Perturbative renormalization of proton observables in lattice QCD using Domain Wall fermions

by

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Abstract

Deep inelastic scattering unambiguously measures hadron observables characterizing the quark-gluon structure of hadrons. The only way to calculate these observables from first principles is lattice QCD. Experiments measure matrix elements of light cone operators $\left\langle P\left|\bar{\psi}\left(\frac{x}{2}\right)\,\Gamma\,\exp\left(\int_{-x/2}^{x/2}A_\mu(y)\,dy^\mu\right)\psi\left(-\frac{x}{2}\right)\right|P'\right\rangle$ where diagonal elements specify the quark density distribution $q(x)$, quark helicity distribution $\Delta q(x)$ and quark transversity distribution $\delta q(x)$. Off-diagonal elements determine form factors and general parton distributions. Due to the Minkowskian nature of these matrix elements, they cannot be evaluated on a Euclidean lattice so one uses the operator product expansion to calculate matrix elements $\langle P|\bar{\psi}(0)\;\Gamma\,D^{\mu_1}\dots D^{\mu_n}\psi(0)|P'\rangle$ which specify moments of these distributions.

In this thesis, renormalization factors have been calculated for local bilinear operators of the form $\bar{\psi}\Gamma D^{\mu_1}\dots D^{\mu_3}\psi$ in a given irreducible representation of hypercubic group as well as mixing coefficients of those operators for low moments of physical interest. In the past, it was only possible to calculate with quark masses such that $m_\pi > 500\text{MeV}$. Now for the first time using Ginsparg-Wilson "Domain Wall" fermions with HYP smearing and full QCD configurations on large lattices make calculations possible in the physical chiral regime.

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Chapter 1

Introduction

In deep inelastic scattering experiments, one can unambiguously measure hadron observables characterizing the quark-gluon structure of hadrons. However, due to the complicated structure of QCD, the only known way to calculate these observables from first principles is lattice QCD. To calculate those observables, the bare operators defined on the lattice need to be renormalized in order to compare results with experiments. The goal of this thesis is to evaluate those renormalization coefficients for Domain Wall (DW) fermions.

1.1 Moments of nucleon light cone quark distributions

This section reviews the theoretical framework for calculating moments of nucleon light cone distributions following the notation and presentations in [1]. Experiments measure matrix elements of light cone operators

$$\left\langle P \middle| \bar{\psi}\left(\frac{x}{2}\right) \Gamma \exp\left(\int_{-x/2}^{x/2} A_\mu(y)\, dy^\mu\right) \psi\left(-\frac{x}{2}\right) \middle| P' \right\rangle \ . \tag{1.1}$$

Due to the Minkowskian nature of these matrix elements, they cannot be evaluated on a Euclidean lattice. By the operator product expansion, moments of the linear combinations of quark and antiquark distributions in the proton

$$\langle x^n \rangle_q \ = \ \int_0^1 \mathrm{d}x x^n (q(x) + (-1)^{n+1} \bar{q}(x)) \tag{1.2}$$

$$\langle x^n\rangle_{\Delta q} = \int_0^1 dx x^n(\Delta q(x)+(-1)^n\Delta\bar{q}(x))$$
$$\langle x^n\rangle_{\delta q} = \int_0^1 dx x^n(\delta q(x)+(-1)^{n+1}\delta\bar{q}(x)),$$

where the quark density, helicity, and transversity distributions

$$\begin{aligned} q &= q_\uparrow + q_\downarrow \\ \Delta q &= q_\uparrow - q_\downarrow \\ \delta q &= q_\top - q_\perp, \end{aligned} \tag{1.3}$$

are related to the following matrix elements of twist-2 operators

$$2\langle x^{n-1}\rangle_{q_r} P_{\mu_1}\cdots P_{\mu_n} \equiv \frac{1}{2}\sum_S \langle PS|\left(\frac{i}{2}\right)^{n-1}\bar{\psi}^r\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\psi^r|PS\rangle \tag{1.4}$$
$$\frac{2}{n+1}\langle x^n\rangle_{\Delta q_r} S_{\{\sigma}P_{\mu_1}\cdots P_{\mu_n\}} \equiv -\langle PS|\left(\frac{i}{2}\right)^{n}\bar{\psi}^r\gamma_5\gamma_{\{\sigma}\overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_n\}}\psi^r|PS\rangle$$
$$\frac{2}{m_N}\langle x^n\rangle_{\delta q_r} S_{[\mu}P_{\{\nu]}P_{\mu_1}\cdots P_{\mu_n\}} \equiv \langle PS|\left(\frac{i}{2}\right)^{n}\bar{\psi}^r\gamma_5\sigma_{\mu\{\nu}\overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_n\}}\psi^r|PS\rangle,$$

Here, $\overleftrightarrow{D} \equiv \overrightarrow{D} - \overleftarrow{D}$, r denotes the quark flavor, x denotes the momentum fraction carried by the quark, $S^2 = m_N^2$, $\{\,\}$ and $[\,]$ denote symmetrization and anti-symmetrization respectively, and the mixed symmetry $[\{\,]\}$ term is first symmetrized and then anti-symmetrized so that it is written explicitly as

$$O^5_{[\sigma\{\mu_1]\cdots\mu_n\}} \equiv \frac{1}{n+1}\left(O^5_{\sigma\mu_1\mu_2\ldots\mu_n} - O^5_{\mu_1\sigma\mu 2\cdots\mu_n} + O^5_{\sigma\mu_2\mu_1\cdots\mu_n} - O^5_{\mu_1\mu_2\sigma\cdots\mu_n} + \cdots\right). \tag{1.5}$$

We note that the odd moments $\langle x^n\rangle_q$ are obtained from the spin-independent structure functions F_1 or F_2 measured in deep inelastic electron or muon scattering

$$\int_0^1 dx x^{n-1} F_1(x,Q^2) = \frac{1}{2}C_n^v(Q^2/\mu^2)\sum_r e_r^2\langle x^{n-1}\rangle_{q_r}(\mu) \tag{1.6}$$
$$\int_0^1 dx x^{n-2} F_2(x,Q^2) = C_n^v(Q^2/\mu^2)\sum_r e_r^2\langle x^{n-1}\rangle_{q_r}(\mu),$$

and even moments of $\langle x^n \rangle_{\Delta q}$ are determined from the spin-dependent structure function g_1

$$\int_0^1 dx x^n g_1(x,Q^2) = \frac{1}{4} C_n^a(Q^2/\mu^2) \sum_r e_r^2 2\langle x^n \rangle_{\Delta q_r}(\mu), \tag{1.7}$$

where e_r is the quark's electric charge, and C_n denotes the Wilson coefficient. Note that the moments $\langle x^n \rangle_q$ and $\langle x^n \rangle_{\Delta q}$ are proportional to the quantities v_{n+1} and a_n defined in Ref. [10]

$$\begin{aligned} \langle x^n \rangle_q &= v_{n+1}^{(q)} \\ \langle x^n \rangle_{\Delta q} &= \frac{1}{2} a_n^{(q)}. \end{aligned} \tag{1.8}$$

In addition, the two spin-dependent structure functions g_1 and g_2 also determine the quantity d_n

$$\frac{1}{n+1} d_n^r S_{[\sigma} P_{\{\mu_1]} \cdots P_{\mu_n\}} \equiv -\langle PS| \left(\frac{i}{2}\right)^n \bar{\psi}^r \gamma_5 \gamma_{[\sigma} \overleftrightarrow{D}_{\{\mu_1]} \cdots \overleftrightarrow{D}_{\mu_n\}} \psi^r |PS\rangle \tag{1.9}$$

which is a twist-three operator and does not have a simple interpretation in terms of parton distribution functions. For Wilson fermions, $\gamma_5 \gamma_{[\sigma} \overleftrightarrow{D}_{\{\mu_1]} \cdots \overleftrightarrow{D}_{\mu_n\}}$ mixes with the lower dimension operator $\frac{1}{a} \gamma_5 \gamma_{[\sigma} \gamma_{\{\mu_1]} \cdots \overleftrightarrow{D}_{\mu_n\}}$, so it is not possible to compare with phenomenological results using the perturbative renormalization constants and mixing coefficients. However, for Domain Wall fermions chiral symmetry protects these operators from mixing.

1.2 Lattice operators

Our objective is to calculate matrix elements of traceless, appropriately symmetrized and anti-symmetrized operators of the general form

$$O \equiv \bar{\psi} \Gamma \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} \psi, \tag{1.10}$$

where $\Gamma = 1, \gamma_5$, or $\gamma_5 \gamma_\sigma$, on a hypercubic lattice to approximate the corresponding continuum operators as accurately as possible. Hence we choose representations of the hypercubic group, H(4) [3], to eliminate operator mixing as much as possible, and after fulfilling this objective, to minimize statistical errors by including as few nonzero components of the nucleon momentum as possible.

Since H(4) is a subgroup of the Lorentz group, irreducible representations of the Lorentz group are in general reducible under the H(4) group, and we choose the representation to optimize the approximation. It is essential to choose a representation that does not mix with lower dimension operators, since the coefficients would increase as $1/a^n$ in the continuum limit. In addition, because of the possible inaccuracy of perturbative mixing coefficients and the difficulty of determining mixing coefficients nonperturbatively, it is desirable to avoid mixing with operators of the same dimension as well. In choosing between operators with the same mixing properties, it is desirable to use a nucleon source with as few non-zero spatial momentum components as possible, since each projection introduces substantial stochastic noise. Since any expectation value of an operator with tensor index j is proportional to P_j (or S_j for spin dependent quark distributions), the nucleon must have an additional momentum component projection for each new distinct tensor index that is added to an operator. Hence, the goal is to limit the number of distinct spatial indices. Eventually, as one proceeds to higher moments of quark distributions, all the space-time indices are exhausted and it becomes impossible to avoid mixing with lower dimension operators.

The representations we have chosen for our operators using these criteria are enumerated in Table 1.1. To illustrate the selection process, we describe selection of the spin-independent operators and analogous analysis yields the remaining operators.

To measure $\langle x \rangle_q$ one needs to calculate matrix elements of the traceless part of the operator $\bar{q}\gamma_{\{\mu}\overleftrightarrow{D}_{\nu\}}q$, which belongs to the representation (1,1) in the continuum decomposition

$$(\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) = (0,0) \oplus (1,0) \oplus (0,1) \oplus (1,1).$$

On the lattice, the nine dimensional representation (1,1) splits into two irreducible representations, $\mathbf{3}_1^+$ and $\mathbf{6}_3^+$, both of which are symmetric and traceless, where the notation for representations is described in the caption of Table 1.1. As a consistency check, it is desir-

observable	H(4)	mixing	$\vec{P}$	lattice operator
$\langle x\rangle_q^{(a)}$	$\mathbf{6}_3^+$	no	1	$\bar{q}\gamma_{\{1}\overleftrightarrow{D}_{4\}}q$
$\langle x\rangle_q^{(b)}$	$\mathbf{3}_1^+$	no	0	$\bar{q}\gamma_4\overleftrightarrow{D}_4q-\frac{1}{3}(\bar{q}\gamma_1\overleftrightarrow{D}_1q+\bar{q}\gamma_2\overleftrightarrow{D}_2q+\bar{q}\gamma_3\overleftrightarrow{D}_3q)$
$\langle x^2\rangle_q$	$\mathbf{8}_1^-$	yes	1	$\bar{q}\gamma_{\{1}\overleftrightarrow{D}_1\overleftrightarrow{D}_{4\}}q-\frac{1}{2}\bar{q}(\gamma_{\{2}\overleftrightarrow{D}_2\overleftrightarrow{D}_{4\}}+\gamma_{\{3}\overleftrightarrow{D}_3\overleftrightarrow{D}_{4\}})q$
$\langle x^3\rangle_q$	$\mathbf{2}_1^+$	no*	1	$\bar{q}\gamma_{\{1}\overleftrightarrow{D}_1\overleftrightarrow{D}_4\overleftrightarrow{D}_{4\}}q+\bar{q}\gamma_{\{2}\overleftrightarrow{D}_2\overleftrightarrow{D}_3\overleftrightarrow{D}_{3\}}q-(3\leftrightarrow 4)$
$\langle 1\rangle_{\Delta q}$	$\mathbf{4}_4^+$	no	0	$\bar{q}\gamma^5\gamma_3q$
$\langle x\rangle_{\Delta q}^{(a)}$	$\mathbf{6}_3^-$	no	1	$\bar{q}\gamma^5\gamma_{\{1}\overleftrightarrow{D}_{3\}}q$
$\langle x\rangle_{\Delta q}^{(b)}$	$\mathbf{6}_3^-$	no	0	$\bar{q}\gamma^5\gamma_{\{3}\overleftrightarrow{D}_{4\}}q$
$\langle x^2\rangle_{\Delta q}$	$\mathbf{4}_2^+$	no	1	$\bar{q}\gamma^5\gamma_{\{1}\overleftrightarrow{D}_3\overleftrightarrow{D}_{4\}}q$
$\langle 1\rangle_{\delta q}$	$\mathbf{6}_1^+$	no	0	$\bar{q}\gamma^5\sigma_{34}q$
$\langle x\rangle_{\delta q}$	$\mathbf{8}_1^-$	no	1	$\bar{q}\gamma^5\sigma_{3\{4}\overleftrightarrow{D}_{1\}}q$
d_1	$\mathbf{6}_1^+$	no**	0	$\bar{q}\gamma^5\gamma_{[3}\overleftrightarrow{D}_{4]}q$
d_2	$\mathbf{8}_1^-$	no**	1	$\bar{q}\gamma^5\gamma_{[1}\overleftrightarrow{D}_{\{3]}\overleftrightarrow{D}_{4\}}q$

Table 1.1: Operators used to measure moments of quark distributions. Different lattice operators corresponding to the same continuum operator are denoted by superscripts a and b. Subscripts of irreducible representations of H(4) distinguish different representations of the same dimensionality and superscripts denote charge conjugation C. In the operator mixing column, no* indicates a case in which mixing generically could exist but vanishes perturbatively for Wilson or overlap fermions and no** indicates perturbative mixing with lower dimension operators for Wilson fermions but no mixing for overlap fermions. The entry in column $\vec{P}$ denotes the number of spatial components of the nucleon momentum, $\vec{P}$, that must be chosen non-zero. Operators requiring one non-zero component have been written for $\vec{P}$ in the 1- direction and $\vec{S}$ in the 3-direction.

able to calculate operators from each representation. For the first operator, denoted $\langle x\rangle_q^{(a)}$, we select the basis vector of $\mathbf{6}_3^+$:

$$\langle P|\bar{q}\gamma_{\{1}\overleftrightarrow{D}_{4\}}q|P\rangle = 2\langle x\rangle_q^{(a)}\cdot P_{\{1}P_{4\}},$$

and for $\langle x\rangle_q^{(b)}$, we choose the basis vector of $\mathbf{3}_1^+$:

$$\langle P|\bar{q}\gamma_4\overleftrightarrow{D}_4q-\frac{1}{3}(\bar{q}\gamma_1\overleftrightarrow{D}_1q+\bar{q}\gamma_2\overleftrightarrow{D}_2q+\bar{q}\gamma_3\overleftrightarrow{D}_3q)|P\rangle = 2\langle x\rangle_q^{(b)}\cdot(P_4P_4-\frac{1}{3}\vec{P}^2).$$

Note that since $\langle x\rangle_q^{(b)}$ involves $\gamma_4\overleftrightarrow{D}_4$, it can be measured with $\vec{P}=0$ whereas since $\langle x\rangle_q^{(a)}$ involves $\gamma_1\overleftrightarrow{D}_4$, it requires a state projected onto non-zero P_1.

For $\langle x^2 \rangle_q$, none of the three (symmetric) representations $\mathbf{4}_1^-$ is appropriate, since they are not traceless and hence mix with lower-dimensional operators. The only representations with two distinct indices are the one $\mathbf{8}_1^+$, which is not symmetric and must therefore be rejected, and the two $\mathbf{8}_1^-$'s

$$\bar{q}\left(\gamma_4 \overleftrightarrow{D}_1 \overleftrightarrow{D}_1 - \frac{1}{2}(\gamma_4 \overleftrightarrow{D}_2 \overleftrightarrow{D}_2 + \gamma_4 \overleftrightarrow{D}_3 \overleftrightarrow{D}_3)\right) q,$$

and

$$\bar{q}\left(\gamma_1 \overleftrightarrow{D}_4 \overleftrightarrow{D}_1 + \gamma_1 \overleftrightarrow{D}_1 \overleftrightarrow{D}_4 - \frac{1}{2}(\gamma_2 \overleftrightarrow{D}_4 \overleftrightarrow{D}_2 + \gamma_2 \overleftrightarrow{D}_2 \overleftrightarrow{D}_4 + \gamma_3 \overleftrightarrow{D}_4 \overleftrightarrow{D}_3 + \gamma_3 \overleftrightarrow{D}_3 \overleftrightarrow{D}_4)\right) q.$$

which mix as discussed in Refs. [4, 5, 5].

For $\langle x^3 \rangle_q$ the following representations have positive charge conjugation and do not mix with lower dimensional operators: $\mathbf{2}_1^+$, $\mathbf{2}_2^+$, $\mathbf{3}_2^+$, $\mathbf{3}_2^+$, $\mathbf{3}_3^+$, $\mathbf{6}_2^+$ and $\mathbf{6}_4^+$. However, the only representations that require a single non-zero momentum component are the two $\mathbf{2}_1^+$'s, which generically could mix with each other but do not mix at the one loop level for Wilson or overlap fermions [5, 2, 7].

Note that in addition to the mixing discussed above, in full QCD there is also mixing between gluonic operators and flavor-singlet fermion operators for moments of the quark density and helicity that will not be considered in this work because we have not yet evaluated lattice matrix elements of the relevant gluon operators.

1.3 Feynman diagrams

There are 6 distinct Feynman diagrams used to calculate renormalization of the self energy, quark currents and twist 2 operators. The first 4 given in the figure (1-1) are also present in continuum while the following two in figure (1-2) exist only on lattice. In the continuum, one can easily evaluate these diagrams analytically in the $\overline{MS}$ scheme as was done in the appendix (B). On the lattice, both the propagators and vertices are much more complicated so in the end, one has to evaluate these amplitudes numerically.

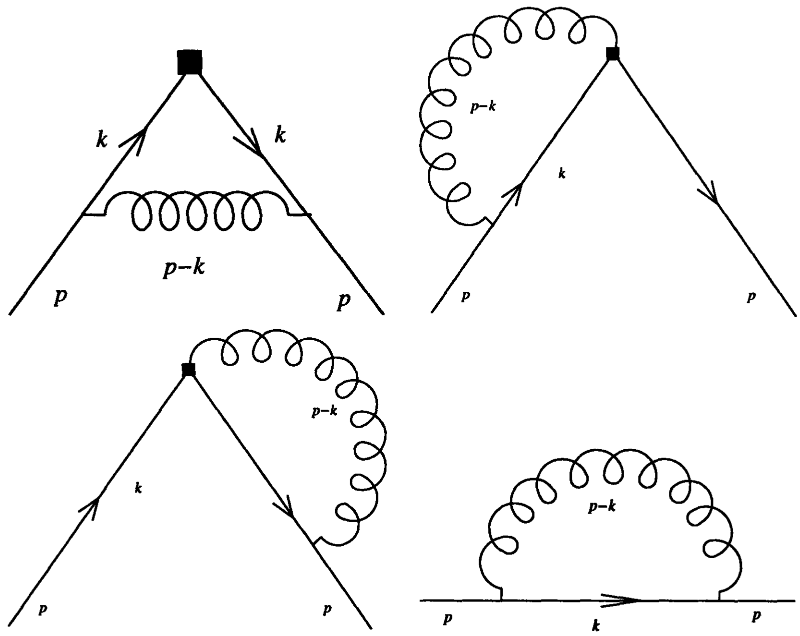

Figure 1-1: Feynman diagrams in the continuum

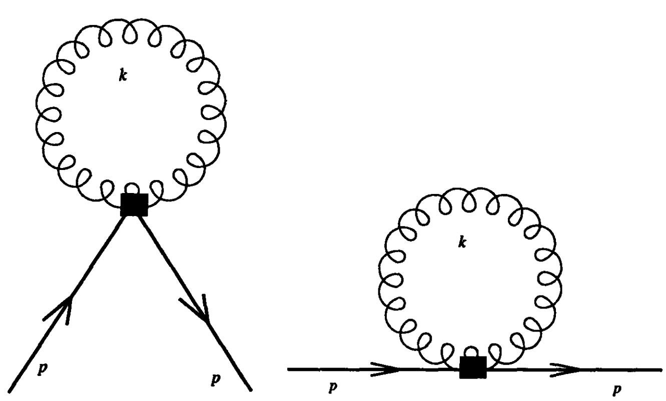

Figure 1-2: Tadpole diagrams which give finite contributions on the lattice

1.4 Perturbative Renormalization

Since the phenomenological light cone quark distributions with which we compare our lattice results are extracted from experimental data using the $\overline{MS}$ renormalization scheme, we will convert our lattice calculations to the $\overline{MS}$ scheme in 1-loop perturbation theory

observable	γ	Σ^{LATT}	$\Sigma^{\overline{MS}}$	$Z(\beta=6.0)$	$Z(\beta=5.6)$
$\langle x\rangle_q^{(a)}$	8/3	-3.16486	-40/9	0.9892	0.9884
$\langle x\rangle_q^{(b)}$	8/3	-1.88259	-40/9	0.9784	0.9768
$\langle x^2\rangle_q$	25/6	-19.57184	-67/9	1.1024	1.1097
$\langle x^3\rangle_q$	157/30	-35.35192	-2216/225	1.2153	1.2307
$\langle 1\rangle_{\Delta q}$	0	15.79628	0	0.8666	0.8571
$\langle x\rangle_{\Delta q}^{(a)}$	8/3	-4.09933	-40/9	0.9971	0.9969
$\langle x\rangle_{\Delta q}^{(b)}$	8/3	-4.09933	-40/9	0.9971	0.9969
$\langle x^2\rangle_{\Delta q}$	25/6	-19.56159	-67/9	1.1023	1.1096
$\langle 1\rangle_{\delta q}$	1	16.01808	-1	0.8563	0.8461
$\langle x\rangle_{\delta q}$	3	-4.47754	-5	0.9956	0.9953
d_1	0	0.36500	0	0.9969	0.9967
d_2	7/6	-15.67745	-35/18	1.1159	1.1242

Table 1.2: Renormalization constants for Wilson fermions

using

$$O_i^{\overline{MS}}(Q^2) = \sum_j \left(\delta_{ij} + \frac{g_0^2}{16\pi^2}\frac{N_c^2-1}{2N_c}\left(\gamma_{ij}^{\overline{MS}}\log(Q^2a^2) - (\Sigma_{ij}^{LATT} - \Sigma_{ij}^{\overline{MS}})\right)\right)\cdot O_j^{LATT}(a^2). \tag{1.11}$$

Previously calculated results from ref. [11, 2] for the anomalous dimensions γ_{ij} and the finite constants Σ_{ij} are given in Table 1.2 for Wilson fermions and the specific operators considered.

The Z factors that convert lattice results to the $\overline{MS}$ scheme at scale $Q^2 = 1/a^2$ are equal to

$$Z(g_0^2 = 6/\beta) = 1 - \frac{g_0^2}{16\pi^2}\frac{4}{3}\left(\Sigma^{LATT} - \Sigma^{\overline{MS}}\right), \tag{1.12}$$

and are tabulated for two typical values of β. Note that all of the moments of quark distributions calculated in this work will be presented at the scale of $\mu^2 = 4$ GeV^2 in the $\overline{MS}$ scheme.

Details of perturbative renormalization may by found in Refs. [8, 11, 5]. The results in Table 1.2 are taken from Refs. [11, 2], in which the renormalization factors of $\langle x\rangle_{\delta q}$ and d_1 for Wilson fermions were calculated for the first time and the remaining operators were checked with earlier results in Refs. [7, 9, 4, 10, 5, 12], revealing a discrepancy in the case of $\langle x^3\rangle_q$. Note that $\Sigma^{\overline{MS}}$ for $\langle x\rangle_{\delta q}$ calculated in this thesis includes a previously omitted term, which yields a result $-13/3$ shown in appendix B.7.

1.5 Lattice and continuum notation

Due to the difference in notation and additional features that arise for Euclidean domain wall fermions compared to familiar textbook treatments of continuum Minkowski electrodynamics, it is useful at the outset to compare the self-energy renormalization in both cases.

$$
\begin{aligned}
S_{ren}(p) &= \frac{iZ_2}{p\cdot\gamma - m_0 + \delta m} && \text{standard QED} \\
S_{ren}(p) &= \frac{Z_q}{ip\cdot\gamma + m_0 Z_W Z_m^{-1}} && \text{lattice DW}
\end{aligned}
\tag{1.13}
$$

The differences in signs and factors of i arise straightforwardly from transforming from Minkowski to Euclidean space-time. What is important to emphasize here is the fact the for domain wall fermions, the wave function renormalization $Z_q = Z_W \times Z_2$ arises from two factors, a five dimensional wave function renormalization Z_2 and a factor Z_W that describes the overlap of the five-dimensional wave function with the physical four-dimensional boundary. This factor Z_W also combines with the mass renormalization factor Z_m^{-1}. Another difference is that for DW fermions on the lattice, the mass m_0 is renormalized multiplicatively due to chiral symmetry, unlike in the continuum where it picks up an additive renormalization since there is no chiral symmetry to protect it. Indeed, for Wilson fermions on the lattice, which also do not have the chiral symmetry, the mass also gets renormalized additively just like in continuum *QED*.

Chapter 2

Domain Wall Fermions

The Lagrangian for the Domain Wall action ("Shamir Domain Wall") is given by [14]

$$\mathcal{L}_{DW} = \mathcal{L}_4 + \mathcal{L}_5 \,. \tag{2.1}$$

The term $\mathcal{L}_4$ is the usual 4D Wilson Lagrangian (in the "minus one" convention; see section (A.2) in the appendix)

$$\begin{aligned} \mathcal{L}_4 &= -\frac{1}{2a}\sum_{\mu}\left[\bar{\psi}(x)(r-\gamma_\mu)U_\mu(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu})(r+\gamma_\mu)U_\mu^\dagger(x)\psi(x)\right] \\ &\quad +\bar{\psi}(x)\left(M+\frac{rd}{a}\right)\psi(x) \end{aligned} \tag{2.2}$$

where $d = 4$ is the dimension of space. The term $\mathcal{L}_5$ is in the 5^{th} dimension obtained from the 5D Wilson action but with covariant derivatives acting only in 4 dimensions (to make things more confusing, this part is in the "plus one" convention)

$$\mathcal{L}_5 = +\frac{1}{2a_5}\left[\bar{\psi}_s(x)(r_5+\gamma_5)\psi_{s+1}(x) + \bar{\psi}_s(x)(r_5-\gamma_5)\psi_{s-1}(x)\right] - \bar{\psi}_s(x)\frac{r_5}{a_5}\psi_s(x) \,. \tag{2.3}$$

Going to momentum space

$$\psi_s(x) = \int\limits_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x}\psi_s(p) \,, \qquad \bar{\psi}_s(x) = \int\limits_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d} e^{-iq\cdot x}\bar{\psi}_s(q) \,, \tag{2.4}$$

and using the formula

$$a^d \sum_x e^{i(p-q)\cdot x} = (2\pi)^d \delta^{(d)}(p-q)\,, \tag{2.5}$$

the DW action becomes

$$S_{DW} \;=\; a^d \sum_x \mathcal{L}_{DW} = \int_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d} \sum_{s,s'} \bar{\psi}_s(p) D_{s,s'}(p) \psi_{s'}(p) \tag{2.6}$$

with the Dirac-Wilson operator (infinite in 5th dimension)

$$D_{s,s'}(p) = \left[i\bar{p}\cdot\gamma + \frac{ra}{2}\hat{p}^2 - \left(\frac{r_5}{a_5} - M\right)\right] \delta_{s,s'} + \frac{r_5+\gamma_5}{2a_5}\delta_{s+1,s'} + \frac{r_5-\gamma_5}{2a_5}\delta_{s-1,s'}\,. \tag{2.7}$$

Although one can use general r_5, usually one picks $r_5 = 1$ to get chirality projectors from r_5 terms

$$D_{s,s'}(p) = \left[i\bar{p}\cdot\gamma - \underbrace{\left\{ \left(\frac{1}{a_5} - M\right) - \frac{ra}{2}\hat{p}^2 \right\}}_{W(p)\equiv b(p)} \right] \delta_{s,s'} + \frac{1}{a_5}\left(P_+\delta_{s+1,s'} + P_-\delta_{s-1,s'}\right)\,, \tag{2.8}$$

where Shamir's [14] $b(p)$ is what Aoki [15] calls $W(p)$. To get the propagator for finite 5th dimension, we multiply D with $\theta(\varepsilon+s)\theta(\varepsilon+s')\theta(N+\varepsilon-s)\theta(N+\varepsilon-s')$ where ε is infinitesimally small value added to avoid the issue of the value of $\theta(0)$.

If we write the Dirac-Wilson operator as

$$D(p) = \left(i\gamma\cdot\bar{p} + W^+\right)P_+ + \left(i\gamma\cdot\bar{p} + W^-\right)P_- \tag{2.9}$$

with

$$[W^\pm]_{s,s'} = \frac{1}{a_5}\delta_{s\pm 1,s'} - W(p)\delta_{s,s'}\,, \qquad \Longrightarrow \qquad (W^\pm)^\dagger = W^\mp\,, \tag{2.10}$$

then it's inverse can be written as

$$\begin{aligned} S_F(p) \;&=\; \left(-i\gamma\cdot\bar{p} + W^-\right)G_+P_+ + \left(-i\gamma\cdot\bar{p} + W^+\right)G_-P_- && (2.11)\\ &=\; -i\gamma\cdot\bar{p}(\sigma_V + \gamma_5\sigma_A) + (\sigma_S + \gamma_5\sigma_P) && (2.12) \end{aligned}$$

where $G = \Omega^{-1} = [DD^\dagger]^{-1}$ and

$$\sigma_V = \frac{1}{2}(G_+ + G_-)\,, \qquad \sigma_A = \frac{1}{2}(G_+ - G_-) \tag{2.13}$$

$$\sigma_S = \frac{1}{2}(W^- G^+ + W^+ G^-)\,, \qquad \sigma_P = \frac{1}{2}(W^- G^+ - W^+ G^-)\,. \tag{2.14}$$

2.1 5-dimensional domain wall propagator

Inverting infinite DW Dirac operator

To invert the infinite Dirac operator (2.8), we use the fact that

$$D^{-1} = D^\dagger G = D^\dagger [DD^\dagger]^{-1} = D^\dagger([D^\dagger]^{-1}[D]^{-1}) \tag{2.15}$$

to calculate the inverse of $\Omega \equiv DD^\dagger$ first, and then multiply it with $D^\dagger$. In Shamir's notation (but with sign conventions from Aoki)

$$\begin{aligned}
[\Omega_0]_{s,s'} &\equiv [DD^\dagger]_{s,s'} = \sum_t \left\{ [i\bar{p}\cdot\gamma - b(p)]\delta_{s,t} + \frac{1}{a_5}\left(P_+\delta_{s+1,t} + P_-\delta_{s-1,t}\right) \right\} \\
&\quad * \left\{ [-i\bar{p}\cdot\gamma - b(p)]\delta_{t,s'} + \frac{1}{a_5}\left(P_+\delta_{t,s'-1} + P_-\delta_{t,s'+1}\right) \right\} \\
&= \left[b^2(p) + \bar{p}^2 + \frac{1}{a_5^2}(P_+ + P_-) \right] \delta_{s,s'} \\
&\qquad + [(+i\bar{p}\cdot\gamma - b(p))P_+ + P_-(-i\bar{p}\cdot\gamma - b(p))]\,\delta_{s,s'+1} \\
&\qquad + [(i\bar{p}\cdot\gamma - b(p))P_- + P_+(-i\bar{p}\cdot\gamma - b(p))]\,\delta_{s,s'-1} \\
&= \left(\frac{1}{a_5^2} + b^2(p) + \bar{p}^2 \right) \delta_{s,s'} - \frac{b(p)}{a_5}\left(\delta_{s,s'+1} + \delta_{s,s'-1}\right) \\
&= \frac{2b(p)}{a_5}\left[\cosh\alpha(p)\delta_{s,s'} - \frac{1}{2}\left(\delta_{s,s'+1} + \delta_{s,s'-1}\right) \right]
\end{aligned} \tag{2.16}$$

with α defined through the relation

$$\cosh\alpha(p) = \frac{\frac{1}{a_5^2} + b^2(p) + \bar{p}^2}{2b(p)/a_5} \tag{2.17}$$

Inverse of Ω is given by

$$[G_0]_{s,s'} = \frac{e^{-\alpha|s-s'|}}{\frac{2b}{a_5}\sinh\alpha} \tag{2.18}$$

Proof:

$$\begin{aligned}
[\Omega_0]_{s,t}[G_0]_{t,s'} &= \sum_t \frac{2b}{a_5}\left(\cosh\alpha\delta_{s,t} - \frac{1}{2}\delta_{s,t-1} - \frac{1}{2}\delta_{s,t+1}\right) * \frac{e^{-\alpha|t-s'|}}{\frac{2b}{a_5}\sinh\alpha} \\
&= \frac{\cosh\alpha e^{-\alpha|s-s'|} - \frac{1}{2}e^{-\alpha|s-s'+1|} - \frac{1}{2}e^{-\alpha|s-s'-1|}}{\sinh\alpha}\,.
\end{aligned} \tag{2.19}$$

For $s-s' \geq 1$

$$e^{-\alpha|s-s'\pm1|} = e^{-\alpha|s-s'|\mp1} \tag{2.20}$$

while for $s-s' \leq -1$

$$e^{-\alpha|s-s'\pm1|} = e^{-\alpha|s-s'|\pm1} \tag{2.21}$$

so for $s \neq s'$

$$\begin{aligned}
[\Omega_0]_{s,t}[G_0]_{t,s'} &= \frac{\cosh\alpha\, e^{-\alpha|s-s'|} - \frac{1}{2}e^{-\alpha(|s-s'|\pm1)} - \frac{1}{2}e^{-\alpha(|s-s'|\mp1)}}{\sinh\alpha} \\
&= \frac{\cosh\alpha\, e^{-\alpha|s-s'|} - \frac{1}{2}e^{-\alpha|s-s'|}(e^{\pm\alpha} + e^{\mp\alpha})}{\sinh\alpha} = 0\,,
\end{aligned} \tag{2.22}$$

while for $s = s'$

$$e^{-\alpha|s-s'\pm1|} = e^{-\alpha(|s-s'|+1)} = e^{-\alpha} \tag{2.23}$$

so

$$[\Omega]_{s,t}[G_0]_{t,s'} = \frac{\cosh\alpha - \frac{1}{2}e^{-\alpha} - \frac{1}{2}e^{-\alpha}}{\sinh\alpha} = 1 \tag{2.24}$$

For $M > 1$, we replace $b(p)$ with $-|b(p)|$; that changes the sign in front of two δ functions in Ω_0 so for general $0 < M < 2$ we may write Ω_0 and $\cosh\alpha$ as

$$\begin{aligned}
[\Omega_0]_{ss'} &= \frac{2|b(p)|}{a_5}\left[\cosh\alpha(p)\delta_{s,s'} \mp \frac{1}{2}\left(\delta_{s,s'+1} + \delta_{s,s'-1}\right)\right] \\
\cosh\alpha(p) &= \frac{\frac{1}{a_5^2} + b^2(p) + \bar{p}^2}{2|b(p)|/a_5}
\end{aligned}$$

$$[G_0]_{s,s'} = \frac{(\pm e^{-\alpha})^{|s-s'|}}{\frac{2|b|}{a_5}\sinh\alpha} \tag{2.25}$$

where $+$ sign applies for $M < 1$ and minus sign applies for $M > 1$.

Semi-infinite DW Dirac propagator

In the semi-infinite case[1]

$$\hat{D}(p) = \theta(-\varepsilon + s)\theta(-\varepsilon + s')D(p) \tag{2.26}$$

the fact that sums over s and s' now go from 0 to ∞ instead of $-\infty$ to ∞ changes the expression for Ω. For $M < 1$ (and therefore $b(p) > 0$) we have

$$\begin{aligned}\hat{\Omega}_{s,s'} &= \sum_{t>0}\left\{[i\bar{p}\cdot\gamma - b(p)]\delta_{s,t} + \frac{1}{a_5}\left(P_+\delta_{s+1,t} + P_-\delta_{s-1,t}\right)\right\} * \\ &\quad \left\{[-i\bar{p}\cdot\gamma - b(p)]\delta_{t,s'} + \frac{1}{a_5}\left(P_+\delta_{t,s'-1} + P_-\delta_{t,s'+1}\right)\right\}.\end{aligned} \tag{2.27}$$

Now the terms $P_-\delta_{s-1,t}$ and $P_-\delta_{t,s'+1}$ do not contribute for the $t = 0$ part of the sum, so we have to subtract their contribution from the Ω:

$$\hat{\Omega}_{s,s'} = [\Omega_0]_{s,s'} - \frac{1}{a_5^2}P_-\,\delta_{s,0}\delta_{s',0}\,. \tag{2.28}$$

The next step is to split G and $\hat{\Omega}$ into $\pm$ parts:

$$\hat{\Omega} = \hat{\Omega}_+P_+ + \hat{\Omega}_-P_-\,, \quad \Longrightarrow \quad \hat{\Omega}_+ = \Omega_0\,, \quad \hat{\Omega}_- = \Omega_0 - \frac{1}{a_5^2}\delta_{s,0}\delta_{s',0}\,. \tag{2.29}$$

Since $P_\pm P_\mp = 0$, we have

$$G = G_+P_+ + G_-P_- \quad \text{with} \quad [G_\pm]_{s,t}[\Omega_\pm]_{t,s'} = \delta_{s,s'}\,. \tag{2.30}$$

[1] Shamir uses $s \geq 0$ range while Aoki uses $s \geq 1$ range for semi-infinite case. I'll use Shamir's convention in the derivation and will only quote results for Aoki's convention

Now, since there are only nearest neighbor interactions, $G_\pm$ must equal G_0 away from the edge. Edge effects can be parametrized as

$$[G_\pm]_{s,s'} = [G_0]_{s,s'} + A_\pm e^{-\alpha(s+s')} . \tag{2.31}$$

To evaluate $A_\pm$ we first compute $\Omega_\pm G_0$ by using the explicit expression for G_0:

$$[\Omega_- G_0]_{s,s'} = \sum_{t>0} \left\{ \frac{2b}{a_5}(\cosh\alpha\,\delta_{s,t} - \frac{1}{2}\delta_{s,t-1} - \frac{1}{2}\delta_{s,t+1}) - \frac{1}{a_5^2}\delta_{s,0}\delta_{t,0} \right\} * \frac{e^{-\alpha|t-s'|}}{\frac{2b}{a_5}\sinh\alpha} . \tag{2.32}$$

For both $s, s' > 0$ this always yields $\delta_{s,s'}$ since edge effects vanish, so we evaluate for $s = 0$

$$\begin{aligned} [\Omega_- G_0]_{0,s'} &= \sum_{t>0} \left\{ \frac{2b}{a_5}(\cosh\alpha\,\delta_{0,t} - \frac{1}{2}\delta_{0,t-1} - \frac{1}{2}\delta_{0,t+1}) - \frac{1}{a_5^2}\delta_{0,0}\delta_{t,0} \right\} * \frac{e^{-\alpha|t-s'|}}{\frac{2b}{a_5}\sinh\alpha} \\ &= \frac{\cosh\alpha\, e^{-\alpha s'} - \frac{1}{2}e^{-\alpha|s'-1|}}{\sinh\alpha} - \frac{e^{-\alpha s'}}{2ba_5\sinh\alpha} \end{aligned} \tag{2.33}$$

For $s' > 0$ we have $|s'-1| = s'-1$ while for $s' = 0$ we have $|s'-1| = 1$, so after some algebra

$$[\Omega_- G_0]_{0,s'} = \delta_{0,s'} + \frac{ba_5 e^{-\alpha} - 1}{2ba_5\sinh\alpha} e^{-\alpha s'} \tag{2.34}$$

which finally yields

$$[\Omega_- G_0]_{s,s'} = \delta_{s,s'} + \frac{ba_5 e^{-\alpha} - 1}{2ba_5\sinh\alpha} e^{-\alpha s'}\delta_{s,0} \tag{2.35}$$

Using the result (2.35) we can calculate

$$\begin{aligned} [\Omega_- G_-]_{s,s'} &\equiv \delta_{s,s'} = [\Omega_-(G_0 + A_- e^{-\alpha(s+s')})]_{s,s'} \\ &= \delta_{s,s'} + \frac{ba_5 e^{-\alpha} - 1}{2ba_5\sinh\alpha} e^{-\alpha s'}\delta_{s,0} + A_- \sum_{t\geq 0} [\Omega_-]_{s,t} e^{-\alpha(t+s')} \\ [\Omega_+ G_+]_{s,s'} &\equiv \delta_{s,s'} = [\Omega_+(G_0 + A_+ e^{-\alpha(s+s')})]_{s,s'} \end{aligned} \tag{2.36}$$

$$= \delta_{s,s'} + \frac{e^{-\alpha}}{2\sinh\alpha} e^{-\alpha s'} \delta_{s,0} + A_+ \sum_{t\geq 0} [\Omega_+]_{s,t} e^{-\alpha(t+s')} \quad (2.37)$$

from which we get the constraints

$$\begin{aligned} \frac{ba_5 e^{-\alpha}-1}{2ba_5\sinh\alpha} e^{-\alpha s'} \delta_{s,0} + A_- \sum_{t\geq 0} [\Omega_-]_{s,t} e^{-\alpha(t+s')} &= 0 \quad (2.38) \\ \frac{e^{-\alpha}}{2\sinh\alpha} e^{-\alpha s'} \delta_{s,0} + A_+ \sum_{t\geq 0} [\Omega_+]_{s,t} e^{-\alpha(t+s')} &= 0 \quad (2.39) \end{aligned}$$

Evaluating those constraints at $s = 0$ gives us equations

$$\begin{aligned} A_-(ba_5 e^{-\alpha} - 1) &= \frac{ba_5 e^{-\alpha}-1}{2ba_5\sinh\alpha} \quad (2.40) \\ A_+ \, ba_5 e^{-\alpha} &= \frac{e^{-\alpha}-1}{2\sinh\alpha} \quad (2.41) \end{aligned}$$

with solutions

$$A_- = \frac{1}{\frac{2b}{a_5}\sinh\alpha} e^{-2\alpha} \frac{ba_5 - e^{\alpha}}{ba_5 - e^{-\alpha}}, \qquad A_+ = \frac{1}{\frac{2b}{a_5}\sinh\alpha} e^{-2\alpha} \quad (2.42)$$

For $M > 1$, again, we replace $b \to -|b|$ and repeat the procedure to get the result

$$\begin{aligned} [G_+]_{s,s'} &= [G_0]_{s,s'} + A_+ (\pm e^{-\alpha})^{s+s'} \quad (2.43) \\ [G_-]_{s,s'} &= [G_0]_{s,s'} + A_- (\pm e^{-\alpha})^{s+s'} \quad (2.44) \\ A_- &= \frac{1}{\frac{2|b|}{a_5}\sinh\alpha} e^{-2\alpha} \frac{|b|a_5 - w^{\alpha}}{|b|a_5 - e^{-\alpha}} \quad (2.45) \\ A_+ &= \frac{1}{\frac{2|b|}{a_5}\sinh\alpha} e^{-2\alpha} \quad (2.46) \end{aligned}$$

Finite DW Dirac operator

In addition to making the lattice finite in the 5th dimension, we also add a mass term

$$\begin{aligned} D_{s,s'}(m) &= \theta(-\varepsilon+s)\theta(-\varepsilon+s')\theta(N+\varepsilon-s)\theta(N+\varepsilon-s')[D_0]_{s,s'} \\ &\quad + m\left(P_-\delta_{s,0}\delta_{s',N} + P_+\delta_{s,N}\delta_{s',0}\right) \quad (2.47) \end{aligned}$$

which yields

$$
\begin{aligned}
[\Omega]_{s,s'} &= [\Omega_0]_{s,s'} - P_-\left(\frac{1}{a_5^2}-m^2\right)\delta_{s,0}\delta_{s',0} - P_+\left(\frac{1}{a_5^2}-m^2\right)\delta_{s,N}\delta_{s',N} \\
&\quad -mb(p)(\delta_{s,0}\delta_{s',N}+\delta_{s,N}\delta_{s',0})\,. \qquad (2.48)
\end{aligned}
$$

This yields extra terms

$$
\begin{aligned}
[\Omega_+]_{s,s'} &= [\Omega_0]_{s,s'} - mb(p)[\delta_{s,0}\delta_{s',N}+\delta_{s,N}\delta_{s',0}] - \left(\frac{1}{a_5^2}-m^2\right)\delta_{s,N}\delta_{s',N} \qquad (2.49)\\
[\Omega_-]_{s,s'} &= [\Omega_0]_{s,s'} - mb(p)[\delta_{s,0}\delta_{s',N}+\delta_{s,N}\delta_{s',0}] - \left(\frac{1}{a_5^2}-m^2\right)\delta_{s,0}\delta_{s',0} \qquad (2.50)\\
& \qquad (2.51)
\end{aligned}
$$

The procedure here is the same as in the semi-infinite case (but somewhat more complicated): evaluate $\Omega_\pm G_0$, then "guess" a proper form of $G_\pm = G_0 + [A]$ and determine what the coefficients are. At the edge $s = 0$:

$$
\begin{aligned}
\sum_t [\Omega_-]_{0,t}[G_0]_{t,s'} &= \sum_{t>0}\left\{\frac{2b}{a_5}(\cosh\alpha\,\delta_{0,t} - \frac{1}{2}\delta_{0,t-1} - \frac{1}{2}\delta_{0,t+1}) - mb[\delta_{0,0}\delta_{s',N}+\delta_{0,N}\delta_{s',0}]\right.\\
&\qquad \left. - \left(\frac{1}{a_5^2}-m^2\right)\delta_{0,0}\delta_{t,0}\right\} * \frac{e^{-\alpha|t-s'|}}{\frac{2b}{a_5}\sinh\alpha}\,.\\
&= A_0\left\{\frac{2b}{a_5}\left(\cosh\alpha e^{-\alpha s'} - \frac{1}{2}e^{-\alpha|s'-1|}\right) - mbe^{-\alpha(N-s')}\right.\\
&\qquad \left. - \left(\frac{1}{a_5^2}-m^2\right)e^{-\alpha s'}\right\} \qquad (2.52)
\end{aligned}
$$

For $s' > 0$ we have $|s'-1| = s'-1$, so

$$
\left(\cosh\alpha e^{-\alpha s'} - \frac{1}{2}e^{-\alpha|s'-1|}\right) = e^{-\alpha s'}\left(\cosh\alpha - \frac{1}{2}e^{\alpha}\right) = e^{-\alpha s'}\frac{1}{2}e^{-\alpha}\,. \qquad (2.53)
$$

For $s' = 0$ $|s'-1| = 1 = s'+1$, so

$$
\left(\cosh\alpha e^{-\alpha s'} - \frac{1}{2}e^{-\alpha|s'-1|}\right) = e^{-\alpha s'}\left(\cosh\alpha - \frac{1}{2}e^{-\alpha}\right) = e^{-\alpha s'}\left(\frac{1}{2}e^{-\alpha} + \sinh\alpha\right)\,. \qquad (2.54)
$$

Hence, the expression (2.52) becomes

$$\sum_t [\Omega_-]_{0,t}[G_0]_{t,s'} = \delta_{0,s'} + A_0 \left\{ e^{-\alpha s'} \left[\frac{b}{a_5} e^{-\alpha} - \left(\frac{1}{a_5^2} - m^2 \right) \right] - mbe^{-\alpha(N-s')} \right\} \quad (2.55)$$

with

$$A_0 = \frac{1}{2\frac{b}{a_5} \sinh \alpha} . \quad (2.56)$$

Repeating the same procedure at the other boundary $s = N$ yields

$$\begin{aligned}
\sum_t [\Omega_-]_{N,s'}[G_0]_{t,s'} &= \sum_{s>0} \left\{ \frac{2b}{a_5} (\cosh \alpha \, \delta_{N,s} - \frac{1}{2}\delta_{N,s-1} - \frac{1}{2}\delta_{N,s+1}) \right. \\
&\quad - mb[\delta_{N,0}\delta_{s,N} + \delta_{N,N}\delta_{s,0}] \\
&\quad \left. - \left(\frac{1}{a_5^2} - m^2 \right) \delta_{N,0}\delta_{t,0} \right\} * \frac{e^{-\alpha|t-s'|}}{\frac{2b}{a_5} \sinh \alpha} . \\
&= A_0 \left\{ \frac{2b}{a_5} \left(\cosh \alpha e^{-\alpha(N-s')} - \frac{1}{2} e^{-\alpha|N-s'-1|} \right) - mbe^{-\alpha s'} \right\} \\
&= \delta_{N,s'} + A_0 \left\{ e^{-\alpha(N-s')} \left[\frac{b}{a_5} e^{-\alpha} \right] - mbe^{-\alpha s'} \right\} , \qquad (2.57)
\end{aligned}$$

The full solution must be of the form

$$\begin{aligned}
G_- &= G_0 + A_- e^{-\alpha(s+s'+c_1)} + A_+ e^{-\alpha(2N-s-s'+c_2)} \\
&\quad + A_m e^{-\alpha(N-s+s'+c_3)} + \bar{A}_m e^{-\alpha(N+s-s'+c_4)} \qquad (2.58)
\end{aligned}$$

where the constants c_i are fixed from the boundary conditions. For a finite lattice, from Shamir [14], $0 \leq s, s' \leq N$ we have the condition

$$\Omega_\pm(0,0) = \Omega_\mp(N,N) , \quad \Longrightarrow \quad \Omega_\pm(s,s') = \Omega_\mp(N-s, N-s') \qquad (2.59)$$

from which we conclude that $A_m = +\bar{A}_m, c_3 = c_4$, and $c_1 = c_2$. From this, we can then choose $c_i = 0$ for all i.

For the lattice Aoki [15] uses, we have $1 \leq s, s' \leq N$ so

$$\Omega_\pm(1,1) = \Omega_\mp(N,N) , \quad \Longrightarrow \quad \Omega_\pm(s,s') = \Omega_\mp(N+1-s, N+1-s') \qquad (2.60)$$

from which we again get $A_m = +\bar{A}_m$ and $c_3 = c_4$, but now $c_2 = c_1 + 2$. A natural choice for the c's is $c_3 = c_4 = -1$, $c_1 = -2$, $c_2 = 0$. It is the solution we get from Shamir's lattice of length $N-1$ after shifting variables $s, s' \to s, s'-1$. Aoki uses the same choice but he takes factors $e^{-\alpha}$ and $e^{-2\alpha}$ from A_m and A_+ and adds them to exponentials $e^{-\alpha(\ldots)}$ which is not incorrect in itself, but the coefficients are then no longer in symmetric form.

Now we plug the expression (2.58) into the equation

$$\sum_t [\Omega_-]_{s,t}[G_-]_{t,s'} \equiv \delta_{s,s'} \tag{2.61}$$

for $s = 0$ and $s = N$:

$$\begin{aligned}
[\Omega_- G_-]_{0,s'} &= [\Omega_-(G_0 - [A])]_{0,s'} = [\Omega_- G_0]_{0,s'} \\
&\quad + \sum_{t>0} \left\{ \frac{2b}{a_5}(\cosh\alpha\,\delta_{0,t} - \frac{1}{2}\delta_{0,t-1} - \frac{1}{2}\delta_{0,t+1}) \right. \\
&\qquad \left. - mb[\delta_{0,0}\delta_{s',N} + \delta_{0,N}\delta_{s',0}] - \left(\frac{1}{a_5^2} - m^2\right)\delta_{0,0}\delta_{t,0} \right\} \\
&\qquad * \left\{ A_- e^{-\alpha(s+s')} + A_+ e^{-\alpha(2N-s-s')} \right. \\
&\qquad \left. + A_m e^{-\alpha(N-s+s')} + \bar{A}_m e^{-\alpha(N+s-s')} \right\} \\
&= [\Omega_- G_0]_{0,s'} + e^{-\alpha s'} \left\{ A_- \left[\frac{b}{a_5} e^{\alpha} - \left(\frac{1}{a_5^2} - m^2\right) - mbe^{-\alpha N} \right] \right. \\
&\qquad \left. + A_m \left[e^{-\alpha N}\left(\frac{b}{a_5} e^{-\alpha} - \left(\frac{1}{a_5^2} - m^2\right)\right) - mb \right] \right\} \\
&\quad + e^{\alpha s'} \left\{ A_+ \left[e^{-\alpha N}\left(\frac{b}{a_5} e^{-\alpha} - \left(\frac{1}{a_5^2} - m^2\right)\right) - mb \right] \right. \\
&\qquad \left. + \bar{A}_m \left[\frac{b}{a_5} e^{\alpha} - \left(\frac{1}{a_5^2} - m^2\right) - mbe^{-\alpha N} \right] \right\}.
\end{aligned} \tag{2.62}$$

If the expression above is to be valid for all s', then the coefficients multiplying $e^{\pm\alpha s'}$ must vanish separately which gives us two equations

$$\begin{aligned}
A_- \left[\frac{b}{a_5} e^{\alpha} - \left(\frac{1}{a_5^2} - m^2\right) - mbe^{-\alpha N} \right] + A_m \left[e^{-\alpha N}\left(\frac{b}{a_5} e^{-\alpha} - \left(\frac{1}{a_5^2} - m^2\right)\right) - mb \right] \\
= -A_0 \left[\frac{b}{a_5} e^{-\alpha} - \left(\frac{1}{a_5^2} - m^2\right) \right]
\end{aligned} \tag{2.63}$$

$$A_+\left[e^{-\alpha N}\left(\frac{b}{a_5}e^{-\alpha}-\left(\frac{1}{a_5^2}-m^2\right)\right)-mb\right]+\bar{A}_m\left[\frac{b}{a_5}e^{\alpha}-\left(\frac{1}{a_5^2}-m^2\right)-mbe^{-\alpha N}\right]$$
$$=A_0\,mb \quad (2.64)$$

where for now we're keeping A_m and $\bar{A}_m$ separate even though they must be equal. That fact will be used in the end as a consistency check if equations were derived correctly.

Repeating the same procedure at the other boundary yields

$$\begin{aligned}
[\Omega_-G_-]_{N,t} &= [\Omega_-(G_0-[A])]_{N,s'}=[\Omega_-G_0]_{N,s'} \\
&\quad+\sum_{t>0}\left\{\frac{2b}{a_5}\left(\cosh\alpha\,\delta_{N,t}-\frac{1}{2}\delta_{N,t-1}-\frac{1}{2}\delta_{N,t+1}\right)\right. \\
&\qquad\left.-mb[\delta_{N,0}\delta_{s',N}+\delta_{N,N}\delta_{s',0}]-\left(\frac{1}{a_5^2}-m^2\right)\delta_{N,0}\delta_{t,0}\right\} \\
&\qquad*\left\{A_-e^{-\alpha(s+s')}+A_+e^{-\alpha(2N-s-s')}\right. \\
&\qquad\left.+A_me^{-\alpha(N-s+s')}+\bar{A}_me^{-\alpha(N+s-s')}\right\} \\
&= [\Omega_-G_0]_{N,s'}+e^{-\alpha s'}\left\{A_-\left[e^{-\alpha N}\frac{b}{a_5}e^{-\alpha}-mb\right]+A_m\left[\frac{b}{a_5}e^{-\alpha}-mbe^{-\alpha N}\right]\right\} \\
&\quad+e^{\alpha s'}\left\{A_+\left[e^{-\alpha N}\frac{b}{a_5}e^{\alpha}-mb\right]+\bar{A}_m\left[\frac{b}{a_5}e^{-\alpha}-mbe^{-\alpha N}\right]\right\}.
\end{aligned} \tag{2.65}$$

which yields a second pair of equations

$$A_-\left[e^{-\alpha N}\frac{b}{a_5}e^{-\alpha}-mb\right]+A_m\left[\frac{b}{a_5}e^{\alpha}-mbe^{-\alpha N}\right] = A_0\,mb \tag{2.66}$$
$$A_+\left[\frac{b}{a_5}e^{\alpha}-mbe^{-\alpha N}\right]+\bar{A}_m\left[e^{-\alpha N}\frac{b}{a_5}e^{-\alpha}-mb\right] = -A_0\frac{b}{a_5}e^{-\alpha} \tag{2.67}$$

If we drop terms that go as $e^{-\alpha N}$ we recover equations (32)-(38) from Shamir [14]. Solving equations (2.63-2.64) and (2.66-2.67) yields

$$A_- = -A_0\frac{\left(\frac{1}{a_5^2}-m^2\right)(1-ba_5e^{-\alpha})}{F_N} \tag{2.68}$$
$$A_+ = -A_0\frac{\left(\frac{1}{a_5^2}-m^2\right)a_5(1-ba_5e^{\alpha})}{F_N}e^{-2\alpha} \tag{2.69}$$
$$\tag{2.70}$$

and

$$A_m = -A_0 \frac{2mb\sinh\alpha - e^{-\alpha(N+1)}\left(\frac{1}{a_5^2}(1-ba_5e^{-\alpha}) - m^2(1-a_5be^{\alpha})\right)}{F_N} e^{-\alpha}, \quad (2.71)$$

where F_N is the denominator

$$F_N = \underbrace{\left[\frac{1}{a_5^2}(1-ba_5e^{\alpha}) - m^2(1-a_5be^{-\alpha})\right]}_{F \text{ in Aoki}} + e^{-\alpha(N+1)}4mb\sinh\alpha$$
$$-e^{-2\alpha(N+1)}\left(\frac{1}{a_5^2}\left(1-ba_5e^{-\alpha}\right) - m^2(1-a_5be^{\alpha})\right). \quad (2.72)$$

As in the infinite and semi-infinite cases, for $M > 1$ we replace $b \to -|b|$ and redo the calculation to get

$$[G_-]_{s,s'} = A_0(\pm e^{-\alpha})^{|s-s'|} + A_-(\pm e^{-\alpha})^{(s+s')} + A_+(\pm e^{-\alpha})^{(2N-s-s')}$$
$$+A_m\{(\pm e^{-\alpha})^{(N-s+s')} + (\pm e^{-\alpha})^{(N+s-s')}\} \quad (2.73)$$

for the $0 \leq s, s' \leq N$ convention with

$$A_- = -A_0 \frac{\left(\frac{1}{a_5^2} - m^2\right)(1-|b|a_5e^{-\alpha})}{F_N} \quad (2.74)$$

$$A_+ = -A_0 \frac{\left(\frac{1}{a_5^2} - m^2\right)(1-|b|a_5e^{\alpha})}{F_N} e^{-2\alpha} \quad (2.75)$$

$$A_m = \mp A_0 \frac{|b|2m\sinh\alpha - (\pm e^{-\alpha})^{(N+1)}\left(\frac{1}{a_5^2}(1-|b|\,a_5e^{-\alpha}) - m^2(1-|b|\,a_5e^{\alpha})\right)}{F_N} e^{-\alpha}$$
$$(2.76)$$

and

$$F_N = \underbrace{\left[\frac{1}{a_5^2}(1-|b|\,a_5e^{\alpha}) - m^2(1-|b|a_5e^{-\alpha})\right]}_{F \text{ in Aoki}} + (\pm e^{-\alpha})^{(N+1)}4m|b|\sinh\alpha$$

$$-(\pm e^{-\alpha})^{2(N+1)}\left(\frac{1}{a_5^2}\left(1-|b|\,a_5 e^{-\alpha}\right)-m^2(1-|b|\,a_5 e^{\alpha})\right). \qquad (2.77)$$

The overall effect for $M>1$ is the same as if we replaced

$$b \to -|b|\ , \qquad e^{\pm\alpha} \to -e^{\pm\alpha} \qquad (2.78)$$

and canceled all extra signs.

Solutions for Aoki case can be obtained from the Shamir lattice of length $N-1$ by replacing arguments s and s' by $s-1$ and $s'-1$. This leads to

$$\begin{aligned}[G_-]_{s,s'} &= A_0(\pm e^{-\alpha})^{|s-s'|}+A_-(\pm e^{-\alpha})^{(s+s'-2)}+A_+(\pm e^{-\alpha})^{(2N-s-s')}\\ &\quad +A_m\{(\pm e^{-\alpha})^{(N-s+s'-1)}+(\pm e^{-\alpha})^{(N+s-s'-1)}\} \qquad (2.79)\end{aligned}$$

Domain Wall propagator in $k\to 0$ limit

In this section we'll work in the $N\to\infty$ limit in which the denominator (2.77) in equations (2.74)-(2.76) equals F. We'll also let $a\to 0$. To take the $m\to 0$ and $p\to 0$ limit, we have to expand everything in powers of p^2 and m^2 and keep only lowest terms. Then

$$b(p)=-\frac{ra}{2}\hat{p}^2+\left(\frac{1}{a_5}-M\right)\to\left(\frac{1}{a_5}-M\right)=\frac{1}{a_5}(1-a_5M)\equiv b_0\equiv\frac{w_0}{a_5}. \qquad (2.80)$$

Expanding $\cosh\alpha$ in powers of p^2:

$$\cosh\alpha=\frac{\frac{1}{a_5^2}+b^2(p)+\bar{p}^2}{2\frac{b(p)}{a_5}}=\frac{\frac{1}{a_5^2}+b_0^2}{2\frac{b_0}{a_5}}+p^2\frac{a_5}{2b_0}=\frac{1+w_0^2}{2w_0}+\frac{p^2a_5^2}{2w_0}. \qquad (2.81)$$

For zero momentum we have

$$\cosh\alpha(0)=\frac{\frac{1}{a_5^2}+b_0}{2\frac{b_0}{a_5}}=\frac{1}{2}\left(\frac{1}{a_5b_0}+a_5b_0\right)=\frac{1+w_0^2}{2w_0}. \qquad (2.82)$$

Since $0 < a_5 M < 1$, we see that $w_0 = 1 - a_5 M < 1$ therefore (for positive α)

$$e^{\alpha_0} = \frac{1}{w_0}, \qquad e^{-\alpha_0} = w_0 \tag{2.83}$$

For $Ma_5 > 1$ we get

$$e^{\alpha_0} = \frac{1}{|w_0|}, \qquad e^{-\alpha_0} = |w_0| \tag{2.84}$$

This gives us other functions of α (to order p^2):

$$\begin{aligned}
\cosh\alpha(p) &= \frac{1+w_0^2}{2w_0} + \frac{p^2 a_5^2}{2w_0} && (2.85)\\
\sinh\alpha(p) &= \sqrt{\cosh^2\alpha - 1} = \frac{1-w_0^2}{2w_0} + \frac{p^2 a_5^2}{2w_0}\frac{1+w_0^2}{1-w_0^2} && (2.86)\\
e^{\alpha(p)} &= \cosh\alpha(p) + \sinh\alpha(p) = \frac{1}{w_0}\left(1 + \frac{p^2 a_5^2}{1-w_0^2}\right) && (2.87)\\
e^{-\alpha(p)} &= \cosh\alpha(p) - \sinh\alpha(p) = w_0\left(1 - \frac{p^2 a_5^2}{1-w_0^2}\right) && (2.88)\\
1 - ba_5 e^{\alpha} &= -\frac{p^2 a_5^2}{1-w_0^2} && (2.89)\\
1 - ba_5 e^{-\alpha} &= 1 - w_0^2\left(1 - \frac{p^2 a_5^2}{1-w_0^2}\right). && (2.90)
\end{aligned}$$

This gives us two poles in A's, one coming from the F term and the other from A_0:

$$\begin{aligned}
A_0^{-1} &= 2\frac{b}{a_5}\sinh\alpha = \frac{2w_0}{a_5^2}\left(\frac{1-w_0^2}{2w_0} + \frac{p^2 a_5^2}{2w_0}\frac{1+w_0^2}{1-w_0^2}\right)\\
&= \frac{1+w_0^2}{1-w_0^2}\left(p^2 + \frac{1}{a_5^2}\frac{(1-w_0^2)^2}{1+w_0^2}\right) && (2.91)\\
F &= \frac{1}{a_5^2}\left(1 - ba_5 e^{\alpha}\right) - m^2\left(1 - a_5 b e^{-\alpha}\right)\\
&= \left(\frac{1}{a_5^2} - \frac{w_0}{a_5^2}\frac{1}{w_0}\left[1 + \frac{p^2 a_5^2}{1-w_0^2}\right] - m^2\left(1 - w_0 * w_0 * \left[1 - \frac{p^2 a_5^2}{1-w_0^2}\right]\right)\right)\\
&= -\left(\frac{p^2(1+m^2 a_5^2 w_0^2)}{1-w_0^2} + m^2\left(1-w_0^2\right)\right)\\
&= -\frac{1+m^2 a_5^2 w_0^2}{1-w_0^2}\left(p^2 + \frac{m^2(1-w_0^2)^2}{1+m^2 a_5^2 w^2}\right). && (2.92)
\end{aligned}$$

The pole from the A_0 is of the order of lattice cutoff $1/a_5$ while the pole from F is proportional to the "mass" parameter m (in small m limit). With these expressions we can expand the propagator for small p:

$$
\begin{aligned}
A_- &= -A_0 \frac{\left(\frac{1}{a_5^2} - m^2\right)(1 - b a_5 e^{-\alpha})}{F} \\
&= -\frac{(1-w_0^2)^2}{\left[(1+w_0^2)\left(p^2 + \frac{1}{a_5^2}\frac{(1-w_0^2)^2}{1+w_0^2}\right)\right]\left[(1+a_5^2 m^2 w_0^2)\left(p^2 + m^2 \frac{(1-w_0^2)^2}{1+m^2 a_5^2 w_0^2}\right)\right]} \\
&\quad * \left(\frac{1}{a_5^2} - m^2\right)\left(\frac{(1-w_0^2)^2}{a_5^2} + p^2\right)\frac{a_5^2}{1-w_0^2} \qquad (2.93) \\
A_+ &= -A_0 \frac{\left(\frac{1}{a_5^2} - m^2\right)(1 - b a_5 e^{\alpha})}{F} e^{-2\alpha} \\
&= -\frac{(1-w_0^2)^2}{\left[(1+w_0^2)\left(p^2 + \frac{1}{a_5^2}\frac{(1-w_0^2)^2}{1+w_0^2}\right)\right]\left[(1+a_5^2 m^2 w_0^2)\left(p^2 + m^2 \frac{(1-w_0^2)^2}{1+m^2 a_5^2 w_0^2}\right)\right]} \\
&\quad * \left(\frac{1}{a_5^2} - m^2\right)\left(\frac{p^2 a_5^2}{1-w_0^2}\right) * w_0^2 \left(1 - 2\frac{p^2 a_5^2}{1-w_0^2}\right) \qquad (2.94) \\
A_m &= -\frac{m a_5 e^{-\alpha}}{F} \\
&= \frac{m a_5 (1-w_0^2)}{\left[(1+w_0^2)\left(p^2 + \frac{1}{a_5^2}\frac{(1-w_0^2)^2}{1+w_0^2}\right)\right]\left[(1+a_5^2 m^2 w_0^2)\left(p^2 + m^2 \frac{(1-w_0^2)^2}{1+m^2 a_5^2 w_0^2}\right)\right]} \\
&\quad \times w_0 \left(1 - \frac{p^2 a_5^2}{1-w_0^2}\right) \qquad (2.95)
\end{aligned}
$$

In the $p \to 0$ limit this becomes (for small m)

$$
\begin{aligned}
A_- &= -\frac{(1-w_0^2)(1-m^2 a_5^2)}{p^2 + m^2(1-w_0^2)^2} \qquad (2.96) \\
A_+ &= -\frac{(1-m^2 a_5^2) w_0^2}{(1-w_0^2)} \frac{p^2 a_5^2}{p^2 + m^2 (1-w_0^2)^2} \qquad (2.97) \\
A_m &= \frac{m a_5 w_0 (1-w_0^2)}{p^2 + m^2(1-w_0^2)^2} \qquad (2.98)
\end{aligned}
$$

so for $m = 0$ only A_- has a pole at $p = 0$

$$A_- = -\frac{1}{p^2} \times (1-w_0^2) \qquad (2.99)$$

This yields IR divergent parts of $G_\pm$

$$
\begin{aligned}
G_-\Big|_{IR} &= e^{-\alpha(s+s'-2)}A_- = -\frac{1}{k^2}\times w_0^{s+s'-2}(1-w_0^2) && (2.100)\\
G_+\Big|_{IR} &= e^{-\alpha(2N-s-s')}A_- = -\frac{1}{k^2}\times w_0^{2N-s-s'}(1-w_0^2) && (2.101)\\
& && (2.102)
\end{aligned}
$$

so IR-divergent part of the DW-fermion propagator is then given by

$$
\begin{aligned}
S_F(k)\Big|_{IR} &= \left(-i\gamma\cdot\bar{k}+W_+\right)P_-G_- + \left(-i\gamma\cdot\bar{k}+W_-\right)P_+G_+ \\
&= -\frac{1}{k^2}\left(1-w_0^2\right)\left\{\left(-i\gamma\cdot\bar{k}\right)P_-w_0^{s+s'-2} + \left(-i\gamma\cdot\bar{k}\right)P_+w_0^{2N-s-s'}\right\} \\
&= \frac{i\gamma\cdot k}{k^2}\left(1-w_0^2\right)\left\{P_-w_0^{s+s'-2}+P_+w_0^{2N-s-s'}\right\} \qquad (2.103)
\end{aligned}
$$

For $Ma_5 > 1$ the only change is to replace $w_0 \to (-|w_0|)$.

2.2 Physical quarks and their propagators

Physical fields are defined on the boundary of the 5^{th} dimension:

$$
q(x) = P_+\psi_1(x) + P_-\psi_N(x)\,, \qquad \bar{q}(x) = \bar{\psi}_1(x)P_- + \bar{\psi}_N(x)P_+\,. \qquad (2.104)
$$

Their propagators in terms of the 5D propagator $S_{st}(p) = \langle\psi_s(-p)\,\bar{\psi}_t(p)\rangle$ are given by

$$
\begin{aligned}
\langle q(-p)\bar{\psi}_s(p)\rangle &= P_+\langle\psi_1(-p)\,\bar{\psi}_t(p)\rangle + P_-\langle\psi_N(-p)\,\bar{\psi}_t(p)\rangle \\
&= \sum_t (P_+\delta_{1,t}+P_-\delta_{N,t})S_{st}(p) && (2.105)\\
\langle\psi_s(-p)\bar{q}(p)\rangle &= \langle\psi_s(-p)\,\bar{\psi}_1(p)\rangle P_- + \langle\psi_s(-p)\,\bar{\psi}_N(p)\rangle P_+ \\
&= \sum_t S_{st}(p)(P_+\delta_{t,N}+P_-\delta_{t,1}) && (2.106)
\end{aligned}
$$

Physical-to-5D propagators

Since these propagators will sit in an expression containing $P_\pm$, it's convenient to commute $P_\pm$ projectors with the γ matrix in S_{st}. Using the fact that

$$P_\pm \gamma_\mu = \gamma_\mu P_\mp \tag{2.107}$$

we get

$$\begin{aligned}
\langle q(-p)\bar{\Psi}_s(p)\rangle &= [-i\bar{p}\cdot\gamma P_- + W^- P_+]_{1t} G^+_{ts} P_+ + [-i\bar{p}\cdot\gamma P_- + W^+ P_+]_{1t} G^-_{ts} P_- \\
&\quad + [-i\bar{p}\cdot\gamma P_+ + W^- P_-]_{Nt} G^+_{ts} P_+ + [-i\bar{p}\cdot\gamma P_+ + W^+ P_-]_{Nt} G^-_{ts} P_- \\
&= -i\bar{p}\cdot\gamma(G^-_{1s}P_- + G^+_{Ns}P_+) + [W^- G_+]_{1s} P_+ + [W^+ G_-]_{Ns} P_- \,. \qquad (2.108)
\end{aligned}$$

To simplify these expressions, we use the definition of the propagator

$$\begin{aligned}
G^\mp_{st} &= A_0 e^{-\alpha|s-t|} + A_\mp e^{-\alpha(s+t-2)} + A_\pm e^{-\alpha(N-s-t)} \\
&\quad + A_m\left(e^{-\alpha(N-s+t-1)} + e^{-\alpha(N+s-t-1)}\right) \qquad (2.109) \\
W^\pm_{st} &= -b(p)\delta_{st} + \frac{1}{a_5}\delta_{s\pm1,t} + m\,\delta_{s,\left[\begin{smallmatrix}N\\1\end{smallmatrix}\right]}\delta_{t,\left[\begin{smallmatrix}1\\N\end{smallmatrix}\right]} \qquad (2.110) \\
& \qquad (2.111)
\end{aligned}$$

to get

$$\begin{aligned}
[W^- G_+]_{1s} &= -b(p)G^+_{1s} + mG^+_{Nt} \qquad (2.112) \\
[W^+ G_-]_{Ns} &= -b(p)G^-_{Ns} + mG^-_{1t}\,. \qquad (2.113)
\end{aligned}$$

Since $G^\pm st = G^\pm ts$, we only need to calculate 4 quantities; furthermore, since $G^+_{s,t} = G^-_{N-s+1,N-t+1}$, we are left with

$$\begin{aligned}
G^-_{1s} &= e^{-\alpha(s-1)}(A_0 + A_- + A_m e^{-\alpha(N-1)}) + e^{-\alpha(N-s)}(A_m + A_+ e^{-\alpha(N-1)}) \qquad (2.114) \\
G^-_{Ns} &= e^{-\alpha(s-1)}(A_m + A_- e^{-\alpha(N-1)}) + e^{-\alpha(N-s)}(A_0 + A_+ + A_m e^{-\alpha(N-1)}) \qquad (2.115) \\
G^+_{1s} &= e^{-\alpha(s-1)}(A_0 + A_+ + A_m e^{-\alpha(N-1)}) + e^{-\alpha(N-s)}(A_m + A_- e^{-\alpha(N-1)}) \qquad (2.116)
\end{aligned}$$

$$G^+_{Ns} = e^{-\alpha(s-1)}(A_m + A_+ e^{-\alpha(N-1)}) + e^{-\alpha(N-s)}(A_0 + A_- + A_m e^{-\alpha(N-1)}) \,. \quad (2.117)$$

To evaluate $A_0 + A_\pm + A_m e^{-\alpha(N-1)}$ and $(A_m + A_\mp e^{-\alpha(N-1)}$ we use their definitions

$$A_- = -\frac{A_0}{F_N}\left(\frac{1}{a_5^2} - m^2\right)\left(1 - b a_5 e^{-\alpha}\right) \quad (2.118)$$

$$A_+ = -\frac{A_0}{F_N}\left(\frac{1}{a_5^2} - m^2\right)\left(1 - b a_5 e^{\alpha}\right) e^{-2\alpha} \quad (2.119)$$

$$A_m = -\frac{A_0}{F_N} e^{-\alpha}\left(2mb\sinh\alpha - e^{-\alpha N}\left[\frac{1}{a_5^2}\left(1 - ba_5 e^{-\alpha}\right) - m^2\left(1 - ba_5 e^{\alpha}\right)\right]\right) \quad (2.120)$$

with $A_0^{-1} = 2b/a_5 \sinh\alpha$ and

$$F_N = \underbrace{\frac{1}{a_5^2}\left(1 - ba_5 e^{\alpha}\right) - m^2\left(1 - ba_5 e^{-\alpha}\right)}_{F} + e^{-\alpha N}\left[4\sinh\alpha\left(\frac{1}{a_5^2} - m^2\right)\right]$$
$$-e^{-2\alpha N}\left[\frac{1}{a_5^2}\left(1 - ba_5 e^{-\alpha}\right) - m^2\left(1 - ba_5 e^{\alpha}\right)\right] \quad (2.121)$$

Up to this point everything was valid for general N; now (for simplicity) we let $N \to \infty$ which means we can neglect all terms proportional to $e^{-\alpha N}$ and we get

$$A_m + A_\mp e^{-\alpha(N-1)} = -\frac{A_0}{F} e^{-\alpha} 2mb\sinh\alpha = -\frac{ma_5 e^{-\alpha}}{F} \quad (2.122)$$

$$A_0 + A_- + A_m e^{-\alpha(N-1)} = \frac{A_0}{F}\left(-\frac{b}{a_5} 2\sinh\alpha\right) = -\frac{1}{F} \quad (2.123)$$

$$A_0 + A_+ + A_m e^{-\alpha(N-1)} = \frac{A_0}{F} 2\sinh\alpha\left(\frac{1}{a_5^2} - m^2 - \frac{b}{a_5} e^{\alpha}\right) e^{-\alpha}\,, \quad (2.124)$$

which yields

$$G^-_{1s} = \frac{1}{F}\left(-e^{-\alpha(s-1)} - ma_5 e^{-\alpha} e^{-\alpha(N-s)}\right) \quad (2.125)$$

$$G^-_{Ns} = \frac{1}{F}\left(-ma_5 e^{-\alpha} e^{-\alpha(s-1)} + \frac{e^{-\alpha}}{b/a_5}\left[\frac{1}{a_5^2} - m^2 - \frac{b}{a_5} e^{\alpha}\right] e^{-\alpha(N-s)}\right) \quad (2.126)$$

$$G^+_{1s} = G^-_{N,N-s+1}$$
$$= \frac{1}{F}\left(-ma_5 e^{-\alpha} e^{-\alpha(N-s)} + \frac{e^{-\alpha}}{b/a_5}\left[\frac{1}{a_5^2} - m^2 - \frac{b}{a_5} e^{\alpha}\right] e^{-\alpha(s-1)}\right) \quad (2.127)$$

$$G^{+}_{Ns} = G^{-}_{1,N-s+1} = \frac{1}{F}\left(-e^{-\alpha(N-s)} - ma_5 e^{-\alpha} e^{-\alpha(s-1)}\right) . \qquad (2.128)$$

The P_- part of the propagator then becomes

$$\begin{aligned}
\langle q(-p)\bar{\psi}_s(p)\rangle P_- &= (-i\bar{p}\cdot\gamma + m)G^{-}_{1s} - bG^{-}_{Ns} \\
&= \frac{(-i\bar{p}\cdot\gamma+m)}{F}\left(-e^{-\alpha(s-1)} - ma_5 e^{-\alpha} e^{-\alpha(N-s)}\right) \\
&\quad -\frac{b}{F}\left(-ma_5 e^{-\alpha} e^{-\alpha(s-1)} + \frac{e^{-\alpha}}{b/a_5}\left[\frac{1}{a_5^2} - m^2 - \frac{b}{a_5}e^{\alpha}\right] e^{-\alpha(N-s)}\right) \\
&= i\bar{p}\cdot\gamma \frac{e^{-\alpha(s-1)} + ma_5 e^{-\alpha} e^{-\alpha(N-s)}}{F} - \frac{m(1-ba_5 e^{-\alpha})e^{-\alpha(s-1)}}{F} \\
&\quad -a_5 e^{-\alpha} \frac{\frac{1}{a_5^2}(1-ba_5 e^{\alpha})}{F} e^{-\alpha(N-s)} . \qquad (2.129)
\end{aligned}$$

By adding and subtracting the term $a_5 e^{-\alpha}\frac{m^2(1-ba_5 e^{-\alpha})}{F} e^{-\alpha(N-s)}$, this simplifies to

$$\begin{aligned}
S^{OUT}_s P_- &\equiv \langle q(-p)\bar{\psi}_s(p)\rangle P_- \\
&= \left[i\bar{p}\cdot\gamma - m(1-ba_5 e^{-\alpha})\right] \frac{e^{-\alpha(s-1)} + ma_5 e^{-\alpha} e^{-\alpha(N-s)}}{F} \\
&\quad -a_5 e^{-\alpha} e^{-\alpha(N-s)} . \qquad (2.130)
\end{aligned}$$

Repeating the same calculation for the P_+ part we get

$$\begin{aligned}
S^{OUT}_s P_+ &\equiv \langle q(-p)\bar{\psi}_s(p)\rangle P_+ = \left(\langle q(-p)\bar{\psi}_s(p)\rangle P_-\right)\Big|_{s\to N-s+1} \\
&= \left[i\bar{p}\cdot\gamma - m(1-ba_5 e^{-\alpha})\right] \frac{e^{-\alpha(N-s)} + ma_5 e^{-\alpha} e^{-\alpha(s-1)}}{F} \\
&\quad -a_5 e^{-\alpha} e^{-\alpha(s-1)} . \qquad (2.131)
\end{aligned}$$

To get the second propagator $\langle\psi_s(-p)\bar{q}(p)\rangle$, we use the definition of the propagator

$$S^{\alpha\beta}_F(p) \equiv \langle 0|\psi_\alpha(-p)\bar{\psi}_\beta(p)|0\rangle \qquad (2.132)$$

and the fact that

$$\left(\psi_\alpha(-p)\bar{\psi}_\beta(p)\right)^\dagger = \gamma_0\psi_\beta(-p)\psi^\dagger_\alpha(p) \qquad (2.133)$$

to get the equation

$$\gamma_0(S_F^{\alpha\beta}(p))^\dagger\gamma_0 \equiv S_F^{\beta\alpha}(p) \tag{2.134}$$

which yields

$$\begin{aligned}
S_s^{IN} &\equiv \langle\psi_s(-p)\bar{q}(p)\rangle \\
&= \gamma_0(\langle q(-p)\bar{\psi}_s(p)\rangle)^\dagger\gamma_0 \\
&= P_-\left(\left[i\bar{p}\cdot\gamma - m(1-ba_5e^{-\alpha})\right]\frac{e^{-\alpha(s-1)}+ma_5e^{-\alpha}e^{-\alpha(N-s)}}{F} - a_5e^{-\alpha}e^{-\alpha(N-s)}\right) \\
&\quad +P_+\left(\left[i\bar{p}\cdot\gamma - m(1-ba_5e^{-\alpha})\right]\frac{e^{-\alpha(N-s)}+ma_5e^{-\alpha}e^{-\alpha(s-1)}}{F} - a_5e^{-\alpha}e^{-\alpha(s-1)}\right)
\end{aligned} \tag{2.135}$$

which agrees with formula (3.3) and (3.4) in Aoki [16]. In general, we can parametrize these formulas as

$$S^{OUT} = -i\bar{p}\cdot\gamma(g_+P_+ + g_-P_-) + (\sigma_+P_+ + \sigma_-P_-) \tag{2.136}$$

$$S^{IN} = (g_-P_+ + g_+P_-)(-i\bar{p}\cdot\gamma) + (\sigma_-P_+ + \sigma_+P_-) \tag{2.137}$$

Repeating the same procedure for $Ma_5 > 1$, we again see that the prescription

$$b \to -|b|\ , \qquad e^{\pm\alpha} \to -e^{\pm\alpha} \tag{2.138}$$

yields the correct result

$$\begin{aligned}
S_s^{OUT} &\equiv \langle q(-p)\bar{\psi}_s(p)\rangle = \left[i\bar{p}\cdot\gamma - m(1-|b|a_5e^{-\alpha})\right]\frac{(\pm e^{-\alpha})^{(s-1)}P_- + (\pm e^{-\alpha})^{(N-s)}P_+}{F} \\
&\quad \pm\left\{\left[i\bar{p}\cdot\gamma - m(1-|b|a_5e^{-\alpha})\right]\frac{ma_5e^{-\alpha}}{F} - a_5e^{-\alpha}\right\} \\
&\quad\ \times\left[(\pm e^{-\alpha})^{(N-s)}P_- + (\pm e^{-\alpha})^{(s-1)}P_+\right] \\
S_s^{IN} &\equiv \langle\psi_s(-p)\bar{q}(p)\rangle = \gamma_0(\langle q(p)\bar{\psi}_s(-p)\rangle)^\dagger\gamma_0 \\
&= \frac{P_-(\pm e^{-\alpha})^{(s-1)} + P_+(\pm e^{-\alpha})^{(N-s)}}{F}\left[i\bar{p}\cdot\gamma - m(1-ba_5e^{-\alpha})\right] \\
&\quad \pm\left[P_-(\pm e^{-\alpha})^{(N-s)} + P_+(\pm e^{-\alpha})^{(s-1)}\right]
\end{aligned}$$

$$\times \left\{ \frac{m a_5 e^{-\alpha}}{F} \left[i\bar{p}\cdot\gamma - m(1 - b a_5 e^{-\alpha}) \right] - a_5 e^{-\alpha} \right\} \tag{2.139}$$

Physical-to-physical quark propagator

The physical-to-physical propagator can easily be obtained from the physical-to-5D:

$$\begin{aligned} \langle q(-p)\bar{q}(p)\rangle &= \langle q(-p)\bar{\psi}_N(p)\rangle P_+ + \langle q(-p)\bar{\psi}_1(p)\rangle P_- & (2.140)\\ &= P_- \langle \psi_1(-p)\bar{q}(p)\rangle + P_+ \langle \psi_N(-p)\bar{q}(p)\rangle & (2.141) \end{aligned}$$

Evaluating any of the two forms and commuting $P_\pm$ matrices either to the left or to the right, we get the formula

$$\begin{aligned} S^{PHYS}(p) &\equiv \langle q(-p)\bar{q}(p)\rangle = \frac{i\bar{p}\cdot\gamma - m(1-|b|\,a_5 e^{-\alpha})}{F} \\ &= \frac{i\bar{p}\cdot\gamma - m(1-|b|\,a_5 e^{-\alpha})}{\frac{1}{a_5^2}(1-|b|\,a_5 e^{\alpha}) - m^2(1-|b|\,a_5 e^{-\alpha})} \end{aligned} \tag{2.142}$$

which agrees with formula (2.27) in Aoki [16]. The inverse of this propagator can be written as

$$S^{-1}(p) = F\,\frac{-i\bar{p}\cdot\gamma - m(1-|b|\,a_5 e^{-\alpha})}{\bar{p}^2 + m^2(1-|b|\,a_5 e^{-\alpha})^2}\,. \tag{2.143}$$

In general we can parametrize this as

$$S_{PHYS}(p) = -i\bar{p}\cdot\gamma\,\sigma_V(p) + \sigma_S(p)\,, \qquad S^{-1}_{PHYS}(p) = i\bar{p}\cdot\gamma A(p) + B(p) \tag{2.144}$$

with

$$A(p) = \frac{\sigma_V}{\bar{p}^2\sigma_V^2 + \sigma_S^2}\,, \qquad B(p) = \frac{\sigma_S}{\bar{p}^2\sigma_V^2 + \sigma_S^2} \tag{2.145}$$

and in this particular case

$$\sigma_V = -\frac{1}{F}\,, \qquad \sigma_S = -m\frac{1 - b a_5 e^{-\alpha(p)}}{F} \tag{2.146}$$

Amputating external legs

Since the propagator $\langle q(-p)\bar{\psi}_s(p)\rangle$ describes propagation from the point s in the 5^{th} dimension to the physical quark, each graph containing this propagator will not have it's external legs amputated. Since we're interested in amputated graphs describing pure operator renormalization, it's convenient to describe the effect of this 5D-to-physical propagation effect by introducing "incoming" and "outgoing" propagators

$$\begin{aligned} \bar{S}_s^{OUT}(p) &\equiv \frac{1}{\langle q(-p)\bar{q}(p)\rangle}\langle q(-p)\bar{\psi}_s(p)\rangle = \frac{1}{S_{PHYS}(p)}S_s^{OUT}(p) \qquad (2.147)\\ \bar{S}_s^{IN}(p) &\equiv \langle \psi_s(-p)\bar{q}(p)\rangle\frac{1}{\langle q(-p)\bar{q}(p)\rangle} = S_s^{IN}(p)\frac{1}{S_{PHYS}(p)}\,. \qquad (2.148)\end{aligned}$$

Evaluation is pretty straightforward:

$$\begin{aligned} \bar{S}_s^{OUT}(p) &= F\frac{-i\bar{p}\cdot\gamma - m(1-ba_5e^{-\alpha})}{\bar{p}^2+m^2(1-ba_5e^{-\alpha})^2}\times\\ &\quad \left\{\left[(i\bar{p}\cdot\gamma - m(1-ba_5e^{-\alpha}))\frac{e^{-\alpha(s-1)}+ma_5e^{-\alpha}e^{-\alpha(N-s)}}{F}\right.\right.\\ &\qquad \left. -a_5e^{-\alpha}e^{-\alpha(N-s)}\right]P_-\\ &\quad +\left[i\bar{p}\cdot\gamma - m(1-ba_5e^{-\alpha}))\frac{e^{-\alpha(N-s)}+ma_5e^{-\alpha}e^{-\alpha(s-1)}}{F}\right.\\ &\qquad \left.\left. -a_5e^{-\alpha}e^{-\alpha(s-1)}\right]P_+\right\} \qquad (2.149)\\ &= \left\{e^{-\alpha(s-1)}+a_5e^{-\alpha}e^{-\alpha(N-s)}\left[m+F\frac{i\bar{p}\cdot\gamma+m(1-ba_5e^{-\alpha})}{\bar{p}^2+m^2(1-ba_5e^{-\alpha})^2}\right]\right\}P_-\\ &\quad +\left\{e^{-\alpha(N-s)}+a_5e^{-\alpha}e^{-\alpha(s-1)}\left[m+F\frac{i\bar{p}\cdot\gamma+m(1-ba_5e^{-\alpha})}{\bar{p}^2+m^2(1-ba_5e^{-\alpha})^2}\right]\right\}P_+ \\ &\qquad (2.150)\end{aligned}$$

$$\begin{aligned} \bar{S}_s^{IN}(p) &= P_+\left\{e^{-\alpha(s-1)}+a_5e^{-\alpha}e^{-\alpha(N-s)}\left[m+F\frac{i\bar{p}\cdot\gamma+m(1-ba_5e^{-\alpha})}{\bar{p}^2+m^2(1-ba_5e^{-\alpha})^2}\right]\right\}\\ &\quad +P_-\left\{e^{-\alpha(N-s)}+a_5e^{-\alpha}e^{-\alpha(s-1)}\left[m+F\frac{i\bar{p}\cdot\gamma+m(1-ba_5e^{-\alpha})}{\bar{p}^2+m^2(1-ba_5e^{-\alpha})^2}\right]\right\}\,. \\ &\qquad (2.151)\end{aligned}$$

Redoing it for $Ma_5 > 1$, these two propagators can be written as

$$\bar{S}_s^{OUT}(p) = -i\bar{p}\cdot\gamma(\bar{g}_+P_+ + \bar{g}_-P_-) + \bar{\sigma}_+P_+ + \bar{\sigma}_-P_- \tag{2.152}$$

$$\bar{S}_s^{IN}(p) = (\bar{g}_-P_+ + \bar{g}_+P_-)(-i\bar{p}\cdot\gamma) + \bar{\sigma}_-P_+ + \bar{\sigma}_+P_- \tag{2.153}$$

with

$$\bar{g}_+(p) = \mathcal{A}(\pm e^{-\alpha})^{(s-1)} \to \mathcal{A}(\pm|w_0|)^{s-1} \tag{2.154}$$

$$\bar{g}_-(p) = \mathcal{A}(\pm e^{-\alpha})^{(N-s)} \to \mathcal{A}(\pm|w_0|)^{N-s} \tag{2.155}$$

$$\bar{\sigma}_+(p) = \mathcal{B}(\pm e^{-\alpha})^{(s-1)} + (\pm e^{-\alpha})^{(N-s)} \to (\pm|w_0|)^{N-s} \tag{2.156}$$

$$\bar{\sigma}_-(p) = \mathcal{B}(\pm e^{-\alpha})^{(N-s)} + (\pm e^{-\alpha})^{(s-1)} \to (\pm|w_0|)^{s-1} \tag{2.157}$$

$$\mathcal{A} = \mp\frac{a_5e^{-\alpha(p)}F(p)}{\bar{p}^2 + m^2\left(1-|b(p)|a_5e^{-\alpha(p)}\right)^2} \to \pm\frac{a_5|w_0|}{1-w_0^2} \tag{2.158}$$

$$\mathcal{B} = \pm a_5me^{-\alpha(p)}\left[1+\frac{F\left(1-|b(p)|a_5e^{-\alpha(p)}\right)}{\bar{p}^2+m^2\left(1-|b(p)|a_5e^{-\alpha(p)}\right)^2}\right] \to 0 \tag{2.159}$$

This formulation is useful since it separates the DW-specific part from the Dirac matrix structure of propagators.

