb{R}\to\mathbb{R}$, one has to note that the solution to the equation $Lf = 0$ is given by

$$f(x) = c_1\exp(-g(x)) + c_2\exp(g(x))\int_0^x \exp\left(2g(y)\right)dy$$

and that for $g(x) = \epsilon x^q$ with $\epsilon = \pm 1$, f is in $L^2(\mathbb{R})$ iff

$$f(x) = \begin{cases} c_1\exp(-x^q) & \text{if } q \text{ is even and } \epsilon = 1 \\ 0 & \text{if } q \text{ is odd or, } q \text{ is even and } \epsilon = -1 \end{cases}$$

which proves that

$$\dim(\ker(L)) = \begin{cases} 1 & \text{if } q \text{ is even and } \epsilon = 1 \\ 0 & \text{if } q \text{ is odd or, } q \text{ is even and } \epsilon = -1 \end{cases}$$

With this, one can finish the computation we were left with in (2.6.15) as

$$\prod_{j=1}^{d}\dim(\ker(L_{c,\Lambda,j})) = \begin{cases} 1 & \text{if } q_j \text{ even, } j = 1,\dots,d \\ & \text{and } \epsilon_j = \begin{cases} -1 & \text{if } j\in\Lambda \\ 1 & \text{if } j\notin\Lambda \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

For an index critical point $c \in Crit$, take the representation in (2.6.1) and denote

$$\Lambda_c = \{j\in\{1,\cdots,d\}, \epsilon_j = -1\} \tag{2.6.16}$$

then we can finally compute

$$\lim_{\alpha\to\infty}\int_{B(0,r)\times B(0,\rho(t,r))} \overline{p}^{\alpha}_{c,\Lambda}(t,x,x)dx = \begin{cases} 1 & \text{if } c \text{ is an index point,} \\ & \text{and } \Lambda = \Lambda_c \\ 0 & \text{otherwise.} \end{cases}$$

From this last relation and Theorem (2.6.11), we get that for a given $k\in\{1,\cdots,d\}$

$$\lim_{\alpha\to\infty}\int_M p^{\alpha}_k(t,x,x)dx = m_k$$

which combined with (2.1.3) ends the proof of Theorem (2.6.3).

The last trick is to use the case when at every critical point the Hessian has the rank at least one. Thus we can replace $\tilde{M} = M \times \mathbb{R}$ and the function $\tilde{h}(x, \xi) = h(x) + |\xi|^2$. This way we do not change any of the information about the cohomology of M and also the index of the function $\tilde{h}$ is the same as the index of the function h. Now we have the rank of $\tilde{h}$ at least one. Of course one has to worry a little bit about the fact that the manifold $\tilde{M}$ is not compact. This is not a real problem since all we need it the compactness of the operators and the estimates on the heat kernel. A careful examination of the techniques involved shows that the whole thing can be carried on. □

Remarks. *1. It seems reasonable that one can prove by these methods the Morse inequalities for an arbitrary function, simply replacing the usual index by Conley index.*

2. *The fact that $\lim_{\alpha\to\infty} Q_k^\alpha(t) = m_k$, can also be written in the following more appealing way:*

$$\lim_{\alpha\to\infty} Tr e^{-t\Box^\alpha} = m_k,$$

for any $t > 0$, or equivalently

$$\lim_{\alpha\to\infty} \int_{[0,\infty)} e^{-ts}\mu_\alpha(ds) = \int_{[0,\infty)} e^{-ts}\mu(ds)$$

where μ_α is the counting measure for the spectrum of $\Box^\alpha$ and μ is the measure with mass M_k concentrated at 0. From this interpretation one can prove that $\mu_\alpha \Rightarrow \mu$ in the weak sense. In particular this implies that, for any interval $[a, b]$ which does not contain 0, the operator $\Box^\alpha$ does not have any eigenvalue in it for large enough α. This proves in particular that there are spectral gaps.

3. *The spectral gaps are important in the Witten-Helffer-Sjöstrand theory. The spectral gap gives a splitting of the De Rham complex in the complex of small eigenvalues and the complex of large eigenvalues. The complex of small eigenvalues reduces in the limit to the complex constructed in terms of the Morse function. Thus the complex in dimension k is generated by the critical points of index k, and the differential is given in terms of the trajectories between the points. The ambition is to recover that on probabilistic grounds. It is not hard to recover the complex itself from the analysis we done so far, because the measures μ_α alluded above are tending to the measure with mass m_k centered at 0. What remains is the differential of this complex. We hope to recover this by looking at the heat kernel not only on the diagonal but also off diagonal. The main idea is to investigate the differential with respect to the first variable of the heat kernel. Appropriately integrating this quantity seems to be the key of recovering the incidence numbers in the Morse-Smale complex.*

Chapter 3

Degenerate Morse Inequalities

Given a compact manifold M endowed with a Bott-Morse function $h : M \to \mathbb{R}$ we will give a proof of degenerate Morse inequalities. The approach is based on heat kernel analysis of various operators, which in the end will prove to give the same result as long as one is able to do computations with respect to one of them.

3.1 Geometric Preparations

In this section we prepare the necessary geometry. As we said in the introduction, the main difficulty comes from the fact that the vector bundles constructed around critical manifolds are not in general flat, the consequence of this is the existence of a nontrivial torsion for the Bismut connection. So, we need to deal with a connection on a manifold with a nontrivial torsion.

3.1.1 The Operator d^∇ and Related Computations

The setup in this section is the following. We are given a Riemannian manifold M endowed with a compatible connection ∇, in the sense that

$$\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle \tag{3.1.1}$$

for any vector fields, X, Y, Z, and a function $h : M \to \mathbb{R}$. We then extend the connection ∇ to a derivation given by (2.1.5). Now let us give the following

Definition 3.1.2.

1. *For an orthonormal basis $((E_j)_x)_{j=1,d}$ at $T_x(M)$, set*

$$d_x^\nabla = \sum_{j=1}^{d} (E_j^*)_x \wedge \nabla_{(E_j)_x}.$$

2. *δ^∇ is the adjoint of d^∇.*

3. *$d^{\nabla,h} = e^{-h} d^\nabla e^h$ and its adjoint, $\delta^{\nabla,h} = e^h \delta^\nabla e^{-h}$.*

4. *The associated Laplacian on forms,* $\square^{\nabla} = d^{\nabla}\delta^{\nabla} + \delta^{\nabla}d^{\nabla}$.

5. *Finally, the Witten Laplacian corresponding to the couple* (∇, h)

$$\square^{\nabla,h} = d^{\nabla,h}\delta^{\nabla,h} + \delta^{\nabla,h}d^{\nabla,h}.$$

Just remark that in fact the definition above does not depend on the basis $((E_j)_x)_{j=1,d}$, and also point out that if we choose a local orthonormal basis, we see that the dependence of d sends smooth forms into smooth forms. Next we collect a number of useful facts in the following proposition.

Proposition 3.1.3. *Denote by T the torsion of the connection ∇. Then we have*

1. $\nabla_X i_Y - i_Y \nabla_X = i_{\nabla_X Y}$, *for any vector fields X, Y.*

2. $\langle \nabla_X \omega, \eta\rangle + \langle \omega, \nabla_X \eta\rangle = X\langle \omega, \eta\rangle$, *for any vector field X and any forms ω, η of the same order.*

3. $\delta_x^{\nabla} = -\sum_{j=1}^d i_{(E_j)_x}\nabla_{(E_j)_x} + \sum_{j,k=1}^d \langle T((E_k)_x, (E_j)_x), (E_k)_x\rangle (E_j)_x$, *for any orthonormal basis $((E_j)_x)_{j=1,d}$ at $T_x(M)$.*

4. *For any form ω,*

$$d^{\nabla,h}\omega = d^{\nabla}\omega + dh \wedge \omega, \quad \delta^{\nabla,h}\omega = \delta^{\nabla}\omega + i_{\mathrm{grad} h}\omega$$

Proof. 1. We start by pointing out that i_X is a anti-derivation on $\wedge(M)$. On the other hand we also have that

$$\begin{aligned}
(\nabla_X i_Y - i_Y \nabla_X)(\omega_1 \wedge \omega_2) &= \nabla_X\left((i_Y\omega_1)\wedge\omega_2\right) + (-1)^{|\omega_1|}\nabla_X\left(\omega_1 \wedge i_Y(\omega_2)\right) \\
&\quad - i_Y((\nabla_X\omega_1)\wedge\omega_2) - i_Y(\omega_1 \wedge (\nabla_Y\omega_2)) \\
&= (\nabla_X i_Y \omega_1)\wedge\omega_2 + (i_Y\omega_1)\wedge(\nabla_X\omega_2) \\
&\quad + (-1)^{|\omega_1|}\omega_1 \wedge (\nabla_X i_Y \omega_2) + (-1)^{|\omega_1|}(\nabla_X\omega_1)\wedge(i_Y\omega_2) \\
&\quad - (i_Y\nabla_X\omega_1)\wedge\omega_2 - (-1)^{\omega_1}(\nabla_X\omega_1)\wedge(i_Y\omega_2) \\
&\quad - (i_Y\omega_1)\wedge(\nabla_X\omega_2) - (-1)^{|\omega_1|}\omega_1\wedge(\imath_y\nabla_X\omega_2) \\
&= ((\nabla_X i_Y - i_Y\nabla_X)\omega_1)\wedge\omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge ((\nabla_X i_Y - i_Y \nabla_X)\omega_2)
\end{aligned}$$

which shows that $\nabla_X i_Y - i_Y \nabla_X$ is also an anti-derivation. Thus, in order to verify the assertion it suffices to check it on 0-forms and on 1-forms. On functions both sides of the identity to be checked are 0. For 1-forms we first point out that it is enough to prove that if, $X = E_j$, $Y = \sum_{k=1}^d \varphi_k E_k$ and $\omega = fE_l^*$, where E_j is a local orthonormal basis around the point we are working near, then $(\nabla_X i_Y - i_Y\nabla_X - i_{\nabla_X Y})\omega = 0$. It suffices to show this when $\omega = E_j$, and this is

$$(\nabla_X i_Y - i_Y \nabla_X - i_{\nabla_X Y})\omega = \sum_{k=1}^{d} \left(\nabla_{E_j} \varphi_k i_{E_k} E_l^* - \varphi_k i_{E_k} \nabla_{E_j} E_k^* - i_{\nabla_{E_j} \varphi_k E_k} E_l^* \right)$$

$$= \sum_{k=1}^{d} \left(E_j(\varphi_k) i_{E_k} E_l^* + \varphi_k \nabla_{E_j} i_{E_k} E_l^* - \varphi_k i_{E_k} \nabla_{E_j} E_l^* - E_j(\varphi_k) i_{E_k} E_l^* - \varphi_k i_{\nabla_{E_j} E_k} E_l^* \right)$$

$$= \sum_{k=1}^{d} \varphi_k \left(\nabla_{E_j} i_{E_k} E_l^* - i_{E_k} \nabla_{E_j} E_l^* - i_{\nabla_{E_j} E_k} E_l^* \right)$$

Now,

$$\nabla_{E_j} E_l^* = \sum_{p=1}^{d} \langle \nabla_{E_j} E_l^*, E_p^* \rangle E_p^* = - \sum_{p=1}^{d} \langle E_l, \nabla_{E_j} E_p \rangle E_p^*, \tag{3.1.4}$$

because from definition, $\langle \nabla_{E_j} E_l^*, E_p^* \rangle = (\nabla_{E_j})(E_p) = -\langle E_l, \nabla_{E_j} E_p \rangle$. Hence,

$$\begin{aligned} &\nabla_{E_j} i_{E_k} E_l^* - i_{E_k} \nabla_{E_j} E_l^* - i_{\nabla_{E_j} E_k} E_l^* \\ &= \nabla_{E_j} \delta_{kl} + i_{E_k} \sum_{p=1}^{d} \langle E_l, \nabla_{E_j} E_p \rangle E_p^* + \sum_{q=1}^{d} \langle E_q, \nabla_{E_j} E_k \rangle i_{E_q} E_l^* \\ &= \sum_{p=1}^{d} \langle E_l, \nabla_{E_j} E_p \rangle \delta_{kp} + \sum_{q=1}^{d} \langle E_q, \nabla_{E_j} E_k \rangle \delta_{ql} \\ &= \langle E_l, \nabla_{E_j} E_k \rangle + \langle E_l, \nabla_{E_j} E_k \rangle \\ &= 0 \end{aligned},$$

where for the last equality we used the compatibility of the connection with the metric.

2. We only have to do it locally, thus we may assume that for a local orthonormal basis E_j around the point in discussion, $\omega = f E_1^* \wedge E_2^* \cdots \wedge E_k^*$, $\eta = g E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^*$. Then, $\nabla_X \omega = X(f) E_1^* \wedge E_2^* \cdots \wedge E_k^* + f \nabla_X (E_1^* \wedge E_2^* \cdots \wedge E_k^*)$ and similarly, $\nabla_X \eta = X(g) E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^* + g \nabla_X (E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^*)$.

From the above, the equality to be proven is:

$$\begin{aligned} &X(f) g \langle E_1^* \wedge E_2^* \cdots \wedge E_k^*, E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^* \rangle \\ &+ f X(g) \langle E_1^* \wedge E_2^* \cdots \wedge E_k^*, E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^* \rangle \\ &+ fg \Big(\langle \nabla_X (E_1^* \wedge E_2^* \cdots \wedge E_k^*), E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^* \rangle \\ &+ \langle E_1^* \wedge E_2^* \cdots \wedge E_k^*, \nabla_X (E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^*) \rangle \Big) \end{aligned}$$

$$= X\Big(fg\langle E_1^* \wedge E_2^* \cdots \wedge E_k^*, E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^*\rangle\Big)$$

and since $X(f)g + fX(g) = X(fg)$ everything reduces to proving that

$$\langle \nabla_X(E_1^* \wedge E_2^* \cdots \wedge E_k^*), E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^*\rangle$$
$$+ \langle E_1^* \wedge E_2^* \cdots \wedge E_k^*, \nabla_X(E_{j_1}^* \wedge E_{j_2}^* \cdots \wedge E_{j_k}^*)\rangle = 0.$$

or equivalently

$$\sum_{p=1}^{k}\Big(\langle E_1^* \wedge \cdots \wedge \nabla_X E_p^* \cdots \wedge E_k^*, E_j^* \wedge E_2^* \cdots \wedge E_k^*\rangle$$
$$+ \langle E_1^* \wedge E_2^* \cdots \wedge E_k^*, E_j^* \wedge \cdots \wedge \nabla_X E_p^* \cdots \wedge E_j^*\rangle\Big) = 0.$$

If the sets $\{1, 2, \cdots, k\}$ and $\{j_1, j_2, \cdots, j_k\}$ differ by more than one element then the sum above is 0. Thus, we may assume that $j_1 = j$ and $j_2 = 2, \cdots j_k = k$. By (3.1.4) we get $\nabla_X E_l^* = -\sum_{q=1}^{d}\langle E_l, \nabla_X E_q\rangle E_q^*$, and if we plug this in the above equality, we need to show

$$-\sum_{p=2}^{k}\sum_{q=2}^{d}\langle E_p, \nabla_X E_q\rangle\langle E_1^* \wedge \cdots \wedge E_q^* \cdots \wedge E_k^*, E_j^* \wedge E_2^* \cdots \wedge E_k^*\rangle$$
$$-\sum_{p=2}^{k}\sum_{q=2}^{d}\langle E_p, \nabla_X E_q\rangle\langle E_1^* \wedge E_2^* \cdots \wedge E_k^*, E_j^* \wedge \cdots \wedge E_q^* \cdots \wedge E_k^*\rangle$$
$$+\sum_{q=1}^{d}\langle E_1, \nabla_X E_q\rangle\langle E_q^* \wedge E_2^* \cdots \wedge E_k^*, E_j^* \wedge E_2^* \cdots \wedge E_k^*\rangle$$
$$+\sum_{q=1}^{d}\langle E_j, \nabla_X E_q\rangle\langle E_1^* \wedge E_2^* \cdots \wedge E_k^*, E_q^* \wedge E_2^* \cdots \wedge E_k^*\rangle = 0.$$

We distinguish now two cases. If $j \neq 1$, the first and the second lines in the above expression are 0, while the third is not 0 if $q = j$ and the forth is not 0 if $q = 1$. In this way the calculation is collapsed to $\langle E_1, \nabla_X E_j\rangle + \langle E_j, \nabla_X E_1\rangle = 0$, which is certainly true by compatibility. The other case is $j = 1$. Then, the forth and third lines are 0 while the the first and second lines are not zero iff $q = p$. But then what is left is a sum over p of terms of the form $\langle E_p, \nabla_X E_p\rangle$, again 0 because of the compatibility.

3. Since ∇ is a derivation we have at first

$$\nabla_{E_j}(E_j^* \wedge \omega) = (\nabla_{E_j} E_j^*) \wedge \omega + E_j^* \wedge (\nabla_{E_j}\omega)$$

for any form ω. Taking duals and using again (3.1.4) we get

$$(E_j^* \wedge \nabla_{E_j})^* = i_{E_j} \nabla_{E_j}^* + i_{\sum_{k=1}^d \langle E_j, \nabla_{E_j} E_k \rangle E_k}$$

On the other hand

$$\int_M \langle \nabla_X \omega, \eta \rangle + \int_M \langle \omega, \nabla_X \eta \rangle = \int_M X \langle \omega, \eta \rangle = -\int_M \operatorname{div}(X) \langle \omega, \eta \rangle,$$

thus

$$\nabla_X^* = -\nabla_X - \operatorname{div}(X). \tag{3.1.5}$$

Continuing the above formula,

$$(E_j^* \wedge \nabla_{E_j})^* = -i_{E_j} \nabla_{E_j} - \operatorname{div}(X) i_{E_j} + \sum_{k=1}^d \langle E_j, \nabla_{E_j} E_k \rangle i_{E_k}.$$

Taking sum over j we arrive at

$$\delta^\nabla = -\sum_{j=1}^d i_{E_j} \nabla_{E_j} - \sum_{j=1}^d \operatorname{div}(E_j) i_{E_j} + \sum_{j,k=1}^d \langle E_k, \nabla_{E_k} E_j \rangle i_{E_j}.$$

Now, $\operatorname{div}(E_j) = \sum_{k=1}^d \langle E_k, [E_k, E_j] \rangle$, and plugging this in the above formula we get

$$\begin{aligned}
\delta^\nabla &= -\sum_{j=1}^d i_{E_j} \nabla_{E_j} - i_{\sum_{j=1}^d \operatorname{div}(E_j) E_j} + \sum_{j,k=1}^d \langle E_k, \nabla_{E_k} E_j \rangle i_{E_j} \\
&= -\sum_{j=1}^d i_{E_j} \nabla_{E_j} + i_{\sum_{j,k=1}^d (\langle E_k, \nabla_{E_k} E_j \rangle i_{E_j} + \langle [E_j, E_k], E_k \rangle E_j)} \\
&= -\sum_{j=1}^d i_{E_j} \nabla_{E_j} + i_{\sum_{j,k=1}^d \langle T(E_k, E_j), E_k \rangle E_j}.
\end{aligned}$$

where in passing to the last row we used that $\langle E_k, \nabla_{E_k} E_j \rangle = \langle E_k, \nabla_{E_j} E_k + [E_k, E_j] + T(E_k, E_j) \rangle$ and that by compatibility $\langle E_k, \nabla_{E_j} E_k \rangle = 0$.

4. This is coming from:

$$\begin{aligned}
d^{\nabla,h} \omega &= e^{-h} \sum_{j=1}^d E_j^* \wedge \nabla_{E_j} e^h \omega = e^{-h} \sum_{j=1}^d E_j^* \wedge \left(e^h E_j(h) \omega + e^h \nabla_{E_j} \omega \right) \\
&= d^\nabla \omega + \sum_{j=1}^d E_j(h) E_j^* \wedge \omega = d^\nabla \omega + dh \wedge \omega.
\end{aligned}$$

5. From formula in 3. we finally get

$$\begin{aligned}\delta^{\nabla,h}\omega &= e^h(-\sum_{j=1}^{d} i_{E_j}\nabla_{E_j}e^{-h}\omega + i_{\sum_{j,k=1}^{d}\langle T(E_k,E_j),E_k\rangle E_j}e^{-h}\omega)\\ &= \sum_{j=1}^{d} i_{E_j}\nabla_{E_j}E_j(h)\omega + \sum_{j=1}^{d} i_{E_j}\nabla_{E_j}\omega + i_{\sum_{j,k=1}^{d}\langle T(E_k,E_j),E_k\rangle E_j}e^{-h}\omega\\ &= \delta^{\nabla}\omega + i_{\text{grad}h}\omega.\end{aligned}$$

□

Next we want to express the operator $\square^{\nabla,h}$ in terms of $\square^{\nabla}$ and some other known quantities. For this we start with the twin of (2.1.4)

$$\begin{aligned}\square^{\nabla,h}\omega &= (d^{\nabla} + dh\wedge)(\delta^{\nabla}\omega + i_{\text{grad}h}\omega) + (\delta^{\nabla} + i_{\text{grad}h})(d^{\nabla}\omega + dh \wedge \omega)\\ &= d^{\nabla}\delta^{\nabla}\omega + d^{\nabla}i_{\text{grad}h}\omega + dh \wedge \delta\omega + dh \wedge i_{\text{grad}h}\omega\\ &\quad + \delta^{\nabla}d^{\nabla}\omega + \delta^{\nabla}dh \wedge \omega + i_{\text{grad}h}d\omega + i_{\text{grad}h}dh \wedge \omega\\ &= \square^{\nabla}\omega + dh \wedge i_{\text{grad}h}\omega + i_{\text{grad}h}dh \wedge \omega\\ &\quad + (d^{\nabla}i_{\text{grad}h} + i_{\text{grad}h}d^{\nabla})\omega + (\delta^{\nabla}dh \wedge + dh \wedge \delta^{\nabla})\omega\\ &= \square^{\nabla}\omega + |\text{grad}h|^2\omega + L^{\nabla}_{\text{grad}h}\omega + (L^{\nabla}_{\text{grad}h})^*\omega,\end{aligned} \tag{3.1.6}$$

with the notation $L^{\nabla}_X = d^{\nabla}i_X + i_Xd^{\nabla}$ where X is a vector field and its corresponding adjoint. We have now

Proposition 3.1.7. *We have*

$$L^{\nabla}_X + (L^{\nabla}_X)^* = -\text{div}(X) + \sum_{j,k=1}^{d}(\langle\nabla_{E_j}X, E_k\rangle + \langle\nabla_{E_k}X, E_j\rangle)E_j^* \wedge i_{E_k}.$$

Proof. Basically it is a copy and paste of the proof of (2.1.7) and instead of the classical identities for the Levi-Civita connection we use the above proposition for the replacements. For completeness, here are the details. We mention that we will chose a local orthonormal basis E_j around a point and we do computations locally. Then,

$$\begin{aligned}L^{\nabla}_X &= \sum_{j=1}^{d}(E_j^* \wedge \nabla_{E_j}i_X + i_XE_j^* \wedge \nabla_{E_j}) = \sum_{j=1}^{d}(E_j^* \wedge \nabla_{E_j}i_X + \langle X, E_j\rangle\nabla_{E_j} - i_XE_j^* \wedge \nabla_{E_j})\\ &= \sum_{j=1}^{d}(E_j^* \wedge i_{\nabla_{E_j}X} + \langle X, E_j\rangle\nabla_{E_j}) = \nabla_X + \sum_{j,k=1}^{d}\langle\nabla_{E_j}X, E_k\rangle E_j^* \wedge i_{E_k}\end{aligned}$$

where we have used the first item in Proposition (3.1.3). Thus, taking the adjoints and reminding that $\nabla_X^* = -\text{div}(X) - \nabla_X$, we get to

$$(L_X^\nabla)^* = \nabla_X^* + \sum_{j,k=1}^{d} \langle \nabla_{E_j} X, E_k \rangle E_k^* \wedge i_{E_j}$$
$$= -\text{div}(X) - \nabla_X + \sum_{j,k=1}^{d} \langle \nabla_{E_j} X, E_k \rangle E_k^* \wedge i_{E_j}$$

Adding up what we got, we arrive at

$$L_X^\nabla + (L_X^\nabla)^* = -\text{div}(X) + \sum_{j,k=1}^{d} (\langle \nabla_{E_j} X, E_k \rangle + \langle \nabla_{E_k} X, E_j \rangle) E_j^* \wedge i_{E_k}.$$

□

Definition 3.1.8. *1. We define the symmetric Hessian* Shess$h$ *of the function h with respect to the connection ∇ by*

$$\langle \text{Shess}_x^\nabla h X_x, Y_x \rangle = \frac{1}{2}(\langle \nabla_{X_x} \text{grad} h, Y_x \rangle + \langle X_x, \nabla_{Y_x} \text{grad} h \rangle)$$

for any point x and any vectors $X_x, Y_x \in T_x(M)$

2. The Hessian with respect to the connection ∇ is given as

$$(\text{hess}_x^\nabla h) X_x = \nabla_{X_x} \text{grad} h$$

3. The ∇-Laplacian on forms is given for any form ω defined around the point x by

$$(\Delta^\nabla \omega)(x) = \sum_{j=1}^{d} \nabla_{(E_j)_x} \nabla_{E_j} \omega - \nabla_{\nabla_{(E_j)_x} E_j} \omega$$

for any orthonormal basis E_j around the point x.

Note that the definition of the ∇-Laplacian does not depend on the choice of the basis E_j and is the same as the one given in (2.2.5) for the Levi-Civita connection.

With this definition we state

Corollary 3.1.9.

$$\Box^{\nabla,h} = \Box^\nabla + |\text{grad} h|^2 - \Delta h + 2\text{Shess}^\nabla h$$

where Shess$h$ *is the extension to forms of the symmetric Hessian given by* (2.1.5).

Proof. One only has to notice that $\Delta h = \text{div}(\text{grad})h$, the rest is just the definition and the above proposition. □

Next we want to find a replacement for the Wietzenböck's formula from the usual case of Levi-Civita connection.

Theorem 3.1.10. *If T, R are the torsion, respectively the curvature of the connection ∇, then for any local orthonormal basis E_j around a point x we have*

$$\Box^\nabla = -\Delta^\nabla - D^*R + \sum_{j,k,l}^{d} \langle T(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k}\nabla_{E_l}$$
$$+ \sum_{j,k=1}^{d} \langle T(E_k, E_j), E_k\rangle \nabla_{E_j} + \sum_{j,k,l=1}^{d} E_j^* \wedge i_{\nabla_{E_j}(\langle T(E_k,E_l),E_k\rangle E_l)}$$

where D^*R is defined in (2.1.5).

Proof. For simplicity we are going to use Einstein's convention. So, any time an index appears twice it is summed over it from 1 to d.

$$\begin{aligned}
d^\nabla\delta^\nabla + \delta^\nabla d^\nabla &= -E_j^* \wedge \nabla_{E_j}(i_{E_k}\nabla_{E_k}) + E_j^* \wedge \nabla_{E_j} i_{\langle T(E_k,E_l),E_k\rangle E_l} \\
&\quad - i_{E_k}\nabla_{E_k}(E_j^* \wedge \nabla_{E_j}) + i_{\langle T(E_k,E_l),E_k\rangle E_l} E_j^* \wedge \nabla_{E_j} \\
&\overset{(3.1.3)\ 1.}{=} -E_j^* \wedge i_{E_k}\nabla_{E_j}\nabla_{E_k} - E_j^* \wedge i_{\nabla_{E_j}E_k}\nabla_{E_k} \\
&\quad - i_{E_k}(\nabla_{E_k}E_j^*) \wedge \nabla_{E_j} - i_{E_k}(E_j^* \wedge \nabla_{E_k}\nabla_{E_j}) \\
&\quad + E_j^* \wedge i_{\nabla_{E_j}(\langle T(E_k,E_l),E_k\rangle E_l)} + E_j^* \wedge i_{\langle T(E_k,E_l),E_k\rangle E_l}\nabla_{E_j} \\
&\quad + \langle T(E_k, E_j), E_k\rangle \nabla_{E_j} - E_j^* \wedge i_{\langle T(E_k,E_l),E_k\rangle E_l}\nabla_{E_j} \\
&\text{notice that, } E_j^* \wedge i_{E_k} + i_{E_k}E_j^* \wedge = \delta_{jk} \\
&= -\nabla_{E_j}\nabla_{E_j} - E_j^* \wedge i_{E_k}(\nabla_{E_j}\nabla_{E_k} - \nabla_{E_k}\nabla_{E_j}) \\
&\quad - \langle \nabla_{E_j}E_k, E_l\rangle E_j^* \wedge i_{E_l}\nabla_{E_k} \\
&\quad - \langle \nabla_{E_k}E_j^*, E_k^*\rangle \nabla_{E_j} + \langle \nabla_{E_k}E_j^*, E_l^*\rangle E_l^* \wedge i_{E_k}\nabla_{E_j} \\
&\quad + E_j^* \wedge i_{\nabla_{E_j}(\langle T(E_k,E_l),E_k\rangle E_l)} + \langle T(E_k, E_j), E_k\rangle \nabla_{E_j}.
\end{aligned}$$

The last line in this computation is the last line in the formula we want to prove. Thus we have to deal with the first three lines in the above formula. Thus, further using $\langle \nabla_{E_j}E_l, E_k\rangle = -\langle \nabla_{E_j}E_k, E_l\rangle$

$$\begin{aligned}
&- \nabla_{E_j}\nabla_{E_j} - E_j^* \wedge i_{E_k}(\nabla_{E_j}\nabla_{E_k} - \nabla_{E_k}\nabla_{E_j}) - \langle \nabla_{E_j}E_k, E_l\rangle E_j^* \wedge i_{E_l}\nabla_{E_k} \\
&\quad = -\Delta^\nabla - \nabla_{\nabla_{E_j}E_j} - E_j^* \wedge i_{E_k}R(E_j, E_k) - E_j^* \wedge i_{E_k}\nabla_{[E_j,E_k]} \\
&\quad\quad - \langle \nabla_{E_j}E_k, E_l\rangle E_j^* \wedge i_{E_l}\nabla_{E_k} + \langle \nabla_{E_j}E_j, E_k\rangle \nabla_{E_k} \\
&\quad\quad - \langle \nabla_{E_l}E_k, E_j\rangle E_l^* \wedge i_{E_k}\nabla_{E_j} \\
&\quad = -\Delta^\nabla - D^*R - \langle [E_j, E_k], E_l\rangle E_j^* \wedge i_{E_k}\nabla_{E_l} \\
&\quad\quad + \langle \nabla_{E_j}E_k, E_l\rangle E_j^* \wedge i_{E_k}\nabla_{E_l} - \langle \nabla_{E_k}E_j, E_l\rangle E_j^* \wedge i_{E_k}\nabla_{E_l} \\
&\quad = -\Delta^\nabla - D^*R + \sum_{j,k,l}^{d} \langle T(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k}\nabla_{E_l}
\end{aligned}$$

and this proves the theorem. □

An immediate corollary of this is

Corollary 3.1.11. *If the torsion T satisfies the condition that for any $X \in T_x(M)$ and any orthonormal basis E_j at $T_x(M)$, $\sum_{j=1}^{d}\langle T(E_j, X), E_j\rangle = 0$, then we get the following shorter formula*

$$\Box^{\nabla} = -\Delta^{\nabla} - D^*R + \sum_{j,k,l}^{d}\langle T(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k}\nabla_{E_l} \tag{3.1.12}$$

In these conditions we obtain the decomposition

$$\Box^{\nabla,h} = -\Delta^{\nabla} + |\mathrm{grad}h|^2 - \Delta h + 2\mathrm{Shess}^{\nabla}h + \sum_{j,k,l}^{d}\langle T(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k}\nabla_{E_l} - D^*R \tag{3.1.13}$$

3.1.2 The Connection on a Vector Bundle

In this section we will construct and analyze Bismut's connection, as defined in [1], on a vector bundle over a Riemannian manifold.

