A Sufficient Condition for the Stability of Interval Matrix Polynomials

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Abstract

There is currently great interest in the root location of sets of scalar polynomials whose coefficients are confined to intervals, and the associated extension to eigenvalues of sets of constant matrices whose coefficients are contained in intervals. A central result for (complex) scalar interval polynomials is a theorem due to Kharitonov [1], which states that each member of a set of such polynomials is stable (or Hurwitz) if and only if *eight* special polynomials from the set are stable. In this note, we examine the case of *interval matrix polynomials*, and provide a Kharitonov-like result for what we term their *strong* stability. This in turn yields a sufficient condition for stability (in the usual sense) of a set of interval matrix polynomials.

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1 Introduction

There is currently great interest in the root location of sets of scalar polynomials whose coefficients are confined to intervals and the extension to sets of constant matrices with elements in intervals, e.g. [2, 3, 4, 5, 6, 7]. Some implications of these sets for robust control are described in [8]. In the scalar case, such sets of polynomials are referred to as *interval polynomials.* A central result concerning interval polynomials is a remarkable theorem due to Kharitonov [1], which states that every member of a set of such monic, complex coefficient, scalar, interval polynomials is Hurwitz or stable (i.e. has all roots in the open left half plane) if and only if *eight* specially chosen polynomials from the set are stable. In this note, we examine the case of polynomials whose *matrix* coefficients are confined to appropriate intervals. We term these polynomials *interval matrix polynomials.*

We begin in Section 2 with an overview and review of the scalar case. We present Kharitonov's result and provide some insight into the geometry of the situation. In Section 3, we discuss a generalization to the matrix case. Whereas most existing generalizations have been concerned with the case of monic *first-order* matrix polynomials, corresponding to the state-space system $\dot{x} = Ax$, our focus will be on the general-order case. While systems may usually be described in state-space form, many systems (e.g. lightly damped structures [9, 10, 11]) are most naturally represented by higher-order descriptions. We define a new and natural notion of matrix interval, together with a conservative notion of stability that we term *strong stability.* These definitions allow us to obtain a Kharitonov-like result for strong stability of a set of interval matrix polynomials. This result, in turn, yields a sufficient condition for (ordinary) stability of a set of interval matrix polynomials.

2 Interval Polynomials

Consider the set $\mathcal{N}$ of n-th degree, monic, complex coefficient polynomials of the form

$$\mathcal{N} = \left\{ s^n + (\alpha_1 + j\beta_1)s^{n-1} + \cdots + (\alpha_n + j\beta_n) \,|\, \alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i],\ \beta_i \in [\underline{\beta}_i, \overline{\beta}_i],\ \alpha_i, \beta_i \in R \right\} \quad (1)$$

which is said to be a set of *interval polynomials*. We term a set of polynomials stable when every member of the set is stable. We will review conditions for when the set $\mathcal{N}$ is stable. These conditions will be generalized in certain ways to the matrix case in Section 3.

We may consider the $2n$-tuple of coefficient components $(\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n)$ as a point in R^{2n}. The set $\mathcal{N}$ then defines a $2n$-dimensional hyper-box whose edges are oriented along the coordinate axes of the space, with each point of the box corresponding to a polynomial of $\mathcal{N}$ (see Figure 1) and conversely. For a given order n, we term the set of coefficient $2n$-tuples corresponding to stable polynomials the *stability domain*. Figure 1 shows the stability domain and a possible interval matrix set $\mathcal{N}$ for the case when $\mathcal{N}$ is composed of real second order polynomials, $p(s) = s^2 + \alpha_1 s + \alpha_2$.

Kharitonov's theorem [1] states that the set $\mathcal{N}$ is stable, i.e. the $2n$-dimensional box of polynomials represented by $\mathcal{N}$ (with 2^{2n} corners) is contained in the stability domain, if and only if the polynomials corresponding to 8 particular corners of the box are stable. In particular, these

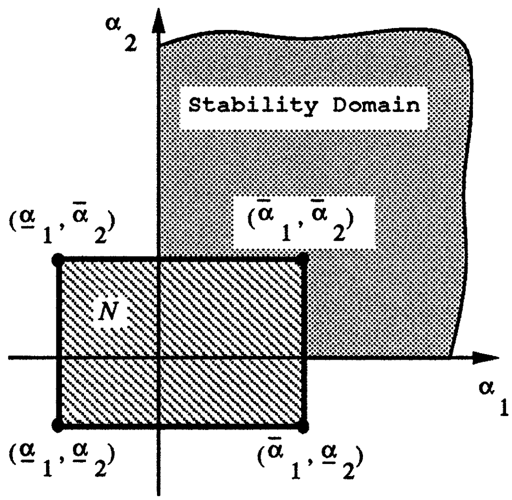

Figure 1: Second order example.