Zero momentum limit of physical propagators

In the limit where $p \to 0$, we have

$$\begin{aligned} b &\to \frac{w_0}{a_5} \\ e^{\mp\alpha} &\to w_0^{\pm 1} \\ F &= \frac{1}{a^2}\left(1-ba_5e^{\alpha}\right) - m^2\left(1-ba_5e^{-\alpha}\right) \to -m^2(1-w_0^2)\,, \end{aligned} \tag{2.160}$$

which yields

$$\bar{S}_s^{OUT}(p) = \left(P_-w_0^{s-1} + P_+w_0^{N-s}\right) - \frac{a_5w_0}{1-w_0^2}ip\cdot\gamma\left(P_-w_0^{N-s}+P_+w_0^{s-1}\right) \tag{2.161}$$

$$\bar{S}_s^{IN}(p) = \left(P_+w_0^{s-1} + P_-w_0^{N-s}\right) - \left(P_+w_0^{N-s}+P_-w_0^{s-1}\right)\frac{a_5w_0}{1-w_0^2}ip\cdot\gamma \tag{2.162}$$

$$S^{PHYS}(p) = \frac{1-w_0^2}{ip\cdot\gamma+m(1-w_0^2)} \tag{2.163}$$

2.3 Sums in 5D

To evaluate DW expressions for physical quarks, one has to add the effect of 5D propagation to external legs; this can be written as

$$I_q = \sum_{s,t=1}^{N} [S_s^{OUT}][I_{s,t}][S_t^{IN}] \tag{2.164}$$

where $S_s^{IN/OUT}$ has the 5D structure

$$S_s^{IN/OUT} \sim const. \times e^{-\alpha_p(s-1)} + const. \times e^{-\alpha_p(N-s)} \tag{2.165}$$

and $I_{s,t}$ contains the propagator S_F with 5D structure

$$\begin{aligned} [G_\pm]_{s,t} &= A_0 e^{-\alpha_k|s-t|} + A_\pm e^{-\alpha_k(s-t-2)} + A_\mp e^{-\alpha_k(2N-s-t)} \\ &\quad + A_m\left(e^{-\alpha_k(N-s+t-1)} + e^{-\alpha_k(N-t+s-1)}\right) \qquad (2.166) \\ W_{st}^{\pm} &= -b(p)\delta_{st} + \frac{1}{a_5}\delta_{s\pm1,t} + m\,\delta_{s,\left[{N \atop 1}\right]}\delta_{t,\left[{1 \atop N}\right]} \qquad (2.167) \end{aligned}$$

to get

$$[W^{\mp}G_\pm]_{s,t} = -b(p)[G_\pm]_{s,t} + \frac{1}{a_5}[G_\pm]_{s\mp1,t} + m\delta_{s,\left[{1 \atop N}\right]}[G_\pm]_{\left[{N \atop 1}\right],t}\,. \tag{2.168}$$

These sums are evaluated exactly both for finite and infinite N in appendix D. In the next subsections we evaluate them for a more general case where we can have three different phase factors $e^{-\alpha_k} \equiv e^{-\alpha}$, $e^{-\alpha_p(out)} \to w_0$ and $e^{-\alpha_p(in)} \to w$.

2.3.1 Sums of $G_\pm$ term in propagator

With the help of formulas from appendix D, we can get the sums over the $G_\pm$ term

$$\begin{aligned}
[G_\pm]_{st} &= A_0 e^{-\alpha|s-t|} + A_\pm e^{-\alpha(s+t-2)} + A_\mp e^{-\alpha(2N-s-t)} \\
&\quad + A_m \left(e^{-\alpha(N-s+t-1)} + e^{-\alpha(N-t+s-1)} \right) \qquad (2.169)
\end{aligned}$$

which yields

$$\sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{st} w_0^{t-1} = \frac{A_0\left(1 - w_0 \frac{2\sinh\alpha}{1-w_0^2}\right)}{(e^{-\alpha}-w_0)(e^{\alpha}-w_0)} + \frac{A_\pm}{(1-w_0 e^{-\alpha})^2} \qquad (2.170)$$

$$\sum_{s,t=1}^{N} w_0^{N-s}[G_\pm]_{st} w_0^{N-s} = \frac{A_0\left(1 - w_0 \frac{2\sinh\alpha}{1-w_0^2}\right)}{(e^{-\alpha}-w_0)(e^{\alpha}-w_0)} + \frac{A_\mp}{(1-w_0 e^{-\alpha})^2} \qquad (2.171)$$

$$\sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{st} w_0^{N-t} = \frac{A_m}{(1-w_0 e^{-\alpha})^2} \qquad (2.172)$$

$$\sum_{s,t=1}^{N} w_0^{N-s}[G_\pm]_{st} w_0^{t-1} = \frac{A_m}{(1-w_0 e^{-\alpha})^2} \qquad (2.173)$$

where $w_0 = e^{-\alpha(p)}|_{p\to 0}$. The sums above are also valid for finite p (with w_0 replaced by $e^{-\alpha(p)}$). Since $w_0 = e^{-\alpha_0}$, we can simplify the first two terms above

$$\begin{aligned}
\sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{st} w_0^{t-1} &= \sum_{s,t=1}^{N} w_0^{N-s}[G_\mp]_{st} w_0^{N-s} = A_0 \left[\frac{1 - \frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0 - \cosh\alpha)} \right. \\
&\quad \left. - \frac{1}{(1-w_0 e^{-\alpha})^2} \frac{\left(\frac{1}{a_5^2} - m^2\right)(1 - b a_5 e^{\pm\alpha}) e^{-\alpha\mp\alpha}}{\frac{1}{a_5^2}(1 - b a_5 e^{\alpha}) - m^2 (1 - b a_5 e^{-\alpha})} \right] \qquad (2.174)
\end{aligned}$$

In the $m \to 0$ limit, this becomes

$$\begin{aligned}
\bar{G}_+ &\equiv \sum_{s,t=1}^{N} w_0^{s-1}[G_+]_{st} w_0^{t-1} = \sum_{s,t=1}^{N} w_0^{N-s}[G_-]_{st} w_0^{N-s} \\
&= A_0 \left(\frac{1 - \frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0 - \cosh\alpha)} - \frac{1}{(e^{\alpha} - w_0)^2} \right) \qquad (2.175)
\end{aligned}$$

$$
\begin{aligned}
\bar{G}_- &\equiv \sum_{s,t=1}^{N} w_0^{s-1}[G_-]_{st} w_0^{t-1} = \sum_{s,t=1}^{N} w_0^{N-s}[G_+]_{st} w_0^{N-s} \\
&= A_0 \left(\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)} - \frac{1}{(e^{\alpha}-w_0)^2}\frac{e^{\alpha}-ba_5}{e^{-\alpha}-ba_5} \right) \qquad (2.176)
\end{aligned}
$$

which agrees with formulas (3.24) and (3.25) in Aoki [16] aside from the factor $1-w_0^2$. Since these two often come in the combination $\sigma_V = (G_+ + G_-)/2$, we can also evaluate that

$$
\begin{aligned}
\bar{\sigma}_V &\equiv \sum_{s,t=1}^{N} w_0^{s-1}[\sigma_V]_{st} w_0^{t-1} = \sum_{s,t=1}^{N} w_0^{N-s}[\sigma_V]_{st} w_0^{N-s} \\
&= A_0 \left(\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)} \right. \\
&\qquad \left. - \frac{e^{-\alpha}}{(1-w_0e^{-\alpha})^2} \frac{\left(\frac{1}{a_5^2}-m^2\right)(\cosh\alpha - ba_5)}{\frac{1}{a_5^2}(1-ba_5e^{\alpha}) - m^2(1-ba_5e^{-\alpha})} \right) \qquad (2.177) \\
&\overset{m\to 0}{\longrightarrow} A_0 \left(\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)} - \frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\frac{\cosh\alpha-ba_5}{1-ba_5e^{\alpha}} \right) \qquad (2.178)
\end{aligned}
$$

For $Ma_5 > 1$ we have

$$
\begin{aligned}
\bar{G}_\pm &= A_0 \left[\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2|w_0|(\cosh\alpha_0-\cosh\alpha)} \right. \\
&\qquad \left. - \frac{1}{(1-|w_0|e^{-\alpha})^2} \frac{\left(\frac{1}{a_5^2}-m^2\right)(1-|b|a_5e^{\pm\alpha})e^{-\alpha\mp\alpha}}{\frac{1}{a_5^2}(1-|b|a_5e^{\alpha}) - m^2(1-|b|a_5e^{-\alpha})} \right]
\end{aligned}
$$

The last two terms are proportional to the mass parameter m. Using the definition of $A_m = -\frac{ma_5}{F}e^{-\alpha}$, we get

$$
\begin{aligned}
\tilde{G}_\pm &= \tilde{\sigma}_V \equiv \sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{st} w_0^{N-t} = \sum_{s,t=1}^{N} w_0^{N-s}[G_\mp]_{st} w_0^{t-1} = \frac{ma_5e^{-\alpha}}{F}\frac{1}{(1-w_0e^{-\alpha})^2} \\
&= \frac{ma_5e^{-\alpha}}{\frac{1}{a_5^2}(1-ba_5e^{\alpha}) - m^2(1-ba_5e^{-\alpha})}\frac{1}{(1-w_0e^{-\alpha})^2}
\end{aligned}
$$

$$= \frac{ma_5 e^{-\alpha}}{\frac{1}{a_5^2}(1 - ba_5 e^{-\alpha})} \frac{1}{(1 - w_0 e^{-\alpha})^2} + O(m^3) \tag{2.179}$$

or for general $0 < Ma_5 < 2$

$$\begin{aligned} \tilde{G}_+ &= \tilde{G}_- = \tilde{\sigma}_V = \pm \frac{ma_5 e^{-\alpha}}{F} \frac{1}{(1 - |w_0| e^{-\alpha})^2} \\ &= \pm \frac{ma_5 e^{-\alpha}}{\frac{1}{a_5^2}(1 - |b| a_5 e^{\alpha}) - m^2 (1 - |b| a_5 e^{-\alpha})} \frac{1}{(1 - |w_0| e^{-\alpha})^2} \end{aligned} \tag{2.180}$$

2.3.2 Sums of $W^{\mp} G_{\pm}$ term in propagator

Since the matrix $W^{\pm}$ can be written as

$$[S_\pm]_{st} \equiv [W^{\mp} G_\pm]_{s,t} = -b(p)[G_\pm]_{s,t} + \frac{1}{a_5}[G_\pm]_{s \mp 1,t} + m\delta_{s,\left[\substack{1\\N}\right]}[G_\pm]_{\left[\substack{N\\1}\right],t} \,. \tag{2.181}$$

we get 3 terms contributing

1. **The first term** was already calculated in previous subsection.

2. **The last term** is straightforward to calculate:

$$\sum_{s,t=1}^{N} w_0^{s-1} m\delta_{s,\left[\substack{1\\N}\right]}[G_\pm]_{\left[\substack{N\\1}\right],t} w_0^{t-1} = m \begin{cases} [G_+]_{Nt} w_0^{t-1} \\ w_0^{N-1} [G_-]_{1t} w_0^{t-1} \end{cases} \tag{2.182}$$

$$\sum_{s,t=1}^{N} w_0^{N-s} m\delta_{s,\left[\substack{1\\N}\right]}[G_\pm]_{\left[\substack{N\\1}\right],t} w_0^{N-s} = m \begin{cases} [G_+]_{Nt} w_0^{N-t} \\ w_0^{N-1} [G_-]_{1t} w_0^{N-t} \end{cases} \tag{2.183}$$

$$\sum_{s,t=1}^{N} w_0^{s-1} m\delta_{s,\left[\substack{1\\N}\right]}[G_\pm]_{\left[\substack{N\\1}\right],t} w_0^{N-t} = m \begin{cases} w_0^{N-1} [G_+]_{Nt} w_0^{t-1} \\ [G_-]_{1t} w_0^{t-1} \end{cases} \tag{2.184}$$

$$\sum_{s,t=1}^{N} w_0^{N-s} m\delta_{s,\left[\substack{1\\N}\right]}[G_\pm]_{\left[\substack{N\\1}\right],t} w_0^{t-1} = m \begin{cases} w_0^{N-1} [G_+]_{Nt} w_0^{N-t} \\ [G_-]_{1t} w_0^{N-t} \end{cases} \tag{2.185}$$

We already have expressions

$$\begin{aligned} G^-_{1t} &= e^{-\alpha(t-1)}(A_0 + A_- + A_m e^{-\alpha(N-1)}) + e^{-\alpha(N-t)}(A_m + A_+ e^{-\alpha(N-1)}) \quad (2.186) \\ G^+_{Nt} &= e^{-\alpha(t-1)}(A_m + A_+ e^{-\alpha(N-1)}) + e^{-\alpha(N-t)}(A_0 + A_- + A_m e^{-\alpha(N-1)}) \quad (2.187) \end{aligned}$$

Summing them over t we get

$$\sum [G_+]_{Nt} w_0^{t-1} = \sum [G_-]_{1t} w_0^{N-t} = \frac{A_m}{1-w_0e^{-\alpha}} \tag{2.188}$$

$$\sum [G_+]_{Nt} w_0^{N-t} = \sum [G_-]_{1t} w_0^{t-1} = \frac{A_0+A_-}{1-w_0e^{-\alpha}} \tag{2.189}$$

so their contributions to $\bar{S}$ and $\tilde{S}$ are (for general $0 < Ma_5 < 2$)

$$\bar{S}_+|_{\text{last term}} = m\frac{A_m}{1-|w_0|\,e^{-\alpha}} = \mp\frac{m^2a_5e^{-\alpha}}{F}\frac{1}{1-|w_0|\,e^{-\alpha}} \tag{2.190}$$

$$\bar{S}_-|_{\text{last term}} = 0 \tag{2.191}$$

$$\tilde{S}_+|_{\text{last term}} = m\frac{A_0+A_-}{1-|w_0|\,e^{-\alpha}} = -\frac{m}{F}\frac{1}{1-|w_0|\,e^{-\alpha}} \tag{2.192}$$

$$\tilde{S}_-|_{\text{last term}} = 0 \tag{2.193}$$

3. **The second term** involves a sum over $G^{\pm}_{s\mp1,t}$; it becomes:

$$\begin{aligned}\sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{s\mp1,t}w_0^{t-1} &= \sum_{s,t=1}^{N} w_0^{N-s}[G_\mp]_{s\pm1,t}w_0^{N-s} \\ &= \frac{A_0\left(e^{\mp\alpha}-w_0^{\pm1}\frac{2w_0\sinh\alpha}{1-w_0^2}\right)}{(e^{-\alpha}-w_0)(e^{\alpha}-w_0)} + \frac{A_\pm e^{\pm\alpha}}{(1-w_0e^{-\alpha})^2}\end{aligned} \tag{2.194}$$

$$\sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{s\mp1,t}w_0^{N-t} = \sum_{s,t=1}^{N} w_0^{N-s}[G_\mp]_{s\pm1,t}w_0^{t-1} = \frac{A_m e^{\pm\alpha}}{(1-w_0e^{-\alpha})^2} \tag{2.195}$$

which can be simplified to

$$\begin{aligned}\sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{s\mp1,t}w_0^{t-1} &= \sum_{s,t=1}^{N} w_0^{N-s}[G_\mp]_{s\pm1,t}w_0^{N-s} \\ &= A_0\left[\frac{e^{\mp\alpha}-w_0^{\pm1}\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0-\cosh\alpha\right)}\right. \\ &\left.-\frac{e^{-\alpha}}{(1-w_0e^{-\alpha})^2}\frac{\left(\frac{1}{a_5^2}-m^2\right)\left(1-ba_5e^{\pm\alpha}\right)}{\frac{1}{a_5^2}\left(1-ba_5e^{\alpha}\right)-m^2\left(1-ba_5e^{-\alpha}\right)}\right]\end{aligned} \tag{2.196}$$

$$
\begin{aligned}
\sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{s\mp 1,t} w_0^{N-t} &= \sum_{s,t=1}^{N} w_0^{N-s}[G_\mp]_{s\pm 1,t} w_0^{t-1} \\
&= \frac{m a_5 e^{-\alpha} e^{\pm\alpha}}{\frac{1}{a_5^2}\left(1-b a_5 e^{\alpha}\right) - m^2\left(1-b a_5 e^{-\alpha}\right)} \frac{1}{(1-w_0 e^{-\alpha})^2} \qquad (2.197)
\end{aligned}
$$

In the $m \to 0$ limit, first term becomes

$$
\begin{aligned}
\sum_{s,t=1}^{N} w_0^{s-1}[G_+]_{s-1,t} w_0^{t-1} &= \sum_{s,t=1}^{N} w_0^{N-s}[G_-]_{s+1,t} w_0^{N-s} \\
&= A_0 \left(\frac{e^{-\alpha} - w_0 \frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0 - \cosh\alpha\right)} - \frac{e^{\alpha}}{\left(e^{\alpha}-w_0\right)^2} \right) \qquad (2.198)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{s,t=1}^{N} w_0^{s-1}[G_-]_{s+1,t} w_0^{t-1} &= \sum_{s,t=1}^{N} w_0^{N-s}[G_+]_{s-1,t} w_0^{N-s} \\
&= A_0 \left(\frac{e^{\alpha} - \frac{1}{w_0} \frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0 - \cosh\alpha\right)} - \frac{e^{-\alpha}}{\left(e^{\alpha}-w_0\right)^2} \frac{e^{\alpha} - b a_5}{e^{-\alpha} - b a_5} \right) \\
& \qquad (2.199)
\end{aligned}
$$

Using these results we get the expression for full $S_\pm \equiv W^{\mp} G_\pm$ terms:

$$
\begin{aligned}
\bar{S}_\pm &\equiv \sum_{s,t=1}^{N} w_0^{s-1}[S_\pm]_{st} w_0^{t-1} = \sum_{s,t=1}^{N} w_0^{N-s}[S_\mp]_{st} w_0^{N-t} \\
&= \sum_{s,t=1}^{N} w_0^{s-1}\left[-b(p) G^\pm_{s,t} + \frac{1}{a_5} G^\pm_{s\mp 1,t}\right] w_0^{t-1} \\
&= A_0 \left[\frac{\left[-b + \frac{1}{a_5} e^{\mp\alpha}\right] - \left[-b + \frac{1}{a_5} w_0^{\pm 1}\right] \frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0 - \cosh\alpha\right)} \right. \\
&\qquad \left. - \frac{e^{-\alpha}}{(1-w_0 e^{-\alpha})^2} \frac{\left(\frac{1}{a_5^2} - m^2\right)\left(1 - b a_5 e^{\pm\alpha}\right)\left(-b e^{\mp\alpha} + \frac{1}{a_5}\right)}{\frac{1}{a_5^2}\left(1-b a_5 e^{\alpha}\right) - m^2\left(1-b a_5 e^{-\alpha}\right)} \right] \\
&= A_0 \left[\frac{\frac{e^{\mp\alpha}}{a_5}\left[1 - b a_5 e^{\pm\alpha}\right] - \frac{w_0^{\pm 1}}{a_5}\left[1 - b a_5 w_0^{\mp 1}\right] \frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0 - \cosh\alpha\right)} \right.
\end{aligned}
$$

$$
-\frac{e^{-\alpha}}{(1-w_0e^{-\alpha})^2}\frac{\left(\frac{1}{a_5^2}-m^2\right)(1-ba_5e^{\pm\alpha})\frac{1}{a_5}(1-ba_5e^{\mp\alpha})}{\frac{1}{a_5^2}(1-ba_5e^{\alpha})-m^2(1-ba_5e^{-\alpha})}\Bigg] \tag{2.200}
$$

from which we get $\sigma_S = (W^- G_+ + W^+ G_-)/2$

$$
\begin{aligned}
\bar{\sigma}_S &\equiv \sum_{s,t=1}^{N} w_0^{s-1}[\sigma_S]_{st}w_0^{t-1} \equiv \sum_{s,t=1}^{N} w_0^{N-s}[\sigma_S]_{st}w_0^{N-t} \\
&= -b(p)\sum_{s,t=1}^{N} w_0^{s-1}[\sigma_V]_{st}w_0^{t-1} + \frac{1}{a_5}\sum_{s,t=1}^{N} w_0^{s-1}[G^+_{s-1,t}+G^-_{s+1,t}]w_0^{t-1} \\
&= -b(p)\sum_{s,t=1}^{N} w_0^{s-1}[\sigma_V]_{st}w_0^{t-1} + \frac{1}{a_5}A_0\left(\frac{\cosh\alpha-\cosh\alpha_0\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)}\right. \\
&\quad \left. -\frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\frac{\left(\frac{1}{a_5^2}-m^2\right)(1-ba_5\cosh\alpha)}{\frac{1}{a_5^2}(1-ba_5e^{\alpha})-m^2(1-ba_5e^{-\alpha})}\right),
\end{aligned} \tag{2.201}
$$

which in the $m \to 0$ limit becomes

$$
\begin{aligned}
\bar{\sigma}_S &= -b(p)\sum_{s,t=1}^{N} w_0^{s-1}[\sigma_V]_{st}w_0^{t-1} + \frac{1}{a_5}A_0\left(\frac{\cosh\alpha-\cosh\alpha_0\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)}\right. \\
&\quad \left. -\frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\frac{1-ba_5\cosh\alpha}{1-ba_5e^{\alpha}}\right).
\end{aligned} \tag{2.202}
$$

Another way of writing the same expression is

$$
\begin{aligned}
\bar{\sigma}_S &= -b(p)\sum_{s,t=1}^{N} w_0^{s-1}[G_+]_{st}w_0^{t-1} + \frac{1}{a_5}A_0\left(\frac{\cosh\alpha-\cosh\alpha_0\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)}\right. \\
&\quad \left. -\frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\right).
\end{aligned} \tag{2.203}
$$

The last term equals $w_0/a_5\,\bar{G}_+$ although it's not obvious:

$$
\begin{aligned}
&\frac{w_0}{a_5}\bar{G}_+ - \frac{1}{a_5}A_0\left(\frac{\cosh\alpha-\cosh\alpha_0\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)} - \frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\right) = \\
&\frac{1}{a_5}A_0\left(\frac{w_0-\cosh\alpha-(w_0-\cosh\alpha_0)\frac{\sinh\alpha}{\sinh\alpha_0}}{(e^{-\alpha}-w_0)(e^{\alpha}-w_0)} - \frac{w_0-e^{\alpha}}{(e^{\alpha}-w_0)^2}\right) =
\end{aligned}
$$

$$\begin{aligned}
\frac{1}{a_5}A_0\left(\frac{w_0-\cosh\alpha+\sinh\alpha_0\frac{\sinh\alpha}{\sinh\alpha_0}}{(e^{-\alpha}-w_0)(e^{\alpha}-w_0)}+\frac{1}{(e^{\alpha}-w_0)}\right) &= \\
\frac{1}{a_5}A_0\left(\frac{w_0-e^{-\alpha}}{(e^{-\alpha}-w_0)(e^{\alpha}-w_0)}+\frac{1}{(e^{\alpha}-w_0)}\right) &= 0 \qquad (2.204)
\end{aligned}$$

so we finally have

$$\bar{\sigma}_S = \left(\frac{w_0}{a_5}-b(p)\right)\bar{G}_+ \qquad (2.205)$$

If we repeat the whole procedure for $Ma_5 > 1$, we end up replacing $b \to |b|$, $w_0 \to |w_0|$ and we pick up an overall minus sign.

$$\begin{aligned}
\bar{S}_\pm \equiv [\pm]A_0 &\left[\frac{\frac{e^{\mp\alpha}}{a_5}[1-|b|a_5e^{\pm\alpha}]-\frac{|w_0|^{\pm 1}}{a_5}\left[1-|b|\,a_5\,|w_0|^{\mp 1}\right]\frac{\sinh\alpha}{\sinh\alpha_0}}{2\,|w_0|\,(\cosh\alpha_0-\cosh\alpha)}\right. \\
&\left.-\frac{e^{-\alpha}}{(1-|w_0|\,e^{-\alpha})^2}\frac{\left(\frac{1}{a_5^2}-m^2\right)(1-|b|\,a_5e^{\pm\alpha})\frac{1}{a_5}(1-|b|\,a_5e^{\mp\alpha})}{\frac{1}{a_5^2}(1-|b|\,a_5e^{\alpha})-m^2(1-|b|\,a_5e^{-\alpha})}\right] \qquad (2.206)
\end{aligned}$$

Finally, we will need to evaluate sums

$$\tilde{S}_\mp \equiv \sum_{s,t} w_0^{N-s}[S_\pm]w_0^{t-1} = \sum_{s,t} w_0^{s-1}[S_\mp]w_0^{N-t}\,. \qquad (2.207)$$

In sums over $G_\pm$ only A_m terms survive; since

$$\sum_{s,t} w_0^{N-s}[G_\pm]_{s\mp\lambda,t}w_0^{t-1} = \frac{A_m e^{\mp\lambda\alpha}}{(1-w_0e^{-\alpha})^2}\,, \qquad (2.208)$$

from which it follows

$$\begin{aligned}
\tilde{S}_\pm &= \left(-b+\frac{e^{\pm\alpha}}{a_5}\right)\frac{A_m}{(1-w_0e^{-\alpha})^2}/ \\
&= \frac{e^{\mp\alpha}}{a_5}\left(1-b\,a_5e^{\pm\alpha}\right)\frac{A_m}{(1-w_0e^{-\alpha})^2} \\
&= \frac{e^{\mp\alpha}}{a_5}\left(1-b\,a_5e^{\pm\alpha}\right)\frac{1}{(1-w_0e^{-\alpha})^2}\frac{ma_5e^{-\alpha}}{\frac{1}{a_5^2}(1-b\,a_5e^{\alpha})-m^2(1-b\,a\,a_5e^{-\alpha})} \qquad (2.209)
\end{aligned}$$

$$\tilde{\sigma}_S \equiv \sum_{s,t} w_0^{N-s} \sigma_S w_0^{t-1} = \sum_{s,t} w_0^{s-1} \sigma_S w_0^{N-t} = \frac{A_m}{(1-w_0 e^{-\alpha})^2} \left(-b + \frac{\cosh\alpha}{a_5}\right) \quad (2.210)$$

For $Ma_5 > 1$, both the first and the second terms pick two $\pm$ factors so the overall result stays the same

$$\tilde{S}_\pm = \left(1 - |b|\, a_5 e^{\pm\alpha}\right) \frac{e^{\mp\alpha}/a_5}{(1-|w_0|\,e^{-\alpha})^2} \frac{m a_5 e^{-\alpha}}{\frac{1}{a_5^2}(1-|b|\,a_5 e^{\alpha}) - m^2(1-|b|\,a_5 e^{-\alpha})} \quad (2.211)$$

$$\tilde{\sigma}_S = \left(-|b| + \frac{\cosh\alpha}{a_5}\right) \frac{1}{(1-|w_0|\,e^{-\alpha})^2} \frac{-m a_5 e^{-\alpha}}{\frac{1}{a_5^2}(1-|b|\,a_5 e^{\alpha}) - m^2(1-|b|\,a_5 e^{-\alpha})} \quad (2.212)$$

Expanding this to the first power in m (as Aoki does) we get

$$\tilde{S}_- = a_5 \frac{m a_5}{(1-|w_0|\,e^{-\alpha})^2} e^{-2\alpha} \quad (2.213)$$

$$\tilde{S}_+ = a_5 \frac{m a_5}{(1-|w_0|\,e^{-\alpha})^2} \frac{1-|b|\,a_5 e^{-\alpha}}{1-|b|\,a_5 e^{\alpha}} \quad (2.214)$$

$$\tilde{\sigma}_S = a_5 \frac{m a_5 e^{-\alpha}}{(1-|w_0|\,e^{-\alpha})^2} \frac{\cosh\alpha - |b|\,a_5}{1-|b|\,a_5 e^{\alpha}} \quad (2.215)$$

$$\tilde{\sigma}_V = \pm a_5 \frac{m a_5}{(1-|w_0|\,e^{-\alpha})^2} \frac{a_5 e^{-\alpha}}{1-|b|\,a_5 e^{\alpha}}\,. \quad (2.216)$$

2.3.3 5D-to-physical propagator sums

When dealing with physical quarks, we also have to evaluate 5D sums of 5D-to-physical propagators

$$S_{OUT}(k) = -i\bar{k}\cdot\gamma\,(g_+(k)P_+ + g_-(k)P_-) + (\sigma_+(k)P_+ + \sigma_-(k)P_-) \quad (2.217)$$

$$S_{IN}(k) = (g_-(k)P_+ + g_+(k)P_-)\,(-i\bar{k}\cdot\gamma) + (\sigma_-(k)P_+ + \sigma_+(k)P_-)\,. \quad (2.218)$$

Since both $g_\pm$ and $\sigma_\pm$ have only $e^{-\alpha p(s-1)}$ and $e^{-\alpha p(N-s)}$ terms, two occurring sums are easy to evaluate:

$$\sum_{s=1}^{N} w_0^{s-1} e^{-\alpha(s-1)} = \sum_{s=1}^{N} w_0^{N-s} e^{-\alpha(N-s)} = \frac{1-(w_0 e^{-\alpha})^N}{1-w_0 e^{-\alpha}} \xrightarrow{N\to\infty} \frac{1}{1-w_0 e^{-\alpha}} \quad (2.219)$$

$$\sum_{s=1}^{N} w_0^{s-1} e^{-\alpha(N-s)} = \sum_{s=1}^{N} w_0^{N-s} e^{-\alpha(s-1)} = \frac{w_0^N - e^{-\alpha N}}{w_0 - e^{-\alpha}} \overset{N\to\infty}{\longrightarrow} 0\,. \tag{2.220}$$

With these, it's easy to evaluate

$$\begin{aligned}
\tilde{g}_+ &\equiv \sum_{s=1}^{N} (\pm|w_0|)^{s-1} g_+ = \sum_{s=1}^{N} (\pm|w_0|)^{N-s} g_- \\
&= \sum_{s=1}^{N} (\pm|w_0|)^{s-1} \left(-\frac{(\pm e^{-\alpha})^{(N-s)} \pm m a_5 e^{-\alpha} (\pm e^{-\alpha})^{(s-1)}}{F} \right) \\
&= \mp \frac{m a_5 e^{-\alpha}}{F} \frac{1}{1-|w_0|\, e^{-\alpha}} && (2.221) \\
\tilde{g}_- &\equiv \sum_{s=1}^{N} (\pm|w_0|)^{s-1} g_- = \sum_{s=1}^{N} (\pm|w_0|)^{N-s} g_+ = -\frac{1}{F}\frac{1}{1-|w_0|\, e^{-\alpha}} && (2.222) \\
\tilde{\sigma}_+ &\equiv \sum_{s=1}^{N} (\pm|w_0|)^{s-1} \sigma_+ = \sum_{s=1}^{N} (\pm|w_0|)^{N-s} \sigma_- \\
&= \sum_{s=1}^{N} (\pm|w_0|)^{s-1} \left(-m(1-|b|\, a_5 e^{-\alpha}) \frac{(\pm e^{-\alpha})^{(N-s)} \pm m a_5 e^{-\alpha} (\pm e^{-\alpha})^{(s-1)}}{F} \right. \\
&\qquad \left. \mp a_5 e^{-\alpha} (\pm e^{-\alpha})^{(s-1)} \right) \\
&= \mp \frac{\frac{1}{a_5} e^{-\alpha} (1-|b|\, a_5 e^{\alpha})}{F} \frac{1}{1-|w_0|\, e^{-\alpha}} && (2.223) \\
\tilde{\sigma}_- &\equiv \sum_{s=1}^{N} (\pm|w_0|)^{s-1} \sigma_- = \sum_{s=1}^{N} (\pm|w_0|)^{N-s} \sigma_+ \\
&= -\frac{m(1-|b|\, a_5 e^{-\alpha})}{F} \frac{1}{1-|w_0|\, e^{-\alpha}}\,. && (2.224)
\end{aligned}$$

2.3.4 *IR* limit of summed propagators

If we expand all these sums around $p \to 0$ (for $m \to 0$), we get

$$\begin{aligned}
\bar{G}_+ &\sim \frac{a_5^2}{1-w_0^2} \left(\frac{1+w_0^2}{(1-w_0^2)^2} - \frac{w_0^2(1-m^2 a_5^2)}{(1-w_0^2)^2} \frac{p^2}{p^2+m^2(1-w_0^2)^2} \right) && (2.225) \\
\bar{G}_- &\sim \frac{a_5^2}{1-w_0^2} \left(\frac{1+w_0^2}{(1-w_0^2)^2} + \frac{\frac{1}{a_5^2} - m^2}{p^2+m^2(1-w_0^2)^2} \right) && (2.226) \\
\bar{S}_+ &\sim -\frac{a_5 w_0 (1-m^2 a_5^2)}{(1-w_0^2)^2} \frac{p^2}{p^2+m^2(1-w_0^2)^2} && (2.227)
\end{aligned}$$

$$
\begin{aligned}
\tilde{S}_- &\sim \frac{a_5}{w_0(1-w_0^2)^2}\left(1+w_0^2-w_0^2(1-m^2a_5^2)\frac{p^2}{p^2+m^2(1-w_0^2)^2}\right) && (2.228)\\
\tilde{G}_\pm &= \tilde{\sigma}_V \sim \frac{ma_5w_0}{p^2+m^2(1-w_0^2)^2} && (2.229)\\
\tilde{S}_- &\sim \left(m\frac{w_0^2a_5^2}{1-w_0^2}p^2+m^2a_5w_0\right)\frac{1}{p^2+m^2(1-w_0^2)^2} && (2.230)\\
\tilde{S}_+ &\sim -m\times\frac{1-w_0^2}{p^2+m^2(1-w_0^2)^2} && (2.231)\\
\tilde{g}_+ &\sim \frac{ma_5w_0}{p^2+m^2(1-w_0^2)^2} && (2.232)\\
\tilde{g}_- &\sim \frac{1}{p^2+m^2(1-w_0^2)^2} && (2.233)\\
\tilde{\sigma}_- &\sim \frac{m}{p^2+m^2(1-w_0^2)^2} && (2.234)\\
\tilde{\sigma}_+ &\sim \frac{a_5w_0}{1-w_0^2}\frac{p^2}{p^2+m^2(1-w_0^2)^2}\,. && (2.235)
\end{aligned}
$$

2.4 Wave function normalization

Let's take another look at the $p \to 0$ limit of the physical-to-physical propagator:

$$
S^{PHYS}(p) = \langle q(-p)\bar{q}(p)\rangle = \frac{1-w_0^2}{ip\cdot\gamma+m(1-w_0^2)} \qquad (2.236)
$$

We can ask ourselves what is the origin of the $1-w_0^2$ factor in the numerator? The answer lies in our definition of the physical field q:

$$
q(x) = P_+\psi_1(x)+P_-\psi_N(x)\,, \qquad \bar{q}(x) = \bar{\psi}_1(x)P_-+\bar{\psi}_N(x)P_+\,. \qquad (2.237)
$$

If we had chosen to redefine our physical field as

$$
q(x) \to \frac{q(x)}{\mathcal{N}} \qquad (2.238)
$$

where $\mathcal{N}$ is some number (real or complex), we would get a result for a physical-to-physical propagator

$$\bar{S}^{PHYS}(p) = \langle q(-p)\bar{q}(p)\rangle = \frac{1}{|\mathcal{N}|^2}\frac{1-w_0^2}{ip\cdot\gamma+m(1-w_0^2)}\,. \tag{2.239}$$

So if we choose $\mathcal{N}=\sqrt{1-w_0^2}$, we get a propagator that has a proper continuum behavior. Furthermore, we can modify the mass term in the Lagrangian $\tilde{m}=m(1-w_0^2)$ to absorb the $(1-w_0^2)$ factor

$$\mathcal{L}_m = \frac{\tilde{m}}{1-w_0^2}\left(\bar{\psi}_1 P_- \psi_N + \bar{\psi}_N P_+ \psi_1\right) = \tilde{m}\bar{q}q \tag{2.240}$$

which yields the tree level propagator in the $p\to 0$ limit

$$\bar{S}^{PHYS}(p) = \frac{1}{ip\cdot\gamma+\tilde{m}}\,. \tag{2.241}$$

Now that we know how to get rid of that term, we can ask ourselves why was that factor there in the first place? Then answer lies in the tree level massless mode of the *5D* propagator

$$\chi_0 = \sqrt{1-w_0^2}\left(P_+ w_0^{s-1}\psi_s + P_- w_0^{N-s}\psi_s\right) = \sqrt{1-w_0^2}\,(q+\ldots) \tag{2.242}$$

where the $\sqrt{1-w_0^2}$ comes from normalization condition $\langle\chi_0|\chi_0\rangle = 1$. Physics in *5D* in the limit when $a\to 0$ is dominated by this massless mode (massive modes with the mass of the order a^{-1} decouple in the $a\to 0$ limit). When we work with the field q instead of the field χ, we have to account for the fact that the overlap between the two differs from one so to extract physical results, we have to include that overlap in the normalization of q.

Alternative way to look at this is to think of it as a form of "tree level" renormalization. When one renormalizes the propagator nonperturbativelly, one imposes conditions that the renormalized propagator obeys the same Euclidean space relations as continuum propagator

$$\lim_{m_{ren}\to 0}\left\{-\frac{i}{12}\mathrm{Tr}\left(\frac{\partial S_{ren}^{-1}(p)}{\partial p\cdot\gamma}\right)_{p^2=\mu^2}\right\} = 1 \tag{2.243}$$

$$\lim_{m_{ren}\to 0} \frac{1}{12 m_{ren}} \mathrm{Tr}\left(S_{ren}^{-1}(p)\right)_{p^2=\mu^2} = 1 \tag{2.244}$$

Perturbativelly, one can impose the same relation at all levels of perturbation theory; if we start with the physical wave function $q(x) = P_+\psi_1(x) + P_-\psi_N(x)$, it leads to "tree level" renormalization

$$\begin{aligned} q(x) &= \sqrt{1-w_0^2}\,(P_+\psi_1(x) + P_-\psi_N(x)) &(2.245)\\ \tilde{m} &= \frac{m}{1-w_0^2}\,. &(2.246)\end{aligned}$$

If we had started with the definitions above, conditions (2.243) and (2.244) would have been trivially satisfied.

2.4.1 Physical propagators

The "new" physical field q definition

$$q(x) = \sqrt{1-w_0^2}\,(P_+\psi_1(x) + P_-\psi_N(x))\,, \qquad \bar{q}(x) = \sqrt{1-w_0^2}\,(\bar{\psi}_1(x)P_- + \bar{\psi}_N(x)P_+)\,, \tag{2.247}$$

is equivalent to replacing

$$q(x) \to \frac{q(x)}{\sqrt{1-w_0^2}} \tag{2.248}$$

so our physical-to-physical propagator picks a factor $1/(1-w_0^2)$, our 5D-to-physical propagators S^{IN} and S^{OUT} pick a factor $1/\sqrt{1-w_0^2}$ while the truncated 5D-to-physical propagator picks a factor $\sqrt{1-w_0^2}$ due to truncating the external propagator. After performing the 5D sums, summed propagators functions $\bar{G}, \tilde{G}, \bar{S}$ and $\tilde{S}$ get an overall $1-w_0^2$ factor while summed propagator functions $\bar{g}, \tilde{g}, \bar{\sigma}$ and $\tilde{\sigma}$ remain unchanged since factors $\sqrt{1-w_0^2}$ from the truncated 5D-to-physical propagator cancels the factor $1/\sqrt{1-w_0^2}$ in the internal 5D-to-physical propagator.

The physical-to-physical propagator is then given by

$$S^{PHYS}(p) \equiv \langle q(-p)\bar{q}(p)\rangle = \frac{1}{1-w_0^2}\frac{i\bar{p}\cdot\gamma - m(1-ba_5e^{-\alpha})}{F}$$

$$= \frac{1}{1-w_0^2} \frac{i\bar{p}\cdot\gamma - m(1-ba_5e^{-\alpha})}{\frac{1}{a_5^2}(1-ba_5e^{\alpha}) - m^2(1-ba_5e^{-\alpha})} \tag{2.249}$$

The effect of external legs on physical propagators is given in

$$\bar{S}_s^{OUT}(p) \equiv \frac{1}{\langle q(-p)\bar{q}(p)\rangle}\langle q(-p)\bar{\psi}_s(p)\rangle \tag{2.250}$$

$$= -i\bar{p}\cdot\gamma(\bar{g}_+P_+ + \bar{g}_-P_-) + \bar{\sigma}_+P_+ + \bar{\sigma}_-P_- \tag{2.251}$$

$$\bar{S}_s^{IN}(p) \equiv \langle\psi_s(-p)\bar{q}(p)\rangle \frac{1}{\langle q(-p)\bar{q}(p)\rangle} \tag{2.252}$$

$$= (\bar{g}_-P_+ + \bar{g}_+P_-)(-i\bar{p}\cdot\gamma) + \bar{\sigma}_-P_+ + \bar{\sigma}_+P_- \tag{2.253}$$

with

$$\bar{g}_+(p) = \mathcal{A}e^{-\alpha(s-1)} \to \sqrt{1-w_0^2}\mathcal{A}w_0^{s-1} \tag{2.254}$$

$$\bar{g}_-(p) = \mathcal{A}e^{-\alpha(N-s)} \to \sqrt{1-w_0^2}\mathcal{A}w_0^{N-s} \tag{2.255}$$

$$\bar{\sigma}_+(p) = \mathcal{B}e^{-\alpha(s-1)} + e^{-\alpha(N-s)} \to \sqrt{1-w_0^2}w_0^{N-s} \tag{2.256}$$

$$\bar{\sigma}_-(p) = \mathcal{B}e^{-\alpha(N-s)} + e^{-\alpha(s-1)} \to \sqrt{1-w_0^2}w_0^{s-1} \tag{2.257}$$

$$\mathcal{A} = -\frac{a_5e^{-\alpha(p)}F(p)}{\bar{p}^2 + m^2\left(1-b(p)a_be^{-\alpha(p)}\right)^2} \to \frac{a_5w_0}{1-w_0^2} \tag{2.258}$$

$$\mathcal{B} = a_5me^{-\alpha(p)}\left[1 + \frac{F\left(1-b(p)a_be^{-\alpha(p)}\right)}{\bar{p}^2 + m^2\left(1-b(p)a_be^{-\alpha(p)}\right)^2}\right] \to 0 \tag{2.259}$$

In the $a \to 0$ limit (and also in the IR limit) this becomes

$$\bar{S}_s^{OUT}(p) = \sqrt{1-w_0^2}\Big\{\left(P_-w_0^{s-1} + P_+w_0^{N-s}\right) - \frac{a_5w_0}{1-w_0^2}ip\cdot\gamma\left(P_-w_0^{N-s} + P_+w_0^{s-1}\right)\Big\} \tag{2.260}$$

$$\bar{S}_s^{IN}(p) = \sqrt{1-w_0^2}\Big\{\left(P_+w_0^{s-1} + P_-w_0^{N-s}\right) - \left(P_+w_0^{N-s} + P_-w_0^{s-1}\right)\frac{a_5w_0}{1-w_0^2}ip\cdot\gamma\Big\} \tag{2.261}$$

$$S^{PHYS}(p) = \frac{1}{ip\cdot\gamma + m(1-w_0^2)} = \frac{1}{ip\cdot\gamma + \tilde{m}} \tag{2.262}$$

2.4.2 Sums over 5D

Same edge contribution

Summed over the 5^{th} dimension, propagator formulas are

$$
\begin{aligned}
\bar{G}_{\pm} &\equiv (1-w_0^2)\sum_{s,t=1}^{N} w_0^{s-1}[G_{\pm}]_{st}w_0^{t-1} = (1-w_0^2)\sum_{s,t=1}^{N} w_0^{N-s}[G_{\mp}]_{st}w_0^{N-s} \\
&= (1-w_0^2)\left\{\frac{A_0\left(1-w_0\frac{2\sinh\alpha}{1-w_0^2}\right)}{(e^{-\alpha}-w_0)(e^{\alpha}-w_0)}+\frac{A_{\pm}}{(1-w_0e^{-\alpha})^2}\right\} \\
&= (1-w_0^2)\left\{A_0\left[\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)}\right.\right. \\
&\quad \left.\left. -\frac{1}{(1-w_0e^{-\alpha})^2}\frac{\left(\frac{1}{a_5^2}-m^2\right)(1-ba_5e^{\pm\alpha})e^{-\alpha\mp\alpha}}{\frac{1}{a_5^2}(1-ba_5e^{\alpha})-m^2(1-ba_5e^{-\alpha})}\right]\right\}
\end{aligned} \tag{2.263}
$$

which in the $m\to 0$ limit become

$$
\bar{G}_{+} = (1-w_0^2)A_0\left(\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)}-\frac{1}{(e^{\alpha}-w_0)^2}\right) \tag{2.264}
$$

$$
\bar{G}_{-} = (1-w_0^2)A_0\left(\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)}-\frac{1}{(e^{\alpha}-w_0)^2}\frac{e^{\alpha}-ba_5}{e^{-\alpha}-ba_5}\right) \tag{2.265}
$$

Their sum $\sigma_V = (G_+ + G)-)/2$ yields

$$
\begin{aligned}
\bar{\sigma}_V &\equiv (1-w_0^2)\sum_{s,t=1}^{N} w_0^{s-1}[\sigma_V]_{st}w_0^{t-1} = (1-w_0^2)\sum_{s,t=1}^{N} w_0^{N-s}[\sigma_V]_{st}w_0^{N-s} \\
&= (1-w_0^2)\left\{A_0\left(\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)}\right.\right. \\
&\quad \left.\left. -\frac{e^{-\alpha}}{(1-w_0e^{-\alpha})^2}\frac{\left(\frac{1}{a_5^2}-m^2\right)(\cosh\alpha-ba_5)}{\frac{1}{a_5^2}(1-ba_5e^{\alpha})-m^2(1-ba_5e^{-\alpha})}\right)\right\}
\end{aligned} \tag{2.266}
$$

$$
\xrightarrow{m\to 0} (1-w_0^2)A_0\left(\frac{1-\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0(\cosh\alpha_0-\cosh\alpha)}-\frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\frac{\cosh\alpha-ba_5}{1-ba_5e^{\alpha}}\right) \tag{2.267}
$$

For $S_\pm \equiv W^\mp G_\pm$ we have

$$
\begin{aligned}
\bar{S}_\pm &\equiv (1-w_0^2)\sum_{s,t=1}^{N} w_0^{s-1}[W^\mp G_\pm]_{st}w_0^{t-1} = (1-w_0^2)\sum_{s,t=1}^{N} w_0^{N-s}[W^\pm G_\mp]_{st}w_0^{N-t} \\
&= (1-w_0^2)\left\{A_0\left[\frac{\frac{e^{\mp\alpha}}{a_5}\left[1-ba_5e^{\pm\alpha}\right]-\frac{w_0^{\pm1}}{a_5}\left[1-ba_5w_0^{\mp1}\right]\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0-\cosh\alpha\right)}\right.\right. \\
&\quad \left.\left.-\frac{e^{-\alpha}}{(1-w_0e^{-\alpha})^2}\frac{\left(\frac{1}{a_5^2}-m^2\right)\left(1-ba_5e^{\pm\alpha}\right)\frac{1}{a_5}\left(1-ba_5e^{\mp\alpha}\right)}{\frac{1}{a_5^2}\left(1-ba_5e^{\alpha}\right)-m^2\left(1-ba_5e^{-\alpha}\right)}\right]\right\} \qquad (2.268)
\end{aligned}
$$

from which we get $\sigma_S = (W^-G_+ + W^+G_-)/2$

$$
\begin{aligned}
\bar{\sigma}_S &= -b(p)(1-w_0^2)\sum_{s,t=1}^{N} w_0^{s-1}[\sigma_V]_{st}w_0^{t-1} + (1-w_0^2)\frac{1}{a_5}\sum_{s,t=1}^{N} w_0^{s-1}[G^+_{s-1,t}+G^-_{s+1,t}]w_0^{t-1} \\
&= -b(p)\bar{\sigma}_V + (1-w_0^2)\left\{\frac{1}{a_5}A_0\left(\frac{\cosh\alpha-\cosh\alpha_0\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0-\cosh\alpha\right)}\right.\right. \\
&\quad \left.\left.-\frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\frac{\left(\frac{1}{a_5^2}-m^2\right)\left(1-ba_5\cosh\alpha\right)}{\frac{1}{a_5^2}\left(1-ba_5e^{\alpha}\right)-m^2\left(1-ba_5e^{-\alpha}\right)}\right)\right\} \qquad (2.269) \\
&\xrightarrow{m\to0} -b(p)\bar{\sigma}_V + (1-w_0^2)\frac{1}{a_5}A_0\left(\frac{\cosh\alpha-\cosh\alpha_0\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0-\cosh\alpha\right)}-\frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\frac{1-ba_5\cosh\alpha}{1-ba_5e^{\alpha}}\right) \\
&= -b(p)\bar{G}_+ + (1-w_0^2)\frac{1}{a_5}A_0\left(\frac{\cosh\alpha-\cosh\alpha_0\frac{\sinh\alpha}{\sinh\alpha_0}}{2w_0\left(\cosh\alpha_0-\cosh\alpha\right)}-\frac{e^{\alpha}}{(e^{\alpha}-w_0)^2}\right) \\
&= \left(\frac{w_0}{a_5}-b(p)\right)\bar{G}_+ \qquad (2.270)
\end{aligned}
$$

Different edge contribution

Sums over different edges yield

$$
\begin{aligned}
\tilde{G}_\pm &\equiv (1-w_0^2)\sum_{s,t=1}^{N} w_0^{s-1}[G_\pm]_{st}w_0^{N-t} = (1-w_0^2)\sum_{s,t=1}^{N} w_0^{N-s}[G_\pm]_{st}w_0^{t-1} \\
&= \tilde{\sigma}_V = (1-w_0^2)\frac{A_m}{(1-w_0e^{-\alpha})^2}
\end{aligned}
$$

$$= (1-w_0^2)\frac{ma_5e^{-\alpha}}{\frac{1}{a_5^2}(1-ba_5e^{\alpha})-m^2(1-ba_5e^{-\alpha})}\frac{1}{(1-w_0e^{-\alpha})^2} \tag{2.271}$$

$$\tilde{S}_{\mp} \equiv (1-w_0^2)\sum_{s,t}w_0^{N-s}[W^{\mp}G_{\pm}]w_0^{t-1} = (1-w_0^2)\sum_{s,t}w_0^{s-1}[W^{\pm}G_{\mp}]w_0^{N-t}$$

$$= (1-w_0^2)\left(-b+\frac{e^{\mp\alpha}}{a_5}+\begin{bmatrix} m \\ 0 \end{bmatrix}\right)\frac{A_m}{(1-w_0e^{-\alpha})^2}$$

$$= \frac{e^{\mp\alpha}}{a_5}\left(1-ba_5e^{\pm\alpha}+\begin{bmatrix} m \\ 0 \end{bmatrix}a_5e^{\pm\alpha}\right)\frac{1-w_0^2}{(1-w_0e^{-\alpha})^2}$$

$$\times\frac{ma_5e^{\alpha}}{\frac{1}{a_5^2}(1-ba_5e^{\alpha})-m^2(1-baa_5e^{-\alpha})} \tag{2.272}$$

$$\tilde{\sigma}_S \equiv (1-w_0^2)\sum_{s,t}w_0^{N-s}\sigma_S w_0^{t-1} = (1-w_0^2)\sum_{s,t}w_0^{s-1}\sigma_S w_0^{N-t}$$

$$= (1-w_0^2)\frac{A_m}{(1-w_0e^{-\alpha})^2}\left(-b+\frac{\cosh\alpha}{a_5}+\frac{m}{2}\right) \tag{2.273}$$

Expanding this to the first power in m (as Aoki does) we get

$$\tilde{S}_- = (1-w_0^2)a_5\frac{ma_5}{(1-w_0e^{-\alpha})^2}e^{-2\alpha} \stackrel{p\to 0}{\longrightarrow} const. \tag{2.274}$$

$$\tilde{S}_+ = (1-w_0^2)a_5\frac{ma_5}{(1-w_0e^{-\alpha})^2}\frac{1-ba_5e^{-\alpha}}{1-ba_5e^{\alpha}} \stackrel{p\to 0}{\longrightarrow} -\frac{\tilde{m}}{p^2} \tag{2.275}$$

$$\tilde{\sigma}_S = (1-w_0^2)a_5\frac{ma_5e^{-\alpha}}{(1-w_0e^{-\alpha})^2}\frac{\cosh\alpha-ba_5}{1-ba_5e^{\alpha}} \stackrel{p\to 0}{\longrightarrow} -\frac{\tilde{m}}{2}\frac{1}{p^2} \tag{2.276}$$

$$\tilde{\sigma}_V = (1-w_0^2)a_5\frac{ma_5}{(1-w_0e^{-\alpha})^2}\frac{a_5e^{-\alpha}}{1-ba_5e^{\alpha}} \stackrel{p\to 0}{\longrightarrow} \frac{\tilde{m}a_5w_0}{1-w_0^2}\frac{1}{p^2} \tag{2.277}$$

Summed 5D-to-physical propagator functions

Summed 5D-to-physical propagator functions are

$$\tilde{g}_+ \equiv \sqrt{1-w_0^2}\sum_{s=1}^{N}w_0^{s-1}g_+ = \sqrt{1-w_0^2}\sum_{s=1}^{N}w_0^{N-s}g_-$$

$$= -\frac{ma_5e^{-\alpha}}{F}\frac{1}{1-w_0e^{-\alpha}} \tag{2.278}$$

$$\tilde{g}_- \equiv \sqrt{1-w_0^2}\sum_{s=1}^{N}w_0^{s-1}g_- = \sqrt{1-w_0^2}\sum_{s=1}^{N}w_0^{N-s}g_+$$

$$= -\frac{1}{F}\frac{1}{1-w_0e^{-\alpha}} \tag{2.279}$$

$$
\begin{aligned}
\tilde{\sigma}_+ &\equiv \sqrt{1-w_0^2}\sum_{s=1}^{N} w_0^{s-1}\sigma_+ = \sqrt{1-w_0^2}\sum_{s=1}^{N} w_0^{N-s}\sigma_- \\
&= -\frac{\frac{1}{a_5}e^{-\alpha}(1-ba_5e^{\alpha})}{F}\frac{1}{1-w_0e^{-\alpha}} \qquad (2.280)\\
\tilde{\sigma}_- &\equiv \sqrt{1-w_0^2}\sum_{s=1}^{N} w_0^{s-1}\sigma_- = \sqrt{1-w_0^2}\sum_{s=1}^{N} w_0^{N-s}\sigma_+ \\
&= -\frac{m(1-ba_5e^{-\alpha})}{F}\frac{1}{1-w_0e^{-\alpha}} \qquad (2.281)
\end{aligned}
$$

IR limit

IR limits of summed propagators are

$$
\begin{aligned}
\bar{G}_+ &\sim a_5^2\left(\frac{1+w_0^2}{(1-w_0^2)^2}-\frac{w_0^2(1-m^2a_5^2)}{(1-w_0^2)^2}\frac{p^2}{p^2+m^2(1-w_0^2)^2}\right) \qquad (2.282)\\
\bar{G}_- &\sim a_5^2\left(\frac{1+w_0^2}{(1-w_0^2)^2}+\frac{\frac{1}{a_5^2}-m^2}{p^2+m^2(1-w_0^2)^2}\right) \qquad (2.283)\\
\bar{S}_+ &\sim -\frac{a_5w_0(1-m^2a_5^2)}{(1-w_0^2)}\frac{p^2}{p^2+m^2(1-w_0^2)^2} \qquad (2.284)\\
\bar{S}_- &\sim \frac{a_5}{w_0(1-w_0^2)}\left(1+w_0^2-w_0^2(1-m^2a_5^2)\frac{p^2}{p^2+m^2(1-w_0^2)^2}\right) \qquad (2.285)\\
\tilde{G}_\pm &= \tilde{\sigma}_V \sim (1-w_0^2)\frac{ma_5w_0}{p^2+m^2(1-w_0^2)^2} = \frac{\tilde{m}a_5w_0}{p^2+\tilde{m}^2} \qquad (2.286)\\
\tilde{S}_- &\sim (1-w_0^2)\left(m\frac{w_0^2a_5^2}{1-w_0^2}p^2+m^2a_5w_0\right)\frac{1}{p^2+m^2(1-w_0^2)^2} \qquad (2.287)\\
\tilde{S}_+ &\sim -m(1-w_0^2)\times\frac{1-w_0^2}{p^2+m^2(1-w_0^2)^2} \qquad (2.288)\\
\tilde{g}_+ &\sim \frac{ma_5w_0}{p^2+m^2(1-w_0^2)^2} = \frac{\tilde{m}a_5w_0}{1-w_0^2}\frac{1}{p^2+\tilde{m}^2} \qquad (2.289)\\
\tilde{g}_- &\sim \frac{1}{p^2+m^2(1-w_0^2)^2} = \frac{1}{p^2+\tilde{m}^2} \qquad (2.290)\\
\tilde{\sigma}_- &\sim \frac{m}{p^2+m^2(1-w_0^2)^2} = \frac{\tilde{m}}{1-w_0^2}\frac{1}{p^2+m^2(1-w_0^2)^2} \qquad (2.291)\\
\tilde{\sigma}_+ &\sim \frac{a_5w_0}{1-w_0^2}\frac{p^2}{p^2+m^2(1-w_0^2)^2} = \frac{a_5w_0}{1-w_0^2}\frac{p^2}{p^2+\tilde{m}^2} \qquad (2.292)
\end{aligned}
$$

2.5 Fat links (Smearing)

The idea here is to replace usual gauge links $U_\mu(x)$ with "fat" or smeared links. In general, we take a "thin" link $U_\mu(x)$ connecting two neighboring points, add longer gauge invariant paths (i.e. paths going through other points) to it and then project the sum back into the $SU(n)$ group:

$$V_\mu(x) = \mathrm{Proj}_n \{U + \sum U \otimes U \otimes U + \dots\} \tag{2.293}$$

where Proj_n projects the sum of links back into the $SU(n)$ group. For infinitesimal steps $V_\mu = 1 + B_\mu$, $U_\mu = 1 + A_\mu$ that projections is not necessary since the product of infinitesimal transformations is in itself an infinitesimal transformation. Then we get

$$B_\mu(x) = \sum_y h_{\mu\nu}(x,y) A_\nu(y) \tag{2.294}$$

which after Fourier transforming becomes a product

$$B_\mu(k) = h_{\mu\nu}(k) A_\nu(k)\,. \tag{2.295}$$

This field redefinition means that the gluon propagator for the B field can be expressed in terms of the gluon propagator for the A field as

$$H_{\mu\nu}(k) = h_{\mu\alpha}(k) \frac{g_{\alpha\beta} - (1-\lambda)\frac{\hat{k}_\alpha \hat{k}_\beta}{\hat{k}^2}}{\hat{k}^2} h_{\beta\nu}(k)\,. \tag{2.296}$$

Since the longitudinal part doesn't depend on the path chosen between two points, smearing doesn't affect that part so finally we have

$$H_{\mu\nu}(k) = \frac{h_{\mu\alpha}(k) h_{\beta\nu}(k) - (1-\lambda)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2}}{\hat{k}^2}\,. \tag{2.297}$$

2.5.1 APE smearing

APE smearing is done by performing n steps of adding "staples" to gauge link

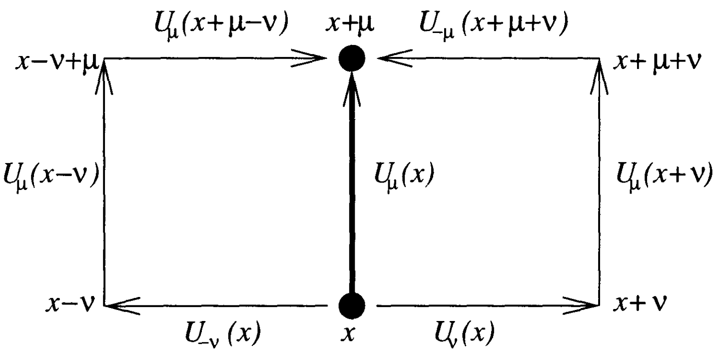

Figure 2-1: APE Smearing: Summation over "staples" in 2D

$$\begin{aligned}
V_\mu^{(n+1)}(x) &= (1+c)V_\mu^{(n)}(x) - \frac{c}{2(d-1)}\sum_{\nu\neq\mu}\Big[V_\nu^{(n)}(x)V_\mu^{(n)}(x+\nu)V_{-\nu}^{(n)}(x+\mu+\nu) \\
&\quad + V_{-\nu}^{(n)}(x)V_\mu^{(n)}(x-\nu)V_\nu^{(n)}(x+\mu-\nu)\Big] \\
&= (1+c)V_\mu^{(n)}(x) - \frac{c}{2(d-1)}\sum_{\nu\neq\mu}\Big[V_\nu^{(n)}(x)V_\mu^{(n)}(x+\nu)V_\nu^{\dagger(n)}(x+\mu) \\
&\quad + V_\nu^{\dagger(n)}(x-\nu)V_\mu^{(n)}(x-\nu)V_\nu^{(n)}(x+\mu-\nu)\Big]
\end{aligned} \tag{2.298}$$

In terms of the generator $A_\mu(x)$ and with

$$\begin{aligned}
U_\mu(x) &\equiv V_\mu^{(0)}(x) = 1 + ig_0aA_\mu(x) \\
V_\mu(x) &\equiv V_\mu^{(1)}(x) = 1 + ig_0aB_\mu(x)
\end{aligned}$$

these become

$$\begin{aligned}
1 + ig_0aB_\mu(x) &= (1+c)(1+ig_0aA_\mu(x)) \\
&\quad - \frac{c}{2(d-1)}\sum_{\nu\neq\mu}\big[1 + ig_0a\big(A_\nu(x) + A_\mu(x+\nu) - A_\nu(x+\mu) \\
&\quad - A_\nu(x-\nu) + A_\mu(x-\nu) + A_\nu(x+\mu-\nu)\big)\big]
\end{aligned} \tag{2.299}$$

from which we get

$$\begin{aligned}
B_\mu(x) &= (1+c)A_\mu(x) - \frac{c}{2(d-1)}\sum_{\nu\neq\mu}\big[A_\nu(x) + A_\mu(x+\nu) - A_\nu(x+\mu) \\
&\quad - A_\nu(x-\nu) + A_\mu(x-\nu) + A_\nu(x+\mu-\nu)\big]\ .
\end{aligned} \tag{2.300}$$

Fourier transforming into momentum space we get

$$\begin{aligned}
B_\mu(k)e^{ik(x+\mu/2)} &= (1+c)A_\mu(k)e^{ik(x+\mu/2)} \\
&\quad -\frac{c}{2(d-1)}\sum_{\nu\neq\mu}\Big[A_\nu(k)e^{ik(x+\nu/2)}+A_\mu(k)e^{ik(x+\nu+\mu/2)} \\
&\qquad -A_\nu(k)e^{ik(x+\mu+\nu/2)}-A_\nu(k)e^{ik(x-\nu/2)} \\
&\qquad +A_\mu(k)e^{ik(x-\nu+\mu/2)}+A_\nu(k)e^{ik(x+\mu-\nu/2)}\Big] . \qquad (2.301)
\end{aligned}$$

Canceling the $e^{ik(x+\mu/2)}$ factors and rearranging things we get

$$\begin{aligned}
B_\mu(k) &= A_\mu(k)-\frac{c}{2(d-1)}\sum_\nu\left(g_{\mu\nu}\hat{k}^2-\hat{k}_\mu\hat{k}_\nu\right)A_\nu(k) \\
&= \sum_\nu\left[\left(1-\frac{c}{2(d-1)}\hat{k}^2\right)\left(g_{\mu\nu}-\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right)+\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right]A_\nu(k) . \qquad (2.302)
\end{aligned}$$

Repeating this step n times we get

$$B_\mu^{(n)}(k) = \sum_\nu\left[\left(1-\frac{c}{2(d-1)}\hat{k}^2\right)^n\left(g_{\mu\nu}-\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right)+\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right]A_\nu(k) \qquad (2.303)$$

from which we can write

$$h_{\mu\nu}=\left[\left(1-\frac{c}{2(d-1)}\hat{k}^2\right)^n\left(g_{\mu\nu}-\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right)+\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right] \qquad (2.304)$$

Absorbing the smearing into redefined gluon propagator, we get the tensor

$$\begin{aligned}
H_{\mu\nu} &= h_{\mu\alpha}h_{\alpha\nu} \\
&= \left[\left(1-\frac{c}{2(d-1)}\hat{k}^2\right)^{2n}\left(g_{\mu\nu}-\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right)+\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right]-(1-\lambda)\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2} \qquad (2.305)
\end{aligned}$$

2.5.2 HYP smearing

HYP smearing is done in 3 APE-like steps:

$$V_\mu(x) \equiv 1+ig_0aB_\mu(x)$$

$$
\begin{aligned}
&= (1-\alpha_1)U_\mu(x) + \frac{\alpha_1}{2(d-1)}\sum_{\nu\neq\mu}\Big[\tilde{V}_{\nu;\mu}(x)\tilde{V}_{\mu;\nu}(x+\nu)\tilde{V}^\dagger_{\nu;\mu}(x+\mu) \\
&\quad +\tilde{V}^\dagger_{\nu;\mu}(x-\nu)\tilde{V}_{\mu;\nu}(x-\nu)\tilde{V}_{\nu;\mu}(x+\mu-\nu)\Big] \qquad (2.306)
\end{aligned}
$$

where the ν after the semicolon in $\tilde{V}_{\mu;\nu}$ means that the fat link $\tilde{V}$ is created by omitting the "staple" in the ν direction. The next step is then

$$
\begin{aligned}
\tilde{V}_{\mu;\nu}(x) &\equiv 1 + ig_0 a\tilde{A}_{\mu;\nu}(x) = (1-\alpha_2)U_\mu(x) \\
&\quad + \frac{\alpha_2}{2(d-2)}\sum_{\rho\neq\mu\nu}\Big[\bar{V}_{\rho;\mu\nu}(x)\bar{V}_{\mu;\rho\nu}(x+\rho)\bar{V}^\dagger_{\rho;\mu\nu}(x+\mu) \\
&\quad +\bar{V}^\dagger_{\rho;\mu\nu}(x-\rho)\bar{V}_{\mu;\rho\nu}(x-\rho)\bar{V}_{\rho;\mu\nu}(x+\mu-\rho)\Big] \qquad (2.307)
\end{aligned}
$$

and finally[2]

$$
\begin{aligned}
\bar{V}_{\mu;\nu}(x) &\equiv 1 + ig_0 a\bar{A}_{\mu;\nu}(x) = (1-\alpha_3)U_\mu(x) \\
&\quad + \frac{\alpha_3}{2(d-3)}\sum_{\sigma\neq\mu\nu\rho}\Big[U_\sigma(x)U_\mu(x+\sigma)U^\dagger_\sigma(x+\mu) \\
&\quad +U^\dagger_\rho(x-\sigma)U_\mu(x-\sigma)U_\sigma(x+\mu-\sigma)\Big]\,. \qquad (2.308)
\end{aligned}
$$

Fourier transforming to momentum space, we get

$$
\begin{aligned}
\bar{A}_{\mu;\nu\rho}(k) &= A_\mu(k) - \alpha_3\left(A_\mu(k) - \frac{1}{d-3}\sum_{\sigma\neq\mu\nu\rho}A_\sigma(k)\right) \\
&\quad -a^2\frac{\alpha_3}{d-3}\sum_{\sigma\neq\mu\nu\rho}\left(\hat{k}^2_\sigma A_\mu(k) - \hat{k}_\mu\hat{k}_\sigma A_\sigma(k)\right) \qquad (2.309) \\
\tilde{A}_{\mu;\nu}(k) &= A_\mu(k) - \alpha_2\left(A_\mu(k) - \frac{1}{d-2}\sum_{\rho\neq\mu\nu}\bar{A}_{\rho;\mu\nu}(k)\right) \\
&\quad -a^2\frac{\alpha_2}{d-2}\sum_{\rho\neq\mu\nu}\left(\hat{k}^2_\rho\bar{A}_{\mu;\nu\rho}(k) - \hat{k}_\mu\hat{k}_\rho\bar{A}_{\rho;\mu\nu}(k)\right) \qquad (2.310) \\
B_\mu(k) &= A_\mu(k) - \alpha_1\left(A_\mu(k) - \frac{1}{d-1}\sum_{\nu\neq\mu}\tilde{A}_{\nu;\mu}(k)\right) \\
&\quad \left(\hat{k}^2_\nu\tilde{A}_{\mu;\nu}(k) - \hat{k}_\mu\hat{k}_\nu\tilde{A}_{\nu;\mu}(k)\right)\,. \qquad (2.311)
\end{aligned}
$$

[2] in 4 dimensions HYP smearing has 3 steps; in d dimensions it would have $d-1$ step

From these we can re-express B_μ as a function of A_μ

$$
\begin{aligned}
B_\mu(k) &= A_\mu(k) - \frac{\alpha_1}{2(d-1)}\left(1+\frac{2\alpha_2}{d-2}\left(1+\frac{\alpha_3}{d-3}\right)\right)a^2\sum_{\nu\neq\mu}\left(\hat{k}_\nu^2 A_\mu - \hat{k}_\mu\hat{k}_\nu A_\nu(k)\right) \\
&+\frac{\alpha_1}{2(d-1)}\frac{\alpha_2}{2(d-2)}\left(1+\frac{2\alpha_3}{d-3}\right)a^4\sum_{\substack{\nu\neq\mu\\ \rho\neq\mu\nu}}\hat{k}_\rho^2\left(\hat{k}_\nu^2 A_\mu - \hat{k}_\mu\hat{k}_\nu A_\nu(k)\right) \\
&-\frac{\alpha_1}{2(d-1)}\frac{\alpha_2}{2(d-2)}\frac{\alpha_3}{2(d-3)}a^6\sum_{\substack{\nu\neq\mu\\ \rho\neq\mu\nu\\ \sigma\neq\mu\nu\rho}}\hat{k}_\sigma^2\hat{k}_\rho^2\left(\hat{k}_\nu^2 A_\mu - \hat{k}_\mu\hat{k}_\nu A_\nu(k)\right)
\end{aligned}
\tag{2.312}
$$

Since gluon lines will be inside loops carrying loop momentum k, after rescaling it $k \to k/a$ we get

$$
\begin{aligned}
B_\mu(k) &= A_\mu(k) - \frac{\alpha_1}{2(d-1)}\left(1+\frac{2\alpha_2}{d-2}\left(1+\frac{\alpha_3}{d-3}\right)\right)\sum_{\nu\neq\mu}\left(\hat{k}_\nu^2 A_\mu - \hat{k}_\mu\hat{k}_\nu A_\nu(k)\right) \\
&+\frac{\alpha_1}{2(d-1)}\frac{\alpha_2}{2(d-2)}\left(1+\frac{2\alpha_3}{d-3}\right)\sum_{\substack{\nu\neq\mu\\ \rho\neq\mu\nu}}\hat{k}_\rho^2\left(\hat{k}_\nu^2 A_\mu - \hat{k}_\mu\hat{k}_\nu A_\nu(k)\right) \\
&-\frac{\alpha_1}{2(d-1)}\frac{\alpha_2}{2(d-2)}\frac{\alpha_3}{2(d-3)}\sum_{\substack{\nu\neq\mu\\ \rho\neq\mu\nu\\ \sigma\neq\mu\nu\rho}}\hat{k}_\sigma^2\hat{k}_\rho^2\left(\hat{k}_\nu^2 A_\mu - \hat{k}_\mu\hat{k}_\nu A_\nu(k)\right)
\end{aligned}
\tag{2.313}
$$

from which we can write

$$
h_{\mu\nu}(k) = g_{\mu\nu}B_\mu + \hat{k}_\mu\hat{k}_\nu(1-g_{\mu\nu})B_{\mu\nu} \tag{2.314}
$$

$$
B_\mu = \left[1-\lambda_1\sum_{\rho\neq\mu}\hat{k}_\rho^2+\lambda_2\sum_{\substack{\rho\neq\mu\\ \sigma\neq\mu\rho}}\hat{k}_\rho^2\hat{k}_\sigma^2-\lambda_3\sum_{\substack{\rho\neq\mu\\ \sigma\neq\mu\rho\\ \lambda\neq\mu\rho\sigma}}\hat{k}_\rho^2\hat{k}_\sigma^2\hat{k}_\lambda^2\right] \tag{2.315}
$$

$$
B_{\mu\nu} = \left[\lambda_1-\lambda_2\sum_{\rho\neq\mu\nu}\hat{k}_\rho^2+\lambda_3\sum_{\substack{\rho\neq\mu\nu\\ \sigma\neq\mu\nu\rho}}\hat{k}_\rho^2\hat{k}_\sigma^2\right] \tag{2.316}
$$

with

$$\lambda_1 = \frac{\alpha_1}{2(d-1)}\left(1+\frac{2\alpha_2}{d-2}\left(1+\frac{\alpha_3}{d-3}\right)\right) = \frac{\alpha_1}{6}\left(1+\alpha_2\left(1+\alpha_3\right)\right) \quad (2.317)$$

$$\lambda_2 = \frac{\alpha_1}{2(d-1)}\frac{\alpha_2}{2(d-2)}\left(1+\frac{2\alpha_3}{d-3}\right) = \frac{\alpha_1\alpha_2}{24}\left(1+2\alpha_3\right) \quad (2.318)$$

$$\lambda_3 = \frac{\alpha_1}{2(d-1)}\frac{\alpha_2}{2(d-2)}\frac{\alpha_3}{2(d-3)} = \frac{\alpha_1\alpha_2\alpha_3}{48} \quad (2.319)$$

The B_μ term contributes only when $\mu=\nu$ and the $B_{\mu\nu}$ term contributes only when $\mu\neq\nu$. For $\mu=\nu$, the gluon propagator tensor $H_{\mu\nu}$ is then given by

$$H_{\mu\mu} = \sum_\rho h_{\mu\rho}h_{\rho\mu} = h_{\mu\mu}h_{\mu\mu} + \sum_{\rho\neq\mu} h_{\mu\rho}h_{\rho\mu} = B_\mu^2 + \hat{k}_\mu^2 \sum_{\rho\neq\mu}\hat{k}_\rho^2 B_{\mu\rho}B_{\rho\mu} \quad (2.320)$$

while for $\mu\neq\nu$

$$\begin{aligned} H_{\mu\nu} &= \sum_\rho h_{\mu\rho}h_{\rho\nu} = h_{\mu\mu}h_{\mu\nu} + h_{\mu\nu}h_{\nu\nu} + \sum_{\rho\neq\mu} h_{\mu\rho}h_{\rho\nu} \\ &= B_{\mu\nu}(B_\mu + B_\nu) + \sum_{\rho\neq\mu}\hat{k}_\rho^2 B_{\mu\rho}B_{\rho\nu}\,. \end{aligned} \quad (2.321)$$

So in general, we can write

$$H_{\mu\nu} = g_{\mu\nu}A_\mu + \hat{k}_\mu\hat{k}_\nu(1-g_{\mu\nu})A_{\mu\nu} \quad (2.322)$$

$$A_\mu = B_\mu^2 + \hat{k}_\mu^2 \sum_{\rho\neq\mu}\hat{k}_\rho^2 B_{\mu\rho}B_{\rho\mu} \quad (2.323)$$

$$A_{\mu\nu} = B_{\mu\nu}(B_\mu + B_\nu) + \sum_{\rho\neq\mu}\hat{k}_\rho^2 B_{\mu\rho}B_{\rho\nu}\,. \quad (2.324)$$

While the only $\hat{k}_\mu$ in A_μ is explicitly visible, $A_{\mu\nu}$ contains "hidden" dependence in $\hat{k}_\mu$ and $\hat{k}_\nu$ since B_μ contains $\hat{k}_\nu$, $B_{\mu\rho}$ contains $\hat{k}_\nu$, etc.

$$B_\mu = 1 - \lambda_1\left(\sum_{\rho\neq\mu\nu}\hat{k}_\rho^2 + \hat{k}_\nu^2\right) + \lambda_2\left(\sum_{\substack{\rho\neq\mu\nu\\ \sigma\neq\mu\rho\nu}}\hat{k}_\rho^2\hat{k}_\sigma^2 + \hat{k}_\nu^2\sum_{\rho\neq\mu\nu}\hat{k}_\rho^2\right)$$

$$-\lambda_3 \hat{k}_\nu^2 \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\nu\rho}} \hat{k}_\rho^2 \hat{k}_\sigma^2 \tag{2.325}$$

so

$$B_\mu + B_\nu = C_{\mu\nu} - \left(\hat{k}_\mu^2 + \hat{k}_\nu^2\right) B_{\mu\nu} \tag{2.326}$$

$$C_{\mu\nu} = 2\left[1 - \lambda_1 \sum_{\rho\neq\mu\nu} \hat{k}_\rho^2 + \lambda_2 \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\rho\nu}} \hat{k}_\rho^2 \hat{k}_\sigma^2\right] \tag{2.327}$$

To simplify the $\sum \rho \hat{k}_\rho^2 B_{\mu\rho} B_{\rho\nu}$ term, we introduce indices α and β so that μ, ν, α, β are all different. Then

$$\sum_{\rho\neq\mu\nu} \hat{k}_\rho^2 = \hat{k}_\alpha^2 + \hat{k}_\beta^2\,, \qquad \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\nu\rho}} \hat{k}_\rho^2 \hat{k}_\sigma^2 = 2\hat{k}_\alpha^2 \hat{k}_\beta^2 \tag{2.328}$$

where the factor 2 in the second term comes from permutations of indices. Then

$$\begin{aligned}
\sum_{\rho\neq\mu\nu} \hat{k}_\rho^2 B_{\mu\rho} B_{\rho\nu} &= \hat{k}_\alpha^2 B_{\mu\alpha} B_{\alpha\nu} + \hat{k}_\beta^2 B_{\mu\beta} B_{\beta\nu} \\
&= \hat{k}_\alpha^2 \left[\lambda_1 - \lambda_2(\hat{k}_\beta^2 + \hat{k}_\nu^2) + \lambda_3(2\hat{k}_\beta^2 \hat{k}_\nu^2)\right] \left[\lambda_1 - \lambda_2(\hat{k}_\beta^2 + \hat{k}_\mu^2) + \lambda_3(2\hat{k}_\beta^2 \hat{k}_\mu^2)\right] \\
&\quad + \hat{k}_\beta^2 \left[\lambda_1 - \lambda_2(\hat{k}_\alpha^2 + \hat{k}_\nu^2) + \lambda_3(2\hat{k}_\alpha^2 \hat{k}_\nu^2)\right] \left[\lambda_1 - \lambda_2(\hat{k}_\alpha^2 + \hat{k}_\mu^2) + \lambda_3(2\hat{k}_\alpha^2 \hat{k}_\mu^2)\right] .
\end{aligned}$$

After expanding the products and collecting terms, we get

$$\sum_{\rho\neq\mu\nu} \hat{k}_\rho^2 B_{\mu\rho} B_{\rho\nu} = (\hat{k}_\mu^2 + \hat{k}_\nu^2) a_{\mu\nu} + (\hat{k}_\mu^2 \hat{k}_\nu^2) b_{\mu\nu} + c_{\mu\nu} \tag{2.329}$$

$$a_{\mu\nu} = -\lambda_1\lambda_2 \sum_{\rho\neq\mu\nu} \hat{k}_\rho^2 + (\lambda_2 + \lambda_1\lambda_3) \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\nu\rho}} \hat{k}_\rho^2 \hat{k}_\sigma^2 - 2\lambda_2\lambda_3 \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\nu\rho}} \hat{k}_\rho^4 \hat{k}_\sigma^2 \tag{2.330}$$

$$b_{\mu\nu} = \lambda_2^2 \sum_{\rho\neq\mu\nu} \hat{k}_\rho^2 - 2\lambda_2\lambda_3 \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\nu\rho}} \hat{k}_\rho^2 \hat{k}_\sigma^2 + 4\lambda_3^2 \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\nu\rho}} \hat{k}_\rho^4 \hat{k}_\sigma^2 \tag{2.331}$$

$$c_{\mu\nu} = \lambda_1^2 - \lambda_1\lambda_2 \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\nu\rho}} \hat{k}_\rho^2 \hat{k}_\sigma^2 + \lambda_2^2 \sum_{\substack{\rho\neq\mu\nu \\ \sigma\neq\mu\nu\rho}} \hat{k}_\rho^4 \hat{k}_\sigma^2 \tag{2.332}$$

so the overall result is

$$A_{\mu\nu} = A^{(0)}_{\mu\nu} + (\hat{k}^2_\mu + \hat{k}^2_\nu) A^{(1)}_{\mu\nu} + \hat{k}^2_\mu \hat{k}^2_\nu A^{(2)}_{\mu\nu} \tag{2.333}$$

with

$$A^{(0)}_{\mu\nu} = (B_{\mu\nu} C_{\mu\nu} + c_{\mu\nu}) , \qquad A^{(1)}_{\mu\nu} = a_{\mu\nu} - B_{\mu\nu} , \qquad A^{(2)}_{\mu\nu} = b_{\mu\nu} \tag{2.334}$$

