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State Feedback ℓ_1-Optimal Controllers can be Dynamic *

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Abstract

This paper considers discrete-time systems with full state feedback, scalar control and scalar disturbance. First, systems with a scalar regulated output are studied (singular problems). It is shown that there is a large class of such systems, characterized by the non-minimum phase zeros of the transfer function from the control to the regulated output, for which the ℓ_1-optimal controller is necessarily dynamic. Moreover, such controllers may have arbitrarily high order. Second, problems with two regulated outputs, one of them being the scalar control sequence, are considered (non-singular problems). It is shown, by means of a fairly general example, that such problems may not have static controllers that are ℓ_1-optimal.

Keywords: State feedback control, BIBO optimal control, Linear optimal control, Rejection of persistent disturbances, Minimax optimal control.

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1 Introduction

Since Dahleh and Pearson ([1],[2]) presented the solution to the ℓ_1 optimal control problem, there has been increasing interest in understanding the basic properties of such problems ([3],[8],[9] and [11]). Considering that in the case of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimization, state feedback optimal controllers have a very special structure (i.e. static), it seems only natural to ask how full state information affects the ℓ_1 optimal solution. In particular, under what conditions (if any) there exist a static linear controller that achieves optimality. This paper presents results regarding this question, for systems with scalar control and scalar disturbance. In particular, two different types of problems within this class of systems are considered: a) those with a scalar regulated output, denoted as *singular problems*, and b) those with two regulated outputs, denoted as *non-singular problems*, where one of the outputs is the scalar control signal. For systems in a), it is shown that there exists a static controller which is ℓ_1-optimal if the non-minimum phase zeros of the transfer function from the control input to the regulated output satisfy a simple algebraic condition. Violating such condition, however, may result in a dynamic ℓ_1-optimal controller of possibly high order (generally when the non-minimum phase zeros are "close" to the unit circle). For systems in b), it is shown by means of a general example that optimal controllers are dynamic in a broad class of cases which are common in control design.

The paper is organized as follows. Section 2 formulates the singular problem along with some basic notation. Sections 3 and 4 present results corresponding to singular problems involving minimum and non-minimum phase plants respectively. Section 5 examines a non-singular problem by means of a general example, followed by the conclusions in Section 6.

2 Problem Formulation

Consider the following state-space minimal realization of a full state feedback system with scalar inputs w (disturbance) and u (control), scalar regulated output z, and measurements x (state vector):

$$\left(\begin{array}{c|cc} A & b_1 & b_2 \\ \hline c_1 & 0 & d_{12} \\ I & 0 & 0 \end{array}\right)$$

where $A \in R^{n\times n}$, b_1 and $b_2 \in R^{n\times 1}$, $c_1 \in R^{1\times n}$, and $d_{12} \in R$. For any internally stabilizing controller k, let $\phi = \{\phi(0), \phi(1), \phi(2), \ldots\}$ denote the closed-loop pulse response sequence from w to z. Then, the problem can be stated as follows:

$$\inf_{k-stab.} \|\phi\|_1 \tag{1}$$

where $\|\phi\|_1 \stackrel{\text{def}}{=} \sum_i |\phi(i)|$. Using standard results in the parameterization of all stabilizing controllers (see [6]), problem (1) can be rewritten as follows:

$$\inf_{q\in \ell_1^{1\times n}} \|t_1 - t_2 * q * t_3\|_1 \tag{2}$$

where $\ell_1^{m\times n}$ indicates the space of all $m \times n$ matrices with entries in ℓ_1 and $*$ denotes convolution. Thus, t_1 and $t_2 \in \ell_1$, and $t_3 \in \ell_1^{n\times 1}$. Let the λ-transform of a right-sided real sequence $h = \{h(0), h(1), h(2), \ldots\}$ be defined as

$$\hat{h}(\lambda) = \sum_{k=0}^{\infty} h(k)\lambda^k$$

where λ represents the unit delay. Then, a state-space realizations for t_1, t_2 and t_3 can be found by using the state-space formulas in [6] with the observer gain matrix, H, equal to $-A$. For this specific choice, the realizations are:

$$\hat{t}_1(\lambda) = \lambda\,[A_f, A\,b_1, c_1 + d_{12}f, c_1 b_1] \tag{3}$$
$$\hat{t}_2(\lambda) = [A_f, b_2, c_1 + d_{12}f, d_{12}] \tag{4}$$
$$\hat{t}_3(\lambda) = [0, b_1, I, 0] = \lambda\, b_1 \tag{5}$$

where $\hat{t}_i(\lambda)$ denotes the λ-transform of t_i, $A_f \stackrel{\text{def}}{=} A + b_2\, f$,

$$[A, B, C, D] \stackrel{\text{def}}{=} \lambda\, C\,(I - \lambda A)^{-1}\, B + D$$

and f is chosen such that all the eigenvalues of A_f are inside the unit disk.