Kharitonov "corner" polynomials of the set $\mathcal{N}$ correspond to the following vectors of coefficients:

$$\begin{array}{lcl}
k_1(s) & \sim & [\ \underline{\alpha}_1 + j\underline{\beta}_1,\ \ \underline{\alpha}_2 + j\overline{\beta}_2,\ \ \overline{\alpha}_3 + j\overline{\beta}_3,\ \ \overline{\alpha}_4 + j\underline{\beta}_4,\ \ \underline{\alpha}_5 + j\underline{\beta}_5,\ \ \underline{\alpha}_6 + j\overline{\beta}_6,\ \ \cdots\ ] \\
k_2(s) & \sim & [\ \underline{\alpha}_1 + j\overline{\beta}_1,\ \ \overline{\alpha}_2 + j\overline{\beta}_2,\ \ \overline{\alpha}_3 + j\underline{\beta}_3,\ \ \underline{\alpha}_4 + j\underline{\beta}_4,\ \ \underline{\alpha}_5 + j\overline{\beta}_5,\ \ \overline{\alpha}_6 + j\overline{\beta}_6,\ \ \cdots\ ] \\
k_3(s) & \sim & [\ \overline{\alpha}_1 + j\underline{\beta}_1,\ \ \underline{\alpha}_2 + j\underline{\beta}_2,\ \ \underline{\alpha}_3 + j\overline{\beta}_3,\ \ \overline{\alpha}_4 + j\overline{\beta}_4,\ \ \overline{\alpha}_5 + j\underline{\beta}_5,\ \ \underline{\alpha}_6 + j\underline{\beta}_6,\ \ \cdots\ ] \\
k_4(s) & \sim & [\ \overline{\alpha}_1 + j\overline{\beta}_1,\ \ \overline{\alpha}_2 + j\underline{\beta}_2,\ \ \underline{\alpha}_3 + j\underline{\beta}_3,\ \ \underline{\alpha}_4 + j\overline{\beta}_4,\ \ \overline{\alpha}_5 + j\overline{\beta}_5,\ \ \overline{\alpha}_6 + j\underline{\beta}_6,\ \ \cdots\ ] \\
k_5(s) & \sim & [\ \underline{\alpha}_1 + j\underline{\beta}_1,\ \ \overline{\alpha}_2 + j\underline{\beta}_2,\ \ \overline{\alpha}_3 + j\overline{\beta}_3,\ \ \underline{\alpha}_4 + j\overline{\beta}_4,\ \ \underline{\alpha}_5 + j\underline{\beta}_5,\ \ \overline{\alpha}_6 + j\underline{\beta}_6,\ \ \cdots\ ] \\
k_6(s) & \sim & [\ \underline{\alpha}_1 + j\overline{\beta}_1,\ \ \underline{\alpha}_2 + j\underline{\beta}_2,\ \ \overline{\alpha}_3 + j\underline{\beta}_3,\ \ \overline{\alpha}_4 + j\overline{\beta}_4,\ \ \underline{\alpha}_5 + j\overline{\beta}_5,\ \ \underline{\alpha}_6 + j\underline{\beta}_6,\ \ \cdots\ ] \\
k_7(s) & \sim & [\ \overline{\alpha}_1 + j\underline{\beta}_1,\ \ \overline{\alpha}_2 + j\overline{\beta}_2,\ \ \underline{\alpha}_3 + j\overline{\beta}_3,\ \ \underline{\alpha}_4 + j\underline{\beta}_4,\ \ \overline{\alpha}_5 + j\underline{\beta}_5,\ \ \overline{\alpha}_6 + j\overline{\beta}_6,\ \ \cdots\ ] \\
k_8(s) & \sim & [\ \overline{\alpha}_1 + j\overline{\beta}_1,\ \ \underline{\alpha}_2 + j\overline{\beta}_2,\ \ \underline{\alpha}_3 + j\underline{\beta}_3,\ \ \overline{\alpha}_4 + j\underline{\beta}_4,\ \ \overline{\alpha}_5 + j\overline{\beta}_5,\ \ \underline{\alpha}_6 + j\overline{\beta}_6,\ \ \cdots\ ]
\end{array} \tag{2}$$

If we denote these Kharitonov corner polynomials by $\mathcal{N}_K$, then Kharitonov's result concerning stability of the set $\mathcal{N}$ is the following:

Theorem 1 (Kharitonov) *The set $\mathcal{N}$ is stable if and only if the set $\mathcal{N}_K$ is stable.*

The original work [1] is in Russian and difficult to understand. An elementary and insightful proof of this result may be found in [12].