Expansion of HYP-smearing tensor

When calculating Feynman diagrams, we typically encounter a situation where we expand the gluon propagator (here in Feynman gauge)

$$\begin{aligned} G_{\mu\nu}(ap-k) &= \frac{H_{\mu\nu}(ap-k)}{\left(\widehat{ap-k}\right)^2 + \mu^2} \\ &= \frac{g_{\mu\nu} A_\mu(ap-k) + \left(\widehat{ap-k}\right)_\mu \left(\widehat{ap-k}\right)_\nu (1-g_{\mu\nu}) A_{\mu\nu}(ap-k)}{\left(\widehat{ap-k}\right)^2 + \mu^2} \end{aligned} \tag{2.335}$$

in power series in a (or equivalently in p_μ). Factors $\left(\widehat{ap-k}\right)_\mu \left(\widehat{ap-k}\right)_\nu$ are straightforward to evaluate (and usually easier to do after contracting with the rest of the amplitude and using trigonometric simplifications) so here we will do only the expansion of A_μ and $A_{\mu\nu}$. Using trigonometric simplifications we get

$$(\widehat{ap-k})^2_\rho = \hat{k}^2_\rho + 2\cos k_\rho (1 - \cos ap_\rho) - 2\bar{k}_\rho \sin ap_\rho = \hat{k}^2_\rho - 2a\,\bar{k}_\rho p_\rho + O(a^2) \,. \tag{2.336}$$

Expanding

$$A_\rho(ap-k) = B^2_\rho(ap-k) + (\widehat{ap-k})^2_\rho \sum_{\sigma\neq\rho} (\widehat{ap-k})^2_\sigma B^2_{\rho\sigma}(ap-k) \tag{2.337}$$

around $a = 0$, we get

$$A_\rho(ap-k) = A_\rho(k) - 2a\, p_\rho \bar{k}_\rho C_\rho(k) - 2a \sum_{\sigma\neq\rho} p_\sigma \bar{k}_\sigma C_{\rho\sigma} \tag{2.338}$$

$$C_\rho = \sum_{\sigma\neq\rho} \hat{k}_\sigma^2 B_{\sigma\rho}^2(k) \tag{2.339}$$

$$C_{\rho\sigma} = -2\left[B_\rho B_{\rho|\sigma} - \frac{1}{2}\hat{k}_\rho^2 B_{\rho\sigma}^2 + \hat{k}_\rho^2 \sum_{\alpha\neq\rho\sigma} \hat{k}_\alpha^2 B_{\rho\alpha} B_{\rho\alpha|\sigma}\right] \tag{2.340}$$

where we have introduced the notation

$$f_{|\sigma} \equiv \frac{\partial f}{\partial \hat{k}_\sigma^2} \tag{2.341}$$

which gives us

$$B_{\rho|\sigma} = \lambda_1 - 2\lambda_2 \sum_{\alpha\neq\rho\sigma} \hat{k}_\alpha^2 + 3\lambda_3 \sum_{\substack{\alpha\neq\rho\sigma \\ \beta\neq\alpha\rho\sigma}} \hat{k}_\alpha^2 \hat{k}_\beta^2 \tag{2.342}$$

$$B_{\rho\alpha|\sigma} = \lambda_2 - 2\lambda_3 \sum_{\beta\neq\alpha\rho\sigma} \hat{k}_\beta^2 \tag{2.343}$$

Using the same approach and the same notation, we expand $A_{\mu\nu}(ap-k)$ as well:

$$\begin{aligned} A_{\mu\nu}(ap-k) &= A_{\mu\nu}^{(0)}(ap-k) + [(\widehat{ap-k})_\mu^2 + (\widehat{ap-k})_\nu^2] A_{\mu\nu}^{(1)}(ap-k) \\ &\quad + [(\widehat{ap-k})_\mu^2 (\widehat{ap-k})_\nu^2]\, A_{\mu\nu}^{(2)}(ap-k) \\ &= A_{\mu\nu}(k) - 2a\, p_\mu \bar{k}_\mu [A_{\mu\nu}^{(1)}(k) + \hat{k}_\nu^2 A_{\mu\nu}^{(2)}(k)] \\ &\quad -2a\, p_\nu \bar{k}_\nu [A_{\mu\nu}^{(1)}(k) + \hat{k}_\mu^2 A_{\mu\nu}^{(2)}(k)] \\ &\quad -2a \sum_{\rho\neq\mu\nu} p_\rho \bar{k}_\rho \left[A_{\mu\nu|\rho}^{(0)} + (\hat{k}_\mu^2 + \hat{k}_\nu^2) A_{\mu\nu|\rho}^{(1)} + \hat{k}_\mu^2 \hat{k}_\nu^2\, A_{\mu\nu|\rho}^{(2)}\right] \end{aligned} \tag{2.344}$$

Chapter 3

Gluon actions

3.1 Wilson gluons

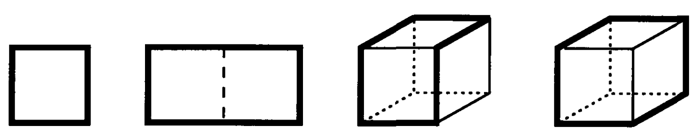

Figure 3-1: Closed gluon loops up to length 6

The simplest gauge-invariant gluon action on a lattice was originally proposed by Wilson. It is the sum off all plaquettes (only closed loop of length 4 that can be constructed on the lattice)

$$S_W = \frac{2N_c}{g_0^2} a^4 \sum_{plaquette} \frac{1}{N_c} \mathrm{ReTr} U_{pl} \tag{3.1}$$

where the plaquette U_{pl} in μ, ν plane is defined as

$$P_{\mu\nu} = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x) \tag{3.2}$$

and U_μ is the gauge link belonging to $SU(N_c)$ group

$$U_\mu(x) = e^{i g_0 a T_b A_\mu^b (x + a\hat{\mu}/2)} \ . \tag{3.3}$$

Expanding the plaquette in terms of field A_μ and commuting all the matrices, the lowest order contribution of a single plaquette has the proper continuum limit

$$\mathrm{Re}\,\mathrm{Tr}\,P_{\mu\nu} = N_c - \frac{1}{2}g_0^2\mathrm{Tr}F_{\mu\nu}^2 + O(a^2)\,, \qquad \text{(no summation)} \tag{3.4}$$

Summing over all plaquettes and taking the $a \to 0$ limit, we recover the continuum action

$$S_{cont} = \frac{1}{2}\mathrm{Tr}\sum_{\mu\nu} F_{\mu\nu}(x)F_{\mu\nu}(x)\,. \tag{3.5}$$

3.2 Improved gluons

The Wilson action has finite-size corrections of order a^2. Symantzik [22, 23] introduced the idea that one could add terms vanishing in the continuum limit which would further improve finite-size corrections to higher order. This idea was extended by Weisz [24, 25] to gluon actions. By combining all closed loops to length 6, one gets the action

$$\begin{aligned} S_{gl} \;=\; & \frac{2N_c}{g_0^2}\left(c_0 \sum_{plaquette}\frac{1}{N_c}\mathrm{ReTr}U_{pl} + c_1 \sum_{rectangle}\frac{1}{N_c}\mathrm{ReTr}U_{rec}\right. \\ & \left. + c_2\sum_{chair}\frac{1}{N_c}\mathrm{ReTr}U_{ch} + c_3 \sum_{parallelogram}\frac{1}{N_c}\mathrm{ReTr}U_{paral}\right)\,. \end{aligned} \tag{3.6}$$

Since the sum has to have the same (correct) continuum limit, expanding the action to lowest order in a we get

$$S_{gl} = \frac{1}{2}(c_0 + 8c_1 + 4(d-2)(2c_2+c_3))\mathrm{Tr}\sum_{\mu\nu} F_{\mu\nu}(x)F_{\mu\nu}(x)\,. \tag{3.7}$$

which gives us the overall constraint

$$c_0 + 8c_1 + 4(d-2)(2c_2+c_3) = 1 \tag{3.8}$$

Different choices of parameters c_i lead to different actions Adding a gauge fixing term to

Action	c_1	c_2	c_3
Symantzik	-1/12	0	0
Iwasaki	-0.331	0	0
Iwasaki'	-0.27	-0.04	-0.04
Wilson	-0.252	$c_2+c_3=-0.17$	
DBW2	-1.40686	0	0

Table 3.1: Various improved actions

the action

$$S_{GF} = a^4 \sum_x \frac{1}{2\lambda} \left[\nabla_\mu A_\mu (x + a\hat{\mu}/2)\right]^2 \tag{3.9}$$

and Fourier-transforming to momentum space, we get the gluon action (to lowest order in A_μ)

$$S_{gl} = \frac{1}{2} \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} \sum_{\mu\nu} A_\mu^a(k) \left[G_{\mu\nu} - \left(1 - \frac{1}{\lambda}\right) \hat{k}_\mu \hat{k}_\nu\right] A_\nu^2(-k) \tag{3.10}$$

with

$$\begin{aligned} G_{\mu\nu} &= \hat{k}_\mu \hat{k}_\nu + \sum_\rho \left(\hat{k}_\rho^2 g_{\mu\nu} - \hat{k}_\mu \hat{k}_\rho g_{\nu\rho}\right) \\ &\quad \times (1 - g_{\mu\rho}) \left[1 - a^2 c_1 (\hat{k}_\mu^2 + \hat{k}_\rho^2) - a^2 (c_2 + c_3)(\hat{k}^2 - \hat{k}_\mu^2 - \hat{k}_\rho^2)\right] \\ &= \hat{k}_\mu \hat{k}_\nu + g_{\mu\nu} \left[(1 - a^2 \hat{k}^2 (c_2 + c_3)) \sum_{\rho \neq \mu} \hat{k}_\rho^2 - a^2 (\sum_\rho \hat{k}_\rho^4 - \hat{k}^2 \hat{k}_\mu^2)(c_1 - c_2 - c_3)\right] \\ &\quad - \hat{k}_\mu \hat{k}_\nu (1 - g_{\mu\nu}) \left[1 - a^2 c_1 (\hat{k}_\mu^2 + \hat{k}_\nu^2) - a^2 (c_2 + c_3) \sum_{\rho \neq \mu\nu} \hat{k}_\rho^2\right] \end{aligned} \tag{3.11}$$

3.2.1 Tadpole improvement

To match lattice results to continuum, we have to match lattice operators to continuum operators as well [26]. For gauge fields this is done by expanding the link U_μ in powers of lattice spacing a

$$U_\mu(x) = e^{iag_0 A_\mu} \to 1 + iag_0 A_\mu(x)\,. \tag{3.12}$$

This expansion is misleading since further corrections do *not* vanish as powers of a; higher order terms contain additional factors of gaA_μ, which if contracted with each other, give ultraviolet divergences that exactly cancel additional powers of a, giving overall suppression by powers of g^2 (not a). These are the QCD tadpole contributions. Since they spoil our intuition on how the lattice theory converges to continuum, we must redefine our formulas for connecting lattice results to continuum. Consider the expectation value of a link variable on lattice; since the link itself is gauge dependent, we define it through a gauge-independent expectation value of a plaquette

$$u_0 = \left\langle \frac{1}{N_c} \mathrm{Tr} U_{plaq} \right\rangle^{1/4} . \tag{3.13}$$

In practice, this number is much smaller then 1; however, in the continuum theory the expectation value of the operator $1 + iag_0 A_\mu$ is 1. This suggests that the appropriate connection between lattice operator and continuum fields is more like

$$U_\mu(x) \to u_0 \left[1 + iag_0 A_\mu(x)\right] . \tag{3.14}$$

3.2.2 MILC gluon lattice action

Lattices used by MILC use tadpole improved Symantzik gauge action [27, 28, 29, 30, 31]

$$\begin{aligned} S_{gl} &= \beta_{pl} \sum_{plaquette} \frac{1}{N_c} \mathrm{ReTr} U_{pl} + \beta_{rect} \sum_{rectangle} \frac{1}{N_c} \mathrm{ReTr} U_{rec} \\ &+ \beta_{par} \sum_{parallelogram} \frac{1}{N_c} \mathrm{ReTr} U_{paral} . \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \beta_{pl} &= \frac{2N_c}{g^2} \frac{5}{3} , \\ \beta_{rec} &= -\frac{\beta_{pl}}{20u_0^2} (1 + 0.4805\alpha_s) , \\ \beta_{par} &= -\frac{\beta_{pl}}{u_0^2} 0.03325\alpha_s \end{aligned} \tag{3.16}$$

and the strong coupling constant is determined through the 1-loop relation

$$\alpha_s = -4\frac{\log u_0}{3.0684}\,. \tag{3.17}$$

and the values β_{pl} and u_0 are taken from [30] and [31] (for $a = 0.13fm$ lattices) In terms

β_{pl}	u_0
8.00	0.8879
7.35	0.8822
7.20	0.8755
7.15	0.8787
6.96	0.8739
6.85	0.8707
6.83	0.8702
6.81	0.8696
6.79	0.8688
6.76	0.8677

Table 3.2: The parameters of form MILC gluon actions.

of coefficients c_i from previous subsection, we have

$$\begin{aligned} c_0 &= \frac{5}{3}\,, \\ c_1 &= -\frac{1}{12u_0^2}\left(1+0.4805\alpha_s\right)\,, \\ c_2 &= 0\,, \\ c_3 &= -\frac{5}{3u_0^2}0.03325\alpha_s\,. \end{aligned} \tag{3.18}$$

Chapter 4

Self energy in DW formulation

Before we start calculating current renormalizations and twist-2 operator renormalizations, we first need to evaluate self-energy renormalization since it will contribute to all other operators.

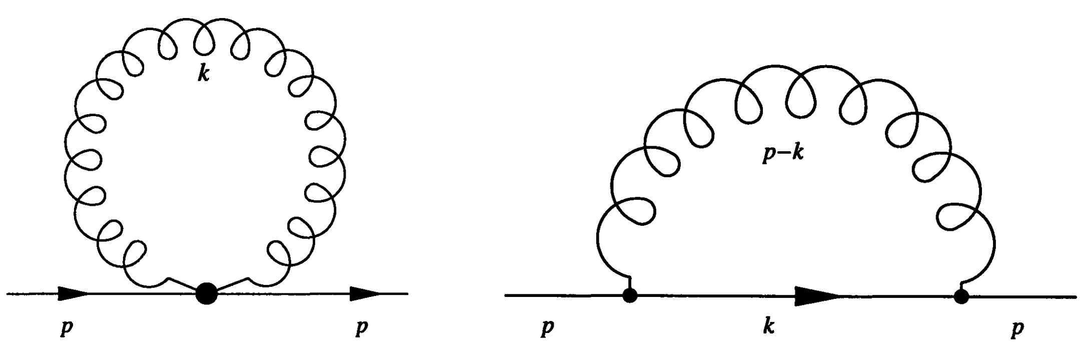

Figure 4-1: Self energy diagrams: Tadpole (left) and Sunset (right).

There are 2 diagrams contributing to the self energy: sunset and tadpole. To evaluate them, we need Feynman rules on the lattice. In the "r=-1" convention (see appendix A.2), the gluon-fermion vertex is

$$\begin{aligned}\left[V^a_\rho\right]_{bc} &= -g_0\,[T^a]_{bc}\left(r\sin\frac{a(k+p)_\rho}{2}+i\gamma_\rho\cos\frac{a(k+p)_\rho}{2}\right)\\ &= -g_0\,[T^a]_{bc}\left(r\frac{a}{2}(\widehat{p+k})_\rho+i\gamma_\rho(\widetilde{p+k})_\rho\right)\,. \qquad (4.1)\end{aligned}$$

The 2-gluon-fermion vertex is given by

$$\left[V^{ab}_{\rho\rho}\right]_{cd} = -\frac{1}{2}ag_0^2\left[\{T^a,T^b\}\right]_{cd}\left(r\cos\frac{a(k+p)_\rho}{2} - i\gamma_\rho\sin\frac{a(k+p)_\rho}{2}\right). \quad (4.2)$$

We can simplify the group structure since it'll be multiplied by δ^{ab}, summed over, and have its trace calculated

$$\frac{1}{2}\mathrm{Tr}\sum_{ab}\delta^{ab}\{T^a,T^b\} = \mathrm{Tr}\sum_{a}T^aT^a = \frac{N^2-1}{2N} = C_F \quad (4.3)$$

we get

$$V^{aa}_{\rho\rho} = -ag_0^2C_F\left(r\cos\frac{a(k+p)_\rho}{2} - i\gamma_\rho\sin\frac{a(k+p)_\rho}{2}\right). \quad (4.4)$$

The gluon propagator is

$$G^{ab}_{\mu\nu}(p-k) = \delta^{ab}H_{\mu\nu}\frac{1}{\sum_\lambda \frac{4}{a^2}\sin^2\frac{a(p-k)_\rho}{2}} = \frac{\delta^{ab}H_{\mu\nu}}{\widehat{(p-k)}^2}, \quad (4.5)$$

with

$$H_{\mu\nu} = h_{\mu\alpha}h_{\alpha\nu} - (1-\lambda)\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2} \quad (4.6)$$

and the fermion propagator is

$$\begin{aligned}
[S_F(k)]_{ss'} &= \left[-i\sum_\mu\gamma_\mu\frac{\sin ak_\mu}{a} + W^-(ak)\right]_{st}[G_+(ak)]_{ts'}P_+ \\
&\quad + \left[-i\sum_\mu\gamma_\mu\frac{\sin ak_\mu}{a} + W^+(ak)\right]_{st}[G_-(ak)]_{ts'}P_- \\
&= \left[-i\gamma\cdot\bar{k} + W^-(ak)\right]_{st}[G_+(ak)]_{ts'}P_+ \\
&\quad + \left[-i\gamma\cdot\bar{k} + W^+(ak)\right]_{st}[G_-(ak)]_{ts'}P_- \qquad (4.7)
\end{aligned}$$

with

$$P_\pm \equiv \frac{1}{2}(1\pm\gamma_5). \quad (4.8)$$

Another way of writing this is

$$[S_F(k)]_{ss'} = -i\gamma\cdot\bar{k}\left(\sigma^V_{ss'}+\gamma_5\sigma^A_{ss'}\right)+\left(\sigma^S_{ss'}+\gamma_5\sigma^P_{ss'}\right) \tag{4.9}$$

where σ denotes matrices in 5^{th} dimension/flavor. Comparing these expressions we get

$$\sigma_V = \frac{G_+ + G_-}{2} \quad , \quad \sigma_A = \frac{G_+ - G_-}{2} \tag{4.10}$$

$$\sigma_S = \frac{W_- G_+ + W^+ G_-}{2} \quad , \quad \sigma_P = \frac{W_- G_+ - W^+ G_-}{2}\ . \tag{4.11}$$

Another useful relation is:

$$(\sigma_V \pm \gamma_5 \sigma_A) = G_+ P_\pm + G_- P_\mp \tag{4.12}$$

$$(\sigma_S \pm \gamma_5 \sigma_P) = W^- G_+ P_\pm + W^+ G_- P_\mp \tag{4.13}$$

Momentum rescaling and Wilson limit

After we rescale the momentum in the integral

$$k_\mu \to \frac{k_\mu}{a}\ , \tag{4.14}$$

we get an overall factor a^{-d} from the $d^d k$ plus some factors from propagators/vertices. Then we have

$$V^a_\rho = -g_0 T^a\left(r\sin\frac{(k+ap)_\rho}{2}+i\gamma_\rho\cos\frac{(k+ap)_\rho}{2}\right)\sim O(a^0) \tag{4.15}$$

$$G^{ab}_{\mu\nu}(p-k) = a^2\delta^{ab}g_{\mu\nu}\frac{1}{\sum\limits_\lambda 4\sin^2\frac{(ap-k)_\rho}{2}}\sim O(a^2) \tag{4.16}$$

$$\begin{aligned} S_F(k) &= a\left\{\left(-i\gamma\cdot\bar{k}+W^-(k)\right)G_+(k)P_+ + \left[-i\gamma\cdot\bar{k}+W^+(k)\right]_{st}G_-(k)P_-\right\} \\ &\sim O(a) \end{aligned} \tag{4.17}$$

where in the fermion propagator we have

$$G_{\pm}(ak) \to a^2\, G_{\pm}(k)\,, \qquad W_{\pm}(ak) \to \frac{1}{a}\, W_{\pm}(k) \tag{4.18}$$

$$\sigma_{V,A}(ak) \to a^2\, \sigma_{V,A}(k)\,, \qquad \sigma_{S,P}(ak) \to a\, \sigma_{S,P}(k)\,. \tag{4.19}$$

To retrieve the Wilson Limit, we have to set $\sigma_A = \sigma_P = 0$,

$$\sigma_V = \frac{1}{\bar{k}^2 + \left(am_0 + \frac{r}{2}\hat{k}^2\right)^2} \tag{4.20}$$

$$\sigma_S = \frac{m_0 a + \frac{r}{2}\hat{k}^2}{\bar{k}^2 + \left(am_0 + \frac{r}{2}\hat{k}^2\right)^2} \tag{4.21}$$

4.1 Sunset diagram

The amplitude for the sunset diagram is given by

$$I_{st}(a,p) = \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} \sum_{\lambda,\rho} G_{\lambda\rho}(p-k) V_\rho(k,p) [S_F(k)]_{st} V_\lambda(p,k)\,. \tag{4.22}$$

Now: we choose to write the gluon propagator as

$$G_{\lambda\rho}(p-k) = \frac{h_\rho h_\lambda}{(\widehat{p-k})^2 + \mu^2} \tag{4.23}$$

where we keep the gluon mass μ finite for now. This way we can do γ-algebra for all terms at once and then later restore smeared/unsmeared/longitudinal limits. After rescaling the integral (4.22) over k, we get

$$\begin{aligned} I(p) &= \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_{\lambda,\rho} G_{\lambda\rho}(ap-k) V_\rho(k,ap) S_F(k) V_\lambda(p,ak) \\ &= g_0^2 C_F \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{a^4} \sum_{\mu} \left[\frac{a^2 h_\rho h_\sigma}{\left(\widehat{ap-k}\right)^2} \right] \left[\frac{r}{2} \left(\widehat{ap+k}\right)_\rho + i\gamma_\rho \left(\widetilde{ap+k}\right)_\rho \right] \\ &\quad \times a \left[(-i\gamma \cdot \bar{k}(G_+ P_+ + G_- P_-) + S_+ P_+ + S_- P_- \right] \end{aligned}$$

$$\times\left[\frac{r}{2}\left(\widehat{ap+k}\right)_\sigma + i\gamma_\sigma\left(\widetilde{ap+k}\right)_\sigma\right] \tag{4.24}$$

$$= \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{g_0^2C_F}{a}\frac{1}{\left(\widehat{ap-k}\right)^2}[a\cdot h + i\gamma\cdot b]$$

$$\times\left[-i\gamma\cdot\bar{k}(G_+P_+ + G_-P_-) + S_+P_+ + S_-P_-\right][a\cdot h + i\gamma\cdot b] \tag{4.25}$$

Now we do the γ-algebra: "a^2" term has no γ matrices so it equals:

$$f_1 = (a\cdot h)^2\left[-i\gamma\cdot\bar{k}G_\pm P_\pm + S_\pm P_\pm\right]\;. \tag{4.26}$$

The "b^2" term has one γ matrix commuted with $P_\pm$, so using $\gamma_\mu P_\pm = P_\mp\gamma_\mu$ we get

$$\begin{aligned} f_2 &= b\cdot\gamma\left[-i\gamma\cdot\bar{k}G_\pm P_\pm + S_\pm P_\pm\right]b\cdot\gamma = \left\{b\cdot\gamma\left[-i\gamma\cdot\bar{k}\right]b\cdot\gamma G_\mp + b^2S_\mp\right\}P_\pm \\ &= \left\{i(b^2\bar{k}\cdot\gamma - 2b\cdot kb\cdot\gamma)G_\mp + b^2S_\mp\right\}P_\pm\;. \end{aligned} \tag{4.27}$$

Finally, "mixed" terms are

$$\begin{aligned} f_3 &= ia\cdot h\left[-i\gamma\cdot\bar{k}G_\pm + S_\pm\right]P_\pm b\cdot\gamma + ib\cdot\gamma\left[-i\gamma\cdot\bar{k}G_\pm + S_\pm\right]P_\pm a\cdot h \\ &= \left\{ia\cdot hb\cdot\gamma(S_\pm + S_\mp) + a\cdot h(\bar{k}\cdot\gamma b\cdot\gamma G_\mp + b\cdot\gamma\bar{k}\cdot\gamma G_\pm)\right\}P_\pm \end{aligned} \tag{4.28}$$

Restoring back definitions of a, b and h, we get

$$\begin{aligned} f_1 + f_2 + f_3 = &\left[h_{\rho\alpha}h_{\alpha\sigma} - (1-\lambda)\frac{\hat{k}_\rho\hat{k}_\sigma}{\hat{k}^2}\right]\left\{\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma(-i\bar{k}\cdot\gamma G_\pm + S_\pm)\right. \\ &+ g_{\rho\sigma}(\widetilde{ap+k})^2_\rho(-i\bar{k}\cdot\gamma G_\mp - S_\mp) \\ &+ \frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma(\bar{k}\cdot\gamma\gamma_\sigma G_\mp + \gamma_\sigma\bar{k}\cdot\gamma G_\pm + i\gamma_\sigma(S_\pm + S_\mp)) \\ &\left.+ 2i(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\gamma_\rho\bar{k}_\sigma G_\mp)\right\}P_\pm\,, \end{aligned} \tag{4.29}$$

from which we get the amplitude I and split it into parts with odd and even number of γ matrices $I = I_{odd} + I_{even}$

$$
\begin{aligned}
I^{\pm}_{odd} &= \frac{h_{\rho\alpha}h_{\alpha\sigma} - (1-\lambda)\frac{\hat{k}_\rho \hat{k}_\sigma}{\hat{k}^2}}{\left(\widehat{ap-k}\right)^2 + \mu^2} \Big\{ -i\bar{k}\cdot\gamma \left(\frac{r^2}{4} (\widehat{ap+k})_\rho (\widehat{ap+k})_\sigma G_\pm + g_{\rho\sigma} (\widetilde{ap+k})^2_\rho G_\mp \right) \\
&\quad + 2i (\widetilde{ap+k})_\rho (\widetilde{ap+k})_\sigma \gamma_\rho \bar{k}_\sigma G_\mp + \frac{r}{2} (\widehat{ap+k})_\rho (\widetilde{ap+k})_\sigma i \gamma_\sigma (S_\pm + S_\mp) \Big\} \qquad (4.30) \\
I^{\pm}_{even} &= \frac{h_{\rho\alpha}h_{\alpha\sigma} - (1-\lambda)\frac{\hat{k}_\rho \hat{k}_\sigma}{\hat{k}^2}}{\left(\widehat{ap-k}\right)^2 + \mu^2} \Big\{ \frac{r^2}{4} (\widehat{ap+k})_\rho (\widehat{ap+k})_\sigma S_\pm - g_{\rho\sigma} (\widetilde{ap+k})^2_\rho S_\mp \\
&\quad + \frac{r}{2} (\widehat{ap+k})_\rho (\widetilde{ap+k})_\sigma (\bar{k}\cdot\gamma\gamma_\sigma G_\mp + \gamma_\sigma \bar{k}\cdot\gamma G_\pm) \Big\} \qquad (4.31)
\end{aligned}
$$

4.1.1 5D sums for physical quarks

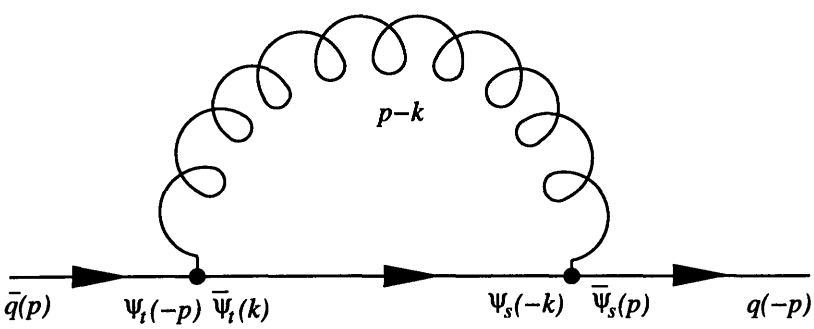

Figure 4-2: Sunset diagram for physical quarks

For physical quarks, we have to take into account that we start and end with quarks defined on the boundary (figure (4-2)) so the total amplitude is given by

$$\bar{I}_q = \langle q(-p)\bar{\psi}_s(p)\rangle I_{st} \langle \psi_t(-p)\bar{q}(p)\rangle \ . \qquad (4.32)$$

However, this amplitude has external legs attached to it so we have to multiply it from the left and from the right by

$$S^{-1}_{phys}(p) = \frac{1}{\langle q(-p)\bar{q}(p)\rangle} \qquad (4.33)$$

Using the terminology from section (2.2), the amputated amplitude is then (expanded to the lowest order in p)

$$\begin{aligned}
I_q(p) &= \bar{S}_s^{OUT} I_{st} \bar{S}_t^{IN} \\
&= (1-w_0^2)\left[-ip\cdot\gamma\mathcal{A}\left(w_0^{N-s}P_- + w_0^{s-1}P_+\right) + \left(w_0^{s-1}P_- + w_0^{N-s}P_+\right)\right]_s I_{st} \\
&\quad \times \left[\left(w_0^{N-t}P_+ + w_0^{t-1}P_-\right)(-ip\cdot\gamma\mathcal{A}) + \left(w_0^{t-1}P_+ + w_0^{N-t}P_-\right)\right]_t . \qquad (4.34)
\end{aligned}$$

Now we use the fact that

$$I_{odd}P_\pm = P_\mp I_{odd}\,, \qquad I_{even}P_\pm = P_\pm I_{even} \qquad (4.35)$$

to get

$$\begin{aligned}
I_{phys}^{odd} &= (-ip\cdot\gamma\mathcal{A})\bar{I}_{odd}^-(-ip\cdot\gamma\mathcal{A}) + \bar{I}_{odd}^+ + (-ip\cdot\gamma\mathcal{A})\tilde{I}_{odd}^- + \tilde{I}_{odd}^+(-ip\cdot\gamma\mathcal{A}) \qquad (4.36)\\
I_{phys}^{even} &= (-ip\cdot\gamma\mathcal{A})\tilde{I}_{even}^+(-ip\cdot\gamma\mathcal{A}) + \tilde{I}_{even}^- + (-ip\cdot\gamma\mathcal{A})\bar{I}_{even}^+ + \bar{I}_{even}^-(-ip\cdot\gamma\mathcal{A}) \qquad (4.37)
\end{aligned}$$

where

$$\begin{aligned}
\bar{I}^\pm &\equiv (1-w_0^2)\sum w_0^{s-1} I^\pm w_0^{t-1} \equiv (1-w_0^2)\sum w_0^{N-s} I^\mp w_0^{N-t}\,, \qquad (4.38)\\
\tilde{I}^\pm &\equiv (1-w_0^2)\sum w_0^{s-1} I^\pm w_0^{N-t} \equiv (1-w_0^2)\sum w_0^{N-s} I^\mp w_0^{t-1} \qquad (4.39)
\end{aligned}$$

and $\mathcal{A} = a_5 w_0/(1-w_0^2) = a_5/(2\sinh\alpha_0)$. To evaluate the renormalization of the self energy, we only need to keep terms up to order p^1:

$$\begin{aligned}
I_{phys} &= \bar{I}_{odd}^+(k) + p_\mu\frac{\partial\bar{I}_{odd}^+(k)}{\partial p_\mu} - i\mathcal{A}\left[p\cdot\gamma\tilde{I}_{odd}^-(k) + \tilde{I}_{odd}^+(k)p\cdot\gamma\right] \\
&\quad + \tilde{I}_{even}^-(k) + p_\mu\frac{\partial\tilde{I}_{even}^-(k)}{\partial p_\mu} - i\mathcal{A}\left[p\cdot\gamma\bar{I}_{even}^+(k) + \bar{I}_{even}^- p\cdot\gamma\right] . \qquad (4.40)
\end{aligned}$$

The terms $p\cdot\gamma\tilde{I}_{odd}^-(k) + \tilde{I}_{odd}^+(k)p\cdot\gamma$, $\bar{I}_{odd}^+(k)$ and $p_\mu\frac{\partial\tilde{I}_{even}^-(k)}{\partial p_\mu}$ vanish after integration since they are odd in k_μ as well so we are left with

$$I_{phys} = \tilde{I}_{even}^-(k) - i\mathcal{A}\left[p\cdot\gamma\bar{I}_{even}^+(k) + \bar{I}_{even}^- p\cdot\gamma\right] + p_\mu \left.\frac{\partial\bar{I}_{odd}^+}{\partial p_\mu}\right|_{p\to 0} . \qquad (4.41)$$

We will evaluate each of them separately.

The first term: $\tilde{I}^{-}_{even}$

The first term is given by

$$\tilde{I}^{-}_{even}(k) = \frac{H_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}_\rho\hat{k}_\sigma\tilde{S}_- - g_{\rho\sigma}\bar{k}^2_\rho\tilde{S}_+ + \frac{r}{2}\hat{k}_\rho\bar{k}_\sigma(\bar{k}\cdot\gamma\gamma_\sigma\tilde{G}_+ + \gamma_\sigma\bar{k}\cdot\gamma\tilde{G}_-)\right\}. \quad (4.42)$$

Since $\tilde{G}_+ = \tilde{G}_- = \tilde{\sigma}_V$, the last term simplifies to $2\bar{k}_\sigma\tilde{\sigma}_V$. Next we use the definition of tensor $H_{\rho\sigma}$

$$H_{\rho\sigma} = g_{\rho\sigma}A_\rho + \hat{k}_\rho\hat{k}_\sigma(1-g_{\rho\sigma})A_{\rho\sigma} \quad (4.43)$$

to get

$$\begin{aligned}\tilde{I}^{-}_{even}(k) &= \sum_\rho\frac{A_\rho}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}^2_\rho\tilde{S}_- - \bar{k}^2_\rho\tilde{S}_+ + r\bar{k}^2_\rho\tilde{\sigma}_V\right\}\\ &+\sum_{\rho\sigma}\frac{A_{\rho\sigma}(1-g_{\rho\sigma})}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\tilde{S}_- + r\hat{k}^2_\rho\bar{k}^2_\sigma\tilde{\sigma}_V\right\}\\ &-\sum_{\rho\sigma}\frac{A_{\rho\sigma}(1-g_{\rho\sigma})}{\hat{k}^2+\mu^2}\left\{g_{\rho\sigma}\bar{k}^2_\sigma\tilde{S}_+\right\}. \end{aligned} \quad (4.44)$$

The last term obviously vanishes for HYP-smearing but will contribute for APE-smearing so we will retain it for now. Since all three propagator functions $\tilde{S}_+$, $\tilde{S}_-$ and $\tilde{\sigma}_V$ are proportional to the mass parameter m, this term contributes to mass renormalization proportional to the mass m. Introducing the parametrization

$$\tilde{I}^{-}_{even} = m\frac{g_0^2C_F}{16\pi^2}\Sigma_2 \quad (4.45)$$

we get

$$\begin{aligned}\Sigma_2 &= \frac{1}{m\pi^2}\int_{-\pi}^{\pi}d^4k\left\{\sum_\rho\frac{A_\rho}{\hat{k}^2+\mu^2}\left(\frac{r^2}{4}\hat{k}^2_\rho\tilde{S}_- - \bar{k}^2_\rho\tilde{S}_+ + r\bar{k}^2_\rho\tilde{\sigma}_V\right)\right.\\ &+\sum_{\rho\sigma}\frac{A_{\rho\sigma}(1-g_{\rho\sigma})}{\hat{k}^2+\mu^2}\left(\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\tilde{S}_- + r\hat{k}^2_\rho\bar{k}^2_\sigma\tilde{\sigma}_V\right)\end{aligned}$$

$$
-\sum_{\rho\sigma}\frac{A_{\rho\sigma}(1-g_{\rho\sigma})}{\hat{k}^2+\mu^2}g_{\rho\sigma}\bar{k}_\sigma^2\tilde{S}_+\Bigg\}\ . \tag{4.46}
$$

The second term: $p\cdot\gamma\bar{I}^+_{even}(k)+\bar{I}^-_{even}p\cdot\gamma$

The second term is given by

$$
\begin{aligned}
p\cdot\gamma\bar{I}^+_{even}(k)+\bar{I}^-_{even}p\cdot\gamma &= p\cdot\gamma\frac{H_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\left(\frac{r^2}{4}\hat{k}_\rho\hat{k}_\sigma-g_{\rho\sigma}\bar{k}_\rho^2\right)(\bar{S}_++\bar{S}_-)\right\}\\
&+\frac{H_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\frac{r}{2}\hat{k}_\rho\bar{k}_\sigma\left(p\cdot\gamma\bar{k}\cdot\gamma\gamma_\sigma\bar{G}_-+p\cdot\gamma\gamma_\sigma\bar{k}\cdot\gamma\bar{G}_+\right.\right.\\
&\left.\left.+\bar{k}\cdot\gamma\gamma_\sigma p\cdot\gamma\bar{G}_++\gamma_\sigma\bar{k}\cdot\gamma p\cdot\gamma\bar{G}_-\right)\right\}\ .
\end{aligned} \tag{4.47}
$$

From the Chisholm identity

$$
\gamma_\mu\gamma_\nu\gamma_\alpha=\gamma_\mu g_{\nu\alpha}-\gamma_\nu g_{\mu\alpha}+\gamma_\alpha g_{\mu\nu}-\varepsilon_{\mu\nu\alpha\beta}\gamma_\beta\gamma_5 \tag{4.48}
$$

we get

$$
\left(p\cdot\gamma\gamma_\sigma\bar{k}\cdot\gamma+\bar{k}\cdot\gamma\gamma_\sigma p\cdot\gamma\right)\bar{G}_+ = 2\left(p\cdot\gamma\bar{k}_\sigma-\gamma_\sigma p\cdot\bar{k}+\bar{k}\cdot\gamma p_\sigma\right)\bar{G}_+ \tag{4.49}
$$

$$
\left(p\cdot\gamma\bar{k}\cdot\gamma\gamma_\sigma+\gamma_\sigma\bar{k}\cdot\gamma p\cdot\gamma\right)\bar{G}_- = 2\left(p\cdot\gamma\bar{k}_\sigma+\gamma_\sigma p\cdot\bar{k}-\bar{k}\cdot\gamma p_\sigma\right)\bar{G}_- \tag{4.50}
$$

so terms with 3 γ matrices add up to

$$
\text{3-}\gamma\text{ term}=2p\cdot\gamma\bar{k}_\sigma(\bar{G}_++\bar{G}_-)+2\left(\bar{k}\cdot\gamma p_\sigma-\gamma_\sigma p\cdot\bar{k}\right)(\bar{G}_+-\bar{G}_-) \tag{4.51}
$$

so

$$
\begin{aligned}
p\cdot\gamma\bar{I}^+_{even}(k)+\bar{I}^-_{even}p\cdot\gamma &= p\cdot\gamma\frac{H_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\left(\frac{r^2}{4}\hat{k}_\rho\hat{k}_\sigma-g_{\rho\sigma}\bar{k}_\rho^2\right)(\bar{S}_++\bar{S}_-)\right.\\
&\left.+r\hat{k}_\rho\bar{k}_\sigma\bar{k}_\sigma(\bar{G}_++\bar{G}_-)\right\}\\
&+\frac{H_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\frac{r}{2}\hat{k}_\rho\bar{k}_\sigma\bar{k}_\alpha\left(\gamma_\alpha p_\sigma-\gamma_\sigma p_\alpha\right)(\bar{G}_+-\bar{G}_-)\right\}\ .
\end{aligned} \tag{4.52}
$$

The second line vanishes for both $g_{\rho\sigma}$ and $\hat{k}_\rho\hat{k}_\sigma$ terms; for $g_{\rho\sigma}$ we have:

$$\begin{aligned}
\frac{g_{\rho\sigma}A_\rho}{\hat{k}^2+\mu^2}\left\{\frac{r}{2}\hat{k}_\rho\bar{k}_\sigma\bar{k}_\alpha\left(\gamma_\alpha p_\sigma-\gamma_\sigma p_\alpha\right)\left(\bar{G}_+-\bar{G}_-\right)\right\} &= \\
\frac{A_\rho}{\hat{k}^2+\mu^2}\left\{\frac{r}{2}\underbrace{\bar{k}_\rho\bar{k}_\alpha}_{symmetric}\underbrace{\left(\gamma_\alpha p_\rho-\gamma_\rho p_\alpha\right)}_{antisymmetric}(\bar{G}_+-\bar{G}_-)\right\} &= 0\,. \qquad (4.53)
\end{aligned}$$

For $\hat{k}_\rho\hat{k}_\sigma$ we have

$$\begin{aligned}
&\frac{\hat{k}_\rho\hat{k}_\sigma(1-g_{\rho\sigma})A_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\frac{r}{2}\hat{k}_\rho\bar{k}_\sigma\bar{k}_\alpha\left(\gamma_\alpha p_\sigma-\gamma_\sigma p_\alpha\right)\left(\bar{G}_+-\bar{G}_-\right)\right\} \\
&=\frac{A_{\rho\sigma}(1-g_{\rho\sigma})}{\hat{k}^2+\mu^2}\left\{\frac{r}{2}\hat{k}_\rho^2\underbrace{\bar{k}_\sigma\bar{k}_\alpha}_{symmetric}\underbrace{\left(\gamma_\alpha p_\sigma-\gamma_\sigma p_\alpha\right)}_{antisymmetric}(\bar{G}_+-\bar{G}_-)\right\}=0\,. \qquad (4.54)
\end{aligned}$$

The final result for the second term is then

$$\begin{aligned}
p\cdot\gamma\bar{I}^+_{even}(k)+\bar{I}^-_{even}p\cdot\gamma &= p\cdot\gamma\sum_\rho\frac{A_\rho}{\hat{k}^2+\mu^2}\left\{\left(\frac{r^2}{4}\hat{k}_\rho^2-\bar{k}_\rho^2\right)(\bar{S}_++\bar{S}_-)\right. \\
&\qquad \left.+r\bar{k}_\rho^2(\bar{G}_++\bar{G}_-)\right\} \\
&\quad +p\cdot\gamma\sum_{\substack{\rho\\ \sigma\neq\rho}}\frac{A_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\left[\frac{r^2}{4}\hat{k}_\rho^2\hat{k}_\sigma^2-g_{\rho\sigma}\bar{k}_\rho^2\right](\bar{S}_++\bar{S}_-)\right. \\
&\qquad \left.+r\hat{k}_\rho^2\bar{k}_\sigma^2(\bar{G}_++\bar{G}_-)\right\}\,. \qquad (4.55)
\end{aligned}$$

Introducing the parametrization

$$-i\mathcal{A}\left(p\cdot\gamma\bar{I}^+_{even}(k)+\bar{I}^-_{even}p\cdot\gamma\right) = -ip\cdot\gamma\frac{g_0^2C_F}{16\pi^2}\Sigma_3\,, \qquad (4.56)$$

we get

$$\begin{aligned}
\Sigma_3 &= \frac{2\mathcal{A}}{a\pi^2}\int_{-\pi}^{\pi}d^4k\left\{\sum_\rho\frac{A_\rho}{\hat{k}^2+\mu^2}\left(\left(\frac{r^2}{4}\hat{k}_\rho^2-\bar{k}_\rho^2\right)\bar{\sigma}_S+r\bar{k}_\rho^2\bar{\sigma}_V\right)\right. \\
&\quad \left.+\sum_{\rho,\sigma}\frac{A_{\rho\sigma}(1-g_{\rho\sigma})}{\hat{k}^2+\mu^2}\left(\left[\frac{r^2}{4}\hat{k}_\rho^2\hat{k}_\sigma^2-g_{\rho\sigma}\bar{k}_\rho^2\right]\bar{\sigma}_S+r\hat{k}_\rho^2\bar{k}_\sigma^2\bar{\sigma}_V\right)\right\} \qquad (4.57)
\end{aligned}$$

Simplifying the $\bar{I}^{+}_{odd}(p)$ term

The last (and the most complicated) term is given by $p_\mu \partial \bar{I}^{+}_{odd}(p)/\partial p_\mu$. Before expanding it, we multiply it with smearing tensor

$$H_{\rho\sigma}(ap-k) = g_{\rho\sigma}A_\rho(ap-k) + (\widehat{ap-k})_\rho(\widehat{ap-k})_\sigma(1-g_{\rho\sigma})A_{\rho\sigma}(ap-k) \qquad (4.58)$$

to get

$$\begin{aligned}
\bar{I}^{+}_{odd}(p) \;=\; & \sum_\rho \frac{A_\rho(ap-k)}{(\widehat{ap-k})^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}(\widehat{ap-k})^2_\rho \bar{G}_+(k) + (\widetilde{ap-k})^2_\rho \bar{G}_-(k)\right] \\
& +2i(\widetilde{ap-k})^2_\rho \gamma_\rho \bar{k}_\rho \bar{G}_-(k) + \frac{r}{2} i (\overline{ap-k})_\rho \gamma_\rho (\bar{S}_+ + \bar{S}_-)\Big\} \\
& + \sum_{\substack{\rho \\ \sigma\neq\rho}} \frac{A_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma(\widehat{ap-k})_\rho(\widehat{ap+k})_\rho(\widehat{ap-k})_\sigma(\widehat{ap+k})_\sigma \bar{G}_+ \\
& +2i(\widehat{ap-k})_\rho(\widetilde{ap+k})_\rho(\widehat{ap-k})_\sigma(\widetilde{ap+k})_\sigma \gamma_\rho \bar{k}_\sigma \bar{G}_- \\
& +\frac{r}{2} i\gamma_\sigma(\widehat{ap-k})_\rho(\widehat{ap+k})_\rho(\widehat{ap-k})_\sigma(\widetilde{ap+k})_\sigma(\bar{S}_+ + \bar{S}_-)\Big\}\,. \qquad (4.59)
\end{aligned}$$

The second term can be simplified using trigonometric identities

$$\sin\frac{x-y}{2}\sin\frac{x+y}{2} = \frac{1}{2}(\cos y - \cos x) \qquad (4.60)$$

$$\sin\frac{x-y}{2}\cos\frac{x+y}{2} = \frac{1}{2}(\sin x - \sin y) \qquad (4.61)$$

from which we get

$$(\widehat{ap-k})_\rho(\widehat{ap+k})_\rho = 2\left(\cos k_\rho - \cos ap_\rho\right) \qquad (4.62)$$

$$(\widehat{ap-k})_\rho(\widetilde{ap+k})_\rho = \left(\sin ap_\rho - \bar{k}_\rho\right) \qquad (4.63)$$

and from which it follows

$$(\widehat{ap-k})_\rho(\widehat{ap+k})_\rho(\widehat{ap-k})_\sigma(\widehat{ap+k})_\sigma = \hat{k}^2_\rho \hat{k}^2_\sigma + O(a^2) \qquad (4.64)$$

$$(\widehat{ap-k})_\rho(\widetilde{ap+k})_\rho(\widehat{ap-k})_\sigma(\widetilde{ap+k})_\sigma = \bar{k}_\rho\bar{k}_\sigma - a(p_\rho\bar{k}_\sigma + p_\sigma\bar{k}_\rho) + O(a^2) \qquad (4.65)$$

$$(\widehat{ap-k})_\rho(\widehat{ap+k})_\rho(\widehat{ap-k})_\sigma(\widetilde{ap+k})_\sigma \;=\; \hat{k}_\rho^2(\bar{k}_\sigma - ap_\sigma) + O(a^2)\,. \tag{4.66}$$

This yields the result

$$\begin{aligned}
\bar{I}^+_{odd}(p) &= \sum_\rho \frac{A_\rho(ap-k)}{(\widehat{ap-k})^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}(\widehat{ap+k})^2_\rho\bar{G}_+(k) + (\widetilde{ap+k})^2_\rho\bar{G}_-(k)\right] \\
&\quad +2i(\widetilde{ap+k})^2_\rho\gamma_\rho\bar{k}_\rho\bar{G}_-(k) + \frac{r}{2}i\,(\overline{ap+k})_\rho\gamma_\rho(\bar{S}_+ + \bar{S}_-)\Big\} \\
&\quad +\sum_{\rho\sigma}\frac{A_{\rho\sigma}(ap-k)(1-g_{\rho\sigma})}{(\widehat{ap-k})^2+\mu^2}\big\{-i\bar{k}\cdot\gamma\big[r^2(\cos k_\rho - \cos p_\rho)(\cos k_\sigma - \cos p_\sigma)\bar{G}_+ \\
&\quad +(\sin p_\rho - \bar{k}_\rho)^2 g_{\rho\sigma}\bar{G}_-\big] + 2i\gamma_\rho\bar{k}_\sigma(\sin p_\rho - \bar{k}_\rho)(\sin p_\sigma - \bar{k}_\sigma)\bar{G}_- \\
&\quad +2ri\gamma_\sigma(\cos k_\rho - \cos p_\rho)(\sin p_\sigma - \bar{k}_\sigma)\bar{\sigma}_S\big\}\,.
\end{aligned} \tag{4.67}$$

The last term $p_\mu\partial\bar{I}^+_{odd}(p)/\partial p_\mu$: part proportional to A_ρ

Expanding the first term we get

$$\begin{aligned}
p_\mu\frac{\partial\bar{I}^+_{odd}}{\partial p_\mu}\bigg|_{A_\rho} &= \sum_\rho\left(\frac{1}{a}\frac{A_\rho(k)}{\hat{k}^2+\mu^2} + \frac{A'_\rho(k)}{\hat{k}^2+\mu^2} + \frac{2p\cdot\bar{k}A_\rho(k)}{(\hat{k}^2+\mu^2)^2}\right) \\
&\quad \Big\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}^2_\rho\bar{G}_+(k) + \tilde{k}^2_\rho\bar{G}_-(k)\right] \\
&\qquad +2i\tilde{k}^2_\rho\gamma_\rho\bar{k}_\rho\bar{G}_-(k) + \frac{r}{2}i\bar{k}_\rho\gamma_\rho(\bar{S}_+(k)+\bar{S}_-(k))\Big\} \\
&\quad +\sum_\rho\frac{A_\rho(k)}{\hat{k}^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\bar{k}_\rho p_\rho\left[\frac{r^2\bar{G}_+(k)-\bar{G}_-(k)}{2}\right] \\
&\qquad -i\gamma_\rho p_\rho\bar{k}^2_\rho\bar{G}_-(k) + i\frac{r}{2}\cos k_\rho\gamma_\rho p_\rho(\bar{S}_+(k)+\bar{S}_-(k))\Big\}
\end{aligned} \tag{4.68}$$

where the a^{-1} term vanishes after integration since it's odd in k_μ. Since the A_ρ doesn't contain any $\hat{k}^2_\rho$, all integrals $\int A_\rho\bar{k}^2_\rho f(\hat{k}^2$ and $\int A_\rho\cos k^2_\rho f(\hat{k}^2$ are the same for all ρ so $p\cdot\gamma$ factors out:

$$\begin{aligned}
&\sum_\rho\frac{A_\rho(k)}{\hat{k}^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\bar{k}_\rho p_\rho\left[\frac{r^2\bar{G}_+(k)-\bar{G}_-(k)}{2}\right] \\
&\qquad -i\gamma_\rho p_\rho\bar{k}^2_\rho\bar{G}_-(k) + i\frac{r}{2}\cos k_\rho\gamma_\rho p_\rho(\bar{S}_+(k)+\bar{S}_-(k))\Big\} \\
&= -ip\cdot\gamma\frac{1}{d}\sum_\rho\frac{A_\rho(k)}{\hat{k}^2+\mu^2}\left\{\bar{k}^2_\rho\left[\frac{r^2\bar{G}_+(k)+\bar{G}_-(k)}{2}\right] - r\cos k_\rho\bar{\sigma}_S(k)\right\}
\end{aligned} \tag{4.69}$$

where after factorizing $p\cdot\gamma$ we have summed over ρ and divided by spacetime dimension d. A similar trick can be used on the $(\hat{k}^2+\mu^2)^{-2}$ term to get

$$
\begin{aligned}
&\sum_\rho \frac{2p\cdot\bar{k}A_\rho(k)}{(\hat{k}^2+\mu^2)^2}\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)+\bar{k}_\rho^2\bar{G}_-(k)\right]\right.\\
&\qquad\left.+2i\bar{k}_\rho^2\gamma_\rho\bar{k}_\rho\bar{G}_-(k)+\frac{r}{2}i\bar{k}_\rho\gamma_\rho(\bar{S}_+(k)+\bar{S}_-(k))\right\}\\
&=-\frac{ip\cdot\gamma}{d}\sum_\rho\frac{2A_\rho(k)}{(\hat{k}^2+\mu^2)^2}\left\{r^2\frac{\hat{k}_\rho^2}{4}\bar{k}^2\bar{G}_+(k)+\bar{k}_\rho^2\left(\bar{k}^2-2\bar{k}_\rho^2\right)\bar{G}_-(k)-r\bar{k}_\rho^2\bar{\sigma}_S(k)\right\}
\end{aligned}
\tag{4.70}
$$

Finally, for the A' term we need to expand the HYP-smearing tensor to first order in p_μ (2.338):

$$
A_\rho(ap-k) \;=\; A_\rho(k)-2a\,p_\rho\bar{k}_\rho C_\rho(k)-2a\sum_{\sigma\neq\rho}p_\sigma\bar{k}_\sigma C_{\rho\sigma} \tag{4.71}
$$

where C_ρ doesn't contain any $\hat{k}_\rho$ and $C_{\rho\sigma}$ doesn't contain any $\hat{k}_\rho$ and $\hat{k}_\sigma$ factors:

$$
\begin{aligned}
&\sum_\rho \frac{A'_\rho(k)}{\hat{k}^2+\mu^2}\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)+\bar{k}_\rho^2\bar{G}_-(k)\right]\right.\\
&\qquad\left.+2i\bar{k}_\rho^2\gamma_\rho\bar{k}_\rho\bar{G}_-(k)+\frac{r}{2}i\bar{k}_\rho\gamma_\rho(\bar{S}_+(k)+\bar{S}_-(k))\right\}\\
&=-2\sum_\rho\frac{\bar{k}_\rho p_\rho C_\rho+\sum_{\sigma\neq\rho}\bar{k}_\sigma p_\sigma C_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)+\bar{k}_\rho^2\bar{G}_-(k)\right]\right.\\
&\qquad\left.+2i\bar{k}_\rho^2\gamma_\rho\bar{k}_\rho\bar{G}_-(k)+\frac{r}{2}i\bar{k}_\rho\gamma_\rho(\bar{S}_+(k)+\bar{S}_-(k))\right\}\,.
\end{aligned}
\tag{4.72}
$$

The term with C_ρ is simple:

$$
\begin{aligned}
&-2\sum_\rho\frac{\bar{k}_\rho p_\rho C_\rho}{\hat{k}^2+\mu^2}\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)+\bar{k}_\rho^2\bar{G}_-(k)\right]\right.\\
&\qquad\left.+2i\bar{k}_\rho^2\gamma_\rho\bar{k}_\rho\bar{G}_-(k)+\frac{r}{2}i\bar{k}_\rho\gamma_\rho(\bar{S}_+(k)+\bar{S}_-(k))\right\}\\
&=-\frac{ip\cdot\gamma}{d}\sum_\rho\frac{-2C_\rho\bar{k}_\rho^2}{\hat{k}^2+\mu^2}\left\{\left[\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)-\bar{k}_\rho^2\bar{G}_-(k)\right]-r\bar{\sigma}_S(k)\right\}
\end{aligned}
\tag{4.73}
$$

The term with $C_{\rho\sigma}$ is given by

$$
\begin{aligned}
&-2\sum_{\rho,\sigma\neq\rho}\frac{\bar{k}_\sigma p_\sigma C_{\rho\sigma}}{\hat{k}^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)+\tilde{k}_\rho^2\bar{G}_-(k)\right] \\
&\qquad +2i\tilde{k}_\rho^2\gamma_\rho\bar{k}_\rho\bar{G}_-(k)+\frac{r}{2}i\bar{k}_\rho\gamma_\rho(\bar{S}_+(k)+\bar{S}_-(k))\Big\} \\
&= -2\sum_{\sigma,\rho\neq\sigma}\frac{C_{\rho\sigma}}{\hat{k}^2+\mu^2}\Big\{-i\bar{k}_\sigma^2\gamma_\sigma p_\sigma\left[\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)+\tilde{k}_\rho^2\bar{G}_-(k)\right] \\
&\qquad +2i\tilde{k}_\rho^2\gamma_\rho\bar{k}_\rho\bar{k}_\sigma p_\sigma\bar{G}_-(k)+ri\bar{k}_\rho\gamma_\rho\bar{k}_\sigma p_\sigma\bar{\sigma}_S(k)\Big\} \\
&= -\frac{ip\cdot\gamma}{d}\sum_{\sigma,\rho\neq\sigma}\frac{-2C_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\bar{k}_\sigma^2\left[\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)+\tilde{k}_\rho^2\bar{G}_-(k)\right]\right\}
\end{aligned}
\tag{4.74}
$$

Collecting all terms, we get

$$
p_\mu\frac{\partial\bar{I}^+_{odd}}{\partial p_\mu}\bigg|_{A_\rho} = -ip\cdot\gamma\frac{g_0^2C_F}{16\pi^2}\Sigma_1^{(a)} \tag{4.75}
$$

with

$$
\begin{aligned}
\Sigma_1^{(a)} &= \frac{1}{d\pi^2}\int_{-\pi}^{\pi}d^4k\Big\{\sum_\rho\frac{A_\rho(k)}{\hat{k}^2+\mu^2}\left(\bar{k}_\rho^2\left[\frac{r^2\bar{G}_+(k)+\bar{G}_-(k)}{2}\right]-r\cos k_\rho\bar{\sigma}_S(k)\right) \\
&\quad +\sum_\rho\frac{2A_\rho(k)}{(\hat{k}^2+\mu^2)^2}\left(r^2\frac{\hat{k}_\rho^2}{4}\bar{k}^2\bar{G}_+(k)+\bar{k}_\rho^2\left(\bar{k}^2-2\tilde{k}_\rho^2\right)\bar{G}_-(k)-r\bar{k}_\rho^2\bar{\sigma}_S(k)\right) \\
&\quad +\sum_\rho\frac{-2C_\rho\bar{k}_\rho^2}{\hat{k}^2+\mu^2}\left(\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)-\tilde{k}_\rho^2\bar{G}_-(k)-r\bar{\sigma}_S(k)\right) \\
&\quad +\sum_{\sigma,\rho\neq\sigma}\frac{-2C_{\rho\sigma}\bar{k}_\sigma^2}{\hat{k}^2+\mu^2}\left(\frac{r^2}{4}\hat{k}_\rho^2\bar{G}_+(k)+\tilde{k}_\rho^2\bar{G}_-(k)\right)\Big\}
\end{aligned}
\tag{4.76}
$$

The last term $p_\mu\partial\bar{I}^+_{odd}(p)/\partial p_\mu$: part proportional to $A_{\rho\sigma}$

Going back to (4.67),

$$
\begin{aligned}
\bar{I}^+_{odd}(p) &= \sum_\rho\frac{A_\rho(ap-k)}{(\widehat{ap-k})^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}(\widehat{ap-k})_\rho^2\bar{G}_+(k)+(\widetilde{ap-k})_\rho^2\bar{G}_-(k)\right] \\
&\qquad +2i(\widetilde{ap-k})_\rho^2\gamma_\rho\bar{k}_\rho\bar{G}_-(k)+\frac{r}{2}i(\overline{ap-k})_\rho\gamma_\rho(\bar{S}_++\bar{S}_-)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\rho\sigma}\frac{A_{\rho\sigma}(ap-k)(1-g_{\rho\sigma})}{(\widehat{ap-k})^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\left[\hat{k}_\rho^2\hat{k}_\sigma^2\bar{G}_+ +(\bar{k}_\rho^2-2ap_\rho\bar{k}_\rho)g_{\rho\sigma}\bar{G}_-\right]\\
&\quad+2i\gamma_\rho\bar{k}_\sigma[\bar{k}_\rho\bar{k}_\sigma-a(p_\rho\bar{k}_\sigma+p_\sigma\bar{k}_\rho)]\bar{G}_-\\
&\quad+\frac{r}{2}i\gamma_\sigma\hat{k}_\rho^2(\bar{k}_\sigma-ap_\sigma)(\bar{S}_+ +\bar{S}_-)\Big\}\,.
\end{aligned}\tag{4.77}
$$

Expanding the second part proportional to $A_{\rho\sigma}$ we get

$$
\begin{aligned}
p_\mu\frac{\partial\bar{I}^+_{odd}}{\partial p_\mu}\bigg|_{A_{\rho\sigma}} &= \sum_{\rho,\sigma\neq\rho}\left(\frac{A'_{\rho\sigma}(k)}{\hat{k}^2+\mu^2}+\frac{2p\cdot\bar{k}A_{\rho\sigma}(k)}{(\hat{k}^2+\mu^2)^2}\right)\Big\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}_\rho^2\hat{k}_\sigma^2\bar{G}_+ +\bar{k}_\rho^2g_{\rho\sigma}\bar{G}_-\right]\\
&\quad+2i\gamma_\rho\bar{k}_\rho\bar{k}_\sigma^2\bar{G}_- +ri\gamma_\sigma\hat{k}_\rho^2\bar{k}_\sigma\bar{\sigma}_S\Big\}\\
&\quad+\sum_{\rho,\sigma\neq\rho}\frac{-2A_{\rho\sigma}(k)}{\hat{k}^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\left[p_\rho\bar{k}_\rho g_{\rho\sigma}\bar{G}_-\right]\\
&\quad+i\gamma_\rho\bar{k}_\sigma[(p_\rho\bar{k}_\sigma+p_\sigma\bar{k}_\rho)]\bar{G}_- +\frac{r}{2}i\gamma_\sigma\hat{k}_\rho^2p_\sigma\bar{\sigma}_S\Big\}\,.
\end{aligned}\tag{4.78}
$$

The last term is the easiest to evaluate, since $g_{\rho\sigma}$ terms cancel and we are left with

$$
p_\mu\frac{\partial\bar{I}^+_{odd}}{\partial p_\mu}\bigg|_{last} = -\frac{ip\cdot\gamma}{d}\sum_{\rho,\sigma\neq\rho}\frac{2A_{\rho\sigma}(k)}{\hat{k}^2+\mu^2}\left\{\bar{k}_\sigma^2\bar{G}_- +\frac{r}{2}\hat{k}_\rho^2\bar{\sigma}_S\right\}\,.\tag{4.79}
$$

The same trick applies to the "second" term as well, so we get

$$
\begin{aligned}
p_\mu\frac{\partial\bar{I}^+_{odd}}{\partial p_\mu}\bigg|_{2^{nd}} &= \sum_{\rho,\sigma\neq\rho}\frac{2p\cdot\bar{k}A_{\rho\sigma}(k)}{(\hat{k}^2+\mu^2)^2}\Big\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}_\rho^2\hat{k}_\sigma^2\bar{G}_+ +\bar{k}_\rho^2g_{\rho\sigma}\bar{G}_-\right]\\
&\quad+2i\gamma_\rho\bar{k}_\rho\bar{k}_\sigma^2\bar{G}_- +ri\gamma_\sigma\hat{k}_\rho^2\bar{k}_\sigma\bar{\sigma}_S\Big\}\\
&= -\frac{ip\cdot\gamma}{d}\sum_{\rho,\sigma\neq\rho}\frac{2A_{\rho\sigma}(k)}{(\hat{k}^2+\mu^2)^2}\Big\{\bar{k}^2\left[\frac{r^2}{4}\hat{k}_\rho^2\hat{k}_\sigma^2\bar{G}_+ +\bar{k}_\rho^2g_{\rho\sigma}\bar{G}_-\right]\\
&\quad-2\bar{k}_\rho^2\bar{k}_\sigma^2\bar{G}_- -r\hat{k}_\rho^2\bar{k}_\sigma^2\bar{\sigma}_S\Big\}\,.
\end{aligned}\tag{4.80}
$$

Finally, the "first" term containing $A'_{\rho\sigma}$ is (again) the most complicated. We use the expansion of $A_{\rho\sigma}$

$$
\begin{aligned}
A'_{\rho\sigma} &= -2\Big[p_\rho\bar{k}_\rho(A^{(1)}_{\rho\sigma}+\hat{k}_\sigma^2A^{(2)}_{\rho\sigma})+p_\sigma\bar{k}_\sigma(A^{(1)}_{\rho\sigma}+\hat{k}_\rho^2A^{(2)}_{\rho\sigma})\\
&\quad+\sum_{\alpha\neq\rho\sigma}p_\alpha\bar{k}_\alpha\left(A^{(0)}_{\rho\sigma|\alpha}+(\hat{k}_\rho^2+\hat{k}_\sigma^2)A^{(1)}_{\rho\sigma|\alpha}+\hat{k}_\rho^2\hat{k}_\sigma^2A^{(2)}_{\rho\sigma|\alpha}\right)\Big]
\end{aligned}\tag{4.81}
$$

where $A^{(i)}_{\rho\sigma|\alpha}$ are derivatives of $A^{(i)}_{\rho\sigma}$ over $\hat{k}^2_\alpha$. The first two terms in $A'_{\rho\sigma}$ yield

$$-2\sum_{\rho,\sigma\neq\rho} \frac{p_\rho \bar{k}_\rho (A^{(1)}_{\rho\sigma} + \hat{k}^2_\sigma A^{(2)}_{\rho\sigma}) + p_\sigma \bar{k}_\sigma (A^{(1)}_{\rho\sigma} + \hat{k}^2_\rho A^{(2)}_{\rho\sigma})}{\hat{k}^2+\mu^2}$$
$$\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right] + 2i\gamma_\rho\bar{k}_\rho\bar{k}^2_\sigma\bar{G}_- + ri\gamma_\sigma\hat{k}^2_\rho\bar{k}_\sigma\bar{\sigma}_S\right\}$$
$$= -2\sum_{\rho,\sigma\neq\rho}\frac{(A^{(1)}_{\rho\sigma} + \hat{k}^2_\sigma A^{(2)}_{\rho\sigma})}{\hat{k}^2+\mu^2}\left\{-i\bar{k}^2_\rho\gamma_\rho p_\rho\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right]\right.$$
$$\left.+2i\gamma_\rho p_\rho\bar{k}^2_\rho\bar{k}^2_\sigma\bar{G}_- + ri\gamma_\sigma p_\sigma g_{\rho\sigma}\hat{k}^2_\rho\bar{k}^2_\sigma\bar{\sigma}_S\right\}$$
$$-2\sum_{\rho,\sigma\neq\rho}\frac{(A^{(1)}_{\rho\sigma} + \hat{k}^2_\rho A^{(2)}_{\rho\sigma})}{\hat{k}^2+\mu^2}\left\{-i\bar{k}^2_\sigma\gamma_\sigma p_\sigma\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right]\right.$$
$$\left.+2i\gamma_\rho p_\rho \sigma g_{\rho\sigma}\bar{k}^2_\rho\bar{k}^2_\sigma\bar{G}_- + ri\gamma_\sigma p_\sigma\hat{k}^2_\rho\bar{k}^2_\sigma\bar{\sigma}_S\right\} \tag{4.82}$$

and

$$p_\mu\frac{\partial\bar{I}^+_{odd}}{\partial p_\mu}\bigg|^{(1,2)}_{A'_{\rho\sigma}} = -\frac{ip\cdot\gamma}{d}\sum_{\rho,\sigma\neq\rho}\frac{-2(A^{(1)}_{\rho\sigma}+\hat{k}^2_\sigma A^{(2)}_{\rho\sigma})}{\hat{k}^2+\mu^2}\left\{\bar{k}^2_\rho\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right]\right.$$
$$\left.-2\bar{k}^2_\rho\bar{k}^2_\sigma\bar{G}_- - rg_{\rho\sigma}\hat{k}^2_\rho\bar{k}^2_\sigma\bar{\sigma}_S\right\}$$
$$-\frac{ip\cdot\gamma}{d}\sum_{\rho,\sigma\neq\rho}\frac{-2(A^{(1)}_{\rho\sigma}+\hat{k}^2_\rho A^{(2)}_{\rho\sigma})}{\hat{k}^2+\mu^2}\left\{\bar{k}^2_\sigma\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right]\right.$$
$$\left.-2g_{\rho\sigma}\bar{k}^2_\rho\bar{k}^2_\sigma\bar{G}_- - r\hat{k}^2_\sigma\bar{k}^2_\rho\bar{\sigma}_S\right\}. \tag{4.83}$$

The last term in $A'_{\rho\sigma}$ yields

$$p_\mu\frac{\partial\bar{I}^+_{odd}}{\partial p_\mu}\bigg|^{(3)}_{A'_{\rho\sigma}} = -2\sum_{\rho,\sigma\neq\rho}\frac{\sum_{\alpha\neq\rho\sigma} p_\alpha\bar{k}_\alpha\left(A^{(0)}_{\rho\sigma|\alpha} + (\hat{k}^2_\rho+\hat{k}^2_\sigma)A^{(1)}_{\rho\sigma|\alpha} + \hat{k}^2_\rho\hat{k}^2_\sigma A^{(2)}_{\rho\sigma|\alpha}\right)}{\hat{k}^2+\mu^2}$$
$$\times\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right] + 2i\gamma_\rho\bar{k}_\rho\bar{k}^2_\sigma\bar{G}_- + ri\gamma_\sigma\hat{k}^2_\rho\bar{k}_\sigma\bar{\sigma}_S\right\}$$
$$= -\frac{ip\cdot\gamma}{d}\sum_{\substack{\rho,\sigma\neq\rho\\ \alpha\neq\rho\sigma}}\frac{-2\left(A^{(0)}_{\rho\sigma|\alpha} + (\hat{k}^2_\rho+\hat{k}^2_\sigma)A^{(1)}_{\rho\sigma|\alpha} + \hat{k}^2_\rho\hat{k}^2_\sigma A^{(2)}_{\rho\sigma|\alpha}\right)}{\hat{k}^2+\mu^2}$$
$$\times\left\{\bar{k}^2_\alpha\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right] - 2g_{\rho\alpha}\bar{k}^2_\rho\bar{k}^2_\sigma\bar{G}_- - rg_{\sigma\alpha}\hat{k}^2_\rho\bar{k}^2_\sigma\bar{\sigma}_S\right\} \tag{4.84}$$

These terms add up to

$$p_\mu \frac{\partial \bar{I}^+_{odd}}{\partial p_\mu}\bigg|_{A'_{\rho\sigma}} = -ip\cdot\gamma \frac{g_0^2 C_F}{16\pi^2}\Sigma_1^{(b)} \tag{4.85}$$

with

$$\begin{aligned}
\Sigma_1^{(b)} = \frac{1}{d\pi^2}\int_{-\pi}^{\pi} d^4k \Bigg\{ & \sum_{\rho,\sigma\neq\rho} \frac{-2(A^{(1)}_{\rho\sigma}+\hat{k}^2_\sigma A^{(2)}_{\rho\sigma})}{\hat{k}^2+\mu^2}\left(\bar{k}^2_\rho\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right]\right. \\
& \left. -2\bar{k}^2_\rho\bar{k}^2_\sigma\bar{G}_- - r g_{\rho\sigma}\hat{k}^2_\rho\bar{k}^2_\sigma\bar{\sigma}_S\right) \\
& + \sum_{\rho,\sigma\neq\rho} \frac{-2(A^{(1)}_{\rho\sigma}+\hat{k}^2_\rho A^{(2)}_{\rho\sigma})}{\hat{k}^2+\mu^2}\left(\bar{k}^2_\sigma\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right]\right. \\
& \left. -2g_{\rho\sigma}\bar{k}^2_\rho\bar{k}^2_\sigma\bar{G}_- - r\hat{k}^2_\sigma\bar{k}^2_\rho\bar{\sigma}_S\right) \\
& + \sum_{\substack{\rho,\sigma\neq\rho\\ \alpha\neq\rho\sigma}} \frac{-2\left(A^{(0)}_{\rho\sigma|\alpha}+(\hat{k}^2_\rho+\hat{k}^2_\sigma)A^{(1)}_{\rho\sigma|\alpha}+\hat{k}^2_\rho\hat{k}^2_\sigma A^{(2)}_{\rho\sigma|\alpha}\right)}{\hat{k}^2+\mu^2} \\
& \times\left(\bar{k}^2_\alpha\left[\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\bar{G}_+ + \bar{k}^2_\rho g_{\rho\sigma}\bar{G}_-\right]\right. \\
& \left. -2g_{\rho\alpha}\bar{k}^2_\rho\bar{k}^2_\sigma\bar{G}_- - r g_{\sigma\alpha}\hat{k}^2_\rho\bar{k}^2_\sigma\bar{\sigma}_S\right)\Bigg\} .
\end{aligned} \tag{4.86}$$

4.1.2 Recovering APE-smearing results

To recover APE-smearing results, we compare formulas for the HYP and APE smearing tensors:

$$\begin{aligned}
H^{APE}_{\mu\nu} &= \left(1+\frac{c}{2(d-1)}\hat{k}^2\right)^{2n}\left(g_{\mu\nu}-\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\right)+\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2} \\
&= g_{\mu\nu}C(k)+\hat{k}_\mu\hat{k}_\nu\left(\frac{1-C(k)}{\hat{k}^2}\right) \qquad (4.87)\\
H^{HYP}_{\mu\nu} &= g_{\mu\nu}A_\mu+\hat{k}_\mu\hat{k}_\nu(1-g_{\mu\nu})A_{\mu\nu} . \qquad (4.88)
\end{aligned}$$

So by replacement

$$A_\rho \to C(k) , \qquad (1-g_{\mu\nu})A_{\mu\nu} \to \left(\frac{1-C(k)}{\hat{k}^2}\right) \tag{4.89}$$

we get the correct limit. Parts with derivatives of A_ρ and $A_{\rho\sigma}$ are a bit more complicated. Comparing expansions of

$$
\begin{aligned}
A'_\rho(k) &= -2a\, p_\rho \bar{k}_\rho C_\rho(k) - 2a \sum_{\sigma\neq\rho} p_\sigma \bar{k}_\sigma C_{\rho\sigma} && (4.90)\\
A'_{\rho\sigma} &= -2a\Big[p_\rho \bar{k}_\rho (A^{(1)}_{\rho\sigma} + \hat{k}^2_\sigma A^{(2)}_{\rho\sigma}) + p_\sigma \bar{k}_\sigma (A^{(1)}_{\rho\sigma} + \hat{k}^2_\rho A^{(2)}_{\rho\sigma}) \\
&\quad + \sum_{\alpha\neq\rho\sigma} p_\alpha \bar{k}_\alpha \left(A^{(0)}_{\rho\sigma|\alpha} + (\hat{k}^2_\rho + \hat{k}^2_\sigma) A^{(1)}_{\rho\sigma|\alpha} + \hat{k}^2_\rho \hat{k}^2_\sigma A^{(2)}_{\rho\sigma|\alpha}\right)\Big] && (4.91)
\end{aligned}
$$

and

$$
\begin{aligned}
-2ap\cdot\bar{k} C'(k) &= -2ap\cdot\bar{k}\, \frac{c}{2(d-1)} 2n \left(1 + \frac{c}{2(d-1)}\hat{k}^2\right)^{2n-1} && (4.92)\\
-2ap\cdot\bar{k}\left(\frac{1-C(k)}{\hat{k}^2}\right)' &= -2ap\cdot\bar{k}\, \frac{C(k) - 1 - \hat{k}^2 C'(k)}{(\hat{k}^2)^2} \\
&= -2ap\cdot\bar{k}\, \frac{(1-2n)\left(1+\frac{c}{2(d-1)}\hat{k}^2\right)^{2n} - 1 + 2n\left(1+\frac{c}{2(d-1)}\hat{k}^2\right)^{2n-1}}{(\hat{k}^2)^2} && (4.93)
\end{aligned}
$$

would suggest one has to set

$$
\begin{aligned}
C'(k) &= C_\rho = C_{\rho\sigma} && (4.94)\\
\bar{C}'(k) &\equiv \left(\frac{1-C(k)}{\hat{k}^2}\right)' = (A^{(1)}_{\rho\sigma} + \hat{k}^2_\sigma A^{(2)}_{\rho\sigma}) = (A^{(1)}_{\rho\sigma} + \hat{k}^2_\rho A^{(2)}_{\rho\sigma}) \\
&= \left(A^{(0)}_{\rho\sigma|\alpha} + (\hat{k}^2_\rho + \hat{k}^2_\sigma) A^{(1)}_{\rho\sigma|\alpha} + \hat{k}^2_\rho \hat{k}^2_\sigma A^{(2)}_{\rho\sigma|\alpha}\right) && (4.95)
\end{aligned}
$$

to recover APE-smearing formulas. That would actually be true if one did it *before* extracting $-ip\cdot\gamma$ and summing indices, but it does not work for final expressions. Luckily, APE spearing results are much easier to just evaluate directly so for the APE smearing we have

$$
\begin{aligned}
&\sum_\rho \frac{A'_\rho(k)}{\hat{k}^2+\mu^2}\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}^2_\rho \bar{G}_+(k) + \bar{k}^2_\rho \bar{G}_-(k)\right] + 2i\bar{k}^2_\rho \gamma_\rho \bar{k}_\rho \bar{G}_-(k) + ri\bar{k}_\rho \gamma_\rho \bar{\sigma}_S(k)\right\} \\
&= -\frac{ip\cdot\gamma}{d}\frac{(-2C'(k))}{\hat{k}^2+\mu^2}\left\{r^2\frac{\hat{k}^2}{4}\bar{k}^2 \bar{G}_+(k) + \left(\bar{k}^2\bar{k}^2 - 2\sum_\rho \bar{k}^2_\rho \bar{k}^2_\rho\right)\bar{G}_-(k) - r\bar{k}^2 \bar{\sigma}_S(k)\right\}. \quad (4.96)
\end{aligned}
$$

and

$$\sum_{\rho,\sigma}\frac{A'_{\rho\sigma}(k)}{\hat{k}^2+\mu^2}\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}_\rho^2\hat{k}_\sigma^2\bar{G}_+ +\bar{k}_\rho^2 g_{\rho\sigma}\bar{G}_-\right]+2i\gamma_\rho\bar{k}_\rho\bar{k}_\sigma^2\bar{G}_- +ri\gamma_\sigma\hat{k}_\rho^2\bar{k}_\sigma\bar{\sigma}_S\right\}$$
$$=\sum_{\rho,\sigma}\frac{-2p\cdot\bar{k}\bar{C}'(k)}{\hat{k}^2+\mu^2}\left\{-i\bar{k}\cdot\gamma\left[\frac{r^2}{4}\hat{k}^4\bar{G}_+ +\bar{k}^2\bar{G}_-\right]+2i\gamma\cdot\bar{k}\bar{k}^2\bar{G}_- +ri\gamma\cdot\bar{k}\hat{k}^2\bar{\sigma}_S\right\}$$
$$=-\frac{ip\cdot\gamma}{d}\sum_{\rho,\sigma}\frac{-2\bar{C}'(k)\bar{k}^2}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}^4\bar{G}_+ -\bar{k}^2\bar{G}_- -r\hat{k}^2\bar{\sigma}_S\right\}. \tag{4.97}$$

4.1.3 No smearing limit

Now if we take the limit $h_{\mu\nu}\to g_{\mu\nu}$ (and choose gauge Feynman gauge $\lambda=1$) we recover Aoki's results without smearing. The first term $\tilde{I}^-_{even}(k)$ behaves like a^{-1} and corresponds to the mass renormalization coefficient

$$\tilde{I}^-_{even}(k) = \frac{g_0^2C_F}{a}\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{1}{\hat{k}^2}\left(r\tilde{\sigma}_V\sum_\mu\sin^2 k_\mu-\sum_\mu\cos^2\frac{k_\mu}{2}\tilde{S}_+ +r^2\sum_\mu\sin^2\frac{k_\mu}{2}\tilde{S}_-\right) \tag{4.98}$$

Using the $m\to 0$ expressions for $\tilde{\sigma}_V$ and $\tilde{S}_\pm$ from section (2.3.2), we get

$$\tilde{I}^-_{even} = \frac{g_0^2C_F}{a}\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{1}{\hat{k}^2}\frac{ma_5}{(1-w_0e^{-\alpha})^2}\left(r\frac{a_5e^{-\alpha}}{1-ba_5e^{\alpha}}\sum_\mu\sin^2 k_\mu\right.$$
$$\left.-\frac{1-ba_5e^{-\alpha}}{1-ba_5e^{\alpha}}\sum_\mu\cos^2\frac{k_\mu}{2}+r^2e^{-2\alpha}\sum_\mu\sin^2\frac{k_\mu}{2}\right), \tag{4.99}$$

which compares to σ_m (3.23) in Aoki. We will parametrize this as

$$\tilde{I}^-_{even}(k) \equiv m\frac{g_0^2C_F}{16\pi^2}\Sigma_2 \tag{4.100}$$

with

$$\Sigma_2=\frac{1}{ma\pi^2}\int_{-\pi}^{\pi}d^4k\frac{1}{\hat{k}^2}\left(r\tilde{\sigma}_V\sum_\mu\sin^2 k_\mu-\sum_\mu\cos^2\frac{k_\mu}{2}\tilde{S}_+ +r^2\sum_\mu\sin^2\frac{k_\mu}{2}\tilde{S}_-\right). \tag{4.101}$$

The remaining two terms renormalize the wave function and the $w = 1 - M$ factor. The second term behaves as a^{-1} and contributes to Z_W:

$$-i\mathcal{A}\left[p\cdot\gamma\bar{I}^{+}_{even} + \bar{I}^{-}_{even}p\cdot\gamma\right] \;=\; -ip\cdot\gamma\frac{g_0^2 C_F}{16\pi^2}\Sigma_3 \tag{4.102}$$

with

$$\Sigma_3 = \frac{2\mathcal{A}}{a\pi^2}\int_{-\pi}^{\pi} d^4k \frac{1}{\hat{k}^2}\left[r\bar{\sigma}_V \sum_\mu \sin^2 k_\mu - \bar{\sigma}_S \sum_\mu \cos k_\mu\right] . \tag{4.103}$$

This compares to term Σ_3 (3.20) and $\bar{\omega}$ (3.22) in Aoki. The last term contributes to both Z_2 and Z_m and has a logarithmic singularity $\log a^2p^2$

$$p_\mu \left.\frac{\partial \bar{I}^{+}_{odd}}{\partial p_\mu}\right|_{p\to 0} \;=\; -ip\cdot\gamma\frac{g_0^2 C_F}{16\pi^2}\Sigma_1 \tag{4.104}$$

with

$$\begin{aligned}\Sigma_1 \;=\; & \frac{1}{d\pi^2}\int_{-\pi}^{\pi} d^4k \left\{ \frac{1}{\hat{k}^2}\left(\frac{1+r^2}{2}\bar{k}^2\bar{\sigma}_V - r\bar{\sigma}_S\sum_\mu \cos k_\mu\right)\right. \\ & \left. -\frac{1}{(\hat{k}^2)^2}\left(2r\bar{k}^2\bar{\sigma}_S + 2\left[\left(2\sum_\rho \bar{k}_\rho^2\bar{k}_\rho^2 - \bar{k}^2\bar{k}^2\right)\bar{G}_- - r^2\bar{k}^2\frac{\hat{k}^2}{4}\bar{G}_+\right]\right)\right\}\end{aligned} \tag{4.105}$$

This compares to Σ_1 (3.18) and I^d (3.21) in Aoki.