The following result, which will be needed in the next section, is proved in [1].

Theorem 2.1 *Assuming $\hat{t}_2(\cdot)$ and $\hat{t}_3(\cdot)$ have no left and right zeros respectively on the unit circle, there exists $q_{opt} \in \ell_1^{1\times n}$ that achieves the optimal norm in problem (2). Moreover, the closed-loop optimal pulse response, $\phi_{opt} = t_1 - t_2 * q_{opt} * t_3$, has finite support.*

3 Singular Problem with Minimum-Phase Plant

This section considers the case where the transfer function from the control input, u, to the regulated output, z, is minimum-phase except for an integer number of unit delays (i.e. zeros at the origin in the λ-plane). It will be assumed throughout that (A, b_2) is reachable.

Theorem 3.1 *For such a system, the static feedback gain, f^*, that places the eigenvalues of $(A + b_2 f^*)$ at the exact location of the minimum-phase zeros of $[A, b_2, c_1, d_{12}]$ and the rest at the origin is ℓ_1-optimal.*

Proof Consider using f^* as the state feedback gain in the parameterization described above. Then, after carrying out all stable pole-zero cancellations,

$$\hat{t}_2(\lambda) = \gamma_r \lambda^r$$

where r is the number of unit delays in $[A, b_2, c_1, d_{12}]$ and γ_r is a scalar depending on r. In what follows, the cases where $r = 0$ and $r > 0$ will be treated separately.

i) If $r = 0$, then $d_{12} \neq 0$, $c_1 + d_{12} f^* = 0$, and $\hat{t}_2(\lambda) = d_{12}$. Also, from equation (3),

$$\hat{t}_1(\lambda) = c_1 b_1 \lambda = (c_1 b_1 + d_{12} f^* b_1 - d_{12} f^* b_1)\lambda = -d_{12} f^* b_1$$

$$\Longrightarrow \hat{\phi}(\lambda) = -d_{12} f^* b_1 \lambda - d_{12} \hat{q}(\lambda) b_1 \lambda$$

Thus, the ℓ_1-optimal solution is given by $\hat{q}_{opt}(\lambda) = -f^*$, and $\hat{\phi}_{opt}(\lambda) = 0$. Furthermore, using the state-space equations in [6] for computing the optimal controller, it can be shown after a little algebra that $\hat{k}_{opt}(\lambda) = f^*$.

ii) If $r > 0$, then $d_{12} = 0$, $c_1 A_{f^*}^r = 0$ by construction, and $\hat{t}_2(\lambda) = c_1 A_{f^*}^{r-1} \lambda^r$. Again, from equation 3,

$$\hat{t}_1(\lambda) = c_1 b_1 \lambda + c_1 A b_1 \lambda^2 + c_1 A_{f^*} A b_1 \lambda^3 + \cdots + c_1 A_{f^*}^{r-1} A b_1 \lambda^{r+1}$$

Therefore, the closed-loop pulse response is given by

$$\begin{aligned} \hat{\phi}(\lambda) &= c_1 b_1 \lambda + c_1 A b_1 \lambda^2 + c_1 A_{f^*} A b_1 \lambda^3 + \cdots \\ &\quad + c_1 A_{f^*}^{r-2} b_1 \lambda^r + c_1 A_{f^*}^{r-1} (A - b_2 \hat{q}(\lambda)) b_1 \lambda^{r+1} \end{aligned}$$

Clearly, q does not affect the first $r+1$ elements of ϕ (i.e. $\phi(i), i=0,1,\ldots,r$). Then, the best possible choice of q, in the sense of minimizing the ℓ_1-norm of ϕ, is the one that makes $\phi(i)=0$ for $i=r+1, r+2, \ldots$, and is achieved by letting $\hat{q}_{opt}(\lambda) = -f^*$, since $\phi(r+1) = c_1 A_{f^*}^r b_1 = 0$. Again, the corresponding ℓ_1-optimal controller is f^*. ∎