We now consider some implications of Theorem 1 for the convexity of certain subsets of the stability domain, as these will guide our later extensions. The stability domain for complex monic polynomials of degree greater than 1 is not convex [13]. In spite of this, Theorem 1 implies that the stability domain *is* convex to perturbations of single coefficient elements (since single-element changes correspond to a 1-dimensional interval box $\mathcal{N}$ with only 2 "corners"). In terms of the coefficient parameter space, such a single element interval set, parallel to the coordinate axes, has a convex intersection with the stability domain. Thus, the coordinate directions appear to be special. Further, since the 8 polynomials in $\mathcal{N}_K$ are the essential ones for stability of the set $\mathcal{N}$, these polynomials must indicate the critical or "narrow" directions of the stability domain, with most of the directions not being binding. These points are illustrated schematically in Figure 2.

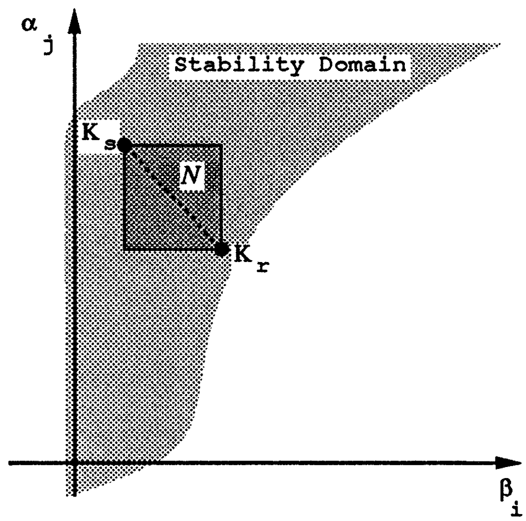

Figure 2: Coordinate convexity.

3 Generalization to Interval Matrix Polynomials

3.1 Matrix Polynomials

We now study complex coefficient monic matrix polynomials of the following form:

$$P(s) = Is^n + P_1 s^{n-1} + P_2 s^{n-2} + \cdots + P_n \tag{3}$$

where the P_i are $m \times m$, possibly complex matrices. The latent roots and associated latent vectors of $P(s)$ are defined as the solutions, λ_i and u_i, to the equation $P(\lambda_i)u_i = 0$, where u_i is a unit vector (without loss of generality). Analogously to the scalar case, the matrix polynomial (3) is termed stable if all of its latent roots lie in the open left half plane. As before, we also term a set of such matrix polynomials stable when every member of the set is stable. The $2nm^2$-tuple of elements corresponding to the real and imaginary parts of the entries of the coefficient matrices P_i may be considered as a point in a $2nm^2$-dimensional space, analogously to the scalar polynomial case. The stability domain is now taken as the region of this space corresponding to combinations of matrices $(P_0, \ldots, P_n)$ that produce stable matrix polynomials $P(s)$. Note that we may uniquely decompose each coefficient P_i as $P_i = \mathcal{A}_i + j\mathcal{B}_i$, where $\mathcal{A}_i = (P_i + P_i^*)/2$ and $\mathcal{B}_i = (P_i - P_i^*)/2j$ are Hermitian matrices (and jB_i is skew-Hermitian). The components $\mathcal{A}_i$ and $\mathcal{B}_i$ can be thought of as serving the role of the real and imaginary parts of the coefficient P_i.