4.1.4 Wilson limit

To get formulas for Wilson fermions, we have to do the following:

1. Replace all propagator functions $G_\pm$ and $g_\pm$ by corresponding Wilson terms

$$G^{\pm}_{st}, g^{\pm}_{s} \to \frac{1}{\bar{k}^2 + \left(am_0 + \frac{r}{2}\hat{k}^2\right)^2} \tag{4.106}$$

$$S^{\pm}_{st}, \sigma^{\pm}_{s} \to \frac{am_0 + \frac{r}{2}\hat{k}^2}{\bar{k}^2 + \left(am_0 + \frac{r}{2}\hat{k}^2\right)^2} \tag{4.107}$$

2. Set the size of 5^{th} dimension equal to 1 $N = 1$, $\mathcal{A} \to 0$ and $w_0 \to 1$. That eliminates the effect of amputating external legs:

$$\bar{S}_s^{OUT} = \left[-ip\cdot\gamma\mathcal{A}\left(w_0^{N-s}P_+ + w_0^{s-1}P_-\right) + \left(w_0^{s-1}P_+ + w_0^{N-s}P_-\right)\right]_s \to 1 \quad (4.108)$$

$$\bar{S}_t^{IN} = \left[\left(w_0^{N-t}P_- + w_0^{t-1}P_+\right)(-ip\cdot\gamma\mathcal{A}) + \left(w_0^{t-1}P_- + w_0^{N-t}P_+\right)\right]_t \to 1 \quad (4.109)$$

In practice, that means that we replace all propagator functions by the corresponding Wilson terms

$$\bar{G}_+ = \bar{G}_- = \tilde{G}_+ = \tilde{G}_- = \bar{g}_+ = \bar{g}_- = \bar{\sigma}_V = \tilde{\sigma}_V \to \frac{1}{\bar{k}^2 + \left(am_0 + \frac{r}{2}\hat{k}^2\right)^2} \quad (4.110)$$

$$\bar{S}_+ = \bar{S}_- = \tilde{S}_+ = \tilde{S}_- = \bar{\sigma}_+ = \bar{\sigma}_- = \bar{\sigma}_S = \tilde{\sigma}_S \to \frac{am_0 + \frac{r}{2}\hat{k}^2}{\bar{k}^2 + \left(am_0 + \frac{r}{2}\hat{k}^2\right)^2} \quad (4.111)$$

4.2 Numerical evaluation of sunset diagram

As we can see, due to complicated smearing tensors, expanding the amplitude in powers of external momentum is quite complicated already in the first order of expansion; when we evaluated twist-2 operators with n derivatives, we have to expand to n^{th} order in p which becomes *really* messy and complicated. Fortunately, there is an alternative solution. One can also evaluate the amplitude directly as a 4-dimensional integral. After rescaling integral (4.22) over k and evaluating the sums in the 5th dimension, to order p_μ^1 we are left with the expression (4.41)

$$I_q = \bar{I}^-_{even}(k) - i\mathcal{A}\left[p\cdot\gamma\bar{I}^+_{even}(k) + \bar{I}^-_{even}p\cdot\gamma\right] + p_\mu \left.\frac{\partial\bar{I}^+_{odd}}{\partial p_\mu}\right|_{p\to 0}. \quad (4.112)$$

In general, the amplitude $I(q)$ is a 4×4 matrix which can be decomposed in terms of γ-matrices

$$I(q) = I_S(p) + I_P(p)\gamma_5 + I_V^\mu(p)\gamma_\mu + I_A^\mu(p)\gamma_\mu\gamma_5 + I_T^{\mu\nu}(p)\sigma_{\mu\nu} \quad (4.113)$$

where

$$\begin{array}{rclrcl} I_S(p) &=& \frac{1}{d}\mathrm{Tr}_D\left[I_q(p)\right] & I_P(p) &=& \frac{1}{d}\mathrm{Tr}_D\left[I_q(p)\gamma_5\right] \\ I_V^\mu(p) &=& \frac{1}{d}\mathrm{Tr}_D\left[I_q(p)\gamma_\mu\right] & I_A^\mu(p) &=& \frac{1}{d}\mathrm{Tr}_D\left[I_q(p)\gamma_5\gamma_\mu\right] \\ I_T^{\mu\nu}(p) &=& \frac{1}{2d}\mathrm{Tr}_D\left[I_q(p)\sigma_{\mu\nu}\right] & & & \end{array} \tag{4.114}$$

We already know for $p \to 0$

$$I_P(p \to 0) = 0\,, \qquad I_A^\mu(p \to 0) = 0\,, \qquad I_T^{\mu\nu}(p \to 0) = 0 \tag{4.115}$$

and

$$I_V^\mu(p \to 0) = -ip_\mu I_V + O(p^3)\,, \tag{4.116}$$

so in the $p \to 0$ limit

$$I_q = I_S(p=0) - ip\cdot\gamma I_V(p=0)\,. \tag{4.117}$$

Since the trace of an odd number of γ matrices vanishes, we can drop it immediately from the expression so we have

$$I_S(p) = \frac{1}{d}\mathrm{Tr}_D\left[\tilde{I}^-_{even}\right] \tag{4.118}$$

$$I_V(p) = \frac{1}{d}\mathrm{Tr}_D\left[\frac{ip\cdot\gamma}{p^2}\left(\bar{I}^+_{odd} - i\mathcal{A}\left(p\cdot\gamma\bar{I}^+_{even}(k) + \bar{I}^-_{even}p\cdot\gamma\right)\right)\right] \tag{4.119}$$

Evaluating these formulas is straightforward:

$$\begin{aligned} I_S(p) = \frac{H_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}&\left[\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\tilde{S}_-\right. \\ &\left. -g_{\rho\sigma}(\widetilde{ap+k})^2_\rho\tilde{S}_+ + r(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\bar{k}_\sigma\tilde{\sigma}_V\right]\,. \end{aligned} \tag{4.120}$$

After inserting the definition of $H_{\rho\sigma}$

$$H_{\rho\sigma}(k) = g_{\rho\sigma}A_\rho + \hat{k}_\rho\hat{k}_\sigma(1-g_{\rho\sigma})A_{\rho\sigma} \tag{4.121}$$

and using trigonometric simplifications (4.62)

$$\begin{aligned}
(\widehat{ap-k})_\rho(\widehat{ap+k})_\rho &= 2\left(\cos k_\rho - \cos ap_\rho\right) && (4.122)\\
(\widehat{ap-k})_\rho(\widetilde{ap+k})_\rho &= \left(\sin ap_\rho - \bar{k}_\rho\right) && (4.123)
\end{aligned}$$

we get

$$\begin{aligned}
I_S(p) &= \sum_\rho \frac{A_\rho(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r^2}{4}(\widehat{ap+k})^2_\rho\tilde{S}_- - (\widetilde{ap+k})^2_\rho\tilde{S}_+ + r(\overline{ap+k})_\rho\bar{k}_\rho\tilde{\sigma}_V\right]\\
&\quad + \sum_{\rho,\sigma\neq\rho}\frac{A_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left[r^2(\cos k_\rho - \cos ap_\rho)(\cos k_\sigma - \cos ap_\sigma)\tilde{S}_-\right.\\
&\qquad \left. - g_{\rho\sigma}(\sin ap_\rho - \bar{k}_\rho)^2\tilde{S}_+ + 2r(\cos k_\rho - \cos ap_\rho)(\sin ap_\sigma - \bar{k}_\sigma)\bar{k}_\sigma\tilde{\sigma}_V\right] && (4.124)
\end{aligned}$$

We split I_V into two parts: one coming from $\bar{I}^+_{odd}$, which contributes to wave function renormalization so is called I_2, and the other coming from the $p\cdot\gamma\bar{I}^+_{even} + \bar{I}^-_{even}p\cdot\gamma$ term, which contributes to the renormalization of w_0 and is called I_w

$$\begin{aligned}
I_2(p) &= \frac{H_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left\{\frac{1}{p^2}\left(p\cdot\bar{k}\left[\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\bar{G}_+ + g_{\rho\sigma}(\widetilde{ap+k})^2_\rho\bar{G}_-\right]\right.\right.\\
&\qquad \left.\left. -2(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma p_\rho\bar{k}_\sigma\bar{G}_- - r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma p_\sigma\bar{\sigma}_S\right)\right\} && (4.125)\\
I_w(p) &= \frac{H_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left\{2\mathcal{A}\left(\left[\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma - g_{\rho\sigma}(\widetilde{ap+k})^2_\rho\right]\bar{\sigma}_S\right.\right.\\
&\qquad \left.\left. + r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\bar{k}_\sigma\bar{\sigma}_V\right)\right\}. && (4.126)
\end{aligned}$$

After inserting the definition of $H_{\rho\sigma}$ and using trigonometric simplifications, we obtain

$$\begin{aligned}
I_2 &= \sum_\rho \frac{A_\rho(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left\{\frac{1}{p^2}\left(p\cdot\bar{k}\left[\frac{r^2}{4}(\widehat{ap+k})^2_\rho\bar{G}_+ + (\widetilde{ap+k})^2_\rho\bar{G}_-\right]\right.\right.\\
&\qquad \left.\left. -2(\widetilde{ap+k})^2_\rho p_\rho\bar{k}_\rho\bar{G}_- - r(\overline{ap+k})_\rho p_\rho\bar{\sigma}_S\right)\right\}\\
&\quad + \sum_{\rho,\sigma\neq\rho}\frac{A_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left\{\frac{1}{p^2}\left(p\cdot\bar{k}\left[r^2(\cos k_\rho - \cos ap_\rho)(\cos k_\sigma - \cos ap_\sigma)\bar{G}_+\right.\right.\right.\\
&\qquad \left. + g_{\rho\sigma}(\sin ap_\rho - \bar{k}_\rho)^2\bar{G}_-\right] - 2p_\rho\bar{k}_\sigma(\sin ap_\rho - \bar{k}_\rho)(\sin ap_\sigma - \bar{k}_\sigma)\bar{G}_-\\
&\qquad \left.\left. -2rp_\sigma(\cos k_\rho - \cos ap_\rho)(\sin ap_\sigma - \bar{k}_\sigma)\bar{\sigma}_S\right)\right\} && (4.127)
\end{aligned}$$

$$
\begin{aligned}
I_w &= 2\mathcal{A}\sum_{\rho}\frac{A_\rho(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left\{\left[\frac{r^2}{4}(\widehat{ap+k})_\rho^2-(\widetilde{ap+k})_\rho^2\right]\bar{\sigma}_S+r(\overline{ap+k})_\rho\bar{k}_\rho\bar{\sigma}_V\right\} \\
&+2\mathcal{A}\sum_{\rho,\sigma\neq\rho}\frac{A_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left\{\left[r^2(\cos k_\rho-\cos ap_\rho)(\cos k_\sigma-\cos ap_\sigma)\right.\right. \\
&\left.\left.-g_{\rho\sigma}(\sin ap_\rho-\bar{k}_\rho)^2\right]\bar{\sigma}_S+2r\bar{k}_\sigma(\cos k_\rho-\cos ap_\rho)(\sin ap_\sigma-\bar{k}_\sigma)\bar{\sigma}_V\right\} \qquad (4.128)
\end{aligned}
$$

Since both I_{odd} and I_{even} have no γ_5 matrices and at most 2 γ_μ matrices, C_P and C_A^μ obviously vanish. However, $\tilde{I}_{even}^-$ has a component proportional to $\sigma_{\mu\nu}$

$$
\begin{aligned}
\tilde{I}_{even}^- &= \frac{H_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\tilde{S}_- - g_{\rho\sigma}(\widetilde{ap+k})_\rho^2\tilde{S}_+\right. \\
&\left.+\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left(\bar{k}_\sigma(\tilde{G}_++\tilde{G}_-)-i\sigma_{\alpha\sigma}\bar{k}_\alpha(\tilde{G}_+-\tilde{G}_-)\right)\right] \qquad (4.129)
\end{aligned}
$$

so we have

$$
\begin{aligned}
I_T^{\mu\nu}(p) &= \frac{H_{\rho\sigma}(ap-k)}{(\widehat{ap-k})^2+\mu^2}\left[-i\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma(\tilde{G}_+-\tilde{G}_-)(\bar{k}_\mu g_{\nu\sigma}-\bar{k}_\nu g_{\mu\sigma})\right] \\
&= \frac{-ir}{(\widehat{ap-k})^2+\mu^2}\frac{\tilde{G}_+-\tilde{G}_-}{2}\left\{\frac{\bar{k}_\mu A_\nu(ap-k)(\overline{ap+k})_\nu-\bar{k}_\nu A_\mu(ap-k)(\overline{ap+k})_\mu}{2}\right. \\
&+(\cos k_\rho-\cos ap_\rho)\left[A_{\rho\nu}(ap-k)\bar{k}_\mu(\sin ap_\nu-\bar{k}_\nu)\right. \\
&\left.\left.-A_{\rho\mu}(ap-k)\bar{k}_\nu(\sin ap_\mu-\bar{k}_\mu)\right]\right\} \qquad (4.130)
\end{aligned}
$$

Evaluated at $p\to 0$, this obviously vanishes due to parity symmetry since $\mu\neq\nu$ and both A_μ and $A_{\mu\nu}$ are even in k_μ.

4.2.1 Results for Wilson fermions

While evaluating the integral for finite p_μ, finiteness of p_μ regulates the gluon propagator automatically so no singularities (which represent themselves as floating point exceptions) are present. However, to evaluate the amplitude at zero momentum numerically, we must regulate both the fermion and gluon propagators with masses m_f and μ. The factor k^2 from the measure d^4k cancels one of k^2 factors in denominator analytically, but when one tries to evaluate the integrand numerically in Cartesian coordinates, that measure factor is not present and one gets floating point exception errors. Therefore we keep both m_f and μ finite and look at the expression as both of them go to zero exponentially.

Error bars are the estimate based on the comparison of 7th degree rule and 5th degree rule used to evaluate the integral. As such, they should be interpreted with caution.

For Wilson fermions, the result is shown in figure (4-3). For values of m_f small

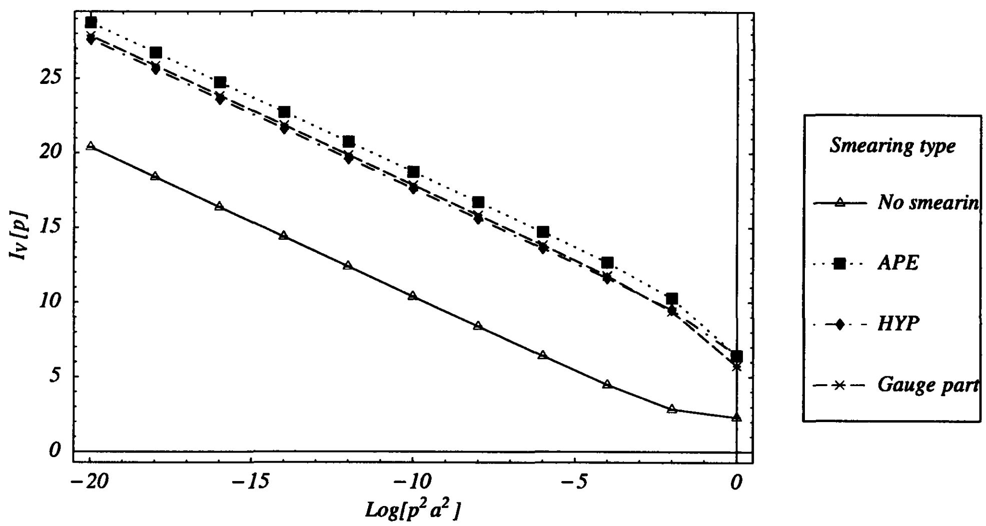

Figure 4-3: Amplitude $I_V(p)$ as a function of $\log p^2a^2$ in the $M \to 0$ limit (for Wilson fermions).

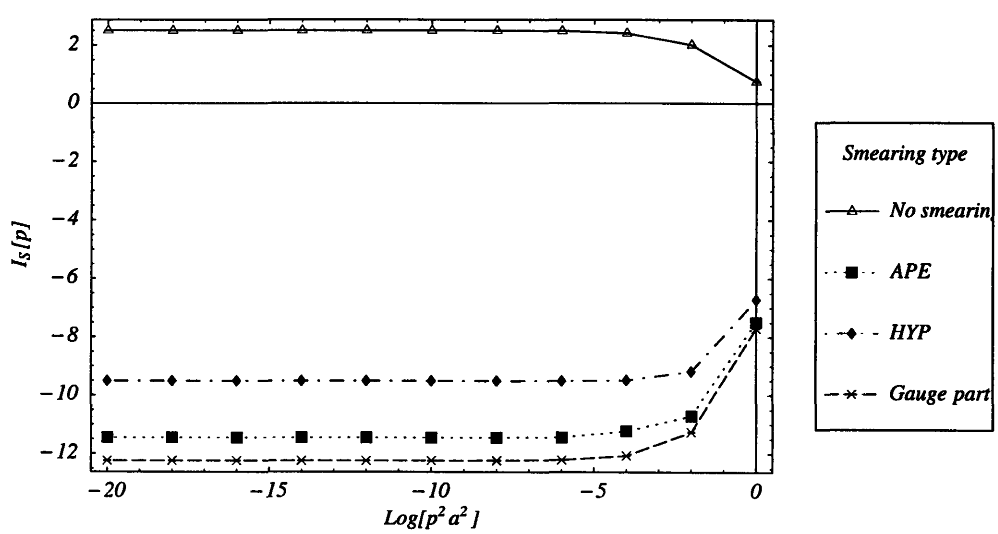

Figure 4-4: Amplitude $I_S(p)$ as a function of $\log p^2a^2$ in the $M \to 0$ limit (for Wilson fermions).

enough, one can clearly see the linear behavior with respect to $\log p^2a^2$. The slope of the plot is 1 as it should be.

4.2.2 Results for Domain Wall fermions

For DW Fermions, things are in principle the same as for the Wilson case. Note that the same C++ programs are used with different file for $G_{\pm}$ and $W^{\mp}G_{\pm}$.

Results for $\tilde{I}^-_{even}$ and I_S.

$\tilde{I}^-_{even}$ evaluated at $p = 0$ is proportional to

$$\tilde{I}^-_{even} = \frac{g_0^2 C_F}{16\pi^2}\left(-4\log\mu^2a^2 + \Sigma_m\right) . \tag{4.131}$$

When the same thing is evaluated for zero gluon mass but finite momentum, we get

$$I_S = \frac{g_0^2 C_F}{16\pi^2}\left(-4\log p^2a^2 + \bar{\Sigma}_m\right) = \frac{g_0^2 C_F}{16\pi^2}\left(-4\log p^2a^2 + (\Sigma_m + 4)\right) . \tag{4.132}$$

due to the different regulator. The slopes of both graphs are -4 as they should be.

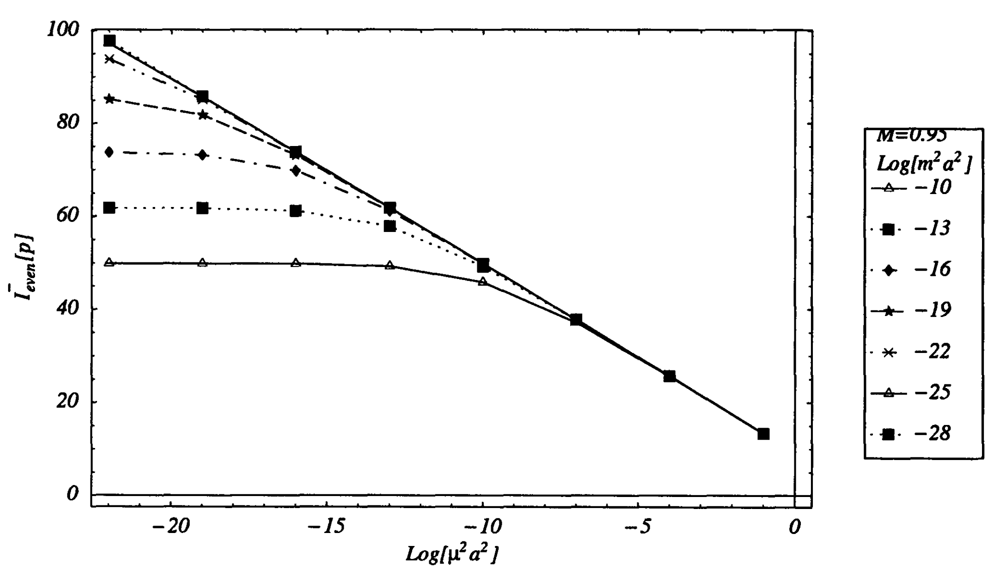

Figure 4-5: Amplitude $\tilde{I}^-_{even}(\mu^2)$ for various values of m_f as a function of μ^2.

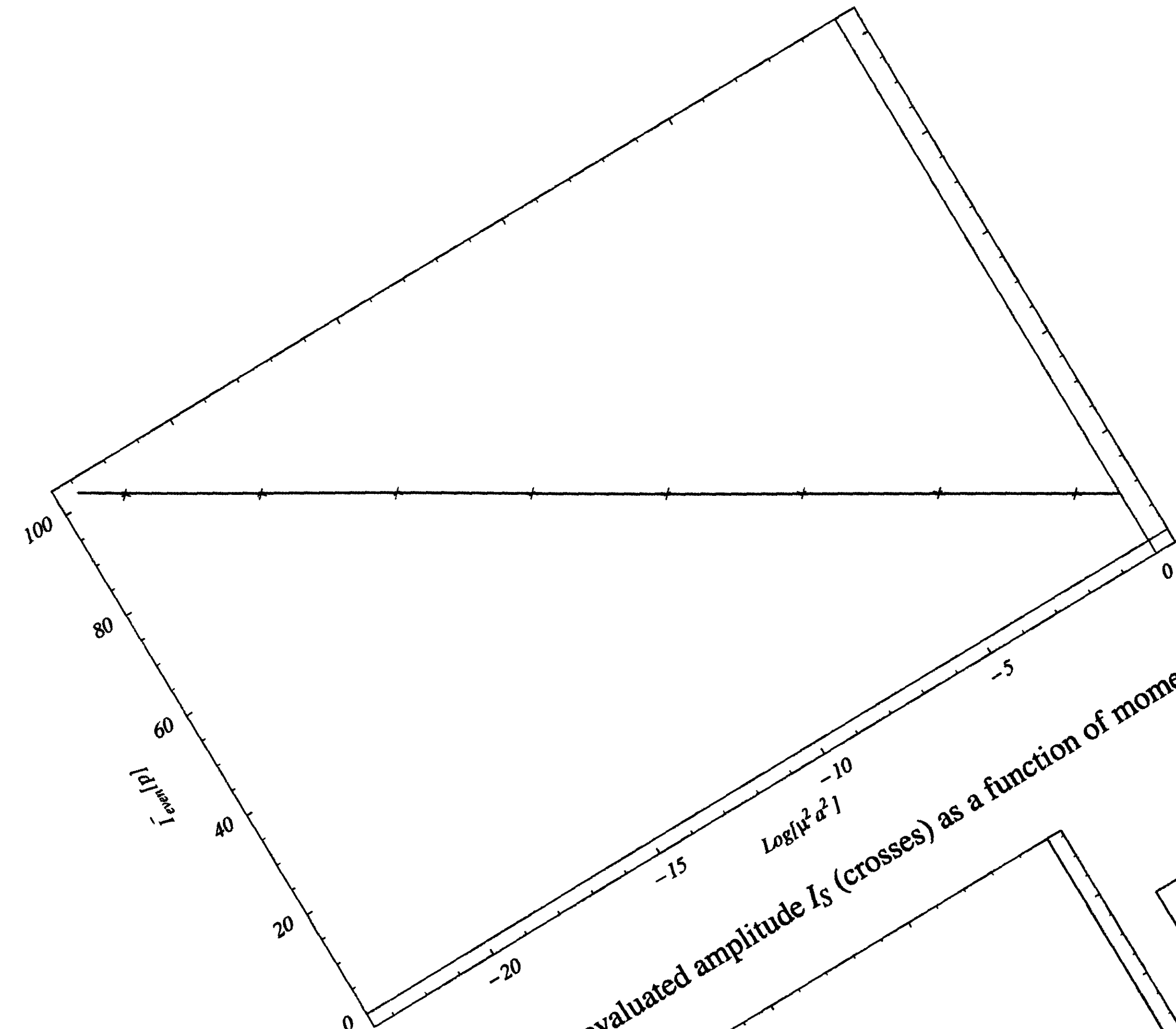

Figure 4-6: Numerically evaluated amplitude I_S (crosses) as a function of momentum p^2.

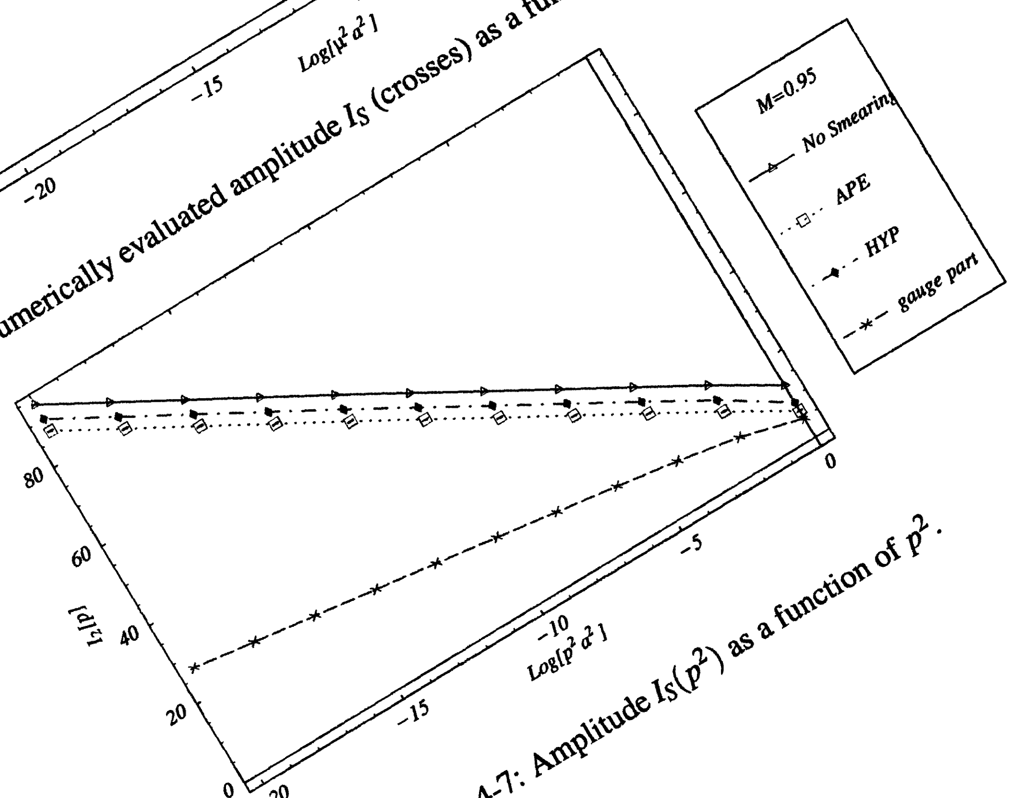

Figure 4-7: Amplitude $I_S(p^2)$ as a function of p^2.

Results for $p_\mu \partial/\partial p_\mu \bar{\Gamma}^+_{odd}$ and I_V.

$p_\mu \partial/\partial p_\mu \bar{\Gamma}^+_{odd}$ evaluated at $p=0$ is proportional to

$$p_\mu \frac{\partial \bar{\Gamma}^+_{odd}}{\partial p_\mu} = \frac{g_0^2 C_F}{16\pi^2}\left(-\log\mu^2 a^2 + \Sigma_2\right).$$

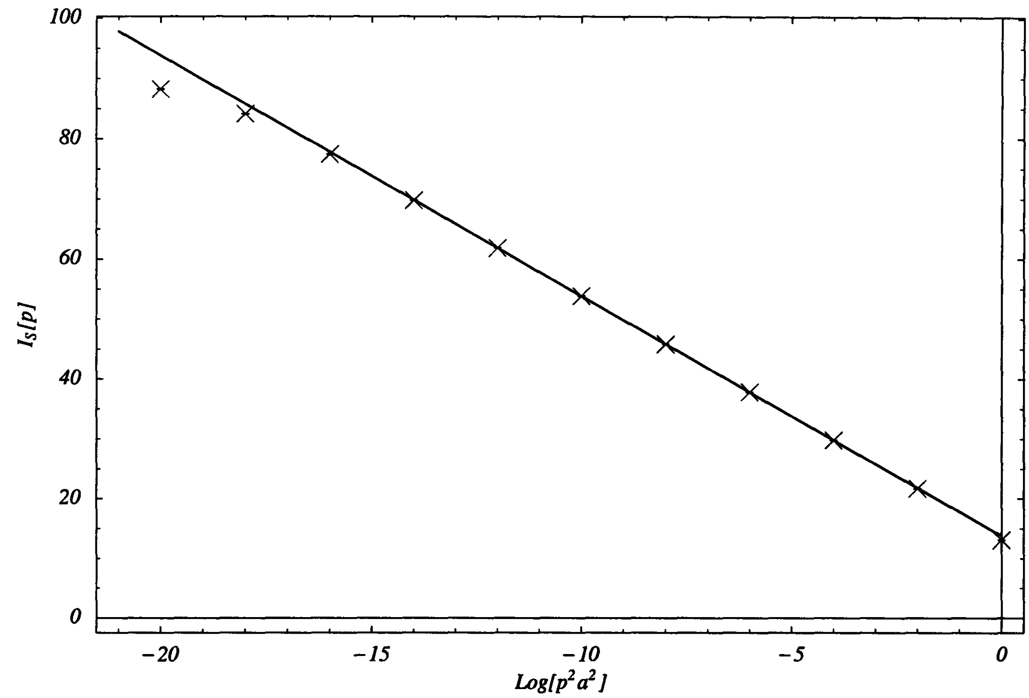

Figure 4-8: Numerically evaluated amplitude I_S (crosses) as a function of momentum p^2, extrapolated to the point $\log p^2a^2 = 0$.

$$I_V = \frac{g_0^2 C_F}{16\pi^2}\left(-4\log p^2a^2 + \bar{\Sigma}_2\right) = \frac{g_0^2 C_F}{16\pi^2}\left(-\log p^2a^2 + (\Sigma_2 + 1)\right) . \quad (4.133)$$

due to the different regulator.

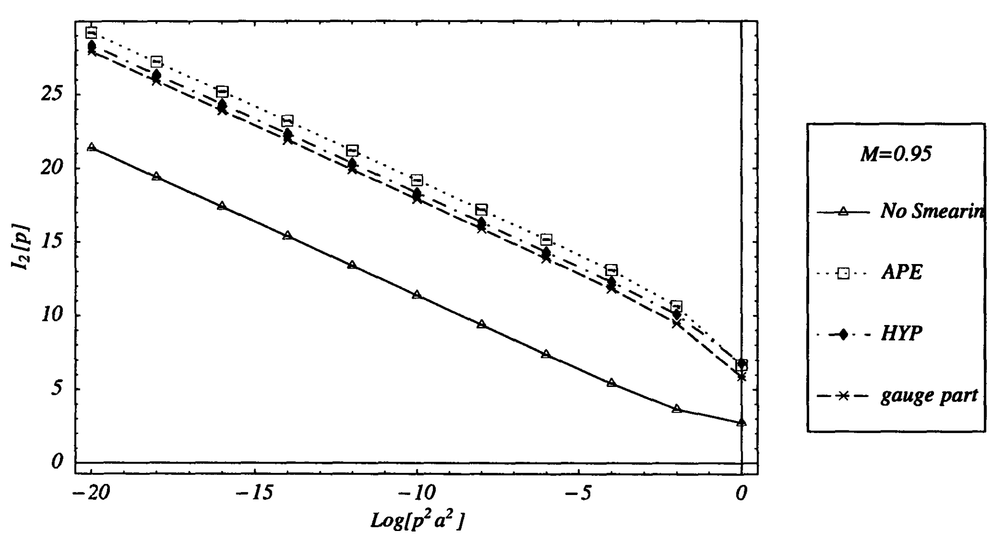

Figure 4-9: Amplitude $I_V(p^2)$ as a function of p^2.

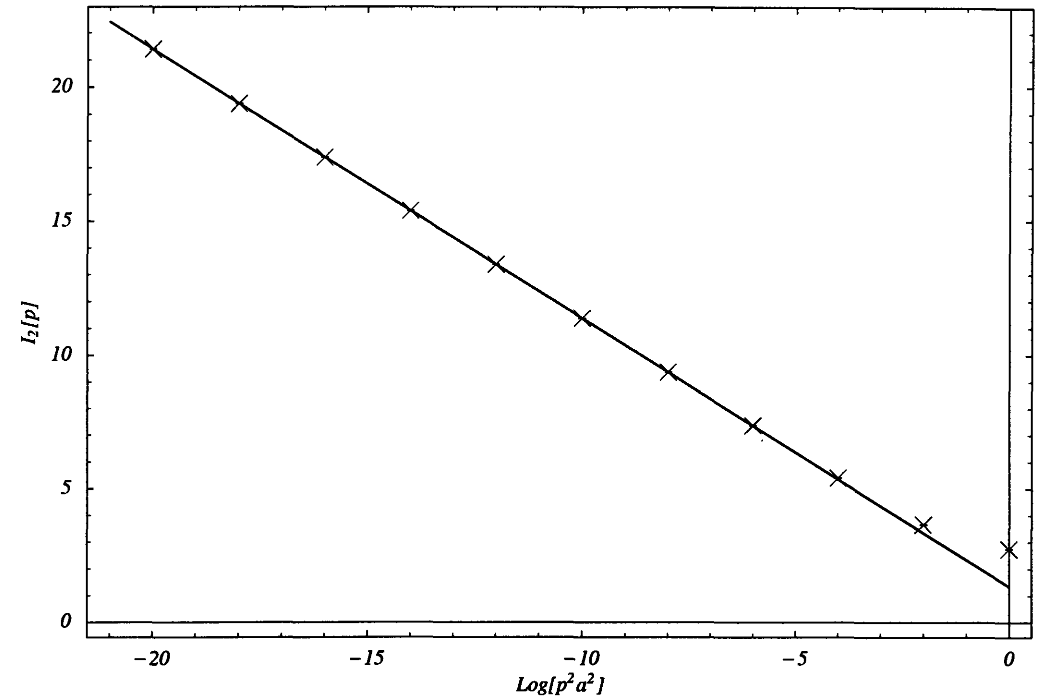

Figure 4-10: Numerically evaluated amplitude I_2 (crosses) as a function of momentum p^2, extrapolated to the point $\log p^2a^2 = 0$.

Results for $p\cdot\gamma\bar{I}^+_{even} + \bar{I}^-_{even}p\cdot\gamma$ and I_W.

$p\cdot\gamma\bar{I}^+_{even} + \bar{I}^-_{even}p\cdot\gamma$ evaluated at $p=0$ is proportional to

$$p\cdot\gamma\bar{I}^+_{even} + \bar{I}^-_{even}p\cdot\gamma \;=\; I_W = \frac{g_0^2C_F}{16\pi^2}\Sigma_w\,. \tag{4.134}$$

and unlike the previous two cases, it doesn't have a logarithmic singularity.

4.3 Tadpole diagram

The amplitude for the tadpole diagram is given by

$$\begin{aligned} I_{st} &= \frac{1}{2}\delta_{st}\int_{-\pi/a}^{\pi/a}\frac{d^dk}{(2\pi)^d}\sum_\rho G_{\lambda\rho}(k)V^{aa}_{\rho\rho}(p,p) \\ &= \frac{\delta_{st}}{2}\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\sum_\rho\left(-ag_0^2C_F\right)\left(r\cos ap_\rho - i\gamma_\rho\sin ap_\rho\right)a^2\frac{H_{\rho\rho}(k)}{\hat{k}^2+\mu^2} \end{aligned}$$

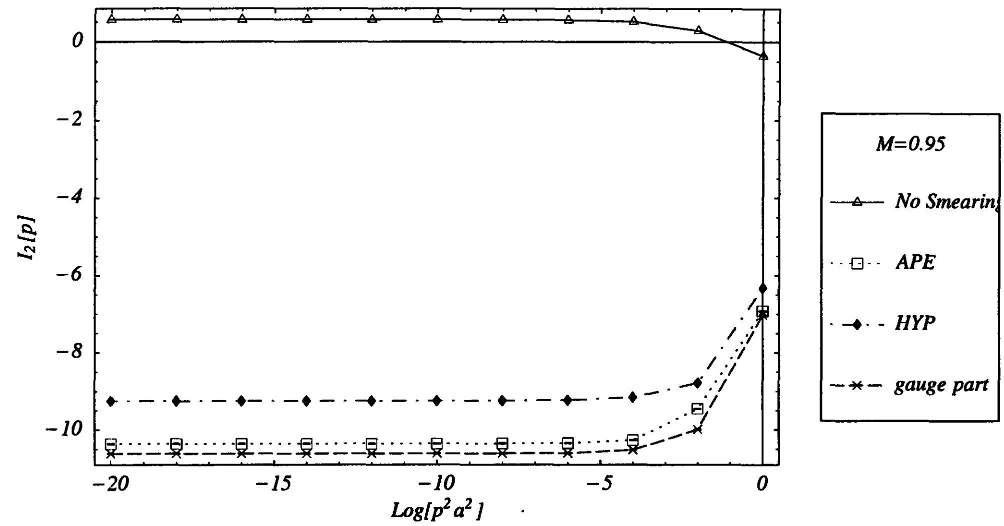

Figure 4-11: Amplitude $I_W(p^2)$ for as a function of p^2.

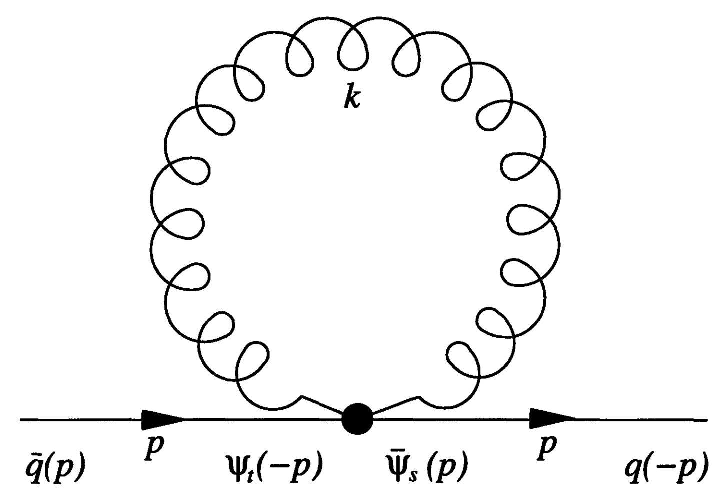

Figure 4-12: Tadpole diagram for physical quarks

$$= \quad -\frac{\delta_{st} g_0^2 C_F}{2} \sum_{\rho} \left(r\frac{1}{a} - i\gamma_\rho p_\rho \right) \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{H_{\rho\rho}(k)}{\hat{k}^2 + \mu^2} \tag{4.135}$$

Since the integral is the same for each ρ, we get

$$I_{st} \quad = \quad -\frac{\delta_{st} g_0^2 C_F}{2} \left(r\frac{d}{a} - i\gamma \cdot p \right) \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{H_{\rho\rho}(k)}{\hat{k}^2 + \mu^2} \qquad \text{(no summation)} \tag{4.136}$$

	no smearing	HYP	APE	gauge-part
T	0.15493	0.05219	0.04202	0.03873
Σ^{tad}	12.2328	4.12076	3.31777	3.058

Table 4.1: Results for tadpole

Since there are no fermion propagators here, 5D sums are straightforward to evaluate:

$$\bar{I} \sim \sum_{s,t=1}^{N} w_0^{s-1}\delta_{st}w_0^{t-1} = \sum_{s=0}^{N-1}(w_0^2)^s = \frac{1-w_0^{2N}}{1-w_0^2} \to \frac{1}{1-w_0^2} \tag{4.137}$$

$$\tilde{I} \sim \sum_{s,t=1}^{N} w_0^{s-1}\delta_{st}w_0^{N-t} = \sum_{s=0}^{N-1} w_0^{N-1} = Nw_0^{N-1} \to 0 \tag{4.138}$$

so the physical amplitude equals

$$\begin{aligned}
I_q(p) &= \bar{S}_s^{OUT} I_{st} \bar{S}_t^{IN} && (4.139)\\
&= (-ip\cdot\gamma\mathcal{A})\bar{I}^-_{odd}(-ip\cdot\gamma\mathcal{A}) + \bar{I}^+_{odd} + (-ip\cdot\gamma\mathcal{A})\bar{I}^+_{even} + \bar{I}^-_{even}(-ip\cdot\gamma\mathcal{A}) && (4.140)\\
&= (1-p^2\mathcal{A}^2)\bar{I}_{odd} - ip\cdot\gamma\mathcal{A}\bar{I}_{even} && (4.141)\\
&= ip\cdot\gamma\frac{g_0^2C_F}{2}\left(1+2r\mathcal{A}\frac{d}{a}\right)\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{H_{\rho\rho}(k)}{\hat{k}^2+\mu^2}\\
&= ip\cdot\gamma\frac{g_0^2C_F}{2}\left(1+2r\mathcal{A}\frac{d}{a}\right)\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{1}{d}\left(\sum_\rho\frac{A_\rho}{\hat{k}^2+\mu^2}+\sum_{\rho\sigma}g_{\rho\sigma}\frac{A_{\rho\sigma}(k)(1-g_{\rho\sigma})\hat{k}_\rho\hat{k}_\sigma}{\hat{k}^2+\mu^2}\right) && (4.142)
\end{aligned}$$

which yields

$$\Sigma_1 = -\frac{1}{2\pi^2}(16\pi^4 T)\,, \qquad \Sigma_3 = -\frac{rd\mathcal{A}}{a\pi^2}(16\pi^4 T) \tag{4.143}$$

where T is the Tadpole contribution

$$T(\mu^2) = \frac{1}{d}\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\left(\sum_\rho\frac{A_\rho}{\hat{k}^2+\mu^2}+\sum_{\rho\sigma}g_{\rho\sigma}\frac{A_{\rho\sigma}(k)(1-g_{\rho\sigma})\hat{k}_\rho\hat{k}_\sigma}{\hat{k}^2+\mu^2}\right)\,. \tag{4.144}$$

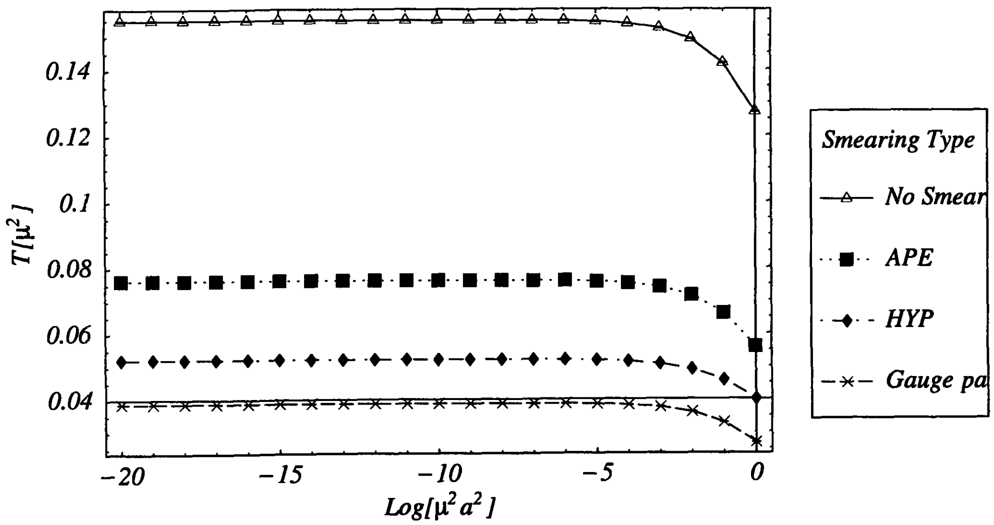

Figure 4-13: Value of tadpole function $T(\mu^2)$ as a function of μ^2.

4.4 Collecting results: self energy renormalization coefficients

At tree level, the physical quark propagator is given by (2.163)

$$S^{PHYS}(p) = \langle q(-p)\bar{q}(p)\rangle_0 = \frac{1}{ip\cdot\gamma+\tilde{m}}\,. \tag{4.145}$$

The one loop correction is then given by

$$\begin{aligned} \langle q(-p)\bar{q}(p)\rangle_1 &= \langle q(-p)\bar{q}(p)\rangle_0 \left(1+I_q\langle q(-p)\bar{q}(p)\rangle_0\right) \\ &= \frac{1}{ip\cdot\gamma+\tilde{m}}\left(1+I_q\frac{1}{ip\cdot\gamma+\tilde{m}}\right) \end{aligned} \tag{4.146}$$

which to the order g^2 equals

$$\langle q(-p)\bar{q}(p)\rangle_1 = \frac{1}{ip\cdot\gamma+\tilde{m}-I_q} \tag{4.147}$$

Parameterizing I_q as

$$I_q = -ip\cdot\gamma A+\tilde{m}B \tag{4.148}$$

we get (to order g^2)

$$\begin{aligned}\langle q(-p)\bar{q}(p)\rangle_1 &= \frac{1}{ip\cdot\gamma(1+A)+\tilde{m}(1-B)} = \frac{(1+A)^{-1}}{ip\cdot\gamma+\tilde{m}(1-B)(1+A)^{-1}} \\ &= \frac{(1-A)}{ip\cdot\gamma+\tilde{m}(1-B)(1-A)} = \frac{(1-A)}{ip\cdot\gamma+\tilde{m}(1-B-A)} \end{aligned} \tag{4.149}$$

which has to equal a generic form

$$\langle q(-p)\bar{q}(p)\rangle_1 = \frac{Z_w Z_2}{ip\cdot\gamma+\tilde{m}Z_w Z_m^{-1}} = \frac{Z_q}{ip\cdot\gamma+\tilde{m}Z_w Z_m^{-1}} \tag{4.150}$$

where to order g^2 we have

$$Z_w = 1+\frac{g^2C_F}{16\pi^2}z_w = 1-\frac{2w_0}{1-w_0^2}\frac{g^2C_F}{16\pi^2}\Sigma_w\,, \tag{4.151}$$

$$Z_2 = 1+\frac{g^2C_F}{16\pi^2}\Sigma_2\,, \tag{4.152}$$

$$Z_m^{-1} = 1-\frac{g^2C_F}{16\pi^2}z_m = 1+\frac{g^2C_F}{16\pi^2}(\Sigma_m-\Sigma_2) \tag{4.153}$$

which yields

$$Z_w Z_2 = 1+\frac{g^2C_F}{16\pi^2}(z_w+z_2) = 1-A\,, \qquad Z_w Z_m^{-1} = 1+\frac{g^2C_F}{16\pi^2}(z_w+z_m) = 1-B-A\,. \tag{4.154}$$

While at first, the separation of the piece in the expression for the self energy proportional to $ip\cdot\gamma$ may seem a bit arbitrary, the origin of two pieces is quite different. Σ_2 comes from the part of $5D$ propagator proportional to $ip\cdot\gamma$ and describes the renormalization of the $5D$ wave function ψ. Part Σ_w comes from the part of the $5D$ propagator proportional to a constant and describes the (additive) renormalization of the $5D$ mass-parameter M or equivalently, additive renormalization of w_0. Overall renormalization of the physical wavefunction q then has two pieces:

$$q_{ren}(x) = Z_q^{1/2}q_0(x)\,, \qquad \text{with } Z_q = Z_2 Z_w\,. \tag{4.155}$$

The first piece, Z_2, describes the renormalization of the $5D$ wavefunctions ψ_1 and ψ_N in the expression

$$q(x) \;=\; \sqrt{1-w_0^2}\,(P_+\psi_1(x)+P_-\psi_N(x)) \tag{4.156}$$

The second piece, Z_w, describes the fact that the overlap between the $5D$ massless mode χ_0 and the physical wavefunction q also changes

$$\langle\chi_0|q_0\rangle \;=\sqrt{1-w_0^2} \to \langle\chi_{ren}|q_{ren}\rangle \;=\sqrt{1-w_R^2} \tag{4.157}$$

Since the massless mode is renormalized as

$$\begin{aligned}\chi_R \;&=\; \sqrt{1-w_R^2}\left(P_+w_R^{s-1}\psi_s^{ren}+P_-w_R^{N-s}\psi_s^{ren}\right)=\sqrt{1-w_R^2}\left[\sqrt{Z_2}(P_+\psi_1+P_-\psi_N)+\ldots\right]\\ &=\; \sqrt{\frac{1-w_R^2}{1-w_0^2}Z_2}\left[\sqrt{1-w_0^2}(P_+\psi_1+P_-\psi_N)+\ldots\right],\end{aligned} \tag{4.158}$$

we can see that the overall renormalization constant Z_q picks up a piece coming from the additive shift of w_0. To order g_0^2 we have

$$\frac{1-w_R^2}{1-w_0^2}=\frac{1-\left(w_0+\frac{g_0^2C_F}{16\pi^2}\Sigma_w\right)^2}{1-w_0^2}=1-\frac{2w_0}{1-w_0^2}\frac{g_0^2C_F}{16\pi^2}\Sigma_w+O(g_0^4)=Z_w+O(g_0^4) \tag{4.159}$$

If we start with large value of w_0, even if the shift $w_R=w_0+\Delta w$ can be described well with 1-loop perturbation theory, using the expression for Z_w expanded to the first order in g_0^2 can give poor results for Z_w as can be seen from the table (4.4). Luckily, we do not have to rely on the perturbation theory alone to evaluate that factor Z_W. Since the shift of zero-mode is universal for all operators, we can evaluate it once nonperturbatively and then use it afterwards to renormalize other matrix elements. There is an exactly conserved $5D$ axial current on the lattice

$$\begin{aligned}\mathcal{A}_\mu(x) \;&=\; \frac{1}{2}\sum_s \mathrm{sign}\left(s-\frac{N-1}{2}\right)\left[\bar\psi_s(x+\hat\mu)(1+\gamma_\mu)U_\mu^\dagger(x)\psi_s(x)\right.\\ &\qquad \left.-\bar\psi_s(x)(1-\gamma_\mu)U_\mu(x)\psi_s(x+\hat\mu)\right]\end{aligned} \tag{4.160}$$

which has the same continuum limit as the local axial current on the lattice

$$A_\mu(x) = \bar{q}(x)\gamma_\mu\gamma_5 q(x)\,. \tag{4.161}$$

Due to the non-local nature of the conserved current, it is complicated to express it in terms of physical fields $q(x)$ defined on the boundary of the lattice so in practice it is easier to use the local (non-conserved) current. But since the $5D$ current is exactly conserved, its renormalization coefficient is exactly equal to one. Then we can evaluate the ratio[1]

$$\frac{\langle \mathcal{A}_\mu(x)\, \bar{q}(y)\gamma_5 q(y)\rangle}{\langle A_\mu(x)\, \bar{q}(y)\gamma_5 q(y)\rangle} = Z_A Z_q \tag{4.162}$$

and use perturbative value of Z_A we can evaluate Z_q nonperturbatively to (relatively) high precision. Since we also need the value of Z_A to extract Z_q, comparison of perturbative an nonperturbative results will be left for later, after we evaluate local current renormalization coefficient Z_A perturbatively.

[1]any renormalization coefficients from the source $\bar{q}(y)\gamma_5 q(y)$ cancel exactly in the numerator and denominator

M	"no smearing"	HYP	APE	"gauge-part"
0.1	13.35206	-1.99177	-3.92017	-4.79048
0.1	13.16034	-2.14575	-4.03369	-4.79048
0.2	13.00997	-2.25961	-4.10974	-4.79048
0.3	12.88293	-2.35157	-4.16660	-4.79048
0.4	12.77305	-2.42809	-4.21094	-4.79048
0.5	12.67728	-2.49224	-4.24571	-4.79048
0.6	12.59396	-2.54610	-4.27291	-4.79048
0.7	12.52220	-2.59084	-4.29440	-4.79048
0.8	12.46168	-2.62717	-4.31080	-4.79048
0.9	12.41249	-2.65543	-4.32268	-4.79048
1	12.37514	-2.67559	-4.33035	-4.79048
1.1	12.35052	-2.68751	-4.33391	-4.79048
1.2	12.34001	-2.69053	-4.33321	-4.79048
1.3	12.34558	-2.68361	-4.32784	-4.79048
1.4	12.36998	-2.66516	-4.31701	-4.79048
1.5	12.41707	-2.63278	-4.29941	-4.79048
1.6	12.49240	-2.58277	-4.27283	-4.79048
1.7	12.60416	-2.50931	-4.23352	-4.79048
1.8	12.76522	-2.40272	-4.17471	-4.79048
1.9	12.99791	-2.24480	-4.08271	-4.79034

Table 4.2: Renormalization coefficient Σ_2 yields the $5D$ wave function renormalization.

M	"no smearing"	HYP	APE	"gauge-part"
0.1	51.43470	6.97653	1.97935	0.00001
0.1	-3.82195	-1.65308	0.26608	-4.79201
0.2	-4.60698	-2.29055	-0.23749	-4.79201
0.3	-5.24244	-2.77267	-0.58900	-4.79201
0.4	-5.80199	-3.17245	-0.86082	-4.79201
0.5	-6.31759	-3.52112	-1.08354	-4.79201
0.6	-6.80737	-3.83592	-1.27373	-4.79201
0.7	-7.28339	-4.12794	-1.44170	-4.79201
0.8	-7.75486	-4.40518	-1.59462	-4.79201
0.9	-8.22961	-4.67409	-1.73797	-4.79201
1	-8.71503	-4.94045	-1.89373	-4.79201
1.1	-9.21872	-5.20976	-2.01406	-4.79201
1.2	-9.74957	-5.48832	-2.15562	-4.79201
1.3	-10.31759	-5.78331	-2.30621	-4.79201
1.4	-10.93598	-6.10409	-2.47257	-4.79201
1.5	-11.62247	-6.46367	-2.66413	-4.79201
1.6	-12.40240	-6.88118	-2.89540	-4.79201
1.7	-13.31492	-7.38769	-3.19082	-4.79201
1.8	-14.42736	-8.03967	-3.59742	-4.79201
1.9	-15.87989	-8.96232	-4.22595	-4.79201

Table 4.3: Renormalization coefficient Σ_m, which yields an additive mass renormalization for Wilson fermions and a multiplicative mass renormalization for DW fermions

	no smearing		HYP		APE		gauge-part	
M	Σ_w	z_w	Σ_w	z_w	Σ_w	z_w	Σ_w	z_w
0.1	51.04818	483.61438	6.70250	63.49735	1.78979	16.95590	-0.00003	-0.00024
0.2	50.74498	225.53326	6.50720	28.92088	1.67079	7.42575	-0.00003	-0.00011
0.3	50.48850	138.59588	6.35360	17.44126	1.58604	4.35382	-0.00003	-0.00007
0.4	50.26641	94.24952	6.22887	11.67914	1.52248	2.85466	-0.00003	-0.00005
0.5	50.07261	66.76348	6.12628	8.16838	1.47403	1.96538	-0.00003	-0.00003
0.6	49.90377	47.52740	6.04182	5.75412	1.43682	1.36840	-0.00003	-0.00002
0.7	49.75818	32.80759	5.97250	3.93791	1.40853	0.92870	-0.00003	-0.00002
0.8	49.63520	20.68133	5.91820	2.46592	1.38744	0.57810	-0.00003	-0.00001
0.9	49.53508	10.00709	5.87652	1.18718	1.37269	0.27731	-0.00003	-0.00001
1.0	49.45884	0.00000	5.84758	0.00000	1.36354	0.00000	-0.00003	-0.00000
1.1	49.40836	-9.98149	5.83158	-1.17810	1.35986	-0.27472	-0.00003	0.00001
1.2	49.38646	-20.57769	5.82928	-2.42887	1.36175	-0.56740	-0.00003	0.00001
1.3	49.39717	-32.56956	5.84218	-3.85198	1.36980	-0.90317	-0.00003	0.00002
1.4	49.44610	-47.09153	5.87268	-5.59303	1.38506	-1.31911	-0.00003	0.00002
1.5	49.54115	-66.05487	5.92466	-7.89954	1.40950	-1.87934	-0.00003	0.00003
1.6	49.69350	-93.17531	6.00415	-11.25778	1.44641	-2.71202	-0.00003	0.00005
1.7	49.91984	-137.03484	6.12087	-16.80237	1.50163	-4.12213	-0.00003	0.00007
1.8	50.24618	-223.31634	6.29182	-27.96364	1.63578	-7.27015	-0.00003	0.00011
1.9	50.71758	-480.48238	6.54946	-62.04756	1.72291	-16.32230	-0.00003	0.00024

Table 4.4: Renormalization coefficient Σ_w, which corresponds to an additive renormalization of the $5D$ mass parameter M. While the shift itself $\frac{g_0^2 C_F}{16\pi^2}\Sigma_w$ is perturbative for all values of M, using the formula $Z_W = 1 - \frac{2w_0}{1-w_0^2}\frac{g_0^2 C_F}{16\pi^2}\Sigma_w$ leads to incorrect results for the full WF renormalization $Z_q = Z_w Z_2$. One has to use the correct expression $Z_w = \frac{1-w_R^2}{1-w_0^2}$ instead.

M	NOS	HYP	APE	GDP	NOS	HYP	APE	GDP
Wilson	1.	1.	1.	1.	1.1	0.97	0.95	0.96
0.1	-4.06	0.45	0.86	1.	-4.46	0.43	0.82	0.96
0.2	-1.41	0.75	0.94	1.	-1.55	0.72	0.89	0.96
0.3	-0.53	0.85	0.96	1.	-0.58	0.82	0.92	0.96
0.4	-0.08	0.9	0.98	1.	-0.08	0.87	0.93	0.96
0.5	0.2	0.93	0.98	1.	0.22	0.9	0.94	0.96
0.6	0.39	0.95	0.99	1.	0.42	0.92	0.94	0.96
0.7	0.53	0.96	0.99	1.	0.58	0.93	0.94	0.96
0.8	0.64	0.98	0.99	1.	0.7	0.94	0.95	0.96
0.9	0.74	0.99	1.	1.	0.81	0.95	0.95	0.96
1.	0.83	1.	1.	1.	0.9	0.96	0.95	0.96
1.1	0.91	1.01	1.	1.	0.99	0.97	0.95	0.96
1.2	0.99	1.02	1.	1.	1.08	0.98	0.96	0.96
1.3	1.08	1.03	1.01	1.	1.18	0.99	0.96	0.96
1.4	1.19	1.04	1.01	1.	1.3	1.01	0.96	0.96
1.5	1.32	1.06	1.02	1.	1.45	1.03	0.97	0.96
1.6	1.51	1.09	1.02	1.	1.65	1.05	0.97	0.96
1.7	1.81	1.14	1.03	1.	1.98	1.1	0.98	0.96
1.8	2.39	1.23	1.06	1.	2.61	1.19	1.01	0.96
1.9	4.09	1.51	1.14	1.	4.49	1.46	1.08	0.96

Table 4.5: Total effect of 5D mass parameter renormalization $Z_W = (1-w_R^2)/(1-w_0^2)$ (left) and the total lattice renormalization coefficient Z_q (right).

Chapter 5

Bilinear operators (quark currents)

We now build upon the methodology established for the self-energy renormalization to calculate the bilinear operators relevant to deep inelastic scattering. In this chapter we concentrate on local quark currents which have no derivative operators.

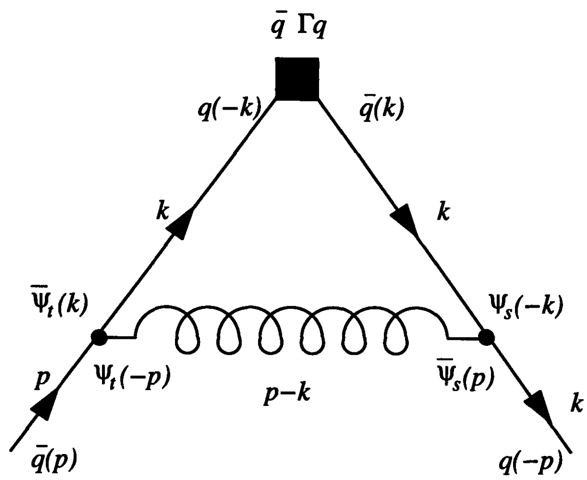

Figure 5-1: Vertex diagram for quark bilinear operators

The $5D$ amplitude for the vertex diagram is given by

$$I_{st} = V_\rho(p,k)S_s^{IN}(k)O(k)S_t^{OUT}(k)V_\lambda(k,p)G_{\rho\lambda}(p-k) \tag{5.1}$$

where $O(k)$ is the Feynman rule for the vertex operator $\bar{q}(k)\Gamma q(k)$ in which Γ is one of Dirac matrices $1,\gamma_5,\gamma_\mu,\gamma_\mu\gamma_5$ or $\sigma_{\mu\nu}$. Dirac algebra for this diagram is the same as for twist-2 operators $\bar{q}\gamma_\mu D_\nu D_\alpha \ldots q$, since each covariant derivative adds only a factor of four-momentum

$\bar{k}_\mu$. The amplitude is then

$$
\begin{aligned}
I_{st} \;=\; & \left[\frac{\delta^{ab} h_\rho h_\sigma}{(\widehat{ap-k})^2+\mu^2}\right]\left[-g_0 T^a_{cd}\left(r\frac{a}{2}(\widehat{ap+k})_\rho + i\gamma_\rho(\widetilde{ap+k})_\rho\right)\right] \\
& \left[(g_- P_+ + g_+ P_-)(-i\bar{k}\cdot\gamma) + (\sigma_- P_+ + \sigma_+ P_-)\right][\Gamma] \\
& \left[(-i\bar{k}\cdot\gamma)(g_+ P_+ + g_- P_-) + (\sigma_+ P_+ + \sigma_- P_-)\right] \\
& \left[-g_0 T^b_{dc}\left(r\frac{a}{2}(\widehat{ap+k})_\sigma + i\gamma_\rho(\widetilde{ap+k})_\sigma\right)\right]. \qquad (5.2)
\end{aligned}
$$

5.1 Scalar and Pseudoscalar current

5.1.1 Amplitude for $5D$ fermions

First, we perform the γ algebra. for $\Gamma = 1, \gamma_5$ we have

$$
\begin{aligned}
[S][O][S] \;=\; & \left[(g_- P_+ + g_+ P_-)(-i\bar{k}\cdot\gamma) + (\sigma_- P_+ + \sigma_+ P_-)\right][\Gamma] \\
& \left[(-i\bar{k}\cdot\gamma)(g_+ P_+ + g_- P_-) + (\sigma_+ P_+ + \sigma_- P_-)\right] \\
=\; & \left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right) P_\pm[\gamma_5]\,. \qquad (5.3)
\end{aligned}
$$

A note for clarification: $\pm$ signs in the subscript denote components multiplying $P_\pm$, while $[\pm]$ signs denote signs for the 1 and γ_5 cases respectively. Using the compact notation for vertices

$$V = a\cdot h + i\bar{b}\cdot\gamma \qquad (5.4)$$

we have

$$
\begin{aligned}
[V][SOS][V] \;=\; & (a\cdot h + i\bar{b}\cdot\gamma)\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm\right. \\
& \left. - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right) P_\pm[\gamma_5](a\cdot h + i\bar{b}\cdot\gamma) \qquad (5.5)
\end{aligned}
$$

which we evaluate term by term. The first one equals

$$\text{“}a^2\text{” term} \;=\; (a\cdot h)^2\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right) P_\pm[\gamma_5]$$

$$
\begin{aligned}
&= H_{\rho\sigma}\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm\right.\\
&\qquad \left.-i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right)[\gamma_5]\,. \qquad (5.6)
\end{aligned}
$$

The second one equals

$$
\begin{aligned}
\text{“}b^2\text{” term} &= (i\bar{b}\cdot\gamma)\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right)P_\pm[\gamma_5](i\bar{b}\cdot\gamma)\\
&= [\pm](i\bar{b}\cdot\gamma)\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right)P_\mp[\gamma_5]\\
&= [\pm]\left\{-\bar{b}^2([\mp]\bar{k}^2 g_\pm g_\mp + \sigma_\pm\sigma_\mp) - i\gamma_\mu\bar{k}_\nu(\bar{b}^2 g_{\mu\nu} - 2\bar{b}_\mu\bar{b}_\nu)\right.\\
&\qquad \left.(g_\mp\sigma_\mp[\pm]\sigma_\mp g_\mp)\right\}P_\pm[\gamma_5]\\
&= [\pm]H_{\rho\sigma}\Big\{-g_{\rho\sigma}(\widetilde{ap+k})^2_\rho([\mp]\bar{k}^2 g_\pm g_\mp + \sigma_\pm\sigma_\mp)\\
&\qquad -i\gamma_\mu\bar{k}_\nu\Big[g_{\mu\nu}g_{\rho\sigma}(\widetilde{ap+k})^2_\rho - 2g_{\rho\mu}g_{\sigma\nu}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\Big]\\
&\qquad (g_\mp\sigma_\mp[\pm]\sigma_\mp g_\mp)\Big\}P_\pm[\gamma_5]\,. \qquad (5.7)
\end{aligned}
$$

Finally, the “mixed” term equals

$$
\begin{aligned}
\text{“mixed” term} &= (i\bar{b}\cdot\gamma)\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right)P_\pm[\gamma_5](a\cdot h)\\
&\quad +(a\cdot h)\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right)P_\pm[\gamma_5](i\bar{b}\cdot\gamma)\\
&= (i\bar{b}\cdot\gamma\, a\cdot h)\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right)P_\pm[\gamma_5]\\
&\quad [\pm](a\cdot h)\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm - i\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\right)(i\bar{b}\cdot\gamma)P_\mp[\gamma_5]\\
&= \left\{(i\bar{b}\cdot\gamma\, a\cdot h)\left([\mp]\bar{k}^2(g_\mp g_\pm[\pm]g_\pm g_\mp) + (\sigma_\mp\sigma_\pm[\pm]\sigma_\pm\sigma_\mp)\right)\right.\\
&\quad \left.+a\cdot h\left(\bar{b}\cdot\gamma\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)[\pm]\bar{k}\cdot\gamma\bar{b}\cdot\gamma(g_\mp\sigma_\mp[\pm]\sigma_\mp g_\mp)\right)\right\}P_\pm[\gamma_5]\\
&= H_{\rho\sigma}\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left\{i\gamma_\sigma\left([\mp]\bar{k}^2(g_\mp g_\pm[\pm]g_\pm g_\mp)\right.\right.\\
&\qquad \left.+(\sigma_\mp\sigma_\pm[\pm]\sigma_\pm\sigma_\mp)\right) + \gamma_\sigma\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)\\
&\qquad \left.[\pm]\bar{k}\cdot\gamma\gamma_\sigma(g_\mp\sigma_\mp[\pm]\sigma_\mp g_\mp)\right\}P_\pm[\gamma_5]\,. \qquad (5.8)
\end{aligned}
$$

Adding them up and separating parts with even and odd number of γ matrices, we get

$$
I^{even}_\pm = \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\left\{\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left([\mp]\bar{k}^2 g_\mp g_\pm + \sigma_\mp\sigma_\pm\right)\right.
$$

$$+g_{\rho\sigma}(\widetilde{ap+k})^2_\rho(\bar{k}^2 g_\pm g_\mp[\mp]\sigma_\pm\sigma_\mp)+\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma$$
$$\times\left(\gamma_\sigma\bar{k}\cdot\gamma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm)+\bar{k}\cdot\gamma\gamma_\sigma([\pm]g_\mp\sigma_\mp+\sigma_\mp g_\mp)\right)\Big\}\,[\gamma_5] \quad (5.9)$$

and

$$\begin{aligned}
I_\pm^{odd} &= \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{-i\bar{k}\cdot\gamma\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma(g_\pm\sigma_\pm[\pm]\sigma_\pm g_\pm) \\
&\quad -i\gamma_\mu\bar{k}_\nu\left[g_{\mu\nu}g_{\rho\sigma}(\widetilde{ap+k})^2_\rho-2g_{\rho\mu}g_{\sigma\nu}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right] \\
&\quad \times([\pm]g_\mp\sigma_\mp+\sigma_\mp g_\mp)+\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma i\gamma_\sigma \\
&\quad \left[-\bar{k}^2([\pm]g_\mp g_\pm+g_\pm g_\mp)+(\sigma_\mp\sigma_\pm[\pm]\sigma_\pm\sigma_\mp)\right]\Big\}\,[\gamma_5] \qquad (5.10)
\end{aligned}$$

5.1.2 Amplitude for physical fermions

The physical amplitude is then obtained after summing in the 5^{th} dimension

$$\begin{aligned}
I_q(p) &= \bar{S}_s^{OUT} I_{st}\bar{S}_t^{IN} \\
&= \left[-ip\cdot\gamma\mathcal{A}\left(w_0^{N-s}P_-+w_0^{s-1}P_+\right)+\left(w_0^{s-1}P_-+w_0^{N-s}P_+\right)\right]_s[I_{st}^+P_++I_{st}^-P_-][\gamma_5] \\
&\quad \times\left[\left(w_0^{N-t}P_++w_0^{t-1}P_-\right)(-ip\cdot\gamma\mathcal{A})+\left(w_0^{t-1}P_++w_0^{N-t}P_-\right)\right]_t . \\
&= I_{phys}^{odd}+I_{phys}^{even} \qquad (5.11) \\
I_{phys}^{odd} &= (-ip\cdot\gamma\mathcal{A})\bar{I}_{odd}^-[\gamma_5](-ip\cdot\gamma\mathcal{A})+\bar{I}_{odd}^+[\gamma_5] \\
&\quad +(-ip\cdot\gamma\mathcal{A})\tilde{I}_{odd}^-[\gamma_5]+\tilde{I}_{odd}^+[\gamma_5](-ip\cdot\gamma\mathcal{A}) \qquad (5.12) \\
I_{phys}^{even} &= (-ip\cdot\gamma\mathcal{A})\bar{I}_{even}^+[\gamma_5](-ip\cdot\gamma\mathcal{A})+\bar{I}_{even}^-[\gamma_5] \\
&\quad +(-ip\cdot\gamma\mathcal{A})\tilde{I}_{even}^+[\gamma_5]+\tilde{I}_{even}^-[\gamma_5](-ip\cdot\gamma\mathcal{A}) , \qquad (5.13)
\end{aligned}$$

where

$$\bar{I}^\pm \equiv \sum w_0^{s-1}I^\pm w_0^{t-1}\equiv\sum w_0^{N-s}I^\mp w_0^{N-t} , \qquad (5.14)$$
$$\tilde{I}^\pm \equiv \sum w_0^{s-1}I^\pm w_0^{N-t}\equiv\sum w_0^{N-s}I^\mp w_0^{t-1} , \qquad (5.15)$$

and we have used the fact

$$I_{odd}P_{\pm} = P_{\mp}I_{odd}\,, \qquad I_{even}P_{\pm} = P_{\pm}I_{even}\,. \tag{5.16}$$

Performing the $5D$ sums, we get

$$\begin{aligned}
\tilde{I}^{even}_{\pm} &= \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma[\mp]g_{\rho\sigma}(\widetilde{ap+k})^2_\rho\right] \\
&\quad \times\left([\mp]\bar{k}^2\tilde{g}_{\mp}\tilde{g}_{\pm}+\tilde{\sigma}_{\mp}\tilde{\sigma}_{\pm}\right)[\gamma_5] && (5.17)\\
\tilde{I}^{even}_{\pm} &= \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\left\{\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left(\tilde{\sigma}^2_{\mp}[\mp]\bar{k}^2\tilde{g}^2_{\mp}\right)\right. \\
&\quad +g_{\rho\sigma}(\widetilde{ap+k})^2_\rho(\bar{k}^2\tilde{g}^2_{\pm}[\mp]\tilde{\sigma}^2_{\pm}) \\
&\quad \left.+r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\bar{k}_\sigma\left(\tilde{g}_{\pm}\tilde{\sigma}_{\mp}[\pm]\tilde{\sigma}_{\pm}\tilde{g}_{\mp}\right)\right\}[\gamma_5] && (5.18)\\
\tilde{I}^{odd}_{\pm} &= \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\left\{-i\bar{k}\cdot\gamma\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left(\tilde{g}_{\pm}\tilde{\sigma}_{\pm}[\pm]\tilde{\sigma}_{\pm}\tilde{g}_{\pm}\right)\right. \\
&\quad -i\gamma_\mu\bar{k}_\nu\left[g_{\mu\nu}g_{\rho\sigma}(\widetilde{ap+k})^2_\rho-2g_{\rho\mu}g_{\sigma\nu}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right] \\
&\quad ([\pm]\tilde{g}_{\mp}\tilde{\sigma}_{\mp}+\tilde{\sigma}_{\mp}\tilde{g}_{\mp})+i\gamma_\sigma\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma \\
&\quad \left.\left[-\bar{k}^2([\pm]\tilde{g}_{\mp}\tilde{g}_{\pm}+\tilde{g}_{\pm}\tilde{g}_{\mp})+(\tilde{\sigma}_{\mp}\tilde{\sigma}_{\pm}[\pm]\tilde{\sigma}_{\pm}\tilde{\sigma}_{\mp})\right]\right\}[\gamma_5] && (5.19)\\
\tilde{I}^{odd}_{\pm} &= \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\left\{-i\bar{k}\cdot\gamma\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left(\tilde{g}_{\pm}\tilde{\sigma}_{\mp}[\pm]\tilde{\sigma}_{\pm}\tilde{g}_{\mp}\right)\right. \\
&\quad -i\gamma_\mu\bar{k}_\nu\left[g_{\mu\nu}g_{\rho\sigma}(\widetilde{ap+k})^2_\rho-2g_{\rho\mu}g_{\sigma\nu}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right] \\
&\quad ([\pm]\tilde{g}_{\mp}\tilde{\sigma}_{\pm}+\tilde{\sigma}_{\mp}\tilde{g}_{\pm})+i\gamma_\sigma\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma \\
&\quad \left.\left[-\bar{k}^2([\pm]\tilde{g}_{\mp}\tilde{g}_{\mp}+\tilde{g}_{\pm}\tilde{g}_{\pm})+(\tilde{\sigma}_{\mp}\tilde{\sigma}_{\mp}[\pm]\tilde{\sigma}_{\pm}\tilde{\sigma}_{\pm})\right]\right\}[\gamma_5]\,. && (5.20)
\end{aligned}$$

To get the physical amplitude, we evaluate this at $p=0$ so we are left with $I_q = \tilde{I}^+_{odd}+\tilde{I}^-_{even}$. It's easy to see that $\tilde{I}^+_{odd}$ vanishes for $p\to 0$ since it's an odd function of k_μ so the physical amplitude is given by

$$\begin{aligned}
I_{S,P} &= \tilde{I}^-_{even} = \frac{H_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}_\rho\hat{k}_\sigma\left(\tilde{\sigma}^2_+[\mp]\bar{k}^2\tilde{g}^2_+\right)+g_{\rho\sigma}\tilde{k}^2_\rho(\bar{k}^2\tilde{g}^2_-[\mp]\tilde{\sigma}^2_-)\right. \\
&\quad \left.+r\hat{k}_\rho\tilde{k}_\sigma\bar{k}_\sigma\left(\tilde{g}_-\tilde{\sigma}_+[\pm]\tilde{\sigma}_-\tilde{g}_+\right)\right\}[\gamma_5]\,, && (5.21)
\end{aligned}$$

which with the help of the 'smearing decomposition

$$H_{\rho\sigma} = g_{\rho\sigma}A_\rho + \hat{k}_\rho\hat{k}_\sigma A_{\rho\sigma} \tag{5.22}$$

can be written as

$$\begin{aligned} I_{S,P} &= \tilde{I}^-_{even} = \frac{A_\rho}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}^2_\rho\,(\tilde{\sigma}^2_+[\mp]\bar{k}^2\tilde{g}^2_+) + \bar{k}^2_\rho(\bar{k}^2\tilde{g}^2_-[\mp]\tilde{\sigma}^2_-)\right. \\ &\qquad \left. + r\bar{k}^2_\rho\,(\tilde{g}_-\tilde{\sigma}_+[\pm]\tilde{\sigma}_-\tilde{g}_+)\right\}[\gamma_5] \\ &\quad + \frac{A_{\rho\sigma}}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\,(\tilde{\sigma}^2_+[\mp]\bar{k}^2\tilde{g}^2_+) + g_{\rho\sigma}\bar{k}^2_\rho(\bar{k}^2\tilde{g}^2_-[\mp]\tilde{\sigma}^2_-)\right. \\ &\qquad \left. + r\hat{k}^2_\rho\bar{k}^2_\sigma\,(\tilde{g}_-\tilde{\sigma}_+[\pm]\tilde{\sigma}_-\tilde{g}_+)\right\}[\gamma_5]\,. \end{aligned} \tag{5.23}$$

5.1.3 No-smearing limit

For DW fermions $\tilde{g}_+, \tilde{\sigma}_- \to 0$ and for no smearing $A_\rho \to 1, A_{\rho\sigma} \to 0$ so we get

$$I_{S,P} = \tilde{I}^-_{even} = \frac{1}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}^2\tilde{\sigma}^2_+ + \bar{k}^2\bar{k}^2\tilde{g}^2_- + r\bar{k}^2\tilde{g}_-\tilde{\sigma}_+\right\}[\gamma_5] \tag{5.24}$$

which agrees with Aoki's [16] results (4.5) and (4.6).

5.2 Vector and Axial vector current

5.2.1 Amplitude for $5D$ fermions

Again, we begin with the γ algebra. For $\Gamma = \gamma_\mu, \gamma_\mu\gamma_5$ we have

$$\bar{k}\cdot\gamma\gamma_\mu[\gamma_5]\bar{k}\cdot\gamma = \mp\gamma_\nu[\gamma_5](g_{\mu\nu}\bar{k}^2 - 2\bar{k}_\mu\bar{k}_\nu)\,, \tag{5.25}$$

so the first term is

$$\begin{aligned} &(g_-P_+ + g_+P_-)(-i\bar{k}\cdot\gamma)\gamma_\mu[\gamma_5](-i\bar{k}\cdot\gamma)(g_+P_+ + g_-P_-) \\ &\qquad = \pm\gamma_\nu[\gamma_5](g_{\mu\nu}\bar{k}^2 - 2\bar{k}_\mu\bar{k}_\nu)g_\pm g_\pm P_\pm \quad . \end{aligned} \tag{5.26}$$

The last term equals

$$(\sigma_- P_+ + \sigma_+ P_-)\gamma_\mu[\gamma_5](\sigma_+ P_+ + \sigma_- P_-) = \gamma_\mu[\gamma_5]\sigma_\pm\sigma_\pm P_\pm\,, \tag{5.27}$$

so the overall $[S][O][S]$ result is

$$\begin{aligned}
[S][O][S] \;&=\; \gamma_\nu[\gamma_5][\pm(g_{\mu\nu}\bar{k}^2 - 2\bar{k}_\mu\bar{k}_\nu)g_\pm g_\pm + g_{\mu\nu}\sigma_\pm\sigma_\pm] \\
&\quad - i\gamma_\mu[\gamma_5]\bar{k}\cdot\gamma\sigma_\mp g_\pm - i\bar{k}\cdot\gamma\gamma_\mu[\gamma_5]g_\mp\sigma_\pm
\end{aligned} \tag{5.28}$$

After doing the algebra, we have the result (which we again split into parts with odd and even number of γ-matrices)

$$\begin{aligned}
I_{st}^{odd} \;=\; & \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{\gamma_\alpha\left(g_{\rho\sigma}g_{\nu\alpha}(\widetilde{ap+k})^2_\rho - 2g_{\rho\nu}g_{\sigma\alpha}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right) \\
& \qquad [(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)g_\mp g_\mp \pm g_{\mu\nu}\sigma_\mp\sigma_\mp] \\
& + \gamma_\nu\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\pm(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)g_\pm g_\pm \pm g_{\mu\nu}\sigma_\pm\sigma_\pm\right] \\
& + \frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[\pm\gamma_\sigma\gamma_\mu\bar{k}\cdot\gamma\sigma_\mp g_\pm + \gamma_\sigma\bar{k}\cdot\gamma\gamma_\mu g_\mp\sigma_\pm\right. \\
& \qquad \left. + \gamma_\mu\bar{k}\cdot\gamma\gamma_\sigma\sigma_\pm g_\mp \pm \bar{k}\cdot\gamma\gamma_\mu\gamma_\sigma g_\pm\sigma_\mp\right]\Big\}\,[\gamma_5]
\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}
I_{st}^{even} \;=\; & \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{ig_{\rho\sigma}(\widetilde{ap+k})^2_\rho\left[\gamma_\mu\bar{k}\cdot\gamma\sigma_\pm g_\mp \pm \bar{k}\cdot\gamma\gamma_\mu g_\pm\sigma_\mp\right] \\
& - 2i(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\gamma_\rho(\gamma_\mu\bar{k}_\sigma - \bar{k}\cdot\gamma g_{\rho\sigma})(\sigma_\pm g_\mp \mp g_\pm\sigma_\mp) \\
& - i\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\pm\gamma_\mu\bar{k}\cdot\gamma\sigma_\mp g_\pm \pm \bar{k}\cdot\gamma\gamma_\mu g_\mp\sigma_\pm\right] \\
& + i\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[(\pm\gamma_\sigma\gamma_\nu g_\pm g_\pm + \gamma_\nu\gamma_\sigma g_\mp g_\mp)(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\right. \\
& \qquad \left. + \gamma_\sigma\gamma_\mu\sigma_\pm\sigma_\pm \pm \gamma_\mu\gamma_\sigma\sigma_\mp\sigma_\mp\right]\Big\}
\end{aligned} \tag{5.30}$$

where we have omitted indices s and t on functions $g_\pm$ and $\sigma_\pm$ with the understanding that the first always carries index s and the second t.