Let E be a ν-dimensional vector bundle over a Riemannian manifold M of dimension m with the projection map $\rho : E \to M$. Assume that E is also endowed with metrics on fibers such that the metric depends smoothly as we move from one point to another in the base manifold M. Also we assume that we are given a compatible connection $\nabla^V : T(M) \times \Gamma(M, E) \to \Gamma(M, E)$ where $\Gamma(M, E)$ stands for the sections in E. By definition ∇^V satisfies the following properties:

1. $X \in T(M) \to \nabla^V_X s \in \Gamma(M, E)$, and $s \in \Gamma(M, E) \to \nabla^V_X s \in \Gamma(M, E)$ are linear maps for any $X \in T(M)$, $s \in \Gamma(M, E)$;
2. $\nabla^V_X(fs) = X(f)s + f\nabla^V_X s$ for any $X \in T(M)$, $s \in \Gamma(M, E)$;
3. $\langle\nabla^V_X s_1, s_2\rangle + \langle s_1, \nabla^V_X s_2\rangle = X\langle s_1, s_2\rangle$ for $X \in T(M)$, $s_1, s_2 \in \Gamma(M, E)$.

Given this connection we will construct a connection on $T(E)$. To start this, we remind here that there is a well defined parallel transportation between fibers of E. This can be done by first observing that if c is a curve in M and $X(t) \in E_{c(t)}$ is a smooth section along c, then one can define $\nabla_{\dot{c}(t)}X = \nabla_{\dot{c}}\tilde{X}(t)$ where $\tilde{X}$ is any smooth extension of X to a neighborhood of $c(t)$. Then, the notion of parallel transport of $X_0 \in E_{c(0)}$ along the curve c is a section $X(t)$ along $c(t)$ such that

$$\nabla_{\dot{c}(t)}X = 0. \tag{3.1.14}$$

Denote $\tau^V_{c|[0,t]}X_0 := X_{c(t)}$ and call it the parallel transportation along the curve c from 0 to t.

Notation. *We will use z to denote a generic point on E and x, a generic point on M, while y will stand for a generic point in a fiber of E. Thus, the point z will be often identified with its corresponding coordinates (x, y) with $x = \rho(z)$, and y its identification as a vector in the fiber E_x.*

Now, given a curve c in M and a point $y \in E_{c(0)}$ we define its lift $\tilde{c}$ to a curve in E starting at y by the prescription that

$$\tilde{c}(t) = \tau^V_{c\restriction[0,t]} y. \tag{3.1.15}$$

Using this lifting we define the lifting of vectors in $T(M)$ to vectors in $T(E)$. For $X_x \in T_x(M)$ take a curve such that $c(0) = x$, $\dot{c}(0) = X_x$ and its lift $\tilde{c}$ starting at $y \in E_x$. Then the lifting of X_x is defined by

$$X^H_z = \dot{\tilde{c}}(0), \tag{3.1.16}$$

and is called the horizontal lift of X_x. We show that this definition is not curve dependent. To do this we are going o express this in local coordinates. Choose a local coordinate system $(x_i)_{i=1,m}$ on $U \subset M$ and $(y_j)_{j=1,\nu}$ a trivialization of E over U. In this coordinates we have

$$\nabla^V_{\left(\frac{\partial}{\partial x_i}\right)_x} \frac{\partial}{\partial y_j} = \overline{\Gamma}^q_{i,j}(x) \frac{\partial}{\partial y_q} \tag{3.1.17}$$

where the summation is made in Einstein's convention. If we represent the curve $\tilde{c}(t) = (c(t), v(t))$, and $c(t) = (c^1(t), \ldots, c^m(t))$, $v(t) = (v^1(t), \ldots, v^\nu(t))$ in these coordinates, then the parallel transportation can be rephrased as

$$\begin{aligned}
\nabla^V_{\dot{c}(t)} v &= \nabla^V_{\dot{c}(t)} v^j \frac{\partial}{\partial y_j} \\
&= \dot{v}^j(t) \frac{\partial}{\partial y_j} + v^j(t) \nabla^V_{\dot{c}(t)} \frac{\partial}{\partial y_j} \\
&= \dot{v}^j(t) \frac{\partial}{\partial y_j} + v^j(t) \dot{c}^i(t) \nabla^V_{\left(\frac{\partial}{\partial x_i}\right)_{c(t)}} \frac{\partial}{\partial y_j} \\
&= \dot{v}^j(t) \frac{\partial}{\partial y_j} + v^j(t) \dot{c}^i(t) \overline{\Gamma}^q_{i,j}(c(t)) \frac{\partial}{\partial y_q} \\
&= 0
\end{aligned}$$

which gives

$$\dot{v}^j(t) + v^p(t) \dot{c}^i(t) \overline{\Gamma}^j_{i,p}(c(t)) = 0$$

for any $j = 1, \nu$. Notice here that the values of $\dot{v}^j(0)$ depend only on the values of $v^j(0)$, $c^j(0)$ and $\dot{c}^j(0)$. In particular at $t = 0$ we get the components of

$$X^H_z = \dot{c}^i(0) \left(\frac{\partial}{\partial x_i}\right)_x + \dot{v}^j(0) \frac{\partial}{\partial y_j}$$

depending only on the values of $c(0)$, $\dot{c}^j(0)$, $v(0)$, thus the lifting is well defined. Moreover we get the formula of the lifting

$$\left(\frac{\partial}{\partial x_i}\right)_z^H = \left(\frac{\partial}{\partial x_i}\right)_x - y^j\Gamma_{i,j}^p(x)\frac{\partial}{\partial y_p} \tag{3.1.18}$$

Denote by $T_z^H(E)$ the space of all horizontal lifts from $T_x(M)$ to $T_z(E)$ and by $T_z^V(E)$ the space of vertical vectors at z in $T_z(E)$, namely the space of tangent vectors of curves starting at z and staying in E_x. From (3.1.18) we see that the maps

$$X \in T_x(M) \to X^H \in T_z^H(E)$$

and

$$X \in E_x \to X^V \in T_z^V(E)$$

are bijective maps, thus we have the following smooth splitting

$$T_z(E) = T_z^H(E) \oplus T_z^V(E).$$

The upshot of this decomposition is that we can lift the metric from $T(M)$ to a metric on $T^H(E)$, and also that the metric in the fiber E_x can be naturally moved to a metric on $T^V(E)$. In this way we construct the metric on $T(E)$ by declaring the subspaces $T_z^H(E)$ and $T_z^V(E)$ to be orthogonal. For a vector $X \in T_z(E)$ we set X^H, X^V the horizontal and vertical components of it. Moreover we will identify these vectors also with the vectors on $T_x(M)$, respectively E_x.

We now proceed to the construction of Bismut's connection on E. We will do this by prescribing the parallel transportation along curves in E. Take a curve α in E and let $\beta = \rho\alpha$ be its projection on M. For a given $X_0 \in T_{\alpha(0)}(E)$ we take its components $X_0^H \in T_{\beta(0)}(M)$ respectively $X_0^V \in E_{\beta(0)}$. Then we take their parallel transport in M and vertically along the curve β, thus obtaining $X_t^H \in T_{\beta(t)}(M)$ $X_t^V \in E_{\beta(t)}$. Then, we obtain a vector field X_t along α with its horizontal and vertical parts X_t^H, X_t^V. We declare this to be the parallel transportation of X_0 along α and we will use the notation $X_t = \tau_{\alpha\restriction[0,t]}X_0$. Now we are ready to define the connection as

Definition 3.1.19 (The connection on $T(E)$). *For $X_z \in T_z(E)$ we set*

$$\nabla_{X_z}Y = \frac{d}{dt}\tau_{\alpha\restriction[t,0]}Y_{\alpha(t)}$$

for any curve α such that $\alpha(0) = z$, $\dot{\alpha}(0) = X_z$ and any vector field Y defined along α.

Proposition 3.1.20. *If Y is a vector field defined around z, then $\nabla_{X_z}Y$ does not depend on the curve α chosen. Moreover, ∇ is a covariant derivative compatible with the metric defined on $T(E)$ and, if ∇^H is the Levi-Civita connection on M, it is given by*

$$\nabla_{X_z}Y = \left[\nabla^H_{X_x^H}Y^H\right]^H + \left[\nabla^V_{X_x^H}Y^V\right]^V + X_z^V(Y^V) \tag{3.1.21}$$

where $X_z^V(Y^V) = \sum_{j=1}^{\nu} X_z^V(f_j)F_j$ *if* $Y^V = \sum_{j=1}^{\nu} f_jF_j$ *is the representation of the vector* Y^V *restricted to* E_x *and* F_j *is any basis in* E_x *(this is nothing but the usual derivative in the Euclidean case).*

Proof. Using the definition of the parallel transportation we have

$$\tau_{\alpha\restriction[t,0]}Y_\alpha(t) = \left[\tau^H_{\beta\restriction[t,0]}Y^H_{\alpha(t)}\right]^H + \left[\tau^V_{\beta\restriction[t,0]}Y^V_{\alpha(t)}\right]^V \tag{*}$$

where $\beta = \rho\alpha$. Fix a basis F_j in E_x, an orthonormal basis E_i at $T_x(M)$ and a geodesic ball $B(x,r)$ in M. Then identify the fibers of E over this ball in M with E_x by using parallel transportation along geodesics radiating from x. Extend E_i, F_j by parallel transport along geodesics and note that we can write $Y = \sum_{i=1}^{m} g_iE_i + \sum_{j=1}^{\nu} f_jF_j$ for some functions $g_i : B(x,r) \to \mathbb{R}$, $f_j : B(x,r) \times E_x \to \mathbb{R}$. Then note that, in this setting, we can write $\alpha(t) = (\beta(t), \gamma(t))$ where γ is the curve in E_x with $\dot\gamma(0) = [\dot\alpha(0)]^V$. So,

$$\tau_{\alpha\restriction[t,0]}Y_\alpha(t) = \sum_{i=1}^{m} g_i(\beta(t))(E_i)_{\beta(t)} + \sum_{j=1}^{\nu} f_j(\beta(t), \gamma(t))F_j$$

and taking derivatives with respect to t at 0 we get (3.1.21). This take care of the well definition of ∇. The compatibility of a connection is equivalent to the fact that the parallel transportation is unitary. This is coming in our case from (*), and the fact that both horizontal and vertical parallel transportations are unitary on their domains of definition. □

Next we want to describe a few properties of this connection. To distinguish among various things we will denote by R^V the curvature of the vertical connection, namely $R^V(X,Y)Z = \nabla^V_X\nabla^V_YZ - \nabla^V_X\nabla^V_YZ - \nabla^V_{[X,Y]}Z$ for X, Y vectors on M and Z a section in E. Also we denote the curvature on M by R^H is the usual curvature tensor for the Levi-Civita on M. We will call ∇^{LC}, the Levi-Civita connection on $T(E)$ with respect to the metric on $T(E)$ and R^{LC}, Ric^{LC}, the curvature, respectively the Ricci tensor on E for the Levi-Civita.

Theorem 3.1.22. *1. The torsion of the connection* ∇ *is given by*

$$T(X,Y) = \left[R^V(X^H,Y^H)y\right]^V \tag{3.1.23}$$

for any X,Y *in* $T_z(E)$ *and* y *is thought as a vector in* $T_z^V(E)$.

2. The curvature of ∇ *is*

$$R(X,Y)Z = \left[R^H(X^H,Y^H)Z^H\right]^H + \left[R^V(X^H,Y^H)Z^V\right]^V \tag{3.1.24}$$

for any X, Y, Z *in* $T_z(E)$.

3. If M *is a compact manifold then there is a constant* $C \geq 0$ *such that*

$$\left|R_z^{LC}(X,Y)Z\right| \leq C(1 + |y|^2)|X|_z|Y|_z|Z|_z \tag{3.1.25}$$

for any X, Y, Z in $T_z(E)$, $z \in E$ and a fixed point o in E. Moreover for $n \geq 1$,

$$\begin{aligned} &\left| \left(\nabla^{LC}_{X_1} \cdots \left(\nabla^{LC}_{X_{n-1}} \left(\nabla^{LC}_{X_n} R^{LC}\right)(X,Y)Z\right) \cdots\right) \right| \\ &\qquad \leq C(1+|y|)^{2+n} |X_1|_z \cdots |X_{n-1}|_z |X_n|_z |X|_z |Y|_z |Z|_z \quad (3.1.26) \end{aligned}$$

for any $X_1, \cdots, X_n, X, Y, Z \in T_z(E)$

4. *The Laplacians on functions with respect to the connection ∇ and with respect to the Levi-Civita connection on E are the same.*

5. *If $h : E \to \mathbb{R}$ is defined by $h(z) = |y|^2$, then the Hessian of h on forms is the same with respect to both connections ∇ and ∇^{LC}. In fact, the Hessian on vector fields with respect to either connection is given by*

$$(\text{hess}_z h)X = 2X^V \quad (3.1.27)$$

for any vector X in $T_z(E)$. We further have on forms

$$(\text{hess}_z h)\omega_z^H \wedge \omega_z^V = 2\deg(\omega_z^V)\omega_z^H \wedge \omega_z^V \quad ((3.1.27)')$$

for any horizontal form ω_z^H and any vertical form ω_z^V, where the deg *stands for the degree of the form.*

Proof. 1. Because of the tensoriality in X, Y it suffices to check it for vertical and horizontal ones. To make the computations somehow nicer we take a normal coordinate system on $U \subset M$ based at the point $x \in M$ and a trivialization of E over U by vertical parallel transportation along geodesics in M. With these preparations we are going to use

$$\begin{aligned} \overline{\Gamma}^p_{i,j}(x) = 0, &\text{ for any allowable } i, j, p, \\ \nabla_{\left(\frac{\partial}{\partial x_i}\right)^H_z} \frac{\partial}{\partial y_j} = 0, &\ \nabla_{\frac{\partial}{\partial y_j}} \left(\frac{\partial}{\partial x_i}\right)^H = 0, \\ \nabla_{\frac{\partial}{\partial y_j}} \left(\frac{\partial}{\partial y_p}\right) = 0, &\ \nabla_{\left(\frac{\partial}{\partial x_i}\right)^H_z} \left(\frac{\partial}{\partial x_k}\right)^H = 0, \end{aligned} \quad (3.1.28)$$

deduced from (3.1.21) and (3.1.18), to make computations in these coordinates. We then take three cases.

(a) If both vectors are vertical,

$$T\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_p}\right) = \nabla_{\frac{\partial}{\partial y_j}} \frac{\partial}{\partial y_p} - \nabla_{\frac{\partial}{\partial y_p}} \frac{\partial}{\partial y_j} - \left[\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_p}\right] = 0,$$

by (3.1.28).

(b) Here we take a vector horizontal and the other vertical,

$$T\left(\left(\frac{\partial}{\partial x_i}\right)_z^H, \frac{\partial}{\partial y_j}\right) = -\left[\frac{\partial}{\partial x_i} - y^q\overline{\Gamma}^p_{i,q}(x)\frac{\partial}{\partial y_p}, \frac{\partial}{\partial y_j}\right] = 0,$$

where we used (3.1.18) and (3.1.28).

(c) For the last case, both vectors horizontal,

$$\begin{aligned}
T\left(\left(\frac{\partial}{\partial x_i}\right)_z^H, \left(\frac{\partial}{\partial x_k}\right)_z^H\right) \quad &= -\left[\left(\frac{\partial}{\partial x_i}\right)^H, \left(\frac{\partial}{\partial x_k}\right)^H\right] \\
= -\left[\frac{\partial}{\partial x_i} - y^q\overline{\Gamma}^p_{i,q}(x)\frac{\partial}{\partial y_p}, \frac{\partial}{\partial x_k} - y^s\overline{\Gamma}^r_{k,s}(x)\frac{\partial}{\partial y_r}\right] \\
= \frac{\partial}{\partial x_i}\left[y^s\overline{\Gamma}^r_{k,s}(x)\frac{\partial}{\partial y_r}\right] \quad - \frac{\partial}{\partial x_k}\left[y^q\overline{\Gamma}^p_{i,q}(x)\frac{\partial}{\partial y_p}\right] \\
= y^s\frac{\partial\overline{\Gamma}^r_{k,s}}{\partial x_i}(x)\frac{\partial}{\partial y_r} - y^q\frac{\partial\overline{\Gamma}^p_{i,q}}{\partial x_k}(x)\frac{\partial}{\partial y_p} \\
= y^q\frac{\partial\overline{\Gamma}^p_{k,q}}{\partial x_i}(x)\frac{\partial}{\partial y_p} - y^q\frac{\partial\overline{\Gamma}^p_{i,q}}{\partial x_k}(x)\frac{\partial}{\partial y_p} \qquad (*)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
R^V\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right)y &= \nabla^V_{\frac{\partial}{\partial x_i}}\nabla^V_{\frac{\partial}{\partial x_k}}y^q\frac{\partial}{\partial y_q} - \nabla^V_{\frac{\partial}{\partial x_k}}\nabla^V_{\frac{\partial}{\partial x_i}}y^q\frac{\partial}{\partial y_q} \\
= \nabla^V_{\frac{\partial}{\partial x_i}}y^q\overline{\Gamma}^r_{k,q}(x)\frac{\partial}{\partial y_r} - \nabla^V_{\frac{\partial}{\partial x_k}}y^q\overline{\Gamma}^r_{i,q}(x)\frac{\partial}{\partial y_r} \\
= y^q\frac{\partial\overline{\Gamma}^r_{k,q}}{\partial x_i}(x)\frac{\partial}{\partial y_r} - y^q\frac{\partial\overline{\Gamma}^r_{i,q}}{\partial x_k}(x)\frac{\partial}{\partial y_r}
\end{aligned}$$

which is the same as the last two lines in (*), thus the end of the case.

2. Because of the tensoriality of both sides it is sufficient to check it for horizontal and vertical vectors. Now we will use two properties deduced from (3.1.21).

$$\nabla_{X^H}Y^V = 0 \qquad (**)$$

for any vertical vector X^H and vertical vector Y^V. Also since we are going to work in local coordinates we note that

$$\nabla_{Y^V}\frac{\partial}{\partial y_j} = 0 \qquad (***)$$

for any vertical vector Y^V and any vector $\frac{\partial}{\partial y_j}$, $j = 1, \nu$.

(a) If $X = \frac{\partial}{\partial y_j}$, $Y = \frac{\partial}{\partial y_p}$, then by (**) $R(X, Y)Z = 0$ for any vector Z.

(b) If $X = \left(\frac{\partial}{\partial x_i}\right)^H$, $Y = \frac{\partial}{\partial y_j}$,

i. and if $Z = \frac{\partial}{\partial y_p}$, then using (**), (***), (3.1.18) and (3.1.21)

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$= -\nabla_{\frac{\partial}{\partial y_j}} \overline{\Gamma}^q_{i,p}(x) \frac{\partial}{\partial y_q} + \nabla_{\overline{\Gamma}^q_{i,r}(x)\left[y^r \frac{\partial}{\partial y_q}, \frac{\partial}{\partial y_j}\right]} \frac{\partial}{\partial y_p} = 0;$$

ii. while if $Z = \left(\frac{\partial}{\partial x_k}\right)^H$, then using again (**) and (3.1.18) we get as above that $R(X,Y)Z = 0$.

(c) If $X = \left(\frac{\partial}{\partial x_i}\right)^H$, $Y = \left(\frac{\partial}{\partial x_k}\right)^H$,

i. and if $Z = \frac{\partial}{\partial y_j}$, then using the fact that $T(X,Y) = -[X,Y]$ is a vertical vector, (**), (***) and (3.1.21)

$$R(X,Y)Z = \left[\nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z\right]^V = \left[R^V(X^H, Y^H) Z^V\right]^V;$$

ii. while if $Z = \left(\frac{\partial}{\partial x_s}\right)^H$, then by the same argument we get that

$$R(X,Y)Z = \left[\nabla^H_X \nabla^H_Y Z - \nabla^H_Y \nabla^H_X Z^H\right]^H = \left[R^H(X^H, Y^H) Z^H\right]^H.$$

Thus in all cases we have checked that

$$R(X,Y)Z = \left[R^V(X^H, Y^H) Z^H\right]^H + \left[R^H(X^H, Y^H) Z^V\right]^V.$$

3. We show first that for any coordinate system on M and a corresponding trivialization of E there is a constant C_U such that

$$\left|R^{LC}_z(X,Y)Z\right| \leq C_U (1 + |y|^2) |X|_z |Y|_z |Z|_z$$

for any $z \in \rho^{-1}U$, $X, Y, Z \in T_z(E)$. For the coordinate system we take the vectors $(\frac{\partial}{\partial x_i})^H_z$, $i = 1, \dots, m$, $\frac{\partial}{\partial y_j}$, $j = 1, \dots, \nu$, as the basis for the space $T_z(E)$. We will denote these generators simply as Z_α, $\alpha = 1, \dots, m + \nu$. Denote by $\Gamma^\gamma_{\alpha,\beta}$ the Christoffel coefficients of the Levi-Civita connection

$$\nabla^{LC}_{Z_\alpha} Z_\beta = \Gamma^\gamma_{\alpha\beta} Z_\gamma$$

We study how these coefficients depend on y. For this purpose we recall here the fact that

$$\langle \nabla^{LC}_{Z_\alpha} Z_\beta, Z_\gamma \rangle = \frac{1}{2} \{ Z_\alpha \langle Z_\beta, Z_\gamma \rangle + Z_\beta \langle Z_\gamma, Z_\alpha \rangle - Z_\gamma \langle Z_\alpha, Z_\beta \rangle$$
$$+ \langle [Z_\alpha, Z_\beta], Z_\gamma \rangle - \langle [Z_\beta, Z_\gamma], Z_\alpha \rangle + \langle [Z_\gamma, Z_\alpha], Z_\beta \rangle \}$$

and from this

$$\Gamma^{\delta}_{\alpha\beta}\langle Z_\delta, Z_\gamma\rangle = \frac{1}{2}\{Z_\alpha\langle Z_\beta, Z_\gamma\rangle + Z_\beta\langle Z_\gamma, Z_\alpha\rangle - Z_\gamma\langle Z_\alpha, Z_\beta\rangle \\ + \langle [Z_\alpha, Z_\beta], Z_\gamma\rangle - \langle [Z_\beta, Z_\gamma], Z_\alpha\rangle + \langle [Z_\gamma, Z_\alpha], Z_\beta\rangle\}.$$

If $g_{\alpha\beta} = \langle Z_\alpha, Z_\beta\rangle$ and $(g^{\alpha\beta})_{\alpha\beta}$ the inverse of the matrix $(g_{\alpha\beta})_{\alpha\beta}$, then we can invert the above formula and arrive at

$$\Gamma^{\delta}_{\alpha\beta} = \frac{1}{2}\{Z_\alpha\langle Z_\beta, Z_\gamma\rangle + Z_\beta\langle Z_\gamma, Z_\alpha\rangle - Z_\gamma\langle Z_\alpha, Z_\beta\rangle \\ + \langle [Z_\alpha, Z_\beta], Z_\gamma\rangle - \langle [Z_\beta, Z_\gamma], Z_\alpha\rangle + \langle [Z_\gamma, Z_\alpha], Z_\beta\rangle\}g^{\gamma\delta}.$$

The dependence on y of the above expression is reduced to the analysis of each bracket. In the first place, one has to notice that by construction, $g_{\alpha\beta}$ does not depend on y, and is a smooth function in x. Also each vector field Z_α has at most a first order behavior in y. On the other hand the commutator of two vectors $[Z_\alpha, Z_\beta]$ is also, as we had seen in the proof of part 1, a polynomial of degree at most 1 in y. Thus all terms above are at most a degree 1 polynomial in y. This implies that all $\Gamma^{\delta}_{\alpha\beta}$ are polynomial of degree at most 1 in y. If we write $[Z_\alpha, Z_\beta] = A^l_{\alpha\beta} Z_l$ then

$$\begin{aligned} R^{LC}(Z_\alpha, Z_\beta)Z_\gamma &= \nabla^{LC}_{Z_\alpha}\nabla^{LC}_{Z_\beta} Z_\gamma - \nabla^{LC}_{Z_\beta}\nabla^{LC}_{Z_\alpha} Z_\gamma - \nabla^{LC}_{[Z_\alpha, Z_\beta]} Z_\gamma \\ &= \nabla^{LC}_{Z_\alpha}\Gamma^m_{\beta\gamma} Z_m - \nabla^{LC}_{Z_\beta}\Gamma^m_{\alpha\gamma} Z_m - A^l_{\alpha\beta}\nabla^{LC}_{Z_l} Z_\gamma \\ &= Z_\alpha(\Gamma^m_{\beta\gamma})Z_m + \Gamma^m_{\beta\gamma}\nabla^{LC}_{Z_\alpha} Z_m \\ &\quad - Z_\beta(\Gamma^m_{\alpha\gamma})Z_m - \Gamma^m_{\alpha\gamma}\nabla^{LC}_{Z_\beta} Z_m - A^l_{\alpha\beta}\Gamma^\delta_{l\gamma} Z_\delta \\ &= Z_\alpha(\Gamma^m_{\beta\gamma})Z_m + \Gamma^m_{\beta\gamma}\Gamma^\delta_{\alpha m} Z_\delta \\ &\quad - Z_\beta(\Gamma^m_{\alpha\gamma})Z_m - \Gamma^m_{\alpha\gamma}\Gamma^\delta_{\beta m} Z_\delta - A^l_{\alpha\beta}\Gamma^\delta_{l\gamma} Z_\delta \end{aligned} \tag{3.1.29}$$

This shows that $\langle R^{LC}(Z_\alpha, Z_\beta)Z_\gamma, Z_\delta\rangle$ is a second order polynomial in y with smooth coefficients in x. Hence, there exists a constant $C_U \geq 0$ such that,

$$\frac{\langle R^{LC}_z(Z_\alpha, Z_\beta)Z_\gamma, Z_\delta\rangle}{1 + |y|^2} \leq C_U |Z_\alpha||Z_\beta||Z_\gamma||Z_\delta|.$$

This in turn implies that, for eventually another constant C_U,

$$\langle R^{LC}_z(Z_1, Z_2)Z_3, Z_4\rangle \leq C_U(1 + |y|^2)|Z_1||Z_2||Z_3||Z_4|$$

for any vector field $Z_1, Z_2, Z_3, Z_4 \in T_z(E)$, $z \in \rho^{-1}U$. Using the compactness of the manifold, from this we can deduce the required inequality for the Levi-Civita curvature. To deal with the estimates on the derivatives, first observe that again by compactness one can do this locally and locally it suffices to prove them for Z_α instead of X_i. To carry this over we apply induction. To pass from step n to $n+1$ we take $n+1$ derivatives in (3.1.29) and use step n.

4. On functions, we have $\Delta^{\nabla} f = \sum_{j=1}^{d} E_j E_j - (\nabla_{E_j} E_j) f$, for any connection ∇ and any local orthonormal basis E_j. Then in order to prove that the Laplacians are the same on functions it suffices to check that for a particular choice of the basis E_j. We will take a basis consisting in vertical and horizontal vectors. For such a basis we will show that $\nabla_{E_j} E_j = \nabla^{LC}_{E_j} E_j$ for any j. Next step is contained in the following

   Lemma 3.1.30. *If M is a Riemannian manifold and ∇ a compatible connection with torsion T, then for any vectors X, Y, Z*

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\{ & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & + \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle \\ & + \langle T(X,Y), Z\rangle - \langle T(Y,Z), X\rangle + \langle T(Z,X), Y\rangle\}\end{aligned} \tag{3.1.31}$$

   Proof. The proof uses exactly the trick proving the similar formula for the Levi-Civita. □

   Coming back to the proof we get that

$$\langle \nabla_{E_j} E_j, E_k\rangle = \langle [E_k, E_j], E_j\rangle + \langle T(E_k, E_j), E_j\rangle$$

   while for the Levi-Civita connection we have

$$\langle \nabla^{LC}_{E_j} E_j, E_k\rangle = \langle [E_k, E_j], E_j\rangle.$$

   The main point here is that by the expression of the torsion we got at the first point in this proof we have $\langle T(E_k, E_j), E_j\rangle = 0$. This can be seen by considering two cases. One is the case E_j vertical, then $T(E_k, E_j) = 0$, the other if E_j is horizontal, in which case we have to point out that $T(E_k, E_j)$ is always vertical, thus $\langle T(E_k, E_j), E_j\rangle = 0$. So we get the equality $\langle \nabla_{E_j} E_j, E_k\rangle = \langle \nabla^{LC}_{E_j} E_j, E_k\rangle$ for any vector E_k, which also ends the proof of this item.

5. We start first by computing the gradient of the function $h(z) = |y|^2$. We are going to show that

$$\operatorname{grad}_z h = 2y \tag{3.1.32}$$

   The proof of this goes in two steps.

   - If X is a horizontal vector field at z then $\langle \operatorname{grad}_z h, X\rangle = 0$. Indeed, if α is a curve on M starting at x and such that $\alpha(0) = \rho_* X$, and β its lift starting at z, then we have

$$\langle \operatorname{grad}_z h, X\rangle = Xh = \frac{d}{dt} h(\beta(t))\big|_{t=0} = \frac{d}{dt} |\beta(t)|^2|_{t=0} = 0$$

     since $\beta(t)$ is obtained by parallel transportation, hence it is of constant length in fibers.

- If X is a vertical vector at z, then take its identification v with a vector in the fiber E_x and consider the curve $\gamma(t) = y + tv$. Then

$$\langle \text{grad}_z h, X \rangle = Xh = \frac{d}{dt} h(\gamma(t))\big|_{t=0} = \frac{d}{dt} |y + tv|^2 \big|_{t=0} = 2\langle y, v \rangle$$

Putting together we obtain the result about the gradient. Returning to the Hessian we start by writing that for any local vectors X, Y

$$\langle (\text{hess}^{\nabla} h) X, Y \rangle = \langle \nabla_X \text{grad} h, Y \rangle$$

$$\langle (\text{hess}^{\nabla^{LC}} h) X, Y \rangle = \langle \nabla^{LC}_X \text{grad} h, Y \rangle.$$

By (3.1.31), the difference between these two quantities is

$$\langle T(X, Y), \text{grad} h \rangle - \langle T(Y, \text{grad} h), X \rangle + \langle T(\text{grad} h, X), Y \rangle.$$

Since grad$h$ is vertical the first two terms in the above expression are zero. The remaining one is reduced to

$$2\langle R^V(X^H, Y^H) y, y \rangle$$

for the computation is made in E_x. But this is 0, since we have in general that

$$\langle R^V(X^H, Y^H) s_1, s_2 \rangle = -\langle R^V(X^H, Y^H) s_2, s_1 \rangle$$

for any sections s_1, s_2 and any curvature of a compatible vertical connection on E. To verify this one has to extend the sections to some local sections, the vectors X^H, Y^H to some local commuting horizontal vectors, and then use the definition of the curvature plus the compatibility of the connection. Finally this brings in the identity $\langle R^V(X^H, Y^H) y, y \rangle = 0$, which shows that the Hessians are the same. Moreover, by definition we have that $\nabla^V_{X^H_z} y = 0$. This together with (3.1.21) give

$$\langle (\text{hess}^{\nabla} h)_z X_z, Y_z \rangle = \langle \nabla_{X_z} \text{grad} h, Y_z \rangle = 2\langle \nabla_{X_z} y, Y_z \rangle = 2\langle X^V_z(y), Y_z \rangle = 2\langle X^V_z, Y_z \rangle$$

which is the end of the proof. □

3.2 Frame Bundles

3.2.1 ∇-Orthonormal Frame Bundle and ∇-Laplacians

In this subsection we are given a d dimensional Riemannian manifold M endowed with a compatible connection ∇.