Corollary 3.1 *The ℓ_1-optimal closed-loop transfer function of the system considered in theorem 3.1 (with $r > 0$) is given by:*

$$\hat{\phi}_{opt}(\lambda) = c_1 \sum_{i=1}^{r} A^{i-1} \lambda^i b_1$$

Proof It follows from the fact that $c_1 A_{f^*}^i b_2 = 0$ for $i = 0, 1, \ldots, r-2$. The details are left to the reader. ∎

Put in words, theorem 3.1 says that there is nothing the controller can do to invert the delays in the system. It can, however, cancel the rest of the dynamics of the system due to the absence of non-minimum phase zeros in the transfer function from the control input to the regulated output. This results in an optimal closed-loop pulse response that is equal to the open loop pulse response in its first $(r+1)$ elements and zero thereafter. It is also worth noting that theorem 3.1 is directly applicable to the discrete-time LQR problem, where $\sum_i \phi_i^2$ is minimized. More precisely, the asymptotic LQR solution (see [7]) where the weight on the control tends to zero (i.e. cheap control problem) is identical to that of theorem 3.1.

4 Singular Problem with Non-minimum Phase Plant

This section considers those cases where $[A, b_2, c_1, d_{12}]$ has r non-minimum phase zeros not necessarily at the origin (i.e. $\lambda = 0$).

Again, we use the same parameterization as in the previous section. That is, we choose f^* to place $(n-r)$ eigenvalues of A_{f^*} at the exact location of the minimum phase zeros of $[A, b_2, c_1, d_{12}]$ and the rest (r) at the origin. Then, from the discussion in section 3, $\hat{t}_1(\lambda)$ is polynomial in λ and of order $(r+1)$,

$\hat{t}_2(\lambda)$ is polynomial too, but of order r, and $\hat{t}_3(\lambda)$ is simply $\lambda\, b_1$. Therefore, the closed-loop transfer function can be written as follows:

$$\hat{\phi}(\lambda) = \left(g_1 \prod_{i=1}^{r} (\lambda - \alpha_i) - g_2 \prod_{j=1}^{r} (\lambda - \beta_j)\, \hat{\tilde{q}}(\lambda) \right) \lambda \stackrel{\text{def}}{=} \hat{\tilde{\phi}}(\lambda)\, \lambda \tag{6}$$

where $g_1, g_2 \in R$, α_i's are the zeros of $\hat{t}_1$, β_j's are the (non-minimum phase) zeros of $\hat{t}_2$ and $[A, b_2, c_1, d_{12}]$, and $\hat{\tilde{q}}(\lambda) \stackrel{\text{def}}{=} \hat{q}(\lambda)\, b_1 \in \ell_1$. Note that $\|\phi\|_1 \equiv \left\|\tilde{\phi}\right\|_1$. Also, by theorem 2.1, $\hat{\tilde{\phi}}_{opt}(\lambda)$ is polynomial in λ, which implies that $\hat{\tilde{q}}_{opt}(\lambda)$ is polynomial in λ. Thus, the optimization problem is equivalent to the following linear programming (primal) problem: for a sufficiently large but finite s,

$$\min_{\tilde{\phi}} \sum_{i=0}^{s} \left|\tilde{\phi}(i)\right| \tag{7}$$

$$s.t. \quad \sum_{i=0}^{s} \tilde{\phi}(i)\, \beta_j^i = g_1 \prod_{i=1}^{r} (\beta_j - \alpha_i) \ , \ j = 1, 2, \ldots, r$$

In the above we have assumed that the β_j's are simple zeros to simplify the formulation of the interpolation conditions. The coming results, however, carry over to the more general case.

The following theorem by Deodhare and Vidyasagar [4] will prove useful. It is stated with no proof.

Theorem 4.1 *The support of $\tilde{\phi}$ in (7), denoted as $(s+1)$, equals the number of constraints r, if*

$$\sum_{i=0}^{r-1} |a_i| < 1 \tag{8}$$

where $\prod_{j=1}^{r} (\lambda - \beta_j) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_1\lambda + a_0$.

Now we are ready to present the next result.