5.2.2 Amplitude for physical quarks

Performing the $5D$ sums yields

$$
\begin{aligned}
\bar{I}_{\pm}^{odd} = \; & \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{\gamma_\alpha\left(g_{\rho\sigma}g_{\nu\alpha}(\widetilde{ap+k})_\rho^2 - 2g_{\rho\nu}g_{\sigma\alpha}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right) \\
& \quad \left[(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\tilde{g}_\mp\tilde{g}_\mp \pm g_{\mu\nu}\tilde{\sigma}_\mp\tilde{\sigma}_\mp\right] \\
& +\gamma_\nu\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\pm(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\tilde{g}_\pm\tilde{g}_\pm + g_{\mu\nu}\tilde{\sigma}_\pm\tilde{\sigma}_\pm\right] \\
& +r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[\gamma_\sigma\bar{k}_\mu(\pm\tilde{\sigma}_\mp\tilde{g}_\pm + \tilde{g}_\mp\tilde{\sigma}_\pm)\right. \\
& \quad \left.+(\bar{k}\cdot\gamma g_{\sigma\mu} - \gamma_\mu\bar{k}_\sigma)(\pm\tilde{\sigma}_\mp\tilde{g}_\pm - \tilde{g}_\mp\tilde{\sigma}_\pm)\right]\Big\}[\gamma_5] \qquad (5.31) \\
\tilde{I}_{\pm}^{odd} = \; & \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{\gamma_\alpha\left(g_{\rho\sigma}g_{\nu\alpha}(\widetilde{ap+k})_\rho^2 - 2g_{\rho\nu}g_{\sigma\alpha}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right) \\
& \quad \left[(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\tilde{g}_\mp\tilde{g}_\pm \pm g_{\mu\nu}\tilde{\sigma}_\mp\tilde{\sigma}_\pm\right] \\
& +\gamma_\nu\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\pm(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\tilde{g}_\pm\tilde{g}_\mp + g_{\mu\nu}\tilde{\sigma}_\pm\tilde{\sigma}_\mp\right] \\
& +\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[\gamma_\sigma\tilde{\sigma}_\mp\tilde{g}_\mp(\bar{k}\cdot\gamma\gamma_\mu \pm \gamma_\mu\bar{k}\cdot\gamma)\right. \\
& \quad \left.+\tilde{\sigma}_\pm\tilde{g}_\pm(\bar{k}\cdot\gamma\gamma_\mu \pm \gamma_\mu\bar{k}\cdot\gamma)\gamma_\sigma\right]\Big\}[\gamma_5] \qquad (5.32) \\
\bar{I}_{\pm}^{even} = \; & \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{ig_{\rho\sigma}(\widetilde{ap+k})_\rho^2\left[\gamma_\mu\bar{k}\cdot\gamma\tilde{\sigma}_\pm\tilde{g}_\mp \pm \bar{k}\cdot\gamma\gamma_\mu\tilde{g}_\pm\tilde{\sigma}_\mp\right] \\
& -2i(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\gamma_\rho(\gamma_\mu\bar{k}_\sigma - \bar{k}\cdot\gamma g_{\sigma\mu})(\tilde{\sigma}_\pm\tilde{g}_\mp \pm \tilde{g}_\pm\tilde{\sigma}_\mp) \\
& -i\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma(\pm\gamma_\mu\bar{k}\cdot\gamma\tilde{\sigma}_\mp\tilde{g}_\pm + \bar{k}\cdot\gamma\gamma_\mu\tilde{g}_\mp\tilde{\sigma}_\pm) \\
& +i\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[(\pm\gamma_\sigma\gamma_\nu\tilde{g}_\pm\tilde{g}_\pm + \gamma_\nu\gamma_\sigma\tilde{g}_\mp\tilde{g}_\mp)(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\right. \\
& \quad \left.+\gamma_\sigma\gamma_\mu\tilde{\sigma}_\pm\tilde{\sigma}_\pm \pm \gamma_\mu\gamma_\sigma\tilde{\sigma}_\mp\tilde{\sigma}_\mp\right]\Big\}[\gamma_5] \qquad (5.33) \\
\tilde{I}_{\pm}^{even} = \; & \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{ig_{\rho\sigma}(\widetilde{ap+k})_\rho^2(\gamma_\mu\bar{k}\cdot\gamma \pm \bar{k}\cdot\gamma\gamma_\mu)\tilde{g}_\pm\tilde{\sigma}_\pm \\
& -2i(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\gamma_\rho(\gamma_\mu\bar{k}_\sigma - \bar{k}\cdot\gamma g_{\sigma\mu})(\tilde{\sigma}_\pm\tilde{g}_\pm \pm \tilde{g}_\pm\tilde{\sigma}_\pm) \\
& -i\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma(\bar{k}\cdot\gamma\gamma_\mu \pm \gamma_\mu\bar{k}\cdot\gamma)\tilde{\sigma}_\mp\tilde{g}_\mp \\
& +i\frac{r}{2}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma(\gamma_\nu\gamma_\sigma \pm \gamma_\sigma\gamma_\nu)\left[\tilde{g}_\pm\tilde{g}_\mp(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\right. \\
& \quad \left.\pm g_{\mu\nu}\tilde{\sigma}_\pm\tilde{\sigma}_\mp\right]\Big\}[\gamma_5]\,. \qquad (5.34)
\end{aligned}
$$

For quark currents $\bar{q}\gamma_\mu[\gamma_5]q$, we evaluate the amplitude at zero external momentum to get $I_q = \bar{I}^+_{odd} + \tilde{I}^-_{even}$. It's easy to see that $\tilde{I}^-_{even}$ vanishes for $p \to 0$ since it's an odd function of k_μ so the physical amplitude is

$$
\begin{aligned}
I^\mu_{V,A} &= \bar{I}^{odd}_+(p=0) = \frac{H_{\rho\sigma}}{\hat{k}^2+\mu^2}\Big\{\gamma_\alpha\left(g_{\rho\sigma}g_{\nu\alpha}\tilde{k}^2_\rho - 2g_{\rho\nu}g_{\sigma\alpha}\tilde{k}_\rho\tilde{k}_\sigma\right) \\
&\qquad \left[(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\tilde{g}_-\tilde{g}_- \underline{\pm g_{\mu\nu}\tilde{\sigma}_-\tilde{\sigma}_-}\right] \\
&\quad +\gamma_\nu\frac{r^2}{4}\hat{k}_\rho\hat{k}_\sigma\left[\underline{\pm(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\tilde{g}_+\tilde{g}_+} + g_{\mu\nu}\tilde{\sigma}_+\tilde{\sigma}_+\right] \\
&\quad +r\hat{k}_\rho\tilde{k}_\sigma\left[\gamma_\sigma\bar{k}\mu(\underline{\pm\tilde{\sigma}_-\tilde{g}_+} + \tilde{g}_-\tilde{\sigma}_+)\right. \\
&\qquad \left.+(\bar{k}\cdot\gamma g_{\sigma\mu} - \gamma_\mu\bar{k}_\sigma)(\underline{\pm\tilde{\sigma}_-\tilde{g}_+} - \tilde{g}_-\tilde{\sigma}_+)\right]\Big\}[\gamma_5], \qquad (5.35)
\end{aligned}
$$

where underlined terms vanish for *DW* fermions since $\tilde{\sigma}_- = \tilde{g}_+ = 0$ in the $m \to 0$ limit. Using the decomposition of the smearing vector

$$
H_{\rho\sigma} = g_{\rho\sigma}A_\rho + \hat{k}_\rho\hat{k}_\sigma A_{\rho\sigma} \qquad (5.36)
$$

we get

$$
\begin{aligned}
I^\mu_{V,A} &= \sum_\rho \frac{\gamma_\mu A_\rho}{\hat{k}^2+\mu^2}\Big\{\tilde{k}^2_\rho\left[\frac{d-4}{d}\bar{k}^2\tilde{g}^2_- \pm \frac{d-2}{d}\tilde{\sigma}^2_-\right] + \frac{4}{d}\tilde{k}^2_\rho\bar{k}^2_\rho\tilde{g}^2_- + \frac{r^2}{4}\hat{k}^2_\rho\left[\pm\frac{d-2}{d}\bar{k}^2\tilde{g}^2_+ + \tilde{\sigma}^2_+\right] \\
&\quad +r\bar{k}^2_\rho\left[\tilde{g}_-\tilde{\sigma}_+ \pm \frac{2-d}{d}\tilde{g}_+\tilde{\sigma}_-\right]\Big\}[\gamma_5] \\
&\quad +\sum_{\rho\sigma}\frac{\gamma_\mu A_{\rho\sigma}}{\hat{k}^2+\mu^2}\Big\{g_{\rho\sigma}\tilde{k}^2_\rho\left[\bar{k}^2\frac{d-4}{d}\tilde{g}^2_- \pm \frac{d-2}{d}\tilde{\sigma}^2_-\right] + \frac{4}{d}\tilde{k}^2_\rho\tilde{k}^2_\sigma\tilde{g}^2_- \\
&\quad +\frac{r^2}{4}\hat{k}^2_\rho\hat{k}^2_\sigma\left[\tilde{\sigma}^2_+ \pm \frac{d-2}{d}\bar{k}^2\tilde{g}^2_+\right] + r\hat{k}^2_\rho\bar{k}^2_\sigma\left[\tilde{g}_-\tilde{\sigma}_+ \pm \frac{2-d}{d}\tilde{\sigma}_-\tilde{g}_+\right]\Big\}[\gamma_5]. \qquad (5.37)
\end{aligned}
$$

Another way to get the same result (which will also be useful for twist 2 operators) is to project out the component proportional to γ_μ

$$
\begin{aligned}
I^\mu_{V,A}(p) &= \frac{1}{d}\mathrm{Tr}_D\left\{\bar{I}^{odd}_+[\gamma_5]\gamma_\mu\right\} \\
&= \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{g_{\mu\alpha}\left(g_{\rho\sigma}g_{\nu\alpha}(\widetilde{ap+k})^2_\rho - 2g_{\rho\nu}g_{\sigma\alpha}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right) \\
&\qquad \left[(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\tilde{g}^2_- \pm g_{\mu\nu}\tilde{\sigma}^2_-\right] \\
&\quad +g_{\mu\nu}\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\pm(\bar{k}^2 g_{\mu\nu} - 2\bar{k}_\mu\bar{k}_\nu)\tilde{g}^2_+ + g_{\mu\nu}\tilde{\sigma}^2_+\right]
\end{aligned}
$$

$$
\begin{aligned}
&+r(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[g_{\sigma\mu}\bar{k}_\mu(\pm\tilde{\sigma}_-\tilde{g}_+ + \tilde{g}_-\tilde{\sigma}_+)\right.\\
&\qquad\left.\left. +(\bar{k}_\mu g_{\sigma\mu} - g_{\mu\mu}\bar{k}_\sigma)(\pm\tilde{\sigma}_-\tilde{g}_+ - \tilde{g}_-\tilde{\sigma}_+)\right]\right\}[\gamma_5]\\
= \;& \frac{1}{(\widehat{ap-k})^2+\mu^2}\left\{H_{\rho\sigma}g_{\rho\sigma}(\widetilde{ap+k})_\rho^2\left[(\bar{k}^2-2\bar{k}_\mu^2)\tilde{g}_-^2 \pm\tilde{\sigma}_-^2\right]\right.\\
&-2H_{\mu\mu}(\widetilde{ap+k})_\mu^2\left[\bar{k}^2\tilde{g}_-^2 \pm\tilde{\sigma}_-^2\right] + 4H_{\rho\mu}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\mu\bar{k}_\rho\bar{k}_\mu\tilde{g}_-^2\\
&+\frac{r^2}{4}H_{\rho\sigma}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\tilde{\sigma}_+^2 \pm(\bar{k}^2-2\bar{k}_\mu^2)\tilde{g}_+^2\right]\\
&+r(\widehat{ap+k})_\rho\Big(H_{\rho\mu}(\widetilde{ap+k})_\mu\bar{k}_\mu(\pm 2g_+\sigma_-)\\
&\qquad\left.+H_{\rho\sigma}(\widetilde{ap+k})_\sigma\bar{k}_\sigma(g_-\sigma_+ \mp g_+\sigma_-)\Big)\right\}[\gamma_5] \qquad (5.38)
\end{aligned}
$$

This result can be used to evaluate the current renormalization with both regularization schemes (finite $\log p^2$ or finite $\log\mu^2$).

5.2.3 No-smearing limit

To compare results with Aoki, we can also take the no-smearing limit by replacing $A_\rho \to 1$ and $A_{\rho\sigma} \to 0$ to get

$$
I_{V,A}^{DW,NOS} = \frac{\gamma_\mu}{\hat{k}^2+\mu^2}\left\{\frac{4}{d}\sum_\rho \bar{k}_\rho^2\bar{k}_\rho^2\tilde{g}_-^2 + \frac{r^2}{4}\hat{k}^2\tilde{\sigma}_+^2 + r\bar{k}^2\tilde{g}_-\tilde{\sigma}_+\right\}[\gamma_5] \qquad (5.39)
$$

which agrees with Aoki's [16] formula (4.5) and (4.6).

5.3 Tensor current

For the $\bar{q}(x)\sigma_{\mu\nu}[\gamma_5]q(x)$ current, everything works the same except that the Dirac algebra is slightly more complicated. The physical amplitude is then obtained after summing in the 5^{th} dimension

$$
\begin{aligned}
I_q(p) &= \bar{S}_s^{OUT} I_{st}\bar{S}_t^{IN}\\
&= \left[-ip\cdot\gamma\mathcal{A}\left(w_0^{N-s}P_- + w_0^{s-1}P_+\right) + \left(w_0^{s-1}P_- + w_0^{N-s}P_+\right)\right]_s\\
&\quad\times[I_{st}^+P_+ + I_{st}^-P_-][\gamma_5]
\end{aligned}
$$

$$
\begin{aligned}
& \times\left[\left(w_0^{N-t}P_+ + w_0^{t-1}P_-\right)(-ip\cdot\gamma\mathcal{A}) + \left(w_0^{t-1}P_+ + w_0^{N-t}P_-\right)\right]_t . \\
= \; & I_{phys}^{odd} + I_{phys}^{even} && (5.40)
\end{aligned}
$$

$$
\begin{aligned}
I_{phys}^{odd} \;=\; & (-ip\cdot\gamma\mathcal{A})\bar{I}_{odd}^{-}[\gamma_5](-ip\cdot\gamma\mathcal{A}) + \bar{I}_{odd}^{+}[\gamma_5] \\
& +(-ip\cdot\gamma\mathcal{A})\tilde{I}_{odd}^{-}[\gamma_5] + \tilde{I}_{odd}^{+}[\gamma_5](-ip\cdot\gamma\mathcal{A}) && (5.41) \\
I_{phys}^{even} \;=\; & (-ip\cdot\gamma\mathcal{A})\tilde{I}_{even}^{+}[\gamma_5](-ip\cdot\gamma\mathcal{A}) + \tilde{I}_{even}^{-}[\gamma_5] \\
& +(-ip\cdot\gamma\mathcal{A})\bar{I}_{even}^{+}[\gamma_5] + \bar{I}_{even}^{-}[\gamma_5](-ip\cdot\gamma\mathcal{A}) && (5.42)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{I}^{\pm} \;&\equiv\; \sum w_0^{s-1} I^{\pm} w_0^{t-1} \equiv \sum w_0^{N-s} I^{\mp} w_0^{N-t} , && (5.43) \\
\tilde{I}^{\pm} \;&\equiv\; \sum w_0^{s-1} I^{\pm} w_0^{N-t} \equiv \sum w_0^{N-s} I^{\mp} w_0^{t-1} && (5.44)
\end{aligned}
$$

and we have used the fact

$$
I_{odd}P_{\pm} = P_{\mp}I_{odd} , \qquad I_{even}P_{\pm} = P_{\pm}I_{even} . \tag{5.45}
$$

To get the physical amplitude for we evaluate this at $p=0$ so we are left with $I_q = \bar{I}_{odd}^{+} + \tilde{I}_{even}^{-}$. For $p\to 0$, $\bar{I}_{odd}^{+}$ vanishes since it's an odd function of k_μ, so the physical amplitude is given by $\tilde{I}_{-}^{even}$

$$
\begin{aligned}
I_T \;=\; & \frac{H_{\rho\sigma}}{(\widehat{ap-k})^2+\mu^2}\Big\{\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left(\sigma_{\mu\nu}\tilde{\sigma}_+^2[\mp][\bar{k}^2\sigma_{\mu\nu} - 2\bar{k}_\mu\sigma_{k\nu} + 2\bar{k}_\nu\sigma_{k\mu}]\tilde{g}_+^2\right) \\
& + \left[g_{\rho\sigma}(\widetilde{ap+k})_\rho^2\sigma_{\mu\nu} + 2(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma(g_{\rho\nu}\sigma_{\sigma\mu} - g_{\rho\mu}\sigma_{\sigma\nu})\right](\bar{k}^2\tilde{g}_-^2[\mp]\tilde{\sigma}_-^2) \\
& +2\Big(g_{\rho\sigma}(\widetilde{ap+k})_\rho^2(\bar{k}_\nu\sigma_{k\mu} - \bar{k}_\mu\sigma_{k\nu}) + 2(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma \\
& \qquad \times[\sigma_{\sigma k}(\bar{k}_\nu g_{\rho\mu} - \bar{k}_\mu g_{\rho\nu}) - \bar{k}_\rho(\bar{k}_\nu\sigma_{\sigma\mu} - \bar{k}_\mu\sigma_{\sigma\nu})]\Big]\tilde{g}_-^2 \\
& +r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[(\tilde{g}_-\tilde{\sigma}_+[\pm]g_+\sigma_-)(\bar{k}_\sigma\sigma_{\mu\nu} + g_{\sigma\nu}\sigma_{k\mu} - g_{\sigma\mu}\sigma_{k\nu})\right. \\
& \qquad \left.+(\tilde{g}_-\tilde{\sigma}_+[\mp]g_+\sigma_-)(\bar{k}_\mu\sigma_{\sigma\nu} - \bar{k}_\nu\sigma_{\sigma\mu})\right]\Big\}[\gamma_5] , && (5.46)
\end{aligned}
$$

where

$$
\sigma_{k\mu} \equiv \sum_\alpha \bar{k}_\alpha\sigma_{\alpha\mu} . \tag{5.47}
$$

To extract the $\sigma_{\alpha\beta}$ component, we multiply by $\sigma_{\alpha\beta}$ and take a trace; the result is then obtained by using the fact that

$$\frac{1}{d}\mathrm{Tr}_D[\sigma_{\mu\nu}\sigma_{\alpha\beta}] = g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\,. \tag{5.48}$$

The final formula is then obtained by replacing $\sigma_{xy} \to g_{x\alpha}g_{y\beta} - g_{x\beta}g_{y\alpha}$ for all x, y in formula (5.46) and will be omitted here.

5.3.1 No-smearing limit

For the no-smearing limit, we replace $H_{\rho\sigma}$ by $g_{\rho\sigma}$ in the $p \to 0$ limit to get

$$\begin{aligned} I_T &= \frac{1}{\hat{k}^2+\mu^2}\Big\{(\bar{k}^2 - 2\bar{k}_\alpha^2 - 2\bar{k}_\beta^2)\sigma_{\alpha\beta} \\ &\qquad \times \left(\left[\bar{k}^2 g_{\mu\alpha}g_{\nu\beta} + 2\bar{k}_\alpha\bar{k}_\nu g_{\mu\beta} - 2\bar{k}_\alpha\bar{k}_\mu g_{\nu\beta}\right]\tilde{g}_-^2 - g_{\mu\alpha}g_{\nu\beta}\tilde{\sigma}_-^2\right) \\ &\quad -\frac{r^2}{4}\hat{k}^2\sigma_{\alpha\beta}\left(\left[\bar{k}^2 g_{\mu\alpha}g_{\nu\beta} + 2\bar{k}_\alpha\bar{k}_\nu g_{\mu\beta} - 2\bar{k}_\alpha\bar{k}_\mu g_{\nu\beta}\right]\tilde{g}_+^2 - g_{\mu\alpha}g_{\nu\beta}\tilde{\sigma}_+^2\right) \\ &\quad + r\sum_\rho \left[\bar{k}_\rho\left(g_{\rho\mu}\sigma_{\nu k} - g_{\rho\nu}\sigma_{\mu k} + \bar{k}_\rho\sigma_{\mu\nu}\right)(\tilde{g}_-\tilde{\sigma}_+ + \tilde{g}_+\tilde{\sigma}_-)\right. \\ &\quad \left. + \left(\bar{k}_\mu\sigma_{\nu\rho} - \bar{k}_\nu\sigma_{\mu\rho}\right)(\tilde{g}_-\tilde{\sigma}_+ - \tilde{g}_+\tilde{\sigma}_-)\right]\Big\}\,. \end{aligned} \tag{5.49}$$

Using the fact that due to parity

$$\int \bar{k}_\mu\bar{k}_\nu f(k^2) = \int \bar{k}^2 g_{\mu\nu} f(k^2) \tag{5.50}$$

and that for domain wall fermions $\tilde{g}_+, \tilde{\sigma}_- \to 0$ we get the result

$$I_T = \frac{\sigma_{\mu\nu}}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}^2\tilde{\sigma}_+^2 + \left[4(\bar{k}_\mu^2\bar{k}_\mu^2 + \bar{k}_\nu^2\bar{k}_\nu^2) + 4(\bar{k}_\mu^2\bar{k}_\nu^2 + \bar{k}_\nu^2\bar{k}_\mu^2)\right]\tilde{g}_-^2 + r\bar{k}^2\tilde{g}_-\tilde{\sigma}_+\right\}\,. \tag{5.51}$$

Since $\mu \neq \nu$, all integrals with $\bar{k}_\mu^2\bar{k}_\nu^2$ are the same so we can use the identity

$$\bar{k}^2\bar{k}^2 = \sum_{\alpha\beta}\bar{k}_\alpha^2\bar{k}_\beta^2 = \sum_\alpha \bar{k}_\alpha^2\bar{k}_\alpha^2 + \sum_{\alpha\neq\beta}\bar{k}_\alpha^2\bar{k}_\beta^2 \tag{5.52}$$

to simplify the coefficient of $\tilde{g}_-^2$ to get

$$I_T = \frac{\sigma_{\mu\nu}}{\hat{k}^2+\mu^2}\left\{\frac{r^2}{4}\hat{k}^2\tilde{\sigma}_+^2 + r\bar{k}^2\tilde{g}_-\tilde{\sigma}_+ + \frac{\tilde{g}_-^2}{3}\left[4\sum_\rho \hat{k}_\rho^2\bar{k}_\rho^2 - \hat{k}^2\bar{k}^2\right]\right\}. \quad (5.53)$$

which agrees with Aoki's [16] formula (4.5) and (4.6).

5.4 Collecting results: renormalization coefficients for currents

At tree level, quark bilinear operators are is given by

$$O_\Gamma = \bar{q}(x)\Gamma q(x)\,, \qquad \text{with} \qquad \Gamma = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}\,. \quad (5.54)$$

We evaluate the one loop correction to the Green's function $\langle O_\Gamma(x) q y \bar{q}(z)\rangle$ in the limit $p, p' \to 0$

$$\langle O_\Gamma q\bar{q}\rangle_{full} = S_q(p)\Gamma\left(1+\frac{g^2 C_F}{16\pi^2}\left[\gamma_\Gamma \log\mu^2 a^2 + \Sigma_\Gamma\right]\right)S_q(p') \quad (5.55)$$

After truncating the external propagators we are left with

$$\langle O_\Gamma q\bar{q}\rangle_1 = \Gamma\left(1+\frac{g^2 C_F}{16\pi^2}\left[\gamma_\Gamma \log\mu^2 a^2 + \Sigma_\Gamma\right]\right) \quad (5.56)$$

which yields the renormalization of the O_γ operator

$$\begin{aligned} Z_\Gamma &= Z_2\left(1+\frac{g^2 C_F}{16\pi^2}\left[\gamma_\Gamma \log\mu^2 a^2 + \Sigma_\Gamma\right]\right) \\ &= \left(1+\frac{g^2 C_F}{16\pi^2}\left[(\gamma_\Gamma+\gamma_2)\log\mu^2 a^2 + \Sigma_\Gamma + \Sigma_2\right]\right) \end{aligned} \quad (5.57)$$

5.5 Comparison with nonperturbative results

Nonperturbative renormalization coefficients for DW fermions has been performed in [13] for the self energy renormalization and for scalar, tensor and axial current. Due to the

M	"no smearing"	HYP	APE	"gauge-part"
Wilson S	2.10001	2.70835	0.83098	4.79201
Wilson P	11.74304	5.00879	1.71069	4.79201
0.1	3.82195	1.65308	-0.26608	4.79201
0.2	4.60698	2.29055	0.23749	4.79201
0.3	5.24244	2.77267	0.58900	4.79201
0.4	5.80199	3.17245	0.86082	4.79201
0.5	6.31849	3.52262	1.08636	4.79201
0.6	6.80737	3.83592	1.27373	4.79201
0.7	7.28339	4.12794	1.44170	4.79201
0.8	7.75486	4.40518	1.59462	4.79201
0.9	8.22978	4.67491	1.74013	4.79201
1	8.71505	4.94040	1.87634	4.79201
1.1	9.21885	5.20976	2.01405	4.79201
1.2	9.74961	5.48830	2.15560	4.79201
1.3	10.31763	5.78331	2.30620	4.79201
1.4	10.93603	6.10413	2.47257	4.79201
1.5	11.62251	6.46369	2.66413	4.79201
1.6	12.40243	6.88117	2.89538	4.79201
1.7	13.31495	7.38768	3.19082	4.79201
1.8	14.42740	8.03967	3.59741	4.79201
1.9	15.87996	8.96229	4.22593	4.79201

Table 5.1: Renormalization coefficient $\Sigma_{S,P}$ for vector and axial vector current renormalization.

M	"no smearing"	HYP	APE	"gauge-part"
Wilson V	7.26539	5.37227	5.01279	4.79201
Wilson A	2.44388	4.22206	4.57294	4.79201
0.1	4.83546	4.79555	4.79261	4.79201
0.2	4.83801	4.79579	4.79265	4.79201
0.3	4.84080	4.79606	4.79270	4.79201
0.4	4.84385	4.79634	4.79275	4.79201
0.5	4.84719	4.79665	4.79280	4.79201
0.6	4.85087	4.79698	4.79286	4.79201
0.7	4.85493	4.79734	4.79291	4.79201
0.8	4.85942	4.79772	4.79297	4.79201
0.9	4.86440	4.79814	4.79306	4.79201
1	4.86995	4.79859	4.79310	4.79201
1.1	4.87616	4.79907	4.79315	4.79201
1.2	4.88313	4.79960	4.79321	4.79201
1.3	4.89101	4.80017	4.79328	4.79201
1.4	4.89995	4.80079	4.79336	4.79201
1.5	4.91017	4.80147	4.79344	4.79201
1.6	4.92194	4.80223	4.79353	4.79201
1.7	4.93559	4.80307	4.79362	4.79201
1.8	4.95159	4.80401	4.79374	4.79201
1.9	4.97055	4.80507	4.79387	4.79201

Table 5.2: Renormalization coefficient $\Sigma_{V,A}$ for vector and axial vector current renormalization.

M	"no smearing"	HYP	APE	"gauge-part"
Wilson	4.16567	5.11003	5.96688	5.79201
0.1	5.17330	5.84304	6.47885	5.79201
0.2	4.91503	5.63088	6.31105	5.79201
0.3	4.70691	5.47052	6.19394	5.79201
0.4	4.52447	5.33764	6.10340	5.79201
0.5	4.35706	5.22183	6.02922	5.79201
0.6	4.19871	5.11733	5.96584	5.79201
0.7	4.04544	5.02047	5.90998	5.79201
0.8	3.89427	4.92857	5.85908	5.79201
0.9	3.74266	4.83949	5.81138	5.79201
1	3.58825	4.75132	5.76533	5.79201
1.1	3.42860	4.66218	5.71951	5.79201
1.2	3.26098	4.57003	5.67242	5.79201
1.3	3.08214	4.47245	5.62231	5.79201
1.4	2.88793	4.36634	5.56695	5.79201
1.5	2.67273	4.24740	5.50321	5.79201
1.6	2.42844	4.10924	5.42624	5.79201
1.7	2.14246	3.94153	5.32789	5.79201
1.8	1.79298	3.72545	5.19251	5.79201
1.9	1.33408	3.41932	4.98317	5.79201

Table 5.3: Renormalization coefficient Σ_T for tensor current renormalization.

different definition of their field q

$$q(x) = P_+\psi_1(x) + P_-\psi_N(x) \tag{5.58}$$

and my definition

$$q(x) = \sqrt{1-w_0^2}\,(P_+\psi_1(x) + P_-\psi_N(x))\ , \tag{5.59}$$

numbers taken from reference [13] have been multiplied with $1/\sqrt{1-w_0^2}$ to account for the difference in notation. Full renormalization coefficients from $\overline{MS}$ to lattice for quark currents, evaluated at $M = 1.8$ are given in the table (5.4). Another comparison that can be

	perturbative	non-perturbative
Z_S	0.79	0.78 ± 0.04
Z_A	0.85	0.93 ± 0.02
Z_T	0.88	1.04 ± 0.11

Table 5.4: Comparison of perturbative and nonperturbative results for quark currents

made is the full lattice wave function renormalization coefficient Z_q. Using two different methods, authors of [13] get

$$\begin{aligned} Z_q^{Ward} &= 2.08 \pm 0.14 \\ Z_q^{hadronic} &= 2.25 \pm 0.06 \end{aligned} \tag{5.60}$$

In the $Z_q^{hadronic}$ they have used the ratio of exactly conserved non-local current and non-conserved local current

$$\frac{\langle \mathcal{A}_\mu(x)\ \bar{q}\gamma_5 q(y)\rangle}{\langle A_\mu(x)\ \bar{q}\gamma_5 q(y)\rangle} = Z_A Z_q = 2.0925 \pm 0.0014 \tag{5.61}$$

to evaluate $Z_q^{hadronic}$ using the non-perturbative value of Z_A. If we repeat their procedure with the perturbative value of Z_A, we get

$$Z_q^{hadronic} = 2.462 \pm 0.002\,, \tag{5.62}$$

which compares with the perturbative value

$$Z_q^{pert} = 2.61 . \tag{5.63}$$

The error in the non-perturbative expression comes completely from the statistical error in $Z_q Z_A$ product and does not account for any systematic error we have introduced by assuming Z_A can be accurately evaluated using the perturbation theory.

As it was noted before, in our evaluation of the perturbative value for Z_W, we have made the assumption that the value of Z_W can be explained as coming from the additive renormalization of w_0. While that is certainly true to the order g_0^2, there is no *a priori* argument that would guarantee that it is true to all orders in g_0. However, since this shift is the same for all matrix elements and operators we consider, we can evaluate it once from the ratio of axial currents and then use the result (5.62) to renormalized all remaining matrix elements.

Chapter 6

Twist 2 operators with $\gamma_\mu[\gamma_5]$ and one derivative

Since domain wall fermions are chiraly invariant, all results presented are the same with or without the γ_5 matrix present. Whenever the numbers are shown for Wilson fermions, they will be calculated without the γ_5 matrix in the operator.

Here we build upon previous two chapters for current and self energy renormalization to calculate renormalization coefficients for twist 2 operators with 1 derivative. While the vertex diagram is very similar to the vertex diagram for local currents, a new feature that appears here for the first time is the sails diagrams.

6.1 Preliminaries

Before we evaluate twist 2 amplitude, we need to do some preliminary work.

6.1.1 Operator vertex

To evaluate twist 2 diagrams, we need to evaluate the vertex for the operator $O_{\mu\nu} = \bar{q}(x)\gamma_\mu D_\nu q(x)$. The derivative operator D_ν on the lattice contains all powers of the gluon field A_μ field so we expand it in powers of g_0. For 1-loop corrections we need only terms up to order g_0^2

$$O_{\mu\nu} = O_{\mu\nu}^{(0)} + g_0 O_{\mu\nu}^{(1)} + g_0^2 O_{\mu\nu}^{(2)} \tag{6.1}$$

Details of the expansion can be found in [17]. The result is

$$O_{\mu\nu}^{(0)} = i\gamma_\mu \bar{k}_\nu \tag{6.2}$$

$$O_{\mu\nu}^{(1)} = T^a i\gamma_\mu \cos\frac{(ap+k)_\nu}{2} \tag{6.3}$$

$$O_{\mu\nu}^{(2)} = -\frac{a^2}{2}\{T^a, T^b\}\bar{p}_\nu \to -\frac{a^2}{2}C_F\gamma_\mu\bar{p}_\nu\,, \tag{6.4}$$

where in the last step we have performed the summation over the group index

$$\mathrm{Tr}\sum_a \delta^{ab}\{T^a, T^b\} = \frac{N_c^2-1}{2N_c} = C_F \tag{6.5}$$

The 0^{th} order $O_{\mu\nu}^{(0)}$ contributes to the vertex diagram; the 1^{st} order contributes to sails diagrams, while the second order contributes to the tadpole diagram.

6.1.2 Amplitude decomposition

The general structure of a particular 1-loop twist-2 diagram for a twist 2 operator is [5]

$$I_{\mu\nu} = \langle q(p)\,|\gamma_\mu D_\nu|\,q(p)\rangle = c_1\gamma_\mu p_\nu + c_2\gamma_\nu p_\mu + c_3 g_{\mu\nu}\gamma_\mu p_\mu + c_4 g_{\mu\nu} p\cdot\gamma + c_5\frac{p_\mu p_\nu}{p^2}p\cdot\gamma \tag{6.6}$$

For operators in the $\mathbf{6}_3^+$ representation, $\mu \neq \nu$ so only terms c_1 and c_2 contribute

$$\langle q(p)\,|\gamma_1 D_4|\,q(p)\rangle = (c_1+c_2)\frac{\gamma_1 p_4 + \gamma_4 p_1}{2} \tag{6.7}$$

On the other hand, for representation $\mathbf{3}_1^+$, c_3 will contribute as well

$$\left\langle q(p)\left|\left[\gamma_4 D_4 - \frac{1}{3}(\gamma_1 D_1 + \gamma_2 D_2 + \gamma_3 D_3)\right]\right|q(p)\right\rangle$$
$$= (c_1+c_2+c_3)\left[\gamma_4 p_4 - \frac{1}{3}(\gamma_1 p_1 + \gamma_2 p_2 + \gamma_3 p_3)\right] \tag{6.8}$$

We can see that the term proportional to $p\cdot\gamma g_{\mu\nu}$ does not contribute so we want to eliminate it. To extract coefficients c_i in the case $\mu \neq \nu$, we multiply the amplitude with γ_α and take a trace to get

$$\frac{1}{d}\mathrm{Tr}\left[I_{\mu\nu}\gamma_\alpha\right] = c_1 g_{\mu\alpha}p_\nu + c_2 g_{\nu\alpha}p_\mu + c_3 g_{\mu\nu}g_{\mu\alpha}p_\mu + c_4 g_{\mu\nu}p_\alpha \tag{6.9}$$

For the $\mathbf{6}_3^+$ representation, we choose $\alpha = \mu,\nu$ and add them up

$$\frac{1}{p_4}\frac{1}{d}\mathrm{Tr}[I_{14}\gamma_1] \;=\; c_1 \tag{6.10}$$
$$\frac{1}{p_1}\frac{1}{d}\mathrm{Tr}[I_{14}\gamma_4] \;=\; c_2\,. \tag{6.11}$$

Alternatively, we can take the symmetrized combination $I_{\mu\nu}+I_{\nu\mu}$ to get[1]

$$\frac{1}{p_4}\frac{1}{d}\mathrm{Tr}[(I_{14}+I_{41})\gamma_1] = c_1 + c_2 \tag{6.12}$$

For the $\mathbf{3}_1^+$ representation, we first choose $\mu=\nu=\alpha$ to get

$$\frac{1}{d}\mathrm{Tr}\left[I_{\mu\mu}\gamma_\mu\right] = (c_1+c_2+c_3+c_4)p_\mu\,. \tag{6.13}$$

To eliminate of the c_4 term, note that if we choose $\mu=\nu\neq\alpha$ (for definiteness, let's pick $\mu=4$ and $\alpha=3$), we get

$$\frac{1}{d}\mathrm{Tr}[I_{44}\gamma_4] = c_1 g_{43}p_4 + c_2 g_{43}p_4 + c_3 g_{44}g_{43}p_4 + c_4 g_{44}p_3 = c_4 p_3 \tag{6.14}$$

so dividing by p_α will give us the c_4 coefficient

$$\frac{1}{p_\alpha}\frac{1}{d}\mathrm{Tr}\left[I_{\mu\mu}\gamma_\alpha\right] \;=\; c_4 \tag{6.15}$$
$$\tag{6.16}$$

[1]Here we choose momentum p_μ to have only p_4 component nonzero so the term $p_\mu p_\nu/p^2\, p\cdot\gamma$ does not contribute; if our 4-momentum had both components p_1 and p_4 nonzero, we'd have to subtract $-2\mathrm{Tr}[I_{14}\gamma_\alpha]$ with $\alpha\neq 1,4$ to cancel the extra contribution

So, for $\mathbf{3}_1^+$ representation the final result is

$$c_1+c_2+c_3=\frac{1}{p_\mu}\frac{1}{d}\mathrm{Tr}\left[I_{\mu\mu}(\gamma_\mu-\gamma_\alpha)\right] , \tag{6.17}$$

where we have chosen $\mu\neq\alpha$ and vector p such that components p_μ and p_α numerically equal. For the example above it would be

$$p_\mu=\left\{0,0,\sqrt{\frac{p^2}{2}},\sqrt{\frac{p^2}{2}}\right\} \tag{6.18}$$

For the operator $\bar{q}\gamma_\mu\gamma_5 D_\nu q$ we multiply by $\gamma_5\gamma_\alpha$ instead of γ_α.

6.2 Vertex diagram

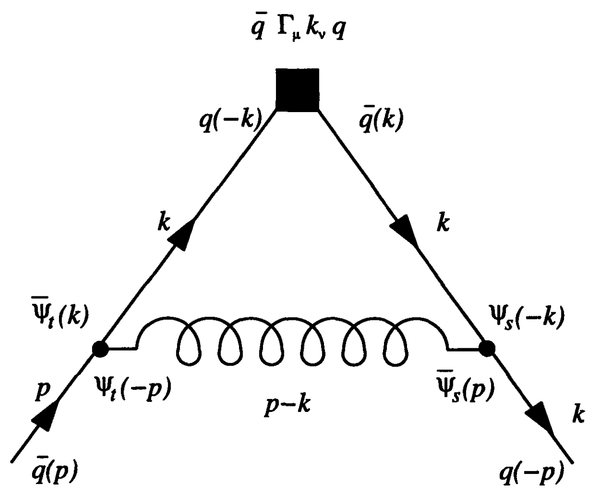

Figure 6-1: Vertex diagram for twist 2 operators

As it was noted in chapter (5), the vertex diagram for twist 2 operators is very similar to the vertex diagram for current renormalizations. Since the only difference is the vertex for the operator $\bar{q}\gamma_\mu D_\nu q$, and it only differs by additional 4-momentum $i\bar{k}_\mu$, the Dirac algebra and group algebra are the same so we can copy the expression for the amplitude (5.31)-(5.34)

$$\bar{I}_\pm^{odd} = \frac{H_{\rho\sigma}\, i\bar{k}_\nu}{(\widehat{ap-k})^2+\mu^2}\left\{\gamma_\alpha\left(g_{\rho\sigma}g_{\lambda\alpha}(\widetilde{ap+k})^2_\rho-2g_{\rho\lambda}g_{\sigma\alpha}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right)\right.$$

$$
\begin{aligned}
& \qquad [(\bar{k}^2 g_{\mu\lambda} - 2\bar{k}_\mu \bar{k}_\lambda)\tilde{g}_\mp \tilde{g}_\mp \pm g_{\mu\lambda}\tilde{\sigma}_\mp \tilde{\sigma}_\mp] \\
& + \gamma_\lambda \frac{r^2}{4}(\widehat{ap+k})_\rho (\widehat{ap+k})_\sigma \left[\pm(\bar{k}^2 g_{\mu\lambda} - 2\bar{k}_\mu \bar{k}_\lambda)\tilde{g}_\pm \tilde{g}_\pm + g_{\mu\lambda}\tilde{\sigma}_\pm \tilde{\sigma}_\pm\right] \\
& + r(\widehat{ap+k})_\rho (\widetilde{ap+k})_\sigma \left[\gamma_\sigma \bar{k}_\mu (\pm\tilde{\sigma}_\mp \tilde{g}_\pm + \tilde{g}_\mp \tilde{\sigma}_\pm)\right. \\
& \qquad \left. + (\bar{k}\cdot\gamma g_{\sigma\mu} - \gamma_\mu \bar{k}_\sigma)(\pm\tilde{\sigma}_\mp \tilde{g}_\pm - \tilde{g}_\mp \tilde{\sigma}_\pm)\right]\Big\}[\gamma_5] \qquad (6.19)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}_\pm^{odd} = & \frac{H_{\rho\sigma} i\bar{k}_\nu}{(\widehat{ap-k})^2 + \mu^2}\Big\{\gamma_\alpha \left(g_{\rho\sigma} g_{\lambda\alpha}(\widetilde{ap+k})^2_\rho - 2 g_{\rho\lambda} g_{\sigma\alpha}(\widetilde{ap+k})_\rho (\widetilde{ap+k})_\sigma\right) \\
& \qquad [(\bar{k}^2 g_{\mu\lambda} - 2\bar{k}_\mu \bar{k}_\lambda)\tilde{g}_\mp \tilde{g}_\pm \pm g_{\mu\lambda}\tilde{\sigma}_\mp \tilde{\sigma}_\pm] \\
& + \gamma_\lambda \frac{r^2}{4}(\widehat{ap+k})_\rho (\widehat{ap+k})_\sigma \left[\pm(\bar{k}^2 g_{\mu\lambda} - 2\bar{k}_\mu \bar{k}_\lambda)\tilde{g}_\pm \tilde{g}_\mp + g_{\mu\lambda}\tilde{\sigma}_\pm \tilde{\sigma}_\mp\right] \\
& + \frac{r}{2}(\widehat{ap+k})_\rho (\widetilde{ap+k})_\sigma \left[\gamma_\sigma \tilde{\sigma}_\mp \tilde{g}_\mp (\bar{k}\cdot\gamma\gamma_\mu \pm \gamma_\mu \bar{k}\cdot\gamma)\right. \\
& \qquad \left. + \tilde{\sigma}_\pm \tilde{g}_\pm (\bar{k}\cdot\gamma\gamma_\mu \pm \gamma_\mu \bar{k}\cdot\gamma)\gamma_\sigma\right]\Big\}[\gamma_5] \qquad (6.20)
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_\pm^{even} = & \frac{H_{\rho\sigma} i\bar{k}_\nu}{(\widehat{ap-k})^2 + \mu^2}\Big\{i g_{\rho\sigma}(\widetilde{ap+k})^2_\rho \left[\gamma_\mu \bar{k}\cdot\gamma\tilde{\sigma}_\pm \tilde{g}_\mp + \bar{k}\cdot\gamma\gamma_\mu \tilde{g}_\pm \tilde{\sigma}_\mp\right] \\
& - 2i(\widetilde{ap+k})_\rho (\widetilde{ap+k})_\sigma \gamma_\rho (\gamma_\mu \bar{k}_\sigma - \bar{k}\cdot\gamma g_{\sigma\mu})(\tilde{\sigma}_\pm \tilde{g}_\mp \pm \tilde{g}_\pm \tilde{\sigma}_\mp) \\
& - i\frac{r^2}{4}(\widehat{ap+k})_\rho (\widehat{ap+k})_\sigma (\pm\gamma_\mu \bar{k}\cdot\gamma\tilde{\sigma}_\mp \tilde{g}_\pm + \bar{k}\cdot\gamma\gamma_\mu \tilde{g}_\mp \tilde{\sigma}_\pm) \\
& + i\frac{r}{2}(\widehat{ap+k})_\rho (\widetilde{ap+k})_\sigma \left[(\pm\gamma_\sigma \gamma_\lambda \tilde{g}_\pm \tilde{g}_\pm + \gamma_\lambda \gamma_\sigma \tilde{g}_\mp \tilde{g}_\mp)(\bar{k}^2 g_{\mu\lambda} - 2\bar{k}_\mu \bar{k}_\lambda)\right. \\
& \qquad \left. + \gamma_\sigma \gamma_\mu \tilde{\sigma}_\pm \tilde{\sigma}_\pm \pm \gamma_\mu \gamma_\sigma \tilde{\sigma}_\mp \tilde{\sigma}_\mp\right]\Big\}[\gamma_5] \qquad (6.21)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}_\pm^{even} = & \frac{H_{\rho\sigma} i\bar{k}_\nu}{(\widehat{ap-k})^2 + \mu^2}\Big\{i g_{\rho\sigma}(\widetilde{ap+k})^2_\rho (\gamma_\mu \bar{k}\cdot\gamma \pm \bar{k}\cdot\gamma\gamma_\mu)\tilde{g}_\pm \tilde{\sigma}_\pm \\
& - 2i(\widetilde{ap+k})_\rho (\widetilde{ap+k})_\sigma \gamma_\rho (\gamma_\mu \bar{k}_\sigma - \bar{k}\cdot\gamma g_{\sigma\mu})(\tilde{\sigma}_\pm \tilde{g}_\pm \pm \tilde{g}_\pm \tilde{\sigma}_\pm) \\
& - i\frac{r^2}{4}(\widehat{ap+k})_\rho (\widehat{ap+k})_\sigma (\bar{k}\cdot\gamma\gamma_\mu \pm \gamma_\mu \bar{k}\cdot\gamma)\tilde{\sigma}_\mp \tilde{g}_\mp \\
& + i\frac{r}{2}(\widehat{ap+k})_\rho (\widetilde{ap+k})_\sigma (\gamma_\lambda \gamma_\sigma \pm \gamma_\sigma \gamma_\lambda) \\
& \qquad \left[\tilde{g}_\pm \tilde{g}_\mp (\bar{k}^2 g_{\mu\lambda} - 2\bar{k}_\mu \bar{k}_\lambda) \pm g_{\mu\lambda}\tilde{\sigma}_\pm \tilde{\sigma}_\mp\right]\Big\}[\gamma_5] \qquad (6.22)
\end{aligned}
$$

6.2.1 Amplitude for physical fermions

The physical amplitude is the sum of odd and even terms,

$$
\begin{aligned}
I_q(p) &= I_{phys}^{odd} + I_{phys}^{even} \qquad (6.23) \\
I_{phys}^{odd} &= (-ip\cdot\gamma\mathcal{A})\bar{I}_{odd}^{-}[\gamma_5](-ip\cdot\gamma\mathcal{A}) + \bar{I}_{odd}^{+}[\gamma_5] \\
& \quad + (-ip\cdot\gamma\mathcal{A})\tilde{I}_{odd}^{-}[\gamma_5] + \tilde{I}_{odd}^{+}[\gamma_5](-ip\cdot\gamma\mathcal{A}) \qquad (6.24)
\end{aligned}
$$

$$
\begin{aligned}
I^{even}_{phys} &= (-ip\cdot\gamma\mathcal{A})\tilde{I}^{+}_{even}[\gamma_5](-ip\cdot\gamma\mathcal{A}) + \tilde{I}^{-}_{even}[\gamma_5] \\
&\quad +(-ip\cdot\gamma\mathcal{A})\bar{I}^{+}_{even}[\gamma_5] + \bar{I}^{-}_{even}[\gamma_5](-ip\cdot\gamma\mathcal{A}) \qquad (6.25)
\end{aligned}
$$

Since we are expanding the amplitude to order p^1, terms with two $p\cdot\gamma$ terms do not contribute; terms $\tilde{I}^{\pm}_{odd}$ and $\bar{I}^{\pm}_{even}$ are evaluated to o^{th} order in p; $\tilde{I}^{\pm}_{odd}$ vanishes since it's odd in k_μ. The term $\tilde{I}^{-}_{even}(p\to 0)$ is even in k_μ which means $\partial\tilde{I}^{-}_{even}/\partial p_\mu$ will be odd and won't contribute. $\bar{I}^{+}_{odd}(p\to 0)$ is odd so it vanishes as well. Hence, we are left with

$$
I_q(p) = p_\alpha \frac{\partial \bar{I}^{+}_{odd}}{\partial p_\alpha}[\gamma_5] + (-ip\cdot\gamma\mathcal{A})\bar{I}^{+}_{even}[\gamma_5] + \bar{I}^{-}_{even}[\gamma_5](-ip\cdot\gamma\mathcal{A})\,. \qquad (6.26)
$$

Since we are evaluating $\bar{I}^{\pm}_{even}$ at zero momentum, after symmetrizing in μ and ν, it must be proportional to

$$
\bar{I}^{\pm}_{even} \sim g_{\mu\nu}\times const. \qquad \Longrightarrow \qquad (-ip\cdot\gamma\mathcal{A})\bar{I}^{+}_{even}[\gamma_5] + \bar{I}^{-}_{even}[\gamma_5](-ip\cdot\gamma\mathcal{A}) \sim p\cdot\gamma g_{\mu\nu} \qquad (6.27)
$$

so it does not contribute. We are now left with

$$
I_q(p) = p_\alpha \frac{\partial \bar{I}^{+}_{odd}}{\partial p_\alpha}[\gamma_5]\,. \qquad (6.28)
$$

Instead of expanding the amplitude, we can evaluate it for finite p and then let p go to zero exponentially. Multiplying with γ_α and taking the trace yields

$$
\begin{aligned}
\frac{1}{d}\mathrm{Tr}_D\left[\bar{I}_{\mu\nu}[\gamma_5]\gamma_\alpha\right] &= \frac{H_{\rho\sigma}\, i\bar{k}_\nu}{(\widehat{ap-k})^2+\mu^2}\Big\{\Big(g_{\rho\sigma}g_{\lambda\alpha}(\widetilde{ap+k})^2_\rho - 2g_{\rho\lambda}g_{\sigma\alpha}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\Big) \\
&\qquad \left[(\bar{k}^2 g_{\mu\lambda} - 2\bar{k}_\mu\bar{k}_\lambda)\tilde{g}^2_- \pm g_{\mu\lambda}\tilde{\sigma}^2_-\right] \\
&\quad + \frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\pm(\bar{k}^2 g_{\mu\alpha} - 2\bar{k}_\mu\bar{k}_\alpha)\tilde{g}^2_+ + g_{\mu\alpha}\tilde{\sigma}^2_+\right] \\
&\quad + r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[g_{\sigma\alpha}\bar{k}_\mu(\tilde{g}_-\tilde{\sigma}_+ \pm \tilde{\sigma}_-\tilde{g}_+)\right. \\
&\qquad \left. -(\bar{k}_\alpha g_{\sigma\mu} - g_{\mu\alpha}\bar{k}_\sigma)(\tilde{g}_-\tilde{\sigma}_+ \mp \tilde{\sigma}_-\tilde{g}_+)\right]\Big\}\,. \qquad (6.29)
\end{aligned}
$$

$\mathbf{6}_3^{\pm}$ representations

The amplitude for the $\mathbf{6}_3^{\pm}$ has two parts

$$I^q_{\{\mu\nu\}} \;=\; \left(\frac{1}{d}\mathrm{Tr}_D\left[\bar{I}_{\mu\nu}[\gamma_5]\gamma_\mu\right] + \frac{1}{d}\mathrm{Tr}_D\left[\bar{I}_{\nu\mu}[\gamma_5]\gamma_\mu\right]\right)\frac{1}{p_\nu}\,. \tag{6.30}$$

The first one is obtained from (6.29) by setting $\alpha = \mu$ and is just the amplitude (5.38) multiplied by $i\bar{k}_\nu$

$$\begin{aligned}
\frac{1}{d}\mathrm{Tr}_D\left[\bar{I}_{\mu\nu}\gamma[5]\gamma_\mu\right] \;=\;& \frac{i\bar{k}_\nu}{(\widehat{ap-k})^2+\mu^2}\Big\{H_{\rho\sigma}g_{\rho\sigma}(\widetilde{ap+k})^2_\rho\left[(\bar{k}^2-2\bar{k}^2_\mu)\tilde{g}^2_- \pm\tilde{\sigma}^2_-\right] \\
&-2H_{\mu\mu}(\widetilde{ap+k})^2_\mu\left[\bar{k}^2\tilde{g}^2_- \pm\tilde{\sigma}^2_-\right] + 4H_{\rho\mu}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\mu\bar{k}_\rho\bar{k}_\mu\tilde{g}^2_- \\
&+\frac{r^2}{4}H_{\rho\sigma}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\tilde{\sigma}^2_+ \pm(\bar{k}^2-2\bar{k}^2_\mu)\tilde{g}^2_+\right] \\
&+r(\widehat{ap+k})_\rho\Big(H_{\rho\mu}(\widetilde{ap+k})_\mu\bar{k}_\mu(\pm 2g_+\sigma_-) \\
&\qquad +H_{\rho\sigma}(\widetilde{ap+k})_\sigma\bar{k}_\sigma(g_-\sigma_+\mp g_+\sigma_-)\Big)\Big\}[\gamma_5]\,,
\end{aligned} \tag{6.31}$$

while the second one is obtained from (6.29) by interchanging μ and ν and then setting $\alpha = \mu$

$$\begin{aligned}
\frac{1}{d}\mathrm{Tr}_D\left[\bar{I}_{\nu\mu}[\gamma_5]\gamma_\mu\right] \;=\;& \frac{i\bar{k}_\mu}{(\widehat{ap-k})^2+\mu^2}\Big\{H_{\rho\sigma}g_{\rho\sigma}(\widetilde{ap+k})^2_\rho\left[-2\bar{k}_\mu\bar{k}_\nu\right]\tilde{g}^2_- \\
&-2H_{\mu\nu}(\widetilde{ap+k})_\mu(\widetilde{ap+k})_\nu\left[\bar{k}^2\tilde{g}^2_- \pm\tilde{\sigma}^2_-\right] \\
&+4H_{\rho\mu}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\mu\bar{k}_\rho\bar{k}_\nu\tilde{g}^2_- \\
&+\frac{r^2}{4}H_{\rho\sigma}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\mp 2\bar{k}_\mu\bar{k}_\nu\right]\tilde{g}^2_+ \\
&+r(\widehat{ap+k})_\rho\Big(H_{\rho\mu}(\widetilde{ap+k})_\mu\bar{k}_\nu(g_-\sigma_+\pm g_+\sigma_-) \\
&\qquad -H_{\rho\nu}(\widetilde{ap+k})_\nu\bar{k}_\mu(g_-\sigma_+\mp g_+\sigma_-)\Big)\Big\}[\gamma_5]\,.
\end{aligned} \tag{6.32}$$

Adding them up we get the amplitude

$$\begin{aligned}
I_q \;=\;& \frac{1}{d}\mathrm{Tr}_D\left[(\bar{I}_{\mu\nu}+\bar{I}_{\nu\mu})\gamma_\mu\right]\frac{1}{p_\nu} \qquad (6.33) \\
=\;& \frac{1}{p_\nu}\frac{1}{(\widehat{ap-k})^2+\mu^2}\Big\{H_{\rho\sigma}g_{\rho\sigma}(\widetilde{ap+k})^2_\rho\bar{k}_\nu\left[(\bar{k}^2-4\bar{k}^2_\mu)\,\tilde{g}^2_- \pm\tilde{\sigma}^2_-\right]
\end{aligned}$$

$$
\begin{aligned}
&-2\left(H_{\mu\mu}(\widetilde{ap+k})^2_\mu \bar{k}_\nu + H_{\nu\mu}(\widetilde{ap+k})_\mu(\widetilde{ap+k})_\nu \bar{k}_\mu\right)\left[\bar{k}^2\tilde{g}_-^2 \pm \tilde{\sigma}_-^2\right] \\
&+8H_{\rho\mu}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\mu \bar{k}_\rho \bar{k}_\mu \bar{k}_\nu \tilde{g}_-^2 \\
&\quad+\frac{r^2}{4}H_{\rho\sigma}\bar{k}_\nu(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\tilde{\sigma}_+^2 \pm \left(\bar{k}^2-4\bar{k}_\mu^2\right)\tilde{g}_+^2\right] \\
&+rH_{\rho\mu}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\mu \bar{k}_\mu \bar{k}_\nu\left[\tilde{g}_-\tilde{\sigma}_+ \pm 3\tilde{g}_+\tilde{\sigma}_-\right] \\
&+r\Big(-H_{\rho\nu}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\nu \bar{k}_\mu^2 \\
&\qquad+H_{\rho\sigma}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma \bar{k}_\nu \bar{k}_\sigma\Big)\left[\tilde{g}_-\tilde{\sigma}_+ \mp \tilde{g}_+\tilde{\sigma}_-\right]\Big\}\,. \qquad (6.34)
\end{aligned}
$$

In the no-smearing limit we get

$$
\begin{aligned}
I_q &= \frac{1}{d}\mathrm{Tr}_D\left[(\bar{I}_{\mu\nu}+\bar{I}_{\nu\mu})\gamma_\mu\right]\frac{1}{p_\nu} \qquad (6.35) \\
&= \frac{1}{p_\nu}\frac{1}{(\widehat{ap-k})^2+\mu^2}\Big\{\left[(\widetilde{ap+k})^2-2(\widetilde{ap+k})^2_\mu\right]\bar{k}_\nu\left[\left(\bar{k}^2-4\bar{k}_\mu^2\right)\tilde{g}_-^2 \pm \tilde{\sigma}_-^2\right] \\
&\quad+\frac{r^2}{4}\bar{k}_\nu(\widehat{ap+k})^2\left[\tilde{\sigma}_+^2 \pm \left(\bar{k}^2-4\bar{k}_\mu^2\right)\tilde{g}_+^2\right] \\
&\quad+r(\overline{ap+k})_\mu \bar{k}_\mu \bar{k}_\nu\left[\tilde{g}_-\tilde{\sigma}_+ \pm 3\tilde{g}_+\tilde{\sigma}_-\right] \\
&\quad+r\left(-(\overline{ap+k})_\nu \bar{k}_\mu^2 + (\overline{ap+k})\cdot\bar{k}\bar{k}_\nu\right)\left[\tilde{g}_-\tilde{\sigma}_+ \mp \tilde{g}_+\tilde{\sigma}_-\right]\Big\}\,. \qquad (6.36)
\end{aligned}
$$

Taking one step further and replacing propagators by Wilson propagators $\tilde{g}_\pm \to 1/\mathcal{D}$, $\tilde{\sigma}_\pm \to \sigma/\mathcal{D}$, for $\bar{q}\gamma_\mu D_\nu q$ we get

$$
\begin{aligned}
I_q &= \frac{1}{p_\nu}\frac{1}{(\widehat{ap-k})^2+\mu^2}\left\{\left[\underbrace{(\widetilde{ap+k})^2+\frac{r^2}{4}(\widehat{ap+k})^2}_{d}-2(\widetilde{ap+k})^2_\mu\right]\bar{k}_\nu\frac{\left(\bar{k}^2-4\bar{k}_\mu^2+\sigma^2\right)}{\mathcal{D}^2}\right. \\
&\quad\left.+r(\overline{ap+k})_\mu \bar{k}_\mu \bar{k}_\nu \frac{4\sigma}{\mathcal{D}^2}\right\}\,. \qquad (6.37)
\end{aligned}
$$

Expanding in p_ν we get

$$
I_q = \frac{-2\bar{k}_\nu^2}{(\hat{k}^2+\mu^2)^2}\left\{\frac{\left[d-2\bar{k}_\mu^2\right]\left(\bar{k}^2-4\bar{k}_\mu^2+\sigma^2\right)+r\bar{k}_\mu^2 4\sigma}{\mathcal{D}^2}\right\}, \qquad (6.38)
$$

where $\sigma = m_0 + r/2\hat{k}^2$ and $\mathcal{D} = \bar{k}^2+\sigma^2$. For massless fermions and massless gluons, this agrees with formula (15.90) in Capitani.

M	"no smearing"	HYP	APE	"gauge-part"
Wilson	2.293	3.162	3.598	1.734
0.1	2.151	2.929	3.523	1.733
0.2	2.075	2.870	3.478	1.733
0.3	2.010	2.824	3.445	1.734
0.4	1.952	2.784	3.418	1.734
0.5	1.899	2.749	3.395	1.734
0.6	1.848	2.717	3.375	1.734
0.7	1.799	2.687	3.357	1.734
0.8	1.751	2.659	3.341	1.734
0.9	1.704	2.632	3.325	1.734
1	1.657	2.606	3.311	1.734
1.1	1.611	2.579	3.296	1.734
1.2	1.563	2.553	3.282	1.734
1.3	1.515	2.526	3.267	1.734
1.4	1.465	2.498	3.251	1.734
1.5	1.412	2.468	3.233	1.734
1.6	1.357	2.435	3.213	1.734
1.7	1.296	2.398	3.189	1.734
1.8	1.229	2.355	3.159	1.734
1.9	1.150	2.299	3.117	1.734

Table 6.1: Renormalization coefficient Σ_{V1} for vertex diagram contribution, in the $\mathbf{6}_3^+$ representation.

3_1^+ representation

In the 3_1^+ representation we have $\mu = \nu$. Using formulas from the beginning of this section, we get

$$I_q = \frac{1}{p_\mu}\frac{1}{d}\mathrm{Tr}\left[I_{\mu\mu}(\gamma_\mu - \gamma_\alpha)\right] . \tag{6.39}$$

where $\mu \neq \alpha$ and the four-vector p has components p_μ and p_α numerically equal. This yields

$$\begin{aligned}
\frac{1}{d}\mathrm{Tr}_D\left[\bar{I}_{\mu\mu}(\gamma_\mu - \gamma_\alpha)\right] \;=\; & \frac{i\bar{k}_\mu}{(\widehat{ap-k})^2+\mu^2}\Big\{H_{\rho\sigma}g_{\rho\sigma}(\widetilde{ap+k})^2_\rho\left[(\bar{k}^2 - 2\bar{k}_\mu(\bar{k}_\mu - \bar{k}_\alpha))\tilde{g}_-^2 \pm \tilde{\sigma}_-^2\right] \\
& -2(\widetilde{ap+k})_\mu\Big(H_{\mu\mu}(\widetilde{ap+k})_\mu - H_{\mu\alpha}(\widetilde{ap+k})_\alpha\Big)\left[\bar{k}^2\tilde{g}_-^2 \pm \tilde{\sigma}_-^2\right] \\
& +4(\widetilde{ap+k})_\rho\bar{k}_\rho\Big(H_{\rho\mu}(\widetilde{ap+k})_\mu - H_{\rho\alpha}(\widetilde{ap+k})_\alpha\Big)\bar{k}_\mu\tilde{g}_-^2 \\
& +\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\tilde{\sigma}_+^2 \pm (\bar{k}^2 - 2\bar{k}_\mu(\bar{k}_\mu - \bar{k}_\alpha))\tilde{g}_+^2\right] \\
& +rH_{\rho\sigma}(\widehat{ap+k})_\rho(\widetilde{ap+k})_\mu H_{\rho\mu}\left[\bar{k}_\mu(\pm 2\tilde{g}_+\tilde{\sigma}_-)\right. \\
& \left.+\bar{k}_\alpha(\tilde{g}_-\tilde{\sigma}_+ \mp \tilde{g}_+\tilde{\sigma}_-)\right] \\
& -r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\alpha H_{\rho\alpha}\bar{k}_\mu(\tilde{g}_-\tilde{\sigma}_+ \pm \tilde{g}_+\tilde{\sigma}_-) \\
& +r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma H_{\rho\sigma}\bar{k}_\sigma(\tilde{g}_-\tilde{\sigma}_+ \mp \tilde{g}_+\tilde{\sigma}_-)\Big\} .
\end{aligned} \tag{6.40}$$

In the no-smearing limit, we have $H_{\mu\nu} = g_{\mu\nu}$ so

$$\begin{aligned}
I_q^{NOS} \;=\; & \frac{1}{ip_\mu}\frac{i\bar{k}_\mu}{(\widehat{ap-k})^2+\mu^2}\Big\{\left[(\widetilde{ap+k})^2 - 2(\widetilde{ap+k})^2_\mu\right]\left[(\bar{k}^2 - 2\bar{k}_\mu(\bar{k}_\mu - \bar{k}_\alpha))\tilde{g}_-^2 \pm \tilde{\sigma}_-^2\right] \\
& +\frac{r^2}{4}(\widehat{ap+k})^2\left[\tilde{\sigma}_+^2 \pm (\bar{k}^2 - 2\bar{k}_\mu(\bar{k}_\mu - \bar{k}_\alpha))\tilde{g}_+^2\right] \\
& +r\bar{k}_\mu\{(\overline{ap+k})_\mu[\pm 2\tilde{g}_+\tilde{\sigma}_-] - (\overline{ap+k})_\alpha[\tilde{g}_-\tilde{\sigma}_+ \pm \tilde{g}_+\tilde{\sigma}_-]\} \\
& +r\left[(\overline{ap+k})\cdot\bar{k} - (\overline{ap+k})_\mu\bar{k}_\alpha\right]\left[\tilde{g}_-\tilde{\sigma}_+ \mp \tilde{g}_+\tilde{\sigma}_-\right]\Big\} .
\end{aligned} \tag{6.41}$$

For Wilson fermions this further simplifies to

$$\begin{aligned}
I_q^{NOS,W} \;=\; & \frac{1}{ip_\mu}\frac{i\bar{k}_\mu}{(\widehat{ap-k})^2+\mu^2}\left\{\left[(\widetilde{ap+k})^2 \pm \frac{r^2}{4}(\widehat{ap+k})^2 - 2(\widetilde{ap+k})^2_\mu\right]\right. \\
& \times\frac{\left[(\bar{k}^2 - 2\bar{k}_\mu(\bar{k}_\mu - \bar{k}_\alpha)) \pm \sigma^2\right]}{\mathcal{D}^2}
\end{aligned}$$

M	"no smearing"	HYP	APE	"gauge-part"
Wilson	3.575	3.483	3.762	1.734
0.1	2.545	3.116	3.635	1.733
0.2	2.493	3.069	3.596	1.733
0.3	2.454	3.034	3.569	1.734
0.4	2.423	3.006	3.548	1.734
0.5	2.397	2.982	3.531	1.734
0.6	2.375	2.962	3.518	1.734
0.7	2.356	2.944	3.506	1.734
0.8	2.341	2.928	3.495	1.734
0.9	2.329	2.914	3.486	1.734
1	2.319	2.901	3.477	1.734
1.1	2.311	2.888	3.468	1.734
1.2	2.306	2.876	3.460	1.734
1.3	2.302	2.865	3.452	1.734
1.4	2.301	2.853	3.443	1.734
1.5	2.302	2.841	3.432	1.734
1.6	2.305	2.827	3.421	1.734
1.7	2.309	2.811	3.406	1.734
1.8	2.314	2.791	3.387	1.734
1.9	2.315	2.763	3.357	1.734

Table 6.2: Renormalization coefficient Σ_{V2} for vertex diagram contribution, in the $\mathbf{3}_1^+$ representation.

$$
\begin{aligned}
&+r\frac{\bar{k}_\mu(\overline{ap+k})_\mu[\pm 2\sigma] - \bar{k}_\mu(\overline{ap+k})_\alpha[\sigma\pm\sigma]}{\mathcal{D}^2} \\
&+r\frac{\left[(\overline{ap+k})\cdot\bar{k} - (\overline{ap+k})_\mu\bar{k}_\alpha\right][\sigma\mp\sigma]}{\mathcal{D}^2}\Bigg\} . \qquad (6.42)
\end{aligned}
$$

6.3 Sails diagram

Since amplitudes for the two sails diagrams are related, we will evaluate them together. They are given by

$$
I_s^{(1)} = G_{\nu\rho}(p-k)V_\rho(p,k)S_s^{IN}O_{\mu\nu} \qquad (6.43)
$$

$$
I_s^{(2)} = G_{\nu\rho}(p-k)O_{\mu\nu}S_s^{OUT}V_\rho(k,p) \qquad (6.44)
$$

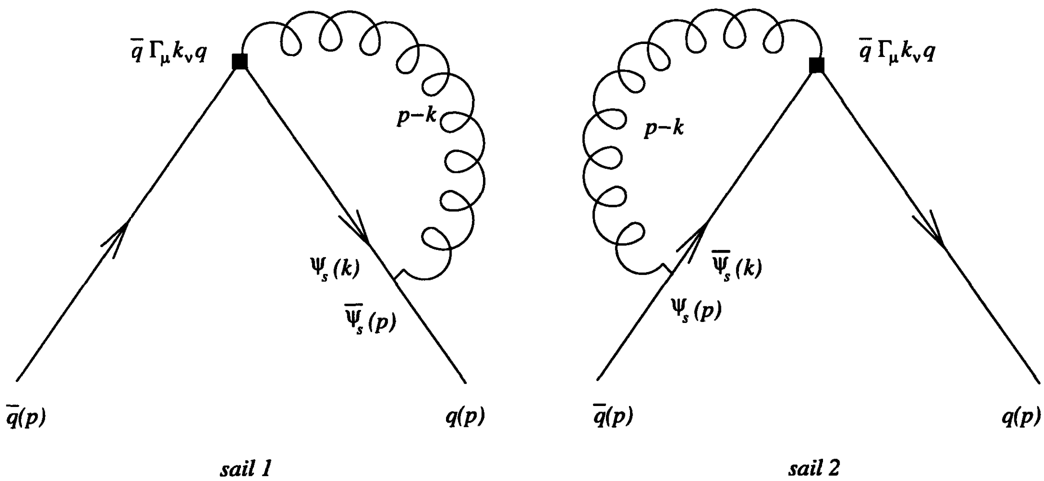

Figure 6-2: Sails diagram for twist 2 operators

Physical amplitudes are then obtained by adding the $5D$-to-physical propagator and amputating the external leg

$$
\begin{aligned}
I_1 &= \bar{S}_s^{OUT} I_s^{(1)} = \bar{S}_s^{OUT} V_\rho(p,k) S_s^{IN} O_{\mu\nu} G_{\nu\rho}(p-k) && (6.45)\\
I_2 &= I_s^{(2)} \bar{S}_s^{IN} = O_{\mu\nu} S_s^{OUT} V_\rho(k,p) G_{\nu\rho}(p-k) \bar{S}_s^{IN} && (6.46)
\end{aligned}
$$

As in the case of vertex diagram, part of $\bar{S}_s^{OUT}$ and $\bar{S}_s^{IN}$ proportional to $p\cdot\gamma$ will give us a contribution proportional to $p\cdot\gamma\, g_{\mu\nu}$ so we can neglect it from the start. That leaves us with

$$
\begin{aligned}
I_1 &= \bar{S}^{OUT}\left[\frac{g^{ab}H_{\nu\rho}}{(\widehat{ap-k})^2+\mu^2}\right]\left[-g_0T^a\left(\frac{r}{2}(\widehat{ap+k})_\rho + i\gamma_\rho(\widetilde{ap+k})_\rho\right)\right]\\
&\quad \left[(g_-P_+ + g_+P_-)\left(-i\bar{k}\cdot\gamma\right) + (\sigma_-P_+ + \sigma_+P_-)\right]\\
&\quad \left[\gamma_\mu(\widetilde{ap+k})_\nu\; ig_0T^b\,[\gamma_5]\right] && (6.47)\\
I_1 &= \left[\frac{g^{ab}H_{\nu\rho}}{(\widehat{ap-k})^2+\mu^2}\right]\left[\gamma_\mu(\widetilde{ap+k})_\nu\; ig_0T^a\,[\gamma_5]\right]\\
&\quad \left[\left(-i\bar{k}\cdot\gamma\right)(g_+P_+ + g_+P_+) + (\sigma_+P_+ + \sigma_-P_-)\right]\\
&\quad \left[-g_0T^b\left(\frac{r}{2}(\widehat{ap+k})_\rho + i\gamma_\rho(\widetilde{ap+k})_\rho\right)\right]\bar{S}_{IN}\,. && (6.48)
\end{aligned}
$$

Contracting with $\bar{S}_{IN,OUT}$, we get

$$
\begin{aligned}
I_1 &= \frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu}{(\widehat{ap-k})^2+\mu^2}\Big[\frac{r}{2}(\widehat{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_+ +\tilde{\sigma}_+\right) \\
&\quad + i\gamma_\rho(\widetilde{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_- +\tilde{\sigma}_-\right)\Big]\gamma_\mu[\gamma_5] \qquad (6.49)\\
I_2 &= \gamma_\mu[\gamma_5]\frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu}{(\widehat{ap-k})^2+\mu^2}\Big[\frac{r}{2}(\widehat{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_+ +\tilde{\sigma}_+\right) \\
&\quad + \left(-i\bar{k}\cdot\gamma\tilde{g}_- +\tilde{\sigma}_-\right)i\gamma_\rho(\widetilde{ap+k})_\rho\Big]\,. \qquad (6.50)
\end{aligned}
$$

Using the relations between γ matrices

$$
a\cdot\gamma b\cdot\gamma c\cdot\gamma - c\cdot\gamma b\cdot\gamma a\cdot\gamma = 2\left(a\cdot\gamma\, b\cdot c - b\cdot\gamma\, a\cdot c + c\cdot\gamma\, a\cdot b\right) \qquad (6.51)
$$

and adding both terms, we get

$$
\begin{aligned}
I_{\mu\nu} &= \frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu}{(\widehat{ap-k})^2+\mu^2}\Big\{-2\gamma_\mu\frac{r}{2}(\widehat{ap+k})_\rho\tilde{\sigma}_+ \\
&\quad -2\left(\bar{k}_\mu\gamma_\rho(\widetilde{ap+k})_\rho - \bar{k}\cdot\gamma g_{\rho\mu}(\widetilde{ap+k})_\mu + \gamma_\mu\bar{k}_\rho(\widetilde{ap+k})_\rho\right)\tilde{g}_- \\
&\quad +\frac{r}{2}i(\widehat{ap+k})_\rho[\bar{k}\cdot\gamma,\gamma_\mu]_\pm\tilde{g}_+ + i(\widetilde{ap+k})_\rho[\gamma_\rho,\gamma_\mu]_\pm\tilde{\sigma}_-\Big\}\,, \qquad (6.52)
\end{aligned}
$$

where $[,]_\pm$ is the commutator/anticommutator of γ matrices. For $p \to 0$, the first two lines are odd while the third one is even, so to order p^1 the third line vanishes due to parity[2]. That gives us the final result

$$
\begin{aligned}
I_{\mu\nu} &= \frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu}{(\widehat{ap-k})^2+\mu^2}\Big\{\gamma_\mu(-r)(\widehat{ap+k})_\rho\tilde{\sigma}_+ \\
&\quad -2\left(\bar{k}_\mu\gamma_\rho(\widetilde{ap+k})_\rho - \bar{k}\cdot\gamma\, g_{\rho\mu}(\widetilde{ap+k})_\mu + \gamma_\mu\bar{k}_\rho(\widetilde{ap+k})_\rho\right)\tilde{g}_-\Big\}\,. \qquad (6.53)
\end{aligned}
$$

[2] To 0^{th} order in p_μ it gives a finite contribution proportional to either $g_{\mu\nu}$ or $\sigma_{\mu\nu}$. The $\sigma_{\mu\nu}$ contribution is killed by symmetrization, while $g_{\mu\nu}$ does not contribute to either representations we are considering

$\mathbf{6}_3^+$ representation

The amplitude for the $\mathbf{6}_3^+$ representation is then

$$\begin{aligned}
I_q &= \frac{1}{d}\mathrm{Tr}_D\left[(\bar{I}_{\mu\nu}+\bar{I}_{\nu\mu})\gamma_\mu\right]\frac{1}{p_\nu} \\
&= \frac{1}{p_\nu}\frac{g_0^2 C_F}{(\widehat{ap-k})^2+\mu^2}\left\{\sum_\rho H_{\nu\rho}(\widetilde{ap+k})_\nu\left[-r(\widehat{ap+k})_\rho\tilde{\sigma}_+ - 2\tilde{g}_-\bar{k}_\rho(\widetilde{ap+k})_\rho\right]\right. \\
&\quad \left. -2\tilde{g}_-(\widetilde{ap+k})_\mu\left[H_{\mu\mu}(\widetilde{ap+k})_\mu\bar{k}_\nu - H_{\mu\nu}(\widetilde{ap+k})_\nu\bar{k}_\mu\right]\right\} \qquad (6.54)
\end{aligned}$$

In the no-smearing limit, this becomes

$$I_q = \frac{1}{p_\nu}\frac{g_0^2 C_F}{(\widehat{ap-k})^2+\mu^2}\left\{-r(\overline{ap+k})_\nu\tilde{\sigma}_+ - 2\left(\bar{k}_\nu(\widetilde{ap+k})_\mu^2+\bar{k}_\nu(\widetilde{ap+k})_\nu^2\right)\tilde{g}_-\right\} \qquad (6.55)$$

which after expansion in p_μ to first order yields

$$\begin{aligned}
I_q &= g_0^2 C_F\left\{\frac{1}{\hat{k}^2+\mu^2}\left[-2r\cos k_\nu\tilde{\sigma}_+ + \bar{k}_\nu^2\tilde{g}_-\right]\right. \\
&\quad \left. -\frac{1}{(\hat{k}^2+\mu^2)^2}\left[r\bar{k}_\nu^2\tilde{\sigma}_+ + 4\tilde{g}_-\bar{k}_\nu^2\left(\bar{k}_\nu^2+\bar{k}_\mu^2\right)\right]\right\}. \qquad (6.56)
\end{aligned}$$

For Wilson fermions, this agrees with Capitani's formula (15.102)

$\mathbf{3}_1^+$ representation

In the $\mathbf{3}_1^+$ representation we have $\mu=\nu$. Using formulas from previous section, we get

$$I_q = \frac{1}{p_\mu}\frac{1}{d}\mathrm{Tr}\left[I_{\mu\mu}(\gamma_\mu-\gamma_\alpha)\right]. \qquad (6.57)$$

where $\mu\neq\alpha$ and the four-vector p has components p_μ and p_α numerically equal. This yields

$$\begin{aligned}
I_q &= \frac{1}{p_\mu}\frac{g_0^2 C_F}{(\widehat{ap-k})^2+\mu^2}\left\{H_{\mu\rho}(\widetilde{ap+k})_\mu\left[-r(\widehat{ap+k})_\rho\tilde{\sigma}_+ - 2\tilde{g}_-\bar{k}_\rho(\widetilde{ap+k})_\rho\right]\right. \\
&\quad \left. -2\tilde{g}_-(\widetilde{ap+k})_\mu\left[H_{\mu\mu}\bar{k}_\alpha(\widetilde{ap+k})_\mu - H_{\mu\alpha}\bar{k}_\mu(\widetilde{ap+k})_\alpha\right]\right\} \qquad (6.58)
\end{aligned}$$

In the no-smearing limit this becomes

$$I_q = \frac{1}{p_\mu} \frac{g_0^2 C_F}{(\widehat{ap-k})^2+\mu^2} \left\{ -r(\overline{ap+k})_\mu \tilde{\sigma}_+ - 2\tilde{g}_-(\widetilde{ap+k})^2_\mu \left[\bar{k}_\mu + \bar{k}_\alpha\right] \right\} \quad (6.59)$$

which after expansion in p_μ yields

$$\begin{aligned} I_q = g_0^2 C_F & \left\{ \frac{1}{\hat{k}^2+\mu^2} \left[\cos k_\mu \tilde{\sigma}_+ + \bar{k}^2_\mu \tilde{g}_-\right] \right. \\ & \left. - \frac{1}{(\hat{k}^2+\mu^2)^2} \left[r\bar{k}^2_\mu \tilde{\sigma}_+ + 4\tilde{g}_- \bar{k}^2_\mu \left(\bar{k}^2_\mu + \bar{k}^2_\alpha\right)\right] \right\} . \end{aligned} \quad (6.60)$$

This is numerically the same as expression (6.56) since indices μ and α can be exchanged in term with $\bar{k}^2_\mu \bar{k}^2_\alpha$. Another way to see this is to observe that the amplitude $I_{\mu\nu}$ has no parts proportional to $g_{\mu\nu}\gamma_\mu p_\mu$ which cause the difference between the two representations

$$\begin{aligned} I_{\mu\nu} = & \frac{g_0^2 C_F}{(\widehat{ap-k})^2+\mu^2} \left\{ \frac{r}{2}(\overline{ap+k})_\nu \tilde{\sigma}_+ \right. \\ & \left. -2\left(\bar{k}_\mu \gamma_\nu (\widetilde{ap+k})^2_\nu - \bar{k}\cdot\gamma\,(\widetilde{ap+k})_\mu (\widetilde{ap+k})_\nu + \gamma_\mu \bar{k}_\nu (\widetilde{ap+k})^2_\rho\right) \tilde{g}_- \right\} . \end{aligned} \quad (6.61)$$

which after expansion in p_μ to first order yields

$$\begin{aligned} I_{\mu\nu} = & \frac{g_0^2 C_F \gamma_\mu p_\nu}{\hat{k}^2+\mu^2} \left\{\cos k_\nu \tilde{\sigma}_+ + \bar{k}^2_\nu \tilde{g}_-\right\} \\ & - \frac{g_0^2 C_F}{(\hat{k}^2+\mu^2)^2} \left\{\gamma_\mu p_\nu r \bar{k}^2_\nu \tilde{\sigma}_+ + 4\tilde{g}_- \left(\gamma_\mu p_\nu \bar{k}^2_\nu \bar{k}^2_\nu + \gamma_\nu p_\mu \bar{k}^2_\mu \bar{k}^2_\nu\right)\right\} . \end{aligned} \quad (6.62)$$

After symmetrization in μ and ν we get expressions (6.56) and (6.60).

Smeared vs. non-smeared operator

In this work, we consider the use of both smeared and unsmeared gauge links in the operator $O_{\mu\nu} = \bar{q}(x)\gamma_\mu D_\nu q(x)$. If one recalls that smearing comes from a form-factor modifying gluon-fermion vertex, then there is no reason that it needs to be included in the guage-link in the external operator. On the other hand, since in the $a \to 0$ limit the gauge link is

$$h_{\rho\sigma} = g_{\rho\sigma} + O(a^2) \ , \tag{6.63}$$

operator $O_{\mu\nu}$ has a correct continuum limit both for smeared and non-smeared link and we can think of it as a sort of improvement of the operator itself. Formulas derived here are correct for both cases, but we have to carefully keep track of indices since in the non-smeared case, tensor $H_{\mu\nu}$ is no longer necessarily symmetric. For the smeared operator case, after including smearing form-factors in the gluon propagator, we get

$$H_{\mu\nu} = h_{\mu\rho} G_{\rho\sigma} h_{\sigma\nu} \ . \tag{6.64}$$

Since both h and G are symmetric, so is H. For the non-smeared case, the gluon propagator gets multiplied only by one h tensor; since the tensor comes from gluon-fermion vertex V, it always comes in the combination $h_{\alpha\rho} V_\rho$ and then gets multiplied by gluon propagator $G_{\nu\alpha}$. In tensor form, this yields

$$H_{\nu\rho} = G_{\nu\alpha} h_{\alpha\rho} = h_{\rho\alpha} G_{\alpha\nu} \tag{6.65}$$

since both G and h are symmetric. Therefore, the correct tensor to use is

$$H_{\nu\rho} = [Gh]_{\nu\rho} = [hG]_{\rho\nu} \ . \tag{6.66}$$

For Wilson gluons $G_{\mu\nu} = g_{\mu\nu}$, so $H_{\mu\nu} = h_{\mu\nu}$ is symmetric and it doesn't make a difference, but for improved gluons one has to be careful.

M	"NOS"	HYP (NSO)	APE (NSO)	HYP (SO)	APE (SO)	"GDP"
Wilson	-5.077	0.712	3.488	2.411	4.999	5.117
0.1	-5.772	0.328	3.242	2.102	4.834	5.117
0.2	-5.604	0.445	3.328	2.202	4.901	5.117
0.3	-5.466	0.532	3.387	2.274	4.941	5.117
0.4	-5.342	0.602	3.430	2.329	4.968	5.117
0.5	-5.227	0.662	3.464	2.375	4.987	5.117
0.6	-5.119	0.714	3.491	2.413	5.002	5.117
0.7	-5.014	0.761	3.514	2.446	5.013	5.117
0.8	-4.911	0.804	3.534	2.475	5.022	5.117
0.9	-4.808	0.844	3.551	2.502	5.030	5.117
1	-4.705	0.881	3.567	2.527	5.037	5.117
1.1	-4.600	0.918	3.583	2.551	5.043	5.117
1.2	-4.492	0.955	3.598	2.576	5.050	5.117
1.3	-4.380	0.993	3.615	2.601	5.058	5.117
1.4	-4.261	1.033	3.633	2.629	5.067	5.117
1.5	-4.133	1.078	3.654	2.660	5.078	5.117
1.6	-3.993	1.129	3.681	2.698	5.094	5.117
1.7	-3.835	1.192	3.717	2.746	5.116	5.117
1.8	-3.650	1.274	3.769	2.811	5.151	5.117
1.9	-3.417	1.394	3.853	2.912	5.213	5.117

Table 6.3: Renormalization coefficient Σ_{V3} for sails diagrams contribution. Notation "SO" and "NSO" means "smeared operator" and "non smeared operator".

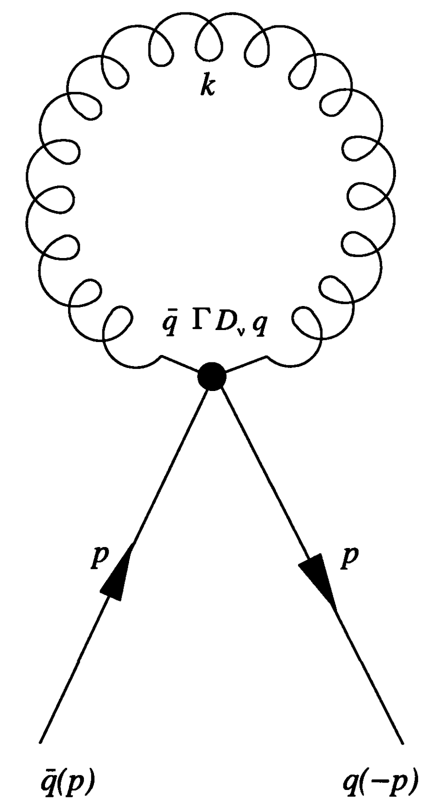

Figure 6-3: Tadpole diagram for twist 2 operators

6.4 Operator tadpole diagram

The amplitude for the tadpole diagram is given by

$$I_q = \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} G_{\nu\nu}(k) O^{aa}_{\mu\nu}(p,p)\,. \tag{6.67}$$

The operator vertex expanded to second order in g_0

$$O^{(2)}_{\mu\nu} = -\frac{a^2}{2}\{T^a, T^b\}\bar{p}_\nu \to -\frac{a^2}{2} C_F \gamma_\mu \bar{p}_\nu \tag{6.68}$$

$$\tag{6.69}$$

is independent of the loop momentum so we are left with the amplitude

$$I_q = -\frac{1}{2} g_0^2 C_F\, i\gamma_\mu p_\nu T = -\frac{g_0^2 C_F}{16\pi^2}\, i\gamma_\mu p_\nu (8\pi^2 T) = i\gamma_\mu p_\nu \frac{g_0^2 C_F}{16\pi^2} \Sigma^{OPtad} \tag{6.70}$$

where T is the tadpole integral

$$T = \lim_{\mu \to 0} T(\mu^2) \quad = \quad \lim_{\mu \to 0} \frac{1}{d} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \sum_{\rho} \frac{H_{\rho\rho}}{\hat{k}^2 + \mu^2} \tag{6.71}$$

and

$$\Sigma^{OPtad} = -8\pi^2 T \tag{6.72}$$

already encountered in the self-energy renormalization.

no smearing	HYP	APE	gauge-part
0.15493	0.05219	0.04202	0.03873

Table 6.4: Results for tadpole integral T

no smearing	HYP(SO)	APE(SO)	HYP(NSO)	APE(NSO)	gauge-part
-12.233	-4.121	-3.318	-12.233	-12.233	-3.058

Table 6.5: Tadpole contribution to twist-2 operators Σ^{OPtad}.

6.5 Collecting results: renormalization coefficients for twist 2 operators with 1 derivative

Finally, collecting results for twist-2 diagrams, we get the formula for renormalization constants

$$\begin{aligned} Z_{\{\mu\nu\}} &= 1 + \frac{g_0^2 C_F}{16\pi^2} \left([\gamma_2 + \gamma^{vert} + \gamma^{sails}] \log p^2/\Lambda^2 + [\Sigma_2 + \Sigma^{vert} + \Sigma^{sails} + \Sigma^{OPtad}] \right) \\ &= 1 + \frac{g_0^2 C_F}{16\pi^2} \left(\frac{8}{3} \log p^2/\Lambda^2 + \Sigma_{\{\mu\nu\}} \right) . \end{aligned} \tag{6.73}$$

M	"NOS"	HYP (SO)	APE (SO)	HYP (NSO)	APE (NSO)	"GDP"
Wilson	-3.16	-2.04	-0.14	-11.85	-10.57	-1.
0.1	-4.19	-2.74	-0.49	-12.62	-11.	-1.
0.2	-4.25	-2.81	-0.55	-12.68	-11.04	-1.
0.3	-4.31	-2.87	-0.6	-12.73	-11.07	-1.
0.4	-4.35	-2.94	-0.64	-12.78	-11.1	-1.
0.5	-4.38	-2.99	-0.68	-12.81	-11.12	-1.
0.6	-4.41	-3.04	-0.71	-12.85	-11.14	-1.
0.7	-4.43	-3.08	-0.74	-12.88	-11.16	-1.
0.8	-4.43	-3.11	-0.77	-12.9	-11.17	-1.
0.9	-4.42	-3.14	-0.79	-12.91	-11.18	-1.
1.	-4.41	-3.16	-0.8	-12.92	-11.19	-1.
1.1	-4.37	-3.18	-0.81	-12.92	-11.19	-1.
1.2	-4.32	-3.18	-0.82	-12.92	-11.19	-1.
1.3	-4.25	-3.18	-0.82	-12.9	-11.18	-1.
1.4	-4.16	-3.16	-0.82	-12.87	-11.17	-1.
1.5	-4.04	-3.13	-0.81	-12.82	-11.15	-1.
1.6	-3.88	-3.07	-0.78	-12.75	-11.11	-1.
1.7	-3.67	-2.99	-0.75	-12.65	-11.06	-1.
1.8	-3.39	-2.86	-0.68	-12.51	-10.98	-1.
1.9	-3.	-2.66	-0.57	-12.28	-10.85	-1.

Table 6.6: Finite part of renormalization coefficient $\Sigma^{latt}_{\{\mu\nu\}}$ in the $\mathbf{6}_3^+$ representation.

M	"NOS"	HYP (SO)	APE (SO)	HYP (NSO)	APE (NSO)	"GDP"
Wilson	1.28	2.4	4.3	-7.41	-6.12	0.
0.1	0.25	1.71	3.95	-8.18	-6.56	0.
0.2	0.19	1.64	3.9	-8.23	-6.59	0.
0.3	0.14	1.57	3.85	-8.28	-6.62	0.
0.4	0.09	1.51	3.8	-8.33	-6.65	0.
0.5	0.06	1.45	3.76	-8.37	-6.68	0.
0.6	0.03	1.41	3.73	-8.4	-6.7	0.
0.7	0.02	1.37	3.7	-8.43	-6.71	0.
0.8	0.01	1.33	3.68	-8.45	-6.72	0.
0.9	0.02	1.3	3.66	-8.47	-6.74	0.
1.	0.04	1.28	3.64	-8.48	-6.74	0.
1.1	0.07	1.27	3.63	-8.48	-6.74	0.
1.2	0.12	1.26	3.63	-8.47	-6.74	0.
1.3	0.19	1.27	3.62	-8.45	-6.73	0.
1.4	0.29	1.28	3.63	-8.42	-6.72	0.
1.5	0.41	1.32	3.64	-8.38	-6.7	0.
1.6	0.57	1.37	3.66	-8.31	-6.67	0.
1.7	0.78	1.46	3.7	-8.21	-6.62	0.
1.8	1.06	1.59	3.76	-8.06	-6.54	0.
1.9	1.44	1.79	3.87	-7.84	-6.4	0.