We begin by recalling the definition of the frame bundle

$$\mathcal{O}(M) = \{(x, e(x)), e(x) = (e_1, \cdots, e_d) \text{ orthonormal basis of } T_x M\}.$$

The vertical space $\mathcal{V}_{\mathfrak{f}}\mathcal{O}(M)$ is defined as in section (2.2). We next turn to defining the horizontal space of $T_{\mathfrak{f}}\mathcal{O}(M)$. To do this we mention only that there is a natural notion of parallel transportation with respect to the connection ∇, and the parallel transportation is an isometry because the connection is compatible with the Riemannian metric. We point out here the main procedure. First, one defines the horizontal lift of a curve in M. This is achieved by taking a curve p in M and defining the horizontal lift starting at $\mathfrak{f} = (p(0), (f_1, \cdots f_d))$ to be the curve $\mathfrak{p}(t) = (p(t), (e_1(t), \cdots e_d(t))$ with $e_k(t) = \tau_{p\restriction[0,t]} f_k$ and τ stands for the parallel transportation with respect to the connection ∇. The horizontal lift of a vector $X_x \in T_x(M)$ is defined as follows. Take any curve p with $p(0) = x$, $\dot{p}(0) = X_x$ and consider its lift $\mathfrak{p}$ starting from $\mathfrak{f}$. Then by definition set $\mathfrak{H}^{\nabla}_{\mathfrak{f}}(X_x) = \dot{\mathfrak{p}}(0)$. Following the same reasoning as in [7, Lemma 8.6] one can show that this is a well defined notion, specifically, this does not depend on the chosen curve and if X is a smooth vector field defined on an open set U around the point x, then the map $\pi^{-1}(U) \ni \mathfrak{f} \to \mathfrak{H}^{\nabla}_{\mathfrak{f}}(X_{\pi\mathfrak{f}}) \in T_{\mathfrak{f}}\mathcal{O}(M)$ is also a smooth map.

As in the standard case we have the splitting

$$T_{\mathfrak{f}}\mathcal{O}(M) = \mathcal{H}^{\nabla}_{\mathfrak{f}}\mathcal{O}(M) \oplus \mathcal{V}_{\mathfrak{f}}\mathcal{O}(M).$$

The canonical vector fields are defined as usually, namely take $\xi \in \mathbb{R}^d$ and set

$$\mathfrak{E}^{\nabla}(\xi)_{\mathfrak{f}} = \mathfrak{H}^{\nabla}_{\mathfrak{f}}(\mathfrak{f}\xi) \tag{3.2.1}$$

We record here the similar facts with those already given in section (2.2). We will point out if necessary the main differences.

Using the definition given in (2.2.2), we get the corresponding version of (2.2.3). For any smooth k-form ω in M defined around $\pi\mathfrak{f}$ we have

$$\widetilde{\nabla_{\mathfrak{f}\xi}\omega} = \mathfrak{E}^{\nabla}(\xi)_{\mathfrak{f}}\tilde{\omega} \tag{3.2.2}$$

or equivalently,

$$\widetilde{\nabla_{X_{\pi\mathfrak{f}}}\omega} = \mathfrak{H}^{\nabla}(X_{\pi\mathfrak{f}})_{\mathfrak{f}}\tilde{\omega} \tag{(3.2.2)'}$$

for any $X_{\pi\mathfrak{f}} \in T_{\pi\mathfrak{f}}(M)$.

The definition of the ∇-Laplacian was given in (3.1.8). We now give the definition of the corresponding of the ∇-Bochner Laplacians as

Definition 3.2.3. *The ∇-Bochner Laplacian Δ_B is*

$$\Delta^{\nabla}_B = \sum_{j=1}^{d} \mathfrak{E}^{\nabla}(e_j)^2,$$

for any orthonormal basis $(e_j)_{j=1,d}$ in $\mathbb{R}^d$.

With the same proof of (2.2.6), we also can show that

Proposition 3.2.4.

$$\widetilde{\Delta^{\nabla}\omega}(\pi\mathfrak{f}) = \Delta^{\nabla}_B\tilde{\omega}(\mathfrak{f})$$

for any smooth form ω locally defined around $\pi\mathfrak{f}$.

3.2.2 An Adapted Orthonormal Frame Bundle

In this section we take a m dimensional Riemannian manifold M and E a ν dimensional vector bundle over M. Assume that the manifold M is endowed with a compatible connection ∇ and that E is an Euclidean vector bundle over M, i.e. we are given a metric on fibers of E and a vertical connection ∇^V compatible with the metric on the fibers.

We set

$$\mathcal{O}(M;E) = \{(x, \mathfrak{e}, \mathfrak{f})\}.$$

where $\mathfrak{e} = (e_1, \cdots, e_d)$, $\mathfrak{f} = (f_1, \cdots, f_k)$ are orthonormal basis of T_xM respectively E_x. Take π the natural projection given by $\pi((x, \mathfrak{e}, \mathfrak{f}) = x$. One can identify the pair $(\mathfrak{e}, \mathfrak{f})$ with an isometry $u : \mathbb{R}^m \oplus \mathbb{R}^\nu \to T_xM \oplus E_x$ that sends $\mathbb{R}^m$ into T_xM and $\mathbb{R}^\nu$ into E_x. We will call an isometry with this property admissible.

The bundle $\mathcal{O}(M;E)$ is a bundle over M with structural group $O(m) \times O(\nu)$. Thus for any $O \in O(m) \times O(\nu)$ define

$$(R_O u)(\xi) = u(O\xi)$$

for any ξ in $\mathbb{R}^{m+\nu}$ and any admissible u. Further for $a \in o(m) \times o(\nu)$ the Lie algebra of $O(m) \times O(\nu)$ we set

$$\lambda(a)_u = \frac{d}{dt} R_{\exp(ta)} u \big|_{t=0}$$

for any admissible u, and call it the vertical vector associated with a. With this notation we define now the vertical subspace $\mathcal{V}_u\mathcal{O}(M;E)$ as the subspace of $\mathcal{O}(M;E)$ consisting in all vectors $\lambda(a)_u$ for $a \in o(m) \times o(\nu)$.

To define the horizontal space we can proceed as in the usual case. Namely take a curve α in M with $\alpha(0) = x$, $u \in \pi^{-1}(x)$ and define its horizontal lift β with $\beta(0) = u$ $\beta(0) = u$ and if we identify $\beta(t)$ with an isometry from $\mathbb{R}^{m+\nu}$ into $T_{\alpha(t)}M \oplus E_{\alpha(t)}$ then,

$$\beta(t)\xi = \tau_{\alpha\restriction[0,t]} u\xi$$

for any $\xi \in \mathbb{R}^{m+\nu}$, where the parallel transportation for a tangent vector at M is transported with respect to the connection ∇ and the vertical transportations is with respect to ∇^V. We now define the horizontal lift of a tangent vector $X_x \in T_xM \oplus E_x$ to a vector $\mathfrak{H}_u(X) \in T_u\mathcal{O}(M;E)$. To do this, take a curve α in M such that $\alpha(0) = x$ and $\dot\alpha(0) = X_x$, and take its lift β starting at u with $\pi(u) = x$, and set

$$\mathfrak{H}_u(X) = \dot\beta(0)$$

the horizontal lift of X at u. This is a well defined notion since it can be shown it does not depend on the curve chosen. Also the map $X \to \mathfrak{H}(X)$ is smooth if the vector X is a smooth one. Denote $\mathcal{H}_u^\nabla\mathcal{O}(M;E)$ the space of all horizontal vector fields.

Definition 3.2.5. *1. The canonical vector fields $\mathfrak{E}(\xi)$ are defined by*

$$\mathfrak{E}^{\nabla}(\xi)_u = \mathfrak{H}_u((u\xi)^T)$$

where T stands for the projection of $u\xi$ on $T(M)$.

2. Set now the ∇-Bochner Laplacian in this context as

$$\Delta^{\nabla}_{B,E} = \sum_{j=1}^{m} \mathfrak{E}^{\nabla}(e_j)^2,$$

for any orthonormal basis $(e_j)_{j=1,m}$ in $\mathbb{R}^m$.

With a few changes in the notation of section (2.2) one can prove the corresponding of (2.2.6) as

Proposition 3.2.6. *For any smooth form ω we have*

$$\widetilde{\Delta^{\nabla}\omega}(\pi u) = \Delta^{\nabla}_{B,E}\tilde{\omega}(u)$$

with the tilde here defined by

$$\tilde{\omega}_u(\xi_1, \cdots, \xi_r) = \omega_{\pi u}((u\xi_1)^T, \cdots, (u\xi_r)^T) \tag{3.2.7}$$

for any $\xi_1, \ldots, \xi_r \in \mathbb{R}^{m+\nu}$. In particular, for any smooth function $f : M \to \mathbb{R}$

$$\Delta^{\nabla} f = \Delta^{\nabla}_{B,E}(f \circ \pi).$$

The proof is the same as the proof of proposition (2.2.6), the only ingredient that one needs is the analog of (2.2.3) or (2.2.3'), here in the form:

$$\widetilde{\nabla_{u\xi}\omega} = \mathfrak{E}^{\nabla}(\xi)_u\tilde{\omega}. \tag{3.2.8}$$

The proof of this is the same as the one for (2.2.3). Once we have this, the proof of (3.2.6) is a repetition of the proof of (2.2.6).

3.2.3 A Split Frame Bundle

In this section we will construct and analyze the frame bundle on a manifold with a particular structure on the tangent bundle. The difference from the classical case is that we will not deal with the Levi-Civita connection but with a connection that has torsion.

In this section we work on a d dimensional Riemannian manifold Z endowed with two smooth distributions $\mathcal{D}$, $\mathcal{E}$ of the tangent bundle, of dimensions m, ν, such that at any point $z \in Z$, $\mathcal{D}_z$, $\mathcal{E}_z$ are orthogonal to each other and they generate the whole space T_zZ.

Definition 3.2.9. *A connection ∇ is said to be compatible with the pair $(\mathcal{D}, \mathcal{E})$ if it is compatible with the metric of Z and if the parallel transportation preserves the distributions $\mathcal{D}$, $\mathcal{E}$. In other words if τ denotes here the parallel transportation operator associated to ∇, then for any curve $\gamma : [a,b] \to Z$, $\tau_{\gamma\restriction[a,b]}\mathcal{D}_{\gamma(a)} \subset \mathcal{D}_{\gamma(b)}$, $\tau_{\gamma\restriction[a,b]}\mathcal{E}_{\gamma(a)} \subset \mathcal{E}_{\gamma(b)}$.*

We mention here that a connection compatible with the Riemannian metric on Z is compatible with the pair $(\mathcal{D}, \mathcal{E})$ iff for any tangent vector $X_z \in T_zZ$, we have that $\nabla_{X_z}\mathcal{D} \subset \mathcal{D}_z$ and also that $\nabla_{X_z}\mathcal{E} \subset \mathcal{E}_z$. From now on in this section, the connection that is going to appear is compatible in the sense given by definition (3.2.9).

We now define $\mathcal{O}(Z; \mathcal{D}, \mathcal{E})$ to be the set of triples $(z, \mathfrak{e}, \mathfrak{f})$ with $z \in Z$, and $\mathfrak{e}, \mathfrak{f}$ orthonormal basis in $\mathcal{D}_z$ respectively $\mathcal{E}_z$. Then, as in the standard case, $\mathcal{O}(Z; \mathcal{D}, \mathcal{E})$ is a principal bundle over Z with the projection $\pi : \mathcal{O}(Z; \mathcal{D}, \mathcal{E}) \to Z$ given by $\pi(z, \mathfrak{e}, \mathfrak{f}) = z$ and structural group $O(m) \times O(\nu) \subset O(d)$.

One can identify the pair $(\mathfrak{e}, \mathfrak{f})$ with an isometry $u : \mathbb{R}^d = \mathbb{R}^m \oplus \mathbb{R}^\nu \to T_zZ$ such that u sends $\mathbb{R}^m$ into $\mathcal{D}_z$ and $\mathbb{R}^\nu$ into $\mathcal{E}_z$. We will call such an isometry $u : \mathbb{R}^d \to T_zZ$ admissible if it has the above property.

If $O \in O(m) \times O(\nu)$ we define

$$(R_O u)(\xi) = u(O\xi)$$

for any ξ in $\mathbb{R}^d$ and any admissible u. For $a \in o(m) \times o(\nu)$ the Lie algebra of $O(m) \times O(\nu)$ we denote

$$\lambda(a)_u = \frac{d}{dt} R_{\exp(ta)} u\Big|_{t=0}$$

for any admissible u, and call it the vertical vector associated with a. With this notation define now the vertical subspace $\mathcal{V}_u\mathcal{O}(Z; \mathcal{D}, \mathcal{E})$ as the space consisting in all vectors $\lambda(a)_u$ for $a \in o(m) \times o(\nu)$.

Next we define the lift of a curve in Z to a curve in $\mathcal{O}(Z; \mathcal{D}, \mathcal{E})$. Take a curve α in Z, with $\alpha(0) = z$, $u \in \mathcal{O}(Z; \mathcal{D}, \mathcal{E})$ with $\pi(u) = z$. Define the lift of α to be the curve β such that $\beta(0) = u$ and if we identify $\beta(t)$ with an isometry from $\mathbb{R}^d$ into $T_{\alpha(t)}Z$ then,

$$\beta(t)\xi = \tau_{\alpha\restriction[0,t]} u\xi$$

for any $\xi \in \mathbb{R}^d$.

Note that β is well define due to the compatibility of ∇ with the pair $(\mathcal{D}, \mathcal{E})$. Also it follows that if the curve α is smooth then the curve β is smooth too.

We now define the horizontal lift of a tangent vector $X \in T_zZ$ to a vector $\mathfrak{H}_u(X) \in T_u\mathcal{O}(Z; \mathcal{D}, \mathcal{E})$. To do this, one takes a curve α in Z such that $\alpha(0) = z$ and $\dot\alpha(0) = X$, take its lift β starting at u with $\pi(u) = z$, and define

$$\mathfrak{H}_u(X) = \dot\beta(0)$$

the horizontal lift of X at u. Remark that if the vector field X is a smooth vector field on a neighborhood U of z then, as in the standard case, (see [7, Lemma 8.6]), one can prove that the vector field $\mathfrak{H}(X)$ is also smooth vector field defined on $\pi^{-1}U$.

Denote $\mathcal{H}_u\mathcal{O}(Z;\mathcal{D},\mathcal{E})$ the space of all horizontal lifts of vectors in $T_{\pi(u)}Z$. Then we have the decomposition

$$T_u\mathcal{O}(Z;\mathcal{D},\mathcal{E}) = \mathcal{H}_u\mathcal{O}(Z;\mathcal{D},\mathcal{E}) \oplus \mathcal{V}_u\mathcal{O}(Z;\mathcal{D},\mathcal{E}). \tag{*}$$

Using this decomposition for tangent vector $\mathfrak{X} \in T_u\mathcal{O}(Z;\mathcal{D},\mathcal{E})$ we denote by $\mathfrak{X}^H$, $\mathfrak{X}^V$ the horizontal respectively the vertical part given by the decomposition in (*).

Definition 3.2.10. *1. The canonical form $\omega : T\mathcal{O}(Z;\mathcal{D},\mathcal{E}) \to o(m) \times o(\nu)$ is given by the prescription that for any $\mathfrak{X} \in T_u\mathcal{O}(Z;\mathcal{D},\mathcal{E})$:*

$$\omega(\mathfrak{X}) = a, \tag{3.2.11}$$

where $a \in o(m) \times o(\nu)$ has the property that $\lambda(a)_u = \mathfrak{X}^V$. Otherwise stated this is completely characterized by the relation

$$\omega(\lambda(a)_u) = a$$

for any $a \in o(m) \times o(\nu)$.

2. *The canonical form $\theta : T\mathcal{O}(Z;\mathcal{D},\mathcal{E}) \to \mathbb{R}^d$ is given by the recipe*

$$\theta(\mathfrak{X}) = u^{-1}(\pi_*\mathfrak{X}) \tag{3.2.12}$$

for any $\mathfrak{X} \in T_u\mathcal{O}(Z;\mathcal{D},\mathcal{E})$.

3. *For a given $\xi \in \mathbb{R}^d$, we define the canonical vector field $\mathfrak{E}(\xi)$ by*

$$\mathfrak{E}(\xi)_u = \mathfrak{H}_u(u\xi). \tag{3.2.13}$$

for any $u \in \mathcal{O}(Z;\mathcal{D},\mathcal{E})$.

We summarize the main properties of these just defined objects.

Proposition 3.2.14. *1. $\theta(\mathfrak{E}(\xi)) = \xi$ for any $\xi \in \mathbb{R}^d$.*

2. *$R_O\mathfrak{E}(\xi) = \mathfrak{E}(O^{-1}\xi)$, for $O \in O(m) \times O(\nu)$, $\xi \in \mathbb{R}^d$.*

3. *If $\xi \neq 0$, then $\mathfrak{E}(\xi)$ never vanishes.*

4. *For $a \in o(k) \times o(m)$, $\xi \in \mathbb{R}^d$, we have $[\lambda(a), \mathfrak{E}(\xi)] = \mathfrak{E}(a\xi)$.*

Proof. 1. Direct from the definitions.

2. $R_O\mathfrak{E}(\xi)$ is a horizontal vector at R_Ou. Because $\pi_*R_O\mathfrak{E}(\xi) = \pi_*\mathfrak{E}(\xi)$ and $R_Ou\xi = uO\xi$, we get that $R_O\mathfrak{E}(\xi)_U = \mathfrak{E}(\xi)_{R_O^{-1}u}$ is the horizontal lift of $R_O^{-1}u\xi = uO^{-1}\xi$. On the other hand, by definition, $\mathfrak{E}(O^{-1}\xi)_u$ is the horizontal lift of $uO^{-1}\xi$, hence $R_O\mathfrak{E}(\xi) = \mathfrak{E}(O^{-1}\xi)$.

3. If $\mathfrak{E}(\xi)_u = 0$, then $0 = \pi_*\mathfrak{E}(\xi)_u = u\xi$ and from here since u is an isometry we get $\xi = 0$.

4. We have $[\lambda(a), \mathfrak{E}(\xi)] = -\frac{d}{dt} R_{\exp(ta)} \mathfrak{E}(\xi)\big|_{t=0} = -\frac{d}{dt}\mathfrak{E}(\exp(ta)\xi)\big|_{t=0} = \mathfrak{E}(a\xi)$. □

Remark 3.2.15. *The canonical vector field $\mathfrak{E}(\xi)$ can be characterized as the unique vector filed $\mathfrak{X} \in T\mathcal{O}(Z;\mathcal{D},\mathcal{E})$ with the property that $\theta(\mathfrak{X}_u) = \xi$ and $\omega(\mathfrak{X}_u) = 0$ for any $u \in \mathcal{O}(Z;\mathcal{D},\mathcal{E})$, $\xi \in \mathbb{R}^d$. This follows from proposition* (3.2.14) *and the fact that $\theta(\mathfrak{X}_u) = 0$ and $\omega(\mathfrak{X}_u) = 0$ iff $\mathfrak{X}_u = 0$.*

We give now the definition of the torsion and curvature forms. These notions are formally taken from [4].

Definition 3.2.16. *1. The* torsion form *is defined as $\Theta = D\theta$, namely*

$$\Theta(\mathfrak{X},\mathfrak{Y}) = d\theta(\mathfrak{X}^H,\mathfrak{Y}^H) = \mathfrak{X}^H\theta(\mathfrak{y}^H) - \mathfrak{Y}^H\theta(\mathfrak{X}^H) - \theta([\mathfrak{X}^H,\mathfrak{Y}^H])$$

2. The curvature form *is defined as $\Omega = D\omega$, or in other words*

$$\Omega(\mathfrak{X},\mathfrak{Y}) = d\omega(\mathfrak{X}^H,\mathfrak{Y}^H) = \mathfrak{X}^H\omega(\mathfrak{y}^H) - \mathfrak{Y}^H\omega(\mathfrak{X}^H) - \omega([\mathfrak{X}^H,\mathfrak{Y}^H]).$$

The next theorem is, for the frame bundle case, given in [4, Theorem 5.1 Chapter 3].

Theorem 3.2.17. *If $X, Y, Z \in T_zZ$ and $\mathfrak{X},\mathfrak{Y} \in T_u\mathcal{O}(Z;\mathcal{D},\mathcal{E})$ for admissible u with $\pi(u) = z$, and such that $\pi_*\mathfrak{X} = X$, $\pi_*\mathfrak{Y} = Y$, then*

$$T(X,Y) = u(\Theta(\mathfrak{X},\mathfrak{Y}))$$

where T is the torsion of the connection ∇,

$$R(X,Y)Z = u(\Omega(\mathfrak{X},\mathfrak{Y})u^{-1}Z)$$

where R denotes here the curvature of the connection ∇ given by $R(X,Y)Z = \nabla_X\nabla_YZ - \nabla_Y\nabla_XZ - \nabla_{[X,Y]}Z$.

Proof. See the proof of [4, Theorem 5.1 Chapter 3]. □

One important corollary of this theorem is the following.

Corollary 3.2.18. *Let $\xi,\eta \in \mathbb{R}^d$ such that $T_{\pi u}(u\xi,u\eta) = 0$, $R_{\pi u}(u\xi,u\eta) = 0$ for any admissible u. Then,*

$$[\mathfrak{E}(\xi),\mathfrak{E}(\eta)] = 0.$$

Proof. Using (3.2.17), one gets first that $\Theta(\mathfrak{E}(\xi),\mathfrak{E}(\eta)) = 0$, $\Omega(\mathfrak{E}(\xi),\mathfrak{E}(\eta)) = 0$. By definition and the fact that $\theta(\mathfrak{E}(\xi)) = \xi$, $\theta(\mathfrak{E}(\eta)) = \eta$, we get that

$$\theta([\mathfrak{E}(\xi),\mathfrak{E}(\eta)]) = -d\theta(\mathfrak{E}(\xi),\mathfrak{E}(\eta)) = -\Theta(\mathfrak{E}(\xi),\mathfrak{E}(\eta)) = 0.$$

In the same way we get that

$$\omega([\mathfrak{E}(\xi),\mathfrak{E}(\eta)]) = -d\omega(\mathfrak{E}(\xi),\mathfrak{E}(\eta)) = -\Omega(\mathfrak{E}(\xi),\mathfrak{E}(\eta)) = 0.$$

From these two equalities we arrive immediately to the conclusion. □

Now we move to the definition of the Laplacians corresponding to the distributions $\mathcal{D}$, $\mathcal{E}$

Definition 3.2.19. *1. Given $z \in Z$, the $\mathcal{D}$-Laplacian is*

$$(\Delta^{\mathcal{D}}\omega)(x) = \sum_{j=1}^{m}(\nabla_{(E_j)_x}\nabla_{E_j}\omega - \nabla_{\nabla_{(E_j)_x}E_j}\omega),$$

where $(E_j)_{j=1,\dots,m} \subset \mathcal{D}$ is an arbitrary local orthonormal basis around z in $\mathcal{D}$ and ω is a local form defined around z.

2. In the same way the $\mathcal{E}$-Laplacian is

$$(\Delta^{\mathcal{E}}\omega)(x) = \sum_{j=1}^{\nu}(\nabla_{(F_j)_x}\nabla_{F_j}\omega - \nabla_{\nabla_{(F_j)_x}F_j}\omega),$$

where $(F_j)_{j=1,\dots,\nu} \subset \mathcal{E}$ is an arbitrary local orthonormal basis around z in $\mathcal{E}$ with the same conditions for ω.

3. The Bochner Laplacian $\Delta_B^{\mathcal{D}}$ is

$$\Delta_B^{\mathcal{D}} = \sum_{j=1}^{m}\mathfrak{E}(e_j)^2,$$

for any orthonormal basis $(e_j)_{j=1,\dots,m}$ in $\mathbb{R}^m$.

4. Similarly, Bochner Laplacian $\Delta_B^{\mathcal{E}}$ is

$$\Delta_B^{\mathcal{E}} = \sum_{j=1}^{\nu}\mathfrak{E}(f_j)^2,$$

for any orthonormal basis $(f_j)_{j=1,\dots,\nu}$ in $\mathbb{R}^\nu$.

Then we can state the analog of (2.2.6)

Proposition 3.2.20. *For any smooth form ω we have*

$$\widetilde{\Delta^{\mathcal{D}}\omega}(\pi u) = \Delta_B^{\mathcal{D}}\tilde{\omega}(u)$$

and similarly

$$\widetilde{\Delta^{\mathcal{E}}\omega}(\pi u) = \Delta_B^{\mathcal{E}}\tilde{\omega}(u)$$

with the same notations given in (2.2.2)

Proof. The proof is basically the same as the one for (2.2.6). The only remark one has to make here is about the choice of the vector fields X and Y we took in there. For the choice of Y we point here the consistency given by the fact that the connection is compatible with the pair $(\mathcal{D}, \mathcal{E})$, thus the parallel transportation along the curve p alluded in there gives a vector field Y that stays in one of the distributions $\mathcal{D}$ or $\mathcal{E}$. The rest is the same. □

Now we can apply all these considerations to the case of a vector bundle. To make things more precisely, we take the manifold Z to be the vector bundle E over the manifold M and the distributions to be $\mathcal{D} = T^H(E)$, $\mathcal{E} = T^V(E)$. If we consider in here the Bismut connection, then we have all the setup in the beginning of this section, namely the compatibility of the connection with the pair $(\mathcal{D}, \mathcal{E})$. To distinguish this particular case from the general case treated in this section we will denote $\mathcal{O}^{H,V}(E) = \mathcal{O}(E; \mathcal{D}, \mathcal{E})$. Also set $\Delta^H = \Delta^{T^H(E)}$, the horizontal Laplacian, $\Delta^V = \Delta^{T^V(E)}$, the vertical Laplacian, $\Delta_B^H = \Delta_B^{T^H(E)}$ and $\Delta_B^V = \Delta_B^{T^V(E)}$. Then we have the following corollary.

Corollary 3.2.21. *In $T\mathcal{O}^{H,V}(E)$, for any $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^\nu$,*

$$[\mathfrak{E}(\xi), \mathfrak{E}(\eta)] = 0.$$

Proof. By theorem (3.1.22) we get that if X, or Y is a vertical vector, then $T(X, Y) = 0$ and $R(X, Y) = 0$. Then applying the above corollary we get the conclusion. □

We record here a rewriting of (3.2.20) on $T\mathcal{O}^{H,V}(E)$.

Corollary 3.2.22. *Within these notations*

$$\widetilde{\Delta^H \omega}(\pi u) = \Delta_B^H \tilde{\omega}(u)$$

and

$$\widetilde{\Delta^V \omega}(\pi u) = \Delta_B^V \tilde{\omega}(u).$$

In particular if $f : E \to \mathbb{R}$, then

$$\Delta^H f = \Delta_B^H (f \circ \pi) \quad \textit{and} \quad \Delta^V f = \Delta_B^V (f \circ \pi).$$

3.3 Brownian Motions and Parallel Transportation

In this section we introduce ∇-Brownian motion with respect to a compatible connection on a Riemannian manifold and we will discuss the representation of the heat kernels on differential forms of the corresponding Laplacians. Also we treat the parallel transportation along ∇-Brownian paths of the basis in a vector bundle. We give the representation of the ∇-Brownian motion on a vector bundle endowed with the Bismut connection.

3.3.1 ∇-Brownian Motion

In this section the ∇-Laplacians are taken on functions only.

Following [7, Theorem 7.2 and 8.33], for a smooth path in $\mathcal{P}(\mathbb{R}^d)$ and $\mathfrak{f} \in \mathcal{O}(M)$ we denote here by $\mathfrak{p}^\nabla(t, \mathfrak{f}, \mathbf{w})$ the solution to

$$\dot{\mathfrak{p}}^\nabla(t, \mathfrak{f}, \mathbf{w}) = \mathfrak{E}^\nabla(\dot{\mathbf{w}}(t))_{\mathfrak{p}^\nabla(t,\mathfrak{f},\mathbf{w})} \quad \text{with} \quad \mathfrak{p}^\nabla(0, \mathfrak{f}, \mathbf{w}) = \mathfrak{f}.$$

Take $p^\nabla(t, \pi\mathfrak{f}, \mathbf{w}) = \pi\mathfrak{p}^\nabla(t, \mathfrak{f}, \mathbf{w})$. Then we have,

Theorem 3.3.1. *Given $x \in M$ and $\mathfrak{f} \in \pi^{-1}(x)$, the martingale problem for $\frac{1}{2}\Delta^\nabla$ starting at x is well posed if and only if the martingale problem for $\frac{1}{2}\Delta_B^\nabla$ starting at $\mathfrak{f}$ is well posed.*

Moreover, if the martingale problem for $\frac{1}{2}\Delta^\nabla$ starting at x is well posed, then the solution to the martingale problem for $\frac{1}{2}\Delta_B^\nabla$ starting at $\mathfrak{f}$ is the $\mathcal{W}_d$-distribution of a measurable map $\mathcal{P}(\mathbb{R}^d) \ni \mathbf{w} \to \mathfrak{p}^\nabla(\cdot, \mathfrak{f}, \mathbf{w}) \in \mathcal{P}(\mathcal{O}(M))$ to which the sequence $\{\mathfrak{p}^\nabla(\cdot, \mathfrak{f}, \mathbf{w}_n)\}_0^\infty$ converges in $\mathcal{P}(\mathcal{O}(M))$ for $\mathcal{W}_d$-almost every $\mathbf{w}$.

Finally if the martingale problem for $\frac{1}{2}\Delta^\nabla$ is well posed on M, then the martingale problem of $\frac{1}{2}\Delta_B^\nabla$ is well posed on $\mathcal{O}(M)$ and there is a measurable $\mathcal{P}(\mathbb{R}^d) \ni \mathbf{w} \to \mathfrak{p}^\nabla(\cdot, *, \mathbf{w}) \in C^{0,\infty}([0, \infty) \times \mathcal{O}(M); \mathcal{O}(M))$ to which $\{\mathfrak{p}^\nabla(\cdot, *, \mathbf{w}_n)\}_0^\infty$ converges in $C^{0,\infty}([0, \infty) \times \mathcal{O}(M); \mathcal{O}(M))$ for $\mathcal{W}_d$-almost every $\mathbf{w}$.

The proof is exactly the proof of [7, Theorem 8.33].