Theorem 4.2 *Let $[A, b_2, c_1, d_{12}]$ have r non-minimum phase zeros, then if (8) is satisfied, f^* is ℓ_1-optimal.*

Proof By theorem 4.1, $\hat{\tilde{\phi}}_{opt}(\lambda)$ is of order $(r-1)$. Then, considering the order of each term in (6), it is clear that $\hat{\tilde{q}}_{opt}(\lambda)$ has to be constant and such that $\tilde{\phi}(r) = 0$. Using the state-space formulas (3), (4) and (5),

$$\begin{aligned} 0 = \tilde{\phi}(r) &= (c_1 + d_{12} f^*) A_{f^*}^{r-1} (A b_1 - b_2 \tilde{q}_{opt}(0)) \\ &= (c_1 + d_{12} f^*) A_{f^*}^{r-1} (A - b_2 q_{opt}(0)) b_1 \end{aligned}$$

But, by construction, $(c_1 + d_{12} f^*) A_{f^*}^r = 0$ due to the stable pole-zero cancellations and the fact that the rest of the poles are placed at the origin. Therefore, $\hat{q}_{opt} = -f^*$ is the required value, and $\hat{k}_{opt} = f^*$. ∎

It remains to consider those cases where the non-minimum phase zeros of $[A, b_2, c_1, d_{12}]$ are such that they violate condition (8). Theorem 4.1 established only a sufficient condition to determine the order of the optimal response. If this condition is violated, the optimal closed-loop response might be of higher order, possibly greater than n, but still polynomial. When that is the case the following theorem applies.

Theorem 4.3 *If the optimal response, $\hat{\phi}_{opt}(\lambda)$, is of order greater than n, then the ℓ_1-optimal controller is necessarily dynamic.*

Proof The highest order polynomial response that a static controller can generate is n, by placing all closed-loop poles of the plant at the origin. Any polynomial response of order greater than n, say N, requires a dynamic compensator of at least order $N - n$. ∎

The following example shows that a large class of state feedback singular problems have this property.

EXAMPLE 1: Consider the following parameterized family of plants (with parameter κ),

$$P_\kappa(\lambda) = \frac{\lambda(\kappa\lambda^2 - 2.5\lambda + 1)}{(1 - 0.2\lambda)(23\lambda^2 - 2.5\lambda + 1)}$$

Assume that the controller has access to the state vector and that the disturbance acts at the plant input. The non-minimum phase zeros relevant to this theory are given by the roots of $\kappa\lambda^2 - 2.5\lambda + 1$, as a function of κ. It is easy to see that for $\kappa > 3.5$ condition (8) is satisfied and the optimal controller is f^*. By applying the methods of [1], it can be shown that for $\kappa = 3.5$ the optimal solution is no longer unique. Actually two possible solutions with

$\|\phi_{opt}\|_1 = 7$ are:

$$\hat{\phi}_{opt_{\kappa=3.5}} = \begin{cases} \lambda - 2.5\lambda^2 + 3.5\lambda^3 \\ \lambda - 1.1\lambda^2 + 4.9\lambda^4 \end{cases}$$

The first is achieved with f^* while the second requires a first order controller. (The non-uniqueness is related to the occurrence of weakly redundant constraints in the linear program.) Note that for this value of κ, the left hand side of (8) is equal to one.

For $1.5 < \kappa < 3.5$ condition (8) is violated and the optimal solution has the following general form:

$$\hat{\phi}_{opt_{1.5<\kappa<3.5}} = \lambda + \phi_\kappa(2)\lambda^2 + \phi_\kappa(N_\kappa)\lambda^{N_\kappa}$$

As $\kappa \searrow 1.5$, one of the non-minimum phase zeros approaches the boundary of the unit disk while $\phi_\kappa(2) \longrightarrow -1.5$, $\phi_\kappa(N_\kappa) \longrightarrow 0.5$, and, most remarkably, $N_\kappa \nearrow \infty$. This implies, by theorem 4.3, that the optimal controller can have arbitrarily large order. For instance, if $\kappa = 1.51$, then

$$\hat{\phi}_{opt_{\kappa=1.51}} \simeq \lambda - 1.4907\lambda^2 + 0.5776\lambda^{12}$$

and the optimal compensator is of order 9. It is also interesting to point out that for $\kappa < 1.5$ one of the non-minimum phase zeros leaves the unit disk and condition (8) is again satisfied. In this case, $\hat{\phi}_{opt_{\kappa<1.5}} = \lambda - 1.5\lambda^2$ and $\hat{k}_{opt} = f^*$. With regard to the optimal norm, it drops from a value arbitrarily close but greater than 3 to a value of 2.5 in the transition.