Table 6.7: Finite part of renormalization coefficient $\Sigma^{latt}_{\{\mu\nu\}} - \Sigma^{MS}_{\{\mu\nu\}}$ in the $\mathbf{6}_3^+$ representation.

M	"NOS"	HYP (SO)	APE (SO)	HYP (NSO)	APE (NSO)	"GDP"
Wilson	-1.88	-1.72	0.02	-11.53	-10.4	-1.
Wilson	-1.88	-1.72	0.02	-11.53	-10.4	-1.
0.1	-3.8	-2.55	-0.38	-12.43	-10.89	-1.
0.2	-3.83	-2.61	-0.43	-12.48	-10.92	-1.
0.3	-3.86	-2.66	-0.47	-12.52	-10.94	-1.
0.4	-3.88	-2.71	-0.51	-12.55	-10.97	-1.
0.5	-3.89	-2.76	-0.55	-12.58	-10.98	-1.
0.6	-3.88	-2.79	-0.57	-12.6	-11.	-1.
0.7	-3.87	-2.82	-0.59	-12.62	-11.01	-1.
0.8	-3.84	-2.85	-0.61	-12.63	-11.01	-1.
0.9	-3.8	-2.86	-0.62	-12.63	-11.02	-1.
1.	-3.74	-2.87	-0.63	-12.63	-11.02	-1.
1.1	-3.67	-2.87	-0.64	-12.61	-11.02	-1.
1.2	-3.58	-2.86	-0.64	-12.59	-11.01	-1.
1.3	-3.47	-2.84	-0.64	-12.56	-10.99	-1.
1.4	-3.32	-2.8	-0.62	-12.51	-10.97	-1.
1.5	-3.15	-2.75	-0.61	-12.45	-10.95	-1.
1.6	-2.93	-2.68	-0.58	-12.36	-10.9	-1.
1.7	-2.65	-2.57	-0.53	-12.24	-10.84	-1.
1.8	-2.3	-2.42	-0.45	-12.07	-10.75	-1.
1.9	-1.84	-2.19	-0.33	-11.82	-10.61	-1.

Table 6.8: Finite part of renormalization coefficient $\Sigma^{latt}_{\{\mu\nu\}}$ in the $\mathbf{3}_1^+$ representation.

M	"NOS"	HYP (SO)	APE (SO)	HYP (NSO)	APE (NSO)	"GDP"
Wilson	2.56	2.73	4.47	-7.09	-5.96	0.
0.1	0.64	1.9	4.06	-7.99	-6.45	0.
0.2	0.61	1.83	4.01	-8.03	-6.47	0.
0.3	0.58	1.78	3.97	-8.07	-6.5	0.
0.4	0.57	1.73	3.93	-8.11	-6.52	0.
0.5	0.56	1.69	3.9	-8.14	-6.54	0.
0.6	0.56	1.65	3.87	-8.16	-6.55	0.
0.7	0.58	1.62	3.85	-8.17	-6.56	0.
0.8	0.6	1.6	3.83	-8.18	-6.57	0.
0.9	0.64	1.58	3.82	-8.19	-6.57	0.
1.	0.7	1.58	3.81	-8.18	-6.57	0.
1.1	0.77	1.57	3.8	-8.17	-6.57	0.
1.2	0.87	1.58	3.8	-8.15	-6.56	0.
1.3	0.98	1.61	3.81	-8.11	-6.55	0.
1.4	1.12	1.64	3.82	-8.07	-6.53	0.
1.5	1.3	1.69	3.84	-8.	-6.5	0.
1.6	1.52	1.77	3.87	-7.92	-6.46	0.
1.7	1.79	1.87	3.91	-7.79	-6.4	0.
1.8	2.14	2.02	3.99	-7.63	-6.31	0.
1.9	2.61	2.25	4.11	-7.38	-6.16	0.

Table 6.9: Finite part of renormalization coefficient $\Sigma^{latt}_{\{\mu\nu\}} - \Sigma^{MS}_{\{\mu\nu\}}$ in the $\mathbf{3}_1^+$ representation.

Chapter 7

Twist 2 operators with $\gamma_\mu[\gamma_5]$ and two derivatives

For the operators with two derivatives, the procedure of evaluating Feynman diagrams is in principle the same as for one derivative, one just has to multiply the proper amplitude with a power of lattice momentum $\bar{k}_\mu$ and external momentum p_μ and add up contributions for sails diagrams where the gluon couples to the different derivative operator. One new feature here is that now we also have tadpole diagrams connecting two different derivative operators in the operator $\bar{q}(x)\gamma_\mu D_\nu D_\alpha q(x)$. Another new feature that arises here is mixing of operators on the lattice which is not present in the continuum. In the continuum, the totally symmetric operator $\bar{q}(x)\gamma_{\{\mu}D_\nu D_{\alpha\}}q(x)$ cannot mix with the mixed-symmetry operator $\bar{q}(x)\gamma_{[\mu}D_{\{\nu}D_{\alpha\}}q(x)$ due to Lorentz invariance. On the lattice, both operators fall into the same representation of $H(4)$ and therefore can (and do) mix.

7.1 Preliminaries

As in the previous section for the operator with one derivative, before we embark on particular Feynman diagrams, we need to expand the operator in powers of g_0 to get proper vertices, and we need to pick particular pieces of the amplitude which will give us the contribution we want and cancel all other unwanted contributions.

7.1.1 Operator vertex

The operator $O_{\mu\nu\alpha} = \bar{q}(x)\gamma_\mu D_\nu D_\alpha q(x)$ yields the following vertices

$$O_{\mu\nu\alpha} = O^{(0)}_{\mu\nu\alpha} + g_0 O^{(1)}_{\mu\nu\alpha} + g_0^2 O^{(2)}_{\mu\nu\alpha} \,, \tag{7.1}$$

with

$$\begin{aligned} O^{(0)}_{\mu\nu\alpha} &= \gamma_\mu i\bar{k}_\nu i\bar{k}_\alpha & (7.2)\\ O^{(1)}_{\mu\nu\alpha} &= T^a\gamma_\mu \left(i\cos\frac{(ap+k)_\nu}{2} A^a_\nu i\bar{k}_\alpha + i\bar{p}_\nu i\cos\frac{(ap+k)_\alpha}{2} A^a_\alpha \right) & (7.3)\\ O^{(2)}_{\mu\nu\alpha} &= \gamma_\mu \left[-\frac{a^2}{2}\{T^a,T^b\}A^a_\nu A^b_\nu \bar{p}_\nu \right] \bar{p}_\alpha + \gamma_\mu \bar{p}_\nu \left[-\frac{a^2}{2}\{T^a,T^b\}A^a_\alpha A^b_\alpha \bar{p}_\alpha \right] \\ &\quad + \gamma_\mu \left[\cos\left(p-\frac{k}{2}\right)_\nu ig_0 T^a A^a_\nu \right] \left[\cos\left(p-\frac{k}{2}\right)_\alpha ig_0 T_b A^b_\alpha \right] \,, & (7.4) \end{aligned}$$

where in the last step we have performed the summation over the group index

$$\mathrm{Tr}\sum_a \delta^{ab}\{T^a,T^b\} = \frac{N_c^2-1}{2N_c} = C_F \,. \tag{7.5}$$

The 0^{th} order $O^{(0)}_{\mu\nu}$ contributes to the vertex diagram; the 1^{st} order contributes to the sails diagrams, while the second order contributes to tadpole diagram.

7.1.2 Amplitude decomposition

The general structure of the operator we are interested here is

$$O_{\{\mu\nu\alpha\}} = \frac{1}{3!}\left(O_{\mu\nu\alpha} + O_{\mu\alpha\nu} + O_{\nu\mu\alpha} + O_{\nu\alpha\mu} + O_{\alpha\mu\nu} + O_{\alpha\nu\mu} \right) \,. \tag{7.6}$$

We consider two representations, $\mathbf{8}_1^-$ and $\mathbf{4}_2^-$:

$$\begin{aligned} O_{\{411\}} &= \frac{1}{3}(O_{411} + O_{141} + O_{114}) \,, & (7.7)\\ O_{\{134\}} &= \frac{1}{3!}(O_{134} + O_{341} + O_{413} + O_{143} + O_{431} + O_{314}) \,. & (7.8) \end{aligned}$$

$\mathbf{8}_1^-$ representation

Unfortunately, in the $\mathbf{8}_1^-$ representation, the operator mixes with the operator with mixed symmetry[1]

$$O_{[\mu\{\nu]\alpha\}} = \left(O_{\mu\nu\alpha} + O_{\mu\alpha\nu} - O_{\nu\mu\alpha} - O_{\nu\alpha\mu}\right) . \tag{7.9}$$

Since we are interested in the operators with $\nu = \alpha$, we get

$$\begin{aligned} O_{\{\mu\nu\nu\}} &= \frac{1}{3}\left(O_{\mu\nu\nu} + O_{\nu\mu\nu} + O_{\nu\nu\mu}\right) & (7.10)\\ O_{[\mu\{\nu]\nu\}} &= \left(2O_{\mu\nu\nu} - O_{\nu\mu\nu} - O_{\nu\nu\mu}\right) . & (7.11)\end{aligned}$$

Going to momentum space, we get the tree-level vertices for these operators

$$\begin{aligned} O_{\{\mu\nu\nu\}} &= \frac{1}{3}\left(\gamma_\mu p_\nu^2 + 2\gamma_\nu p_\mu p_\nu\right) & (7.12)\\ O_{[\mu\{\nu]\nu\}} &= 2\left(\gamma_\mu p_\nu^2 - \gamma_\nu p_\mu p_\nu\right) . & (7.13)\end{aligned}$$

For 1-loop level corrections of the operator $O_{\mu\nu\alpha}$, aside from the 3 terms which are Lorentz-invariant, we have many other terms which break Lorentz invariance and lead to mixing of the operators above. However, since we are only interested in the operator with $\nu = \alpha$, and $\mu \neq \nu$, we can drop most of them and we will keep only the ones that contribute. That leaves us with the expression for the 1 loop correction

$$\begin{aligned} I_{\{\mu\nu\nu\}} &= \frac{A_1}{3}\gamma_\mu p_\nu^2 + A_2\frac{2}{3}\gamma_\nu p_\mu p_\nu + E_1\gamma_\mu g_{\nu\nu} + E_2 p\cdot\gamma p_\mu g_{\nu\nu} + E_3\frac{p\cdot\gamma p_\mu p_\nu p_\nu}{p^2} & (7.14)\\ I_{[\mu\{\nu]\nu\}} &= 2B_1\gamma_\mu p_\nu^2 - 2B_2\gamma_\nu p_\mu p_\nu + F_1\gamma_\mu g_{\nu\nu} + F_2 p\cdot\gamma p_\mu g_{\nu\nu} & (7.15)\end{aligned}$$

Terms E_1 and F_1 describe mixing with the lower-dimensional operator $\bar{q}\gamma_\mu q$, but the mixing terms cancel for operators $O_{\{411\}} - \frac{1}{2}(O_{\{422\}} + O_{\{433\}})$ and $O_{[4\{1]1\}} - \frac{1}{2}(O_{[4\{2]2\}} + O_{[4\{3]3\}})$. Terms proportional to E_2 and F_2 also cancel; if we pick components of 4-momentum p to be numerically equal, terms proportional to E_3 will cancel as well and we are left only with terms that we want. If $A_1 = A_2$ and $B_1 = B_2$, then the operators will not mix; otherwise they

[1] For compactness of the notation, I will also use the subscript "S" to denote the symmetric piece, and subscript "H" to denote the piece with mixed symmetry

will. To extract the remaining coefficients A_i and B_i, first we pick 4-momentum p such that the components α and ν equal (in practice it is even better for numerical evaluation if they can be set to zero) and then we multiply with the appropriate γ matrix and take a trace

$$
\begin{aligned}
\frac{A_1}{3} &= \frac{1}{p_\nu^2}\frac{1}{d}\mathrm{Tr}_D\left[(I_{\{\mu\nu\nu\}} - I_{\{\mu\alpha\alpha\}})\gamma_\mu\right] \\
\frac{2A_2}{3} &= \frac{1}{p_\mu p_\nu}\frac{1}{d}\mathrm{Tr}_D\left[(I_{\{\mu\nu\nu\}} - I_{\{\mu\alpha\alpha\}})\gamma_\nu\right] \\
2B_1 &= \frac{1}{p_\nu^2}\frac{1}{d}\mathrm{Tr}_D\left[(I_{[\mu\{\nu]\nu\}} - I_{[\mu\{\alpha]\alpha\}})\gamma_\mu\right] \\
-2B_2 &= \frac{1}{p_\mu p_\nu}\frac{1}{d}\mathrm{Tr}_D\left[(I_{[\mu\{\nu]\nu\}} - I_{[\mu\{\alpha]\alpha\}})\gamma_\nu\right]
\end{aligned}
\tag{7.16}
$$

Another way of rewriting these coefficients is

$$
I_{\{\mu\nu\nu\}} = \frac{g_0^2 C_F}{16\pi^2}\left\{\frac{1}{3}\left(\gamma_\mu p_\nu^2 + 2\gamma_\nu p_\mu p_\nu\right)\Sigma_{SS} + 2\left(\gamma_\mu p_\nu^2 - \gamma_\nu p_\mu p_\nu\right)\Sigma_{SH}\right\} \tag{7.17}
$$

$$
I_{[\mu\{\nu]\nu\}} = \frac{g_0^2 C_F}{16\pi^2}\left\{2\left(\gamma_\mu p_\nu^2 - \gamma_\nu p_\mu p_\nu\right)\Sigma_{HH} + \frac{1}{3}\left(\gamma_\mu p_\nu^2 + 2\gamma_\nu p_\mu p_\nu\right)\Sigma_{HS}\right\} \tag{7.18}
$$

where Σ are our mixing coefficients

$$
\bar{\Sigma}_{SS} = \frac{A_2 - A_1}{3}\,, \quad \bar{\Sigma}_{SH} = (2A_2 - A_1)\,, \quad \bar{\Sigma}_{HH} = \frac{2B_1 + B_2}{3}\,, \quad \bar{\Sigma}_{HS} = 2(B_1 - B_2) \tag{7.19}
$$

and $\bar{\Sigma} = (g_0^2 C_F)/(16\pi^2)\,\Sigma$.

$\mathbf{4_2^-}$ representation

In the $\mathbf{4_2^-}$ representation, there can be no mixing since all indices are different. So if we take the operator

$$
O_{\{\mu\nu\alpha\}} = \frac{1}{3!}\left(O_{\mu\nu\alpha} + O_{\mu\alpha\nu} + O_{\nu\mu\alpha} + O_{\nu\alpha\mu} + O_{\alpha\mu\nu} + O_{\alpha\nu\mu}\right)\,. \tag{7.20}
$$

(with all three indices different) and Fourier transform to momentum space, we get the tree-level vertex

$$O_{\{\mu\nu\alpha\}} = \frac{1}{3}\left(\gamma_\mu p_\nu p_\alpha + \gamma_\nu p_\mu p_\alpha + \gamma_\alpha p_\mu p_\nu\right) \tag{7.21}$$

The 1-loop level correction of the operator $O_{\{\mu\nu\alpha\}}$ is then

$$I_{\{\mu\nu\alpha\}} = \frac{g_0^2 C_F}{16\pi^2}\frac{1}{3}\left(\gamma_\mu p_\nu p_\alpha + \gamma_\nu p_\mu p_\alpha + \gamma_\alpha p_\mu p_\nu\right)\Sigma_{\{\mu\nu\alpha\}} \tag{7.22}$$

and to extract Σ_3 we simply multiply by γ_μ, take a trace, and divide by $p_\nu p_\alpha$ (of course, we must pick the four vector p_μ to have at least two components nonzero).

7.2 Vertex diagram

The only difference between contributions $I_{\mu\nu}$ and $I_{\mu\nu\alpha}$ is an additional factor $i\bar{k}_\alpha$ since the photon propagator does not connect to the operator vertex. This will change behavior of integrals under parity; however, expansion in p also changes parity, we end up with

$$I_q(p) = I^{odd}_{phys} + I^{even}_{phys} \tag{7.23}$$

$$\begin{aligned} I^{odd}_{phys} &= (-ip\cdot\gamma\mathcal{A})\bar{I}^-_{odd}[\gamma_5](-ip\cdot\gamma\mathcal{A}) + \bar{I}^+_{odd}[\gamma_5] \\ &\quad + (-ip\cdot\gamma\mathcal{A})\bar{I}^-_{odd}[\gamma_5] + \bar{I}^+_{odd}[\gamma_5](-ip\cdot\gamma\mathcal{A}) \end{aligned} \tag{7.24}$$

$$\begin{aligned} I^{even}_{phys} &= (-ip\cdot\gamma\mathcal{A})\bar{I}^+_{even}[\gamma_5](-ip\cdot\gamma\mathcal{A}) + \bar{I}^-_{even}[\gamma_5] \\ &\quad + (-ip\cdot\gamma\mathcal{A})\bar{I}^+_{even}[\gamma_5] + \bar{I}^-_{even}[\gamma_5](-ip\cdot\gamma\mathcal{A})\,. \end{aligned} \tag{7.25}$$

After expanding in powers of p and eliminating integrals odd under parity, we are left with

$$\begin{aligned} I_q(p) &= (-ip\cdot\gamma\mathcal{A})\bar{I}^-_{odd}[\gamma_5](-ip\cdot\gamma\mathcal{A}) + p_\alpha p_\beta \frac{\partial^2 \bar{I}^+_{odd}}{\partial p_\alpha \partial p_\beta}[\gamma_5] \\ &\quad + (-ip\cdot\gamma\mathcal{A})p_\alpha \frac{\partial}{\partial p_\alpha}\bar{I}^+_{even}[\gamma_5] + p_\alpha \frac{\partial}{\partial p_\alpha}\bar{I}^-_{even}[\gamma_5](-ip\cdot\gamma\mathcal{A}) \end{aligned} \tag{7.26}$$

At zero momentum, the first term $\bar{I}^{-}_{odd}$ must be proportional to $\gamma_\mu g_{\nu\alpha}$ plus other permutations of indices. After substituting $p\cdot\gamma\gamma_\mu p\cdot\gamma = -p^2\gamma_\mu + 2p\cdot\gamma p_\mu$, we see that it's contribution will cancel for $\mathbf{8}_1^-$ representation. The second line also vanishes. The $\bar{I}^{\pm}_{even}$ terms have an even number of γ matrices, and expanding to first order in p, the result must be proportional to either $g_{\mu\nu}p_\alpha$, $\sigma_{\mu\nu}p_\alpha$ or $g_{\mu\nu}\sigma_{\alpha\rho}p_\rho$ (plus perturbations of indices). The $\sigma_{\mu\nu}$ term vanishes after symmetrization while the other two give contributions proportional to $p\cdot\gamma g_{\mu\nu}p_\alpha$ which vanish for $\bar{I}^{\pm}_{even}$ representation.

Multiplying the amplitude with γ_κ and taking the trace,

$$
\begin{aligned}
J_{\mu\nu\alpha\kappa} &\equiv \frac{1}{d}\mathrm{Tr}_D\left[\bar{I}_{\mu\nu\alpha}\gamma_\kappa\right] \\
&= g_0^2 C_F \int_{-\pi/a}^{\pi/a} \frac{d^dk}{(2\pi)^d}\frac{H_{\rho\sigma}\, i\bar{k}_\nu\, i\bar{k}_\alpha}{(\widehat{ap-k})^2+\mu^2}\Big\{\Big(g_{\rho\sigma}g_{\lambda\kappa}(\widetilde{ap+k})^2_\rho \\
&\quad -2g_{\rho\lambda}g_{\sigma\kappa}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\Big)\left[(\bar{k}^2 g_{\mu\lambda}-2\bar{k}_\mu\bar{k}_\lambda)\tilde{g}_-^2 \pm g_{\mu\lambda}\tilde{\sigma}_-^2\right] \\
&\quad +\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\pm(\bar{k}^2 g_{\mu\kappa}-2\bar{k}_\mu\bar{k}_\kappa)\tilde{g}_+^2 + g_{\mu\kappa}\tilde{\sigma}_+^2\right] \\
&\quad +r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\Big[g_{\sigma\kappa}\bar{k}_\mu(\tilde{g}_-\tilde{\sigma}_+ \pm \tilde{\sigma}_-\tilde{g}_+) \\
&\qquad -(\bar{k}_\kappa g_{\sigma\mu}-g_{\mu\kappa}\bar{k}_\sigma)(\tilde{g}_-\tilde{\sigma}_+ \mp \tilde{\sigma}_-\tilde{g}_+)\Big]\Big\}\ .
\end{aligned}
\tag{7.27}
$$

Coefficients $A_{1,2}$ and $B_{1,2}$ and $\Sigma_{\{\mu\nu\alpha\}}$ from previous section are then easily evaluated

$$
\begin{aligned}
\frac{A_1}{3} &= \frac{1}{p_\nu^2}\frac{1}{d}\mathrm{Tr}_D\left[(J_{\{\mu\nu\nu\}}-J_{\{\mu\alpha\alpha\}})\gamma_\mu\right] && (7.28)\\
\frac{2A_2}{3} &= \frac{1}{p_\mu p_\nu}\frac{1}{d}\mathrm{Tr}_D\left[(J_{\{\mu\nu\nu\}}-J_{\{\mu\alpha\alpha\}})\gamma_\nu\right] && (7.29)\\
2B_1 &= \frac{1}{p_\nu^2}\frac{1}{d}\mathrm{Tr}_D\left[(J_{[\mu\{\nu]\nu\}}-J_{[\mu\{\alpha]\alpha\}})\gamma_\mu\right] && (7.30)\\
-2B_2 &= \frac{1}{p_\mu p_\nu}\frac{1}{d}\mathrm{Tr}_D\left[(J_{[\mu\{\nu]\nu\}}-J_{[\mu\{\alpha]\alpha\}})\gamma_\nu\right]\ . && (7.31)\\
\Sigma_{\{\mu\nu\alpha\}} &= \frac{1}{p_\nu p_\alpha}\frac{1}{d}\mathrm{Tr}_D\left[J_{\{\mu\nu\alpha\}}\gamma_\mu\right]\ . && (7.32)
\end{aligned}
$$

M	NOS	HYP(SO)	APE(SO)	Gauge Pt.
Wilson	1.35	1.91	2.41	0.67
0.1	1.18	1.76	2.36	0.66
0.2	1.13	1.7	2.32	0.67
0.3	1.09	1.66	2.29	0.67
0.4	1.05	1.63	2.26	0.67
0.5	1.02	1.6	2.24	0.67
0.6	0.99	1.57	2.22	0.67
0.7	0.97	1.55	2.2	0.67
0.8	0.94	1.53	2.18	0.67
0.9	0.92	1.5	2.16	0.67
1.	0.89	1.48	2.15	0.67
1.1	0.87	1.46	2.13	0.67
1.2	0.85	1.44	2.12	0.67
1.3	0.82	1.42	2.1	0.67
1.4	0.8	1.39	2.09	0.67
1.5	0.77	1.37	2.07	0.67
1.6	0.74	1.34	2.05	0.67
1.7	0.71	1.31	2.03	0.67
1.8	0.67	1.27	2.	0.67
1.9	0.62	1.23	1.95	0.67

Table 7.1: Vertex Diagram contribution Σ_{SS}^{vertex} to the finite part of renormalization of the symmetric tensor $\bar{q}\gamma_{\{\mu}D_{\nu}D_{\nu\}}$ in the $\mathbf{8}_1^-$ representation.

7.3 Sails diagram

Similar to the 1-derivative case, sails diagrams can be written as

$$I_s^{(1)} = G_{\nu\rho}(p-k)V_\rho(p,k)S_s^{IN}O_{\mu\nu\alpha}^{(\nu)} + G_{\alpha\rho}(p-k)V_\rho(p,k)S_s^{IN}O_{\mu\nu\alpha}^{(\alpha)} \tag{7.33}$$

$$I_s^{(2)} = G_{\nu\rho}(p-k)O_{\mu\nu\alpha}^{(\nu)}S_s^{OUT}V_\rho(k,p) + G_{\alpha\rho}(p-k)O_{\mu\nu\alpha}^{(\alpha)}S_s^{OUT}V_\rho(k,p) \tag{7.34}$$

where the superscript for operator O denotes the gauge link to which the gluon propagator couples. Physical amplitudes are then obtained by adding the $5D$-to-physical propagator and amputating the external leg

$$\begin{aligned} I_1 &= \bar{S}_s^{OUT}I_s^{(1)} = \bar{S}_s^{OUT}V_\rho(p,k)S_s^{IN}O_{\mu\nu\alpha}^{(\nu)}G_{\nu\rho}(p-k) \\ &\quad + \bar{S}_s^{OUT}V_\rho(p,k)S_s^{IN}O_{\mu\nu\alpha}^{(\alpha)}G_{\alpha\rho}(p-k) \end{aligned} \tag{7.35}$$

M	NOS	HYP(SO)	APE(SO)	Gauge Pt.
Wilson	-0.06	-0.03	-0.02	0.
0.1	-0.05	-0.04	-0.02	-0.01
0.2	-0.05	-0.04	-0.02	-0.01
0.3	-0.05	-0.04	-0.02	-0.01
0.4	-0.05	-0.04	-0.02	-0.01
0.5	-0.05	-0.04	-0.02	-0.01
0.6	-0.05	-0.03	-0.02	0.
0.7	-0.05	-0.03	-0.02	0.
0.8	-0.05	-0.03	-0.02	0.
0.9	-0.05	-0.03	-0.01	0.
1.	-0.05	-0.03	-0.01	0.
1.1	-0.05	-0.03	-0.01	0.01
1.2	-0.05	-0.03	-0.01	0.01
1.3	-0.05	-0.03	-0.01	0.01
1.4	-0.05	-0.03	-0.01	0.01
1.5	-0.05	-0.03	0.	0.02
1.6	-0.05	-0.03	0.	0.02
1.7	-0.05	-0.02	0.	0.02
1.8	-0.05	-0.02	0.	0.03
1.9	-0.05	-0.02	0.	0.03

Table 7.2: Vertex Diagram contribution Σ^{vertex}_{SH} to the mixing between symmetric tensor $\bar{q}\gamma_{\{\mu}D_{\nu}D_{\nu\}}$ in the $\mathbf{8}^-_1$ representation and mixed-symmetry tensor $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\nu\}}$

$$\begin{aligned} I_2 &= I^{(2)}_s \bar{S}^{IN}_s = O^{(\nu)}_{\mu\nu\alpha} S^{OUT}_s V_\rho(k,p) G_{\nu\rho}(p-k) \bar{S}^{IN}_s \\ &\quad + O^{(\alpha)}_{\mu\nu\alpha} S^{OUT}_s V_\rho(k,p) G_{\alpha\rho}(p-k) \bar{S}^{IN}_s \end{aligned} \tag{7.36}$$

As in the case of the vertex diagram, part of $\bar{S}^{OUT}_s$ and $\bar{S}^{IN}_s$ proportional to $p\cdot\gamma$ will give us a contribution proportional to $p\cdot\gamma\, g_{\mu\nu}$ so we can neglect it from the start. That leaves us with

$$\begin{aligned} I_1 &= \bar{S}^{OUT} \left[\frac{g^{ab} H_{\nu\rho}}{(\widehat{ap-k})^2+\mu^2} \right] \left[-g_0 T^a \left(\frac{r}{2} (\widehat{ap+k})_\rho + i\gamma_\rho (\widetilde{ap+k})_\rho \right) \right] \\ &\quad \left[(g_- P_+ + g_+ P_-) \left(-i\bar{k}\cdot\gamma \right) + (\sigma_- P_+ + \sigma_+ P_-) \right] \left[\gamma_\mu (\widetilde{ap+k})_\nu \, i g_0 T^b \, i\bar{k}_\alpha [\gamma_5] \right] \\ &\quad + \bar{S}^{OUT} \left[\frac{g^{ab} H_{\alpha\rho}}{(\widehat{ap-k})^2+\mu^2} \right] \left[-g_0 T^a \left(\frac{r}{2} (\widehat{ap+k})_\rho + i\gamma_\rho (\widetilde{ap+k})_\rho \right) \right] \\ &\quad \left[(g_- P_+ + g_+ P_-) \left(-i\bar{k}\cdot\gamma \right) + (\sigma_- P_+ + \sigma_+ P_-) \right] \end{aligned}$$

M	"NOS"	HYP	APE	"GDP"
Wilson	1.215	1.964	2.537	0.881
0.1	1.180	1.866	2.513	0.881
0.2	1.120	1.810	2.465	0.881
0.3	1.071	1.765	2.428	0.881
0.4	1.028	1.725	2.398	0.881
0.5	0.988	1.690	2.371	0.881
0.6	0.952	1.658	2.348	0.881
0.7	0.917	1.627	2.326	0.881
0.8	0.883	1.598	2.307	0.881
0.9	0.850	1.569	2.288	0.881
1	0.817	1.541	2.269	0.881
1.1	0.784	1.513	2.251	0.881
1.2	0.751	1.485	2.233	0.881
1.3	0.717	1.456	2.214	0.881
1.4	0.681	1.426	2.194	0.881
1.5	0.643	1.393	2.173	0.881
1.6	0.602	1.358	2.148	0.881
1.7	0.557	1.318	2.120	0.881
1.8	0.503	1.270	2.085	0.881
1.9	0.437	1.210	2.036	0.881

Table 7.3: Vertex Diagram contribution $\Sigma^{vertex}_{\{\mu\nu\alpha\}}$ to renormalization of the tensor $\bar{q}\gamma_{\{\mu}D_\nu D_{\alpha\}}$ in the $\mathbf{4}_2^-$ representation.

$$\left[\gamma_\mu i\bar{p}_\nu(\widetilde{ap+k})_\alpha\, ig_0T^b\,[\gamma_5]\right] \tag{7.37}$$

$$\begin{aligned}
I_2 \;=\; & \left[\frac{g^{ab}H_{\nu\rho}}{(\widehat{ap-k})^2+\mu^2}\right]\left[\gamma_\mu(\widetilde{ap+k})_\nu\, ig_0T^a\, i\bar{k}_\alpha\,[\gamma_5]\right] \\
& \left[(-i\bar{k}\cdot\gamma)\,(g_+P_+ + g_+P_+) + (\sigma_+P_+ + \sigma_-P_-)\right] \\
& \left[-g_0T^b\left(\frac{r}{2}(\widehat{ap+k})_\rho + i\gamma_\rho(\widetilde{ap+k})_\rho\right)\right]\bar{S}_{IN} \\
+ & \left[\frac{g^{ab}H_{\alpha\rho}}{(\widehat{ap-k})^2+\mu^2}\right]\left[\gamma_\mu ip_\nu(\widetilde{ap+k})_\alpha\, ig_0T^a\,[\gamma_5]\right] \\
& \left[(-i\bar{k}\cdot\gamma)\qquad (g_+P_+ + g_+P_+) + (\sigma_+P_+ + \sigma_-P_-)\right] \\
& \left[-g_0T^b\left(\frac{r}{2}(\widehat{ap+k})_\rho + i\gamma_\rho(\widetilde{ap+k})_\rho\right)\right]\bar{S}_{IN}\,.
\end{aligned} \tag{7.38}$$

Contracting with $\bar{S}_{IN,OUT}$, we get

$$\begin{aligned}
I_1 &= \frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu\, i\bar{k}_\alpha}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r}{2}(\widehat{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_+ + \tilde{\sigma}_+\right)\right.\\
&\quad \left.+i\gamma_\rho(\widetilde{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_- + \tilde{\sigma}_-\right)\right]\gamma_\mu[\gamma_5]\\
&\quad +i\bar{p}_\nu\frac{g_0^2 C_F H_{\alpha\rho}(\widetilde{ap+k})_\alpha}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r}{2}(\widehat{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_+ + \tilde{\sigma}_+\right)\right.\\
&\quad \left.+i\gamma_\rho(\widetilde{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_- + \tilde{\sigma}_-\right)\right]\gamma_\mu[\gamma_5] \qquad (7.39)\\
I_2 &= \gamma_\mu[\gamma_5]\frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu\, i\bar{k}_\alpha}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r}{2}(\widehat{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_+ + \tilde{\sigma}_+\right)\right.\\
&\quad \left.+\left(-i\bar{k}\cdot\gamma\tilde{g}_- + \tilde{\sigma}_-\right)i\gamma_\rho(\widetilde{ap+k})_\rho\right]\\
&\quad +\gamma_\mu[\gamma_5]i\bar{p}_\nu\frac{g_0^2 C_F H_{\alpha\rho}(\widetilde{ap+k})_\alpha}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r}{2}(\widehat{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_+ + \tilde{\sigma}_+\right)\right.\\
&\quad \left.+\left(-i\bar{k}\cdot\gamma\tilde{g}_- + \tilde{\sigma}_-\right)i\gamma_\rho(\widetilde{ap+k})_\rho\right]. \qquad (7.40)
\end{aligned}$$

We see that the amplitude is just a sum of two parts

$$I_{\mu\nu\alpha} = \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\left(I_{\mu\nu}i\bar{k}_\alpha + I_{\mu\alpha}i\bar{p}_\nu\right) \qquad (7.41)$$

where $I_{\mu\nu}$ is the sail amplitude with 1 derivative. The remaining part (symmetrization, projecting out a piece proportional to a specific γ matrix) is the same as for vertex diagram.

7.4 Operator tadpole diagram

Operator tadpole for this diagram is more complicated since we can have gluons from one gauge link connecting to another gauge link. The Operator vertex expanded to second order in g_0 is:

$$\begin{aligned}
O^{(2)}_{\mu\nu\alpha} &= \gamma_\mu\left[-\frac{a^2}{2}\{T^a,T^b\}A^a_\nu A^b_\nu\bar{p}_\nu\right]\bar{p}_\alpha + \gamma_\mu\bar{p}_\nu\left[-\frac{a^2}{2}\{T^a,T^b\}A^a_\alpha A^b_\alpha\bar{p}_\alpha\right]\\
&\quad +\gamma_\mu\left[\cos\left(p-\frac{k}{2}\right)_\nu T^a A^a_\nu\right]\left[\cos\left(p-\frac{k}{2}\right)_\alpha T_b A^b_\alpha\right]
\end{aligned}$$

M	NOS	HYP(SO)	APE(SO)	Gauge Pt.
Wilson	-6.72	-0.13	6.43	6.93
0.1	-7.55	1.98	6.18	6.93
0.2	-7.34	2.12	6.27	6.93
0.3	-7.17	2.22	6.33	6.93
0.4	-7.03	2.3	6.38	6.93
0.5	-6.89	2.36	6.41	6.93
0.6	-6.77	2.42	6.43	6.93
0.7	-6.64	2.47	6.45	6.93
0.8	-6.53	2.51	6.46	6.93
0.9	-6.41	2.55	6.48	6.93
1.	-6.3	2.58	6.49	6.93
1.1	-6.18	2.62	6.5	6.93
1.2	-6.06	2.65	6.52	6.93
1.3	-5.93	2.69	6.52	6.93
1.4	-5.8	2.73	6.54	6.93
1.5	-5.66	2.77	6.55	6.93
1.6	-5.5	2.83	6.58	6.93
1.7	-5.32	2.89	6.61	6.93
1.8	-5.11	2.99	6.66	6.93
1.9	-7.49	3.13	6.76	6.93

Table 7.4: Sails Diagram Σ_{SS}^{sails} contribution to the finite part of the renormalization of the symmetric tensor $\bar{q}\gamma_{\{\mu}D_\nu D_{\nu\}}$ in the $\mathbf{8}_1^-$ representation.

$$\begin{aligned}
&= \gamma_\mu\left[-\frac{a^2}{2}\{T^a,T^b\}A_\nu^a A_\nu^b \bar{p}_\nu\right]\bar{p}_\alpha+\gamma_\mu\bar{p}_\nu\left[-\frac{a^2}{2}\{T^a,T^b\}A_\alpha^a A_\alpha^b \bar{p}_\alpha\right] \\
&\quad +a^2\gamma_\mu\left[-\cos^2\frac{k_\nu}{2}\frac{1}{2}(p_\alpha^2+p_\nu^2)g_{\nu\alpha}+\frac{\hat{k}_\nu\hat{k}_\alpha}{4}p_\nu p_\alpha\right]g_{\nu\alpha}T^aA_\nu^aT_bA_\alpha^b \\
&= \gamma_\mu\left[-\frac{a^2}{2}C_FA_\nu^aA_\nu^b\bar{p}_\nu\right]\bar{p}_\alpha+\gamma_\mu\bar{p}_\nu\left[-\frac{a^2}{2}C_FA_\alpha^aA_\alpha^b\bar{p}_\alpha\right] \\
&\quad +a^2\gamma_\mu\left[\left(\frac{\hat{k}_\nu^2}{4}-1\right)g_{\nu\alpha}+\frac{\hat{k}_\nu\hat{k}_\alpha}{4}\right]p_\nu p_\alpha C_FA_\nu^aA_\alpha^b\,, \qquad (7.42)
\end{aligned}$$

which leads to the tadpole contribution

$$I_q = i\gamma_\mu p_\nu p_\alpha \frac{g_0^2C_F}{16\pi^2}\Sigma^{OPtad} \qquad (7.43)$$

M	NOS	HYP(SO)	APE(SO)	Gauge Pt.
Wilson	0.21	0.52	0.08	0.07
0.1	0.24	0.11	0.1	0.08
0.2	0.23	0.11	0.09	0.08
0.3	0.23	0.11	0.09	0.08
0.4	0.22	0.1	0.09	0.08
0.5	0.22	0.1	0.09	0.08
0.6	0.21	0.1	0.08	0.07
0.7	0.21	0.09	0.08	0.07
0.8	0.2	0.09	0.08	0.07
0.9	0.2	0.09	0.08	0.07
1.	0.19	0.09	0.07	0.07
1.1	0.19	0.08	0.07	0.06
1.2	0.18	0.08	0.07	0.06
1.3	0.18	0.08	0.07	0.06
1.4	0.17	0.07	0.06	0.06
1.5	0.16	0.07	0.06	0.05
1.6	0.16	0.06	0.06	0.05
1.7	0.15	0.06	0.06	0.05
1.8	0.14	0.06	0.05	0.05
1.9	0.57	0.05	0.05	0.04

Table 7.5: Sails Diagram contribution Σ_{SH}^{sails} to the mixing between symmetric tensor $\bar{q}\gamma_{\{\mu}D_{\nu}D_{\nu\}}$ in the $\mathbf{8}_1^-$ representation and mixed-symmetry tensor $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\nu\}}$

where T is the tadpole integral

$$\begin{aligned}
\Sigma^{OPtad} &= 16\pi^2 \lim_{\mu\to 0}^{2} \int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \frac{-\frac{1}{2}(H_{\nu\nu}+H_{\alpha\alpha}) + g_{\nu\alpha}\left(\frac{\hat{k}_\nu^2}{4}-1\right)H_{\nu\alpha} + \frac{1}{4}\hat{k}_\nu H_{\nu\alpha}\hat{k}_\alpha}{\hat{k}^2+\mu^2} \\
&= \lim_{\mu\to 0} 16\pi^2 \int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \frac{\frac{1}{d}\sum_\rho\{-H_{\rho\rho}(1+g_{\nu\alpha}) + \frac{g_{\nu\alpha}}{4}\hat{k}_\rho H_{\rho\rho}\hat{k}_\rho\} + \frac{1}{4}\hat{k}_\nu H_{\nu\alpha}\hat{k}_\alpha}{\hat{k}^2+\mu^2}
\end{aligned} \tag{7.44}$$

$\mathbf{8}_1^-$ representation

In terms of symmetric and hybrid combinations this becomes

$$I_{\{\mu\nu\nu\}} = \frac{1}{d}\sum_\rho\left(-2H_{\rho\rho} + \frac{1}{2}\hat{k}_\rho^2 H_{\rho\rho}\right)\frac{\gamma_\mu p_\nu^2}{3}$$

M	"NOS"	HYP	APE	"GDP"
Wilson	-7.598	1.855	5.926	6.494
0.1	-8.463	1.422	5.679	6.494
0.2	-8.250	1.561	5.776	6.494
0.3	-8.075	1.661	5.836	6.494
0.4	-7.920	1.739	5.878	6.494
0.5	-7.778	1.804	5.908	6.494
0.6	-7.644	1.858	5.931	6.494
0.7	-7.515	1.906	5.949	6.494
0.8	-7.389	1.949	5.963	6.494
0.9	-7.265	1.987	5.976	6.494
1	-7.140	2.024	5.987	6.494
1.1	-7.013	2.059	5.998	6.494
1.2	-6.883	2.094	6.009	6.494
1.3	-6.748	2.131	6.021	6.494
1.4	-6.604	2.170	6.035	6.494
1.5	-6.450	2.215	6.053	6.493
1.6	-6.280	2.268	6.076	6.493
1.7	-6.088	2.336	6.110	6.493
1.8	-5.861	2.428	6.162	6.494
1.9	-5.572	2.568	6.252	6.493

Table 7.6: Sails diagram contribution $\Sigma^{sails}_{\{\mu\nu\alpha\}}$ to renormalization of the tensor $\bar{q}\gamma_{\{\mu}D_\nu D_{\alpha\}}$ in the $\mathbf{4}_2^-$ representation.

$$+\left(-\frac{1}{d}\sum_\rho H_{\rho\rho}+\frac{1}{4}\hat{k}_\mu H_{\mu\nu}\hat{k}_\nu\right)\frac{2\gamma_\nu p_\mu p_\nu}{3} \tag{7.45}$$

$$\begin{aligned}I_{[\mu\{\nu]\nu\}} &= \frac{1}{d}\sum_\rho\left(-2H_{\rho\rho}+\frac{1}{2}\hat{k}_\rho^2 H_{\rho\rho}\right)2\gamma_\mu p_\nu^2 \\ &\quad -\left(-\frac{1}{d}\sum_\rho H_{\rho\rho}+\frac{1}{4}\hat{k}_\mu H_{\mu\nu}\hat{k}_\nu\right)2\gamma_\nu p_\mu p_\nu\,.\end{aligned} \tag{7.46}$$

We can decompose operators O_{411} and $O_{141}+O_{114}$ in terms of symmetric operator $O_{\{411\}}$ and the operator $O_{[4\{1]1\}}$ with mixed symmetry, and vice versa:

$$O_{\{411\}} = \frac{1}{3}\left(O_{411}+[O_{141}+O_{114}]\right) \tag{7.47}$$

$$O_{[4\{1]1\}} = 2O_{411}-[O_{141}+O_{114}] \tag{7.48}$$

$$O_{411} = O_{\{411\}}+\frac{1}{3}O_{[4\{1]1\}} \tag{7.49}$$

$$[O_{141}+O_{114}] = 2O_{\{411\}}-\frac{1}{3}O_{[4\{1]1\}}\,, \tag{7.50}$$

which can be used to obtain the result in terms of symmetric and hybrid combination

$$\begin{aligned} I_{\{\mu\nu\nu\}} &= \frac{1}{3}\left(\frac{1}{d}\sum_{\rho}\left[-4+\frac{\hat{k}_\rho^2}{2}\right]H_{\rho\rho}+\frac{1}{2}\hat{k}_\mu H_{\mu\nu}\hat{k}_\nu\right)O_{\{\mu\nu\nu\}} \\ &\quad +\frac{1}{9}\left(\frac{1}{d}\sum_{\rho}\left[-1+\frac{\hat{k}_\rho^2}{2}\right]H_{\rho\rho}-\frac{1}{4}\hat{k}_\mu H_{\mu\nu}\hat{k}_\nu\right)O_{[\mu\{\nu]\nu\}} \qquad (7.51) \\ I_{[\mu\{\nu]\nu\}} &= 2\left(\frac{1}{d}\sum_{\rho}\left[-1+\frac{1}{2}\hat{k}_\rho^2\right]H_{\rho\rho}-\frac{1}{4}\hat{k}_\mu H_{\mu\nu}\hat{k}_\nu\right)O_{\{\mu\nu\nu\}} \\ &\quad +\frac{1}{3}\left(\frac{1}{d}\sum_{\rho}\left[-5+\frac{\hat{k}_\rho^2}{2}\right]H_{\rho\rho}+\frac{1}{4}\hat{k}_\mu H_{\mu\nu}\hat{k}_\nu\right)O_{[\mu\{\nu]\nu\}}\,. \qquad (7.52) \end{aligned}$$

For Wilson gluons without smearing this becomes

$$\begin{aligned} \Sigma^{OPtad} &= \lim_{\mu\to 0}16\pi^2\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{-(1+g_{\nu\alpha})+\frac{g_{\nu\alpha}}{2d}\hat{k}^2}{\hat{k}^2+\mu^2} \\ &= \lim_{\mu\to 0}16\pi^2\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\left(\frac{g_{\nu\alpha}}{2d}-\frac{1}{\hat{k}^2+\mu^2}\right) \\ &= 16\pi^2\left\{-(1+g_{\nu\alpha})Z_0+\frac{g_{\nu\alpha}}{2d}\right\}\,. \qquad (7.53) \end{aligned}$$

For the symmetric and mixed symmetry contributions $I_{\{411\}}$ and $I_{[4\{1]1\}}$ we get

$$\begin{aligned} I_{\{411\}} &= i\frac{g_0^2C_F}{16\pi^2}\left\{\frac{1}{3}\left[(2\pi^2-32\pi^2Z_0)\gamma_4 p_1^2+2(-16\pi^2Z_0)\gamma_1 p_1 p^4\right]\right\} \qquad (7.54) \\ I_{[4\{1]1\}} &= i\frac{g_0^2C_F}{16\pi^2}\left\{2(2\pi^2-32\pi^2Z_0)\gamma_4 p_1^2-2(-16\pi^2Z_0)\gamma_1 p_1 p^4\right\}\,, \qquad (7.55) \end{aligned}$$

or in terms of symmetric operator $O_{\{411\}}$ and mixed symmetry operator $O_{[4\{1]1\}}$

$$\begin{aligned} I_{\{411\}} &= \frac{g_0^2C_F}{16\pi^2}\left\{\Sigma_{SS}O_{\{411\}}+\Sigma_{SH}O_{[4\{1]1\}}\right\} \qquad (7.56) \\ I_{[4\{1]1\}} &= \frac{g_0^2C_F}{16\pi^2}\left\{\Sigma_{HH}O_{[4\{1]1\}}+\Sigma_{HS}O_{\{411\}}\right\} \qquad (7.57) \end{aligned}$$

with

$$\Sigma_{SS} = \frac{1}{3}\left(2\pi^2 - 64\pi^2 Z_0\right) \quad (7.58)$$

$$\Sigma_{SH} = \frac{1}{9}\left(2\pi^2 - 16\pi^2 Z_0\right) \quad (7.59)$$

$$(7.60)$$

$$\Sigma_{HS} = 4\pi^2 - 32\pi^2 Z_0 \quad (7.61)$$

$$\Sigma_{HH} = \frac{1}{3}\left(4\pi^2 - 80\pi^2 Z_0\right) , \quad (7.62)$$

which agrees with Capitani [2] and Göckler *et. al.*[5].

	no smearing	HYP	APE	gauge-part
Σ_{SS}	-26.0415	-6.9261	-4.9806	-4.2937
Σ_{SH}	-0.5259	-0.1895	-0.1263	-0.0726

Table 7.7: Tadpole contribution to symmetric and hybrid part Σ^{OPtad}.

$\mathbf{4_2^-}$ representation

Since all indices are different here, the result is simply

$$\begin{aligned}\Sigma^{OPtad} &= 16\pi^2 \lim_{\mu\to 0}^{2} \int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \frac{-H_{\rho\rho} + \frac{1}{4}\hat{k}_\rho H_{\rho\sigma}\hat{k}_\sigma}{\hat{k}^2+\mu^2} \\ &= \lim_{\mu\to 0} 16\pi^2 \int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \frac{-\frac{1}{d}\sum_\rho H_{\rho\rho} + \frac{1}{4}\hat{k}_\rho H_{\rho\sigma}\hat{k}_\sigma}{\hat{k}^2+\mu^2} \end{aligned} \quad (7.63)$$

with $\rho \neq \sigma$ in the second term.

	no smearing	HYP	APE	gauge-part
T	-0.15493	-0.04025	-0.02914	-0.02582
Σ^{OPTad}	-24.4656	-6.35603	-4.6016	-4.07733

Table 7.8: Tadpole contribution for $\bar{q}\gamma_{\{\mu}D_\nu D_{\alpha\}}$ operator in $\mathbf{4_2^-}$ representation.

7.5 Collecting results: renormalization coefficients for twist 2 operators with 2 derivatives

Finally, collecting results for twist-2 diagrams, we get the formula for renormalization constants

$$O^{ren}_{\{\mu\nu\alpha\}} = Z_{\{\mu\nu\alpha\}} O^{tree}_{\{\mu\nu\alpha\}} + Z_{[\mu\{\nu]\alpha\}} O^{tree}_{[\mu\{\nu]\alpha\}} \tag{7.64}$$

with

$$Z_{\{\mu\nu\alpha\}} = 1 + \frac{g_0^2 C_F}{16\pi^2} z_S \qquad Z_{[\mu\{\nu]\alpha\}} = \frac{g_0^2 C_F}{16\pi^2} z_H \tag{7.65}$$

and

$$\begin{aligned} z_S &= Z_{\{\mu\nu\alpha\},\{\mu\nu\alpha\}} = \left([\gamma_2 + \gamma^{vert} + \gamma^{sails}] \log p^2/\Lambda^2 + [\Sigma_2 + \Sigma^{vert} + \Sigma^{sails} + \Sigma^{OPtad}]\right) \\ &= \left(-\frac{26}{6} \log p^2/\Lambda^2 + \Sigma_{\{\mu\nu\nu\}}\right) . \end{aligned} \tag{7.66}$$

$$\begin{aligned} z_H &= Z_{\{\mu\nu\alpha\},[\mu\{\nu]\alpha\}} = \left([\gamma^{vert} + \gamma^{sails}] \log p^2/\Lambda^2 + [\Sigma^{vert} + \Sigma^{sails} + \Sigma^{OPtad}]\right) \\ &= \left(-2 \log p^2/\Lambda^2 + \Sigma_{[\mu\{\nu]\nu\}}\right) . \end{aligned} \tag{7.67}$$

M	NOS	HYP(SO)	APE(SO)	Gauge Pt.
Wilson	-19.56	-8.64	-1.56	-1.50
0.1	-20.75	-6.83	-1.97	-1.50
0.2	-20.74	-6.86	-2.	-1.50
0.3	-20.74	-6.89	-2.03	-1.50
0.4	-20.74	-6.92	-2.06	-1.50
0.5	-20.73	-6.95	-2.08	-1.50
0.6	-20.72	-6.98	-2.11	-1.50
0.7	-20.7	-7.	-2.13	-1.50
0.8	-20.67	-7.02	-2.15	-1.50
0.9	-20.62	-7.03	-2.16	-1.50
1.	-20.57	-7.04	-2.18	-1.50
1.1	-20.5	-7.04	-2.18	-1.50
1.2	-20.41	-7.03	-2.18	-1.50
1.3	-20.3	-7.	-2.18	-1.50
1.4	-20.17	-6.97	-2.17	-1.50
1.5	-20.01	-6.92	-2.16	-1.50
1.6	-19.81	-6.84	-2.12	-1.50
1.7	-19.55	-6.73	-2.08	-1.50
1.8	-19.21	-6.57	-1.99	-1.50
1.9	-21.41	-6.32	-1.85	-1.50

Table 7.9: Finite part $\Sigma_{\{\mu\nu\nu\}}$ of the renormalization coefficient for operator $\bar{q}\gamma_{\{\mu}D_\nu D_{\nu\}}$ in the $\mathbf{8}_1^-$ representation.

M	NOS	HYP(SO)	APE(SO)	Gauge Pt.
Wilson	-0.37	0.3	-0.06	0.
0.1	-0.33	-0.12	-0.06	0.
0.2	-0.34	-0.12	-0.06	0.
0.3	-0.34	-0.12	-0.06	0.
0.4	-0.35	-0.12	-0.06	0.
0.5	-0.35	-0.13	-0.06	0.
0.6	-0.36	-0.13	-0.06	0.
0.7	-0.36	-0.13	-0.06	0.
0.8	-0.37	-0.13	-0.06	0.
0.9	-0.37	-0.13	-0.06	0.
1.	-0.38	-0.14	-0.06	0.
1.1	-0.38	-0.14	-0.07	0.
1.2	-0.39	-0.14	-0.07	0.
1.3	-0.4	-0.14	-0.07	0.
1.4	-0.4	-0.14	-0.07	0.
1.5	-0.41	-0.15	-0.07	0.
1.6	-0.42	-0.15	-0.07	0.
1.7	-0.43	-0.15	-0.07	0.
1.8	-0.44	-0.16	-0.07	0.
1.9	-0.01	-0.16	-0.07	0.

Table 7.10: Finite part $\Sigma_{[\mu\{\nu]\nu\}}$ of the mixing between symmetric tensor $\bar{q}\gamma_{\{\mu}D_{\nu}D_{\nu\}}$ in the $\mathbf{8}_1^-$ representation and mixed-symmetry tensor $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\nu\}}$

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-19.	-6.03	-1.56	-1.5
0.1	-20.09	-6.71	-1.94	-1.5
0.2	-20.09	-6.74	-1.97	-1.5
0.3	-20.09	-6.78	-2.	-1.5
0.4	-20.08	-6.82	-2.04	-1.5
0.5	-20.08	-6.85	-2.07	-1.5
0.6	-20.06	-6.89	-2.1	-1.5
0.7	-20.04	-6.91	-2.12	-1.5
0.8	-20.01	-6.94	-2.14	-1.5
0.9	-19.97	-6.96	-2.16	-1.5
1.	-19.91	-6.97	-2.18	-1.5
1.1	-19.84	-6.97	-2.19	-1.5
1.2	-19.76	-6.97	-2.19	-1.5
1.3	-19.65	-6.95	-2.19	-1.5
1.4	-19.52	-6.93	-2.19	-1.5
1.5	-19.36	-6.88	-2.17	-1.5
1.6	-19.15	-6.81	-2.15	-1.5
1.7	-18.89	-6.71	-2.1	-1.5
1.8	-18.56	-6.56	-2.03	-1.5
1.9	-18.1	-6.32	-1.9	-1.5

Table 7.11: Finite part of the renormalization of the tensor $\bar{q}\gamma_{\{\mu}D_{\nu}D_{\alpha\}}$ in the $\mathbf{4}_2^-$ representation.

Chapter 8

Twist 2 operators with $\gamma_\mu[\gamma_5]$ and three derivatives

Aside from the mixing issue which is not present here, the procedure to calculate the renormalization coefficient of the operator $\bar{q}\gamma_\mu D_\nu D_\alpha D_\beta q(x)$ is completely analogous to the previous chapter where we evaluated renormalization of the equivalent operator with two derivatives.

8.1 Preliminaries

8.1.1 Operator vertex

As before, we expand the operator $O_{\mu\nu\alpha\beta} = \bar{q}(x)\gamma_\mu D_\nu D_\alpha D_\beta q(x)$ in a power series in g_0

$$O_{\mu\nu\alpha\beta} = O^{(0)}_{\mu\nu\alpha\beta} + g_0 O^{(1)}_{\mu\nu\alpha\beta} + g_0^2 O^{(2)}_{\mu\nu\alpha\beta} \tag{8.1}$$

and get the following vertices

$$\begin{aligned} O^{(0)}_{\mu\nu\alpha\beta} &= \gamma_\mu i\bar{k}_\nu\, i\bar{k}_\alpha\, i\bar{k}_\beta & (8.2)\\ O^{(1)}_{\mu\nu\alpha\beta} &= T^a\gamma_\mu \left(i\cos\frac{(ap+k)_\nu}{2} A^a_\nu\, i\bar{k}_\alpha\, i\bar{k}_\beta + i\bar{p}_\nu i\cos\frac{(ap+k)_\alpha}{2} A^a_\alpha\, i\bar{k}_\beta \right.\\ &\qquad \left. + i\bar{p}_\nu\, i\bar{p}_\alpha i\cos\frac{(ap+k)_\beta}{2} A^a_\beta \right) & (8.3)\end{aligned}$$

$$
\begin{aligned}
O^{(2)}_{\mu\nu\alpha\beta} &= \gamma_\mu\left[-\frac{a^2}{2}C_F A^a_\nu A^b_\nu \bar{p}_\nu\right]\bar{p}_\alpha\bar{p}_\beta + \gamma_\mu\bar{p}_\nu\left[-\frac{a^2}{2}C_F A^a_\alpha A^b_\alpha \bar{p}_\alpha\right]\bar{p}_\beta \\
&\quad +\gamma_\mu\bar{p}_\nu\bar{p}_\alpha\left[-\frac{a^2}{2}C_F A^a_\beta A^b_\beta \bar{p}_\beta\right] \\
&\quad +C_F\gamma_\mu\left[\cos\left(p-\frac{k}{2}\right)_\nu A^a_\nu\right]\left[\cos\left(p-\frac{k}{2}\right)_\alpha A^b_\alpha\right]\bar{p}_\beta \\
&\quad +C_F\gamma_\mu\bar{p}_\nu\left[\cos\left(p-\frac{k}{2}\right)_\alpha A^a_\alpha\right]\left[\cos\left(p-\frac{k}{2}\right)_\beta A^b_\beta\right] \\
&\quad +C_F\gamma_\mu\left[\cos\left(p-\frac{k}{2}\right)_\nu A^a_\nu\right](\overline{p+k})_\alpha\left[\cos\left(p-\frac{k}{2}\right)_\beta A^b_\beta\right] . \qquad (8.4)
\end{aligned}
$$

Just as in the last few chapters, in the last step we have performed the summation over the group index

$$
\mathrm{Tr}\sum_a \delta^{ab}\{T^a,T^b\} = \frac{N_c^2-1}{2N_c} = C_F . \qquad (8.5)
$$

8.1.2 Amplitude decomposition

For a general set of indices, there are $4! = 24$ permutations of indices; if we set two of them equal in pairs, we get

$$
O_{\{\mu\mu\nu\nu\}} = \frac{1}{6}\left(O_{\mu\mu\nu\nu} + O_{\mu\nu\mu\nu} + O_{\mu\nu\nu\mu} + O_{\nu\mu\mu\nu} + O_{\nu\mu\nu\mu} + O_{\nu\nu\mu\mu}\right) \qquad (8.6)
$$

which yields a tree-level operator vertex

$$
O_{\{\mu\mu\nu\nu\}} = \frac{1}{2}\left(\gamma_\mu p_\mu p_\nu^2 + \gamma_\nu p_\nu p_\mu^2\right) . \qquad (8.7)
$$

The amplitude will have the form

$$
\begin{aligned}
I_{\{\mu\mu\nu\nu\}} &= A_1\gamma_\mu p_\mu p_\nu^2 + A_2\gamma_\nu p_\nu p_\mu^2 + B_1\gamma_\mu p_\mu g_{\nu\nu} + B_2\gamma_\nu p_\nu g_{\mu\mu} + B_3 p\cdot\gamma g_{\mu\mu}g_{\nu\nu} \\
&\quad + B_4 p\cdot\gamma p_\mu p_\mu g_{\nu\nu} + B_5 p\cdot\gamma p_\nu p_\nu g_{\mu\mu} + B_6\frac{p\cdot\gamma p_\mu p_\mu p_\nu p_\nu}{p^2} \qquad (8.8)
\end{aligned}
$$

To extract coefficients of $\gamma_\mu p_\mu p_\nu^2$ and $\gamma_\nu p_\nu p_\mu^2$, we need to cancel unwanted contributions first. If we evaluate $Tr_D[I_{\{\mu\mu\nu\nu\}}(\gamma_\mu - \gamma_\alpha)]$ with 3 components $p_\mu = p_\nu = p_\alpha$ equal (and $\mu \neq \alpha \neq \nu$) and the fourth component vanishing $p_\beta = 0$, the trace will kill terms proportional to A_2 and B_2. The terms proportional to B_3, B_4, B_5 and B_6 will be proportional to $(p_\mu - p_\alpha)$ and will cancel as well. The only surviving contributions are A_1 and B_1.

To subtract out the term proportional to B_1, we have to subtract the trace $Tr_D[I_{\{\mu\mu\beta\beta\}}(\gamma_\mu - \gamma_\alpha)]$. As in the previous trace, terms proportional to B_3, B_4, B_5 and B_6 will be proportional to $(p_\mu - p_\alpha)$ and will cancel. Terms proportional to A_2 and B_2 vanish since the trace of γ matrices is zero, and the term proportional to A_1 will vanish since it's proportional to $p_\mu p_\beta^2$ and $p_\beta = 0$. The only surviving term is the B_1 term, which exactly cancels the B_1 term in the first trace. This yields

$$A_1 = \frac{1}{p_\mu^3}\frac{1}{d}\mathrm{Tr}_D\left[(I_{\{\mu\mu\nu\nu\}} - I_{\{\mu\mu\beta\beta\}})(\gamma_\mu - \gamma_\alpha)\right] \tag{8.9}$$

To extract the A_2 piece, we repeat the procedure above with γ_μ replaced by γ_ν to get

$$A_2 = \frac{1}{p_\mu^3}\frac{1}{d}\mathrm{Tr}_D\left[(I_{\{\mu\mu\nu\nu\}} - I_{\{\mu\mu\beta\beta\}})(\gamma_\nu - \gamma_\alpha)\right] \tag{8.10}$$

If there is no mixing, we must have $A_1 = A_2$ which holds both for Wilson and domain wall fermions.

While the expressions for A_1 and A_2 above are correct, to obtain the final answer, we need to exactly cancel terms proportional to B_1 which are several orders of magnitude bigger then terms proportional to A_1. The ratio of terms proportional to A_1 and B_1 is of the order of p^2. Since we are interested in the $p \to 0$ limit, we want to make p exponentially small so in practice the B_1 term is 5-10 orders of magnitude bigger than the A_1 term. This means that to obtain precision $1 : 10^n$ for the term A_1, we need to calculate the whole integral to a precision from $1 : 10^{n+5}$ to $1 : 10^{n+10}$ which, of course, takes *huge* amounts of time which we'd prefer to avoid if possible. In the expressions above, those B_1 contributions come from different areas of integration, so while the integrals do cancel, *integrands* do not. To make that happen, we can use the $H(4)$ symmetry of the integral to rotate the second term in the (ν, β) plane

$$A_1 = \frac{1}{p_\mu^3}\frac{1}{d}\mathrm{Tr}_D\left[(I_{\{\mu\mu\nu\nu\}}(p) - I_{\{\mu\mu\nu\nu\}}(p'))(\gamma_\mu - \gamma_\alpha)\right], \tag{8.11}$$

with

$$p=\{0,p_\alpha,p_\nu,p_\mu\}\,, \qquad \text{and} \qquad p'=\{p_\nu,p_\alpha,0,p_\mu\}\,. \tag{8.12}$$

That way we maximize the cancellation of two large contributions on the level of integrands themselves, so the numerical procedure of evaluating those integrals is much faster.

8.2 Vertex diagram

As in the case of 2 derivatives, the only difference between contributions $I_{\mu\nu}$ and $I_{\mu\nu\alpha\beta}$, is an additional factor $i\bar{k}_\alpha i\bar{k}_\beta$ since the photon propagator does not connect to the operator vertex. The amplitude is then obtained by expanding $\bar{I}^+_{odd}$ in power series

$$I_q(p) \;=\; p_\alpha p_\beta p_\gamma \frac{\partial^3 \bar{I}^+_{odd}}{\partial p_\alpha \partial p_\beta \partial p_\gamma}[\gamma_5]\,. \tag{8.13}$$

Multiplying the amplitude with γ_κ and taking the trace

$$\begin{aligned}
J_{\mu\nu\alpha\beta\kappa} \;&\equiv\; \frac{1}{d}\mathrm{Tr}_D\left[\bar{I}_{\mu\nu\alpha\beta}\gamma_\kappa\right] \\
&=\; g_0^2 C_F \int\limits_{-\pi/a}^{\pi/a} \frac{d^dk}{(2\pi)^d}\frac{H_{\rho\sigma}\, i\bar{k}_\nu\, i\bar{k}_\alpha\, i\bar{k}_\beta}{(\widetilde{ap-k})^2+\mu^2}\Big\{\Big(g_{\rho\sigma}g_{\lambda\kappa}(\widetilde{ap+k})^2_\rho \\
&\qquad -2g_{\rho\lambda}g_{\sigma\kappa}(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\Big)\left[(\bar{k}^2 g_{\mu\lambda}-2\bar{k}_\mu\bar{k}_\lambda)\tilde{g}_-^2 \pm g_{\mu\lambda}\tilde{\sigma}_-^2\right] \\
&\qquad +\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\left[\pm(\bar{k}^2 g_{\mu\kappa}-2\bar{k}_\mu\bar{k}_\kappa)\tilde{g}_+^2 + g_{\mu\kappa}\tilde{\sigma}_+^2\right] \\
&\qquad +r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[g_{\sigma\kappa}\bar{k}_\mu(\tilde{g}_-\tilde{\sigma}_+ \pm \tilde{\sigma}_-\tilde{g}_+)\right. \\
&\qquad\qquad \left.-(\bar{k}_\kappa g_{\sigma\mu}-g_{\mu\kappa}\bar{k}_\sigma)(\tilde{g}_-\tilde{\sigma}_+ \mp \tilde{\sigma}_-\tilde{g}_+)\right]\Big\}\,.
\end{aligned} \tag{8.14}$$

The amplitude is then given by

$$\Sigma_{\{\mu\mu\nu\nu\}} \;=\; \frac{1}{p_\mu^3}\left[(J_{\{\mu\mu\nu\nu\}\mu}(p)-J_{\{\mu\mu\nu\nu\}\mu}(p')) - (J_{\{\mu\mu\nu\nu\}\alpha}(p)-J_{\{\mu\mu\nu\nu\}\alpha}(p'))\right] \tag{8.15}$$

with

$$p=\{0,p_\mu,p_\mu,p_\mu\}\,, \qquad \text{and} \qquad p'=\{p_\mu,p_\mu,0,p_\mu\}\,. \tag{8.16}$$

Numerical results for vertex diagrams are given in the table (8.1). For Wilson fermions, they agree with results published in Capitani [2], but do not agree with Göckler *et. al.*[5].

M	NOS	HYP(SO)	APE(SO)	GDP.
Wilson	0.84	1.19	1.67	0.06
0.1	0.75	1.12	1.65	0.06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.6	0.41	0.78	1.36	0.06
1.7	0.39	0.75	1.33	0.06
1.8	0.36	0.72	1.31	0.06
1.9	0.31	0.67	1.27	0.06

Table 8.1: Vertex Diagram contribution to the 3-derivative operator

8.3 Sails diagram

The sails diagram for the 3-derivative operator can be easily constructed from diagrams for 1 and 2 derivative operators. Since gauge links to the left of the one coupling to the gluon give a factor $i\bar{p}_\mu$ of external momentum, and the ones on the right give a factor $i\bar{k}_\mu$ of internal momentum, the amplitude is given by

$$I_{\mu\nu\alpha\beta} = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} (I_{\mu\nu} i\bar{k}_\alpha i\bar{k}_\beta + i\bar{p}_\nu I_{\mu\alpha} i\bar{k}_\beta + i\bar{p}_\nu i\bar{p}_\alpha I_{\nu\beta}) \,, \tag{8.17}$$

where $I_{\mu\nu}$ is the sail amplitude with 1 derivative. The remaining part (symmetrization, projecting out a piece proportional to a specific γ matrix) is the same as for vertex diagram. Results for sails diagrams are given in the table (8.2). For Wilson fermions, they agree with results published in Göckler *et. al.*[5], but do not agree with Capitani [2].

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-5.84	2.69	7.19	7.97
0.1	-6.71	2.22	6.92	7.97
⋮	⋮	⋮	⋮	⋮
1.6	-4.58	3.11	7.34	7.97
1.7	-4.39	3.18	7.38	7.97
1.8	-4.17	3.28	7.44	7.97
1.9	-3.88	3.43	7.54	7.97

Table 8.2: Sails diagram contribution to renormalization of the $\bar{q}\gamma_{\{\mu}D_\mu D_\nu D_{\nu\}}q$ operator.

8.4 Operator tadpole diagram

The operator vertex expanded to second order in g_0 yields

$$\begin{aligned}
O^{(2)}_{\mu\nu\alpha\beta} &= \gamma_\mu\left[-\frac{a^2}{2}C_F A^a_\nu A^b_\nu \bar{p}_\nu\right]\bar{p}_\alpha\bar{p}_\beta + \gamma_\mu\bar{p}_\nu\left[-\frac{a^2}{2}C_F A^a_\alpha A^b_\alpha \bar{p}_\alpha\right]\bar{p}_\beta \\
&\quad + \gamma_\mu\bar{p}_\nu\bar{p}_\alpha\left[-\frac{a^2}{2}C_F A^a_\beta A^b_\beta \bar{p}_\beta\right] \\
&\quad + C_F\gamma_\mu\left[\cos\left(p-\frac{k}{2}\right)_\nu A^a_\nu\right]\left[\cos\left(p-\frac{k}{2}\right)_\alpha A^b_\alpha\right]\bar{p}_\beta \\
&\quad + C_F\gamma_\mu\bar{p}_\nu\left[\cos\left(p-\frac{k}{2}\right)_\alpha A^a_\alpha\right]\left[\cos\left(p-\frac{k}{2}\right)_\beta A^b_\beta\right] \\
&\quad + C_F\gamma_\mu\left[\cos\left(p-\frac{k}{2}\right)_\nu A^a_\nu\right](\overline{p-k})_\alpha\left[\cos\left(p-\frac{k}{2}\right)_\beta A^b_\beta\right] \qquad (8.18) \\
&= \gamma_\mu\left[-\frac{a^2}{2}C_F A^a_\nu A^b_\nu p_\nu\right]p_\alpha p_\beta + \gamma_\mu p_\nu\left[-\frac{a^2}{2}C_F A^a_\alpha A^b_\alpha p_\alpha\right]p_\beta \\
&\quad + \gamma_\mu p_\nu p_\alpha\left[-\frac{a^2}{2}C_F A^a_\beta A^b_\beta p_\beta\right] \\
&\quad + a^2 C_F\gamma_\mu p_\nu p_\alpha p_\beta\left\{\left[\left(\frac{\hat{k}^2_\nu}{4}-1\right)g_{\nu\alpha}+\frac{\hat{k}_\nu\hat{k}_\alpha}{4}\right]A^a_\nu A^b_\alpha\right. \\
&\quad + \left[\left(\frac{\hat{k}^2_\alpha}{4}-1\right)g_{\alpha\beta}+\frac{\hat{k}_\alpha\hat{k}_\beta}{4}\right]A^a_\alpha A^b_\beta \\
&\quad + \left(\left[\left(\frac{\hat{k}^2_\nu}{4}-1\right)g_{\nu\beta}+\frac{\hat{k}_\nu\hat{k}_\beta}{4}\right]\cos k_\alpha\right. \\
&\quad \left.\left. + g_{\nu\alpha}\bar{k}_\alpha\bar{k}_\nu\frac{\hat{k}_\beta}{2} + g_{\alpha\beta}\bar{k}_\alpha\frac{\hat{k}_\nu}{2}\bar{k}_\beta\right)A^a_\nu A^b_\alpha\right\}, \qquad (8.19)
\end{aligned}$$

which yields the tadpole contribution

$$I_q = i\gamma_\mu p_\nu p_\alpha p_\beta \frac{g_0^2 C_F}{16\pi^2} \Sigma^{OPtad} , \tag{8.20}$$

where Σ^{OPtad} is the tadpole integral

$$\begin{aligned}
\Sigma^{OPtad} = & \; 16\pi^2 \lim_{\mu\to 0} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ -\frac{1}{2}(H_{\nu\nu} + H_{\alpha\alpha} + H_{\beta\beta} + 2g_{\nu\alpha}H_{\nu\alpha} + 2g_{\alpha\beta}H_{\alpha\beta} \right. \\
& + 2g_{\nu\beta}H_{\nu\beta}(1 - \hat{k}_\alpha^2/2)) + \frac{1+g_{\nu\alpha}}{4}\hat{k}_\nu H_{\nu\alpha}\hat{k}_\alpha + \frac{1+g_{\alpha\beta}}{4}\hat{k}_\alpha H_{\alpha\beta}\hat{k}_\beta \\
& + \frac{1+g_{\nu\beta}}{4}\hat{k}_\nu H_{\nu\beta}\hat{k}_\beta \left(1 - \frac{\hat{k}_\alpha^2}{2}\right) + g_{\nu\alpha}\bar{k}_\alpha \bar{k}_\nu H_{\nu\beta}\frac{\hat{k}_\beta}{2} \\
& \left. + g_{\alpha\beta}\bar{k}_\alpha \frac{\hat{k}_\nu}{2} H_{\nu\beta}\bar{k}_\beta \right\} / \{\hat{k}^2 + \mu^2\}
\end{aligned} \tag{8.21}$$

$$\begin{aligned}
= & \; 16\pi^2 \lim_{\mu\to 0} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ -\frac{1}{2}(H_{\nu\nu} + H_{\alpha\alpha} + H_{\beta\beta} + 2g_{\nu\alpha}H_{\nu\alpha} + 2g_{\alpha\beta}H_{\alpha\beta} \right. \\
& + 2g_{\nu\beta}H_{\nu\beta} + \frac{1+g_{\nu\alpha}}{4}\hat{k}_\nu H_{\nu\alpha}\hat{k}_\alpha + \frac{1+g_{\alpha\beta}}{4}\hat{k}_\alpha H_{\alpha\beta}\hat{k}_\beta + \frac{1+g_{\nu\beta}}{4}\hat{k}_\nu H_{\nu\beta}\hat{k}_\beta \\
& + g_{\nu\beta}H_{\nu\beta}\frac{\hat{k}_\alpha^2}{2} - \frac{1+g_{\nu\beta}}{4}\hat{k}_\nu H_{\nu\beta}\hat{k}_\beta \frac{\hat{k}_\alpha^2}{2} \\
& \left. + g_{\nu\alpha}\bar{k}_\alpha \bar{k}_\nu H_{\nu\beta}\frac{\hat{k}_\beta}{2} + g_{\alpha\beta}\bar{k}_\alpha \frac{\hat{k}_\nu}{2} H_{\nu\beta}\bar{k}_\beta \right\} / \{\hat{k}^2 + \mu^2\} .
\end{aligned} \tag{8.22}$$

First two lines in the equation above are symmetric and yield the contribution

$$\Sigma_{sym}^{OPtad} = 16\pi^2 \lim_{\mu\to 0} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \left(-\frac{5}{2} + \frac{\hat{k}_\mu^2}{2} \right) H_{\mu\mu} + \frac{1}{2}\hat{k}_\mu H_{\mu\nu}\hat{k}_\nu \right\} / \{\hat{k}^2 + \mu^2\} . \tag{8.23}$$

Second two lines are not, so we take the symmetrized average $(\Sigma_{\mu\nu\nu} + \Sigma_{\nu\mu\nu} + \Sigma_{\nu\nu\mu})/3$

$$\Sigma_{\mu\nu\nu} = 16\pi^2 \lim_{\mu\to 0} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \hat{k}_\mu H_{\mu\nu}\hat{k}_\nu \left(-\frac{\hat{k}_\nu^2}{8} + \frac{\bar{k}_\nu^2}{2} \right) \right\} / \{\hat{k}^2 + \mu^2\} \tag{8.24}$$

$$\Sigma_{\nu\mu\nu} = 16\pi^2 \lim_{\mu\to 0} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ H_{\mu\mu} \left(\frac{\hat{k}_\nu^2}{2} - \frac{\hat{k}_\nu^2\hat{k}_\mu^2}{4} \right) \right\} / \{\hat{k}^2 + \mu^2\} \tag{8.25}$$

$$\Sigma_{\nu\nu\mu} = 16\pi^2 \lim_{\mu\to 0} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \hat{k}_\mu H_{\mu\nu} \hat{k}_\nu \left(-\frac{\hat{k}_\nu^2}{8} + \frac{\tilde{k}_\nu^2}{2} \right) \right\} / \{\hat{k}^2 + \mu^2\} \,, \tag{8.26}$$

which yields the symmetrized operator $O_{\{\mu\mu\nu\nu\}}$

$$\begin{aligned} \Sigma^{OPtad} &= 16\pi^2 \lim_{\mu\to 0} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \left\{ \left(-\frac{5}{2} + \frac{\hat{k}_\mu^2}{2} \right) H_{\mu\mu} + \frac{1}{2} \hat{k}_\mu H_{\mu\nu} \hat{k}_\nu \right. \\ &\left. + \frac{1}{3} \left[H_{\mu\mu} \frac{\hat{k}_\nu^2}{2} \left(1 - \frac{\hat{k}_\mu^2}{2} \right) + \hat{k}_\mu H_{\mu\nu} \hat{k}_\nu \left(1 - \frac{1}{2} \hat{k}_\nu^2 \right) \right] \right\} / \{\hat{k}^2 + \mu^2\} \,. \end{aligned} \tag{8.27}$$

For Wilson gluons without smearing this becomes

$$\Sigma^{OPtad} = 16\pi^2 \left\{ -\left(\frac{3}{2} + g_{\nu\alpha} + g_{\alpha\beta} + g_{\nu\beta} \right) Z_0 + \frac{g_{\nu\alpha} + g_{\alpha\beta} + 2g_{\nu\beta}}{2d} - g_{\nu\beta} Z_1 \right\} \,, \tag{8.28}$$

which agrees with Capitani [2] (but does not with Göckler *et. al.*[5]).

	no smearing	HYP	APE	gauge-part
T	-0.25659	-0.06094	-0.04012	-0.03208
Σ^{OPTad}	-40.5191	-14.4033	-11.095	-5.06609

Table 8.3: Tadpole contribution for $\bar{q}\gamma_{\{\mu} D_\mu D_\nu D_{\nu\}} q$ operator.

8.5 Collecting results: renormalization coefficients for twist 2 operators with 3 derivatives

Finally, collecting results for twist-2 diagrams, we get the formula for renormalization constants

$$\begin{aligned} Z^{ren}_{\{\mu\mu\nu\nu\}} &= 1 + \frac{g_0^2 C_F}{16\pi^2} \left([\gamma_2 + \gamma^{vert} + \gamma^{sails}] \log p^2 a^2 + \Sigma^{vert} + \Sigma^{sails} + \Sigma^{OPtad}] \right) \\ & 1 + \frac{g_0^2 C_F}{16\pi^2} \left(\gamma_{\{\mu\mu\nu\nu\}} \log p^2 a^2 + \Sigma_{\{\mu\mu\nu\nu\}} \right) \end{aligned} \tag{8.29}$$

with

$$\gamma_{\{\mu\mu\nu\nu\}} = \frac{157}{30} \tag{8.30}$$

and $\Sigma_{\{\mu\mu\nu\nu\}}$ given in table (8.4). While for Wilson fermions every diagram agrees either with Göckler *et. al.* [5], or with Capitani [2], final results do not agree with either since they do not agree on any single diagram.

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-33.67	-14.02	-7.65	-1.83
0.1	-34.81	-14.71	-8.06	-1.83
⋮	⋮	⋮	⋮	⋮
1.6	-33.69	-14.6	-8.17	-1.83
1.7	-33.42	-14.48	-8.11	-1.83
1.8	-33.07	-14.31	-8.03	-1.83
1.9	-32.59	-14.05	-7.87	-1.83

Table 8.4: Finite part $\Sigma_{\{\mu\mu\nu\nu\}}$ of the renormalization coefficient for operator $\bar{q}\gamma_{\{\mu}D_\nu D_\nu D_{\nu\}}$.

Chapter 9

Twist 2 operator $\bar{q}[\gamma_5]\sigma_{\mu\{\nu}D_{\alpha\}}q$

9.1 Preliminaries

Aside from slightly different γ-algebra, the procedure for calculating the amplitude for this operator is almost exactly the same as for $\bar{q}\gamma_\mu D_\nu q$ operator. Expansion of the operator vertex is the same with γ_μ being replaced by $\sigma_{\mu\nu}$

$$O_{\mu\nu\alpha} = O^{(0)}_{\mu\nu} + g_0 O^{(1)}_{\mu\nu} + g_0^2 O^{(2)}_{\mu\nu} \tag{9.1}$$

$$O^{(0)}_{\mu\nu\alpha} = i\sigma_{\mu\nu}\bar{k}_\alpha \tag{9.2}$$

$$O^{(1)}_{\mu\nu\alpha} = T^a i\sigma_{\mu\nu}\cos\frac{(ap+k)_\alpha}{2} \tag{9.3}$$

$$O^{(2)}_{\mu\nu\alpha} = -\frac{a^2}{2}\{T^a, T^b\}\sigma_{\mu\nu}\bar{p}_\alpha \to -\frac{a^2}{2}C_F\sigma_{\mu\nu}\bar{p}_\alpha\,. \tag{9.4}$$

The operator tadpole diagram does not depend on the γ matrix in the vertex so we can just copy the result

$$I_q = -\frac{1}{2}g_0^2 C_F\, i\sigma_{\mu\nu}p_\alpha T = -\frac{g_0^2 C_F}{16\pi^2}\, i\sigma_{\mu\nu}p_\alpha(8\pi^2 T) = i\sigma_{\mu\nu}p_\alpha\,\frac{g_0^2 C_F}{16\pi^2}\Sigma^{OPtad} \tag{9.5}$$

where T is the tadpole integral

$$T = \lim_{\mu\to 0} T(\mu^2) = \lim_{\mu\to 0}\frac{1}{d}\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\sum_\rho\frac{H_{\rho\rho}}{\hat{k}^2+\mu^2} \tag{9.6}$$

and

$$\Sigma^{OPtad} = -8\pi^2 T \tag{9.7}$$

already encountered in the self-energy renormalization.

	no smearing	HYP	APE	gauge-part
T	0.15493	0.05219	0.04202	0.03873
Σ^{OPTad}	-12.233	-4.121	-3.318	-3.058

Table 9.1: Results for tadpole integral T and Σ^{OPtad}.

To extract other contributions, we again use the fact that the general structure of a particular 1-loop twist-2 diagram for a twist 2 operator is

$$\begin{aligned} I_{\mu\{\nu\alpha\}} &= \langle q(p)\,|\sigma_{\mu\{\nu}D_{\alpha\}}|\,q(p)\rangle = c_1\sigma_{\mu\nu}p_\alpha + c_2\sigma_{\mu\alpha}p_\nu \\ &\quad + c_3\sigma_{\mu\rho}g_{\nu\alpha}p_\rho + c_4\sigma_{\nu\rho}g_{\mu\alpha}p_\rho + c_5\sigma_{\mu\nu}g_{\nu\alpha}p_\mu + c_5\sigma_{\mu\nu}g_{\mu\alpha}p_\nu \end{aligned} \tag{9.8}$$

so to extract the factor we need, we multiply by with appropriate σ matrix and take a trace

$$c_1 = \frac{1}{p_\alpha}\frac{1}{d}\mathrm{Tr}_D\left[I_{\mu\{\nu\alpha\}}\sigma_{\mu\nu}\right] \tag{9.9}$$

9.2 Vertex diagram

Again, just as in the case of $\gamma_\mu D_\nu$ operator, the amplitude for the vertex diagram for the $\sigma_{\mu\nu}D_\alpha$ operator is obtained from the amplitude for the current $\bar{q}\sigma_{\mu\nu}q$ by just multiplying with $i\bar{k}_\alpha$ factor.

For the $\bar{q}(x)\sigma_{\mu\nu}q(x)$ current, everything works the same except that the Dirac algebra is slightly more complicated. The physical amplitude for $\bar{q}(x)\sigma_{\mu\nu}q(x)$ is then given by

$$\begin{aligned} I^{\mu\nu\alpha} &= \frac{H_{\rho\sigma}i\bar{k}_\alpha}{(\widehat{ap-k})^2+\mu^2}\Big\{\frac{r^2}{4}(\widehat{ap+k})_\rho(\widehat{ap+k})_\sigma\,(\sigma_{\mu\nu}\tilde{\sigma}_+^2 \\ &\qquad [\mp][\bar{k}^2\sigma_{\mu\nu} - 2\bar{k}_\mu\sigma_{k\nu} + 2\bar{k}_\nu\sigma_{k\mu}]\tilde{g}_+^2) \\ &\quad + \Big[g_{\rho\sigma}(\widetilde{ap+k})_\rho^2\sigma_{\mu\nu} + 2(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma(g_{\rho\nu}\sigma_{\sigma\mu} - g_{\rho\mu}\sigma_{\sigma\nu})\Big] \\ &\qquad (\bar{k}^2\tilde{g}_-^2[\mp]\tilde{\sigma}_-^2) \end{aligned}$$

$$+2\left(g_{\rho\sigma}(\widetilde{ap+k})^2_\rho(\bar{k}_\nu\sigma_{k\mu}-\bar{k}_\mu\sigma_{k\nu})+2(\widetilde{ap+k})_\rho(\widetilde{ap+k})_\sigma\right.$$
$$\times[\sigma_{\sigma k}(\bar{k}_\nu g_{\rho\mu}-\bar{k}_\mu g_{\rho\nu})-\bar{k}_\rho(\bar{k}_\nu\sigma_{\sigma\mu}-\bar{k}_\mu\sigma_{\sigma\nu})]]\,\tilde{g}_-^2$$
$$+r(\widehat{ap+k})_\rho(\widetilde{ap+k})_\sigma\left[(\tilde{g}_-\tilde{\sigma}_+[\pm]g_+\sigma_-)(\bar{k}_\sigma\sigma_{\mu\nu}+g_{\sigma\nu}\sigma_{k\mu}-g_{\sigma\mu}\sigma_{k\nu})\right.$$
$$\left.+(\tilde{g}_-\tilde{\sigma}_+[\mp]g_+\sigma_-)(\bar{k}_\mu\sigma_{\sigma\nu}-\bar{k}_\nu\sigma_{\sigma\mu})]\right\}[\gamma_5]\,, \tag{9.10}$$

where

$$\sigma_{k\mu}\equiv\sum_\lambda\bar{k}_\lambda\sigma_{\lambda\mu}\,. \tag{9.11}$$

To get the total contribution for the vertex diagram, we symmetrize in ν and α, multiply with $\sigma_{\mu\nu}$, take a trace and divide by p_α and get the amplitude in the table (9.2).