We will call from here ∇-Brownian motion starting at x, the $\mathcal{W}_d$-distribution $\mu_x^{\nabla,M}$ of

$$\mathcal{P}(\mathbb{R}^d) \ni \mathbf{w} \to p^\nabla(\cdot, x, \mathbf{w}) \in \mathcal{P}(\mathcal{O}(M))$$

Also denote by $\mu_\mathfrak{f}^{\nabla,\mathcal{O}(M)}$ the $\mathcal{W}_d$-distribution of

$$\mathcal{P}(\mathbb{R}^d) \ni \mathbf{w} \to \mathfrak{p}^\nabla(\cdot, \mathfrak{f}, \mathbf{w}) \in \mathcal{P}(\mathcal{O}(M))$$

Corollary 3.3.2. *If ∇_1, ∇_2 are two compatible connections such that the associated Laplacians coincide on functions then the martingale problem is well posed for any of the Laplacians if it is well posed for one of them. In this situation, ∇_1-Brownian motion is the same as ∇_2-Brownian motion.*

3.3.2 About the Vector Bundle Case

In here we shortly discuss the analogous of Theorem (3.3.1) on a vector bundle. More precisely we take the situation from sections (3.2.2) and (3.2.3). Then we have the following theorem, in the same spirit with theorem (3.3.1).

With the notations in section (3.2.2), set $\mathfrak{p}_E^\nabla(t, u, \mathbf{w})$ for a smooth path $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d)$, to be the solution to

$$\dot{\mathfrak{p}}_E^\nabla(t, u, \mathbf{w}) = \mathfrak{E}^\nabla(\dot{\mathbf{w}}(t))_{\mathfrak{p}_E^\nabla(t,u,\mathbf{w})} \quad \text{with} \quad \mathfrak{p}_E^\nabla(0, u, \mathbf{w}) = u.$$

Set $p^\nabla(t, \pi u, \mathbf{w}) = \pi\mathfrak{p}_E^\nabla(t, u, \mathbf{w})$

Then we have the following result.

Theorem 3.3.3. *Given $x \in M$ and $u \in \pi^{-1}(x)$, the martingale problem for $\frac{1}{2}\Delta^{\nabla}$ starting at x is well posed if and only if the martingale problem for $\frac{1}{2}\Delta^{\nabla}_{B,E}$ starting at u is well posed.*

Moreover, if the martingale problem for $\frac{1}{2}\Delta^{\nabla}$ starting at x is well posed, then, the solution to the martingale problem for $\frac{1}{2}\Delta^{\nabla}_{B,E}$ starting at u is the $\mathcal{W}_d$-distribution of a measurable map $\mathcal{P}(\mathbb{R}^d) \ni \mathbf{w} \to \mathfrak{p}^{\nabla}_E(\cdot, u, \mathbf{w}) \in \mathcal{P}(\mathcal{O}(M;E))$ to which the sequence $\{\mathfrak{p}^{\nabla}_E(\cdot, u, \mathbf{w}_n)\}_0^{\infty}$ converges in $\mathcal{P}(\mathcal{O}(M;E))$ for $\mathcal{W}_d$-almost every $\mathbf{w}$.

Finally if the martingale problem for $\frac{1}{2}\Delta^{\nabla}$ is well posed on M, then the martingale problem of $\frac{1}{2}\Delta^{\nabla}_{B,E}$ is well posed on $\mathcal{O}(M;E)$ and there is a measurable $\mathcal{P}(\mathbb{R}^d) \ni \mathbf{w} \to \mathfrak{p}^{\nabla}_E(\cdot, *, \mathbf{w}) \in C^{0,\infty}([0,\infty) \times \mathcal{O}(M;E); \mathcal{O}(M;E))$ to which $\{\mathfrak{p}^{\nabla}_E(\cdot, *, \mathbf{w}_n)\}_0^{\infty}$ converges in $C^{0,\infty}([0,\infty) \times \mathcal{O}(M;E); \mathcal{O}(M;E))$ for $\mathcal{W}_d$-almost every $\mathbf{w}$.

Proof. By (3.2.6) the only problem with the existence of the solution to the martingale problem is the explosion. Because a path $\mathfrak{p}$ in $\mathcal{P}(\mathcal{O}(M;E))$ explodes if and only if the projected path $\pi(\mathfrak{p})$ explodes, what is left is covered entirely by [7, Theorem 7.2]. □

Now we refer to the situation at the end of section (3.2.3) for notations. For a smooth path $\mathbf{w}' \in \mathcal{P}(\mathbb{R}^m)$ define $\mathfrak{p}^h(t, u, \mathbf{w}')$ to be the solution to

$$\dot{\mathfrak{p}}^h(t, u, \mathbf{w}') = \mathfrak{E}^{\nabla}(\dot{\mathbf{w}}'(t))_{\mathfrak{p}^h(t,u,\mathbf{w}')} \quad \text{with} \quad \mathfrak{p}^h(0, u, \mathbf{w}') = u,$$

and for a smooth $\mathbf{w}'' \in \mathcal{P}(\mathbb{R}^{\nu})$ define $\mathfrak{p}^v(t, u, \mathbf{w}'')$ to be the solution to

$$\dot{\mathfrak{p}}^v(t, u, \mathbf{w}'') = \mathfrak{E}^{\nabla}(\dot{\mathbf{w}}''(t))_{\mathfrak{p}^v(t,u,\mathbf{w}'')} \quad \text{with} \quad \mathfrak{p}^v(0, u, \mathbf{w}'') = u.$$

Set now, $p^h(t, z, \mathbf{w}') = \pi(\mathfrak{p}^h(t, u, \mathbf{w}'))$, $p^v(t, z, \mathbf{w}') = \pi(\mathfrak{p}^v(t, u, \mathbf{w}'))$.

For a smooth path in $\mathcal{P}(\mathbb{R}^d)$ and $u \in \mathcal{P}(\mathcal{O}^{H,V}(M)))$ we denote by $\mathfrak{p}^{\nabla}(t, u, \mathbf{w})$ the solution to

$$\dot{\mathfrak{p}}^{\nabla}(t, u, \mathbf{w}) = \mathfrak{E}^{\nabla}(\dot{\mathbf{w}}(t))_{\mathfrak{p}^{\nabla}(t,u,\mathbf{w})} \quad \text{with} \quad \mathfrak{p}^{\nabla}(0, u, \mathbf{w}) = u.$$

Take then $p^{\nabla}(t, \pi u, \mathbf{w}) = \pi \mathfrak{p}^{\nabla}(t, u, \mathbf{w})$.

Then we can state the theorem in this framework.

Theorem 3.3.4.

1. *Given $z \in E$ and $u \in \pi^{-1}(z)$, the martingale problem for $\frac{1}{2}\Delta^H$ ($\frac{1}{2}\Delta^V$) starting at z is well posed if and only if the martingale problem for $\frac{1}{2}\Delta^H_B$ ($\frac{1}{2}\Delta^H_B$) starting at u is well posed.*

2. *In the case of well posedness at z or u, the solution to the martingale problem for $\frac{1}{2}\Delta^{H(V)}_B$ starting at u is the $\mathcal{W}_{m(\nu)}$-distribution of a measurable map $\mathcal{P}(\mathbb{R}^{m(\nu)}) \ni \mathbf{w} \to \mathfrak{p}^{h(v)}(\cdot, u, \mathbf{w}'('')) \in \mathcal{P}(\mathcal{O}^{H,V}(M))$ to which the sequence $\{\mathfrak{p}^{h(v)}_E(\cdot, u, \mathbf{w}'('')_n)\}_0^{\infty}$ converges in $\mathcal{P}(\mathcal{O}^{H,V}(M))$ for $\mathcal{W}_{m(\nu)}$-almost every $\mathbf{w}'('')$.*

3. *If the martingale problems for $\frac{1}{2}\Delta^H$ and $\frac{1}{2}\Delta^V$ are well posed on E then the martingale problems for $\frac{1}{2}\Delta^H_B$ and $\frac{1}{2}\Delta^V_B$ are well posed on $\mathcal{P}(\mathcal{O}^{H,V}(M))$ and there are measurable maps*

$$\mathcal{P}(\mathbb{R}^m) \ni \mathbf{w}' \to \mathfrak{p}^h(\cdot, *, \mathbf{w}') \in C^{0,\infty}([0,\infty) \times \mathcal{O}^{H,V}(M); \mathcal{O}^{H,V}(M))$$

and

$$\mathcal{P}(\mathbb{R}^\nu) \ni \mathbf{w}'' \to \mathfrak{p}^v(\cdot, *, \mathbf{w}'') \in C^{0,\infty}([0,\infty) \times \mathcal{O}^{H,V}(M); \mathcal{O}^{H,V}(M))$$

to which $\{\mathfrak{p}^h_E(\cdot, *, \mathbf{w}'_n)\}_0^\infty$ *respectively* $\{\mathfrak{p}^v_E(\cdot, *, \mathbf{w}''_n)\}_0^\infty$ *converge in* $C^{0,\infty}([0,\infty) \times \mathcal{O}^{H,V}(M); \mathcal{O}^{H,V}(M))$ *for* $\mathcal{W}_{m(\nu)}$*-almost every* $\mathbf{w}'('')$.

4. *Moreover in the case the martingale problems for* $\frac{1}{2}\Delta^H$ *and* $\frac{1}{2}\Delta^V$ *are well posed on* E*, then the martingale problem for* $\frac{1}{2}\Delta^\nabla$ *is well posed on* E *and*

$$\mathfrak{p}^\nabla(t, u, (\mathbf{w}', \mathbf{w}'')) = \mathfrak{p}^h(t, \mathfrak{p}^v(t, u, \mathbf{w}''), \mathbf{w}') = \mathfrak{p}^v(t, \mathfrak{p}^h(t, u, \mathbf{w}'), \mathbf{w}'') \tag{3.3.5}$$

for $\mathcal{W}_d$*-almost every path* $\mathbf{w} = (\mathbf{w}', \mathbf{w}'')$.

Proof. The proofs of 1., 2. and 3. follow like the proof of Theorem [7, 7.2] and the observation that the explosion of the path on the orthonormal level occurs iff the projected path in the manifold E explodes.

To prove part 4. we state and prove here the following elementary fact.

Lemma 3.3.6. *Let* N *be a manifold and* $[0,\infty) \ni t \to A(t) \in TN$, $[0,\infty) \ni t \to B(t) \in TN$ *two families of piecewise constant vector fields such that* $[A(t), B(s)] = 0$ *for any t,s. If* $\varphi_A(t,x) : [0,\infty) \times N \to N$ *and* $\varphi_B(t,x) : [0,\infty) \times N \to N$ *are the continuous piecewise differentiable solutions to*

$$\dot\varphi_A(t,x) = (A(t))_{\varphi_A(t,x)} \quad \textit{with} \quad \varphi_A(0,x) = x$$

and

$$\dot\varphi_B(t,x) = (B(t))_{\varphi_B(t,x)} \quad \textit{with} \quad \varphi_B(0,x) = x$$

then $\varphi(t,x) = \varphi_A(t, \varphi_B(t,x))$ *is the continuous piecewise differentiable solution to*

$$\dot\varphi(t,x) = (A(t,x) + B(t,x))_{\varphi(t,x)} \quad \textit{with} \quad \varphi(0,x) = x. \tag{*}$$

Proof of Lemma. If $A(t), B(t)$ are constant on the interval $[0,a]$ then everything follows from the standard case of differential geometry. Thus, $\varphi(t,x) = \varphi_A(t, \varphi_B(t,x))$ for $t \in [0,a]$.

Take the maximal $\overline{a}$ with the property that on the interval $[0,\overline{a})$ we have the desired property for φ. Assume $\overline{a} < \infty$. We show that there is an interval $[0,b]$ with $[0,\overline{a}) \subset [0,b]$ such that on this interval φ satisfies (*).

Indeed take b such that for $t \in [\overline{a}, b]$ the vector $A(t), B(t)$ are constant. Then consider α_A, α_B the solutions on $[0, b-\overline{a}]$ to

$$\begin{cases} \dot\alpha_A(t,x) = (A(\overline{a}))_{\alpha_A(t,x)} \\ \alpha_A(0,x) = \varphi_A(\overline{a}, x) \end{cases}$$

$$\begin{cases} \dot\alpha_B(t,x) = (B(\overline{a}))_{\alpha_B(t,x)} \\ \alpha_B(0,x) = \varphi_B(\overline{a}, x) \end{cases}$$

and set $\beta(t,x) = \alpha_A(t, \alpha_B(t,x))$. By a simple calculation one sees that

$$\begin{cases} \dot\beta(t,x) = (A(\overline{a}) + B(\overline{a}))_{\beta(t,x)} \\ \beta(0,x) = \varphi(\overline{a}, x) \end{cases}$$

for any $t \in [0, b-\overline{a}]$. Extending through

$$\varphi(t,x) = \begin{cases} \varphi(t,x) & t \in [0,\overline{a}] \\ \beta(t-\overline{a},x) & t \in [\overline{a}, b] \end{cases}$$

we get a contradiction. Thus $\overline{a} = \infty$ and the Lemma. □

Returning to the proof of the Theorem, we only have to mention here that for a path $\mathbf{w} = (\mathbf{w}', \mathbf{w}'')$, by (3.2.21) $\mathfrak{E}(\mathbf{w}'_n)$ and $\mathfrak{E}(\mathbf{w}''_n)$ satisfy the conditions in the Lemma. Because $\mathfrak{p}^h$ and $\mathfrak{p}^v$, play the rôle of φ_A and φ_B while $\mathfrak{E}(\mathbf{w}_n) = \mathfrak{E}(\mathbf{w}'_n) + \mathfrak{E}(\mathbf{w}''_n)$ and by definition $\mathfrak{p}^\nabla(t,u,\mathbf{w}_n)$ is the flow of $\mathfrak{E}(\mathbf{w}_n)$, we conclude that

$$\mathfrak{p}^\nabla(t,u,(\mathbf{w}'_n,\mathbf{w}''_n)) = \mathfrak{p}^h(t, \mathfrak{p}^v(t,u,\mathbf{w}''_n), \mathbf{w}'_n) = \mathfrak{p}^v(t, \mathfrak{p}^h(t,u,\mathbf{w}'_n), \mathbf{w}''_n).$$

From here the convergence in 3. finishes the argument. □

3.3.3 Parallel Transportation

As in the usual case one can interpret the parallel transportation along the ∇-Brownian paths in the following way. For a given $x \in M$, $X_x \in T_x(M)$ we identify $\mathfrak{p}(t)(\mathfrak{f}^{-1}X_x)$ with $\tau^\nabla_{\pi(\mathfrak{p})\restriction[0,t]}X_x$, $\mu_\mathfrak{f}^{\nabla,\mathcal{O}(M)}$-almost everywhere. Thus, the $\mu_\mathfrak{f}^{\nabla,\mathcal{O}(M)}$-distribution of

$$\mathcal{P}(\mathcal{O}(M)) \ni \mathfrak{p} \to \mathfrak{p}(t)(\mathfrak{f}^{-1}X_x) \in T_{\pi(\mathfrak{p}(t))}(M)$$

is the same as the $\mu_x^{\nabla,M}$-distribution of

$$\mathcal{P}(M)) \ni p \to \tau^\nabla_{p\restriction[0,t]}X_x \in T_{p(t)}(M).$$

With the notations in section (3.2.2) and Theorem (3.3.3) let M be a Riemannian manifold endowed with a compatible connection and E an Euclidean vector bundle over M. If the martingale problem for Δ^∇ on M is well posed, then there is a natural notion of parallel transportation of sections in E along paths on M. To be more precise, we remind that if $\mathbf{w}$ is a smooth path, $\mathfrak{p}_E^\nabla(t,u,\mathbf{w})\xi$ is nothing but parallel transportation of the vector $u\xi \in T_xM \oplus E_x$ along the path $p^\nabla(t,x,\mathbf{w})$. On the other hand because the $\mathcal{W}_d$-distribution of $\mathcal{P}(\mathbb{R}^d) \ni \mathbf{w} \to p^\nabla(\cdot,x,\mathbf{w}) \in \mathcal{P}(M)$ is the solution to the martingale problem for $\frac{1}{2}\Delta^\nabla$ starting at x, we can say that for any $(X_x, s_x) \in T_xM \oplus E_x$

$$\tau^\nabla_{p(\cdot,x,\mathbf{w})\restriction[0,t]}(X_x, s_x) = \mathfrak{p}_E^\nabla(t,u,\mathbf{w})u^{-1}(X_x,s_x).$$

is the parallel transportation along the path $p(\cdot,x,\mathbf{w})$. In particular we have a well defined notion of parallel transportation along ∇-Brownian paths in M for sections in E.

Definition 3.3.7. *For a ∇-Brownian path φ starting at x in M, we denote the parallel transportation of $s_x \in E_x$ along φ by*

$$\tau^V_{\varphi\restriction[0,t]} s_x.$$

3.3.4 The Brownian Motion on a Vector Bundle

By Theorem (3.3.4) it suffices to check that the martingale problems for Δ^H and Δ^V are well posed. More than that we will get a representation of the ∇-Brownian motion once we have the representation of the horizontal and vertical motions.

In the first place, fix a point $z \in E$. By definition $\Delta^V = \sum_{j=1}^{\nu}(\nabla_{(F_j)_x}\nabla_{F_j} - \nabla_{\nabla_{(F_j)_x}F_j})$ where $(F_j)_{j=1,\nu} \subset T^V(E)$ is an orthonormal basis around z. Now using (3.1.21), the vertical Laplacian on functions is $\Delta^V = \sum_{j=1}^{\nu}(F_j)_x(F_j)$. This immediately implies that if we identify the fiber E_x with $\mathbb{R}^\nu$ using the basis F_j, then the vertical Laplacian is precisely the usual Laplacian in $\mathbb{R}^\nu$. Hence the vertical Brownian motion is the Euclidean Brownian motion in the fiber E_x starting at y. This takes care of the vertical Brownian motion.

For the horizontal motion, we mention that we work with Levi-Civita connection on M.

Proposition 3.3.8. *Assume that the martingale problem at x for $\frac{1}{2}\Delta_M$ is well posed and take the associated map $p_M(t,x,\mathbf{w}')$ with $\mathbf{w}' \in \mathcal{P}(\mathbb{R}^d)$. Then*

$$p^H(t,z,\mathbf{w}') = \tau^V_{p_M(\cdot,x,\mathbf{w}')\restriction[0,t]} y.$$

Proof. Take an isometry $\mathfrak{f} : \mathbb{R}^d \to T_x(M)$ and an the isometry $u : \mathbb{R}^d \to T^H(E_z)$ given by $\mathfrak{f}$ composed with the identification of $T_x(M)$ with the horizontal space $T^H(E_z)$. Then for a smooth path $\mathbf{w}'$ in $\mathbb{R}^d$, $p_M(t,x,\mathbf{w}')$ is the path φ given by

$$\dot{\varphi}(t) = \tau_{\varphi\restriction[0,t]}\mathfrak{f}\dot{\mathbf{w}}'(t) \quad \text{with} \quad \varphi(0) = x$$

where here the parallel transport stands for the one on M, while the path $p^H(t,z,\mathbf{w}')$ is the path ψ given by

$$\dot{\psi}(t) = \tau^{\nabla}_{\psi\restriction[0,t]} u\dot{\mathbf{w}}'(t) \quad \text{with} \quad \psi(0) = z$$

where here the parallel transportation is the one on E with respect to the Bismut connection. Since $u\dot{\mathbf{w}}'(t)$ is horizontal and is canonically identified with $\mathfrak{f}\dot{\mathbf{w}}'$, the characterization of the Bismut connection gives that

$$\rho(\psi(t)) = \varphi(t) \quad \text{and} \quad \psi(t) = \tau^V_{\varphi\restriction[0,t]} y.$$

From here and the fact that the martingale problem for $\frac{1}{2}\Delta_M$ is well posed, we get the conclusion by taking limits of $\mathbf{w}'_n$ for a generic path. □

Putting together the characterizations of the vertical and horizontal motions, (3.3.2), point (4) of Theorem (3.1.22) and definition (3.3.7) we get

Theorem 3.3.9. *If $p(t,z,\mathbf{w})$ stands for the associated object constructed with respect to the Laplacian of the Levi-Civita connection, then for $\mathcal{W}_{m+\nu}$-almost every $\mathbf{w} = (\mathbf{w}', \mathbf{w}'')$ path in $\mathcal{P}(\mathbb{R}^{m+\nu})$*

$$p(t,z,\mathbf{w}) = \left(p_{M_i}(t,x,\mathbf{w}'), \tau^V_{p_M(\cdot,x,\mathbf{w}')\restriction[0,t]}(y + \mathbf{w}''(t))\right)$$

where we identify the fiber E_x with $\mathbb{R}^\nu$ by an orthonormal basis.

3.4 Heat Kernel Estimates and Comparisons

3.4.1 The Boundedness

The setup in this case is the following. The class of the operators acting on forms for which we prove the boundedness estimates is

$$L^\alpha = -\Delta^\nabla + \alpha^2|\mathrm{grad}h|^2 - \alpha\Delta h + 2\alpha(\mathrm{hess}h) + \sum_{j=1}^{d} B(E_j)\nabla_{E_j} + C \tag{3.4.1}$$

where the data obeys:

1. The connection ∇ satisfies

   (a) compatibility with the metric on M;

   (b) ∇-Laplacian on functions is the same as the standard Laplacian;

   (c) the Hessian of the function h is the same as the Hessian with respect to the Levi-Civita connection.

2. $B_z(X_z) = (D^*S_0)_z(X_z) + (D^*S_1)_z(X_z) + \cdots + (D^*S_k)_z(X_z)$ in the notations of Definition 1.5 with the crucial supplementary condition that $B_z(X_z)$ is skew symmetric for any $z \in M$, $X_z \in T_z(M)$;

3. $C = D^*T_0 + D^*T_1 + \cdots + D^*T_l$ for some smooth tensors T_i.

Take the heat kernel of the operator $\frac{1}{2}L^\alpha$,

$$p^{L^\alpha}(t,z_1,z_2) : \wedge_{z_2}(M) \to \wedge_{z_1}(M) \tag{3.4.2}$$

and denote

$$Q^{L^\alpha}(t) = \int_M \mathrm{Tr}\, p^{L^\alpha}(t,z,z)dz. \tag{3.4.3}$$

Using the expression for the operator L^α we can give the expression of the heat kernel (cf. Appendix B) as

$$p^{L^\alpha}(t, z_1, z_2) = \mathbb{E}^{\mathcal{W}_d}\left[U^\alpha(t, z_1, \mathbf{w})\tau^{\nabla}_{p(\cdot, z_1, \mathbf{w})\restriction[t,0]}\delta_{z_2}(p(t, z_1, \mathbf{w}))\right] \tag{3.4.4}$$

where

$$\begin{cases} dU^\alpha(t, z, \mathbf{w}) = U^\alpha(t, z, \mathbf{w})\Big(-\frac{\alpha^2}{2}|\text{grad}h(p(t, z, \mathbf{w}))|^2 I\ dt + \frac{\alpha}{2}\Delta h(p(t, z, \mathbf{w}))I\ dt \\ \qquad - \alpha\left(\tau^{\nabla}_{p(\cdot, z, \mathbf{w})\restriction[0,t]}\right)^{-1}(\text{hess}_{p(t,z,\mathbf{w})}h)\ \tau^{\nabla}_{p(\cdot, z, \mathbf{w})\restriction[0,t]}dt \\ \qquad + C^\nabla(t, z, \mathbf{w})dt + \sum_{j=1}^{d} B_j^\nabla(t, z, \mathbf{w})d\mathbf{w}_j(t)\Big) \\ U^\alpha(0, z, \mathbf{w}) = \text{Id}_{\bigwedge_z(M)} \end{cases}$$

or

$$p^{L^\alpha}(t, z_1, z_2) = \mathbb{E}^{\mathcal{W}_d}\left[e^{A(\alpha,t,\mathbf{w})}V^\alpha(t, z_1, \mathbf{w})\tau^{\nabla}_{p(\cdot, z_1, \mathbf{w})\restriction[t,0]}\delta_{z_2}(p(t, z_1, \mathbf{w}))\right] \tag{3.4.5}$$

with

$$A(\alpha, t, \mathbf{w}) = -\frac{\alpha^2}{2}\int_0^t |\text{grad}h(p(s, z, \mathbf{w}))|^2 ds + \frac{\alpha}{2}\int_0^t \Delta h(p(s, z, \mathbf{w}))ds$$

and

$$\begin{cases} dV^\alpha(t, z, \mathbf{w}) = V^\alpha(t, z, \mathbf{w})\Big(- \alpha\left(\tau^{\nabla}_{p(\cdot, z, \mathbf{w})\restriction[0,t]}\right)^{-1}(\text{hess}_{p(t,z,\mathbf{w})}h)\ \tau^{\nabla}_{p(\cdot, z, \mathbf{w})\restriction[0,t]}dt \\ \qquad + C^\nabla(t, z, \mathbf{w})dt + \sum_{j=1}^{d} B_j^\nabla(t, z, \mathbf{w})d\mathbf{w}_j(t)\Big) \\ V^\alpha(0, z, \mathbf{w}) = \text{Id}_{\bigwedge_z(M)} \end{cases}$$

Conform (B.1.6) the basic estimate of the heat kernel on k-forms as:

$$\|p_k^{L^\alpha}(t, z_1, z_2)\|_{z_2, z_1} \le C\mathbb{E}^{\mathcal{W}_d}\left[e^{\overline{A}(\alpha,t,z_1,\mathbf{w})}\delta_{z_2}(p(t, z_1, \mathbf{w}))\right] \tag{3.4.6}$$

for a constant depending on B, C in the definition of L^α and with $\overline{A}(\alpha, t, z, \mathbf{w})$ defined as

$$-\frac{\alpha^2}{2}\int_0^t |\text{grad}h(p(s, z, \mathbf{w}))|^2 ds + \frac{\alpha}{2}\int_0^t \Delta h(p(s, z, \mathbf{w}))ds + \alpha\int_0^t f(p(s, z, \mathbf{w}))ds$$

for a smooth function f such that

$$\begin{aligned} &f(z) = \nu_i^- \text{ for } z \text{ close to the critical submanifold } M_i, \\ &-\text{hess}_z h|_{\bigwedge_z^k(M)} \le f(x)Id_{\bigwedge_z^k(M)} \text{ for any } z \in M, k = 1, \cdots, d. \end{aligned} \tag{3.4.7}$$

Using this first estimate, the estimation of the heat kernel $p_k^{L\alpha}(t, z_1, z_2)$ can be done by estimating the quantity

$$\mathcal{P}^\alpha(t, z_1, z_2) = \mathbb{E}^{\mathcal{W}_d}\left[e^{\overline{A}(\alpha,t,z_1,\mathbf{w})}\delta_{z_2}(p(t, z_1, \mathbf{w}))\right] \tag{3.4.8}$$

on the right hand side of (3.4.6).

The first result of this section is the following.

Theorem 3.4.9 (The Away Case). *For $r > 0$ small enough, set*

$$\Omega_r = \{z \in M, \text{dist}(z, M_i) > r, \textit{ for all } i = 1, \cdots, l\}.$$

Then, there exist constants $\alpha_0(t, r) > 0$, $C_1(t, r) > 0$, $C_2(t, r) > 0$ depending on t and r such that for any $\alpha \geq \alpha_0(t, r)$,

$$\mathcal{P}^\alpha(t, z_1, z_2) \leq C_1(t, r)e^{-C_2(t,r)\alpha}$$

uniformly for $z_1 \in \Omega_r$ and $z_2 \in M$.

Sketch of the Proof. Because the proof is merely a repetition of the argument we gave in the non-degenerate case, we will point only the main steps. First we apply the integration by parts and Hölder's inequality to the right hand side of (3.4.6) to show that for any $\eta > 0$, there is a polynomial P_η in α such that

$$\mathcal{P}^\alpha(t, z_1, z_2) \leq P_\eta(\alpha)\left\{\mathbb{E}^{\mu_{z_1}^M}\left[\exp\left(\int_0^t H_\eta^\alpha(v, \varphi)dv\right)\right]\right\}^{\frac{1}{1+\eta}}$$

uniformly in $z_1 \in \Omega_r$, $z_2 \in M$, with the notation

$$H_\eta^\alpha(v, \varphi) = (1 + \eta)\left(-\frac{\alpha^2}{2}|\text{grad}h(\varphi(v))|^2 + \frac{\alpha}{2}\Delta h(\varphi(v)) + \alpha f(\varphi(v))\right). \tag{3.4.10}$$

Hence we need to estimate

$$q^\alpha(t, z) = \mathbb{E}^{\mu_z^M}\left[\exp\left(\int_0^t H_\eta^\alpha(v, \varphi)dv\right)\right]. \tag{3.4.11}$$

To do this, we use the same iterative argument as in the non-degenerate case. We can run all the arguments replacing the critical points in there with the critical submanifolds and the balls around critical points with the ball bundles $B(M_i, r) = \{z \in E_i; |z| \leq r\}$. Everything works just as in the non-degenerate case, the only thing one has to worry about is the identification of the function $u : \mathbb{R}_+ \times B(M_i, r) \longrightarrow \mathbb{R}$ given by

$$u_\eta^\alpha(s, z) = \int \exp\left(\int_0^{\tau(\psi)\wedge s} H_\eta^\alpha(v, \psi)dv\right)\mu_z^M(d\psi) \tag{3.4.12}$$

with τ the exit time from the ball bundle $B(M_i, r)$. Because the function h near M_i is $\frac{1}{2}\left(|y^+|^2 - |y^-|^2\right)$ and the Hessian is given by the formulas in (3.1.22), an easy calculation gives

$$H_\eta^\alpha(v,\varphi) = (1+\eta)\left(-\frac{\alpha^2}{2}|\varphi^V(v))|^2 + \frac{\alpha\nu_i}{2}\right).$$

Now we use the representation of the Brownian motion on the bundle E_i given by Theorem (3.3.9) as the parallel transportation of the vertical Brownian motion versus the horizontal Brownian motion. Since the parallel transportation is an isometry in fibers, an immediate consequence is that,

$$u_\eta^\alpha(s,z) = u_{\eta,r}^\alpha(s,y)$$

with

$$u_{\eta,r}^\alpha(s,y) = \mathbb{E}^{\mathcal{W}_{\nu_i}}\left[e^{\int_0^{\tau_r(\mathbf{w}'')\wedge s}(1+\eta)\left(-\frac{\alpha^2}{2}|y+\mathbf{w}''(v)|^2+\frac{\alpha\nu_i}{2}\right)dv}\right]$$

where we identified the fiber $(E_i)_x$ with $\mathbb{R}^{\nu_i}$ and τ_r is the exit time from the ball of radius r. This is exactly the same thing as in the non-degenerate case, this time taken on the fiber. □

For the close to the critical set case we follow the same route as in the non-degenerate case in section (2.4) to reduce the estimation to one on the vector bundles E_i.

To deal with the near to the critical set case, we make some notations. On each vector bundle E_i we denote

$$\overline{\mathcal{P}}_i^\alpha(t,z_1,z_2) = \mathbb{E}^{\mathcal{W}_d}\left[e^{-\frac{\alpha^2}{2}\int_0^t|y_1+\mathbf{w}''(s)|^2ds+\frac{\alpha t\nu_i}{2}}\delta_{z_2}(p(t,z_1,\mathbf{w}))\right] \tag{3.4.13}$$

with $\mathbf{w}''$ standing for the Brownian motion starting at 0 on the fiber $(E_i)_{x_1}$ (see Theorem (3.3.9) for details).