3.2 Interval Sets for Matrices

Here we generalize the concept of the interval set $\mathcal{N}$ to the matrix case in an appropriate way. To do this we need to define precisely what we mean by inclusion of a matrix coefficient in an interval. Unlike the scalar case, there are a variety of senses in which a matrix A may be considered included in an interval defined by two other matrices, $\underline{A}$ and $\overline{A}$. Certainly one sense is elementwise inclusion, i.e. for a real matrix A, we write $A \in [\underline{A}, \overline{A}]$ if $\underline{A}_{k\ell} \leq A_{k\ell} \leq \overline{A}_{k\ell}$, where $A_{k\ell}$ is element (k, ℓ) of A, and similarly for $\underline{A}_{k\ell}$ and $\overline{A}_{k\ell}$ [14, 15, 5]. In the parameter space, the resulting interval matrix sets will be the Cartesian product of the corresponding component intervals. These interval matrix sets will again be boxes, as shown in Figure 3. While this definition produces a box (with 2^{nm^2} corners!), the box is specified in terms of the matrix *elements* and so does not lead to results recreating the flavor of the scalar ones.

The preceding element-based definition is used almost universally in the current interval matrix

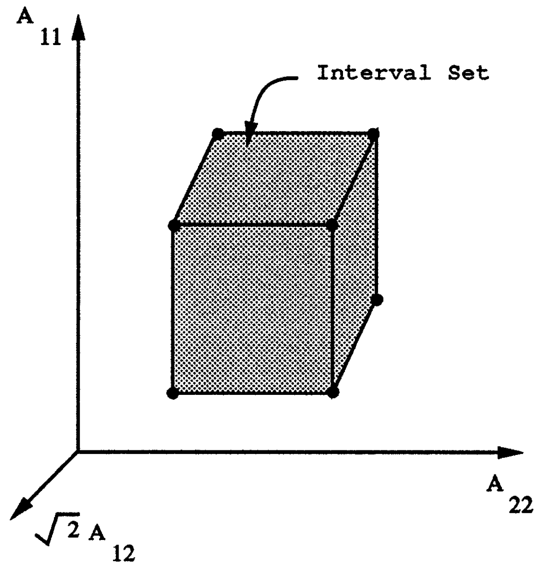

Figure 3: An interval matrix box.

literature. Although it is a fruitful definition for special classes of matrices, such as positive matrices [2], it has not yielded a necessary and sufficient condition comparable to Theorem 1 [16, 7]. In fact, many natural conjectures turn out to be false [17, 18]. As a consequence, existing results typically provide *sufficiency* conditions for the stability of monic *first-order* interval matrix polynomials [14, 15].

Instead of the above element-based definition of matrix intervals, we propose more fundamentally matrix-based ones. For example, we might consider the commonly used sense of inclusion based on the notion of positive definiteness. Here the Hermitian matrix A is considered to be in the interval defined by $\underline{A}$ and $\overline{A}$ if $(A - \underline{A})$ and $(\overline{A} - A)$ are positive semi-definite (PSD) [19]; the interval exists if and only if $\overline{A} - \underline{A}$ is PSD. Still another idea is found by considering convex combinations of the end points, $\underline{A}$ and $\overline{A}$. In this definition, A is considered to lie in the interval defined by $\underline{A}$ and $\overline{A}$ when $A = \gamma\underline{A} + (1-\gamma)\overline{A}$ for some $0 \leq \gamma \leq 1$. These two notions share the common property that $A \in [\underline{A}, \overline{A}]$ implies the scalar relation $u^*\underline{A}u \leq u^*Au \leq u^*\overline{A}u$ for all complex unit vectors u. This will turn out to be the basic property that we desire in a definition of matrix interval sets, providing a tie between the matrix and scalar cases.

We will term an interval constraint $A \in [\underline{A}, \overline{A}]$ on the Hermitian matrix A to be a *quadratic interval* constraint if, for each complex unit vector u, $\{u^*Au \mid A \in [\underline{A}, \overline{A}]\}$ equals the closed interval whose endpoints are $u^*\underline{A}u$ and $u^*\overline{A}u$. In particular note that this type of matrix interval implies that $u^*\underline{A}u \leq u^*Au \leq u^*\overline{A}u$ for any u. We restrict ourselves in the rest of this paper to such quadratic intervals of matrices.

3.3 Interval Matrix Polynomials

We may now define the interval matrix polynomial set $\mathcal{M}$ used in the rest of this work as follows:

Definition 1 (Interval Matrix Set $\mathcal{M}$) *Let $\mathcal{M}$ denote the set of monic, n-th degree, complex coefficient, matrix polynomials of the form:*

$$\mathcal{M} = \left\{Is^n + (\mathcal{A}_1 + j\mathcal{B}_1)s^{n-1} + \cdots + (\mathcal{A}_n + j\mathcal{B}_n) \mid \mathcal{A}_i \in [\underline{\mathcal{A}}_i, \overline{\mathcal{A}}_i],\ \mathcal{B}_i \in [\underline{\mathcal{B}}_i, \overline{\mathcal{B}}_i]\right\} \tag{4}$$

where $\mathcal{A}_i$ and $\mathcal{B}_i$ are Hermitian matrices satisfying quadratic interval constraints, with $\mathcal{A}_i$ and $j\mathcal{B}_i$ being, respectively, the Hermitian and skew-Hermitian parts of the matrix coefficient P_i.