	"no smearing"	HYP	APE	"gauge-part"
Wilson	0.98014	2.13774	2.96046	1.23372
0.1	1.90770	2.64477	3.33276	1.23374
0.2	1.76095	2.51270	3.22136	1.23374
0.3	1.64295	2.41133	3.13997	1.23369
0.4	1.54147	2.32526	3.07445	1.23373
0.5	1.44988	2.24870	3.01883	1.23373
0.6	1.36472	2.17981	2.97042	1.23373
0.7	1.28452	2.11511	2.92657	1.23373
0.8	1.20770	2.05477	2.88654	1.23373
0.9	1.13251	1.99549	2.84852	1.23373
1	1.05821	1.93705	2.81166	1.23372
1.1	0.98356	1.87853	2.77139	1.23354
1.2	0.90774	1.81872	2.73824	1.23374
1.3	0.82918	1.75607	2.69939	1.23374
1.4	0.74646	1.68932	2.65735	1.23380
1.5	0.65740	1.61649	2.61002	1.23374
1.6	0.55892	1.53379	2.55438	1.23373
1.7	0.44612	1.43651	2.48543	1.23377
1.8	0.31031	1.31562	2.39757	1.23374
1.9	0.13237	1.15193	2.26853	1.23370

Table 9.2: Vertex diagram amplitude for the operator $\bar{q}(x)[\gamma_5]\sigma_{\mu\{\nu}D_{\alpha\}}q(x)$.

9.3 Sails diagram

The procedure here is similar to the case with γ_μ matrix. Since amplitudes for two sails diagrams are related, we will evaluate them together. They are given by

$$I_s^{(1)} = G_{\alpha\rho}(p-k)V_\rho(p,k)S_s^{IN}O_{\mu\nu\alpha} \tag{9.12}$$

$$I_s^{(2)} = G_{\alpha\rho}(p-k)O_{\mu\nu\alpha}S_s^{OUT}V_\rho(k,p)\,. \tag{9.13}$$

Physical amplitudes are then obtained by adding the $5D$-to-physical propagator and amputating the external leg

$$I_1 = \bar{S}_s^{OUT}I_s^{(1)} = \bar{S}_s^{OUT}V_\rho(p,k)S_s^{IN}O_{\mu\nu\alpha}G_{\alpha\rho}(p-k) \tag{9.14}$$

$$I_2 = I_s^{(2)}\bar{S}_s^{IN} = O_{\mu\nu\alpha}S_s^{OUT}V_\rho(k,p)G_{\alpha\rho}(p-k)\bar{S}_s^{IN} \tag{9.15}$$

As in the case of the vertex diagram, part of $\bar{S}_s^{OUT}$ and $\bar{S}_s^{IN}$ proportional to $p\cdot\gamma$ will not contribute so we can neglect it from the start. That leaves us with

$$\begin{aligned} I_1 &= \bar{S}^{OUT}\left[\frac{g^{ab}H_{\alpha\rho}}{(\widehat{ap-k})^2+\mu^2}\right]\left[-g_0T^a\left(\frac{r}{2}(\widehat{ap+k})_\rho+i\gamma_\rho(\widetilde{ap+k})_\rho\right)\right] \\ &\quad \left[(g_-P_++g_+P_-)\left(-i\bar{k}\cdot\gamma\right)+(\sigma_-P_++\sigma_+P_-)\right] \\ &\quad \left[\sigma_{\mu\nu}(\widetilde{ap+k})_\nu\, ig_0T^b\,[\gamma_5]\right] \end{aligned} \tag{9.16}$$

$$\begin{aligned} I_1 &= \left[\frac{g^{ab}H_{\alpha\rho}}{(\widehat{ap-k})^2+\mu^2}\right]\left[\sigma_{\mu\nu}(\widetilde{ap+k})_\nu\, ig_0T^a\,[\gamma_5]\right] \\ &\quad \left[\left(-i\bar{k}\cdot\gamma\right)(g_+P_++g_+P_+)+(\sigma_+P_++\sigma_-P_-)\right] \\ &\quad \left[-g_0T^b\left(\frac{r}{2}(\widehat{ap+k})_\rho+i\gamma_\rho(\widetilde{ap+k})_\rho\right)\right]\bar{S}_{IN}\,. \end{aligned} \tag{9.17}$$

Contracting with $\bar{S}_{IN,OUT}$, we get

$$\begin{aligned} I_1 &= \frac{g_0^2C_FH_{\alpha\rho}(\widetilde{ap+k})_\alpha}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r}{2}(\widehat{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_++\tilde{\sigma}_+\right)\right. \\ &\quad \left.+i\gamma_\rho(\widetilde{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_-+\tilde{\sigma}_-\right)\right]\sigma_{\mu\nu}[\gamma_5] \end{aligned} \tag{9.18}$$

$$
\begin{aligned}
I_2 &= \sigma_{\mu\nu}[\gamma_5]\frac{g_0^2 C_F H_{\alpha\rho}(\widetilde{ap+k})_\alpha}{(\widehat{ap-k})^2+\mu^2}\left[\frac{r}{2}(\widehat{ap+k})_\rho\left(-i\bar{k}\cdot\gamma\tilde{g}_+ + \tilde{\sigma}_+\right)\right. \\
&\left. + \left(-i\bar{k}\cdot\gamma\tilde{g}_- + \tilde{\sigma}_-\right) i\gamma_\rho(\widetilde{ap+k})_\rho\right] . \qquad (9.19)
\end{aligned}
$$

Using the relations between γ matrices

$$
\gamma_\rho\bar{k}\cdot\gamma\sigma_{\mu\nu} + \sigma_{\mu\nu}\bar{k}\cdot\gamma\gamma_\rho = 2\left(\bar{k}_\rho\sigma_{\mu\nu} + \bar{k}_\nu\sigma_{\mu\rho} - \bar{k}_\mu\sigma_{\nu\rho} + g_{\nu\rho}\sigma_{k\mu} - g_{\mu\rho}\sigma_{k\nu}\right) , \qquad (9.20)
$$

and adding both terms, we get

$$
\begin{aligned}
I_{\mu\nu\alpha} &= \frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu}{(\widehat{ap-k})^2+\mu^2}\left\{-2\sigma_{\mu\nu}\frac{r}{2}(\widehat{ap+k})_\rho\tilde{\sigma}_+\right. \\
&-2(\widetilde{ap+k})_\rho\left(\bar{k}_\rho\sigma_{\mu\nu} + \bar{k}_\nu\sigma_{\mu\rho} - \bar{k}_\mu\sigma_{\nu\rho} + g_{\nu\rho}\sigma_{k\mu} - g_{\mu\rho}\sigma_{k\nu}\right)\tilde{g}_- \\
&\left. +\frac{r}{2}i(\widehat{ap+k})_\rho[\bar{k}\cdot\gamma,\sigma_{\mu\nu}]_\pm\tilde{g}_+ + i(\widetilde{ap+k})_\rho[\gamma_\rho,\sigma_{\mu\nu}]_\pm\tilde{\sigma}_-\right\} , \qquad (9.21)
\end{aligned}
$$

where $[,]_\pm$ is the commutator/anticommutator of γ matrices. For $p \to 0$, first two lines are odd while the third one is even. To order the p^1 the second line vanishes due to parity. That gives us the final result

$$
\begin{aligned}
I_{\mu\nu\alpha} &= \frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu}{(\widehat{ap-k})^2+\mu^2}\left\{-2\sigma_{\mu\nu}\frac{r}{2}(\widehat{ap+k})_\rho\tilde{\sigma}_+\right. \\
&\left. -2(\widetilde{ap+k})_\rho\left(\bar{k}_\rho\sigma_{\mu\nu} + \bar{k}_\nu\sigma_{\mu\rho} - \bar{k}_\mu\sigma_{\nu\rho} + g_{\nu\rho}\sigma_{k\mu} - g_{\mu\rho}\sigma_{k\nu}\right)\tilde{g}_-\right\} . \qquad (9.22)
\end{aligned}
$$

Symmetrizing in ν and α, multiplying with $\sigma_{\mu\nu}$, taking a trace and dividing by p_α we get the expression for the amplitude given in table (9.3).

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-5.07710	2.41098	4.99918	5.11699
0.1	-5.77155	2.10147	4.83422	5.11677
0.2	-5.60444	2.20219	4.90055	5.11676
0.3	-5.46570	2.27371	4.94089	5.11674
0.4	-5.34201	2.32922	4.96806	5.11696
0.5	-5.22749	2.37446	4.98743	5.11700
0.6	-5.11881	2.41260	5.00219	5.11677
0.7	-5.01375	2.44565	5.01304	5.11680
0.8	-4.91068	2.47505	5.02205	5.11671
0.9	-4.80824	2.50185	5.02970	5.11671
1	-4.70522	2.52700	5.03658	5.11671
1.1	-4.60042	2.55132	5.04319	5.11678
1.2	-4.49251	2.57568	5.05005	5.11683
1.3	-4.38003	2.60104	5.05764	5.11684
1.4	-4.26108	2.62859	5.06666	5.11680
1.5	-4.13319	2.65996	5.07814	5.11681
1.6	-3.99293	2.69759	5.09360	5.11671
1.7	-3.83489	2.74550	5.11590	5.11671
1.8	-3.64967	2.81120	5.15080	5.11671
1.9	-3.41727	2.91220	5.21316	5.11686

Table 9.3: Sails diagram contribution to the renormalization coefficient for the operator $\bar{q}\sigma_{\mu\{\nu}D_{\alpha\}}q$.

9.4 Collecting results: renormalization coefficients for twist 2 operators with $\sigma_{\mu\nu}$

Finally, collecting results for all twist-2 diagrams, we get the formula for renormalization constants

$$
\begin{aligned}
Z_{\mu\{\nu\alpha\}} &= 1+\frac{g_0^2C_F}{16\pi^2}\left([\gamma_2+\gamma^{vert}+\gamma^{sails}]\log p^2/\Lambda^2+[\Sigma_2+\Sigma^{vert}+\Sigma^{sails}+\Sigma^{OPtad}]\right) \\
&= 1+\frac{g_0^2C_F}{16\pi^2}\left(3\log p^2/\Lambda^2+\Sigma_{\mu\{\nu\alpha\}}\right) . \qquad (9.23)
\end{aligned}
$$

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-4.48	-3.06	-0.78	-1.5
0.1	-4.44	-3.02	-0.68	-1.5
0.2	-4.57	-3.17	-0.81	-1.5
0.3	-4.67	-3.29	-0.9	-1.5
0.4	-4.76	-3.4	-0.99	-1.5
0.5	-4.83	-3.49	-1.06	-1.5
0.6	-4.89	-3.58	-1.12	-1.5
0.7	-4.94	-3.65	-1.17	-1.5
0.8	-4.97	-3.72	-1.22	-1.5
0.9	-5.	-3.78	-1.26	-1.5
1.	-5.	-3.83	-1.3	-1.5
1.1	-5.	-3.88	-1.34	-1.5
1.2	-4.98	-3.92	-1.36	-1.5
1.3	-4.94	-3.95	-1.39	-1.5
1.4	-4.88	-3.97	-1.41	-1.5
1.5	-4.79	-3.98	-1.43	-1.5
1.6	-4.67	-3.97	-1.44	-1.5
1.7	-4.52	-3.95	-1.45	-1.5
1.8	-4.31	-3.9	-1.44	-1.5
1.9	-4.02	-3.8	-1.42	-1.5

Table 9.4: Finite part $\Sigma_{\mu\{\nu\alpha\}}$ of the lattice renormalization coefficient Z for operator $\bar{q}\sigma_{\mu\nu}D_\alpha q$

Chapter 10

Twist 3 operators d_1 and d_2

The procedure to calculate renormalization coefficients for twist 3 operators $\bar{q}\gamma_{[\mu}[\gamma_5]D_{\nu]}q$ and $\bar{q}\gamma_{[\mu}[\gamma_5]D_{\{\nu]}D_{\alpha\}}q$ is very similar to the procedure for symmetric twist 2 operators $\bar{q}\gamma_{\{\mu}[\gamma_5]D_{\nu\}}q$; and $\bar{q}\gamma_{\{\mu}[\gamma_5]D_\nu D_{\alpha\}}q$. Aside from anti-symmetrization (as opposed to symmetrization), the only difference comes from the mixing piece coming from sails diagrams. For sails diagrams, the amplitude is given by

$$
\begin{aligned}
I_{\mu\nu} \;=\;& \frac{g_0^2 C_F H_{\nu\rho}(\widetilde{ap+k})_\nu}{(\widehat{ap-k})^2+\mu^2}\Big\{-2\gamma_\mu\frac{r}{2}(\widehat{ap+k})_\rho\tilde{\sigma}_+ \\
& -2\left(\bar{k}_\mu\gamma_\rho(\widetilde{ap+k})_\rho-\bar{k}\cdot\gamma g_{\rho\mu}(\widetilde{ap+k})_\mu+\gamma_\mu\bar{k}_\rho(\widetilde{ap+k})_\rho\right)\tilde{g}_- \\
& +\frac{r}{2}i(\widehat{ap+k})_\rho[\bar{k}\cdot\gamma,\gamma_\mu]_\pm\tilde{g}_+ + i(\widetilde{ap+k})_\rho[\gamma_\rho,\gamma_\mu]_\pm\tilde{\sigma}_-\Big\}\,.
\end{aligned}
\tag{10.1}
$$

The first two lines are the same as for twist 2 operators. The third line, when expanded in powers of p gives a non-vanishing contribution of order p^0 and vanishing contribution of the order p^1. For twist 2 operators, it was either proportional to $g_{\mu\nu}$ (without the γ_5 matrix) which vanished for $\mathbf{6}_3^+$ representation and canceled out in $\mathbf{3}_1^+$ representation, or to $\sigma_{\mu\nu}$ (with the γ_5 matrix) which vanished after symmetrization in μ and ν. Here we are anti-symmetrizing in μ and ν so without the γ_5 matrix, the symmetric contribution proportional to $g_{\mu\nu}$ cancels out, but the antisymmetric piece proportional to $\sigma_{\mu\nu}$ gives us mixing with the lower-dimensional operator $\bar{q}\sigma_{\mu\nu}q$:

$$I^{mix}_{[\mu\nu]} = \frac{g_0^2 C_F H_{\nu\rho}\widetilde{(ap+k)}_\nu}{(\widehat{ap-k})^2+\mu^2}\left\{\frac{r}{2}i(\widehat{ap+k})_\rho[\bar{k}\cdot\gamma,\gamma_\mu]_-\tilde{g}_+ + i(\widetilde{ap+k})_\rho[\gamma_\rho,\gamma_\mu]_-\tilde{\sigma}_-\right\} \tag{10.2}$$

$$= \frac{g_0^2 C_F H_{\nu\rho}\tilde{k}_\nu}{\hat{k}^2+\mu^2}\left\{\frac{r}{4}\hat{k}_\rho\bar{k}_\alpha\sigma_{\alpha\mu}\tilde{g}_+ + \frac{\tilde{k}_\rho}{2}\sigma_{\rho\mu}\tilde{\sigma}_-\right\} \sim \sigma_{\mu\nu}\,. \tag{10.3}$$

However, for domain wall fermions both $\tilde{g}_+$ and $\tilde{\sigma}_-$ vanish in the $m \to 0$ limit so the mixing term vanishes as well. For the operator with 2 derivatives, $\bar{q}\gamma_{[\mu}[\gamma_5]D_{\{\nu]}D_{\alpha\}}q$, the same arguments apply since the integrand is just multiplied with one power of p_α or $\bar{k}_\alpha$.

10.1 Twist 3 operator $\bar{q}\gamma_{[\mu}[\gamma_5]D_{\nu]}q$

It was shown in chapter 6 that the general structure of a particular 1-loop diagram is given by

$$I_{\mu\nu} = \langle q(p)\,|\gamma_\mu D_\nu|\,q(p)\rangle = c_1\gamma_\mu p_\nu + c_2\gamma_\nu p_\mu + c_3 g_{\mu\nu}\gamma_\mu p_\mu + c_4 g_{\mu\nu} p\cdot\gamma + c_5\frac{p_\mu p_\nu}{p^2}p\cdot\gamma \tag{10.4}$$

Since $\mu \neq \nu$ in the $\mathbf{6}_3^-$ representation, terms with c_3, c_4 and c_5 do not contribute after anti-symmetrization, so the amplitude is given by

$$I_{[\mu\nu]} = c_1 - c_2 = \frac{1}{p_\nu}\frac{1}{d}\mathrm{Tr}\left[(I_{\mu\nu}-I_{\nu\mu})\gamma_\mu\right]\,. \tag{10.5}$$

where formulas for calculating c_1 and c_2 have been derived in chapter 6. The results for vertex, sails and tadpole diagrams, as well as the total finite contribution are given in tables 10.1, 10.2, 10.3, and 10.4, respectively.

10.2 Twist 3 operator $\bar{q}\gamma_{[\mu}[\gamma_5]D_{\{\nu]}D_{\alpha\}}q$

For this operator we repeat the procedure from section 7 for twist 2 operator $\bar{q}\gamma_{[\mu}[\gamma_5]D_{\nu]}q$ in $\mathbf{4}_2^-$ representation, the only difference is that we first anti-symmetrize in μ and ν and then

M	NOS	HYP	APE	GDP
Wilson	2.591	1.305	0.525	0.813
0.1	2.475	1.581	0.880	1.475
0.2	2.464	1.550	0.829	1.314
0.3	2.453	1.519	0.777	1.180
0.4	2.442	1.488	0.724	1.062
0.5	2.430	1.455	0.671	0.952
0.6	2.418	1.422	0.616	0.849
0.7	2.406	1.388	0.560	0.750
0.8	2.392	1.353	0.503	0.652
0.9	2.379	1.316	0.445	0.556
1	2.364	1.279	0.385	0.459
1.1	2.349	1.240	0.323	0.360
1.2	2.332	1.200	0.259	0.259
1.3	2.315	1.158	0.192	0.153
1.4	2.297	1.114	0.123	0.041
1.5	2.277	1.068	0.050	-0.079
1.6	2.256	1.020	-0.027	-0.212
1.7	2.232	0.969	-0.109	-0.362
1.8	2.206	0.916	-0.196	-0.538
1.9	2.178	0.859	-0.290	-0.761

Table 10.1: Finite part of the vertex diagram contribution to operator $\bar{q}\gamma_{[\mu}D_{\nu]}q$.

symmetrize in ν and α

$$I_{[\mu\{\nu]\alpha\}} = \frac{1}{p_\nu p_\alpha}\frac{1}{d}\mathrm{Tr}_D\left[(I_{\mu\nu\alpha}+I_{\mu\alpha\nu}-I_{\nu\mu\alpha}-I_{\nu\alpha\mu})\gamma_\mu\right]\,. \tag{10.6}$$

The results for vertex, sails and tadpole diagrams, as well as the total finite contribution are given in tables 10.5, 10.6, 10.7, and 10.8, respectively.

M	NOS	HYP	APE	GDP
Wilson	-1.845	6.128	8.101	7.037
0.1	-3.771	4.516	6.763	6.374
0.2	-3.294	4.942	7.136	6.536
0.3	-2.903	5.280	7.422	6.670
0.4	-2.557	5.571	7.662	6.788
0.5	-2.241	5.832	7.872	6.898
0.6	-1.942	6.073	8.063	7.001
0.7	-1.655	6.300	8.240	7.100
0.8	-1.375	6.519	8.408	7.198
0.9	-1.099	6.732	8.570	7.440
1	-0.821	6.942	8.731	7.391
1.1	-0.540	7.155	8.891	7.490
1.2	-0.251	7.371	9.056	7.591
1.3	0.050	7.597	9.227	7.697
1.4	0.369	7.835	9.410	7.809
1.5	0.712	8.094	9.610	7.930
1.6	1.090	8.382	9.835	8.062
1.7	1.519	8.715	10.099	8.212
1.8	2.026	9.117	10.427	8.389
1.9	2.670	9.648	10.874	8.612

Table 10.2: Finite part of the sails diagram contribution to operator $\bar{q}\gamma_{[\mu}D_{\nu]}q$.

	NOS	HYP	APE	GDP
T	0.15493	0.05219	0.04202	0.03873
Σ^{tad}	12.2328	4.12076	3.31777	3.058

Table 10.3: Finite part of the tadpole diagram contribution to operator $\bar{q}\gamma_{[\mu}D_{\nu]}q$.

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	0.37	-0.18	-0.11	0.
0.1	-1.87	-1.67	-1.21	0.
0.2	-1.55	-1.39	-0.96	0.
0.3	-1.3	-1.17	-0.79	0.
0.4	-1.07	-0.99	-0.64	0.
0.5	-0.87	-0.83	-0.52	0.
0.6	-0.66	-0.67	-0.41	0.
0.7	-0.46	-0.52	-0.31	0.
0.8	-0.25	-0.38	-0.22	0.
0.9	-0.04	-0.23	-0.13	0.
1.	0.19	-0.08	-0.03	0.
1.1	0.43	0.09	0.06	0.
1.2	0.69	0.26	0.16	0.
1.3	0.98	0.45	0.27	0.
1.4	1.3	0.66	0.4	0.
1.5	1.67	0.91	0.54	0.
1.6	2.11	1.2	0.72	0.
1.7	2.62	1.55	0.94	0.
1.8	3.26	2.01	1.24	0.
1.9	4.11	2.64	1.68	0.

Table 10.4: Finite part of the lattice renormalization coefficient for operator $\bar{q}\gamma_{[\mu}D_{\nu]}q$

M	NOS	HYP	APE	GDP
Wilson	0.989	0.740	0.411	0.585
0.1	0.974	0.907	0.664	1.172
0.2	0.982	0.897	0.634	1.023
0.3	0.985	0.882	0.601	0.902
0.4	0.986	0.865	0.565	0.797
0.5	0.987	0.846	0.528	0.701
0.6	0.986	0.827	0.490	0.612
0.7	0.984	0.806	0.452	0.527
0.8	0.982	0.784	0.412	0.445
0.9	0.979	0.762	0.372	0.365
1	0.976	0.739	0.330	0.285
1.1	0.973	0.715	0.288	0.204
1.2	0.970	0.690	0.244	0.121
1.3	0.966	0.664	0.199	0.035
1.4	0.963	0.638	0.153	-0.055
1.5	0.961	0.612	0.105	-0.153
1.6	0.959	0.584	0.055	-0.262
1.7	0.958	0.557	0.003	-0.386
1.8	0.961	0.531	-0.050	-0.534
1.9	0.970	0.509	-0.103	-0.726

Table 10.5: Finite part of the vertex diagram contribution to operator $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q$.

M	NOS	HYP	APE	GDP
Wilson	-4.049	5.335	8.453	7.788
0.1	-6.389	3.431	6.924	7.200
0.2	-5.810	3.949	7.360	7.350
0.3	-5.328	4.355	7.697	7.471
0.4	-4.906	4.700	7.973	7.576
0.5	-4.520	5.005	8.211	7.672
0.6	-4.159	5.284	8.424	7.761
0.7	-3.814	5.544	8.618	7.846
0.8	-3.478	5.792	8.800	7.928
0.9	-3.148	6.032	8.974	8.008
1	-2.817	6.268	9.143	8.089
1.1	-2.483	6.504	9.312	8.169
1.2	-2.140	6.744	9.483	8.252
1.3	-1.783	6.993	9.662	8.338
1.4	-1.405	7.257	9.853	8.429
1.5	-0.997	7.544	10.064	8.527
1.6	-0.547	7.866	10.303	8.635
1.7	-0.036	8.240	10.589	8.759
1.8	0.573	8.701	10.951	8.909
1.9	1.354	9.320	11.459	9.100

Table 10.6: Finite part of the sails diagram contribution to operator $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q$.

M	NOS	HYP	APE	GDP
T	-0.15493	-0.04025	-0.02914	-0.02582
Σ^{OPTad}	-24.4656	-6.35603	-4.6016	-4.07733

Table 10.7: Finite part of the tadpole diagram contribution to operator $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q$.

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-15.67	-3.77	-1.16	-0.5
0.1	-18.22	-5.66	-2.55	-0.5
0.2	-17.78	-5.27	-2.22	-0.5
0.3	-17.43	-4.97	-1.97	-0.5
0.4	-17.11	-4.72	-1.77	-0.5
0.5	-16.82	-4.5	-1.61	-0.5
0.6	-16.54	-4.29	-1.46	-0.5
0.7	-16.27	-4.1	-1.33	-0.5
0.8	-16.	-3.91	-1.2	-0.5
0.9	-15.72	-3.72	-1.08	-0.5
1.	-15.43	-3.52	-0.96	-0.5
1.1	-15.13	-3.32	-0.84	-0.5
1.2	-14.8	-3.11	-0.71	-0.5
1.3	-14.44	-2.88	-0.57	-0.5
1.4	-14.04	-2.63	-0.41	-0.5
1.5	-13.58	-2.33	-0.23	-0.5
1.6	-13.06	-1.99	-0.02	-0.5
1.7	-12.44	-1.57	0.26	-0.5
1.8	-11.67	-1.03	0.62	-0.49
1.9	-10.64	-0.27	1.17	-0.5

Table 10.8: Finite part of the renormalization of the operator $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q$.

Chapter 11

Renormalization coefficients for MILC lattices

This section repeats the calculations of previous sections for all operators we have calculated so far, but now for MILC lattices. Numbers quoted here are evaluated for $\beta = 6.96$ and $u_0 = 0.8739$. For these parameters we have

$$\frac{g_0^2 C_F}{16\pi^2} = \frac{1}{82.4309} \tag{11.1}$$

and for the 1-loop g evaluated from the value of the plaquette

$$\frac{g^2 C_F}{16\pi^2} = \frac{1}{53.6372} \,. \tag{11.2}$$

Full renormalization coefficients Z_i are evaluated from finite parts Σ_i using the formula

$$Z_i = 1 + \frac{g^2 C_F}{16\pi^2}\left(\gamma_i \log \Lambda^2 a^2 + \Sigma_i\right) \,, \tag{11.3}$$

where the scale Λ has been chosen to make the logarithmic term vanish

$$\Lambda = \frac{1}{a} \,. \tag{11.4}$$

One specific feature of the MILC gluon action is that the action coefficients (see chapter 3) themselves depend on the value of the plaquette u_0 which in itself depends on g_0 and g (3.17)

$$\alpha_s = \frac{g^2}{4\pi} = -4\frac{\log u_0}{3.0684} \,. \tag{11.5}$$

For the Wilson gluon actions (as well as some improved actions) this was not the case, so all the g dependence was in the the gluon-fermion vertex and could therefore be factored out. For the MILC gluon action we have to numerically evaluate all the coefficients separately for each different β. Since in practice we always work with M in the range $1.6 \leq M \leq 1.9$, we do not quote numbers for the full range of allowed values of M.

M	NOS	HYP	APE	GDP	NOS	HYP	APE	GDP
Wilson	6.92	-3.87	-5.46	-4.79	1.08	0.95	0.93	0.94
0.1	6.74	-4.02	-5.57	-4.79	1.08	0.95	0.93	0.94
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.6	6.12	-4.43	-5.8	-4.79	1.07	0.95	0.93	0.94
1.7	6.22	-4.36	-5.76	-4.79	1.08	0.95	0.93	0.94
1.8	6.37	-4.26	-5.71	-4.79	1.08	0.95	0.93	0.94
1.9	6.59	-4.11	-5.62	-4.79	1.08	0.95	0.93	0.94

Table 11.1: Finite part of the lattice self-energy renormalization (left) and the full lattice renormalization coefficient Z_2 (right)

M	NOS	HYP	APE	GDP
Wilson	0.	0.	0.	0.
0.1	36.21	5.64	1.7	0.
⋮	⋮	⋮	⋮	⋮
1.6	35.01	4.99	1.42	0.
1.7	35.21	5.09	1.42	0.
1.8	35.5	5.25	1.5	0.
1.9	35.92	5.5	1.63	0.

Table 11.2: Finite part of the lattice renormalization coefficient Σ_W which contributes to the additive renormalization of M and w_0

M	NOS	HYP	APE	GDP	NOS	HYP	APE	GDP
Wilson	1.	1.	1.	1.	1.08	0.95	0.93	0.94
0.1	-4.18	0.33	0.8	1.	-4.52	0.31	0.75	0.94
0.2	-1.47	0.69	0.91	1.	-1.58	0.66	0.85	0.94
0.3	-0.56	0.82	0.95	1.	-0.6	0.77	0.88	0.94
0.4	-0.1	0.88	0.97	1.	-0.1	0.83	0.9	0.94
0.5	0.18	0.91	0.98	1.	0.2	0.86	0.91	0.94
0.6	0.38	0.94	0.98	1.	0.41	0.89	0.91	0.94
0.7	0.52	0.96	0.99	1.	0.56	0.9	0.92	0.94
0.8	0.64	0.97	0.99	1.	0.68	0.92	0.92	0.94
0.9	0.73	0.98	1.	1.	0.79	0.93	0.93	0.94
1.	0.82	1.	1.	1.	0.88	0.94	0.93	0.94
1.1	0.91	1.01	1.	1.	0.97	0.95	0.93	0.94
1.2	0.99	1.02	1.01	1.	1.06	0.96	0.93	0.94
1.3	1.08	1.03	1.01	1.	1.16	0.98	0.94	0.94
1.4	1.19	1.05	1.01	1.	1.28	0.99	0.94	0.94
1.5	1.33	1.07	1.02	1.	1.42	1.02	0.95	0.94
1.6	1.51	1.11	1.03	1.	1.63	1.05	0.96	0.94
1.7	1.81	1.16	1.05	1.	1.95	1.1	0.97	0.94
1.8	2.4	1.27	1.08	1.	2.58	1.21	1.01	0.94
1.9	4.13	1.61	1.19	1.	4.46	1.53	1.1	0.94

Table 11.3: Total effect of 5D mass parameter renormalization $Z_W = (1-w_R^2)/(1-w_0^2)$ (left) and the total lattice renormalization coefficient $Z_q = Z_2 Z_W$ (right)

M	NOS	HYP	APE	GDP
Wilson A	11.37	1.91	0.62	0.
Wilson V	15.14	2.95	1.04	0.
0.1	13.06	2.28	0.72	0.
⋮	⋮	⋮	⋮	⋮
1.6	12.5	1.87	0.49	0.
1.7	12.61	1.94	0.53	0.
1.8	12.77	2.04	0.59	0.
1.9	13.	2.19	0.68	0.

Table 11.4: Finite part of the lattice renormalization coefficient $Z_{V,A}$

M	NOS	HYP	APE	GDP
Wilson S	13.38	2.67	-0.74	1.
Wilson P	20.93	4.75	0.1	1.
0.1	14.08	1.44	-1.93	1.
⋮	⋮	⋮	⋮	⋮
1.6	21.25	5.99	0.91	1.
1.7	22.17	6.54	1.24	1.
1.8	23.32	7.26	1.69	1.
1.9	24.86	8.29	2.39	1.

Table 11.5: Finite part of the lattice renormalization coefficient $Z_{S,P}$

M	NOS	HYP	APE	GDP
Wilson	11.21	1.3	0.53	1.
0.1	12.	1.87	0.93	1.
⋮	⋮	⋮	⋮	⋮
1.6	8.78	-0.23	-0.34	1.
1.7	8.62	-0.32	-0.4	1.
1.8	8.44	-0.43	-0.47	1.
1.9	8.22	-0.58	-0.59	1.

Table 11.6: Finite part of the lattice renormalization coefficient Z_T

M	NOS	HYP(SO)	APE(SO)	HYP(NSO)	APE(NSO)	GDP
Wilson	-2.96	-1.83	-0.03	-8.74	-7.53	-1.
0.1	-3.98	-2.49	-0.37	-9.47	-7.95	-1.
⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.6	-3.86	-2.82	-0.65	-9.61	-8.07	-1.
1.7	-3.68	-2.74	-0.62	-9.52	-8.02	-1.
1.8	-3.44	-2.62	-0.55	-9.38	-7.94	-1.
1.9	-3.1	-2.42	-0.44	-9.17	-7.81	-1.

Table 11.7: Finite part of the lattice renormalization coefficient $Z_{6_3^+}$ for the operator $\bar{q}(x)\gamma_{\{\mu}D_{\nu\}}q(x)$ in the 6_3^+ representation.

M	NOS	HYP(SO)	APE(SO)	HYP(NSO)	APE(NSO)	GDP
Wilson	-2.11	-1.58	0.11	-8.5	-7.4	-1.
0.1	-3.69	-2.34	-0.28	-9.32	-7.86	-1.
⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.6	-3.17	-2.51	-0.48	-9.31	-7.9	-1.
1.7	-2.95	-2.42	-0.44	-9.2	-7.84	-1.
1.8	-2.65	-2.28	-0.37	-9.05	-7.75	-1.
1.9	-2.26	-2.07	-0.25	-8.82	-7.61	-1.

Table 11.8: Finite part of the lattice renormalization coefficient $Z_{3_1^+}$ for the operator $\bar{q}(x)\gamma_{\{\mu}D_{\nu\}}q(x)$ in the 3_1^+ representation.

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-18.4	-5.84	-1.46	-1.49
0.1	-19.51	-6.55	-1.86	-1.49
⋮	⋮	⋮	⋮	⋮
1.6	-18.73	-6.56	-2.01	-1.49
1.7	-18.51	-6.46	-1.96	-1.49
1.8	-20.81	-6.31	-1.88	-1.49
1.9	-17.79	-6.07	-1.75	-1.49

Table 11.9: Finite part of the renormalization coefficient for the operator $\bar{q}(x)\gamma_{\{\mu}D_{\nu}D_{\alpha\}}q(x)$ in the 8_1^- representation ($\nu = \alpha$).

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-0.42	-0.13	-0.06	0.
0.1	-0.39	-0.12	-0.05	0.
⋮	⋮	⋮	⋮	⋮
1.6	-0.46	-0.15	-0.07	0.
1.7	-0.47	-0.15	-0.07	0.
1.8	-0.04	-0.16	-0.07	0.
1.9	-0.49	-0.16	-0.07	0.

Table 11.10: Finite part of the mixing between the symmetric operator $\bar{q}(x)\gamma_{\{\mu}D_{\nu}D_{\alpha\}}q(x)$ and the mixed-symmetry operator $\bar{q}(x)\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q(x)$ (both in the 8_1^- representation, $\nu = \alpha$).

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-17.63	-5.73	-1.45	-1.49
0.1	-18.66	-6.39	-1.83	-1.49
⋮	⋮	⋮	⋮	⋮
1.6	-17.9	-6.49	-2.03	-1.5
1.7	-17.68	-6.4	-1.99	-1.5
1.8	-17.38	-6.25	-1.91	-1.49
1.9	-16.97	-6.03	-1.78	-1.5

Table 11.11: Finite part of the renormalization coefficients for the operator $\bar{q}(x)\gamma_{\{\mu}D_{\nu}D_{\alpha\}}q(x)$ in the 4_2^- representation ($\mu \neq \nu \neq \alpha$).

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-30.14	-8.8	-2.75	-1.83
0.1	-31.21	-9.47	-3.16	-1.83
⋮	⋮	⋮	⋮	⋮
1.6	-30.24	-9.38	-3.26	-1.83
1.7	-29.98	-9.26	-3.21	-1.83
1.8	-29.67	-9.1	-3.12	-1.83
1.9	-29.23	-8.85	-2.98	-1.83

Table 11.12: Finite part of the lattice renormalization coefficient for the operator $\bar{q}(x)\gamma_{\{\mu}D_{\nu}D_{\alpha}D_{\beta\}}q(x)$ in the 2_1^+ representation ($\mu = \nu$, $\alpha = \beta$).

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-5.03	-2.98	-0.71	-1.5
0.1	-5.02	-2.93	-0.62	-1.5
⋮	⋮	⋮	⋮	⋮
1.6	-5.41	-3.87	-1.37	-1.5
1.7	-5.28	-3.85	-1.37	-1.5
1.8	-5.1	-3.81	-1.37	-1.5
1.9	-4.86	-3.72	-1.35	-1.5

Table 11.13: Finite part of the lattice renormalization coefficient for the operator $\bar{q}\sigma_{\mu\{\nu}D_{\alpha\}}q$

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-0.48	-0.24	-0.12	0.
0.1	-2.56	-1.7	-1.2	0.
⋮	⋮	⋮	⋮	⋮
1.6	1.27	1.11	0.7	0.
1.7	1.76	1.45	0.92	0.
1.8	2.37	1.89	1.21	0.
1.9	3.18	2.51	1.65	0.

Table 11.14: Finite part of the lattice renormalization coefficient for the twist 3 operator $\bar{q}\gamma_{[\mu}D_{\nu]}q$

M	NOS	HYP(SO)	APE(SO)	GDP
Wilson	-14.43	-3.6	-1.12	-0.5
0.1	-16.88	-5.45	-2.49	-0.5
⋮	⋮	⋮	⋮	⋮
1.6	-11.94	-1.85	0.01	-0.5
1.7	-11.35	-1.44	0.28	-0.5
1.8	-10.62	-0.92	0.64	-0.49
1.9	-9.65	-0.18	1.18	-0.5

Table 11.15: Finite part of the lattice renormalization coefficient for the twist 3 operator $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q$

operator	$H(4)$	NOS	HYP	APE
$\bar{q}[\gamma_5]q$	$1_1^{\pm}$	0.792	0.981	1.046
$\bar{q}[\gamma_5]\gamma_\mu q$	$4_4^{\mp}$	0.847	0.976	0.994
$\bar{q}[\gamma_5]\sigma_{\mu\nu}q$	$6_1^{\mp}$	0.883	0.992	0.993
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_{\nu\}}q$	$6_3^{\pm}$	0.991	0.979	0.954
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_{\nu\}}q$	$3_1^{\pm}$	0.982	0.975	0.951
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_\nu D_{\alpha\}}q$	$8_1^{\mp}$	1.134	0.988	0.934
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_\nu D_{\alpha\}}q$	mixing	5.71×10^{-3}	1.88×10^{-3}	8.21×10^{-4}
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_\nu D_{\alpha\}}q$	$4_2^{\mp}$	1.124	0.987	0.934
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_\nu D_\alpha D_{\beta\}}q$	$2_1^{\pm}$	1.244	0.993	0.919
$\bar{q}[\gamma_5]\sigma_{\mu\{\nu}D_{\alpha\}}q$	$8_1^{\pm}$	1.011	0.994	0.964
$\bar{q}[\gamma_5]\gamma_{[\mu}D_{\nu]}q$	$6_1^{\mp}$	0.979	0.982	0.989
$\bar{q}[\gamma_5]\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q$	$8_1^{\pm}$	0.955	0.959	0.965

Table 11.16: Full $\overline{MS}$ to lattice renormalization coefficients for $M = 1.7$ and tree-level g_0. By chiral symmetry matrix elements are the same (except for parity) with and without γ_5, and this is indicated by the $[\gamma_5]$ notation where the upper parity arises in the absence of γ_5.

operator	$H(4)$	NOS	HYP	APE
$\bar{q}[\gamma_5]q$	$1_1^{\pm}$	0.68	0.971	1.07
$\bar{q}[\gamma_5]\gamma_\mu q$	$4_4^{\mp}$	0.765	0.964	0.99
$\bar{q}[\gamma_5]\sigma_{\mu\nu}q$	$6_1^{\mp}$	0.821	0.987	0.989
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_{\nu\}}q$	$6_3^{\pm}$	0.986	0.968	0.929
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_{\nu\}}q$	$3_1^{\pm}$	0.972	0.962	0.925
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_\nu D_{\alpha\}}q$	$8_1^{\mp}$	1.206	0.982	0.898
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_\nu D_{\alpha\}}q$	mixing	8.78×10^{-3}	2.88×10^{-3}	1.26×10^{-3}
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_\nu D_{\alpha\}}q$	$4_2^{\mp}$	1.191	0.98	0.898
$\bar{q}[\gamma_5]\gamma_{\{\mu}D_\nu D_\alpha D_{\beta\}}q$	$2_1^{\pm}$	1.375	0.989	0.876
$\bar{q}[\gamma_5]\sigma_{\mu\{\nu}D_{\alpha\}}q$	$8_1^{\pm}$	1.018	0.991	0.945
$\bar{q}[\gamma_5]\gamma_{[\mu}D_{\nu]}q$	$6_1^{\mp}$	0.967	0.973	0.983
$\bar{q}[\gamma_5]\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q$	$8_1^{\pm}$	0.931	0.937	0.947

Table 11.17: Full $\overline{MS}$ to lattice renormalization coefficients for $M = 1.7$ and 1-loop expression for g. By chiral symmetry matrix elements are the same (except for parity) with and without γ_5, and this is indicated by the $[\gamma_5]$ notation where the upper parity arises in the absence of γ_5.

Chapter 12

Summary and conclusions

This thesis calculates renormalization coefficients for the self energy and quark bilinear operators on the lattice. While in the end we are interested in MILC gluon actions and Domain Wall fermions with HYP smearing, all calculations have been performed for several different actions on the lattice. All calculations have been performed in a general gauge. Results obtained lead to several main conclusions

- ◇ All the formulas derived are formulated in modular form where different modules contain details about the gluon action, fermion action, smearing, etc. This allows us to evaluate those formulas for different actions and different parameters and therefore check different pieces separately against published results. In the end, even though there are no published results for the particular combination of actions and operators that we are interested in, we were able to check every potential source of errors in the calculation, which provides a strong test of the accuracy of our results.
- ◇ Self energy renormalization can be large and essentially non-perturbative due to effects of the nonperturbative renormalization of the light $5D$ quark mode on the lattice. However, this effect is the same for all operators considered, and can be evaluated from a ratio of matrix elements for the exactly conserved non-local current and the local non-conserved current on the lattice.
- ◇ Results for the Wilson gluon action and for improved actions are qualitatively the same. Improvement does not have a significant effect on renormalization of twist 2

operators since differences are of the order of several percent or less.

- Smearing *does* make a significant difference. Both HYP and APE smeared renormalization coefficients are much closer to 1 then unsmeared coefficients, both for Wilson and Domain Wall fermions, which provides a strong evidence that renormalization effects are under control for smeared actions.

Appendix A

Notation

A.1 Sin-momenta

Throughout this paper we have used the notation

$$\hat{k}_\mu \equiv \frac{\sin\frac{ak_\mu}{2}}{a/2}, \qquad \hat{k}^2 = \sum_\lambda \left(\frac{\sin\frac{ak_\mu}{2}}{a/2}\right)^2 \tag{A.1}$$

$$\bar{k}_\mu \equiv \frac{\sin ak_\mu}{a}, \qquad \bar{k}^2 = \sum_\lambda \left(\frac{\sin ak_\mu}{a}\right)^2 \tag{A.2}$$

$$\tilde{k}_\mu \equiv \cos\frac{ak_\mu}{2}, \qquad \tilde{k}^2 = \sum_\lambda \left(\cos\frac{ak_\mu}{2}\right)^2 \tag{A.3}$$

After rescaling the integration variable, the same notation is used for dimensionless momentum as well ($a \to 1$). Whenever an explicit factor of a appears, it means we are working with dimensionless momentum; for example

$$\overline{(ap+k)}_\mu \equiv \sin(ap_\mu + k_\mu)\,, \qquad \widehat{(ap+k)}_\mu \equiv 2\sin\frac{ap_\mu + k_\mu}{2}\,, \qquad \text{etc.} \tag{A.4}$$

Table A.1 gives a list of acronyms used throughout the thesis.

acronym	stands for
NOS	no smearing
HYP	HYP smearing
APE	APE spearing
GDP	gauge dependent part
SO	smeared operator link
NSO	non-smeared operator link

Table A.1: Table of acronyms

A.2 Conventions

Unfortunately, there are several conventions used in perturbative lattice literature. There are two main issues: sign of r and Fourier transforms. To ensure "reflection positivity", one has to choose $r = \pm 1$, depending on the choice of the Lagrangian. For the "minus 1" convention, the Lagrangian is given by

$$\begin{aligned} \mathcal{L}_- &= -\frac{1}{2a}\sum_\mu \left[\bar{\psi}(x)(r-\gamma_\mu)U_\mu(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu})(r+\gamma_\mu)U^\dagger_\mu(x)\psi(x)\right] \\ &\quad +\bar{\psi}(x)\left(M+\frac{rd}{a}\right)\psi(x) \end{aligned} \tag{A.5}$$

and it becomes the proper Lagrangian for $r = -1$. This convention is used by Aoki and by Capitani (although Capitani claims that $r = 1$). The second convention is the "plus 1" convention

$$\begin{aligned} \mathcal{L}_+ &= \frac{1}{2a}\sum_\mu \left[\bar{\psi}(x)(r+\gamma_\mu)U_\mu(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu})(r-\gamma_\mu)U^\dagger_\mu(x)\psi(x)\right] \\ &\quad +\bar{\psi}(x)\left(M-\frac{rd}{a}\right)\psi(x) \end{aligned} \tag{A.6}$$

where the proper physics is obtained by setting $r = 1$. This is what Shamir uses.

The second issue concerns Fourier transforms. The original Wilson papers as well as

Shamir's paper use the convention

$$\psi_s(x) = \int_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d} e^{-ip\cdot x} \psi_s(p) \,, \qquad \bar{\psi}_s(x) = \int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d} e^{iq\cdot x} \bar{\psi}_s(q) \,, \tag{A.7}$$

to go to momentum space. This yields the Dirac operator

$$D(p) = S_F^{-1}(p) = -i\bar{p}\cdot\gamma \mp \frac{ra}{2}\hat{p}^2 + M \tag{A.8}$$

for $r = \pm 1$ convention. Aoki and Capitani use the convention

$$\psi_s(x) = \int_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \psi_s(p) \,, \qquad \bar{\psi}_s(x) = \int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d} e^{-iq\cdot x} \bar{\psi}_s(q) \,, \tag{A.9}$$

which yields the Dirac operator

$$D(p) = S_F^{-1}(p) = i\bar{p}\cdot\gamma \mp \frac{ra}{2}\hat{p}^2 + M \tag{A.10}$$

for $r = \pm 1$ convention. The fermion-gluon vertex also depends on the choice of "r" convention; for the "$r = \mp 1$" convention we have

$$\begin{aligned} \left[V_\rho^a\right]_{bc} &= -g_0 \left[T^a\right]_{bc} \left(\pm r \sin\frac{a(k+p)_\rho}{2} + i\gamma_\rho \cos\frac{a(k+p)_\rho}{2} \right) \\ &= -g_0 \left[T^a\right]_{bc} \left(\pm r \frac{a}{2} (\widehat{p+k})_\rho + i\gamma_\rho (\widetilde{p+k})_\rho \right) \,, \end{aligned} \tag{A.11}$$

Appendix B

$\overline{MS}$ results

B.1 Useful formulas

Here we collect some useful formulas. Feynman parameters are defined by

$$\frac{1}{a^n b^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 \frac{x^{n-1}(1-x)^{m-1}}{[ax+b(1-x)]^{n+m}} dx \tag{B.1}$$

$$\frac{1}{abc} = \int_0^1 dx \int_0^1 dy \frac{2x}{[a(1-x)+bxy+xx(1-y)]^3} . \tag{B.2}$$

Some d-dimensional integrals:

$$\int \frac{d^d d}{(2\pi)^d} \frac{1}{[k^2+\Delta]^n} = \frac{\Gamma\left(n-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2} \tag{B.3}$$

$$\int \frac{d^d d}{(2\pi)^d} \frac{k^2}{[k^2+\Delta]^n} = \frac{d}{2} \frac{\Gamma\left(n-\frac{d}{2}-1\right)}{(4\pi)^{d/2}\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2-1} \tag{B.4}$$

$$\int \frac{d^d d}{(2\pi)^d} \frac{\left(k^2\right)^2}{[k^2+\Delta]^n} = \frac{d(d+2)}{4} \frac{\Gamma\left(n-\frac{d}{2}-2\right)}{(4\pi)^{d/2}\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2-2} . \tag{B.5}$$

Expansion of Γ pole around $\varepsilon = 0$:

$$\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(n)} \left(\frac{1}{\Delta}\right)^{2-d/2} = \frac{1}{(4\pi)^2} \left[\frac{1}{\varepsilon} + \log 4\pi - \gamma_E - \log\Delta\right] \tag{B.6}$$

where $\varepsilon = 2 - d/2$. Some formulas for integration of symmetric terms in d-dimensions:

$$\int k^\mu k^\nu f(k^2)\, d^4k = \frac{g^{\mu\nu}}{d} \int k^d f(k^2)\, d^4k \tag{B.7}$$

$$\int k^\mu k^\nu k^\alpha k^\beta f(k^2)\, d^4k = \frac{g^{\mu\nu} g^{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\alpha\nu}}{d(d+2)} \int (k^2)^2 f(k^2)\, d^dk\,. \tag{B.8}$$

B.2 Self energy

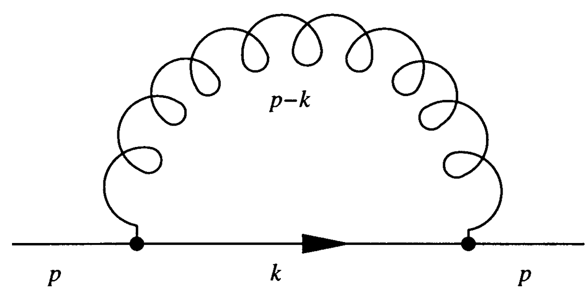

Figure B-1: Self energy in $\bar{MS}$ scheme.

In the $\overline{MS}$ scheme the amplitude is given by

$$\begin{aligned} I &= \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2 + \mu^2} \sum_\rho (-ig_0 T_a \gamma_\rho) \frac{-ik\cdot\gamma + m}{k^2 + m^2} (-ig_0 T_a \gamma_\rho) \\ &= -g_0^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{1}{p^2 + k^2 - 2p\cdot k + \mu^2} \frac{ik\cdot\gamma(d-2) + md}{k^2 + m^2} \end{aligned} \tag{B.9}$$

Using the Feynman parameters (B.1) to change the denominator

$$(p^2 + k^2 - 2p\cdot k + \mu^2)(k^2 + m^2) \to [k - xp]^2 + \underbrace{p^2 x(1-x) + m^2(1-x) + \mu^2 x}_{\Delta} \tag{B.10}$$

and shift the integration variable $k \to k + xp$ to get

$$\begin{aligned} I &= -g_0^2 C_F \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{i(k\cdot\gamma + xp\cdot\gamma)(d-2) + md}{[k^2 + \Delta]^2} \\ &= -g_0^2 C_F \int_0^1 dx (ixp\cdot\gamma(d-2) + md) \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 + \Delta]^2} \end{aligned}$$

$$= -g_0^2 C_F \int_0^1 dx\,[ixp\cdot\gamma(d-2)+md]\,\frac{1}{(4\pi)^2}\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log\Delta\right] \quad \text{(B.11)}$$

Now, we can choose two different ways of regulating this expression; first one is to let $m \to 0$ and $\mu \to 0$ and leave the momentum finite:

$$\begin{aligned} I &= -\frac{g_0^2 C_F}{16\pi^2}\left(ip\cdot\gamma\frac{d-2}{2}\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log p^2+2\right]\right. \\ &\quad \left. +md\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log p^2+2\right]\right) \\ &= -\frac{g_0^2 C_F}{16\pi^2}\left(ip\cdot\gamma\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log p^2+1\right]\right. \\ &\quad \left. +4m\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log p^2+\frac{3}{2}\right]\right) . \end{aligned} \quad \text{(B.12)}$$

The other is to set $m \to 0$ and $p \to 0$ and to keep μ finite

$$\begin{aligned} I &= -\frac{g_0^2 C_F}{16\pi^2}\left(ip\cdot\gamma\frac{d-2}{2}\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log \mu^2+\frac{1}{2}\right]\right. \\ &\quad \left. +md\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log \mu^2+1\right]\right) \\ &= -\frac{g_0^2 C_F}{16\pi^2}\left(ip\cdot\gamma\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log \mu^2-\frac{1}{2}\right]\right. \\ &\quad \left. +4m\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log \mu^2+\frac{1}{2}\right]\right) . \end{aligned} \quad \text{(B.13)}$$

So to switch from one regularization to another, one has to replace $-\log p^2 \to -\log\mu^2 - 3/2$ in the $ip\cdot\gamma$ term and $-d\log p^2 \to -d(\log\mu^2+1)$ in the mass term. The gauge-part of the propagator

$$G_{\rho\sigma}(k) = \frac{g_{\rho\sigma}-(1-\lambda)\frac{k_\rho k_\sigma}{k^2}}{k^2+\mu^2} \quad \text{(B.14)}$$

yields a contribution to the amplitude

$$I^g = -\frac{(1-\lambda)}{(p-k)^2+\mu^2}\left(-ig_0T_a\gamma_\rho\right)\frac{-ik\cdot\gamma+m}{k^2+m^2}\left(-ig_0T_a\gamma_\sigma\right)\frac{(p-k)_\rho(p-k)_\sigma}{(p-k)^2} . \quad \text{(B.15)}$$

Performing the γ-algebra

$$(p-k)\cdot\gamma(-ik\cdot\gamma+m)(p-k)\cdot\gamma=(p-k)^2(ik\cdot\gamma+m)-2i(p-k)\cdot\gamma(p-k)\cdot k \quad \text{(B.16)}$$

we can split the integral in two parts:

$$\begin{aligned} I_1^g &= -(1-\lambda)\int\frac{d^dk}{(2\pi)^d}\frac{-g_0^2C_F}{(p-k)^2+\mu^2}\frac{ik\cdot\gamma+m}{k^2+m^2} \\ I_2^g &= -(1-\lambda)\int\frac{d^dk}{(2\pi)^d}\frac{-g_0^2C_F}{(p-k)^2+\mu^2}\frac{1}{(p-k)^2}\frac{-2i(p-k)\cdot\gamma(p-k)\cdot k}{k^2+m^2}\,. \end{aligned} \quad \text{(B.17)}$$

Using the Feynman parametrization and shifting the integration variable, we get

$$\begin{aligned} I_1^g &= -(1-\lambda)\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+\Delta)^2}\left(-g_0^2C_F(i(k\cdot\gamma+xp\cdot\gamma)+m\right) \\ &= -(1-\lambda)\frac{-g_0^2C_F}{16\pi^2}\int_0^1 dx\,(ixp\cdot\gamma+m)\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\log\Delta\right] \\ &= -(1-\lambda)\frac{-g_0^2C_F}{16\pi^2}\left(ip\cdot\gamma\frac{1}{2}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log\mu^2\end{Bmatrix}+\begin{Bmatrix}2\\ 1/2\end{Bmatrix}\right]\right. \\ &\qquad \left.+m\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log\mu^2\end{Bmatrix}+\begin{Bmatrix}2\\ 1\end{Bmatrix}\right]\right)\,. \end{aligned} \quad \text{(B.18)}$$

For the second term we have to use the Feynman parametrization with 3 factors; this changes the denominator

$$\left[(p-k)^2+\mu^2\right](p-k)^2\left[k^2+m^2\right]\to\left(\left[k-p(1-x(1-y))\right]^2+\Delta\right)^3 \quad \text{(B.19)}$$

where

$$\Delta=p^2(1-x(1-y))x(1-y)+\mu^2(1-x)+m^2x(1-y)\,. \quad \text{(B.20)}$$

Shifting the integration variable $k\to k+p(1-x(1-y))$, the second part becomes

$$\begin{aligned} I_2^g &= -(1-\lambda)\int_0^1 dy\int_0^1 2x\,dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+\Delta)^3}(-g_0^2C_F)\left(-2i\left[p\cdot\gamma x(1-y)+k\cdot\gamma\right]\right. \\ &\qquad \left.\left[p^2x(1-y)(1-x(1-y))-k^2-k\cdot p(1-2x(1-y))\right]\right) \end{aligned}$$

$$
\begin{aligned}
&= -(1-\lambda)(-g_0^2 C_F)\int_0^1 dy \int_0^1 2x dx \int \frac{d^d k}{(2\pi)^d}\frac{1}{(k^2+\Delta)^3} \\
&\quad \left(-2ip\cdot\gamma\{x(1-y)\, p^2 x(1-y)(1-x(1-y))\right. \\
&\quad \left. -k^2\left[x(1-y)-\frac{1-2x(1-2y)}{d}\right]\}\right) \\
&= -(1-\lambda)\frac{-g_0^2 C_F}{16\pi^2}\int_0^1 dy\int_0^1 2x dx \left(\frac{-2ip\cdot\gamma x(1-y)\, p^2 x(1-y)(1-x(1-y))}{2\Delta}\right. \\
&\quad \left. -\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\log\Delta\right]\left[x(1-y)-\frac{1-2x(1-2y)}{d}\right]\right) \\
&= -(1-\lambda)\frac{-g_0^2 C_F}{16\pi^2}\, ip\cdot\gamma\left[\begin{Bmatrix}-1/3\\0\end{Bmatrix}\right. \\
&\quad \left. +\left(\frac{1}{2}-\frac{\varepsilon}{3}\right)\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log\mu^2\end{Bmatrix}\right]+\begin{Bmatrix}2/3\\13/12\end{Bmatrix}\right] \\
&= -(1-\lambda)\frac{-g_0^2 C_F}{16\pi^2}\, ip\cdot\gamma\left[\frac{1}{2}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log\mu^2\end{Bmatrix}\right]+\begin{Bmatrix}0\\3/4\end{Bmatrix}\right]. \qquad \text{(B.21)}
\end{aligned}
$$

Adding them up, we get

$$
\begin{aligned}
I^g &= -(1-\lambda)\frac{-g_0^2 C_F}{16\pi^2}\left(ip\cdot\gamma\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log\mu^2\end{Bmatrix}+\begin{Bmatrix}1\\1\end{Bmatrix}\right]\right. \\
&\quad \left. +m\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log\mu^2\end{Bmatrix}+\begin{Bmatrix}2\\1\end{Bmatrix}\right]\right). \qquad \text{(B.22)}
\end{aligned}
$$

From this we see that the finite part of the wave-function renormalization term Σ_2 is unchanged when we switch from $\log p^2$ regularization to $\log\mu^2$ regularization, while for the mass-renormalization we need to substitute $\log p^2 \to \log\mu^2+1$ or $\log\mu^2 \to \log p^2-1$.

To get renormalization constants for self energy, the one loop contribution

$$
\Sigma_1 = \frac{g_0^2 C_F}{16\pi^2}(ip\cdot\gamma\Sigma_2+m\Sigma_m) \qquad \text{(B.23)}
$$

has to be inserted into the propagator

$$
S_1 = \frac{1}{ip\cdot\gamma+m}+\frac{1}{ip\cdot\gamma+m}\Sigma_1\frac{1}{ip\cdot\gamma+m}\approx\frac{1}{ip\cdot\gamma+m-\Sigma_1} \qquad \text{(B.24)}
$$

which yields

$$S_1 = \frac{Z_2}{ip\cdot\gamma + mZ_m^{-1}} \tag{B.25}$$

with renormalization coefficients

$$Z_2 = 1 + \frac{g_0^2 C_F}{16\pi^2}\Sigma_2\,, \qquad Z_m^{-1} = 1 + \frac{g_0^2 C_F}{16\pi^2}(-\Sigma_m + \Sigma_2)\,. \tag{B.26}$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
Σ_2	-1	-1	1/2
Σ_m	-4	-6	-2
Z_2	-1	-1	1/2
Z_m^{-1}	3	5	5/2

B.3 Quark bilinears

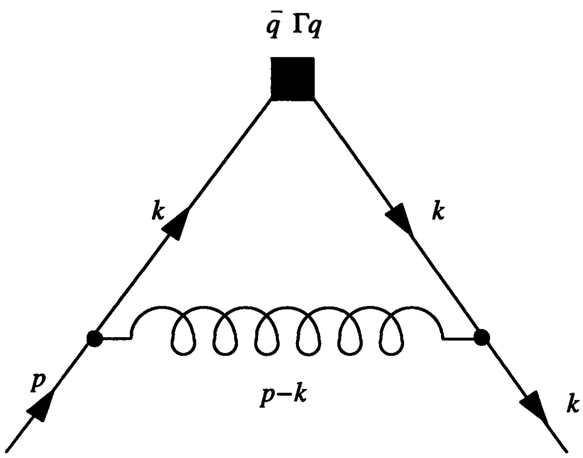

Figure B-2: Vertex diagram for quark bilinear operators

B.3.1 S,P currents

The amplitude for S and P currents in the $\overline{MS}$ scheme is given by

$$I_{S,P} = \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \sum_\rho (-ig_0 T_a \gamma_\rho) \frac{-ik\cdot\gamma+m}{k^2+m^2}[1,\gamma_5]\frac{-ik\cdot\gamma+m}{k^2+m^2}(-ig_0 T_a \gamma_\rho) \tag{B.27}$$

Color matrices add up to $g_0^2 C_F$, while the Dirac algebra yields

$$(-ik\cdot\gamma+m)[1](-ik\cdot\gamma+m) = (-k^2+m^2) - 2ik\cdot\gamma m \tag{B.28}$$

$$(-ik\cdot\gamma+m)[\gamma_5](-ik\cdot\gamma+m) = (k^2+m^2)\gamma_5 \tag{B.29}$$

so

$$\gamma_\rho(-ik\cdot\gamma+m)[1](-ik\cdot\gamma+m)\gamma_\rho = d(-k^2+m^2) - 2i(2-d)k\cdot\gamma m \tag{B.30}$$

$$\gamma_\rho(-ik\cdot\gamma+m)[\gamma_5](-ik\cdot\gamma+m)\gamma_\rho = d(-k^2-m^2)\gamma_;. \tag{B.31}$$

In the massless limit both expressions reduce to $-dk^2$ which yields the amplitude

$$\begin{aligned}
I_{S,P} &= (-g_0^2 C_F) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \frac{-d}{k^2} \\
&= (-g_0^2 C_F) \int_0^2 dx \int \frac{d^dk}{(2\pi)^d} \frac{-d}{k^2+\Delta} \\
&= \frac{g_0^2 C_F (4-2\varepsilon)}{16\pi^2} \int_0^2 dx \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log\Delta\right] \\
&= \frac{g_0^2 C_F (4-2\varepsilon)}{16\pi^2} \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \left\{\begin{matrix} 2-\log p^2 \\ 1-\log\mu^2 \end{matrix}\right\}\right] \\
&= \frac{g_0^2 C_F}{16\pi^2} 4 \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \left\{\begin{matrix} 3/2-\log p^2 \\ 1/2-\log\mu^2 \end{matrix}\right\}\right] \\
&= \frac{g_0^2 C_F}{16\pi^2} \left[4\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \left\{\begin{matrix} \log p^2 \\ \log\mu^2 \end{matrix}\right\}\right) + \left\{\begin{matrix} 6 \\ 2 \end{matrix}\right\}\right]
\end{aligned} \tag{B.32}$$

To get the renormalization coefficients $Z_{S,P}$ we need to subtract the Z_2 wave function renormalization which yields current renormalization coefficients

$$Z_{V,A} = 1 + \frac{g_0^2 C_F}{16\pi^2}\left([\gamma_2 + \gamma_{S,P}]\log Q^2/\Lambda^2 + [z_2 + \Sigma_{S,P}]\right) . \tag{B.33}$$

function	γ	Σ_P/z_P	Σ_μ/z_μ
vertex diag. 1	4	6	2
vertex diag. γ_5	4	6	2
Z_S	3	5	5/2
Z_P	3	5	5/2

B.3.2 V,A currents

In the $\overline{MS}$ scheme the amplitude is given by

$$I_{V,A} = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\sum_\rho(-ig_0T_a\gamma_\rho)\frac{-ik\cdot\gamma+m}{k^2+m^2}\gamma_\mu\frac{-ik\cdot\gamma+m}{k^2+m^2}(-ig_0T_a\gamma_\rho) \tag{B.34}$$

Color matrices add up to $g_0^2C_F$, while the Dirac algebra yields

$$(-ik\cdot\gamma+m)\gamma_\mu(-ik\cdot\gamma+m) = \gamma_\mu(k^2+m^2) - 2ik_\mu(-ik\cdot\gamma+m) \tag{B.35}$$

so

$$\gamma_\rho(-ik\cdot\gamma+m)\gamma_\mu(-ik\cdot\gamma+m)\gamma_\rho = (2-d)\gamma_\mu(k^2+m^2) - 2ik_\mu(-ik\cdot\gamma(d-2)+md)\,, \tag{B.36}$$

which yields the expression

$$\begin{aligned} I &= -(I_1+I_2) \\ I_1 &= g_0^2C_F\int\frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\frac{(2-d)\gamma_\mu}{k^2+m^2} \end{aligned}$$

$$I_2 = g_0^2 C_F \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \frac{-2ik_\mu(-ik\cdot\gamma(d-2)+md)}{(k^2+m^2)^2}. \tag{B.37}$$

Using the Feynman parameters (B.1) to change the denominator

$$(p^2+k^2-2p\cdot k+\mu^2)(k^2+m^2) \to [k-xp]^2 + \underbrace{p^2x(1-x)+m^2(1-x)+\mu^2 x}_{\Delta} \tag{B.38}$$

and shift the integration variable $k \to k+xp$ to get

$$\begin{aligned} I_1 &= -g_0^2 C_F \gamma_\mu (d-2) \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{1}{[k^2+\Delta]^2} \\ &= -\frac{g_0^2 C_F}{16\pi^2} \gamma_\mu (d-2) \int_0^1 dx \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi - \log\Delta\right] \end{aligned} \tag{B.39}$$

which yields the result

$$\begin{aligned} I_1 &= -\frac{g_0^2 C_F}{16\pi^2} \gamma_\mu (d-2) \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi + \begin{Bmatrix} 2-\log p^2 \\ 1-\log \mu^2 \end{Bmatrix}\right] \\ &= -\frac{g_0^2 C_F}{16\pi^2} \gamma_\mu \, 2 \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi + \begin{Bmatrix} 1-\log p^2 \\ 0-\log \mu^2 \end{Bmatrix}\right]. \end{aligned} \tag{B.40}$$

The second term becomes

$$\begin{aligned} I_2 &= g_0^2 C_F \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx\, 2(1-x) \frac{-2i\left(k_\mu + xp_\mu\right)\left(i[k\cdot\gamma + xp\cdot\gamma](2-d)+md\right)}{[k^2+\Delta]^3} \\ &= g_0^2 C_F \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx\, 2(1-x) \frac{2k_\mu k\cdot\gamma(d-2)}{[k^2+\Delta]^3} + O(p) \\ &= g_0^2 C_F \frac{2(d-2)}{d} \gamma_\mu \int_0^1 dx\, 2(1-x) \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{[k^2+\Delta]^3} + O(p) \\ &= \frac{g_0^2 C_F}{16\pi^2} \frac{2(d-2)}{d} \gamma_\mu \int_0^1 dx\, 2(1-x) \frac{d}{4} \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi - \log\Delta\right] + O(p) \\ &= \frac{g_0^2 C_F}{16\pi^2} \frac{(d-2)}{2} \gamma_\mu \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi + \begin{Bmatrix} 2-\log p^2 \\ \frac{3}{2}-\log \mu^2 \end{Bmatrix}\right] + O(p) \end{aligned}$$

$$= \frac{g_0^2 C_F}{16\pi^2}\gamma_\mu \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi + \left\{ \begin{array}{c} 1-\log p^2 \\ \frac{1}{2} - \log \mu^2 \end{array} \right\}\right] + O(p)\,, \tag{B.41}$$

where we have neglected terms linear in m and p. Adding those two contributions up we get

$$\begin{aligned} \Sigma_{V,A} &= \frac{g_0^2 C_F}{16\pi^2}\frac{(d-2)}{2}\gamma_\mu \left[\left(\frac{1}{2-d/2} - \gamma_E + \log 4\pi\right) - \left\{ \begin{array}{c} \log p^2 - 2 \\ \log \mu^2 - \frac{1}{2} \end{array} \right\}\right] \\ &= \frac{g_0^2 C_F}{16\pi^2}\gamma_\mu \left[\left(\frac{1}{2-d/2} - \gamma_E + \log 4\pi\right) + \left\{ \begin{array}{c} 1-\log p^2 \\ -\frac{1}{2} - \log \mu^2 \end{array} \right\}\right]. \end{aligned} \tag{B.42}$$

For massless fermions, results are the same for $\bar{q}\gamma_\mu\gamma_5 D_\nu q$ operator. To get the renormalization coefficients $Z_{V,A}$ we need to subtract the Z_2 wave function renormalization which yields current renormalization coefficients

$$Z_{V,A} = 1 + \frac{g_0^2 C_F}{16\pi^2}\left([\gamma_2 + \gamma_{V,A}]\log Q^2/\Lambda^2 + [z_2 + \Sigma_{V,A}]\right)\,. \tag{B.43}$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
vertex diag. γ_μ	1	1	-1/2
vertex diag. $\gamma_\mu\gamma_5$	1	1	-1/2
Z_V	0	0	0
Z_A	0	0	0

B.3.3 *T* current

Amplitude for T current in the $\overline{MS}$ scheme is given by

$$I_T = \int \frac{d^d k}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\sum_\rho (-ig_0 T_a \gamma_\rho)\frac{-ik\cdot\gamma + m}{k^2+m^2}\sigma_{\mu\nu}[\gamma_5]\frac{-ik\cdot\gamma+m}{k^2+m^2}(-ig_0 T_a\gamma_\rho) \tag{B.44}$$

As for other currents, color matrices add up to $g_0^2 C_F$; Dirac algebra yields

$$(-ik\cdot\gamma)\sigma_\mu\nu[\gamma_5](-ik\cdot\gamma) = \pm(-k^2\sigma_\mu\nu + 2(k_\mu\sigma_{k\nu} - k_\nu\sigma_{k\mu}))[\gamma_5] \tag{B.45}$$

where we have used the commutator between γ and σ matrices

$$\left[\sigma_{\mu\nu},\gamma_\rho\right] = -2i(g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu)\,. \tag{B.46}$$

Using that same relation and the definition of a product of two γ matrices

$$\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu} \tag{B.47}$$

we can evaluate

$$\begin{aligned}
\gamma_\rho\sigma_{\mu\nu}\gamma_\rho &= \gamma_\rho\left\{\gamma_\rho\sigma_{\mu\nu} - 2i(g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu)\right\} \\
&= d\sigma_{\mu\nu} - 2i(g_{\rho\mu}\gamma_\rho\gamma_\nu - g_{\rho\nu}\gamma_\rho\gamma_\mu) \\
&= d\sigma_{\mu\nu} - 2i(g_{\rho\mu}[g_{\rho\nu} - i\sigma_{\rho\nu}] - g_{\rho\nu}[g_{\rho\nu} - i\sigma_{\rho\nu}]) \\
&= (d-4)\sigma_{\mu\nu}
\end{aligned} \tag{B.48}$$

to get the expression for the amplitude

$$I_T = \int \frac{d^dk}{(2\pi)^d}\frac{-g_0^2C_F(d-4)}{(p-k)^2+\mu^2}\frac{-k^2\sigma_{\mu\nu}+2k_\mu k_\alpha\sigma_{\alpha\nu}-2k_\nu k_\alpha\sigma_{\alpha\mu}}{(k^2)^2}\,. \tag{B.49}$$

Using Feynman parameters we get and shifting the integration variable $k \to x + kp$ we get

$$\begin{aligned}
I_T &= (d-4)\int_0^1 2(1-x)\,dx\int\frac{d^dk}{(2\pi)^d}\frac{-g_0^2C_F}{(k^2+\Delta)^3}\left\{-(k+xp)^2\sigma_{\mu\nu}\right. \\
&\qquad \left. +2(k+xp)_\mu(k+xp)_\alpha\sigma_{\alpha\nu} - 2(k+xp)_\nu(k+xp)_\alpha\sigma_{\alpha\mu}\right\} \\
&= (d-4)\int_0^1 2(1-x)\,dx\int\frac{d^dk}{(2\pi)^d}\frac{-g_0^2C_F}{(k^2+\Delta)^3}\left\{-(k^2+x^2p^2)\sigma_{\mu\nu}\right. \\
&\qquad \left. +2\left(\frac{k^2}{d}g_{\mu\alpha}+x^2p_\mu p_\alpha\right)\sigma_{\alpha\nu} - 2\left(\frac{k^2}{d}g_{\nu\alpha}+x^2p_\nu p_\alpha\right)\sigma_{\alpha\mu}\right\} \\
&= (d-4)\int_0^1 2(1-x)\,dx\int\frac{d^dk}{(2\pi)^d}\frac{-g_0^2C_F}{(k^2+\Delta)^3}\left\{k^2\frac{4-d}{d}\sigma_{\mu\nu}\right. \\
&\qquad \left. -x^2\left(p^2\sigma_{\mu\nu} - 2p_\mu p_\alpha\sigma_{\alpha\nu} + 2p_\nu p_\alpha\sigma_{\alpha\mu}\right)\right\}\,.
\end{aligned} \tag{B.50}$$

Evaluating d-dimensional integrals and replacing $d = 4 - 2\varepsilon$ we get

$$\begin{aligned} I_T &= \frac{g_0^2 C_F}{16\pi^2} 2\varepsilon \int_0^1 2(1-x)\,dx \left\{ \frac{\varepsilon}{2}\sigma_{\mu\nu}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log\Delta\right] \right. \\ &\left. - \frac{x^2}{2\Delta}\left(p^2\sigma_{\mu\nu} - 2p_\mu p_\alpha \sigma_{\alpha\nu} + 2p_\nu p_\alpha \sigma_{\alpha\mu}\right) \right\} \end{aligned} \tag{B.51}$$

which vanishes for $\varepsilon \to 0$ since the $1/\varepsilon$ term is multiplied by ε^2 and the finite term is multiplied by ε.