Theorem 3.4.14 (The Near Case). *With the notation in* (3.4.8), *for small* $r>0$ *and any* $t>0$, *there are constants* $C_1(t,r)>0$, $C_2(t,r)>0$, $\alpha(t,r)>0$ *such that for any* $z\in B(M_i,r)=\{z\in E_i \mid \operatorname{dist}(z,M_i)\le r\}$ *and* $\alpha\ge\alpha(t,r)$,

$$\left|\mathcal{P}^\alpha(t,z,z)-\overline{\mathcal{P}}_i^\alpha(t,z,z)\right| \le C_1(t,r)e^{-\alpha C_2(t,r)}. \tag{3.4.15}$$

Proof. The proof is on the same ideas we used to handle the non-degenerate case. Here are some adjustments to that procedure due to the geometry of the vector bundles.

First, we repeat the basic step of integrating by parts to write

$$\mathcal{P}^\alpha(t,z,z) = \mathbb{E}^{\mathcal{W}_d}\left[T(\alpha,z,\mathbf{w})e^{\overline{A}(\alpha,t,z,\mathbf{w})}\right] \tag{3.4.16}$$

where (cf. Theorem (A.2.6)) $T(\alpha,z,\mathbf{w})$ is polynomial in α with coefficients in all

$L^p(\mathcal{W}_d)$ for any $p > 1$. We next split this integral into two parts

$$\mathbb{E}^{\mathcal{W}_d}\left[T(\alpha,z,\mathbf{w})e^{\overline{A}(\alpha,t,z,\mathbf{w})}\right] = \mathbb{E}^{\mathcal{W}_d}\left[T(\alpha,z,\mathbf{w})e^{\overline{A}(\alpha,t,z,\mathbf{w})}, \tau_{2r}(p(\cdot,z,\mathbf{w})) \geq t\right]$$
$$+ \mathbb{E}^{\mathcal{W}_d}\left[T(\alpha,z,\mathbf{w})e^{\overline{A}_k(\alpha,t,z,\mathbf{w})}, \tau_{2r}(p(\cdot,z,\mathbf{w})) < t\right]$$
$$= I_z^{int}(\alpha) + I_z^{ext}(\alpha) \tag{3.4.17}$$

For the second integral one can use Hölder's inequality to justify that for a positive $\eta > 0$ there is a constant $C_\eta > 0$,

$$I_z^{ext}(\alpha) \leq C_\eta \left\{q_{ext}^\alpha(t,z)\right\}^{1/(1+\eta)}$$

where

$$q_{ext}^\alpha(t,z) = \mathbb{E}^{\mu_z^M}\left[\exp\Big(\int_0^t H_\eta^\alpha(v,\varphi)dv\Big), \tau_{2r}(\varphi) < t\right].$$

Hence, estimates on the size of $I_z^{ext}(\alpha)$ come down to estimates on the size of $q_{ext}^\alpha(t,z)$. This is done in the non-degenerate case in Proposition (2.4.4). We now shortly describe what the adjustment is here. The iteration works as the iteration in the non-degenerate case with the only change that we discussed in the proof of the above theorem. The last bit invoked in the proof of (2.4.4) is the estimation of the last term of the iteration to an estimation of an initial-boundary problem (see (2.4.6)). In our case here we have to write the last quantity in the iteration as

$$u_\eta^\alpha(s,z) = \mathbb{E}^{\mu_z^M}\left[\exp\Big(\int_0^{\tau_{2r}\wedge s} H_\eta^\alpha(v,\varphi)dv\Big), \tau_{2r}(\varphi) < s\right].$$

The last thing here is the description of the Brownian motion in the vector bundle, and an argument as the one in the proof of the above theorem to show that $u_\eta^\alpha(s,z) = u_{\eta,r}^\alpha(s,y)$, with

$$u_{\eta,r}^\alpha(s,y) = \mathbb{E}^{\mathcal{W}_{\nu_i}}\left[e^{\int_0^{\tau_{2r}(\mathbf{w}'')\wedge s}(1+\eta)\left(-\frac{\alpha^2}{2}|y+\mathbf{w}''(v)|^2+\frac{\alpha\nu_i}{2}\right)dv}, \tau_{2r}(\mathbf{w}'') < s\right].$$

This last integral can be identified with the solution on the appropriate Euclidean space of (2.4.6), we can use again the estimates given in Lemma (2.3.30) to finally show that

$$I_z^{ext}(\alpha) \leq c_1(t,r)e^{-\alpha c_2(t,r)} \tag{3.4.18}$$

for some constants $c_1(t,r) > 0$, $c_2(t,r) > 0$ and for all $\alpha \geq \alpha(t,r)$, $z \in B(M_i,r)$.

Now we turn to the integral $I_z^{ext}(\alpha)$. For this, we replace the manifold M with the manifold E_i and the function h with the function $h(z) = \frac{1}{2}\left(|y^+|^2 - |y^-|^2\right)$. Notice here that despite the fact E_i is a noncompact manifold, the theory from Appendix A justifies the integration by parts. Indeed, the curvature of the Levi-Civita connection is well behaved (cf. (3.1.26)). The key observation is that the factors T in the expressions of $I_z^{ext}(\alpha)$, on M and E_i coincide if the path does not leave the ball

$B(M_i, 2r)$ and the same is true for $\overline{A}(\alpha, t, z, \mathbf{w})$. Thus we can say that the integral $I_z^{ext}(\alpha)$ can be thought of as the integral coming from the computation of $\overline{\mathcal{P}}_i^\alpha$ on E_i. Also on E_i one can prove the same estimates for the corresponding $I_z^{int}(\alpha)$, so that in the end, using again the representation of the Brownian motion on E_i we get

$$\left|\mathcal{P}^\alpha(t,z,z) - \overline{\mathcal{P}}_i^\alpha(t,z,z)\right| \leq C_1(t,r)e^{-\alpha C_2(t,r)}.$$

□

Now everything is reduced to the estimation of $\overline{\mathcal{P}}_i^\alpha(t,z,z)$.

Proposition 3.4.19. *There exist constants $C_i(t) > 0$, $i = 1, \cdots, l$, depending only on t and the data of the submanifolds M_i, such that for any $z \in E_i$,*

$$\overline{\mathcal{P}}_i^\alpha(t,z,z) \leq C_i(t)\left(\frac{\alpha}{\pi\left(1 - e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\alpha \tanh(t\alpha/2)|y|^2}. \tag{3.4.20}$$

In particular

$$\int_{B(M_i,r)} \overline{\mathcal{P}}_i^\alpha(t,z,z)dz \leq C_i(t)vol(M_i), \tag{3.4.21}$$

where $vol(N)$ denotes the volume of the manifold N with respect to the metric on it.

Proof. Recalling the representation of the Brownian motion given by Theorem (3.3.9), we write for any compactly supported function $f : E_i \to \mathbb{R}$,

$$\begin{aligned}
&\int_{E_i} \overline{\mathcal{P}}_{k,i}^\alpha(t,z,u)f(u)du \\
&\quad = \mathbb{E}^{\mathcal{W}_d}\left[e^{-\frac{\alpha^2}{2}\int_0^t |y+\mathbf{w}''(s)|^2 ds + \frac{\alpha t \nu_i}{2}} f(p_{M_i}(t,x,\mathbf{w}'), \tau^V_{p_{M_i}(\cdot,x,\mathbf{w}')\restriction[0,t]}(y + \mathbf{w}''(t)))\right] \\
&\quad = \mathbb{E}^{\mathcal{W}_{d-\nu_i}}\left[\int_{(E_i)_x} \mathcal{Q}_i^\alpha(t,y,y_1) f(p_{M_i}(t,x,\mathbf{w}'), \tau^V_{p_{M_i}(\cdot,x,\mathbf{w}')\restriction[0,t]}(y_1)dy_1\right]
\end{aligned} \tag{3.4.22}$$

with $\mathcal{Q}_i^\alpha(t,y,y')$ the heat kernel for the Hermite like operator $\Delta + \frac{\alpha^2}{2}|y|^2 - \frac{\alpha\nu_i}{2}$ on $\mathbb{R}^{\nu_i}$, given by the formula (see for example [9, page 390])

$$\mathcal{Q}_i^\alpha(t,y,y') = \left(\frac{\alpha}{\pi\left(1 - e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\frac{\alpha \coth(t\alpha))}{2}\left(|y|^2 - \frac{2(y,y')}{\cosh(t\alpha)} + |y'|^2\right)}. \tag{3.4.23}$$

From this from this formula, if $y_1 \in B(y', \rho)$, then

$$\begin{aligned}
&\left(\frac{\alpha}{\pi\left(1 - e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\frac{\alpha \coth(t\alpha))}{2}\left(|y|^2 - \frac{2|y|(|y'|+\rho)}{\cosh(t\alpha)} + |y'|^2 - \rho|y'| - \rho^2\right)} \\
&\qquad\qquad \leq \mathcal{Q}_i^\alpha(t,y,y') \\
&\qquad\qquad \left(\frac{\alpha}{\pi\left(1 - e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\frac{\alpha \coth(t\alpha))}{2}\left(|y|^2 + \frac{2|y|(|y'|+\rho)}{\cosh(t\alpha)} + |y'|^2 + \rho|y'| + \rho^2\right)}.
\end{aligned} \tag{3.4.24}$$

In particular if the function f has support in $B(z',\rho)$ then by the fact that the parallel transportation is an isometry in fibers, we have the inequalitites

$$\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\frac{\alpha\coth(t\alpha))}{2}\left(|y|^2-\frac{2|y|(|y'|+\rho)}{\cosh(t\alpha)}+|y'|^2-\rho|y'|-\rho^2\right)}$$
$$\cdot\mathbb{E}^{\mathcal{W}_{d-\nu_i}}\left[\int_{(E_i)_x} f(p_{M_i}(t,x,\mathbf{w}'),\tau^V_{p_{M_i}(\cdot,x,\mathbf{w}')\restriction[0,t]}(y_1)dy_1\right]$$
$$\leq\int_{E_i}\overline{\mathcal{P}}^{\alpha}_{k,i}(t,z,u)f(u)du$$
$$\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\frac{\alpha\coth(t\alpha))}{2}\left(|y|^2+\frac{2|y|(|y'|+\rho)}{\cosh(t\alpha)}+|y'|^2+\rho|y'|+\rho^2\right)}$$
$$\cdot\mathbb{E}^{\mathcal{W}_{d-\nu_i}}\left[\int_{(E_i)_x} f(p_{M_i}(t,x,\mathbf{w}'),\tau^V_{p_{M_i}(\cdot,x,\mathbf{w}')\restriction[0,t]}(y_1)dy_1\right].$$

Now for a given point z' in E_i we choose the local trivialization of the bundle by parallel transportation. Then we can choose a smooth compactly supported approximation f^v_n in the distributional sense to the delta function $\delta_{y'}$ in $(E_i)_{x'}$.We extend this in an obvious way to nearby fibers. Choose now an approximation f^h_n with support in a small neigborhood of x' to the delta function $\delta_{x'}$ on M_i. Finally one can define $f_n(x,y)=f^h_n(x)f^v_n(y)$. Then using this choice in the above inequality and letting n tend to infinity, then ρ tend to 0, we get

$$\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\frac{\alpha\coth(t\alpha))}{2}\left(|y|^2-\frac{2|y||y'|}{\cosh(t\alpha)}+|y'|^2\right)}p_{M_i}(t,x,x')$$
$$\leq\overline{\mathcal{P}}^{\alpha}_{k,i}(t,z,z')$$
$$\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\frac{\alpha\coth(t\alpha))}{2}\left(|y|^2+\frac{2|y||y'|}{\cosh(t\alpha)}+|y'|^2\right)}p_{M_i}(t,x,x'). \quad (3.4.25)$$

To prove (3.4.21), one has to integrate the estimation we have already gotten to justify that

$$\int_{B(M_i,r)}\overline{\mathcal{P}}^{\alpha}_{i}(t,z,z)dz\leq C_i vol(M_i)\int_{|y|\leq r}\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\alpha\tanh(t\alpha/2)|y|^2}dy$$

and then changing $y\longrightarrow\sqrt{\frac{1}{\alpha\tanh t\alpha/2}}y$ one gets

$$\int_{B(M_i,r)}\overline{\mathcal{P}}^{\alpha}_{i}(t,z,z)dz\leq C_i vol(M_i)\left(\frac{1}{\pi\left(1+e^{-t\alpha}\right)^2}\right)^{\frac{\nu_i}{2}}\int_{|y|\leq r\sqrt{\alpha\tanh t\alpha/2}}dy$$

which ends the proof. □

Putting together Theorem (3.4.9), Theorem (3.4.14) and the Proposition (3.4.19) we prove the following main result of this section

Corollary 3.4.26 (The Boundedness). *Assume we have continuous families $u \in [0,1] \to \nabla_u$, $u \in [0,1] \to B_u$ and $u \in [0,1] \to C_u$ such that for each $u \in [0,1]$ they satisfy the requirements at the beginning of this section. If L_u^α is the corresponding operator, then, there exist constants $K(t, B, C, M_i)$ and α_0 depending on t, $\sup_{u\in[0,1]} \|B_u\|$, $\sup_{u\in[0,1]} \|C_u\|$ and the heat kernels of the submanifolds M_i such that*

$$\int_M \sup_{u\in[0,1]} \left\|p_k^{L_u^\alpha}(t,z,z)\right\|_{z,z} dz \leq K(t,B,C) \tag{3.4.27}$$

for all $\alpha \geq \alpha_0$.

Proof. The proof consists in putting together the theorems proved so far in this section. To wit, we first use (B.1.2) together with (A.2.11) as we used in (3.4.6) to deduce that

$$\sup_{u\in[0,1]} \left\|p_k^{L_u^\alpha}(t,z,z)\right\|_{z,z} \leq K\mathbb{E}^{\mathcal{W}_d}\left[e^{\overline{A}(\alpha,t,z,\mathbf{w})}\delta_z(p(t,z,\mathbf{w}))\right] \tag{*}$$

with the same $\overline{A}$ as in the (3.4.6). We mention here that the constant K depends on the bounds on B_u and C_u one gets from (B.1.2) and (A.2.11). Here the existence of K is due to the fact that B_u and C_u depend continuously on u.

Now choose a particular r_0 small enough. Next step is an invoice of (3.4.9) with r replaced by r_0, to get the exponential decay of the integral

$$\int_{\Omega_{r_0}} \mathcal{P}^\alpha(t,z,z)dz \leq K_1 e^{-\alpha K_2} \tag{**}$$

with $\Omega_{r_0} = \{z \in M \mid \mathrm{dist}(x, M_i) \geq r_0 \text{ for all } i = 1, \cdots, l\}$.

Next we use (3.4.21) to get to

$$\int_{B(M_i,r_0)} \overline{\mathcal{P}}_i^\alpha(t,z,z)dz \leq C_i vol(M_i). \tag{***}$$

Now the result follows by putting together Theorem (3.4.14), (*), (**) and (***). □

3.4.2 The Comparison

In this section we show how based on the results in the previous section, one can prove that for certain operators the integrals of the heat kernels are close to each other. In this section we will see that the crucial role will be played by the fact that the Bismut connection and the Levi-Civita connection are close to each other.

We begin with a choice of a cut-off function φ on M which, for a small enough r_0 is 0 outside the ball $B(Cr(h), r_0)$ of the critical set, 1 on $B(Cr(h), r_0/2)$ and $0 \leq \varphi \leq 1$.

Also we assume that r_0 is sufficiently small so that $B(Cr(h), 4r_0)$ is in the union of the sets $\mathcal{V}_i$ defined in (1.0.4). For a small enough $r > 0$, denote

$$\varphi_r(z_i) = \varphi\left(x_i, \frac{y_i}{r}\right). \tag{3.4.28}$$

for $z_i \in E_i$ close to M_i. Then, $\text{supp}(\varphi_r) \subset \cup_{i=1}^{l} B(M_i, r)$.

Definition 3.4.29.

1. *Define the connection ∇^r on M by*

$$\nabla_z^r = \varphi_r(z)\left(\nabla_i^B\right)_z + (1 - \varphi_r(z))\,\nabla_z^{LC}$$

if $z \in E_i$, where ∇^B stands for the Bismut connection. Set T^r and R^r the torsion respectively the curvature tensors of ∇^r.

2. *Define the operators*

$$\diamond_r^\alpha = d^{\nabla^r,\alpha h}\delta^{\nabla^r,\alpha h} + \delta^{\nabla^r,\alpha h}d^{\nabla^r,\alpha h}$$

where the operators $d^{\nabla^r,\alpha h}$ and $\delta^{\nabla^r,\alpha h}$ are defined in (3.1.2).

The next proposition contains the key relationships between these connections.

Proposition 3.4.30.

1. *There exists a constant $C > 0$ such that for any unit vectors $X, Y \in T_z(M)$,*

$$\left|\left(\nabla_X^r - \nabla_X^{LC}\right)Y\right| \leq C\varphi_r(z)r.$$

2. *The torsion of the connection ∇^r is given by*

$$T_z^r(X,Y) = \varphi_r(z)\left[R_z^V(X^H, Y^H)y_i\right]$$

for any $X, Y \in T_z(E_i)$ where the curvature on the right is the vertical curvature on the bundle E_i. In particular, there is a constant $C > 0$ so that for any unit vectors $X, Y \in T_z(M)$,

$$|T_z^r(X,Y)| \leq C\varphi_r(z)r.$$

3. *There is a constant $C > 0$ such that for any unit vectors $X, Y, Z \in T_z(M)$*

$$|R^r(X,Y)Z - R(X,Y)Z| \leq C\varphi_r(z)r.$$

4. *The Laplacian on functions of the connection ∇^r is the same as the usual Laplacian on functions.*

5. *The Hessian of the function h is the same in either connection ∇^r or ∇^{LC}.*

6. *There exists a constant $C > 0$ such that for any form $\omega \in \bigwedge(M)$ and $\alpha > 0$,*

$$\left|\left(d^{\nabla^r,\alpha h} - d^{\alpha h}\right)\omega\right| \leq C\varphi_r r\left|\omega\right| \tag{3.4.31}$$

and

$$\left|\left(\delta^{\nabla^r,\alpha h} - \delta^{\alpha h}\right)\omega\right| \leq C\varphi_r r\left|\omega\right|. \tag{3.4.32}$$

Proof. The main fact is that the difference between the connection ∇^r and ∇^{LC} is a tensor that can be expressed in terms of the torsion of the Bismut connection.

Denote $S^r = \nabla^r - \nabla^{LC}$. Using (3.1.30) we have a characterization of this tensor as

$$\langle S^r_{X_z}Y_z, Z_z\rangle = \frac{\varphi_r(z)}{2}\left\{\langle T_z(X_z,Y_z),Z_z\rangle - \langle T_z(Y_z,Z_z),X_z\rangle + \langle T_z(Z_z,X_z),Y_z\rangle\right\} \tag{3.4.33}$$

for any vectors $X_z, Y_z, Z_z \in T_z(M)$ with T_z the torsion of the Bismut connection on E_i if $z \in E_i$. Further by (3.1.23) we see that

$$\left\|S^r_{X_z}\right\| \leq C\varphi_r(z)r\left|X_z\right| \tag{3.4.34}$$

where the constant C depends on the bounds of the vertical curvature on the bundles E_i. Using this observation, 1. follows immediately. The second statement is nothing but a reflection of the fact that the torsion of the Levi-Civita connection is 0, the definition of ∇^r and (3.1.23).

For the third item, a simple calculation shows that

$$R^r(X,Y) = \varphi_r^2 R^{\nabla^B}(X,Y) + (1-\varphi_r)^2 R^{\nabla^{LC}} - \varphi_r(1-\varphi_r)(S_X S_Y - S_Y S_X). \tag{3.4.35}$$

where ∇^B is the Bismut connection on the bundles. From this it follows the estimate about the difference of the curvatures.

Recalling the definition of the Laplacian, we have

$$\left(\Delta^{\nabla^r}h\right)(z) = \sum_{i=1}^{d}\left((F_i)_z F_i h - \nabla^r_{(F_i)_z}F_i h\right)$$

for any local orthonormal basis F_i around the point z. Now using the definition of ∇ and Theorem (3.1.22) item (4) we get the statement in (4).

Using the same argument we can prove that the Hessians are equal. To wit, we observe that the Hessians of a function is (see for definitions (3.1.8)) is linear as a function of the connection. Thus invoking again Theorem (3.1.22) we get the statement in item (5).

To prove the last statement, first observe that by Definition (3.1.2) and Proposition (3.1.3) we have

$$d^{\nabla^r,\alpha h} - d^{\alpha h} = \sum_{i=1}^{d}(F_i^*)_z \wedge S^r_{(F_i)_z}$$

From this and the discussion at the beginning of the proof, the statement about d's follows immediately. The statement about δ's follows the same way from Definition (3.1.2), Proposition (3.1.3) (items 3,4) and the estimate on the torsion of ∇^r and S^r. $\square$

First Comparison. The next Theorem is the first step toward the comparison of the integrals of the heat kernels for operators $\square_r^\alpha$ and $\square^\alpha$, the latter one being defined by (1.0.2).

Theorem 3.4.36. *There exist, a constant $C(t) > 0$ depending on t and the data of the manifold M but independent of r, and $\alpha(t,r) > 0$ depending on t, r such that for $\alpha \geq \alpha(t,r)$ we have*

$$\left| \int_M \operatorname{Tr} p_k^{\square^\alpha}(t,z,z)dz - \int_M \operatorname{Tr} p_k^{\Diamond_r^\alpha}(t,z,z)dz \right| \leq C(t)r. \tag{3.4.37}$$

Proof. First, exactly the way we did in deducing the formula (2.1.2), we have the following two identities:

$$\int_M \operatorname{Tr} p_k^{\Diamond_r^\alpha}(t,z,z)dz = \operatorname{Tr} e^{-t(\Diamond_r^\alpha)_k/2} = \sum_{i=0}^{\infty} e^{-t\lambda_i^r(\alpha)/2}$$

and

$$\int_M \operatorname{Tr} p_k^{\square^\alpha}(t,z,z)dz = \operatorname{Tr} e^{-t\square_k^\alpha/2} = \sum_{i=0}^{\infty} e^{-t\lambda_i(\alpha)/2}$$

where $\lambda_i^r(\alpha)$ and $\lambda_i(\alpha)$ are the eigenvalues arranged increasingly of the operators $\Diamond_r^\alpha$ and $\square^\alpha$ acting on H_k^2 (see the beginning of section (2.1.1)).

The idea is to compare the eigenvalues by using the min-max principle. The min-max principle gives the eigenvalues of an elliptic operator second order operator L on k-forms bounded from below, by the formula

$$\lambda_i(L) = \inf_{\substack{V \subset H_k^2 \\ \dim(V)=i}} \max \left\{ \int_M \langle (Lu)(z), u(z) \rangle dz \mid u \in V, \ \|u\| = 1 \right\}$$

where $\|\cdot\|$ here and during this proof, stands for the L^2 norm of sections in $L^2(\bigwedge^k(M))$. To apply this formula to the operators $\square_k^\alpha$ and $(\Diamond_r^\alpha)_k$ we first remark that for $u \in H_k^2$,

$$\langle \Diamond_r^\alpha u, u \rangle = \left| d^{\nabla^r, \alpha h} u \right|^2 + \left| \delta^{\nabla^r, \alpha h} u \right|^2$$

and alike

$$\langle \square^\alpha u, u \rangle = \left| d^{\alpha h} u \right|^2 + \left| \delta^{\alpha h} u \right|^2.$$

From this we get the inequality

$$\begin{aligned}
&\left|\langle\Diamond_r^\alpha u,u\rangle^{1/2}-\langle\Box^\alpha u,u\rangle^{1/2}\right|\\
&\qquad\le\left|\left(\left|d^{\nabla^r,\alpha h}u\right|^2+\left|\delta^{\nabla^r,\alpha h}u\right|^2\right)^{1/2}-\left(\left|d^{\alpha h}u\right|^2+\left|\delta^{\alpha h}u\right|^2\right)^{1/2}\right|\\
&\qquad\le\left|d^{\nabla^r,\alpha h}u-d^{\alpha h}u\right|+\left|\delta^{\nabla^r,\alpha h}u-\delta^{\alpha h}u\right|
\end{aligned}$$

where we have used the elementary inequality for real numbers,

$$|\left(x^2+y^2\right)^{\frac{1}{2}}-\left(z^2+t^2\right)^{\frac{1}{2}}|\le|x-z|+|y-t|$$

to justify the passage to the last line. Now invoking the inequalities (3.4.31) and (3.4.32) we arrive at the inequality

$$\left|\langle\Diamond_r^\alpha u,u\rangle^{1/2}-\langle\Box^\alpha u,u\rangle^{1/2}\right|\le Cr \tag{*}$$

for any u with $\|u\|=1$ and any $\alpha>0$.

Armed with (*), we can prove that

$$\left|\sqrt{\lambda_i^r(\alpha)}-\sqrt{\lambda_i(\alpha)}\right|\le Cr$$

for any positive α and any integer $i\ge 0$. Next, we point the elementary inequality for positive numbers

$$\left|e^{-x}-e^{-y}\right|\le\sqrt{2}\left|\sqrt{x}-\sqrt{y}\right|\left(e^{-x/2}+e^{-y/2}\right)$$

to use it in proving that

$$\left|\sum_{i=0}^{\infty}e^{-t\lambda_i^r(\alpha)/2}-\sum_{i=0}^{\infty}e^{-t\lambda_i(\alpha)/2}\right|\le Cr\sum_{i=0}^{\infty}\left(e^{-t\lambda_i^r(\alpha)/4}+e^{-t\lambda_i(\alpha)/4}\right).$$

This last inequality can be written in terms of heat kernels as

$$\begin{aligned}
&\left|\int_M\operatorname{Tr}p_k^{\Diamond_r^\alpha}(t,z,z)dz-\int_M\operatorname{Tr}p_k^{\Box^\alpha}(t,z,z)dz\right|\\
&\qquad\qquad\le Cr\left(\int_M\operatorname{Tr}p_k^{\Diamond_r^\alpha}(t/2,z,z)dz+\int_M\operatorname{Tr}p_k^{\Box^\alpha}(t/2,z,z)dz\right).
\end{aligned}$$

Now we want to prove that the last part in this inequality is bounded when α is large. To do this we want to apply the boundedness that is granted by Corollary (3.4.26). In order to be able to apply the alluded result, one has to check the conditions required. Basically one has to make sure that the operators we are dealing with have the form (3.4.1). Now, this is granted to us thanks to Proposition (3.4.30) item (2) and the decomposition formula (3.1.13), provided that we prove that for any vector

$X \in T_z(M)$,

$$\sum_{j,k,l}^{d} \langle T^r(E_j, E_k), X\rangle E_j^* \wedge i_{E_k}$$

is a skew symmetric operator on forms. To check this last thing we need only to say that $T^r(E_j, E_k) = -T^r(E_k, E_j)$, implies the desired property. Now using Corollary (3.4.26) for $t/2$ instead of t we end the proof. $\square$

As a remark here we give the expressions for the operators $\square^\alpha$ and $\diamond_r^\alpha$. From (3.1.13) and (3.4.35)

$$\begin{aligned}
&\square^\alpha = -\Delta + \alpha^2 |\text{grad} h|^2 - \alpha\Delta h + 2\alpha \text{hess}^\nabla h - D^* R, \\
&\diamond_r^\alpha = -\Delta^{\nabla^r} + \alpha^2 |\text{grad} h|^2 - \alpha\Delta h + 2\alpha \text{hess} h + \varphi_r^2 D^* R^{\nabla^B} + (1-\varphi_r)^2 D^* R \\
&+ \varphi_r \sum_{j,k,l}^{d} \langle T(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k} \nabla_{E_l}^r - \varphi_r(1-\varphi_r) \sum_{j,k}^{d} (S_{E_j} S_{E_k} - S_{E_k} S_{E_j}) E_j^* \wedge i_{E_k}
\end{aligned} \tag{3.4.38}$$

where here R stands for the curvature of the Levi-Civita connection and T for the torsion of the Bismut connection.

Second Comparison. The first comparison left with the operators in (3.4.38). Next in line is the reduction from the operator $\diamond_r^\alpha$ to the operator

$$\square_r^\alpha = -\Delta^{\nabla^r} + \alpha^2 |\text{grad} h|^2 - \alpha\Delta h + 2\alpha \text{hess} h + \varphi_r^2 D^* R^{\nabla^B} + (1-\varphi_r)^2 D^* R. \tag{3.4.39}$$

The improvement consists in the fact that the first order operator together with a 0^{th} order operator disappear, thus making the operator more amenable to computations.

Theorem 3.4.40. *There exist, a constant $C(t) > 0$ depending on t and the data of the manifold M but independent of r, and $\alpha(t, r) > 0$ depending on t, r such that for $\alpha \geq \alpha(t, r)$ we have*

$$\left| \int_M \text{Tr}\, p_k^{\diamond_r^\alpha}(t, z, z) dz - \int_M \text{Tr}\, p_k^{\square_r^\alpha}(t, z, z) dz \right| \leq C(t) r. \tag{3.4.41}$$

Proof. First, the operator $\varphi_r D^* R^{\nabla^B}$ acting on forms is a self-adjoint operator. This can be easily seen because we have at our disposal precise formulas given in Theorem (3.1.22) item (3.1.24). On the other hand we know that the curvature $D^* R$ is self-adjoint.

These remarks together with [8, Corollary 2.10] prove that the operator $\square_r^\alpha$ is self-adjoint. Take the first order operator

$$\Lambda = \varphi_r \sum_{j,k,l=1}^{d} \langle T(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k} \nabla^r_{E_l} - \varphi_r(1-\varphi_r) \sum_{j,k=1}^{d} (S_{E_j} S_{E_k} - S_{E_k} S_{E_j}) E_j^* \wedge i_{E_k}$$
$$= \sum_{j=1}^{d} \Xi_1(E_j) \nabla^r_{E_j} + \Xi_0 \tag{3.4.42}$$

with the notation

$$\Xi_1(X) = \varphi_r \sum_{k,l=1}^{d} \langle T(E_k, E_l), X\rangle E_k^* \wedge i_{E_l}$$

and

$$\Xi_0 = -\varphi_r(1-\varphi_r) \sum_{j,k=1}^{d} (S_{E_j} S_{E_k} - S_{E_k} S_{E_j}) E_j^* \wedge i_{E_k}.$$

Notice that Λ is formally self-adjoint, a very important property in what follows.

Consider the family of operators

$$L_u^\alpha = \square_r^\alpha + u\Lambda \quad \text{for} \quad 0 \le u \le 1.$$

Note that $L_0^\alpha = \square_r^\alpha$, $L_1^\alpha = \diamond_r^\alpha$ and L_u^α is a formally self-adjoint operator.