Similar behavior has been reported in [10], for the case of sensitivity minimization through output feedback. The above example shows that the nature of such solutions have comparable characteristics even under full state feedback.

5 A Non-Singular Problem

So far we have considered problems with a scalar regulated output. One could argue that sensitivity minimization problems, such as the one in the above example, where a measure of the control effort is not included in the cost functional (i.e. singular problems), may have peculiar solutions that could hide the structure of the more general non-singular case. To clarify this point, we will consider a variation of the above example by including the control effort in the cost functional. That is,

$$\inf_{k-stab.} \left\| \begin{array}{c} \phi_1 \\ \gamma\, \phi_2 \end{array} \right\|_1 \stackrel{\text{def}}{=} \inf_{k-stab.} max(\|\phi_1\|_1, \gamma\, \|\phi_2\|_1) \tag{9}$$

where ϕ_1 represents the closed-loop map from the disturbance to the output of the plant, ϕ_2 represents the closed-loop map from the disturbance to the control input, and γ is a positive scalar weight. The fact that there are two regulated outputs and only a scalar control makes this problem of the *bad rank* class (see [2] and [8]). This implies that a linear programming formulation of the solution will have, in general, an infinite number of non-zero variables and active constraints (theorem 2.1 no longer holds) making the exact solutions difficult to obtain. For the following example, however, it can be shown that the optimal response has finite support, and that an exact solution can be computed ([11],[5]).

EXAMPLE 2: Consider problem (9) for the parameterized family of plants of example 1. If $\kappa = 2$ and $\gamma = 0.1$ then the exact ℓ_1-optimal solution can be shown to be

$$\hat{\phi}_1(\lambda) = \lambda - \frac{887}{558}\lambda^2 + \frac{631}{558}\lambda^4 + \frac{308}{558}\lambda^5$$

$$\hat{\phi}_2(\lambda) = -\frac{998.6}{558}\lambda + \frac{11895.4}{558}\lambda^2 + \frac{8955.4}{558}\lambda^3 + \frac{1282.2}{558}\lambda^4 - \frac{708.4}{558}\lambda^5$$

where $\|\phi_1\|_1 = \gamma\,\|\phi_2\| = 1192/279 \simeq 4.2724$, and the optimal controller is dynamic and of second order.

It is also interesting to consider the singular problem corresponding to this example (i.e. $\kappa = 2$ and $\gamma = 0$). The optimal solution is given by

$$\hat{\phi}_1(\lambda) = \lambda - \frac{90}{68}\lambda^2 + \frac{128}{68}\lambda^5$$

$$\hat{\phi}_2(\lambda) = -\frac{103.6}{68}\lambda + \frac{1446}{68}\lambda^2 + \frac{1394.4}{68}\lambda^3 + \frac{1136}{68}\lambda^4 - \frac{294.4}{68}\lambda^5$$

where $\|\phi_1\|_1 = 286/68 \simeq 4.2059$ while $\|\phi_2\|_1 = 4374.4/68 \simeq 64.3294$ is clearly larger. In fact, the above solution is valid for $\gamma \in [0, 286/4374.4]$ since for any γ in such interval $\|\phi_1\|_1 \geq \gamma\,\|\phi_2\|_1$. Moreover, for any such γ, the ℓ_1-optimal controller is dynamic and of second order since the optimal ϕ_1 is polynomial and of fifth order.

All this indicates that given a non-singular problem, the optimal controller may very well be dynamic, whether or not the two regulated outputs impose conflicting goals. Further, it can be shown that even when the corresponding singular problem has a static optimal controller, the non-singular problem may require a dynamic one.

6 Concluding Remarks

This paper presented a study of the ℓ_1 optimization problem for systems with full state feedback, scalar disturbance and scalar control. Two classes of problems were considered: a) singular problems with a scalar regulated output, and b) non-singular problems with two regulated outputs, one of them being the control sequence. The main purpose of the study was to determine whether or not there is always a static controller which is ℓ_1-optimal. First, it was shown that for a large class of relevant problems the ℓ_1-optimal controller is necessarily dynamic and that the order of the controller can be arbitrarily high. Although the systems in question were simple, it is safe to conclude that more complex MIMO state feedback ℓ_1 optimization problems will also have this characteristic in general. Second, it was shown that singular problems satisfying equation 8, which only involves the non-minimum phase zeros of the transfer function from the control to the regulated output, always have a static optimal controller, and that such gain can be easily computed without solving a standard ℓ_1 problem.

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