The matrix set $\mathcal{M}$ is our generalization of the scalar interval polynomial set $\mathcal{N}$. Note that $\mathcal{M}$ again defines a box, this time with respect to axes defined by the chosen quadratic interval constraint. The corners of this box are defined directly in terms of the extremes of the Hermitian and skew-Hermitian parts of the coefficient matrices P_i. Thus these matrix components naturally generalize the role played by the real and imaginary coefficient components in the scalar case. In particular, this matrix box has 2^{2n} corners, exactly like the scalar case, and independent of the matrix coefficient size m.

In the parameter space, the interval sets $\mathcal{M}$ will be the Cartesian product of the corresponding component intervals. The form of these component intervals will vary greatly, depending on what quadratic interval constraint is chosen. For example, for the case of a PSD interval, where we allow A such that $(A-\underline{A})$ and $(\overline{A}-A)$ are both PSD, the resulting component interval is the intersection of 2 cones, as shown for the 2×2 case in Figure 4. In contrast, if we choose the definition of intervals where $A=\gamma\underline{A}+(1-\gamma)\overline{A}$ for some $0\leq\gamma\leq 1$, we obtain a line in parameter space (see Figure 5 for the 2×2 case). The resulting interval is "thin" but we no longer have any restrictions on the end points, $\overline{A}$ and $\underline{A}$. The overall set $\mathcal{M}$ is then the Cartesian product of such component interval sets.

3.4 Strongly Stable Systems

Now that we have a natural notion of the matrix interval polynomial set $\mathcal{M}$ as given by (4), we need an associated definition of stability. Obtaining exact yet simple necessary and sufficient conditions

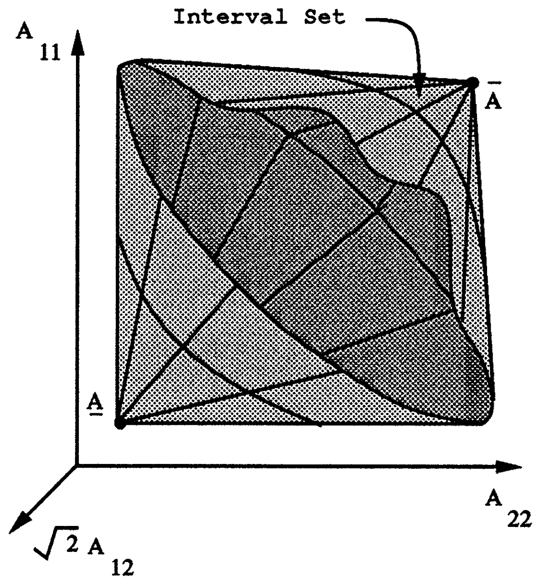

Figure 4: A positive definite interval.

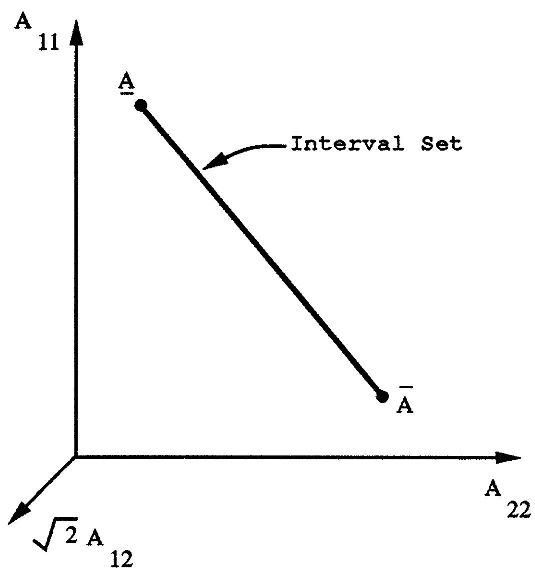

Figure 5: A convex combination interval.

for the stability (in the usual sense) of the set $\mathcal{M}$ directly in terms of the coefficients P_i appears extremely difficult, if not impossible. In what follows we focus on preserving the simplicity of the scalar result in Theorem 1 at the cost of a more restrictive, though still useful, notion of stability.