Before starting the real proof, we note that (see for example [8, Proposition 2.44]), the trace of a trace class operator $\mathfrak{X}$ on k-forms with kernel $\mathfrak{x}_k(z_1, z_2)$ is given by

$$\mathrm{Tr}(\mathfrak{X}) = \int_M \mathrm{Tr}\mathfrak{x}_k(z,z)dz.$$

We want to use Duhamel's formula to estimate the size of the difference of the integrals of the heat kernels. More precisely, start with

$$\int_M \mathrm{Tr}\, p_k^{\diamond_r^\alpha}(t,z,z)dz - \int_M \mathrm{Tr}\, p_k^{\square_r^\alpha}(t,z,z)dz = \mathrm{Tr} e^{-t(\diamond_r^\alpha)_k/2} - \mathrm{Tr} e^{-t(\square_r^\alpha)_k/2}$$
$$= \int_0^1 \frac{\partial}{\partial u} \mathrm{Tr}\left(e^{-t(L_u^\alpha)_k/2}\right) du. \tag{3.4.43}$$

Then, using the consequence of Duhamel's formula from [8, Corollary 2.50], one gets the identity

$$\frac{\partial}{\partial u} \mathrm{Tr}\left(e^{-t(L_u^\alpha)_k/2}\right) = -\frac{t}{2} \mathrm{Tr}\left(\frac{\partial L_u^\alpha}{\partial u} e^{-t(L_u^\alpha)_k/2}\right) = -\frac{t}{2} \mathrm{Tr}\left(\Lambda e^{-t(L_u^\alpha)_k/2}\right).$$

Thus, everything comes down to the estimation of $\mathrm{Tr}\left(\Lambda e^{-t(L_u^\alpha)_k/2}\right)$. Now, the kernel of the operator $\Lambda e^{-t(L_u^\alpha)_k/2}$ is $\Lambda_1 p_k^{L_u^\alpha}(t, z_1, z_2)$ (with the meaning that Λ_1 is the action on the first variable evaluated at z_1), so

$$\operatorname{Tr}\left(\Lambda e^{-t(L_u^\alpha)_k/2}\right) = \int_M \operatorname{Tr} \Lambda_1 p_k^{L_u^\alpha}(t,\cdot,z)dz.$$

We also use the fact that

$$\operatorname{Tr}\left(\Lambda e^{-t(L_u^\alpha)_k/2}\right) = \operatorname{Tr}\left(e^{-t(L_u^\alpha)_k/2}\Lambda\right)$$

and the fact that the kernel of the latter one is $\Lambda_2^* p_k^{L_u^\alpha}(t,z_1,z_2)$ (with the meaning that Λ_2 is the action on the second variable evaluated at z_2) together with self-adjointness to justify that

$$\operatorname{Tr}\left(e^{-t(L_u^\alpha)_k/2}\Lambda\right) = \int_M \operatorname{Tr} \Lambda_2 p_k^{L_u^\alpha}(t,z,z)dz.$$

Summing things up we arrive at

$$\operatorname{Tr}\left(\Lambda e^{-t(L_u^\alpha)_k/2}\right) = \frac{1}{2}\int_M \operatorname{Tr}\ (\Lambda_1+\Lambda_2)\, p_k^{L_u^\alpha}(t,z,z)dz.$$

Choose a ρ smaller than the injectivity radius of M. Then choose a finite covering of M with finite balls B_β, of radius ρ, $\beta = 1,\cdots,N$. For each β choose a smooth orthonormal basis $\left(F_j^\beta\right)_{j=1,d}$ in B_β. Extend the orthonormal basis $\left(F_j^\beta\right)_{j=1,d}$ to orthonormal basis $\left(F_J^\beta\right)_{J=1,\binom{d}{k}}$ of $\bigwedge^k(M)$ over the set B_β. Take $(\psi_\beta)_{\beta=1,N}$ a partition of unity subordinated to $(B_\beta))_{\beta=1,N}$. Then, write

$$\begin{aligned}
\int_M \operatorname{Tr}\ (\Lambda_1+\Lambda_2)\, p_k^{L_u^\alpha}(t,z,z)dz &= \sum_{\beta=1}^N \int_{B_\beta} \psi_\beta(z)\operatorname{Tr}\ (\Lambda_1+\Lambda_2)\, p_k^{L_u^\alpha}(t,z,z)dz \\
&= \sum_{\beta=1}^N \sum_{J=1}^{\binom{d}{k}} \int_{B_\beta} \langle \psi_\beta(z) F_J^\beta(z), \left((\Lambda_1+\Lambda_2)\, p_k^{L_u^\alpha}(t,z,z)\right) F_J^\beta(z)\rangle dz \\
&= \sum_{\beta=1}^N \sum_{J=1}^{\binom{d}{k}} \sum_{j=1}^{d} \int_{B_\beta} \langle \psi_\beta(z) F_J^\beta(z), \Xi_1(F_j)\nabla_{F_j}^r \left(p_k^{L_u^\alpha}(t,z,z) F_J^\beta(z)\right)\rangle dz \\
&\quad - \sum_{\beta=1}^N \sum_{J=1}^{\binom{d}{k}} \sum_{j=1}^{d} \int_{B_\beta} \langle \psi_\beta(z) F_J^\beta(z), \Xi_1(F_j) p_k^{L_u^\alpha}(t,z,z) \nabla_{F_j}^r F_J^\beta(z)\rangle dz \\
&\quad + \sum_{\beta=1}^N \sum_{J=1}^{\binom{d}{k}} \int_{B_\beta} \langle \psi_\beta(z) F_J^\beta(z), \left(\Xi_0 p_k^{L_u^\alpha}(t,z,z) + p_k^{L_u^\alpha}(t,z,z)\Xi_0\right) F_J^\beta(z)\rangle dz.
\end{aligned} \tag{3.4.44}$$

Replacing $\Xi_1(F_j)\nabla_{F_j}^r$ by $\Lambda - \Xi_0$, and using the fact that Λ is self-adjoint we continue with

$$\int_{B_\beta} \langle \psi_\beta(z) F_J^\beta(z), \Xi_1(F_j) \nabla^r_{F_j} \left(p_k^{L_u^\alpha}(t,z,z) F_J^\beta(z) \right) \rangle dz$$
$$= \int_{B_\beta} \langle \Lambda \psi_\beta(z) F_J^\beta(z), p_k^{L_u^\alpha}(t,z,z) F_J^\beta(z) \rangle dz$$
$$- \int_{B_\beta} \langle \psi_\beta(z) F_J^\beta(z), \Xi_0 \left(p_k^{L_u^\alpha}(t,z,z) F_J^\beta(z) \right) \rangle dz.$$

Finally replacing back Λ by $\Xi_1(F_j)\nabla^r_{F_j} + \Xi_0$, the expression above is

$$\int_{B_\beta} \langle \Xi_1(F_j) \nabla^r_{F_j} \left(\psi_\beta(z) F_J^\beta(z) \right), p_k^{L_u^\alpha}(t,z,z) F_J^\beta(z) \rangle dz$$
$$+ \int_{B_\beta} \langle \Xi_0 \psi_\beta(z) F_J^\beta(z), p_k^{L_u^\alpha}(t,z,z) F_J^\beta(z) \rangle dz$$
$$- \int_{B_\beta} \langle \psi_\beta(z) F_J^\beta(z), \Xi_0 \left(p_k^{L_u^\alpha}(t,z,z) F_J^\beta(z) \right) \rangle dz.$$

Returning with these at (3.4.44) we freed the heat kernel of derivatives. Moreover among the quantities involved, for each integral we have at least one Ξ_0 or Ξ_1 as one term. Now we can estimate

$$\left| \int_M \mathrm{Tr} \, (\Lambda_1 + \Lambda_2) \, p_k^{L_u^\alpha}(t,z,z) dz \right| \leq C \int_M (\|\Xi_0(z)\| + \|\Xi_1(z)\|) \left\| p_k^{L_u^\alpha}(t,z,z) \right\| dz$$

where the constant C depends on the covering chosen and on the partition chosen. Using the definition of Ξ_0 and Ξ_1 together with the estimates in (2) and (3.4.34) we arrive at

$$\left| \int_M \mathrm{Tr} \, (\Lambda_1 + \Lambda_2) \, p_k^{L_u^\alpha}(t,z,z) dz \right| \leq Cr \int_M \left\| p_k^{L_u^\alpha}(t,z,z) \right\| dz$$

This last relation, Theorem (3.4.26) and (3.4.43) end the proof the the comparison. □

We can summarize the results of the two comparison steps so far in

Corollary 3.4.45. *There exist, a constant $C(t) > 0$ depending on t and the data of the manifold M but independent of r, and $\alpha(t,r) > 0$ depending on t, r such that for $\alpha \geq \alpha(t,r)$ we have*

$$\left| \int_M \mathrm{Tr} \, p_k^\alpha(t,z,z) dz - \int_M \mathrm{Tr} \, p_k^{\square_r^\alpha}(t,z,z) dz \right| \leq C(t) r. \qquad (3.4.46)$$

3.5 The Proof of the Degenerate Morse Inequalities

In this section we prove Morse inequalities in the degenerate case. So far we were able to prove the comparison which allows to perform the last computation. The advantage of the comparison is that the concrete computations for the operator $\Box_r^\alpha$ are more amenable to the geometry at hand. Basically the main property this operator has is that near critical submanifolds the curvature term is only the curvature of the Bismut connection which behaves nicely under parallel transportation with respect to the Bismut connection.

The main idea is to estimate the heat kernel on the complementary of the set $B(Cr(h), r/2)$ and on the set $B(Cr(h), r/2)$.

The estimation of the heat kernel on the complementary of the set $B(Cr(h), r/2)$ is clear as a result of Theorem (3.4.9).

For the closed case we run exactly the same argument as the one in the proof of Theorem (3.4.14) to justify that for $z \in B(M_i, r/2)$ there are constants $C_1(t,r) > 0$, $C_2(t,r) > 0$ and $\alpha(t,r) > 0$ depending on r and t so that for $\alpha \geq \alpha(t,r)$ and $z \in B(M_i, r/2)$

$$\left\| p_k^{\Box_r^\alpha}(t,z,z) - \overline{p}_k^{\Box_i^\alpha}(t,z,z) \right\| \leq C_1(t,r) e^{-\alpha C_2(t,r)} \tag{3.5.1}$$

where here $\overline{p}_k^{\Box_i^\alpha}(t,z,z)$ stands for the heat kernel of the operator

$$\Box_i^\alpha = -\Delta^{\nabla^B} + \alpha^2 |y|^2 - \alpha(\nu_i^+ - \nu_i^-) + 2\alpha \mathrm{hess} h + D^* R^{\nabla^B} \tag{3.5.2}$$

on $\bigwedge^k(E_i)$ with

$$h(z) = \frac{1}{2}\left(|y^+|^2 - |y^-|^2 \right).$$

The heat kernel for this operator has the form

$$p_k^{\Box_i^\alpha}(t, z_1, z_2) = \mathbb{E}^{\mu_{z_1}}\left[e^{-\frac{\alpha^2}{2}\int_0^t |\varphi(s)|^2 ds + \frac{\alpha t(\nu_i^+ - \nu_i^-)}{2}} V_k^\alpha(t,\varphi) \tau^{\nabla^B}_{\varphi\restriction[t,0]} \delta_{z_2}(\varphi(t)) \right] \tag{3.5.3}$$

where where μ_z is the Wiener measure on $\mathbb{E}_i$ starting at z and $V^\alpha(s,\varphi)$ is the solution to the ODE on $\bigwedge_{\varphi(0)}(E_i)$

$$\begin{cases} \dot{V}^\alpha(s,\varphi) = V^\alpha(s,\varphi)\left(\tau^{\nabla^B}_{\varphi\restriction[s,0]} \left(-\alpha \mathrm{hess}_{\varphi(s)} h + \frac{1}{2} D^* R^{\nabla^B}_{\varphi(s)} \right) \tau^{\nabla^B}_{\varphi\restriction[0,s]} \right) \\ V_k^\alpha(0,\varphi) = \mathrm{Id}_{\bigwedge_{\varphi(0)}(M)} \end{cases}$$

and $V_k^\alpha(s,\varphi)$ is the restriction of $V^\alpha(s,\varphi)$ to $\bigwedge^k_{\varphi(0)}(E_i)$. In order to better understand this on forms, we first write the following decomposition

$$\bigwedge\nolimits_z^k(E_i) = \bigoplus_{\substack{q+q^++q^-=k\\ 0\le q\le \dim(M_i),\\ 0\le q^+\le\nu_i^+,0\le q^-\le\nu_i^-}} \bigwedge\nolimits_x^q(M_i)\wedge\bigwedge\nolimits_x^{q^+}(E_i^+)\wedge\bigwedge\nolimits_x^{q^-}(E_i^-). \tag{3.5.4}$$

For shortness, denote $\bigwedge_z^{q,q^+,q^-}(E_i) = \bigwedge_x^q(M_i)\wedge\bigwedge_x^{q^+}(E_i^+)\wedge\bigwedge_x^{q^-}(E_i^-)$.

Conform (3.1.27), the Hessian hess$h : T_z(E_i) \to T_z(E_i)$ is block diagonal. More precisely we write (see for details Section (3.1.2)) $T_z(E_i) = (E_i^+)_x \oplus (E_i^-)_x \oplus T_x(M_i)$ and in this decomposition the Hessian has the following matrix block decomposition

$$\begin{bmatrix} I & 0 & 0\\ 0 & -I & 0\\ 0 & 0 & 0\end{bmatrix} \tag{3.5.5}$$

and from here it follows that the Hessian commutes with the parallel transportation with respect to the Bismut connection. The Hessian has a particular form for $\omega \in \bigwedge_z^{q,q^+,q^-}(E_i)$, namely

$$\text{hess}h\,\omega = (q^+ - q^-)\omega \tag{3.5.6}$$

Particularly, the Hessian of h preserves all $\bigwedge_z^{q,q^+,q^-}(E_i)$. The crucial property of the Bismut connection is that it preserves the horizontal and the vertical spaces, and in particular the parallel transportation does not mixes up these spaces. On the other hand using the definition given in (2.1.5) and (3.1.24) we have the following formula

$$\begin{aligned}
&\tau^{\nabla^B}_{\varphi\restriction[s,0]} D^* R^{\nabla^B}_{\varphi(s)}\left(\tau^{\nabla^B}_{\varphi\restriction[0,s]}\right) = \\
&\sum_{j,k,l,m}^{\dim(M_i)} \langle R^{\nabla^{LC}}_{\varphi(s)}(F_j(s,\varphi), F_k(s,\varphi))F_l(s,\varphi), F_m(s,\varphi)\rangle \left(F_j^*\wedge i_{F_k}\right)\circ\left(F_l^*\wedge i_{F_m}\right)\\
&+\sum_{j,k}^{\dim(M_i)}\sum_{l,m}^{\nu_i^+} R^{\nabla^B}_{\varphi(s)}(F_j(s,\varphi), F_k(s,\varphi))F_l^+(s,\varphi), F_m^+(s,\varphi)\rangle \left(F_j^*\wedge i_{F_k}\right)\circ\left((F_l^+)^*\wedge i_{F_m^+}\right)\\
&+\sum_{j,k}^{\dim(M_i)}\sum_{l,m}^{\nu_i^-} R^{\nabla^B}_{\varphi(s)}(F_j(s,\varphi), F_k(s,\varphi))F_l^-(s,\varphi), F_m^-(s,\varphi)\rangle \left(F_j^*\wedge i_{F_k}\right)\circ\left((F_l^-)^*\wedge i_{F_m^-}\right)
\end{aligned} \tag{3.5.7}$$

where $(F_j)_{j=1,\dim(M_i)}$ is any orthonormal basis of $T_{\rho_i(\varphi(0))}(M_i)$, and $(F_j^\pm)_{j=1,\nu_i^\pm}$ are orthonormal basis of $(E_i^\pm)_{\rho_i(\varphi(0))}$, with the notation $F_j(s,\varphi) = \tau^{\nabla^B}_{\varphi\restriction[0,s]}F_j$ and the alike, the parallel transportation with respect to the Bismut connection along the path φ.

Observe from this, that the curvature term in (3.5) also preserves the spaces $\bigwedge_z^{q,q^+,q^-}(E_i)$. As a consequence, $V_k^\alpha(s,\varphi)$ also preserves the spaces $\bigwedge_{\varphi(0)}^{q,q^+,q^-}(E_i)$. Thus, $V^\alpha(s,\varphi)$ acting on $\bigwedge_{\varphi(0)}^k(E_i)$ is a diagonal map when is taken with respect to the decomposition (3.5.4). We denote $V^{\alpha}_{q,q^+,q^-}(s,\varphi)$ the restriction of $V^{\alpha}(s,\varphi)$ to the subspace $\bigwedge^{q,q^+,q^-}_{\varphi(0)}(E_i)$.

It follows in particular that $p_k^{\square_i^\alpha}(t,z_{1,2})$ sends $\bigwedge^{q,q^+,q^-}_{z_2}(E_i)$ to $\bigwedge^{q,q^+,q^-}_{z_1}(E_i)$. Call this restriction $p^{\square_i^\alpha}_{q,q^+,q^-}(t,z_1,z_2)$ and point out that we have a similar expression, with the replacement of V_k^α by V^α_{q,q^+,q^-},

$$p^{\square_i^\alpha}_{q,q^+,q^-}(t,z_1,z_2) = \mathbb{E}^{\mu_{z_1}}\left[e^{-\frac{\alpha^2}{2}\int_0^t|\varphi(s)|^2ds+\frac{\alpha t(\nu_i^+-\nu_i^-)}{2}}V^\alpha_{q,q^+,q^-}(t,\varphi)\tau^{\nabla^B}_{\varphi\restriction[t,0]}\delta_{z_2}(\varphi(t))\right]. \tag{3.5.8}$$

Hence $\operatorname{Tr} p_k^{\square_i^\alpha}(t,z,z)$, the trace of the heat kernel is

$$\operatorname{Tr} p_k^{\square_i^\alpha}(t,z,z) = \sum_{\substack{q+q^++q^-=k\\ 0\le q\le \dim(M_i),\\ 0\le q^+\le\nu_i^+,0\le q^-\le\nu_i^-}} \operatorname{Tr} p^{\square_i^\alpha}_{q,q^+,q^-}(t,z,z). \tag{3.5.9}$$

Next, we notice that on $\bigwedge^{q,q^+,q^-}_{\varphi(0)}(E_i)$, the Hessian and the curvature terms appearing in (3.5) commute. Then, because of (3.5.6) we get

$$V^\alpha_{q,q^+,q^-}(t,\varphi) = e^{-\alpha t(q^+-q^-)}W_{q,q^+,q^-}(t,\varphi)$$

with $W_{q,q^+,q^-}(t,\varphi)$ the solution to the ODE on $\bigwedge^{q,q^+,q^-}_{\varphi(0)}(E_i)$

$$\begin{cases} \dot W_{q,q^+,q^-}(s,\varphi) = W_{q,q^+,q^-}(s,\varphi)\left(\tau^{\nabla^B}_{\varphi\restriction[s,0]}D^*R^{\nabla^B}_{\varphi(s)}\tau^{\nabla^B}_{\varphi\restriction[0,s]}\right)\\ W_{q,q^+,q^-}(0,\varphi) = \operatorname{Id}_{\bigwedge^{q,q^+,q^-}_{\varphi(0)}(M)}. \end{cases} \tag{3.5.10}$$

Therefore $p^{\square_i^\alpha}_{q,q^+,q^-}(t,z_1,z_2)$ equals the following expression

$$e^{\frac{\alpha t(\nu_i^+-\nu_i^--2q^++2q^-)}{2}}\mathbb{E}^{\mu_{z_1}}\left[e^{-\frac{\alpha^2}{2}\int_0^t|\varphi(s)|^2ds}W_{q,q^+,q^-}(t,\varphi)\tau^{\nabla^B}_{\varphi\restriction[t,0]}\delta_{z_2}(\varphi(t))\right]. \tag{3.5.11}$$

Now we want to estimate each term in (3.5.9). This is done in the following theorem which is the backbone of this section.

Theorem 3.5.12. *If* $0<q^+$ *or* $q^-<\nu_i^-$, *then there exists* $C(t,r)>0$, *depending on* t *and* r, *such that for any* $z_1,z_2\in E_i$, $|z_2|\le r$,

$$\left\|p^{\square_i^\alpha}_{q,q^+,q^-}(t,z_1,z_2)\right\|_{z_2,z_1} \le C\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\alpha t(q^++\nu_i^--q^-)}e^{-\alpha\tanh(t\alpha/2)|y_1|^2}.$$

If $q^+=0$ *and* $q^-=\nu_i^-$, *then for* $z\in B(M_i,r)$ *and some constant* $C>0$ *depending* *on t, r,*

$$\operatorname{Tr} p^{\square_i^\alpha}_{q,0,\nu_i^-}(t,z,z) = \mathcal{O}(e^{-C\alpha})\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\alpha\tanh(t\alpha/2)|y|^2}$$
$$+\,\mathbb{E}^{\mu_x^{M_i}}\left[\operatorname{Tr}\left(W_q^-(t,\psi)\tau^-_{\psi\restriction[t,0]}\right)\delta_x(\psi(t))\right]\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}} e^{-\alpha\tanh(t\alpha/2)|y|^2}$$

where the integration is taken with respect to the Wiener measure on M_i, τ^- here stands for the parallel transportation on $\bigwedge^q_{\psi(0)}(M_i)\otimes\bigwedge^{\nu_i^-}\left((E_i^-)_{\psi(0)}\right)$ with respect to the connection $\nabla^{i,-} = \nabla^{LC}\otimes\nabla^V$ and $W_q^-(t,\psi)$ is the solution to the ODE on $\bigwedge^q_{\psi(0)}(M_i)\otimes\bigwedge^{\nu_i^-}\left((E_i^-)_{\psi(0)}\right)$ of

$$\begin{cases} \dot{W}_q^-(s,\psi) = W_q^-(s,\psi)\left(\tau^-_{\psi\restriction[s,0]}D^*R^-_{\psi(s)}\tau^-_{\psi\restriction[0,s]}\right) \\ W_q^-(0,\psi) = \mathrm{Id}_{\bigwedge^q_{\psi(0)}(M_i)\otimes\bigwedge^{\nu_i^-}\left((E_i^-)_{\psi(0)}\right)}. \end{cases} \tag{3.5.13}$$

Proof. For the first case we can use the (B.1.6) to derive a heat kernel comparison based on bounds of the ODE involved in. Therefore in this case one can easily get the estimate by saying that $W_{q,q^+,q^-}(t,\varphi)\tau^{\nabla^B}_{\varphi\restriction[t,0]}$ is bounded by a constant depending on the manifold M_i and t, r. Thus

$$\left\|p^{\square_i^\alpha}_{q,q^+,q^-}(t,z_1,z_2)\right\|_{z_2,z_1} \le Ce^{\frac{\alpha t(\nu_i^+-\nu_i^--2q^++2q^-)}{2}}\mathbb{E}^{\mu_{z_1}}\left[e^{-\frac{\alpha^2}{2}\int_0^t|\varphi(s)|^2ds}\delta_{z_2}(\varphi(t))\right].$$

The integral can be recognized in terms of $\overline{\mathcal{P}}$ of (3.4.15). Using the estimates in Proposition (3.4.19) we get the required inequality at once.

For the second part, one has to notice that almost everything comes from the observation that the action of $\tau^{\nabla^B}_{\varphi\restriction[s,0]}D^*R^{\nabla^B}_{\varphi(s)}\tau^{\nabla^B}_{\varphi\restriction[0,s]}$ as given in (3.5.7) on $\bigwedge_z^{q,0,\nu_i^-}(E_i)$ is reduced to

$$\sum_{j,k,l,m}^{\dim(M_i)}\langle R^{\nabla^{LC}}_{\varphi(s)}(F_j(s,\varphi),F_k(s,\varphi))F_l(s,\varphi),F_m(s,\varphi)\rangle\left(F_j^*\wedge i_{F_k}\right)\circ\left(F_l^*\wedge i_{F_m}\right).$$

Indeed, what one has to check is the fact the the other two lines in (3.5.7) are 0 on $\bigwedge_z^{q,0,\nu_i^-}(E_i)$, which for the second line is clear since any form in the space at discussion here is a linear combinations of elementary forms $F^*_{j_1}\wedge\cdots\wedge F^*_{j_q}\wedge(F_1^-)^*\wedge\cdots\wedge(F^-_{\nu_i^-})^*$, while for the third line in (3.5.7) one has to observe that because

$$(Z^-,T^-)\to\langle R^{\nabla^B}_{\varphi(s)}(F_j(s,\varphi),F_k(s,\varphi))Z^-,T^-\rangle$$

is an anti-symmetric map, the third line contains only terms of the form $\left(F_j^*\wedge\ i_{F_k}\right)\circ\left((F_l^-)^*\wedge\ i_{F_m^-}\right)$ with $l\neq m$, thus its action on the elementary form is 0.

From this discussion, $W_{q,0,\nu_i^-}(t,\varphi)$ acts only on the horizontal part of $\bigwedge^{q,0,\nu_i^-}(E_i)$.

Now looking at $W_{q,0,\nu_i^-}(t,\varphi)\tau^{\nabla^B}_{\varphi\restriction[t,0]}$, and taking into account that $\bigwedge^{\nu_i^-}(E_i^-)$ is one dimensional, a little thinking leads to the conclusion that the trace of $W_{q,0,\nu_i^-}(t,\varphi)\tau^{\nabla^B}_{\varphi\restriction[t,0]}$ is the same as the trace of $W_q^-(t,\psi)\tau^-_{\psi\restriction[t,0]}$.

So far we obtained the equality

$$\operatorname{Tr} p^{\square_i^\alpha}_{q,0,\nu_i^-}(t,z,z) = \mathbb{E}^{\mu_z}\left[e^{-\frac{\alpha^2}{2}\int_0^t|\varphi(s)|^2ds+\frac{\alpha t\nu_i}{2}}\operatorname{Tr}\left(W_q^-(t,\psi)\tau^-_{\psi\restriction[t,0]}\right)\delta_z(\phi(t))\right]$$

with ψ the projection of the path ϕ on the submanifold M_i.

In order to obtain the estimates on the heat kernel we try to make a comparison similar to the one used in the proof of (3.4.19). Denote first

$$\mathfrak{q}(t,z,u) = \mathbb{E}^{\mu_z}\left[e^{-\frac{\alpha^2}{2}\int_0^t|\varphi(s)|^2ds+\frac{\alpha t\nu_i}{2}}W_q^-(t,\psi)\tau^-_{\psi\restriction[t,0]}\delta_u(\phi(t))\right]$$

Basically we start with a function f with compact support and then write the equality

$$\begin{aligned}\int_{E_i}\mathfrak{q}(t,z,u)f(u)du &= \mathbb{E}^{\mu_z}\left[e^{-\frac{\alpha^2}{2}\int_0^t|\varphi(s)|^2ds+\frac{\alpha t\nu_i}{2}}W_q^-(t,\psi)\tau^-_{\psi\restriction[t,0]}f(\phi(t))\right]\\ &= \mathbb{E}^{\mu_x^{M-i}}\left[W_q^-(t,\psi)\tau^-_{\psi\restriction[t,0]}\int_{(E_i)_x}\mathcal{Q}_i^\alpha(t,y,y_1)f\left(\psi(t),\tau^V_{\psi\restriction[0,t]}(y_1)\right)dy_1\right].\end{aligned} \quad (3.5.14)$$

From here, the idea is to use the same kind of reasoning as in the proof of Proposition (3.4.19). To wit a little bit, we point out that for $|y| = |y'| \le r$ we have the following inequality

$$|\mathcal{Q}_i^\alpha(t,y,y') - \mathcal{Q}_i^\alpha(t,y,y)| \le Ce^{-C\alpha}\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}}e^{-\alpha\tanh(t\alpha/2)|y|^2}.$$

From here, a similar choice of the approximate identity as the one in the proof of Proposition (3.4.19) gives the last ingredients for the required estimates. □

From this theorem, with integration on $B(M_i, r/2)$ and elementary computations, one has that

$$\begin{aligned}&\left|\int_{B(M_i,r/2)}\operatorname{Tr} p^{\square_i^\alpha}_{q,q^+,q^-}(t,z,z)dz\right|\\ &\quad\le C\left(\frac{\alpha}{\pi\left(1-e^{-2t\alpha}\right)}\right)^{\frac{\nu_i}{2}}e^{-\alpha t(q^++\nu_i^--q^-)}\int_{|y|\le r/2}e^{-\alpha\tanh(t\alpha/2)|y|^2}dy\\ &\quad\le Ce^{-\alpha t(q^++\nu_i^--q^-)}.\end{aligned} \quad (3.5.15)$$

Therefore it tends to zero in the case $0 < q^+$ or $q^- < \nu_i^-$. On the other hand, in the remaining case, an integration on $B(M_i, r/2)$ followed by the change of variable

$y \to \sqrt{\frac{1}{\alpha \tanh(t\alpha/2)}}y$ and some simple calculations gives

$$\int_{B(M_i,r/2)} \operatorname{Tr} p^{\square_i^{\alpha}}_{q,0,\nu_i^-}(t,z,z)dz$$
$$= \mathcal{O}(e^{-C\alpha}) + \int_{M_i} \mathbb{E}^{\mu_x^{M_i}} \left[\operatorname{Tr} \left(W_q^-(t,\psi)\tau^-_{\psi\restriction[t,0]} \right) \delta_x(\psi(t)) \right] dx \quad (3.5.16)$$

for a constant depending on t and r.

Our next goal is to interpret $\mathbb{E}^{\mu_x^{M_i}} \left[\operatorname{Tr} \left(W_q^-(t,\psi)\tau^-_{\psi\restriction[t,0]} \right) \delta_x(\psi(t)) \right]$. The idea of doing this is to take the bundles $\mathcal{F}_x^q = \bigwedge_x^q(M_i) \otimes \bigwedge^{\nu_i^-}((E_i^-)_x)$. Topologically, these bundle are the same as $\bigwedge_x^q(T(M_i) \otimes o(E_i^-))$, with $o(E_i^-)$ the orientation bundle of E_i^-. On these topological bundles one can define the differential on forms as in [5, Chapter I, section 7]. Thus their cohomology is well defined and is by definition $H(M_i; o(E_i^-))$.

From the analytical side, one can realize this by putting a connection on them and representing the differential in terms of this connection. The natural connection on this is the tensor connection, called $\nabla^{i,-}$ of the Levi-Civita and the vertical connection on $\bigwedge^{\nu_i^-}((E_i^-)_x)$. Remark here that the latter connection is flat, and the differential operator can be written as

$$d_{M_i}^-(\omega \otimes s) = \left((F_j)^* \wedge \nabla_{F_j}\omega\right) \otimes s + (F_j)^* \wedge \omega \otimes \left(\nabla_{F_j}^V s\right).$$

for an orthonormal basis (F_j) in M_i. Thus, we get the complex

$$0 \to \mathcal{F}^0 \xrightarrow[d_{M_i}^-]{} \mathcal{F}^1 \xrightarrow[d_{M_i}^-]{} \cdots \xrightarrow[d_{M_i}^-]{} \mathcal{F}^{\dim(M_i)} \xrightarrow[d_{M_i}^-]{} 0$$

with the cohomology $H(M_i; o(E_i^-))$.

Now the bundles $\mathcal{F}^q$ inherit the natural metric form the bundles $\bigwedge_x^q(M)$ and $\bigwedge^{\nu_i^-}((E_i^-)_x)$. Using this metric we can define the adjoint of $d_{M_i}^-$, $\delta_{M_i}^-$. From here, the idea is to express the heat kernel of the operator

$$\square^{i,-} = d_{M_i}^-\delta_{M_i}^- + \delta_{M_i}^- d_{M_i}^-, \quad (3.5.17)$$

with the notation $\square_q^{i,-}$ for this operator on q-forms. First one has the similar Wietzenböck formula in this framework as

$$\square^{i,-} = -\Delta^{i,-} + D^*(R^-)$$

where $\Delta^{i,-}$, R^- is the Laplacian and the curvature of the connection $\nabla^{i,-}$. Then, the heat kernel of this operator can be written as

$$p_q^{i,-}(t,x,y) = \mathbb{E}^{\mu_x^{M_i}} \left[W_q^-(t,\psi)\delta_y(\psi(t)) \right].$$

Taking its trace on the diagonal and integrating over M_i one gets

$$\int_{M_i} \operatorname{Tr} p_q^{i,-}(t,x,x)dx = \operatorname{Tr} e^{-t\Box_q^{i,-}}$$

and with this and (3.5.16) one arrives to

$$\int_{B(M_i,r/2)} \operatorname{Tr} p_{q,0,\nu_i^-}^{\Box_i^\alpha}(t,z,z)dz = \mathcal{O}(e^{-C\alpha}) + \operatorname{Tr} e^{-t\Box_q^{i,-}}. \tag{3.5.18}$$

Finally, putting this together with (3.5.9), Theorems (3.5.12) and (3.4.9), (3.5.1) and Corollary (3.4.45) one proves that there are constants $C_1(t)$ depending only on t and $C_2(t,r) > 0$, $C_3(t,r) > 0$, $\alpha(t,r) > 0$ depending on t, r so that

$$\left| \int_M \operatorname{Tr} p_k^\alpha(t,z,z)dz - \sum_{i=1}^{l} \operatorname{Tr} e^{-t\Box_{k-\nu_i^-}^{i,-}} \right| \leq C(t)r + C_2(t,r)e^{-\alpha C_2(t,r)} \tag{3.5.19}$$

for all $\alpha \geq \alpha(t,r)$.

Letting α tend to infinity we have

$$\limsup_{\alpha\to\infty} \left| \int_M \operatorname{Tr} p_k^\alpha(t,z,z)dz - \sum_{i=1}^{l} \operatorname{Tr} e^{-t\Box_{k-\nu_i^-}^{i,-}} \right| \leq C(t)r.$$

Since this is true for any $r > 0$ and the left hand side does not depend on r, we get the following theorem.

Theorem 3.5.20. *For any $t > 0$,*

$$\lim_{\alpha\to\infty} \int_M \operatorname{Tr} p_k^\alpha(t,z,z)dz = \sum_{i=1}^{l} \operatorname{Tr} e^{-t\Box_{k-\nu_i^-}^{i,-}}.$$

Therefore, by (2.1.3), *for any $t > 0$*

$$m_k(t) - m_{k-1}(t) + \cdots + (-1)^k m_0(t) \geq B_k - B_{k-1} + \cdots (-1)^k B_0$$

with

$$m_k(t) = \sum_{i=1}^{l} \operatorname{Tr} e^{-t\Box_{k-\nu_i^-}^{i,-}}.$$

Now, if we let t tend to ∞ and use Hodge theory for this situation we finally arrive at

Theorem 3.5.21 (Degenerate Morse Inequalities).

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq B_k - B_{k-1} + \cdots (-1)^k B_0 \tag{3.5.22}$$

where

$$m_k = \sum_{i=1}^{l} \dim H^{k-\nu_i^-}(M_i; o(E_i^-)),$$

with $H^k(M_i; o(E^r-_i))$ *standing for the dimension of the cohomology group of* M_i *twisted by the orientation bundle of* E_i^-. *This inequality becomes equality for* $k = d$.

Appendix A

The Malliavin Calculus

A.1 Integration by Parts I

This section is heavily based on [7, Chapter 10]. In order to avoid repetition we will refer to the proofs in there for those that are essentially the same.

In this section we assume that M is a complete Riemannian manifold with the Ricci curvature satisfying:

$$-C(1+\text{dist}(x,o)^2)|X|_x^2 \leq Ric_x(X_x,X_x) \tag{A.1.1}$$

for some positive constants $C \geq 0$ and for any $x \in M$, $X_x \in T_x(M)$ where o is a fixed reference point.

Consider a smooth map $\mathfrak{G} \in C^\infty(\mathcal{O}(M), \text{Hom}(\mathbb{R}^d, \text{End}(\mathbb{R}^d)))$. Our perturbation is going to depend on this map and here we shortly describe how. Our choice of Ξ that appears in [7, Chapter 10] is simpler. We take $h \in H^1(\mathbb{R}^d)$, this is, we take $h : [0,t] \to \mathbb{R}^d$ such that there is a function $\dot{\mathbf{h}} \in L^2([0,\infty];\mathbb{R}^d)$ with $\mathbf{h}(s) = \int_0^s \dot{\mathbf{h}}(\sigma)d\sigma$ and set $\Xi_s(t,\mathfrak{f},\mathbf{w}) = \mathbf{h}(t)$. We add to equations in [7, 10.7] the equation for a map $\mathfrak{O}_s(t,\mathfrak{f},\mathbf{w}) \in \text{End}(\mathbb{R}^d)$ given by

$$\dot{\mathfrak{O}}_s(t,\mathfrak{f},\mathbf{w}) = \mathfrak{O}_s(t,\mathfrak{f},\mathbf{w})\mathfrak{G}_{\mathfrak{P}_s(t,\mathfrak{f},\mathbf{w})}(\dot{\mathbf{W}}(t,\mathfrak{f},\mathbf{w})). \tag{A.1.2}$$

The space $\mathcal{H}^{(m)}$ in paragraph 10.2.3 of the invoiced reference is here replaced by

$$\mathcal{H}^{(m)}(M) = \mathcal{O}(M) \times W^{(m)}(o(\mathbb{R}^d)) \times W^{(m)}(\mathbb{R}^d) \times W^{(m)}(\text{End}(\mathbb{R}^d)).$$

First we choose an orthonormal basis in $\mathbb{R}^d$ and the vector fields

$$\begin{aligned}
\mathfrak{F}(t,\mathbf{h}) &: \mathcal{H}^{(1)}(M) \to W^1(\mathcal{O}(M)) \\
\mathbf{H}(t,\mathbf{h}) &: \mathcal{H}^{(1)}(M) \to W^{(1)}(\mathbb{R}^d) \\
\Omega_k(t,\mathbf{h}) &: \mathcal{H}^{(1)}(M) \to W^{(1)}(o(\mathbb{R}^d)) \\
\Lambda_k(t,\mathbf{h}) &: \mathcal{H}^{(1)}(M) \to W^{(1)}(\text{End}(\mathbb{R}^d))
\end{aligned}$$

given by

$$\begin{aligned}
\mathfrak{F}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o}) &= \mathfrak{F}_s(\mathfrak{f},a,\mathbf{h}(t))\\
\mathbf{H}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o}) &= \int_0^s O_{\sigma,s}(a)\dot{\mathbf{h}}(\sigma)d\sigma\\
\Omega_0(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o}) &= \Omega_{\mathfrak{F}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o})}(\mathbf{H}(t,\mathbf{h})_s(\mathfrak{f},a,v),\mathbf{h}(t))\\
\Omega_k(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o}) &= \Omega_{\mathfrak{F}(t,\mathbf{h})_s(\mathfrak{f},a,v)}(O_s(a)e_k,\mathbf{h}(t)), \qquad 1\le k\le d\\
\Lambda_0(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o}) &= \mathfrak{o}_s\mathfrak{S}_{\mathfrak{F}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o})}(\mathbf{H}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o}))\\
\Lambda_k(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o}) &= \mathfrak{o}_s\mathfrak{S}_{\mathfrak{F}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o})}(O_s(a)e_k), \qquad 1\le k
\end{aligned} \tag{A.1.3}$$

for $s\in[0,1]$ and $(\mathfrak{f},a,v)\in\mathcal{H}^{(1)}(M)$. With these definitions at hand we continue by defining the vector fields, $\{\widehat{\mathfrak{X}_0(t,\mathbf{h})},\cdots,\{\widehat{\mathfrak{X}_d(t,\mathbf{h})}\}$ on $\mathcal{H}^{(m)}(M)$ so that

$$\widehat{\mathfrak{X}_0(t,\mathbf{h})}_{(\mathfrak{f},a,v,\mathfrak{o})} = (\partial_{\Omega_0(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o})})_a + (\partial_{\mathbf{H}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o})})_v + (\partial_{\Lambda_0(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o})})_\mathfrak{o} \tag{A.1.4}$$

and for $1\le k\le d$,

$$\widehat{\mathfrak{X}_k(t,\mathbf{h})}_{(\mathfrak{f},a,v,\mathfrak{o})} = \mathfrak{E}(O(a)e_k)_\mathfrak{f} + (\partial_{\Omega_k(t,\mathbf{h})(\mathfrak{f},a,v,\mathfrak{o})})_a + (\partial_{O(a)e_k})_v + (\partial_{\Lambda_k(t,\mathbf{h})(\mathfrak{f},a,v,\mathfrak{o})})_\mathfrak{o}. \tag{A.1.5}$$

We define now for a given $\xi\in\mathbb{R}^d$ the vector field

$$B(\xi):\mathcal{O}(M)\times\mathbb{R}^d\times\mathrm{End}(\mathbb{R}^d)\to\mathrm{End}(\mathbb{R}^d)$$

by

$$B(\xi)(\mathfrak{f},v,\mathfrak{o}) = \mathfrak{o}\mathfrak{S}_\mathfrak{f}(\xi).$$

For a given $\xi\in\mathbb{R}^d$ define the vector field $\mathfrak{Y}(\xi)$, $1\le k\le d$ on $\mathcal{O}(M)\times\mathbb{R}^d\times\mathrm{End}(\mathbb{R}^d)$ by

$$\mathfrak{Y}(\xi) = \mathfrak{E}(\xi)_\mathfrak{f} + (\partial_\xi)_v + \left(\partial_{B(\xi)(\mathfrak{f},v,\mathfrak{o})}\right)_\mathfrak{o}. \tag{A.1.6}$$

Then, one can prove the following Lemma, which is the analog of Lemma 10.22 in the cited reference.

Lemma A.1.7. *If h is a smooth function, then for each piecewise smooth function $\mathbf{w}\in\mathcal{P}(\mathbb{R}^d)$, and $\mathfrak{f}\in\mathcal{O}(M)$, there is a unique piecewise smooth path*

$$t\in[0,\infty)\to(\mathfrak{p}(t),A(t),\mathbf{w}(t),\mathfrak{O}(t))\in\mathcal{H}^{(1)}(M)$$

which is the integral curve of the time-dependent vector field

$$t\to\widehat{\mathfrak{X}_0(t,\mathbf{h})}+\sum_{k=1}^d\langle\dot{\mathbf{w}}(t),e_k\rangle_{\mathbb{R}^d}\widehat{\mathfrak{X}_k(t,\mathbf{h})}$$

starting at $(\mathfrak{f},0,0,I)$. In fact, for each $m\ge1$, this path is piecewise smooth map into

$\mathcal{H}^{(m)}(M)$. *Next, set*

$$\mathfrak{P}_s(t) = \mathfrak{F}(t,\mathbf{h})_s(\mathfrak{p}(t), A(t), \mathbf{w}(t), \mathfrak{O}(t)). \tag{A.1.8}$$

Then

$$\begin{aligned}
\dot{\mathfrak{P}}_s(t) &= \mathfrak{E}(\dot{\mathbf{W}}_s(t))_{\mathfrak{P}_s(t)} \\
\mathfrak{P}'_s(t) &= \mathfrak{E}(\mathbf{h}(t))_{\mathfrak{P}_s(t)} + \lambda(A_s(t))_{\mathfrak{P}_s(t)} \\
\frac{d}{ds}\dot{\mathbf{W}}_s(t) &= \dot{\mathbf{h}}(t) - A_s(t)\dot{\mathbf{W}}_s(t) \quad \textit{with} \quad \mathbf{W}_0 \ = \mathbf{w} \\
\dot{A}_s(t) &= \Omega_{\mathfrak{P}_s(t)}(\dot{\mathbf{W}}_s(t), \mathbf{h}(t)) \\
\dot{\mathfrak{O}}_s(t) &= \mathfrak{O}_s(t)\mathfrak{S}_{\mathfrak{P}_s(t)}(\dot{\mathbf{W}}_s(t)).
\end{aligned} \tag{A.1.9}$$

In particular,

$$\begin{aligned}
(\mathfrak{F}(t,\mathbf{h})_s)_*\widehat{\mathfrak{X}_0(t,\mathbf{h})}_{(\mathfrak{f},a,v,\mathfrak{o})} &= \mathfrak{E}((\mathbf{H}(t,\mathbf{h}))_s(\mathfrak{f},a,v))_{\mathfrak{F}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o})} \\
(\mathfrak{F}(t,\mathbf{h})_s)_*\widehat{\mathfrak{X}_k(t,\mathbf{h})}_{(\mathfrak{f},a,v,\mathfrak{o})} &= \mathfrak{E}(O_s(a)e_k)_{\mathfrak{F}(t,\mathbf{h})_s(\mathfrak{f},a,v,\mathfrak{o})}, \quad 1 \le k \le d.
\end{aligned} \tag{A.1.10}$$

Proof. This is entirely based on [7, Lemma 10.11] and the reasoning given in the proof of [7, Lemma 10.22]. □

Our next goal is to get to the perturbed Brownian motion. We put this in the following Theorem, which is basically the same as [7, Theorem 10.26].

Theorem A.1.11. *Assume that* $\mathbf{h}$ *is a smooth function and that* $\mathfrak{f} \in \mathcal{O}(M)$ *is given. For each* $n \in \mathbb{N}$, *use* $\mathbf{w}_n$ *to denote the polygonal approximation to* $\mathbf{w}$ *and*

$$t \in [0,\infty) \to (\mathfrak{p}(t,\mathfrak{f},\mathbf{w}_n), A(t,\mathfrak{f},\mathbf{w}_n), \mathbf{W}(t,\mathfrak{f},\mathbf{w}_n), \mathfrak{O}(t,\mathfrak{f},\mathbf{w}_n)) \in \bigcap_{m\ge 1} \mathcal{H}^{(m)}(M)$$

to denote the integral curve of

$$t \to \widehat{\mathfrak{X}_0(t,\mathbf{h})} + \sum_{k=1}^{d} \langle \dot{\mathbf{w}}_n(t), e_k\rangle_{\mathbb{R}^d} \widehat{\mathfrak{X}_k(t,\mathbf{h})} \quad \textit{starting at} \quad (\mathfrak{f},0,0,I).$$

Then, there exists a $\{\overline{\mathcal{B}}_t : t \ge 0\}$ *-progressively measurable map*

$$(t,\mathbf{w}) \in [0,\infty) \times \mathcal{P}(\mathbb{R}^d) \to \\ (\mathfrak{p}(t,\mathfrak{f},\mathbf{w}), A(t,\mathfrak{f},\mathbf{w}), \mathbf{W}(t,\mathfrak{f},\mathbf{w}), \mathfrak{O}(t,\mathfrak{f},\mathbf{w})) \in \bigcap_{m\ge 1} \mathcal{H}^{(m)}(M)$$

such that, for each $m \ge 1$,

$$(\mathfrak{p}(t,\mathfrak{f},\mathbf{w}_n), A(t,\mathfrak{f},\mathbf{w}_n), \mathbf{W}(t,\mathfrak{f},\mathbf{w}_n), \mathfrak{O}(t,\mathfrak{f},\mathbf{w}_n)) \to \\ (\mathfrak{p}(t,\mathfrak{f},\mathbf{w}), A(t,\mathfrak{f},\mathbf{w}), \mathbf{W}(t,\mathfrak{f},\mathbf{w}), \mathfrak{O}(t,\mathfrak{f},\mathbf{w})) \textit{ in } \mathcal{H}^{(m)}(M)$$

for $\mathcal{W}_d$-almost every $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d)$. Moreover, the $\mathcal{W}_d$-distribution $\widehat{\mathbb{P}^{\mathbf{h}}_{\mathfrak{f}}}$ of

$$\mathbf{w} \in \mathcal{P}(\mathbb{R}^d) \to (\mathfrak{p}(t,\mathfrak{f},\mathbf{w}), A(t,\mathfrak{f},\mathbf{w}), \mathbf{W}(t,\mathfrak{f},\mathbf{w}), \mathfrak{O}(t,\mathfrak{f},\mathbf{w})) \in \mathcal{P}(\bigcap_{m\geq 1} \mathcal{H}^{(m)}(M))$$

solves the martingale problem starting at $(\mathfrak{f},0,0,I)$ for

$$\widehat{\mathcal{L}^{\mathbf{h}}_t} = \widehat{\mathfrak{X}_0(t,\mathbf{h})} + \frac{1}{2}\sum_{k=1}^{d} \widehat{\mathfrak{X}_k(t,\mathbf{h})}^2. \tag{A.1.12}$$

Finally after changes on a set of $\mathcal{W}_d$-measure zero we have

$$\mathbf{w} \to (\mathfrak{p}(\cdot,*,\mathbf{w}), A(\cdot,*,\mathbf{w}), \mathbf{W}(\cdot,*,\mathbf{w}), \mathfrak{O}(\cdot,*,\mathbf{w})) \\ \in C^{0,\infty}\left([0,\infty)\times\mathcal{O}(M); \bigcap_{m\geq 1}\mathcal{H}^{(m)}(M)\right).$$

Proof. The proof goes on the same line as [7, Theorem 10.26], the only difference is that we have to use the non-explosion and the estimates given by [7, 8.62] for the case the Ricci tensor is bounded below by the squared distance. □

As an immediate corollary we give here

Corollary A.1.13. *Assuming that $\mathbf{h}$ is a smooth function, $\mathfrak{f} \in \mathcal{O}(M)$ is given, and $\mathfrak{P}_s$ is defined as in (A.1.8), then $\widehat{\mathbb{P}^{\mathbf{h}}_{\mathfrak{f}}}$-almost surely*

$$\mathfrak{P}'_s(t) = \mathfrak{E}(\mathbf{h}(t))_{\mathfrak{P}_s(t)} + \lambda(A_s(t))_{\mathfrak{P}_s(t)}, \quad (s,t)\in[0,1]\times[0,\infty). \tag{A.1.14}$$

Moreover, for each $s \in [0,1]$ and $\varphi \in C^2_c(\mathbb{R}^d \times \mathcal{O}(M) \times \mathrm{End}(\mathbb{R}^d))$,

$$\varphi(\mathbf{W}_s(t),\mathfrak{P}_s(t),\mathfrak{O}_s(t)) - \int_0^t \Big(\mathfrak{Y}(b^{\mathbf{h}}_s(\tau,\mathfrak{P},A)) \\ + \sum_{k=1}^{d}\frac{1}{2}(\mathfrak{Y}(e_k))^2\Big)\varphi(\mathbf{W}_s(\tau),\mathfrak{P}_s(\tau),\mathfrak{O}_s(\tau))d\tau$$

is a $\widehat{\mathbb{P}^{\mathbf{h}}_{\mathfrak{f}}}$-martingale, for

$$b^{\mathbf{h}}_s(\tau,\mathfrak{P},A) = \int_0^s O_{\sigma,s}(A(\tau))\left(\dot{h}(\tau) + \frac{1}{2}\mathfrak{R}_{\mathfrak{P}_\sigma(\tau)}h(\tau)\right)d\sigma. \tag{A.1.15}$$

Proof. Clearly, (A.1.14) follows from (A.1.9). From the martingale property part, we need to show that

$$\left[\widehat{\mathcal{L}^{\mathbf{h}}_{\tau}}\varphi\right](\mathbf{W}_s(\tau), \mathfrak{F}(\tau, \mathbf{h})_s(\mathfrak{p}(\tau), A(\tau), \mathbf{w}(\tau), \mathfrak{O}_s(\tau)))$$
$$= \left(\mathfrak{Y}(b^{\mathbf{h}}_s(\tau, \mathfrak{P}, A)) + \sum_{k=1}^{d} \frac{1}{2}\left(\mathfrak{Y}(e_k)\right)^2\right)\varphi(\mathbf{W}_s(\tau), \mathfrak{P}_s(\tau), \mathfrak{O}_s(\tau)).$$

Using (A.1.10) and (A.1.6), we can write for $1 \leq k \leq d$,

$$\widehat{\mathfrak{X}_k(\tau, \mathbf{h})}_{(\mathfrak{f},a,v,\mathfrak{o})}\varphi(v, \mathfrak{F}(\tau, \mathbf{h})_s(\mathfrak{f}, a, v, \mathfrak{o}), \mathfrak{o})$$
$$= \left(\mathfrak{Y}(O_s(a)e_k)\varphi\right)(\mathfrak{f}, \mathfrak{F}(\tau, \mathbf{h})_s(\mathfrak{f}, a, v, \mathfrak{o}), \mathfrak{o}),$$

and so

$$\widehat{\mathfrak{X}_k(\tau, \mathbf{h})}^2\varphi(\mathbf{W}_s, \mathfrak{F}(\tau, \mathbf{h})_s(\mathfrak{p}(\tau), A(\tau), \mathbf{w}(\tau)), \mathfrak{O}_s)$$
$$= \left(\mathfrak{Y}(c^{\mathbf{h}}_s(\tau, \mathfrak{P}, A)) + \sum_{k=1}^{d} \frac{1}{2}\left(\mathfrak{Y}(O_s(a)e_k)\right)^2\right)\varphi(\mathbf{W}_s(\tau), \mathfrak{P}_s(\tau), \mathfrak{O}_s(\tau)),$$

where (cf. [7, 10.10] and (A.1.5)), $c^{\mathbf{h}}_{s,k}(\tau, \mathfrak{P}, A)$ is taken equal to

$$-\int_0^s O_{\sigma,s}(A(\tau))\Omega_k(\tau, \mathbf{h})_\sigma(\mathfrak{p}(\tau), A(\tau), \mathbf{W}(\tau))O_\sigma(A(\tau))e_k \, d\sigma.$$

Finally, we point out that by the last equation of (A.1.3)

$$-\sum_{k=1}^{d} \Omega_k(\tau, \mathbf{h})_\sigma(\mathfrak{p}(\tau), A(\tau), \mathbf{W}(\tau))O_\sigma(A(\tau))e_k = \mathfrak{R}_{\mathfrak{P}_s(\tau)}h(\tau)$$

while

$$\sum_{k=1}^{d} \left(\mathfrak{Y}(O_s(A(\tau))e_k)\right)^2 = \sum_{k=1}^{d} \left(\mathfrak{Y}(e_k)\right)^2.$$

By (A.1.10),

$$\widehat{\mathfrak{X}_0(\tau, \mathbf{h})}_{(\mathfrak{f},a,v,\mathfrak{o})}\varphi(v, \mathfrak{F}(\tau, \mathbf{h})_s(\mathfrak{f}, a, v, \mathfrak{o}), \mathfrak{o})$$
$$= \left(\mathfrak{Y}(\mathbf{H}(\tau, \mathbf{h})_s(\mathfrak{f}, a, v, \mathfrak{o}))\varphi\right)(\mathfrak{f}, \mathfrak{F}(\tau, \mathbf{h})_s(\mathfrak{f}, a, v, \mathfrak{o}), \mathfrak{o}),$$

and from this and (A.1.3) the result follows. □

The next step is to find the Radon-Nikodym factor that eliminates the "drift" term in the expression of the operator above. In order to get started, for any $R > 0$, choose a compactly supported smooth function $\psi_R : \mathcal{O}(M) \to \mathbb{R}$, such that ψ_R is 1 on $U_{2R} = \pi^{-1}(B(o, 2R))$, where $B(o, 2R)$ is the ball of radius $2R$ in M and π is the projection from $\mathcal{O}(M)$ to M. For $\mathfrak{P}(t, \mathfrak{f}, \mathbf{w})$ and $A(t, \mathfrak{f}, \mathbf{w})$ given by (A.1.8) and

Theorem (A.1.11), set

$$E_s^{\mathbf{h},R}(t,\mathfrak{P},A) = \exp\left(-\int_0^t \langle b_s^{\mathbf{h},R}(\tau,\mathfrak{P},A), d\mathbf{w}(\tau)\rangle - \frac{1}{2}\int_0^t \left|b_s^{\mathbf{h},R}(\tau,\mathfrak{P},A)\right|^2 d\tau\right) \tag{A.1.16}$$

with $d\mathbf{w}$ taken in the sense of Itô and

$$b_s^{\mathbf{h},R}(\tau,\mathfrak{P},A) = \int_0^s \psi_R(\mathfrak{P}_\sigma(\tau)) O_{\sigma,s}(A(\tau)) \left(\dot{h}(\tau) + \frac{1}{2}\mathfrak{R}_{\mathfrak{P}_\sigma(\tau)} h(\tau)\right) d\sigma.$$

Theorem A.1.17. *For any function* $\varphi \in C_c^2(\mathbb{R}^d \times \mathcal{O}(M) \times \mathrm{End}(\mathbb{R}^d))$,

$$E_s^{\mathbf{h},R}(t,\mathfrak{P},A)\left[\varphi(\mathbf{W}_s(t,\mathfrak{f},\mathbf{w}), \mathfrak{P}_s(t), \mathfrak{O}_s(t,\mathfrak{f},\mathbf{w})) - \int_0^t \left(\mathfrak{Y}(d_s^{\mathbf{h},R}(\tau,\mathfrak{P},A)) + \sum_{k=1}^d \frac{1}{2}(\mathfrak{Y}(e_k))^2\right)\varphi(\mathbf{W}_s(\tau,\mathfrak{f},\mathbf{w}), \mathfrak{P}_s(\tau,\mathfrak{f},\mathbf{w}), \mathfrak{O}_s(\tau,\mathfrak{f},\mathbf{w}))d\tau\right] \tag{A.1.18}$$

is a $\mathcal{W}_d$*-martingale, for*

$$d_s^{\mathbf{h},R}(t,\mathfrak{P},A) = \int_0^s \left(1 - \psi_R\left(\mathfrak{P}_\sigma(\tau)\right)\right) O_{\sigma,s}(A(\tau)) \left(\dot{h}(\tau) + \frac{1}{2}\mathfrak{R}_{\mathfrak{P}_\sigma(\tau)} h(\tau)\right) d\sigma.$$

Proof. To show that (A.1.18) is a $\mathcal{W}_d$-martingale, we use Itô's formula and the reasoning given in the proof of (A.1.13) to get first that

$$\varphi(\mathbf{W}_s(t,\mathfrak{f},\mathbf{w}), \mathfrak{P}_s(t), \mathfrak{O}_s(t,\mathfrak{f},\mathbf{w})) - \int_0^t \left(\mathfrak{Y}(d_s^{\mathbf{h},R}(\tau,\mathfrak{P},A)) + \sum_{k=1}^d \frac{1}{2}(\mathfrak{Y}(e_k))^2\right)\varphi(\mathbf{W}_s(\tau,\mathfrak{f},\mathbf{w}), \mathfrak{P}_s(\tau,\mathfrak{f},\mathbf{w}), \mathfrak{O}_s(\tau,\mathfrak{f},\mathbf{w}))d\tau$$
$$= \sum_{k=1}^d \int_0^t \mathfrak{Y}(e_k)\varphi(\mathbf{W}_s(\tau,\mathfrak{f},\mathbf{w}), \mathfrak{P}_s(\tau,\mathfrak{f},\mathbf{w}) d\langle \mathbf{w}(\tau), e_k\rangle.$$

From here, using

$$dE_s^{\mathbf{h},R}(t,\mathfrak{P},A) = -E_s^{\mathbf{h},R}(t,\mathfrak{P},A)\langle b_s^{\mathbf{h},R}(\tau,\mathfrak{P},A), d\mathbf{w}\rangle$$

and Itô's formula one can prove the rest. □

From this result we get the following Corollary.

Corollary A.1.19. *Denote by* $\zeta_R : \mathcal{P}(\mathcal{O}(M)) \to [0,\infty)$, *the first exit time from the set* $U_R = \pi^{-1}B(o,R)$. *Then, for large enough* R, $s \in [0,1]$ *and* $\Phi : \mathcal{P}(\mathbb{R}^d) \times$

$\mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(\mathrm{End}(\mathbb{R}^d)) \to \mathbb{R}$ *which is bounded and* $\mathcal{F}_t$*-measurable*

$$\mathbb{E}^{\mathcal{W}_d}\left[E_s^{\mathbf{h},R}(\cdot \wedge \zeta_R)\Phi(\mathbf{W}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w})), \mathfrak{P}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{O}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\right] = \mathbb{E}^{\mathcal{W}_d}\left[\Phi(\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\right] \quad (A.1.20)$$

where $\mathfrak{o}(t, \mathfrak{f}, \mathbf{w}) = \mathfrak{O}_0(t, \mathfrak{f}, \mathbf{w})$. *Therefore,*

$$\frac{d}{ds}\mathbb{E}^{\mathcal{W}_d}\left[\Phi(\mathbf{W}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w})), \mathfrak{P}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{O}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\right]\Big|_{s=0} = \mathbb{E}^{\mathcal{W}_d}\left[\Phi(\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\int_0^{t\wedge\zeta_R}\langle \dot{h}(\tau) + \frac{1}{2}\mathfrak{R}_{\mathfrak{p}(\tau)}h(\tau), d\mathbf{w}(\tau)\rangle\right]. \quad (A.1.21)$$

and if the function Φ *is also smooth, then*

$$\mathbb{E}^{\mathcal{W}_d}\left[[X(\mathbf{h})\Phi](\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\right] = \mathbb{E}^{\mathcal{W}_d}\left[\Phi(\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\int_0^{t\wedge\zeta_R}\langle \dot{h}(\tau) + \frac{1}{2}\mathfrak{R}_{\mathfrak{p}(\tau)}h(\tau), d\mathbf{w}(\tau)\rangle\right], \quad (A.1.22)$$

where

$$[X(\mathbf{h})\Phi](\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w})) = \frac{d}{ds}\Phi(\mathbf{W}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w})), \mathfrak{P}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{O}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\Big|_{s=0}. \quad (A.1.23)$$

Moreover if Φ, Ψ *are two bounded* $\mathcal{F}_t$*-measurable smooth functions then*

$$s \to \mathbb{E}^{\mathcal{W}_d}\big[\Phi(\mathbf{W}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w})), \mathfrak{P}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{O}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w})) \cdot \Psi(\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\big]$$

is differentiable at 0 *if and only if*

$$s \to \mathbb{E}^{\mathcal{W}_d}\big[\Phi(\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w})) \cdot \Psi(\mathbf{W}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w})), \mathfrak{P}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{O}_s(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\big]$$

is, in which case we have the following integration by parts formula

$$\begin{aligned}&\mathbb{E}^{\mathcal{W}_d}\big[[X(\mathbf{h})\Phi](\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\Psi(\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\big] \\ &= -\mathbb{E}^{\mathcal{W}_d}\big[\Phi(\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))[X(\mathbf{h})\Psi](\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\big] \\ &\quad + \mathbb{E}^{\mathcal{W}_d}\big[(\Phi\Psi)(\mathbf{w}, \mathfrak{p}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot \wedge \zeta_R, \mathfrak{f}, \mathbf{w}))\int_0^{t\wedge\zeta_R}\langle \dot{\mathbf{h}}(\tau) + \frac{1}{2}\mathfrak{R}_{\mathfrak{p}(\tau)}\mathbf{h}(\tau), d\mathbf{w}(\tau)\rangle\big].\end{aligned} \quad (A.1.24)$$

Proof. Using Theorem (A.1.17), the second equation in (A.1.9), and the continuity of the function $\mathbf{h}$, one can show that for large R, $(s,\tau) \in [0,1] \times [0,t] \to \mathfrak{P}_s(\tau \wedge \zeta_R, \mathfrak{f}, \mathbf{w})$ is staying inside U_{2R} and that under the probability $\mathbb{Q}^{\mathbf{h},R}_{s,\mathfrak{f}}$ on $\mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(\mathrm{End}(\mathbb{R}^d))$ defined by

$$\mathbb{Q}^{\mathbf{h},R}_{s,\mathfrak{f}}(C) = \mathbb{E}^{\mathcal{W}_d}\left[E^{\mathbf{h},R}_s(t,\mathfrak{P},A), (\mathbf{W}_s(t,\mathfrak{f},\mathbf{w}), \mathfrak{P}_s(t,\mathfrak{f},\mathbf{w}), \mathfrak{O}_s(t,\mathfrak{f},\mathbf{w})) \in C\right],$$

$$\varphi(\mathbf{W}_s(t\wedge\zeta_R), \mathfrak{P}_s(t\wedge\zeta_R), \mathfrak{O}_s(t\wedge\zeta_R)) - \frac{1}{2}\int_0^{t\wedge\zeta_R} \sum_{k=1}^{d} (\mathfrak{Y}(e_k))^2\, \varphi(\mathbf{W}_s(\tau), \mathfrak{P}_s(\tau), \mathfrak{O}_s(\tau))d\tau$$

is a $\mathbb{Q}^{\mathbf{h},R}_{s,\mathfrak{f}}$-martingale for any function $\varphi \in C^2_c(\mathbb{R}^d \times \mathcal{O}(M) \times \mathcal{P}(\mathrm{End}(\mathbb{R}^d))$. On the other hand one can show that the martingale problem for the operator $\sum_{k=1}^{d} (\mathfrak{Y}(e_k))^2$ on $\mathbb{R}^d \times \mathcal{O}(M) \times \mathrm{End}(\mathbb{R}^d)$ is well posed, or equivalently stated $\mathbf{w} \to (\mathbf{w}, \mathfrak{p}(\cdot,\mathfrak{f},\mathbf{w}), \mathfrak{o}(\cdot,\mathfrak{f},\mathbf{w}))$ does not explode. For more reference one can look at the proof of [7, Theorem 10.26]. Hence, using [7, Theorem 4.37] we get that $\mathbb{Q}^{\mathbf{h},R}_{s,\mathfrak{f}} \restriction \mathcal{F}_{\zeta_R}$ is the same as the distribution of $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d) \to (\mathbf{w}, \mathfrak{p}(t,\mathfrak{f},\mathbf{w}), \mathfrak{o}(\cdot,\mathfrak{f},\mathbf{w})) \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathcal{O}(M) \times \mathcal{P}(\mathrm{End}(\mathbb{R}^d)))$ restricted to $\mathcal{F}_{\zeta_R}$. The rest is just a simple application of [6, Theorem 3.3.5] and standard measure theory. □

A.2 Integration by parts II

In this section we try to make use of the formulas we got in so far. Especially the idea is to show that stopping time can be eliminated under suitable conditions.