Definition 2 (Strongly Stable Matrix Polynomials) *The matrix polynomial $P(s)$ of (3) is termed strongly stable if the following derived scalar equation is stable for all unit-magnitude complex vectors u:*

$$p(u,s) = u^* P(s) u = s^n + p_1(u)s^{n-1} + p_2(u)s^{n-2} + \ldots + p_n(u) \qquad \textit{where } p_i(u) = u^* P_i u. \tag{5}$$

A set of matrix polynomials is termed strongly stable if all of its elements are strongly stable.

The reason for the terminology is that stability of $p(s,u)$ for all u is only a sufficient condition for stability of $P(s)$. Thus, strongly stable matrix polynomials are stable, but not vice versa.

3.5 A Kharitonov Result for Strong Stability of Interval Matrix Polynomials

With the concepts above, we may now invoke Kharitonov's scalar polynomial result to derive useful results for the *matrix* case. We show that working with the Hermitian and skew-Hermitian parts of the coefficient matrices for the set of strongly stable matrix polynomials produces results similar to those found for the scalar polynomial case. In particular, we show that the set $\mathcal{M}$ in (4) is strongly stable if and only if 8 corners of the set are strongly stable.

Consider the application of Definition 2 to the interval matrix polynomial set $\mathcal{M}$ given in (4).

For each fixed unit vector u, let $\underline{\alpha}_i(u) \equiv u^* \underline{\mathcal{A}}\, u$, $\overline{\alpha}_i(u) \equiv u^* \overline{\mathcal{A}}\, u$ and similarly for $\underline{\beta}_i(u)$, $\overline{\beta}_i(u)$. For each such u, define the set of scalar interval polynomials $\mathcal{N}_{\mathcal{M}}(u)$ associated with the set $\mathcal{M}$ as follows:

$$\mathcal{N}_{\mathcal{M}}(u) = \left\{ s^n + (\alpha_1 + j\beta_1)s^{n-1} + \cdots + (\alpha_n + j\beta_n) \mid \alpha_i \in [\underline{\alpha}_i(u), \overline{\alpha}_i(u)],\ \beta_i \in [\underline{\beta}_i(u), \overline{\beta}_i(u)] \right\} \quad (6)$$

The interval matrix set $\mathcal{M}$ is evidently strongly stable if and only if the scalar sets $\mathcal{N}_{\mathcal{M}}(u)$ are stable for all unit vectors u. Now, consider the 8 generalized Kharitonov corner matrix polynomials of the set $\mathcal{M}$ that correspond to the following vectors of coefficients:

$$\begin{array}{lcl}
K_1(s) & \sim & [\ \underline{\mathcal{A}}_1 + j\underline{\mathcal{B}}_1,\ \ \underline{\mathcal{A}}_2 + j\overline{\mathcal{B}}_2,\ \ \overline{\mathcal{A}}_3 + j\overline{\mathcal{B}}_3,\ \ \overline{\mathcal{A}}_4 + j\underline{\mathcal{B}}_4,\ \ \underline{\mathcal{A}}_5 + j\underline{\mathcal{B}}_5,\ \ \underline{\mathcal{A}}_6 + j\overline{\mathcal{B}}_6,\ \ \cdots\ ] \\
K_2(s) & \sim & [\ \underline{\mathcal{A}}_1 + j\overline{\mathcal{B}}_1,\ \ \overline{\mathcal{A}}_2 + j\overline{\mathcal{B}}_2,\ \ \overline{\mathcal{A}}_3 + j\underline{\mathcal{B}}_3,\ \ \underline{\mathcal{A}}_4 + j\underline{\mathcal{B}}_4,\ \ \underline{\mathcal{A}}_5 + j\overline{\mathcal{B}}_5,\ \ \overline{\mathcal{A}}_6 + j\overline{\mathcal{B}}_6,\ \ \cdots\ ] \\
K_3(s) & \sim & [\ \overline{\mathcal{A}}_1 + j\underline{\mathcal{B}}_1,\ \ \underline{\mathcal{A}}_2 + j\underline{\mathcal{B}}_2,\ \ \underline{\mathcal{A}}_3 + j\overline{\mathcal{B}}_3,\ \ \overline{\mathcal{A}}_4 + j\overline{\mathcal{B}}_4,\ \ \overline{\mathcal{A}}_5 + j\underline{\mathcal{B}}_5,\ \ \underline{\mathcal{A}}_6 + j\underline{\mathcal{B}}_6,\ \ \cdots\ ] \\
K_4(s) & \sim & [\ \overline{\mathcal{A}}_1 + j\overline{\mathcal{B}}_1,\ \ \overline{\mathcal{A}}_2 + j\underline{\mathcal{B}}_2,\ \ \underline{\mathcal{A}}_3 + j\underline{\mathcal{B}}_3,\ \ \underline{\mathcal{A}}_4 + j\overline{\mathcal{B}}_4,\ \ \overline{\mathcal{A}}_5 + j\overline{\mathcal{B}}_5,\ \ \overline{\mathcal{A}}_6 + j\underline{\mathcal{B}}_6,\ \ \cdots\ ] \\
K_5(s) & \sim & [\ \underline{\mathcal{A}}_1 + j\underline{\mathcal{B}}_1,\ \ \overline{\mathcal{A}}_2 + j\underline{\mathcal{B}}_2,\ \ \overline{\mathcal{A}}_3 + j\overline{\mathcal{B}}_3,\ \ \underline{\mathcal{A}}_4 + j\overline{\mathcal{B}}_4,\ \ \underline{\mathcal{A}}_5 + j\underline{\mathcal{B}}_5,\ \ \overline{\mathcal{A}}_6 + j\underline{\mathcal{B}}_6,\ \ \cdots\ ] \\
K_6(s) & \sim & [\ \underline{\mathcal{A}}_1 + j\overline{\mathcal{B}}_1,\ \ \underline{\mathcal{A}}_2 + j\underline{\mathcal{B}}_2,\ \ \overline{\mathcal{A}}_3 + j\underline{\mathcal{B}}_3,\ \ \overline{\mathcal{A}}_4 + j\overline{\mathcal{B}}_4,\ \ \underline{\mathcal{A}}_5 + j\overline{\mathcal{B}}_5,\ \ \underline{\mathcal{A}}_6 + j\underline{\mathcal{B}}_6,\ \ \cdots\ ] \\
K_7(s) & \sim & [\ \overline{\mathcal{A}}_1 + j\underline{\mathcal{B}}_1,\ \ \overline{\mathcal{A}}_2 + j\overline{\mathcal{B}}_2,\ \ \underline{\mathcal{A}}_3 + j\overline{\mathcal{B}}_3,\ \ \underline{\mathcal{A}}_4 + j\underline{\mathcal{B}}_4,\ \ \overline{\mathcal{A}}_5 + j\underline{\mathcal{B}}_5,\ \ \overline{\mathcal{A}}_6 + j\overline{\mathcal{B}}_6,\ \ \cdots\ ] \\
K_8(s) & \sim & [\ \overline{\mathcal{A}}_1 + j\overline{\mathcal{B}}_1,\ \ \underline{\mathcal{A}}_2 + j\overline{\mathcal{B}}_2,\ \ \underline{\mathcal{A}}_3 + j\underline{\mathcal{B}}_3,\ \ \overline{\mathcal{A}}_4 + j\underline{\mathcal{B}}_4,\ \ \overline{\mathcal{A}}_5 + j\overline{\mathcal{B}}_5,\ \ \underline{\mathcal{A}}_6 + j\overline{\mathcal{B}}_6,\ \ \cdots\ ]
\end{array} \quad (7)$$

For a fixed u we know from Kharitonov's Theorem that the scalar interval polynomial set $\mathcal{N}_{\mathcal{M}}(u)$ is stable if and only if the 8 scalar polynomials given by $u^* K_i(s)\, u$ are stable. Thus, the set $\mathcal{M}$ is strongly stable if and only if all of the $u^* K_i(s)\, u$ are stable for each u.

Let $\mathcal{M}_K$ denote the 8 generalized Kharitonov corner matrix polynomials in (7). We then have that the matrix interval set $\mathcal{M}$ is strongly stable if and only if the corner set $\mathcal{M}_K$ is strongly stable. This is the sought for generalization of Kharitonov's Theorem:

Theorem 2 (Strong Stability of Matrix Polynomials) *The set $\mathcal{M}$ is strongly stable if and only if the set $\mathcal{M}_K$ is strongly stable.*

Corollary 1 (Stability of Interval Matrix Polynomials) *The set $\mathcal{M}$ is stable if the eight elements of the set $\mathcal{M}_K$ are strongly stable.*

The Hermitian and skew-Hermitian parts, $\mathcal{A}_i$ and $j\mathcal{B}_i$, of the matrix coefficients P_i are the basic building blocks of the set of strongly stable systems, and they play the role of the coefficient coordinate axes. The coordinate convexity observed in the scalar polynomial case can now be generalized as follows:

Corollary 2 (Coordinate Convexity) *The domain of strongly stable matrix polynomials under single matrix coefficient component perturbation is convex with respect to any quadratic interval definition applied to that component (the Hermitian and skew-Hermitian parts of a coefficient are considered to be separate components).*

4 Concluding Comments

The conditions (5) for a system to be strongly stable are stated in terms of a test over *all* unit vectors u. Simpler *sufficiency* conditions for a matrix polynomial to be strongly stable may sometimes be stated directly in terms of the coefficient matrices, by combining properties of the components $\mathcal{A}_i$ and $\mathcal{B}_i$ with stability conditions for the corresponding scalar polynomial. For example, consider the case of a second-order matrix polynomial in (3), with $n = 2$:

$$P(s) = Is^2 + (\mathcal{A}_1 + j\mathcal{B}_1)\, s^2 + (\mathcal{A}_2 + j\mathcal{B}_2) \tag{8}$$

The corresponding second-order *scalar* polynomial is

$$p(u, s) = s^2 + (\alpha_1 + j\beta_1)s + (\alpha_2 + j\beta_2)$$

and is stable (e.g. from the Routh-Hurwitz criterion) if and only if

$$\begin{aligned} \alpha_1\beta_1\beta_2 + \alpha_1^2\alpha_2 - \beta_2^2 &> 0 \\ \alpha_1 &> 0 \end{aligned} \tag{9}$$

Combining the above observations with Definition 2, we may show that (8) is strongly stable if:

$$\begin{aligned} \underline{\lambda}^2(\mathcal{A}_1)\underline{\lambda}(\mathcal{A}_2) &> \overline{\lambda}^2(\mathcal{B}_2) \\ \underline{\lambda}(\mathcal{A}_1) &> 0 \\ \mathcal{B}_1 &= 0 \end{aligned} \tag{10}$$

where $\underline{\lambda}(\cdot)$, $\overline{\lambda}(\cdot)$ denote the minimum and maximum eigenvalue of the argument, respectively. Here we have used Rayleigh quotient bounds on the quadratic form of a Hermitian matrix $\mathcal{A}$, $\underline{\lambda}(\mathcal{A}) \leq u^*\mathcal{A}u \leq \overline{\lambda}(\mathcal{A})$. In a forthcoming note [20] we exploit such conditions to obtain guidelines for stability analysis and control design for second-order matrix systems.

All our results have used the concept of strongly stable systems. We have been unable to show similar results when strong stability is replaced by stability in the usual sense. It is not yet clear how restrictive the concept of strong stability is, though the indications are that it is not overly so. For instance, consider the case where the coefficients $P_i = \mathcal{A}_i + j\mathcal{B}_i$ in (8) are real, so that $\mathcal{A}_i$ and $j\mathcal{B}_i$ are real and contain the symmetric and skew-symmetric parts of the coefficient P_i respectively. This situation often arises in problems of classical mechanics, aerodynamics, and robotic systems [13, 10, 11, 21, 22]. The matrices P_1 and P_2 are then usually known as the damping and stiffness matrices, respectively, and reflect physical properties of the structure under consideration. A classical result for these types of systems is the Kelvin-Tait-Chetaev (KTC) Theorem [23, 24]. This result states that if $\mathcal{B}_2 = 0$ and $\mathcal{A}_2$ is positive definite (so P_2 is symmetric and positive definite) then the system (8) is stable if and only if $\mathcal{A}_1$ (the symmetric part of P_1) is positive definite.

Applying our Definition 2 to this case where $\mathcal{B}_2 = 0$ and $\mathcal{A}_2$ is positive definite, we find that (8) is strongly stable if and only if $\mathcal{A}_1$ is positive definite. In particular, this implies that (8) is stable if $\mathcal{A}_1$ is positive definite. Thus, for this classical case of real interest our sufficiency condition for stability given in Definition 2 actually coincides with the true condition for stability. Other results of this type have appeared in the literature [9] and suggest that the notion of strongly stability is not overly restrictive, at least for many systems of interest.

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