We begin with the following definition.

Definition A.2.1. *We say that a map $\mathcal{A} : \mathcal{O}(M) \times N \to V$, where N is a smooth Riemannian manifold and V is a normed vector space, has at most polynomial growth in all its derivatives if for any positive integer n, there are $N_n \geq 0$, $C_n \geq 0$ with the property that for any vector fields $\mathfrak{X}_1, \mathfrak{X}_2, \cdots, \mathfrak{X}_n \in \{(X,Y) \in T\mathcal{O}(M) \times TN \,|\, |\pi_* X| \leq 1, |Y| \leq 1\}$ and any $(\mathfrak{f}, y) \in \mathcal{O}(M) \times N$,*

$$\left\| (\mathfrak{X}_1)_{(\mathfrak{f},y)} \mathfrak{X}_2 \cdots \mathfrak{X}_n \mathcal{A} \right\| \leq C_n (1 + \mathrm{dist}(\pi\mathfrak{f}, o))^{N_n} \qquad \text{(A.2.2)}$$

with $\|\cdot\|$ standing for the norm on the vector space V.

Assumption 1. *In this section we assume that the map $\mathfrak{S}$ we considered is a map in $C^\infty(\mathcal{O}(M), \mathrm{Hom}(\mathbb{R}^d, o(d)))$. Also we assume that $\mathfrak{S}$ and Ω are both of at most polynomial growth in all their derivatives in the appropriate sense.*

These assumptions will make the integrability conditions sufficient to pass to the limit in Corollary (A.1.19).

We will work out here a case which is sufficient for our needs, nevertheless it can be extended to other situation as well. For what we need we take a vector space V

and a solution to the ODE

$$\begin{cases} \dot{\mathfrak{U}}_t(\mathfrak{p}, \mathfrak{o}) = \mathfrak{U}_t(\mathfrak{p}, \mathfrak{o})\mathfrak{A}(\mathfrak{p}(t), \mathfrak{o}(t)) \\ \mathfrak{U}_0 = I_V. \end{cases} \tag{A.2.3}$$

for given paths $\mathfrak{p}, \mathfrak{o}$.

Assumption 2. *We assume that $\mathfrak{A} : \mathcal{O}(M) \times \mathrm{End}(\mathbb{R}^d) \to \mathrm{End}(V)$ is a smooth map of at most polynomial growth in all its derivatives. We assume that there are, a smooth function $\phi : \mathcal{O}(M) \to \mathbb{R}$ and a constant $C > 0$ such that for any $\mathfrak{f}$*

$$\langle \mathfrak{A}(\mathfrak{f}, \mathfrak{o})\xi, \xi \rangle \leq \phi(\mathfrak{f})|\xi|^2, \text{ for } \xi \in V, \tag{A.2.4}$$

$$\phi(\mathfrak{f}) \leq C\mathrm{dist}(\pi\mathfrak{f}, o). \tag{A.2.5}$$

The main result is the following.

Theorem A.2.6. *Suppose $f : M \to \mathbb{R}$ is a compactly supported smooth function, $F : \mathcal{O}(M) \to \mathbb{R}$ its lift to the orthonormal frame bundle, $G \in C^\infty(\mathrm{End}(\mathbb{R}^d, o(\mathbb{R}^d), \mathrm{End}(V))$ with at most polynomial growth in all its derivatives and $\mathfrak{X}_1, \cdots, \mathfrak{X}_n$, smooth horizontal vector fields on $\mathcal{O}(M)$. Then there exists a function $\Psi(t) \in \bigcap_{p>0} L^p(\mathcal{W}_d, V)$ so that*

$$\begin{aligned} \mathbb{E}^{\mathcal{W}_d} \left[\mathfrak{U}_t(\mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot, \mathfrak{f}, \mathbf{w}))G(\mathfrak{o}(t, \mathfrak{f}, \mathbf{w}) \left(\mathfrak{X}_1 \cdots \mathfrak{X}_n F\right) (\mathfrak{p}(t, \mathfrak{f}, \mathbf{w}))\right] \\ = \mathbb{E}^{\mathcal{W}_d} \left[\Psi(t, \mathbf{w})F(\mathfrak{p}(t, \mathfrak{f}, \mathbf{w}))\right]. \end{aligned} \tag{A.2.7}$$

If, $\mathfrak{f} \in \Gamma \subset \mathcal{O}(M)$, $\tau_\Gamma(\mathbf{w}) = \inf\{t > 0, \mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w}) \in \Gamma^c\}$, then $\Psi(t)$ on the set $\{t \leq \tau_\Gamma\}$ depends only on $\mathfrak{S} \upharpoonright \Gamma$, $\Omega \upharpoonright \Gamma$, $\mathfrak{A} \upharpoonright (\Gamma \times \mathbb{R}^d)$ and the support of f. Moreover, there exists $\mathfrak{T}(t) \in \bigcap_{p>0} L^p(\mathcal{W}_d)$ with the property

$$\|\Psi(t, \mathbf{w})\| \leq \mathfrak{T}(t, \mathbf{w}) \exp\left(\int_0^t \phi(\mathfrak{p}(\sigma, \mathfrak{f}, \mathbf{w}))d\sigma\right). \tag{A.2.8}$$

In addition assume that the function f depends on a parameter $z \in N$ with support in $K \subset M$ a compact set for all z, where N is a smooth manifold and the function $\mathfrak{A}$ also depends on this parameter $z \in N$ so that the conditions in (A.2.4) *are fulfilled uniformly, then the maps*

$$(\mathfrak{f}, z) \in \mathcal{O}(M) \times N \to \Psi(t)$$

and

$$(\mathfrak{f}, z) \in \mathcal{O}(M) \times N \to \mathbb{E}^{\mathcal{W}_d} \left[\Psi(t, \mathbf{w})F(\mathfrak{p}(t, \mathfrak{f}, \mathbf{w}))\right]$$

can be taken to be smooth maps.

Proof. The idea is to use the integration by parts as we established for the stopped perturbation and then to show that we have enough integrability conditions to let the stopping time go to infinity.

First notice that because the function F has compact support, we can assume that the vector fields $\mathfrak{X}_i$ have compact support. Next notice that for a compactly supported horizontal vector field $\mathfrak{X}$, there are compactly supported smooth functions a_i, $i = 1, \cdots, d$, such that $\mathfrak{X}_{\mathfrak{f}} = \sum_{k=1}^{d} a_i(\mathfrak{f})\mathfrak{E}(e_k)_{\mathfrak{f}}$. This reduces the problem to the problem when all vector fields are of the form $a\mathfrak{E}(e_k)$, where a is a compactly supported function.

The next observation is the fact that for any function $a : \mathcal{O}(M) \to \mathbb{R}$

$$[X(\mathbf{h}_k)\Phi_a](\mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w})) = \left(\mathfrak{E}(e_k)a + \lambda\left(\int_0^t \Omega_{\mathfrak{p}(\tau,\mathfrak{f},\mathbf{w})}(e_l, \mathbf{h}_k(\tau)) \circ d\mathbf{w}_l(\tau)\right) a\right)(\mathfrak{p}(t, \mathfrak{f}, \mathbf{w}))$$

where $\mathbf{h}_k(s) = \frac{s}{t}e_k$ for $s \geq 0$ and $\Phi_a(\mathfrak{p}) = a(\mathfrak{p}(t))$. For the function F, there is no λ part so that applying this repeatedly, we can freed the function F from all the derivatives. We also point here that the derivatives on the left are easily controllable for functions with compact support.

Now we are left with the estimation of the derivatives of stochastic differential equations. Here we mention only two cases. One is when we have a purely stochastic integral, for example like the one involving the Ricci curvature, where we say that

$$X(\mathbf{h}_{l_m}) \cdots X(\mathbf{h}_{l_1}) \int_0^t \langle \dot{\mathbf{h}}_k(\tau) + \frac{1}{2}\mathfrak{R}_{\mathfrak{p}(\tau,\mathfrak{f},\mathbf{w})}\mathbf{h}_k(\tau), d\mathbf{w}(\tau)\rangle = \frac{1}{2}\int_0^t \langle \left(\mathfrak{E}(e_{l_m})_{\mathfrak{p}(\tau,\mathfrak{f},\mathbf{w})} \cdots \mathfrak{E}(e_{l_1})\mathfrak{R}\right) \mathbf{h}_k(\tau), d\mathbf{w}(\tau)\rangle. \quad \text{(A.2.9)}$$

The other case is the case when we have to estimate the derivatives of $G(\mathfrak{o}(t, \mathfrak{f}, \mathbf{w}))$ or $\mathfrak{U}_t(\mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w}), \mathfrak{o}(\cdot, \mathfrak{f}, \mathbf{w}))$. In this case we use induction to step from one case to another.

We deal first with the case of $\mathfrak{o}(t, \mathfrak{f}, \mathbf{w})$. We start pointing out that because $\mathfrak{S}$ is skew-symmetric, the solution for $\mathfrak{o}(t, \mathfrak{f}, \mathbf{w})$ is orthogonal matrix. Set

$$\mathfrak{o}^m(t) = X(\mathbf{h}_{l_m}) \cdots X(\mathbf{h}_{l_1})\mathfrak{o}.$$

The equation for $\mathfrak{o}_t^1$ is given by

$$\begin{aligned}\dot{\mathfrak{o}}^1(t, \mathfrak{f}, \mathbf{w}) &= \mathfrak{o}^1(t, \mathfrak{f}, \mathbf{w})\mathfrak{S}_{\mathfrak{p}(t,\mathfrak{f},\mathbf{w})}(\circ d\mathbf{w}_s(t, \mathfrak{f}, \mathbf{w})) \\ &+ \mathfrak{o}(t, \mathfrak{f}, \mathbf{w})\left(\mathfrak{E}(e_k) + \lambda\left(\int_0^t \Omega_{\mathfrak{p}(\tau,\mathfrak{f},\mathbf{w})}(\circ d\mathbf{w}(\tau), \mathbf{h}_k(\tau))\right)\right)_{\mathfrak{p}(t,\mathfrak{f},\mathbf{w})} \mathfrak{S}_{\cdot}(\circ d\mathbf{w}(t, \mathfrak{f}, \mathbf{w})) \\ &+ \mathfrak{o}(t, \mathfrak{f}, \mathbf{w})\mathfrak{S}_{\mathfrak{p}(t,\mathfrak{f},\mathbf{w})}\left(\dot{\mathbf{h}}(t)dt - \int_0^t \Omega_{\mathfrak{p}(\tau,\mathfrak{f},\mathbf{w})}(\circ d\mathbf{w}(\tau), \mathbf{h}_k(\tau)) \circ d\mathbf{w}(t, \mathfrak{f}, \mathbf{w})\right). \quad \text{(A.2.10)}\end{aligned}$$

In general the solution to the stochastic differential equation

$$dX_t = X_t\mathfrak{S}_{\mathfrak{p}(t,\mathfrak{f},\mathbf{w})}(\circ d\mathbf{w}_s(t, \mathfrak{f}, \mathbf{w})) + dZ(t)$$

in the Stratonovich form is given by

$$\int_0^t Z(\tau)\mathfrak{o}(\tau,\mathfrak{f},\mathbf{w})^*\mathfrak{o}(t,\mathfrak{f},\mathbf{w}).$$

From here, using Burkholder's inequality, the assumptions in the beginning of this paragraph and the estimates in [7, Theorem 8.62], we deduce that the $\mathfrak{o}^1(t)$ is in $\bigcap_{p>0} L^p(\mathcal{W}_d, V)$.

In the same way we can estimate higher derivatives. The same thing works, with the only change that at step k the equation satisfied by $\mathfrak{o}^k$ is written in terms of the lower derivatives and derivatives of $\mathfrak{S}$ and the curvature, we already know how to control.

We point out the general structure of these derivatives, namely they are iterated Stratonovich integrals where the integrands are polynomially bounded in terms of the distance function on M.

In the case of $\mathfrak{U}_t$, set

$$\mathfrak{U}_t^m = X(\mathbf{h}_{l_m})\cdots X(\mathbf{h}_{l_1})\mathfrak{U}_t.$$

We give now a Lemma that shows us how one can estimate the size of $\mathfrak{U}_t$.

Lemma A.2.11. *Let $(H, \langle,\rangle)$ be a finite dimensional vector space with an inner product. Let $s \in [0,\infty) \to X_s, Y_s \in \mathrm{End}(H)$ be two continuous maps. If W_s is the solution to*

$$\dot{W}_s = W_s X_s + Y_s$$

then,

$$\|W_s\|^2 \le e^{2\int_0^s f(\sigma)d\sigma}\left(\|W_0\|^2 + \int_0^s \|Y_\sigma\|^2 e^{-\int_0^\sigma (2f(\tau)+1)d\tau}\right)$$

for any s and for any local integrable function f with the property that $\langle X_s\xi,\xi\rangle \le f(s)|\xi|^2$ for $s \ge 0$.

Proof. The proof follows from the fact that $\dot{W}_s^* = X_s^* W_s^* + Y_s^*$ and that for any vector $\xi \in H$ we have

$$\frac{d}{dt}|W_s^*\xi|^2 = 2\langle W_s^*\xi, X_s^*W_s^*\xi + Y_s^*\xi\rangle \le (2f(s)+1)|W_s^*\xi|^2 + |Y_s^*\xi|^2.$$

From here a standard argument ends the proof. □

Now using this Lemma with $Y = 0$ and $W_0 = I$ one can get the first estimates on $\mathfrak{U}_t$ as

$$\mathfrak{U}_t(\mathfrak{p}(\cdot,\mathfrak{f},\mathbf{w}),\mathfrak{o}(\cdot,\mathfrak{f},\mathbf{w})) \le C\exp\left(\int_0^t \phi(\mathfrak{p}(\sigma,\mathfrak{f},\mathbf{w}))d\sigma\right).$$

Next, taking the first derivative we get that

$$\begin{cases} d\mathfrak{U}_t^1 = \mathfrak{U}_t^1\mathfrak{A}(\mathfrak{p}(t,\mathfrak{f},\mathbf{w}),\mathfrak{o}(t,\mathfrak{f},\mathbf{w})) + \mathfrak{U}_t\mathfrak{A}^1(\mathfrak{p}(t,\mathfrak{f},\mathbf{w}),\mathfrak{o}(t,\mathfrak{f},\mathbf{w})) \\ \mathfrak{U}_0^1 = 0 \end{cases} \qquad (*)$$

where as in the equation (A.2.10) $\mathfrak{A}^1$ is written in terms of the derivatives of the vector field $\mathfrak{A}$ the curvature and the derivatives of o. To estimate this first derivative we use the above Lemma with $Y = \mathfrak{U}_t\mathfrak{A}^1(\mathfrak{p}(t,\mathfrak{f},\mathbf{w}),\mathfrak{o}(t,\mathfrak{f},\mathbf{w}))$ together with the estimate on the $\mathfrak{U}$ to get that

$$\|\mathfrak{U}_t^1\| \leq \mathfrak{T}_t^1 \exp\left(\int_0^t \phi(\mathfrak{p}(\sigma,\mathfrak{f},\mathbf{w}))d\sigma\right).$$

For higher derivatives we proceed in basically the same way. We point only that we get some estimates of the form

$$\|\mathfrak{U}_t^k\| \leq \mathfrak{T}_t^k \exp\left(\int_0^t \phi(\mathfrak{p}(\sigma,\mathfrak{f},\mathbf{w}))d\sigma\right)$$

where $\mathfrak{T}_t^k$ is a multiple integral (involving eventually stochastic integrals) with each integrand a polynomially bounded quantity in terms of the distance along the path $p(\cdot,x,\mathbf{w})$.

Using [7, Theorem 8.46] one can justify the integrability of the exponential of the distance function, $\exp\left(\int_0^t C\mathrm{dist}(\mathfrak{p}(\sigma,\mathfrak{f},\mathbf{w}))d\sigma\right)$ for any positive constant C.

The proof ends with the remark that we can make the integration by parts with the stopped path and then, because all our things are integrable, one can pass R to infinity.

The rest of the proof is easy and is left to the reader. □

Remark A.2.12. *One can extend the validity of this theorem if one replaces the ordinary differential equation* (A.2.3) *corresponding to* $\mathfrak{U}$ *by a stochastic differential equation of the form*

$$\begin{cases} \dot{\mathfrak{U}}_t(\mathbf{w}) = \mathfrak{U}_t(\mathbf{w})\left(\mathfrak{A}(\mathfrak{p}(t,\mathfrak{f},\mathbf{w}),\mathfrak{o}(t,\mathfrak{f},\mathbf{w})) + \sum_{i=1}^d \mathfrak{B}_i(\mathfrak{p}(t,\mathfrak{f},\mathbf{w}),\mathfrak{o}(t,\mathfrak{f},\mathbf{w}))d\langle\mathbf{w}(t),e_i\rangle\right) \\ \mathfrak{U}_0 = I_V, \end{cases}$$

with $\mathfrak{A},\mathfrak{B}_i$, $i = 1,\cdots,d$ *polynomially bounded in all thir derivatives and* $\mathfrak{B}_i$ *skew-symmetric maps.*

As a typical application of this theorem we point out the following case. Take a Riemannian manifold and a compatible connection ∇ which has the same Laplacian on functions as the usual Laplacian. Assuming that the torsion of ∇ and the curvature of the Levi-Civita connection have at most polynomial growth at infinity, one can apply the machine developed here to study integrals of the form

$$\mathbb{E}^{\mathcal{W}_d}\left[U(t,x,\mathbf{w})\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\delta_y(p(t,x,\mathbf{w}))\right]$$

where τ^{∇} is the parallel transportation with respect to the connection ∇ and U satisfies a differential equation of the following form

$$\begin{cases} dU(t,x,\mathbf{w}) = U(t,x,\mathbf{w})\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}A(p(t,x,\mathbf{w}))\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[0,t]}dt \\ U(0,x,\mathbf{w}) = \mathrm{Id}_{\Lambda_x(M)}. \end{cases}$$

To give a hint here we only point out that in order to make this amenable by the methods we developed here we can denote by

$$o(t,x,\mathbf{w}) = \tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]}$$

where here the τ is for the parallel translation with respect to the Levi-Civita connection. Then we can rewrite everything in terms of the parallel transportation with respect to Levi-Civita connection and this map o and lift things on the orthonormal frame bundle. The last piece is the writing of the delta function in the distibutional sense as

$$X_1 X_2 \cdots X_d u$$

for a continuous function u (this a local Euclidean problem!). Taking a smooth approximation f_n of u, we get an approximation of the delta function by smooth functions. Then we just use the above theorem to justify the existence of the integral with resprect to the delta function.

Appendix B

About Semigroups and Heat Kernels

B.1 General Results

The theme of this section is to give some statements about the existence of the heat kernel of some operators. We also show here how one can get bound one heat kernels by looking at the semigroup.

In this section we assume that we have a Riemannian manifold M with the curvature at most polynomially growing at infinity and with

$$-C(1+\mathrm{dist}(z,o)^2)|X|_z^2 \leq Ric_z(X_z,X_z).$$

This suffices for the existence of the Brownian motion on M.

We take the class of operators on forms on M

$$L = -\Delta^{\nabla} + \sum_{j=1}^{d} B(E_j)\nabla_{E_j} + C \tag{B.1.1}$$

where the data satisfies:

1. The connection ∇ obeys
   (a) compatibility with the metric on M;
   (b) ∇-Laplacian on functions is the same as the standard Laplacian;
2. $B_z(X_z) = (D^*S_0)_z(X_z) + (D^*S_1)_z(X_z) + \cdots + (D^*S_k)_z(X_z)$ in the notations of Definition 1.5 with the crucial supplementary condition that $B_z(X_z)$ is skew symmetric for any $z \in M$, $X_z \in T_z(M)$;
3. $C = D^*T_0 + D^*T_1 + \cdots + D^*T_l$ for some smooth tensors T_i.

4. there is constant K such that

$$\langle C_z X_z + \sum_{i=1}^{n} B(E_i)_z^2 X_z, X_z \rangle \leq K|X_z|^2.$$

We have the following Lemma.

Lemma B.1.2. *Let $(H, \langle , \rangle)$ be a finite dimensional vector space endowed with an inner product and X_s, Y_s^i, $i = 1, \cdots, n$ locally bounded progressive measurable processes $End(H)$-valued such that Y_s^i is skew symmetric for $i = 1, \cdots, n$. Let V_s be the solution to the stochastic differential equation*

$$\begin{cases} dV_s = V_s(X_s ds + \sum_{i=1}^n Y_s^i d\mathbf{w}_i(s)) \\ V_0 = Id, \end{cases}$$

where $d\mathbf{w}_i$ stands for the Ito stochastic differential. If T_s is the solution to the Stratonovich equation

$$\begin{cases} dT_s = T_s \circ d \sum_{i=1}^n \int_0^s Y_\sigma^i d\mathbf{w}_i(\sigma) \\ T_0 = Id, \end{cases}$$

and W the solution to the ODE

$$\begin{cases} \dot{W}_s = W_s T_s \left(X_s - \frac{1}{2} \sum_{i=1}^n (Y_s^i)^2 \right) T_s^{-1} \\ W_0 = Id. \end{cases}$$

then T_s is unitary for any s and

$$V_s = W_s T_s.$$

Thus, estimates on the size of V_s reduces to estimates on the size of W_s.

Proof. Rewrite the equation for V in the Stratonovich form:

$$\begin{cases} dV_s = V_s X_s ds + V_s \circ d \sum_{i=1}^n \int_0^s Y_\sigma^i d\mathbf{w}_i(\sigma) - \frac{1}{2} d\langle V_s, \int_0^s \sum_{i=1}^n Y_\sigma^i d\mathbf{w}_i(\sigma) \rangle \\ V_0 = Id. \end{cases}$$

Now because $V_s = Id + \int_0^s U_\sigma \sum_{i=1}^n Y_\sigma^i d\mathbf{w}_i(\sigma) + \int_0^s V_\sigma X_\sigma d\sigma$, we get that

$$\begin{aligned} \langle V_s, \int_0^s \sum_{i=1}^n Y_\sigma^i d\mathbf{w}_i(\sigma) \rangle &= \langle \int_0^s U_\sigma \sum_{i=1}^n Y_\sigma^i d\mathbf{w}_i(\sigma), \int_0^s \sum_{j=1}^n Y_\sigma^j d\mathbf{w}_j(\sigma) \rangle \\ &= \sum_{i=1}^n \int_0^s U_\sigma Y_\sigma^i Y_\sigma^i d\sigma \\ &= \sum_{i=1}^n \int_0^s U_\sigma (Y_\sigma^i)^2 d\sigma. \end{aligned}$$

Then the equation for V is

$$dV_s = V_s\left(\circ d\sum_{i=1}^{n}\int_0^s Y_\sigma^i d\mathbf{w}_i(\sigma) + \left(X_s - \frac{1}{2}\sum_{i=1}^{n}(Y_s^i)^2\right)ds\right).$$

On the other hand taking T we can see that

$$\begin{cases} dT_s^* = \left(\circ d\sum_{i=1}^n \int_0^s (Y_\sigma^i)^* d\mathbf{w}_i(\sigma)\right)T_s^* \\ T_0^* = Id, \end{cases}$$

and then

$$\begin{aligned} \circ d\left(T_sT_s^*\right) &= \left(\circ dT_s\right)T_s^* + T_s\circ dT_s^* \\ &= T_s\left(\circ d\sum_{i=1}^{n}\int_0^s Y_\sigma^i d\mathbf{w}_i(\sigma)\right)T_s^* + T_s\left(\circ d\sum_{i=1}^{n}\int_0^s (Y_\sigma^i)^* d\mathbf{w}_i(\sigma)\right)T_s^* \\ &= 0 \end{aligned}$$

because $(Y_\sigma^i)^* = -Y_\sigma^i$. This proves that T_s is an isometry.

Now if W is the solution to the ODE above then:

$$\begin{aligned} \circ dW_sT_s &= \left(\circ dW_s\right)T_s + W_s\circ dT_s \\ &= W_s\left(T_s\left(X_s - \frac{1}{2}\sum_{i=1}^{n}(Y_s^i)^2\right)T_s^{-1}\right)T_s + W_sT_s\circ d\sum_{i=1}^{n}\int_0^s (Y_\sigma^i)^* d\mathbf{w}_i(\sigma) \\ &= W_sT_s\left(\circ d\sum_{i=1}^{n}\int_0^s Y_\sigma^i d\mathbf{w}_i(\sigma) + \left(X_s - \frac{1}{2}\sum_{i=1}^{n}(Y_s^i)^2\right)ds\right) \end{aligned}$$

with the starting $W_0T_0 = Id$. Thus by uniqueness we get $V_s = W_sT_s$. □

Theorem B.1.3. *The semigroup of the operator exists and we have the following formula. For any compactly supported function,*

$$\left(\mathbf{P}_t^L\omega\right)(z) = \mathbb{E}^{\mathcal{W}_d}\left[U(t,z,\mathbf{w})\tau^{\nabla}_{p(\cdot,z,\mathbf{w})\restriction[t,0]}\omega(p(t,z,\mathbf{w}))\right]$$

where U satisfying

$$\begin{cases} dU(t,z,\mathbf{w}) = U(t,z,\mathbf{w})\left(C(t,z,\mathbf{w})dt + \sum_{j=1}^{d} B_j(t,z,\mathbf{w})d\mathbf{w}_j(t)\right) \\ U(0,z,\mathbf{w}) = \mathrm{Id}_{\Lambda_z(M)}. \end{cases}$$

with

$$C(t,z,\mathbf{w}) = \tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}C_{p(t,x,\mathbf{w})}\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[0,t]},\ \textit{and} \tag{B.1.4}$$

$$B_j(t,x,\mathbf{w}) = \tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}B\left(\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[0,t]}E_j\right)_{p(t,x,\mathbf{w})}\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[0,t]} \tag{B.1.5}$$

In this case the heat kernel $p^L(t, z_1, z_2) : \bigwedge_{z_2}^k M \to \bigwedge_{z_1}^k M$ *of such operator exists. Moreover, if the tensors* B *and* C *have at most polynomial growth at infinity in all their derivatives, then the heat kernel has an expression via the Malliavin calculus*

$$p^L(t, z_1, z_2) = \mathbb{E}^{\mathcal{W}_d}\left[U^\alpha(t, z_1, \mathbf{w})\tau^{\nabla}_{p(\cdot, z_1, \mathbf{w})\restriction[t,0]}\delta_{z_2}(p(t, z_1, \mathbf{w}))\right].$$

Proof. Exactly as in section (2.2.2), we can prove that the expression of the semigroup is given by that formula. Here we have to point out that in order to justify the integrability, we use (B.1.2) together with (A.2.11) to show that U is a bounded map.

Now, the only thing that needs some explanation is that about the existence of heat kernels. For this we mention that the proof outlined in [7, Theorem 6.25] works in this case as well because the semigroup in discussion here enjoys the same properties as the one dealt with in the reference. □

The following proposition shows how one can compare heat kernels based on comparison of the semigroups.

Proposition B.1.6. *Let* L *be an operator in the class described. Assume that there is a smooth function* $\phi : M \to \mathbb{R}$ *bounded from above such that with*

$$\langle C_z X_z + \sum_{i=1}^n B(E_i)_z^2 X_z, X_z\rangle \le \phi(z)|X_z|^2$$

for any $z \in M$*. Then*

$$\|p^L(t, z_1, z_2)\|_{z_2, z_1} \le p^\phi(t, z_1, z_2)$$

where the last one is the heat kernel of the operator $\Delta + \phi$ *on functions.*

Proof. Using the expression for the semigroups we get the bounds on the semigroups. Then by taking an approximate identity that tends to the delta function in distributions, it is an easy step to the heat kernel estimates. □

Remark B.1.7. *Note here that we do not need the representation via the Malliavin calculus for the heat kernels in this proposition. What we need is the existence of the heat kernels and the integral representation of the semigroups.*

Bibliography

[1] Jean-Michel Bismut. The Witten complex and the degenerate Morse inequalities. *J. Differential Geom.*, 23(3):207–240, 1986.

[2] Elton P. Hsu. *Stochastic Analysis on Manifolds*, volume 38 of *Graduate Studies in Mathematics*. American Mathematical Society, 2002.

[3] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, reprint edition, 2002.

[4] Shoshichi Kobayashi & Katsumi Nomizu. *Foundations of Differential Geometry*, volume I, II of *Wiley Classics Library*. John Wiley & Sons, Inc., 1996. reprint of the 1969 original.

[5] Raul Bott & Loring Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, 1982.

[6] Daniel W. Stroock. *A Concise Introduction to the Theory of Integration*. Birkhäuser Boston, Inc., second edition, 1994.

[7] Daniel W. Stroock. *An Introduction to the Analysis of Paths on a Riemannian Manifold*, volume 74 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2000.

[8] Nicole Berline, Ezra Getzler & Michele Vergne. *Heat Kernels and Dirac Operators*. Springer-Verlag, second edition, 1996.

[9] Nobuyuki Ikeda & Shinzo Watanabe. *Stochastic Differential Equations and Diffusion Processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., 1981.

[10] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692, 1983.