As for other currents, to get the renormalization coefficient Z_T we need to subtract the Z_2 wave function renormalization which yields current renormalization coefficients

$$Z_T = 1 + \frac{g_0^2 C_F}{16\pi^2}\left([\gamma_2 + \gamma_T]\log Q^2/\Lambda^2 + [z_2 + \Sigma_T]\right) . \tag{B.52}$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
vertex diag. $\sigma_{\mu\nu}$	0	0	0
Z_T	-1	-1	1/2

B.4 Twist 2 operators $\gamma_\mu D_\nu$

We are interested in two operators, the symmetric combination $\bar{q}(x)\gamma_{\{\mu}D_{\nu\}}q(x)$ and the antisymmetric $\bar{q}(x)\gamma_{[\mu}D_{\nu]}q(x)$. Their tree-level contributions are given by

$$\langle \bar{q}(x)\gamma_{\{\mu}D_{\nu\}}q(x)\rangle = \frac{\gamma_\mu i p_\nu + \gamma_\nu i p_\mu}{2} \qquad \langle \bar{q}(x)\gamma_{[\mu}D_{\nu]}q(x)\rangle = \gamma_\mu i p_\nu - \gamma_\nu i p_\mu \tag{B.53}$$

B.4.1 Vertex diagram

Amplitude for the vertex diagram is essentially the same as for V and A currents multiplied by k_ν

$$
\begin{aligned}
I_{\mu\nu} &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \sum_{\rho} (-ig_0 T_a \gamma_\rho) \frac{-ik\cdot\gamma+m}{k^2+m^2} \gamma_\mu i k_\nu \frac{-ik\cdot\gamma+m}{k^2+m^2} (-ig_0 T_a \gamma_\rho) \\
&= I_1 + I_2 \qquad \text{(B.54)}
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= g_0^2 C_F \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \frac{(2-d)\gamma_\mu i k_\nu}{k^2} \\
I_2 &= g_0^2 C_F \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \frac{-2ik_\mu(-ik\cdot\gamma(d-2))ik_\nu}{(k^2)^2} . \qquad \text{(B.55)}
\end{aligned}
$$

Using the Feynman parameters (B.1) and shifting the integration variable $k \to k + xp$ we get

$$
\begin{aligned}
I_1 &= -g_0^2 C_F \gamma_\mu (d-2) \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{i(k_\nu + x p_\nu)}{[k^2+\Delta]^2} \\
&= -\frac{g_0^2 C_F}{16\pi^2} \gamma_\mu i p_\nu (d-2) \int_0^1 x\, dx \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi - \log\Delta \right] \qquad \text{(B.56)}
\end{aligned}
$$

which yields the result

$$
\begin{aligned}
I_1 &= \frac{g_0^2 C_F}{16\pi^2} \gamma_\mu i p_\nu \frac{d-2}{2} \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \left\{ \begin{array}{c} 2 - \log p^2 \\ 1/2 - \log\mu^2 \end{array} \right\} \right] \\
&= \frac{g_0^2 C_F}{16\pi^2} \gamma_\mu i p_\nu \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \left\{ \begin{array}{c} 1 - \log p^2 \\ -1/2 - \log\mu^2 \end{array} \right\} \right] . \qquad \text{(B.57)}
\end{aligned}
$$

The second term becomes

$$
\begin{aligned}
I_2 &= g_0^2 C_F \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx\, 2(1-x) \frac{-2i\left(k_\mu + x p_\mu\right) i \left(k_\nu + x p_\nu\right) \left(i[k\cdot\gamma + xp\cdot\gamma](2-d) + md\right)}{[k^2+\Delta]^3} \\
&= g_0^2 C_F \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx\, 2(1-x) \left(\frac{2(k_\mu k_\nu + x^2 p_\nu p_\mu) i x p\cdot\gamma(d-2)}{[k^2+\Delta]^3} \right. \\
&\quad \left. + \frac{2ix(k_\mu p_\nu + k_\nu p_\mu) k\cdot\gamma(d-2)}{[k^2+\Delta]^3} \right) \qquad \text{(B.58)}
\end{aligned}
$$

The first term yields contributions proportional to $g_{\mu\nu} p \cdot \gamma$ and $p_\mu p_\nu / p^2 \; p \cdot \gamma$ so we drop them; second term yields

$$
\begin{aligned}
I_2 &= (-g_0^2 C_F) \frac{2(d-2)}{d} \int_0^1 dx\, 2(1-x) \int \frac{d^d k}{(2\pi)^d} \frac{2x(\gamma_\mu i p_\nu + \gamma_\nu i p_\mu) k^2}{[k^2+\Delta]^3} \\
&= \frac{g_0^2 C_F}{16\pi^2} \frac{2(d-2)}{d} (\gamma_\mu i p_\nu + \gamma_\nu i p_\mu) \int_0^1 dx\, 2(1-x) \frac{d}{4} x \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log \Delta \right] \\
&= \frac{g_0^2 C_F}{16\pi^2} (\varepsilon - 1)(\gamma_\mu i p_\nu + \gamma_\nu i p_\mu) \left[\frac{1}{3} \left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \left\{ \begin{array}{c} \log p^2 \\ \log \mu^2 \end{array} \right\} \right) - \left\{ \begin{array}{c} 5/9 \\ 5/18 \end{array} \right\} \right] \\
&= -\frac{g_0^2 C_F}{16\pi^2} (\gamma_\mu i p_\nu + \gamma_\nu i p_\mu) \left[\frac{1}{3} \left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \left\{ \begin{array}{c} \log p^2 \\ \log \mu^2 \end{array} \right\} \right) + \left\{ \begin{array}{c} 2/9 \\ -1/18 \end{array} \right\} \right] \quad \text{(B.59)}
\end{aligned}
$$

Symmetrizing in μ and ν and adding those two contributions up we get

$$
I_{\{\mu\nu\}} = \frac{g_0^2 C_F}{16\pi^2} \left[\frac{1}{3} \left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \left\{ \begin{array}{c} \log p^2 \\ \log \mu^2 \end{array} \right\} \right) + \left\{ \begin{array}{c} 5/9 \\ -7/18 \end{array} \right\} \right] \quad \text{(B.60)}
$$

For the antisymmetric combination, second term doesn't contribute so we are left with

$$
I_{[\mu\nu]} = \frac{g_0^2 C_F}{16\pi^2} \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \left\{ \begin{array}{c} 1 - \log p^2 \\ -1/2 - \log \mu^2 \end{array} \right\} \right] \quad \text{(B.61)}
$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{vert}_{\{\mu\nu\}}$	1/3	5/9	-7/18
$\Sigma^{vert}_{[\mu\nu]}$	1	1	-1/2

B.4.2 Sails

The amplitude for two sails diagrams is

$$
I_1 = \int \frac{d^d k}{(2\pi)^d} \frac{g_{\rho\nu}}{(p-k)^2 + \mu^2} \left(-i g_0 T_a \gamma_\rho\right) \frac{-ik \cdot \gamma + m}{k^2 + m^2} \left(i g_0 T_a \gamma_\mu\right) [\gamma_5] \quad \text{(B.62)}
$$

$$I_2 = \int \frac{d^d k}{(2\pi)^d} \frac{g_{\rho\nu}}{(p-k)^2+\mu^2} \left(ig_0 T_a \gamma_\mu\right) [\gamma_5] \frac{-ik\cdot\gamma+m}{k^2+m^2} \left(-ig_0 T_a \gamma_\rho\right) , \tag{B.63}$$

so the total amplitude for two sails diagrams is

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{g_0^2 C_F}{(p-k)^2+\mu^2} \frac{\gamma_\mu(-ik\cdot\gamma\pm m)\gamma_\nu + \gamma_\nu(-ik\cdot\gamma+m)\gamma_\mu}{k^2+m^2} [\gamma_5] . \tag{B.64}$$

For the massless fermions both terms (with and without γ_5) are the same so we'll drop it from now on. Using Feynman parameters we get the amplitude

$$\begin{aligned} I &= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{g_{\rho\nu}}{(k^2+\Delta)^2} (-i) \left(\gamma_\mu(k\cdot\gamma + xp\cdot\gamma)\gamma_\nu + \gamma_\nu(k\cdot\gamma+xp\cdot\gamma)\gamma_\mu\right) \\ &= \frac{g_0^2 C_F}{16\pi^2} \int_0^1 dx \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log\Delta\right] (-ix) \left(\gamma_\mu p\cdot\gamma\gamma_\nu + \gamma_\nu p\cdot\gamma\gamma_\mu\right) . \end{aligned} \tag{B.65}$$

Using the relation between γ matrices

$$\gamma_\mu\gamma_\alpha\gamma_\nu + \gamma_\nu\gamma_\alpha\gamma_\mu = 2(\gamma_\mu g_{\alpha\nu} + \gamma_\nu g_{\alpha\mu} - \gamma_\alpha g_{\mu\nu}) , \tag{B.66}$$

we get the amplitude

$$\begin{aligned} I &= \frac{g_0^2 C_F}{16\pi^2} \int_0^1 dx \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log\Delta\right] (-2x) \left(\gamma_\mu i p_\nu + \gamma_\nu i p_\mu - ip\cdot\gamma g_{\mu\nu}\right) \\ &= -\frac{g_0^2 C_F}{16\pi^2} \left(\gamma_\mu i p_\nu + \gamma_\nu i p_\mu - ip\cdot\gamma g_{\mu\nu}\right) \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \left\{ \begin{array}{c} 2-\log p^2 \\ 1/2 - \log\mu^2 \end{array} \right\}\right] . \end{aligned} \tag{B.67}$$

Symmetrizing in μ and ν yields a factor of 2 while anti-symmetrizing cancels it so we get the final result

$$I^{sails}_{\{\mu\nu\}} = \frac{g_0^2 C_F}{16\pi^2} \left(\gamma_\mu i p_\nu + \gamma_\nu i p_\mu\right) \left[-2\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \left\{ \begin{array}{c} \log p^2 \\ \log\mu^2 \end{array} \right\}\right) - \left\{ \begin{array}{c} 4 \\ 1 \end{array} \right\}\right] . \tag{B.68}$$

$$I^{sails}_{[\mu\nu]} = 0 \tag{B.69}$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{sails}_{\{\mu\nu\}}$	-2	-4	-1
$\Sigma^{sails}_{[\mu\nu]}$	0	0	0

B.4.3 Collecting results

Finally, collecting results for twist-2 diagrams, we get the formula for renormalization constants

$$Z_{\mu\nu} = 1 + \frac{g_0^2 C_F}{16\pi^2}\left([\gamma_2 + \gamma^{vert} + \gamma^{sails}]\log Q^2/\Lambda^2 + [\Sigma_2 + \Sigma^{vert} + \Sigma^{sails}]\right) . \tag{B.70}$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{vert}_{\{\mu\nu\}}$	1/3	5/9	-7/18
$\Sigma^{sails}_{\{\mu\nu\}}$	-2	-4	-1
$\Sigma^{vert}_{[\mu\nu]}$	1	1	-1/2
$\Sigma^{sails}_{[\mu\nu]}$	0	0	0
Σ_2	-1	-1	1/2
$Z_{\{\mu\nu\}}$	-8/3	-40/9	-16/18
$Z_{[\mu\nu]}$	0	0	0

B.5 Twist 2 operators $\gamma_\mu D_\nu D_\alpha$

The tree-level expectation values of operators we are interested in are given by

$$\begin{aligned} \left\langle \bar{q}(x)\gamma_{\{\mu}D_\nu D_{\alpha\}}q(x)\right\rangle &= \frac{\gamma_\mu ip_\nu ip_\alpha + \gamma_\nu ip_\mu ip_\alpha + \gamma_\alpha ip_\mu ip_\nu}{3} & \text{(B.71)} \\ \left\langle \bar{q}(x)\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q(x)\right\rangle &= \gamma_\mu ip_\nu ip_\alpha - \gamma_\nu ip_\mu ip_\alpha & \text{(B.72)} \end{aligned}$$

B.5.1 Vertex diagram

The amplitude for the vertex diagram is essentially the same as for $\gamma_\mu D_\nu$ operator, multiplied with k_α

$$
\begin{aligned}
I_{\mu\nu\alpha} &= \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \sum_\rho (-ig_0 T_a \gamma_\rho) \frac{-ik\cdot\gamma+m}{k^2+m^2} \gamma_\mu ik_\nu ik_\alpha \frac{-ik\cdot\gamma+m}{k^2+m^2} (-ig_0 T_a \gamma_\rho) \\
&= I_1 + I_2 \qquad \text{(B.73)}
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= g_0^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \frac{(2-d)\gamma_\mu ik_\nu ik_\alpha}{k^2} \\
I_2 &= g_0^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \frac{-2ik_\mu(-ik\cdot\gamma(d-2))ik_\nu ik_\alpha}{(k^2)^2} . \qquad \text{(B.74)}
\end{aligned}
$$

Using the Feynman parameters (B.1) and shifting the integration variable $k \to k + xp$ we get

$$
\begin{aligned}
I_1 &= -g_0^2 C_F \gamma_\mu (d-2) \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{i(k_\nu + xp_\nu) i(k_\alpha + xp_\alpha)}{[k^2+\Delta]^2} \\
&= -g_0^2 C_F \gamma_\mu (d-2) \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{(k_\nu k_\alpha + x^2 p_\nu p_\alpha)}{[k^2+\Delta]^2} \\
&= -\frac{g_0^2 C_F}{16\pi^2} \gamma_\mu p_\nu p_\alpha (d-2) \int_0^1 x^2\, dx \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi - \log\Delta\right] \\
&\quad + \frac{g_0^2 C_F}{16\pi^2} \gamma_\mu g_{\nu\alpha} (d-2) \int_0^1 dx \frac{\Delta}{2} \left[\frac{1}{2-d/2} + 1 - \gamma_E + \log 4\pi - \log\Delta\right] \qquad \text{(B.75)}
\end{aligned}
$$

which yields the result

$$
\begin{aligned}
I_1 &= -\frac{g_0^2 C_F}{16\pi^2} \gamma_\mu p_\nu p_\alpha \left(\frac{2}{3}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \begin{Bmatrix} \log p^2 \\ \log \mu^2 \end{Bmatrix}\right] + \begin{Bmatrix} 7/9 \\ -4/9 \end{Bmatrix}\right) \\
&\quad + \frac{g_0^2 C_F}{16\pi^2} \gamma_\mu g_{\nu\alpha} \left(\begin{Bmatrix} p^2/6 \\ \mu^2/2 \end{Bmatrix}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \begin{Bmatrix} \log p^2 \\ \log \mu^2 \end{Bmatrix}\right] + \begin{Bmatrix} p^2\, 5/18 \\ \mu^2/4 \end{Bmatrix}\right) . \\
&\qquad \text{(B.76)}
\end{aligned}
$$

The second term becomes

$$
\begin{aligned}
I_2 &= g_0^2 C_F 2(d-2) \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx\, 2(1-x) \\
&\quad \times \frac{(k_\mu + x p_\mu)(k_\nu + x p_\nu)(k\cdot\gamma + x p\cdot\gamma)(k_\alpha + x p_\alpha)}{[k^2+\Delta]^3}
\end{aligned}
\tag{B.77}
$$

The term with 4 loop momenta k yields

$$
\begin{aligned}
I_{2,4} &= g_0^2 C_F 2(d-2) \int_0^1 dx\, 2(1-x) \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu k\cdot\gamma k_\alpha}{[k^2+\Delta]^3} \\
&= g_0^2 C_F 2(d-2) \int_0^1 dx\, 2(1-x) \int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^2}{[k^2+\Delta]^3} \frac{\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} + \gamma_\alpha g_{\mu\nu}}{d(d+2)} \\
&= g_0^2 C_F 4(1-\varepsilon) \frac{\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} + \gamma_\alpha g_{\mu\nu}}{d(d+2)} \\
&\quad \int_0^1 dx\, 2(1-x) \left[\frac{d(d+2)}{4} \frac{\Gamma(-1+\varepsilon)}{(4\pi)^{d/2}\Gamma(3)} \left(\frac{1}{\Delta}\right)^{-1+\varepsilon} \right] \\
&= \frac{g_0^2 C_F}{16\pi^2} (1-\varepsilon) \left(\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} + \gamma_\alpha g_{\mu\nu}\right) \\
&\quad \int_0^1 dx\, 2(1-x) \frac{-\Delta}{2} \left[\frac{1}{\varepsilon} + 1 - \gamma_E + \log 4\pi - \log\Delta \right] \\
&= \frac{g_0^2 C_F}{16\pi^2} \left(\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} + \gamma_\alpha g_{\mu\nu}\right) \\
&\quad \int_0^1 dx\, (1-x)(-\Delta) \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log\Delta \right] \\
&= \frac{g_0^2 C_F}{16\pi^2} \frac{\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} + \gamma_\alpha g_{\mu\nu}}{3} \\
&\quad \left(\begin{Bmatrix} -p^2/4 \\ -\mu^2/2 \end{Bmatrix} \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \begin{Bmatrix} \log p^2 \\ \log \mu^2 \end{Bmatrix} \right] + \begin{Bmatrix} p^2\, 5/12 \\ \mu^2\, 5/12 \end{Bmatrix} \right)
\end{aligned}
\tag{B.78}
$$

Terms with 2 loop momenta k simplify to

$$
(\ldots) = \frac{k^2}{d} x^2 S_{\mu\nu\alpha}
\tag{B.79}
$$

with

$$S_{\mu\nu\alpha} = \gamma_\mu p_\nu p_\alpha + \gamma_\nu p_\mu p_\alpha + \gamma_\alpha p_\mu p_\nu + p\cdot\gamma\left(p_\mu g_{\nu\alpha} + p_\nu g_{\mu\alpha} + p_\alpha g_{\mu\nu}\right) \tag{B.80}$$

so the amplitude becomes

$$\begin{aligned}
I_{2,2} &= g_0^2 C_F 2(d-2)\int_0^1 dx\, 2(1-x)\int \frac{d^dk}{(2\pi)^d}\frac{x^2k^2/d}{[k^2+\Delta]^3}\, S_{\mu\nu\alpha} \\
&= g_0^2 C_F 2(d-2)\int_0^1 dx\, 2(1-x)\frac{x^2}{d}S_{\mu\nu\alpha}\left[\frac{d}{2}\frac{\Gamma(\varepsilon)}{(4\pi)^{d/2}\Gamma(3)}\left(\frac{1}{\Delta}\right)^{\varepsilon}\right] \\
&= \frac{g_0^2C_F}{16\pi^2}(1-\varepsilon)S_{\mu\nu\alpha}\int_0^1 dx\, 2(1-x)x^2\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\log\Delta\right] \\
&= \frac{g_0^2C_F}{16\pi^2}S_{\mu\nu\alpha}\left(\frac{1}{6}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\left\{\begin{array}{c}\log p^2\\ \log\mu^2\end{array}\right\}\right]+\left\{\begin{array}{c}1/9\\ -5/72\end{array}\right\}\right)\,.
\end{aligned} \tag{B.81}$$

Finally, the term with 4 external momenta p yields

$$\begin{aligned}
I_{2,0} &= g_0^2C_F 2(d-2)\int_0^1 dx\, 2(1-x)\int\frac{d^dk}{(2\pi)^d}x^4\frac{p_\mu p_\nu p\cdot\gamma p_\alpha}{[k^2+\Delta]^3} \\
&= g_0^2C_F 4(1-\varepsilon)p\cdot\gamma p_\mu p_\nu p_\alpha\int_0^1 dx\, 2(1-x)x^4\left[\frac{\Gamma(1)}{(4\pi)^{d/2}\Gamma(3)}\left(\frac{1}{\Delta}\right)\right] \\
&= \frac{g_0^2C_F}{16\pi^2}(1-\varepsilon)p\cdot\gamma p_\mu p_\nu p_\alpha\int_0^1 dx\, 4(1-x)\frac{1}{\Delta} \\
&= \frac{g_0^2C_F}{16\pi^2}(1-\varepsilon)p\cdot\gamma p_\mu p_\nu p_\alpha\left\{\begin{array}{c}1/p^2\\ 1/(5\mu^2)\end{array}\right\}\,.
\end{aligned} \tag{B.82}$$

Collecting all terms together and symmetrizing, we get

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{vert}_{\{\mu\nu\alpha\}}$	1/6	4/9	-17/72
$\Sigma^{vert}_{[\mu\{\nu]\alpha\}}$	2/3	7/9	-4/9

B.5.2 Sails

The amplitude for two sails diagrams is

$$
\begin{aligned}
I_1 &= \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \left(-ig_0 T_a \gamma_\rho\right) \frac{-ik\cdot\gamma+m}{k^2+m^2} \gamma_\mu \\
&\quad \left[\left(ig_0 T_a g_{\nu\rho}\right) ik_\alpha + ip_\nu \left(ig_0 T_a g_{\alpha\rho}\right)\right] [\gamma_5] \qquad \text{(B.83)} \\
I_2 &= \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2+\mu^2} \gamma_\mu \left[\left(ig_0 T_a g_{\nu\rho}\right) ik_\alpha + ip_\nu \left(ig_0 T_a g_{\alpha\rho}\right)\right] [\gamma_5] \\
&\quad \frac{-ik\cdot\gamma+m}{k^2+m^2} \left(-ig_0 T_a \gamma_\rho\right) \qquad \text{(B.84)}
\end{aligned}
$$

so the total amplitude for two sails diagrams is

$$
\begin{aligned}
I &= \int \frac{d^dk}{(2\pi)^d} \frac{g_0^2 C_F}{(p-k)^2+\mu^2} \frac{\left(\gamma_\mu k\cdot\gamma\gamma_\nu + \gamma_\nu k\cdot\gamma\gamma_\mu\right) k_\alpha}{k^2+m^2} [\gamma_5] \\
&\quad + \int \frac{d^dk}{(2\pi)^d} \frac{g_0^2 C_F}{(p-k)^2+\mu^2} \frac{\left(\gamma_\mu k\cdot\gamma\gamma_\alpha + \gamma_\alpha k\cdot\gamma\gamma_\mu\right) p_\nu}{k^2+m^2} [\gamma_5] \qquad \text{(B.85)}
\end{aligned}
$$

The second term is just the amplitude for $\gamma_\mu D_\alpha$ multiplied by p_ν so we can just copy the result

$$
I = \frac{g_0^2 C_F}{16\pi^2} \left(\gamma_\mu p_\alpha + \gamma_\alpha p_\mu - p\cdot\gamma g_{\mu\alpha}\right) p_\nu \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \left\{ \begin{array}{c} 2-\log p^2 \\ 1/2 - \log\mu^2 \end{array} \right\}\right]. \qquad \text{(B.86)}
$$

Symmetrizing in μ and ν and α yields a factor of 2 while so we get the final result

$$
\begin{aligned}
I^{(2)}_{\{\mu\nu\alpha\}} &= \frac{g_0^2 C_F}{16\pi^2} \frac{\gamma_\mu ip_\nu ip_\alpha + \gamma_\nu ip_\mu ip_\alpha + \gamma_\alpha ip_\mu ip_\nu}{3} \\
&\quad \left[-2\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \left\{ \begin{array}{c} \log p^2 \\ \log\mu^2 \end{array} \right\}\right) - \left\{ \begin{array}{c} 4 \\ 1 \end{array} \right\}\right]. \qquad \text{(B.87)} \\
I^{(2)}_{[\mu\{\nu]\alpha\}} &= \frac{g_0^2 C_F}{16\pi^2} \left(\gamma_\mu ip_\nu ip_\alpha - \gamma_\nu ip_\mu ip_\alpha\right) \\
&\quad \frac{1}{2}\left[-\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \left\{ \begin{array}{c} \log p^2 \\ \log\mu^2 \end{array} \right\}\right) - \left\{ \begin{array}{c} 2 \\ 1/2 \end{array} \right\}\right]. \qquad \text{(B.88)}
\end{aligned}
$$

The first term after Feynman parametrization and integration variable shift becomes

$$
I_1 = \int \frac{d^dk}{(2\pi)^d} \frac{g_0^2 C_F}{(p-k)^2+\mu^2} \frac{\left(\gamma_\mu k\cdot\gamma\gamma_\nu + \gamma_\nu k\cdot\gamma\gamma_\mu\right) k_\alpha}{k^2+m^2} [\gamma_5]
$$

$$
\begin{aligned}
&= g_0^2 C_F \int_0^1 dx \frac{\left(\gamma_\mu(k\cdot\gamma + xp\cdot\gamma)\gamma_\nu + \gamma_\nu(k\cdot\gamma + xp\cdot\gamma)\gamma_\mu\right)(k_\alpha + xp_\alpha)}{[k^2+\Delta]^2} \\
&= g_0^2 C_F \int_0^1 dx \frac{\left(\gamma_\mu k\cdot\gamma\gamma_\nu + \gamma_\nu k\cdot\gamma\gamma_\mu\right)k_\alpha + x^2\left(\gamma_\mu p\cdot\gamma\gamma_\nu + \gamma_\nu p\cdot\gamma\gamma_\mu\right)p_\alpha}{[k^2+\Delta]^2} \\
&= 2g_0^2 C_F \int_0^1 dx \frac{\left(\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} - \gamma_\alpha g_{\mu\nu}\right)k^2/d + x^2\left(\gamma_\mu p_\nu + \gamma_\nu p_\mu - p\cdot\gamma g_{\mu\nu}\right)p_\alpha}{[k^2+\Delta]^2} .
\end{aligned}
\tag{B.89}
$$

The second term becomes

$$
\begin{aligned}
I_{1,2} &= 2\frac{g_0^2 C_F}{16\pi^2}\left(\gamma_\mu p_\nu + \gamma_\nu p_\mu - p\cdot\gamma g_{\mu\nu}\right)p_\alpha \int_0^1 dx x^2 \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log\Delta\right] \\
&= \frac{g_0^2 C_F}{16\pi^2}\left(\gamma_\mu p_\nu + \gamma_\nu p_\mu - p\cdot\gamma g_{\mu\nu}\right)p_\alpha \\
&\qquad \left(\frac{2}{3}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \begin{Bmatrix}\log p^2 \\ \log \mu^2\end{Bmatrix}\right] + \begin{Bmatrix}13/9 \\ 2/9\end{Bmatrix}\right) ,
\end{aligned}
\tag{B.90}
$$

while the first one becomes

$$
\begin{aligned}
I_{1,1} &= 2g_0^2 C_F \left(\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} - \gamma_\alpha g_{\mu\nu}\right)\frac{1}{d}\int_0^1 dx \left[\frac{d}{2}\frac{\Gamma(-1+\varepsilon)}{(4\pi)^{2-\varepsilon}\Gamma(2)}\left(\frac{1}{\Delta}\right)^{-1+\varepsilon}\right] \\
&= g_0^2 C_F \left(\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} - \gamma_\alpha g_{\mu\nu}\right)\int_0^1 dx \left((-\Delta)\left[\frac{1}{\varepsilon} + 1 - \gamma_E + \log 4\pi - \log\Delta\right]\right) \\
&= g_0^2 C_F \left(\gamma_\mu g_{\nu\alpha} + \gamma_\nu g_{\mu\alpha} - \gamma_\alpha g_{\mu\nu}\right)\int_0^1 dx \left(-\frac{1}{6}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log p^2\right] - \frac{4}{9}\right) .
\end{aligned}
\tag{B.91}
$$

Symmetrization and anti-symmetrization give us results

$$
\begin{aligned}
I^{(1)}_{\{\mu\nu\alpha\}} &= \frac{g_0^2 C_F}{16\pi^2}\frac{\gamma_\mu ip_\nu ip_\alpha + \gamma_\nu ip_\mu ip_\alpha + \gamma_\alpha ip_\mu ip_\nu}{3} \\
&\qquad \times\left(-\frac{4}{3}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \begin{Bmatrix}\log p^2 \\ \log\mu^2\end{Bmatrix}\right] - \begin{Bmatrix}26/9 \\ 4/9\end{Bmatrix}\right)
\end{aligned}
$$

$$I^{(1)}_{[\mu\{\nu]\alpha\}} = \frac{g_0^2 C_F}{16\pi^2}\left(\gamma_\mu ip_\nu ip_\alpha - \gamma_\nu ip_\mu ip_\alpha\right)$$
$$\times\frac{1}{2}\left(-\frac{2}{3}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log\mu^2\end{Bmatrix}\right]-\begin{Bmatrix}13/9\\ 2/9\end{Bmatrix}\right). \quad \text{(B.92)}$$

Adding those contributions up, we get

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{sails}_{\{\mu\nu\alpha\}}$	-10/3	-62/9	-13/9
$\Sigma^{sails}_{[\mu\{\nu]\alpha\}}$	-5/6	-31/18	-13/36

B.5.3 Collecting results

Finally, collecting results for twist-2 diagrams, we get the formula for renormalization constants

$$Z_{\mu\nu\alpha} = 1+\frac{g_0^2 C_F}{16\pi^2}\left(\left[\gamma_2+\gamma^{vert}+\gamma^{sails}\right]\log Q^2/\Lambda^2+\left[\Sigma_2+\Sigma^{vert}+\Sigma^{sails}\right]\right). \quad \text{(B.93)}$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{vert}_{\{\mu\nu\alpha\}}$	1/6	4/9	-17/72
$\Sigma^{sails}_{\{\mu\nu\alpha\}}$	-10/3	-62/9	-13/9
$\Sigma^{vert}_{[\mu\{\nu]\alpha\}}$	2/3	7/9	-4/9
$\Sigma^{sails}_{[\mu\{\nu]\alpha\}}$	-5/6	-31/18	-13/36
Σ_2	-1	-1	1/2
$Z_{\{\mu\nu\alpha\}}$	-25/6	-67/9	-85/72
$Z_{[\mu\{\nu]\alpha\}}$	-7/6	-35/18	-11/36

B.6 Twist 2 operators $\gamma_\mu D_\nu D_\alpha D_\beta$

The tree-level expectation values of operators we are interested in are given by

$$\langle \bar{q}(x)\gamma_{\{\mu}D_\nu D_\alpha D_{\beta\}}q(x)\rangle = \left(\gamma_\mu ip_\nu ip_\alpha ip_\beta+\gamma_\nu ip_\mu ip_\alpha ip_\beta\right.$$

$$+\gamma_\alpha ip_\mu ip_\nu ip_\beta + \gamma_\alpha ip_\mu ip_\nu ip_\alpha)/4\,. \qquad \text{(B.94)}$$

B.6.1 Vertex diagram

The amplitude for the vertex diagram is essentially the same as for $\gamma_\mu D_\nu$ operator, multiplied with k_α

$$\begin{aligned} I_{\mu\nu\alpha\beta} &= \int \frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\sum_\rho(-ig_0T_a\gamma_\rho)\frac{-ik\cdot\gamma+m}{k^2+m^2}\gamma_\mu ik_\nu ik_\alpha ik_\beta\frac{-ik\cdot\gamma+m}{k^2+m^2}(-ig_0T_a\gamma_\rho) \\ &= I_1+I_2\,, \end{aligned} \qquad \text{(B.95)}$$

where

$$\begin{aligned} I_1 &= g_0^2C_F\int\frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\frac{(d-2)\gamma_\mu ik_\nu ik_\alpha ik_\beta}{k^2} \\ I_2 &= g_0^2C_F\int\frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\frac{-2k_\mu k\cdot\gamma(d-2)ik_\nu ik_\alpha ik_\beta}{(k^2)^2}\,. \end{aligned} \qquad \text{(B.96)}$$

Using the Feynman parameters (B.1) and shifting the integration variable $k\to k+xp$ we get

$$\begin{aligned} I_1 &= g_0^2C_F\gamma_\mu(d-2)\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\frac{i(k_\nu+xp_\nu)i(k_\alpha+xp_\alpha)i(k_\beta+xp_\beta)}{[k^2+\Delta]^2} \\ &= g_0^2C_F\gamma_\mu(d-2)\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\frac{x(ik_\nu ik_\alpha ip_\beta+ik_\nu ip_\alpha ik_\beta+ip_\nu ik_\alpha ik_\beta)+x^3ip_\mu ip_\nu ip_\alpha}{[k^2+\Delta]^2} \\ &= \frac{g_0^2C_F}{16\pi^2}\gamma_\mu ip_\nu ip_\alpha ip_\beta(d-2)\int_0^1 x^3\,dx\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log\Delta\right] \\ &\quad+\frac{g_0^2C_F}{16\pi^2}\gamma_\mu(g_{\nu\alpha}ip_\beta+g_{\nu\beta}ip_\alpha+g_{\alpha\beta}ip_\nu)(d-2) \\ &\quad\times\int_0^1 dx\,x\frac{\Delta}{2}\left[\frac{1}{2-d/2}+1-\gamma_E+\log 4\pi-\log\Delta\right]\,, \end{aligned} \qquad \text{(B.97)}$$

which yields the result

$$
\begin{aligned}
I_1 &= \frac{g_0^2 C_F}{16\pi^2}\gamma_\mu i p_\nu i p_\alpha i p_\beta \left(\frac{1}{2}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log \mu^2\end{Bmatrix}\right]+\begin{Bmatrix}2/3\\ -3/8\end{Bmatrix}\right)\\
&+\frac{g_0^2 C_F}{16\pi^2}\gamma_\mu (g_{\nu\alpha} i p_\beta + g_{\nu\beta} i p_\alpha + g_{\alpha\beta} i p_\nu)\\
&\quad\times\left(\begin{Bmatrix}p^2/12\\ \mu^2/3\end{Bmatrix}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log \mu^2\end{Bmatrix}\right]-\begin{Bmatrix}p^2/36\\ \mu^2/9\end{Bmatrix}\right) . \qquad \text{(B.98)}
\end{aligned}
$$

The second term becomes

$$
\begin{aligned}
I_2 &= -g_0^2 C_F 2(d-2)\int \frac{d^d k}{(2\pi)^d}\int_0^1 dx\, 2(1-x)\left\{(k_\mu + x p_\mu)(k\cdot\gamma + x p\cdot\gamma)\right.\\
&\quad \left. i(k_\nu + x p_\nu)\, i(k_\alpha + x p_\alpha)\, i(k_\beta + x p_\beta)\right\} / \left\{[k^2+\Delta]^3\right\} . \qquad \text{(B.99)}
\end{aligned}
$$

The term with 4 loop momenta k yields

$$
\begin{aligned}
I_{2,4} &= -g_0^2 C_F 2(d-2)\int_0^1 dx\, 2(1-x)\int \frac{d^d k}{(2\pi)^d} x\left(k_\mu k\cdot\gamma i k_\nu i k_\alpha i p_\beta + k_\mu k\cdot\gamma i k_\nu i p_\alpha i k_\beta\right.\\
&\quad \left. + k_\mu k\cdot\gamma i p_\nu i k_\alpha i k_\beta + k_\mu p\cdot\gamma i k_\nu i k_\alpha i k_\beta + p_\mu k\cdot\gamma i k_\nu i k_\alpha i k_\beta\right) / [k^2+\Delta]^3 . \qquad \text{(B.100)}
\end{aligned}
$$

Using the formula (B.8) we get

$$
\begin{aligned}
I_{2,4} &= -g_0^2 C_F 2(d-2)\int_0^1 dx\, 2(1-x)\int \frac{d^d k}{(2\pi)^d}\frac{(k^2)^2}{[k^2+\Delta]^3}\frac{x i^2 S_{\mu\nu\alpha\beta}}{d(d+2)}\\
&= g_0^2 C_F 4(1-\varepsilon)\frac{S_{\mu\nu\alpha\beta}}{d(d+2)}\int_0^1 dx\, 2(1-x)\, x\left[\frac{d(d+2)}{4}\frac{\Gamma(-1+\varepsilon)}{(4\pi)^{d/2}\Gamma(3)}\left(\frac{1}{\Delta}\right)^{-1+\varepsilon}\right]\\
&= \frac{g_0^2 C_F}{16\pi^2}(1-\varepsilon) S_{\mu\nu\alpha\beta}\int_0^1 dx\, 2(1-x)\, x\frac{-\Delta}{2}\left[\frac{1}{\varepsilon}+1-\gamma_E+\log 4\pi-\log\Delta\right]\\
&= \frac{g_0^2 C_F}{16\pi^2} S_{\mu\nu\alpha\beta}\int_0^1 dx\,(1-x)(-x\Delta)\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\log\Delta\right]\\
&= \frac{g_0^2 C_F}{16\pi^2} S_{\mu\nu\alpha\beta}\left(\begin{Bmatrix}p^2/30\\ \mu^2/12\end{Bmatrix}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\begin{Bmatrix}\log p^2\\ \log \mu^2\end{Bmatrix}\right]-\begin{Bmatrix}p^2\, 47/300\\ \mu^2\, 7/144\end{Bmatrix}\right) ,
\end{aligned}
$$

(B.101)

where $S_{\mu\nu\alpha\beta}$ is the symmetric tensor

$$S_{\mu\nu\alpha\beta} = ip_{\{\mu}\gamma_\nu g_{\alpha\beta\}} + ip\cdot\gamma g_{\{\mu\nu}g_{\alpha\beta\}}\ . \tag{B.102}$$

Terms with 2 loop momenta k simplify to

$$(\ldots) = \frac{k^2}{d}x^3\bar{S}_{\mu\nu\alpha\beta} \tag{B.103}$$

with different $\bar{S}$

$$\bar{S}_{\mu\nu\alpha\beta} = \gamma_{\{\mu}ip_\nu ip_\alpha ip_{\beta\}} + ip\cdot\gamma ip_{\{\mu}ip_\nu g_{\alpha\beta\}} \tag{B.104}$$

so the amplitude becomes

$$\begin{aligned}
I_{2,2} &= -g_0^2C_F2(d-2)\int_0^1 dx\,2(1-x)\int\frac{d^dk}{(2\pi)^d}\frac{x^3k^2/d}{[k^2+\Delta]^3}\bar{S}_{\mu\nu\alpha\beta} \\
&= -g_0^2C_F2(d-2)\int_0^1 dx\,2(1-x)\frac{x^3}{d}\bar{S}_{\mu\nu\alpha\beta}\left[\frac{d}{2}\frac{\Gamma(\varepsilon)}{(4\pi)^{d/2}\Gamma(3)}\left(\frac{1}{\Delta}\right)^{\varepsilon}\right] \\
&= -\frac{g_0^2C_F}{16\pi^2}(1-\varepsilon)\bar{S}_{\mu\nu\alpha\beta}\int_0^1 dx\,2(1-x)x^3\left[\frac{1}{\varepsilon}-\gamma_E+\log4\pi-\log\Delta\right] \\
&= \frac{g_0^2C_F}{16\pi^2}\bar{S}_{\mu\nu\alpha\beta}\left(-\frac{1}{10}\left[\frac{1}{\varepsilon}-\gamma_E+\log4\pi-\begin{Bmatrix}\log p^2\\ \log\mu^2\end{Bmatrix}\right]+\begin{Bmatrix}-11/150\\ 9/200\end{Bmatrix}\right).
\end{aligned} \tag{B.105}$$

Finally, the term with 4 external momenta p yields

$$\begin{aligned}
I_{2,0} &= g_0^2C_F2(d-2)\int_0^1 dx\,2(1-x)\int\frac{d^dk}{(2\pi)^d}x^5\frac{ip\cdot\gamma p_\mu p_\nu p_\alpha p_\beta}{[k^2+\Delta]^3} \\
&= g_0^2C_F4ip\cdot\gamma p_\mu p_\nu p_\alpha p_\beta\int_0^1 dx\,2(1-x)x^5\left[\frac{\Gamma(1)}{(4\pi)^{d/2}\Gamma(3)}\left(\frac{1}{\Delta}\right)\right] \\
&= \frac{g_0^2C_F}{16\pi^2}ip\cdot\gamma p_\mu p_\nu p_\alpha p_\beta\int_0^1 dx\,4(1-x)\frac{x^5}{\Delta} \\
&= \frac{g_0^2C_F}{16\pi^2}ip\cdot\gamma p_\mu p_\nu p_\alpha p_\beta\begin{Bmatrix}-4/(5p^2)\\ -2/(15\mu^2)\end{Bmatrix}\ .
\end{aligned} \tag{B.106}$$

Collecting all terms together and symmetrizing, we get

$$I_{\{\mu\nu\alpha\beta\}} = \frac{g_0^2 C_F}{16\pi^2}\frac{\gamma_{\{\mu}ip_\nu ip_\alpha ip_{\beta\}}}{4}\left(\frac{1}{10}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\left\{\begin{array}{c}\log p^2\\ \log\mu^2\end{array}\right\}\right]+\left\{\begin{array}{c}28/75\\ -31/200\end{array}\right\}\right) \quad (B.107)$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{vert}_{\{\mu\nu\alpha\beta\}}$	1/10	28/75	-31/200

B.6.2 Sails

The amplitude for two sails diagrams is

$$\begin{aligned} I_1 &= \int\frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\left(-ig_0T_a\gamma_\rho\right)\frac{-ik\cdot\gamma+m}{k^2+m^2}\gamma_\mu\left[\left(ig_0T_ag_{\nu\rho}\right)ik_\alpha ik_\beta\right.\\ &\quad\left.+ip_\nu\left(ig_0T_ag_{\alpha\rho}\right)ik_\beta+ip_\nu ip_\alpha\left(ig_0T_ag_{\beta\rho}\right)\right][\gamma_5] \quad (B.108)\\ I_2 &= \int\frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\gamma_\mu\left[\left(ig_0T_ag_{\nu\rho}\right)ik_\alpha ik_\beta\right.\\ &\quad\left.+ip_\nu\left(ig_0T_ag_{\alpha\rho}\right)ik_\beta+ip_\nu ip_\alpha\left(ig_0T_ag_{\beta\rho}\right)\right][\gamma_5]\frac{-ik\cdot\gamma+m}{k^2+m^2}\left(-ig_0T_a\gamma_\rho\right), \quad (B.109)\end{aligned}$$

so the total amplitude for two sails diagrams is

$$I_{\mu\nu\alpha\beta} = -\int\frac{d^dk}{(2\pi)^d}\frac{g_0^2C_F}{(p-k)^2+\mu^2}\frac{\left(\gamma_\mu ik\cdot\gamma\gamma_\nu+\gamma_\nu ik\cdot\gamma\gamma_\mu\right)ik_\alpha ik_\beta}{k^2+m^2}[\gamma_5]+ip_\nu I_{\mu\alpha\beta}\,. \quad (B.110)$$

The second term is just the amplitude for $\gamma_\mu D_\alpha D_\beta$ multiplied by ip_ν so we can just copy the result

$$\begin{aligned} I &= \frac{g_0^2C_F}{16\pi^2}\left(\gamma_\mu ip_\beta+\gamma_\beta p_\mu-p\cdot\gamma g_{\mu\beta}\right)ip_\nu ip_\alpha\\ &\qquad\times\left(-\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\left\{\begin{array}{c}\log p^2\\ \log\mu^2\end{array}\right\}-\left\{\begin{array}{c}2\\ 1/2\end{array}\right\}\right]\right)\\ &\quad+\frac{g_0^2C_F}{16\pi^2}\left(\gamma_\mu ip_\alpha+\gamma_\alpha p_\mu-p\cdot\gamma g_{\mu\alpha}\right)ip_\nu ip_\beta\end{aligned}$$

$$\times\left(-\frac{2}{3}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\left\{\begin{array}{c}\log p^2\\ \log\mu^2\end{array}\right\}-\left\{\begin{array}{c}13/9\\ 2/9\end{array}\right\}\right]\right)\,. \qquad \text{(B.111)}$$

Symmetrizing in μ and ν and α yields a factor of 2 while so we get the final result

$$I^{(2)}_{\{\mu\nu\alpha\beta\}} = \frac{g_0^2 C_F}{16\pi^2}\frac{\bar{S}_{\mu\nu\alpha\beta}}{4}\left[-\frac{10}{3}\left(\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\left\{\begin{array}{c}\log p^2\\ \log\mu^2\end{array}\right\}\right)-\left\{\begin{array}{c}62/9\\ 13/9\end{array}\right\}\right]\,. \qquad \text{(B.112)}$$

First term after Feynman parametrization and integration variable shift becomes

$$\begin{aligned}
I_1 &= -\int\frac{d^dk}{(2\pi)^d}\frac{g_0^2 C_F}{(p-k)^2+\mu^2}\frac{(\gamma_\mu ik\cdot\gamma\gamma_\nu+\gamma_\nu ik\cdot\gamma\gamma_\mu)\,ik_\alpha ik_\beta}{k^2+m^2}[\gamma_5]\\
&= -g_0^2 C_F\int_0^1 dx\frac{(\gamma_\mu i(k\cdot\gamma+xp\cdot\gamma)\gamma_\nu+\gamma_\nu i(k\cdot\gamma+xp\cdot\gamma)\gamma_\mu)\,i(k_\alpha+xp_\alpha)i(k_\beta+xp_\beta)}{[k^2+\Delta]^2}\\
&= -g_0^2 C_F\int_0^1 dx\left\{x\left(\gamma_\mu ik\cdot\gamma\gamma_\nu+\gamma_\nu ik\cdot\gamma\gamma_\mu\right)\left(ik_\alpha ip_\beta+ip_\alpha ik_\beta\right)\right.\\
&\qquad\left.+x\left(\gamma_\mu ip\cdot\gamma\gamma_\nu+\gamma_\nu ip\cdot\gamma\gamma_\mu\right)\left(ik_\alpha ik_\beta+x^2 ip_\alpha ip_\beta\right)\right\}/\left\{[k^2+\Delta]^2\right\}\,. \qquad \text{(B.113)}
\end{aligned}$$

The term with 3 external momenta p becomes

$$\begin{aligned}
I_{1,2} &= -2\frac{g_0^2 C_F}{16\pi^2}\left(\gamma_\mu ip_\nu+\gamma_\nu ip_\mu-ip\cdot\gamma g_{\mu\nu}\right)ip_\alpha ip_\beta\int_0^1 dx x^3\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\log\Delta\right]\\
&= -\frac{g_0^2 C_F}{16\pi^2}\left(\gamma_\mu ip_\nu+\gamma_\nu ip_\mu-ip\cdot\gamma g_{\mu\nu}\right)ip_\alpha ip_\beta\\
&\qquad\times\left(\frac{1}{2}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\left\{\begin{array}{c}\log p^2\\ \log\mu^2\end{array}\right\}\right]+\left\{\begin{array}{c}7/6\\ 1/8\end{array}\right\}\right)\,, \qquad \text{(B.114)}
\end{aligned}$$

while the term with 1 external momentum becomes

$$\begin{aligned}
I_{1,1} &= 2g_0^2 C_F\left\{\left(\gamma_\mu g_{\nu\alpha}+\gamma_\nu g_{\mu\alpha}-\gamma_\alpha g_{\mu\nu}\right)ip_\beta+\left(\gamma_\mu g_{\nu\beta}+\gamma_\nu g_{\mu\beta}-\gamma_\beta g_{\mu\nu}\right)ip_\alpha\right.\\
&\qquad\left.+\left(\gamma_\mu p_\nu+\gamma_\nu p_\mu-p\cdot\gamma g_{\mu\nu}\right)g_{\alpha\beta}\right\}\frac{1}{d}\int_0^1 dx\left[\frac{d}{2}\frac{\Gamma(-1+\varepsilon)}{(4\pi)^{2-\varepsilon}\Gamma(2)}\left(\frac{1}{\Delta}\right)^{-1+\varepsilon}\right]\\
&= g_0^2 C_F(\ldots)\int_0^1 dx x\left((-\Delta)\left[\frac{1}{\varepsilon}+1-\gamma_E+\log 4\pi-\log\Delta\right]\right)
\end{aligned}$$

$$= \frac{g_0^2 C_F}{16\pi^2}(\ldots)\int_0^1 dx\,x\left(-\frac{1}{6}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\log p^2\right]-\frac{4}{9}\right)$$

$$= \frac{g_0^2 C_F}{16\pi^2}(\ldots)\left(\left\{\begin{matrix} p^2/12 \\ \mu^2/3 \end{matrix}\right\}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\log p^2\right]+\left\{\begin{matrix} 5p^2/36 \\ 13\mu^2/12 \end{matrix}\right\}\right) . \quad \text{(B.115)}$$

Symmetrizing and adding them up gives us the final result

$$I_{\{\mu\nu\alpha\beta\}} = \frac{g_0^2 C_F}{16\pi^2}\frac{\gamma_{\{\mu} i p_\nu i p_\alpha i p_{\beta\}}}{4}\left(-\frac{26}{6}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\left\{\begin{matrix} \log p^2 \\ \log \mu^2 \end{matrix}\right\}\right]-\left\{\begin{matrix} 83/9 \\ 61/36 \end{matrix}\right\}\right) . \quad \text{(B.116)}$$

Adding those contributions up, we get

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{sails}_{\{\mu\nu\alpha\}}$	-26/6	-83/9	-61/36

B.6.3 Collecting results

Finally, collecting results for twist-2 diagrams, we get the formula for renormalization constants

$$Z_{\mu\nu\alpha\beta} = 1+\frac{g_0^2 C_F}{16\pi^2}\left([\gamma_2+\gamma^{vert}+\gamma^{sails}]\log Q^2/\Lambda^2+[\Sigma_2+\Sigma^{vert}+\Sigma^{sails}]\right) . \quad \text{(B.117)}$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{vert}_{\{\mu\nu\alpha\beta\}}$	1/10	28/75	-31/200
$\Sigma^{sails}_{\{\mu\nu\alpha\beta\}}$	-26/6	-83/9	-61/36
Σ_2	-1	-1	1/2
$Z_{\{\mu\nu\alpha\beta\}}$	-157/30	-2216/225	-2429/1800

B.7 $\bar{q}\sigma_{\mu\nu}D_\alpha q$ operator

B.7.1 Vertex diagram

The amplitude for $\bar{q}\sigma_{\mu\nu}D_\alpha q$ current in the $\overline{MS}$ scheme is given by

$$
\begin{aligned}
I_{\mu\nu\alpha} &= \int \frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2+\mu^2}\sum_\rho[-ig_0T_a\gamma_\rho]\frac{-ik\cdot\gamma+m}{k^2+m^2} \\
&\quad \left[\sigma_{\mu\nu}[\gamma_5]ik_\alpha\right]\frac{-ik\cdot\gamma+m}{k^2+m^2}\left[-ig_0T_a\gamma_\rho\right] .
\end{aligned} \tag{B.118}
$$

As for other currents, color matrices add up to $g_0^2C_F$; Dirac algebra yields

$$
(-ik\cdot\gamma)\sigma_{\mu\nu}[\gamma_5](-ik\cdot\gamma) = \pm(-k^2\sigma_{\mu\nu}+2(k_\mu\sigma_{k\nu}-k_\nu\sigma_{k\mu}))[\gamma_5] , \tag{B.119}
$$

where we have used the commutator between γ and σ matrices

$$
\left[\sigma_{\mu\nu},\gamma_\rho\right] = -2i(g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu) . \tag{B.120}
$$

Using that same relation and the definition of a product of two γ matrices

$$
\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu} , \tag{B.121}
$$

we can evaluate

$$
\begin{aligned}
\gamma_\rho\sigma_{\mu\nu}\gamma_\rho &= \gamma_\rho\left\{\gamma_\rho\sigma_{\mu\nu} - 2i(g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu)\right\} \\
&= d\sigma_{\mu\nu} - 2i(g_{\rho\mu}\gamma_\rho\gamma_\nu - g_{\rho\nu}\gamma_\rho\gamma_\mu) \\
&= d\sigma_{\mu\nu} - 2i(g_{\rho\mu}[g_{\rho\nu} - i\sigma_{\rho\nu}] - g_{\rho\nu}[g_{\rho\nu} - i\sigma_{\rho\nu}]) \\
&= (d-4)\sigma_{\mu\nu}
\end{aligned} \tag{B.122}
$$

to get the expression for the amplitude

$$
I_{\mu\nu\alpha} = \int \frac{d^dk}{(2\pi)^d}\frac{-g_0^2C_F(d-4)}{(p-k)^2+\mu^2}\frac{-k^2\sigma_{\mu\nu}+2k_\mu k_\rho\sigma_{\rho\nu}-2k_\nu k_\rho\sigma_{\rho\mu}}{(k^2)^2}ik_\alpha . \tag{B.123}
$$

Using Feynman parameters and shifting the integration variable $k \to x + kp$ we get

$$
\begin{aligned}
I_{\mu\nu\alpha} &= (d-4)\int_0^1 2(1-x)\,dx \int \frac{d^dk}{(2\pi)^d}\frac{-g_0^2 C_F}{(k^2+\Delta)^3}\{-(k+xp)^2\sigma_{\mu\nu} \\
&\quad +2(k+xp)_\mu(k+xp)_\rho\sigma_{\rho\nu} - 2(k+xp)_\nu(k+xp)_\rho\sigma_{\rho\mu}\} \\
&\quad \times i(k_\alpha + xp_\alpha) \\
&= (d-4)\int_0^1 2(1-x)\,dx \int \frac{d^dk}{(2\pi)^d}\frac{-ig_0^2 C_F}{(k^2+\Delta)^3}\{[-(k^2+x^2p^2)\sigma_{\mu\nu} \\
&\quad +2\left(\frac{k^2}{d}g_{\mu\rho}+x^2p_\mu p_\rho\right)\sigma_{\rho\nu} - 2\left(\frac{k^2}{d}g_{\nu\rho}+x^2p_\nu p_\rho\right)\sigma_{\rho\mu}\Bigg]xp_\alpha \\
&\quad +\Bigg[-2x\frac{k^2}{d}p_\alpha\sigma_{\mu\nu} + 2x\frac{k^2}{d}\left(g_{\mu\alpha}p_\rho + g_{\alpha\rho}p_\mu\right)\sigma_{\rho\nu} \\
&\quad -2x\frac{k^2}{d}\left(g_{\nu\alpha}p_\rho + g_{\alpha\rho}p_\nu\right)\sigma_{\rho\mu}\Bigg]\Bigg\} \\
&= (d-4)\int_0^1 2(1-x)\,dx \int \frac{d^dk}{(2\pi)^d}\frac{-ig_0^2 C_F}{(k^2+\Delta)^3}\Bigg\{k^2\frac{4-d}{d}\sigma_{\mu\nu}p_\alpha \\
&\quad -x^2\left(p^2\sigma_{\mu\nu} - 2p_\mu p_\rho\sigma_{\rho\nu} + 2p_\nu p_\rho\sigma_{\rho\mu}\right)p_\alpha \\
&\quad +2x\frac{k^2}{d}\left(\sigma_{\mu\nu}p_\alpha + p_\mu\sigma_{\alpha\nu} - p_\nu\sigma_{\alpha\mu}\right)\Bigg\}\,.
\end{aligned} \tag{B.124}
$$

The first two terms in the last expression vanish; the first one is logarithmically divergent but has two powers of ε multiplying it, while the second is constant multiplied with one power of ε. So the only term that contributes is the last one. Evaluating d-dimensional integrals and replacing $d = 4 - 2\varepsilon$ we get

$$
\begin{aligned}
I_{\mu\nu\alpha} &= \frac{ig_0^2 C_F}{16\pi^2}2\varepsilon\int_0^1 2(1-x)\,dx\,\frac{2x}{d}\left(\sigma_{\mu\nu}p_\alpha + p_\mu\sigma_{\alpha\nu} - p_\nu\sigma_{\alpha\mu}\right) \\
&\quad \times\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log\Delta\right] \\
&= \frac{ig_0^2 C_F}{16\pi^2}\int_0^1 2(1-x)\,dx\,\frac{2x}{d}\,2\left(\sigma_{\mu\nu}p_\alpha + p_\mu\sigma_{\alpha\nu} - p_\nu\sigma_{\alpha\mu}\right) \\
&= \frac{g_0^2 C_F}{16\pi^2}\frac{1}{3}\left(\sigma_{\mu\nu}ip_\alpha - \sigma_{\nu\alpha}ip_\mu + \sigma_{\mu\alpha}ip_\nu\right)
\end{aligned} \tag{B.125}
$$

Symmetrizing in ν and α, we get

$$I_{\mu\{\nu\alpha\}} = \frac{g_0^2 C_F}{16\pi^2}\frac{2}{3}\left(\sigma_{\mu\nu} i p_\alpha + \sigma_{\mu\alpha} i p_\nu\right) \tag{B.126}$$

B.7.2 Sails

The amplitude for two sails diagrams is

$$I_1 = \int \frac{d^d k}{(2\pi)^d}\frac{g_{\rho\alpha}}{(p-k)^2+\mu^2}\left(-ig_0 T_a \gamma_\rho\right)\frac{-ik\cdot\gamma}{k^2}\left(ig_0 T_a\right)\sigma_{\mu\nu}[\gamma_5] \tag{B.127}$$

$$I_2 = \int \frac{d^d k}{(2\pi)^d}\frac{g_{\rho\alpha}}{(p-k)^2+\mu^2}\left(ig_0 T_a\right)\sigma_{\mu\nu}[\gamma_5]\frac{-ik\cdot\gamma}{k^2}\left(-ig_0 T_a \gamma_\rho\right) \tag{B.128}$$

so the total amplitude for two sails diagrams is

$$I = \int \frac{d^d k}{(2\pi)^d}\frac{g_0^2 C_F}{(p-k)^2+\mu^2}\frac{\sigma_{\mu\nu}(-ik\cdot\gamma)\gamma_\alpha + \gamma_\alpha(-ik\cdot\gamma)\sigma_{\mu\nu}}{k^2}[\gamma_5]\,. \tag{B.129}$$

Dirac algebra simplifies to

$$\sigma_{\mu\nu} k\cdot\gamma\gamma_\alpha + \gamma_\alpha k\cdot\gamma\sigma_{\mu\nu} = 2\left(k_\alpha\sigma_{\mu\nu} + k_\nu\sigma_{\mu\alpha} - k_\mu\sigma_{\nu\alpha} + g_{\nu\alpha}\sigma_{k\mu} - g_{\mu\alpha}\sigma_{k\nu}\right)\,. \tag{B.130}$$

Using Feynman parameters, shifting the integration variable and dropping odd terms in k we get the amplitude

$$\begin{aligned}
I &= -ig_0^2 C_F\int_0^1 dx\int\frac{d^d k}{(2\pi)^d}\frac{1}{(k^2+\Delta)^2}2x \\
&\quad\times\left(p_\alpha\sigma_{\mu\nu} + p_\nu\sigma_{\mu\alpha} - p_\mu\sigma_{\nu\alpha} + g_{\nu\alpha}\sigma_{p\mu} - g_{\mu\alpha}\sigma_{p\nu}\right) \\
&= \frac{-ig_0^2 C_F}{16\pi^2}\int_0^1 dx\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - \log\Delta\right]2x \\
&\quad\times\left(p_\alpha\sigma_{\mu\nu} + p_\nu\sigma_{\mu\alpha} - p_\mu\sigma_{\nu\alpha} + g_{\nu\alpha}\sigma_{p\mu} - g_{\mu\alpha}\sigma_{p\nu}\right) \\
&= -\frac{g_0^2 C_F}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \left\{\begin{array}{c} 2-\log p^2 \\ 1/2 - \log\mu^2\end{array}\right\}\right] \\
&\quad\times\left(p_\alpha\sigma_{\mu\nu} + p_\nu\sigma_{\mu\alpha} - p_\mu\sigma_{\nu\alpha} + g_{\nu\alpha}\sigma_{p\mu} - g_{\mu\alpha}\sigma_{p\nu}\right)\,.
\end{aligned} \tag{B.131}$$

Symmetrizing in ν and α yields a factor of 2 so we get the final result

$$\begin{aligned} I^{sails}_{\mu\{\nu\alpha\}} &= \frac{g_0^2 C_F}{16\pi^2}\left[-2\left(\frac{1}{\varepsilon}-\gamma_E+\log 4\pi-\left\{\begin{array}{c}\log p^2\\ \log\mu^2\end{array}\right\}\right)-\left\{\begin{array}{c}4\\ 1\end{array}\right\}\right] \\ &\times\left(\sigma_{\mu\nu}ip_\alpha+\sigma_{\mu\alpha}ip_\nu+\sigma_{p\mu}g_{\nu\alpha}-\frac{\sigma_{p\nu}g_{\mu\alpha}+\sigma_{p\alpha}g_{\mu\nu}}{2}\right) . \end{aligned} \quad \text{(B.132)}$$

B.7.3 Collecting results

Finally, collecting results for twist-2 diagrams, we get the formula for renormalization constants

$$Z_{\mu\nu} = 1+\frac{g_0^2 C_F}{16\pi^2}\left([\gamma_2+\gamma^{vert}+\gamma^{sails}]\log Q^2/\Lambda^2+[\Sigma_2+\Sigma^{vert}+\Sigma^{sails}]\right) . \quad \text{(B.133)}$$

function	γ	Σ_p/z_p	Σ_μ/z_μ
$\Sigma^{vert}_{\mu\{\nu\alpha\}}$	0	2/3	2/3
$\Sigma^{sails}_{\mu\{\nu\alpha\}}$	-2	-4	-1
Σ_2	-1	-1	1/2
$Z_{\mu\{\nu\alpha\}}$	-3	-13/3	1/6

B.8 Final $\overline{MS}$ results

Here we collect results from previous section; all results are written in the form

$$I = \frac{g_0^2 C_F}{16\pi^2}\left(\gamma_i \log Q^2/\Lambda^2+\Sigma_i\right) \quad \text{(B.134)}$$

for individual diagrams and

$$Z_i = 1+\frac{g_0^2 C_F}{16\pi^2}\left(\gamma_i \log Q^2/\Lambda^2+z_i\right) \quad \text{(B.135)}$$

for renormalization coefficients, where the logarithmic term can come either from finite external momentum p, $\log p^2/\Lambda^2$ or from finite gluon mass μ, $\log\mu^2/\Lambda^2$.

function	γ	Σ_p/z_p	Σ_μ/z_μ
Self Energy			
Σ_2	-1	-1	1/2
Σ_m	-4	-6	-2
Z_2	-1	-1	1/2
Z_m^{-1}	3	5	5/2
Currents			
Σ_S	4	6	2
Σ_P	4	6	2
Σ_V	1	1	-1/2
Σ_A	1	1	-1/2
Σ_T	0	0	0
Z_S	3	5	5/2
Z_P	3	5	5/2
Z_V	0	0	0
Z_A	0	0	0
Z_T	-1	-1	1/2
Twist 2 $\bar{q}\gamma_{\{\mu}D_{\nu\}}q$			
Σ^{vert}	1/3	5/9	-7/18
Σ^{sails}	-2	-4	-1
Σ_2	-1	-1	1/2
$Z_{\{\mu\nu\}}$	-8/3	-40/9	-16/18
Twist 3 $\bar{q}\gamma_{[\mu}D_{\nu]}q$			
Σ^{vert}	1	1	-1/2
Σ^{sails}	0	0	0
Σ_2	-1	-1	1/2
$Z_{[\mu\nu]}$	0	0	0

function	γ	Σ_p/z_p	Σ_μ/z_μ
Twist 2 $\bar{q}\gamma_{\{\mu}D_\nu D_{\alpha\}}q$			
$\Sigma^{vert}_{\{\mu\nu\alpha\}}$	1/6	4/9	-17/72
$\Sigma^{sails}_{\{\mu\nu\alpha\}}$	-10/3	-62/9	-13/9
Σ_2	-1	-1	1/2
$Z_{\{\mu\nu\alpha\}}$	-25/6	-67/9	-85/72
Twist 3 $\bar{q}\gamma_{[\mu}D_{\{\nu]}D_{\alpha\}}q$			
$\Sigma^{vert}_{[\mu\{\nu]\alpha\}}$	2/3	7/9	-4/9
$\Sigma^{sails}_{[\mu\{\nu]\alpha\}}$	-5/6	-31/18	-13/36
Σ_2	-1	-1	1/2
$Z_{[\mu\{\nu]\alpha\}}$	-7/6	-35/18	-11/36
Twist 2 $\bar{q}\gamma_{\{\mu}D_\nu D_\alpha D_{\beta\}}q$			
$\Sigma^{vert}_{\{\mu\nu\alpha\beta\}}$	1/10	28/75	-31/200
$\Sigma^{sails}_{\{\mu\nu\alpha\beta\}}$	-26/6	-83/9	-61/36
Σ_2	-1	-1	1/2
$Z_{\{\mu\nu\alpha\beta\}}$	-157/30	-2216/225	-2429/1800
Twist 2 $\bar{q}\sigma_{\mu\{\nu}D_{\alpha\}}q$			
$\Sigma^{vert}_{\mu\{\nu\alpha\}}$	0	2/3	2/3
$\Sigma^{sails}_{\mu\{\nu\alpha\}}$	-2	-4	-1
Σ_2	-1	-1	1/2
$Z_{\mu\{\nu\alpha\}}$	-3	-13/3	1/6

Table B.1: Final results for all operators in $\overline{MS}$ scheme.

Appendix C

IR singularity: extracting the $\log p^2$ term

The expression for the self energy is given by formula (4.22):

$$I(a,p) = \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} \sum_{\lambda,\rho} G_{\lambda\rho}(p-k) V_\rho(k,p) S_F(k) V_\lambda(p,k) \,. \tag{C.1}$$

It depends on external momentum p_μ and the lattice size a. Now, we are interested in taking the continuum limit $a \to 0$ of that expression to match it against the $\overline{MS}$ scheme (or something else if necessary). To do that, we expand it in external momentum:

$$I(a,p) = \frac{1}{a} I^{(-1)} + p \cdot I^{(0)} + a p^2 I^{(1)} + a^2 p^3 I^{(2)} + \dots \,, \tag{C.2}$$

where we have explicitly shown a and p behavior; if all coefficients $I^{(k)}$ were finite, that would be the end of the story; we'd take the $a \to 0$ limit, keep $I^{(-1)}$ and $I^{(0)}$ and discard all the rest. There would be no need to do *anything* else. However, since the self energy has an IR singularity, coefficient $I^{(0)}$ is also singular so we have to evaluate it differently. There are (at least) two ways to do it.

1. **Reisz theorem and continuum limit**: Approach used by Capitani.

   The essence of this approach is this: take the integral $I(a,p)$ and expand it in powers of a (or equivalently of p_μ:

$$J = I(p=0) + p_\mu \partial I/\partial p_\mu + \ldots \tag{C.3}$$

Take as many terms as needed to get to the term that goes as a^0. Then evaluate these integrals using some IR-regulator (fermion and/or gluon mass for example):

$$J = (const.1) \times \log \frac{m^2}{a^2} + (const.2) \tag{C.4}$$

Then write the original expression as

$$I(p) = (I(p) - J) + J\,. \tag{C.5}$$

The difference $(I-J)$ is now UV finite and therefore according to the theorem of Reisz (see [17] section 15) can be evaluated in the continuum limit, so evaluate it there

$$I(p) - J = \int_{-\infty}^{\infty} (\ldots)\, d^d k = (const.3) \times \log \frac{p^2}{m^2} + (const.4) \tag{C.6}$$

If everything was done correctly, then $(const.1) = (const.3)$ and the result is

$$I(p) = (const.1) \times \log p^2 a^2 + (const.5) \tag{C.7}$$

While this is very straightforward and well defined, it has the drawback that the continuum integral $(I(p) - J$ must be something that we can evaluate analytically, so it works in Wilson case but not in the Domain Wall case.

2. **Subtraction of divergence**: Approach used by Aoki.

Aoki uses a different approach to handle the divergence. The key step to understand this method is to understand that the logarithmic singularity comes from the lower boundary of integration. For the continuum term, it goes like this:

$$\begin{aligned}
\lim_{p\to 0}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2k^2} &= \lim_{p\to 0}\int \frac{d^dk}{(2\pi)^d}\int_0^1 dx\frac{1}{([k-(1-x)p]^2+x(1-x)p^2)^2} \\
&= \lim_{p\to 0}\int_0^1 dx\int \frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2} = \lim_{p\to 0}\int_0^1 dx\int_0^\infty \frac{\pi^2 l^2\, dl^2}{(l^2+\Delta)^2} \\
&= \lim_{p\to 0}\int_0^1 dx\int_\Delta^\infty \frac{\pi^2(l^2-\Delta)\, dl^2}{(l^2)^2} \\
&= \lim_{p\to 0}\int_0^1 dx\pi^2\log l^2\Big|_{p^2x(1-x)}^{\infty}
\end{aligned} \tag{C.8}$$

The infinite part is regularized by dimensional regularization (or some other method) and the $\log p^2$ part obviously comes from lower integration boundary. As long as the integrand behaves like in the example above, there will be $\log p^2$ singularity for $p\to 0$, regardless of what the upper bound behavior is (or in another words, same singularity for lattice integrals as well as continuum ones).

This is how it applies to lattice integrals. We start with the lattice expression (4.22):

$$I(a,p) = \int_{-\pi/a}^{\pi/a}\frac{d^dk}{(2\pi)^d}\sum_{\lambda,\rho}G_{\lambda\rho}(p-k)V_\rho(k,p)S_F(k)V_\lambda(p,k) .= \int_{-\pi/a}^{\pi/a}\frac{d^dk}{(2\pi)^d}f(a,p;k) \tag{C.9}$$

where we use the notation $f(a,p;k)$ for the integrand. Now, the integrand $f(a,p;k)$ has a logarithmic singularity for $p\to 0$:

$$\lim_{p,k\to 0} f(a,p;k) = \underbrace{\frac{1}{(p-k)^2}}_{\substack{\text{from gluon}\\ \text{propagator}}}\ \underbrace{i\gamma\cdot k}_{\substack{\text{from Dirac}\\ \text{algebra}}}\ \underbrace{\frac{(B_+P_+ + B_-P_-)}{k^2}}_{\substack{\text{from fermion}\\ \text{propagator}}} \tag{C.10}$$

If we add and subtract another function that has the same behavior around zero momentum $g(a,p;k)$, but is easier to evaluate analytically, we get

$$I(a,p) = \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} (f(a,p;k) - g(a,p;k)) + \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} g(a,p;k) \,. \tag{C.11}$$

By construction, we can evaluate $\int g(a,p;k)$ analytically so we're left with $\int (f-g)$. But now $f(a,p;k) - g(a,p;k)$ is regular (again by construction) so we can expand it in powers of p and integrate term by term. Since all terms are finite, we simply set $a \to 0$ and that's it!

The choice of $g(a,p;k)$ is somewhat arbitrary; any function that has the correct logarithmic behavior will do.

C.1 Continuum limit for Wilson fermions

The amplitude is given by $I(p)$ (4.22). To evaluate it we use the technique described above of adding and subtracting the same term

$$I(p) = (I(p) - J) + J \tag{C.12}$$

where we have chosen $J = I(p=0) + p_\mu \partial I / \partial p_\mu$ in such a way to make $(I-J)$ UV finite and therefore according to the theorem of Reisz (see [17] section 15) can be evaluated in the continuum limit. All continuum terms in Capitani seem to contain only the $I(p)$ in the continuum limit which is *not* UV finite. Terms $I(p=0)$ vanish due to the symmetry while terms $p_\mu \partial I(0)/\partial p_\mu$ for all graphs Capitani does contribute only as $\log m^2$ if one uses mass-regularization for IR poles.

In the continuum limit, the finite part $I-J$ becomes

$$I - J = -g_0^2 C_F \left\{ \frac{1}{(p-k)^2 + \mu^2} - \frac{1}{k^2 + \mu^2} - \frac{2p \cdot k}{(k^2 + \mu^2)^2} \right\} \sum_\rho \gamma_\rho \frac{-ik \cdot \gamma + m}{k^2 + m^2} \gamma_\rho \tag{C.13}$$

where we have restored gluon mass μ and fermion mass m to handle the IR singularities. The second term in curly braces vanishes since the integrand is odd under parity. Dirac algebra reduces that factor to

$$\sum_\rho \gamma_\rho \frac{-ik\cdot\gamma+m}{k^2+m^2}\gamma_\rho = \frac{ik\cdot\gamma(d-2)+md}{k^2+m^2}\,, \tag{C.14}$$

so we are left with two terms:

$$\begin{aligned} A_1 &= -g_0^2 C_F \int \frac{d^dk}{(2\pi)^d}\frac{1}{p^2+k^2-2p\cdot k+\mu^2}\frac{ik\cdot\gamma(d-2)+md}{k^2+m^2} & \text{(C.15)}\\ A_2 &= -g_0^2 C_F \int \frac{d^dk}{(2\pi)^d}\frac{-2p\cdot k}{(k^2+\mu^2)^2}\frac{ik\cdot\gamma(d-2)+md}{k^2+m^2}\,. & \text{(C.16)}\end{aligned}$$

In the first term we use the formula (B.1) to change the denominator

$$(k^2+m^2)(p^2+k^2-2p\cdot k+\mu^2) \to [k-(1-x)p]^2+\underbrace{p^2x(1-x)+m^2x+\mu^2(1-x)}_{\Delta} \tag{C.17}$$

and shift the integration variable $k \to k+(1-x)p$ to get

$$\begin{aligned} A_1 &= -g_0^2 C_F \int \frac{d^dk}{(2\pi)^d}\int_0^1 dx \frac{i(k\cdot\gamma+(1-x)p\cdot\gamma)(d-2)+md}{[k^2+\Delta]^2} \\ &= -g_0^2 C_F \int_0^1 dx(i(1-x)p\cdot\gamma(d-2)+md)\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2+\Delta]^2} \\ &= -g_0^2 C_F \int_0^1 dx[i(1-x)p\cdot\gamma(d-2)+md]\frac{1}{(4\pi)^2}\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log\Delta\right] \\ &= -g_0^2 C_F ip\cdot\gamma\frac{d-2}{2}\Bigg[\frac{1}{2-d/2}-\gamma_E+\log 4\pi \\ &\quad +\underbrace{\left\{\left(\frac{m^2}{p^2}+2\right)\left(1+\frac{m^2\log m^2}{p^2}\right)+\left(1+\frac{m^2}{p^2}\right)^2\log\left(m^2+p^2\right)\right\}}_{-\log p^2+2+\ldots}\Bigg]\,. \end{aligned} \tag{C.18}$$

Evaluating the integral in the $\mu \to 0$ and $m \to 0$ limit, we have $\Delta = x(1-x)p^2$ so

$$A_1 = -ip\cdot\gamma\, g_0^2 C_F \frac{d-2}{2}\left[\frac{1}{2-d/2}-\gamma_E+\log 4\pi-\log p^2+2\right]\,. \tag{C.19}$$

In the second integral, we use the formula (B.7) to get

$$
\begin{aligned}
A_2 &= -g_0^2 C_F \int \frac{d^d k}{(2\pi)^d} \frac{-2p\cdot k}{(k^2+\mu^2)^2} \frac{ik\cdot\gamma(d-2)+md}{k^2+m^2} \\
&= -ip\cdot\gamma\, g_0^2 C_F \frac{d-2}{d} \int \frac{d^d k}{(2\pi)^d} \frac{-2k^2}{(k^2+\mu^2)^2(k^2+m^2)} \,.
\end{aligned}
\tag{C.20}
$$

Then we use the Feynman parameter (B.1) to modify the denominator

$$
(k^2+\mu^2)^2(k^2+m^2) \to k^2+m^2x+\mu^2(1-x)\,,
\tag{C.21}
$$

to get

$$
\begin{aligned}
A_2 &= ip\cdot\gamma\, g_0^2 C_F \frac{2(d-2)}{d} \int_0^1 dx\, 2x \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{[k^2+\Delta]^3} \,. \\
&= ip\cdot\gamma\, g_0^2 C_F \frac{2(d-2)}{d} \int_0^1 dx\, 2x \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(3)} \frac{d}{2} \left(\frac{1}{\Delta}\right)^{2-d/2} \\
&= ip\cdot\gamma\, g_0^2 C_F \frac{d-2}{2} \int_0^1 dx\, 2x \frac{1}{(4\pi)^2} \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi - \log\Delta\right] \\
&= ip\cdot\gamma\, g_0^2 C_F \frac{d-2}{2} \frac{1}{(4\pi)^2} \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi \right. \\
&\qquad \left. \underbrace{+\left\{\frac{1}{2} - \frac{\mu^2+m^2\log m^2}{m^2-\mu^2} + \frac{\mu^2}{(m^2-\mu^2)^2}\left(m^2\log m^2 - \mu^2\log\mu^2\right)\right\}}_{\to\frac{1}{2}-\log m^2}\right] ,
\end{aligned}
\tag{C.22}
$$

which in the $\mu \to 0$ limit becomes

$$
A_2 = ip\cdot\gamma\, g_0^2 C_F \frac{d-2}{2} \frac{1}{(4\pi)^2} \left[\frac{1}{2-d/2} - \gamma_E + \log 4\pi + \frac{1}{2} - \log m^2\right] .
\tag{C.23}
$$

Combining the two terms together, we get

$$
I - J = ip\cdot\gamma \frac{g_0^2 C_F}{(4\pi)^2} \frac{d-2}{2} \left[\log\frac{p^2}{m^2} - \frac{3}{2}\right]
\tag{C.24}
$$

which compares to formula (15.127) in Capitani.

Repeating the same procedure for Domain Wall case doesn't work since the continuum integral cannot be solved analytically (and there is also a question of weather the Reisz if Reisz theorem applies.)

C.2 Subtracting the divergence for Domain Wall fermions

One logical choice is the following: since the complicated behavior in the continuum limit comes from the Domain Wall propagator, we use the substitution

$$S_{DW}(k) = \underbrace{S_{DW}(k) - (B_+P_+ + B_-P_-)\bar{S}_W(k)}_{\bar{S}_{DW}} + (B_+P_+ + B_-P_-)S_W(k) \tag{C.25}$$

where $\bar{S}_W$ is the part of Wilson propagator proportional to the γ_μ matrix[1] and $B_\pm$ are constants chosen to cancel the IR divergence of the expression (they are determined by expanding the S_{DW} around zero momentum). The first term is then regular and one can do a simple expansion in p_μ and a and keep only non-vanishing terms. The second term is then the usual Wilson propagator so one can either calculate it with the Reisz theorem or one can use the existing results from literature. The big advantage of this approach is that one can use the same expressions for all coefficients by simply changing the definition of $G_\pm$ and $W^\mp G_\pm$ (in practice that means just relinking the numerical code to calculate with different propagators).

What Aoki does is a bit different; instead of just changing the propagator, he subtracts a continuum expression

$$\frac{1}{(p-k)^2}\sum_{\rho,a}\mathrm{Tr}\left(-ig_0T^a\gamma_\rho\right)\frac{-ik\cdot\gamma(B_+P_+ + B_-P_-)+m}{k^2}\left(-ig_0T^a\gamma_\rho\right) \tag{C.26}$$

which is then multiplied with $\theta(\pi^2 - k^2)$ to keep the integral restricted to the first Brillouin

[1] Since we are adding and subtracting the same thing, in principle it should not matter if we add/subtract only the γ_μ part or the whole Wilson propagator, as long as it doesn't introduce any new singularities (which must cancel analytically but might could mean trouble numerically

zone. One then has to evaluate

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(ap-k)^2+\mu^2} \frac{-(2-d)i\gamma\cdot k + md}{k^2+m^2}\theta(\Lambda^2-k^2)$$

$$= 2\int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{-i(2-d)\gamma\cdot k+md}{(k-apx)^2+a^2p^2x(1-x)+\mu^2 x+m^2(1-x)}\theta(\Lambda^2-k^2)\,. \quad \text{(C.27)}$$

in the limit $\mu \to 0$, $m \to 0$ and $a \to 0$. (Λ is the cutoff.) To extract the $\log p^2$ behavior, we can work with $m = \mu = 0$ immediately, but for numerics it's better to keep μ and m finite and then look at the behavior of the expression as they both go to zero. Then we have to evaluate (after variable shift)

$$I = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{-i(2-d)\gamma\cdot(k+apx)+md}{(k^2+\lambda^2)^2}\theta(\Lambda^2-a^2p^2+a2p\cdot k-k^2)\,. \quad \text{(C.28)}$$

In the $a \to 0$ limit, the integration limit becomes $\theta(\Lambda^2-k^2)$ and the term proportional to $\gamma\cdot k$ vanishes due to symmetry so we're left with

$$I = \int_0^1 dx \int_0^{\Lambda^2} \frac{\pi^2 k^2\, dk^2}{(2\pi)^d} \frac{-i(2-d)p\cdot\gamma ax+md}{(k^2+\lambda^2)^2}$$

$$= \frac{1}{16\pi^2}\int_0^1 dx\,(-i(2-d)p\cdot\gamma ax+md)\left(\int_0^{\Lambda^2}\frac{dk^2}{k^2+\lambda^2} - \lambda^2\int_0^{\Lambda^2}\frac{dk^2}{(k^2+\lambda^2)^2}\right)$$

$$= \frac{1}{16\pi^2}\int_0^1 dx(-i(2-d)p\cdot\gamma ax+md)\left(\log\frac{\Lambda^2+\lambda^2(x)}{\lambda^2}+\frac{\lambda^2(x)}{\Lambda^2+\lambda^2(x)}-1\right)$$

$$= \frac{1}{16\pi^2}\int_0^1 dx\,(-i(2-d)p\cdot\gamma ax+md)$$

$$\times\left(\log\frac{\Lambda^2+a^2p^2x(1-x)}{a^2p^2x(1-x)}+\frac{a^2p^2x(1-x)}{\Lambda^2+a^2p^2x(1-x)}-1\right)$$

$$= \frac{i\gamma\cdot p+md}{16\pi^2}\left(\log\Lambda^2-\log a^2p^2+1\right)+O(a) \quad \text{(C.29)}$$

Expanding the same integral around zero p but nonzero μ and m yields

$$I = \frac{ip\cdot\gamma + md}{16\pi^2} \frac{m^2 \log\frac{m^2+\Lambda^2}{m^2} - \mu^2 \log\frac{\mu^2+\Lambda^2}{\mu^2}}{m^2-\mu^2} \tag{C.30}$$

which in the $\mu = 0$, $m \to 0$ limit becomes

$$I = \frac{ip\cdot\gamma + md}{16\pi^2}\left(\log\Lambda^2 - \log m^2\right) \tag{C.31}$$

Aoki uses the value $\Lambda = \pi$ which is the largest cutoff allowed that keeps the lattice integrand in the first Brillouin zone. This yields formula (34) in Aoki [15]

$$I^{\pm}_{\log} = \frac{1}{16\pi^2}[B_{\mp}]_{st}\left(\log\pi^2 + 1 - \log p^2a^2\right) \tag{C.32}$$

for DW fermions, while for Wilson fermions one has the same formula but without factors $B_{\pm}$. Aoki's formula has a number "1/2" instead of "1" in parenthesis; definitions of $B_{\pm}$ are

$$\begin{aligned} B_+ &= (1-w_0^2)w_0^{2N-s-t} \\ B_- &= (1-w_0^2)w_0^{s+t-2} \end{aligned} \tag{C.33}$$

where $B_{\pm} = C_{\mp}$ in Aoki's notation.

Throughout this thesis we use the method similar to Aoki's method. The idea of the method is the same, add and subtract a piece that has exactly the same logarithmic behavior (so that the difference of the original term and the added term is finite), and which can be evaluated analytically on the lattice. The problem with Aoki's choice is that the added term has a discontinuous θ function which requires a large number of subdivisions of the integrand to evaluate the integral. The choice used in these calculations is

$$\begin{aligned} I_{regulator}(p,\mu) &= \lim_{m\to 0}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(\widehat{ap-k})^2 + a^2\mu^2}\frac{1}{\hat{k}^2 + a^2m^2} \\ &= -\log p^2a^2 + F_0 - \gamma_E + 2 + O(\mu^2) \qquad \text{(C.34)} \\ &= -\log\mu^2a^2 + F_0 - \gamma_E + 1 + O(p^2) \qquad \text{(C.35)} \end{aligned}$$

which is smooth throughout the first Brillouin zone and converges much faster.

Appendix D

5D sums of exponentials appearing in propagators

D.1 $e^{-\alpha|s\pm\lambda-t|}$ terms

We have to carefully examine cases $M<1$ ($w_0>0$) and $M>1$ ($w_0<0$). For $M<1$

$$\begin{aligned}\sum_{s=1}^{N} w_0^{s-1} e^{-\alpha|s\mp\lambda-t|} &= \sum_{s=1}^{t\pm\lambda} w_0^{s-1} e^{-\alpha(t-s\pm\lambda)} + \sum_{s=t\pm\lambda+1}^{N} w_0^{s-1} e^{-\alpha(s\mp\lambda-t)} \\ &= \frac{e^{-\alpha(t\pm\lambda)}}{w_0} \sum_{s=1}^{t\pm\lambda} (e^{\alpha} w_0)^s + w_0^{t\pm\lambda+1} \sum_{s'=1}^{N-(t\pm\lambda)} \left(e^{-\alpha} w_0\right)^{s'} . \end{aligned} \tag{D.1}$$

Using the formula

$$\sum_{n=1}^{N} x^n = x \sum_{n=0}^{N-1} x^n = x \frac{1-x^N}{1-x} \tag{D.2}$$

which is valid for both $x>0$ and $x<0$ we get

$$\begin{aligned}\sum_{s=1}^{N} w_0^{s-1} e^{-\alpha|s\mp\lambda-t|} &= \frac{e^{-\alpha(t\pm\lambda)}}{w_0} e^{\alpha} w_0 \frac{1-(e^{\alpha}w_0)^N}{1-e^{\alpha}w_0} + w_0^{t\pm\lambda+1} e^{-\alpha} w_0 \frac{1-(e^{-\alpha}w_0)^N}{1-e^{-\alpha}w_0} \\ &= \frac{e^{-\alpha(t\pm\lambda)} - w_0^{t\pm\lambda}}{e^{-\alpha}-w_0} + \frac{w_0^{t\pm\lambda} - (e^{-\alpha}w_0)^N e^{\alpha(t\pm\lambda)}}{e^{\alpha}-w_0} . \end{aligned} \tag{D.3}$$

where λ can be 1 or 0. Repeating the same procedure for w_0^{N-s} we get

$$\begin{aligned}
\sum_{s=1}^{N} w_0^{N-s} e^{-\alpha|s\mp\lambda-t|} &= \sum_{s=1}^{t\pm\lambda} w_0^{N-s} e^{-\alpha(t-s\pm\lambda)} + \sum_{s=t\pm\lambda+1}^{N} w_0^{N-s} e^{-\alpha(s\mp\lambda-t)} \\
&= w_0^N \left(\frac{e^{-\alpha(t\pm\lambda)}}{w_0} \sum_{s=1}^{t\pm\lambda} \left(e^{\alpha} w_0^{-1}\right)^s \right. \\
&\qquad \left. + w_0^{t\pm\lambda+1} \sum_{s'=1}^{N-(t\pm\lambda)} \left(e^{-\alpha} w_0^{-1}\right)^{s'} \right) \\
&= w_0^N \left(\frac{e^{-\alpha(t\pm\lambda)}}{w_0} e^{\alpha} w_0^{-1} \frac{1-(e^{\alpha} w_0^{-1})^N}{1-e^{\alpha} w_0^{-1}} \right. \\
&\qquad \left. + w_0^{t\pm\lambda+1} e^{-\alpha} w_0^{-1} \frac{1-(e^{-\alpha} w_0^{-1})^N}{1-e^{-\alpha} w_0^{-1}} \right) \\
&= w_0^N \left(e^{\alpha} \frac{e^{-\alpha(t\pm\lambda)} - w_0^{-(t\pm\lambda)}}{w_0 - e^{\alpha}} \right. \\
&\qquad \left. + e^{-\alpha} \frac{w_0^{-(t\pm\lambda)} - (e^{-\alpha} w_0^{-1})^N e^{\alpha(t\pm\lambda)}}{w_0 - e^{-\alpha}} \right) . \qquad \text{(D.4)}
\end{aligned}$$

From these two we can evaluate double sums

$$\begin{aligned}
\sum_{s,t=1}^{N} w_0^{s-1} e^{-\alpha|s\mp\lambda-t|} w^{t-1} &= \sum_{t=1}^{N} \left(\frac{e^{-\alpha(t\pm\lambda)} - w_0^{t\pm\lambda}}{e^{-\alpha} - w_0} + \frac{w_0^{t\pm\lambda} - (e^{-\alpha} w_0)^N e^{\alpha(t\pm\lambda)}}{e^{\alpha} - w_0} \right) w^{t-1} \\
&= \sum_{t=1}^{N} \left(\frac{e^{\mp\lambda\alpha} w^{-1} (e^{-\alpha} w)^t - w_0^{\pm\lambda} w^{-1} (w w_0)^t}{e^{-\alpha} - w_0} \right. \\
&\qquad \left. + \frac{w_0^{\pm\lambda} w^{-1} (w w_0)^t - (e^{-\alpha} w_0)^N e^{\pm\lambda\alpha} w^{-1} (e^{\alpha} w)^t}{e^{\alpha} - w_0} \right) \\
&= \frac{1}{e^{-\alpha} - w_0} \left\{ e^{\mp\lambda\alpha} w^{-1} e^{-\alpha} w \frac{1-(e^{-\alpha} w)^N}{1-e^{-\alpha} w} \right. \\
&\qquad \left. - w_0^{\pm\lambda} w^{-1} w w_0 \frac{1-(w w_0)^N}{1-w w_0} \right\} \\
&\quad + \frac{1}{e^{\alpha} - w_0} \left\{ w_0^{\pm\lambda} w^{-1} w w_0 \frac{1-(w w_0)^N}{1-w w_0} \right. \\
&\qquad \left. - e^{\pm\lambda\alpha} w^{-1} e^{\alpha} w \frac{1-(e^{\alpha} w)^N}{1-e^{\alpha} w} \right\} \\
&= \frac{e^{\mp\lambda\alpha}(1-e^{-\alpha N} w^N)}{(e^{-\alpha} - w_0)(e^{\alpha} - w)} + \frac{e^{\pm\lambda\alpha}(w^N - e^{-\alpha N}) w_0^N}{(e^{\alpha} - w_0)(e^{-\alpha} - w)}
\end{aligned}$$

$$-\frac{2\sinh\alpha\, w_0^{\pm\lambda+1}}{(e^{\alpha}-w_0)(e^{-\alpha}-w_0)}\frac{1-(ww_0)^N}{1-ww_0} \tag{D.5}$$

Since all $e^{-\alpha}, w, w_0 < 1$, in the $N \to \infty$ limit we have

$$\sum_{s,t=1}^{N} w_0^{s-1} e^{-\alpha|s\mp\lambda-t|} w^{t-1} = \frac{e^{\mp\lambda\alpha}}{(e^{-\alpha}-w_0)(e^{\alpha}-w)} - \frac{2\sinh\alpha}{(e^{\alpha}-w_0)(e^{-\alpha}-w_0)}\frac{w_0^{\pm\lambda+1}}{1-ww_0}\,, \tag{D.6}$$

where λ equals zero or one. For $M > 1$, exponential terms in propagator change pick a minus sign and so does the w_0

$$e^{-\alpha|s-s'|} \to (\pm e^{-\alpha})^{|s-s'|}\,,\ \text{etc.} \qquad w_0^{s-1} \to (\pm w_0)^{s-1}\,,\ \text{etc.} \tag{D.7}$$

for $M > 1$ we get the same formula with $w_0 \to |w_0|$ and overall $(\pm 1)^{\lambda}$ factor. The second double sum equals

$$\begin{aligned}
\sum_{s,t=1}^{N} w_0^{s-1} e^{-\alpha|s\mp\lambda-t|} w^{N-t} &= \sum_{t=1}^{N}\left(\frac{e^{-\alpha(t\pm\lambda)}-w_0^{t\pm\lambda}}{e^{-\alpha}-w_0} + \frac{w_0^{t\pm\lambda}-(e^{-\alpha}w_0)^N e^{\alpha(t\pm\lambda)}}{e^{\alpha}-w_0}\right) w^{N-t} \\
&= \sum_{t=1}^{N}\left(\frac{e^{\mp\lambda\alpha}(e^{-\alpha}w^{-1})^t - w_0^{\pm\lambda}(w_0/w)^t}{e^{-\alpha}-w_0}\right. \\
&\quad \left. + \frac{w_0^{\pm\lambda}(w_0/w)^t - (e^{-\alpha}w_0)^N e^{\pm\lambda\alpha}(e^{\alpha}w^{-1})^t}{e^{\alpha}-w_0}\right) w^N \\
&= \frac{w^N}{e^{-\alpha}-w_0}\left\{e^{\mp\lambda\alpha} e^{-\alpha} w^{-1}\frac{1-(e^{-\alpha}w^{-1})^N}{1-e^{-\alpha}w^{-1}}\right. \\
&\quad \left. - w_0^{\pm\lambda}(w_0/w)\frac{1-(w_0/w)^N}{1-w_0/w}\right\} \\
&\quad + \frac{w^N}{e^{\alpha}-w_0}\left\{w_0^{\pm\lambda}\, w_0/w\frac{1-(w_0/w)^N}{1-w_0/w}\right. \\
&\quad \left. - e^{\pm\lambda\alpha}(e^{-\alpha}w_0)^N e^{\alpha} w^{-1}\frac{1-(e^{\alpha}w^{-1})^N}{1-e^{\alpha}w^{-1}}\right\} \\
&= \frac{1}{e^{-\alpha}-w_0}\left\{e^{\mp\lambda\alpha} e^{-\alpha}\frac{w^N - e^{-\alpha N}}{w-e^{-\alpha}}\right. \\
&\quad \left. - w_0^{\pm\lambda} w^N \frac{w_0}{w}\frac{1-(w_0/w)^N}{1-w_0/w}\right\} \\
&\quad + \frac{w^N}{e^{\alpha}-w_0}\left\{w_0^{\pm\lambda} w^N \frac{w_0}{w}\frac{1-(w_0/w)^N}{1-w_0/w}\right.
\end{aligned}$$

$$-e^{\pm\lambda\alpha}e^{\alpha}\frac{e^{-\alpha N}(ww_0)^N-w_0^N}{w-e^{\alpha}}\Bigg\}\,, \tag{D.8}$$

which vanishes in the $N\to\infty$ limit[1]. The last one is

$$\begin{aligned}
\sum_{s,t=1}^{N} w_0^{N-s}e^{-\alpha|s\mp\lambda-t|}w^{N-t} &= \sum_{t=1}^{N}(ww_0)^N\left(e^{\alpha}\frac{e^{-\alpha(t\pm\lambda)}-w_0^{-(t\pm\lambda)}}{w_0-e^{\alpha}}\right.\\
&\qquad\left.+e^{-\alpha}\frac{w_0^{-(t\pm\lambda)}-(e^{-\alpha}w_0^{-1})^N e^{\alpha(t\pm\lambda)}}{w_0-e^{-\alpha}}\right)w^{-t}\\
&= \sum_{t=1}^{N}(ww_0)^N\left(\frac{e^{\alpha\mp\lambda\alpha}(e^{-\alpha}w^{-1})^t-e^{\alpha}w_0^{\mp\lambda}(ww_0)^{-t}}{w_0-e^{\alpha}}\right.\\
&\qquad\left.+\frac{e^{-\alpha}w_0^{\mp\lambda}(ww_0)^{-t}-(e^{-\alpha}w_0^{-1})^N e^{-\alpha}e^{\pm\lambda\alpha}(e^{\alpha}w^{-1})^t}{w_0-e^{-\alpha}}\right)\\
&= \frac{(ww_0)^N}{w_0-e^{\alpha}}\left[e^{\alpha}e^{\mp\lambda\alpha}e^{-\alpha}w^{-1}\frac{1-(e^{-\alpha}w^{-1})^N}{1-e^{-\alpha}w^{-1}}\right.\\
&\qquad\left.-e^{\alpha}w_0^{\mp\lambda}\frac{1-(ww_0)^{-N}}{ww_0-1}\right]\\
&\quad+\frac{(ww_0)^N}{w_0-e^{-\alpha}}\left[e^{-\alpha}w_0^{\mp\lambda}\frac{1-(ww_0)^{-N}}{ww_0-1}\right.\\
&\qquad\left.-(e^{-\alpha}w_0^{-1})^N e^{-\alpha}e^{\pm\lambda\alpha}e^{\alpha}w^{-1}\frac{1-(e^{\alpha}w^{-1})^N}{1-e^{\alpha}w^{-1}}\right]\\
&= \frac{e^{\mp\lambda\alpha}\left((ww_0)^N-e^{-\alpha N}w_0^N\right)}{(w_0-e^{\alpha})(w-e^{-\alpha})}+\frac{e^{\pm\lambda\alpha}\left(1-e^{-\alpha N}w^N\right)}{(w_0-e^{-\alpha})(w-e^{\alpha})}\\
&\quad-\frac{2\sinh\alpha}{(w_0-e^{\alpha})(w_0-e^{-\alpha})}w_0^{\mp\lambda+1}\frac{1-(ww_0)^N}{1-ww_0}\,,
\end{aligned} \tag{D.9}$$

which in the $N\to\infty$ limit becomes

$$\sum_{s,t=1}^{N} w_0^{N-s}e^{-\alpha|s\mp\lambda-t|}w^{N-t}=\frac{e^{\pm\lambda\alpha}}{(w_0-e^{-\alpha})(w-e^{\alpha})}-\frac{2\sinh\alpha}{(w_0-e^{\alpha})(w_0-e^{-\alpha})}\frac{w_0^{\mp\lambda+1}}{1-ww_0}\,. \tag{D.10}$$

For $M>1$ we get the same formula with $w_0\to|w_0|$ and overall $(\pm1)^{\lambda}$ factor.

[1] $\frac{w_0}{w}\frac{1-(w_0/w)^N}{1-w_0/w}\to N$ in the $N\to\infty$ limit.

D.2 Other terms

Coefficients multiplying $A_\pm$ and A_m come as products of terms $e^{-\alpha(s-1)}$ and $e^{-\alpha(N-s)}$; since they factor nicely into s and t dependent terms, we can factor the whole double sum. We have

$$\sum_{s=1}^{N} w_0^{s-1} e^{-\alpha(s-1)} = \sum_{s'=0}^{N-1} (w_0 e^{-\alpha})^{s'} = \frac{1-w_0^N e^{-\alpha N}}{1-w_0 e^{-\alpha}} \tag{D.11}$$

$$\begin{aligned}\sum_{s=1}^{N} w_0^{N-s} e^{-\alpha(s-1)} &= w_0^N e^{\alpha} \sum_{s=1}^{N} (w_0^{-1} e^{-\alpha})^s = w_0^N e^{\alpha} w_0^{-1} e^{-\alpha} \frac{1-(w_0^{-1}e^{-\alpha})^N}{1-w_0^{-1}e^{-\alpha}} \\ &= \frac{w_0^N - e^{-\alpha N}}{w_0 - e^{-\alpha}}\end{aligned} \tag{D.12}$$

$$\begin{aligned}\sum_{s=1}^{N} w_0^{s-1} e^{-\alpha(N-s)} &= \frac{e^{-\alpha N}}{w_0} \sum_{s=1}^{N} (w_0 e^{\alpha})^s = \frac{e^{-\alpha N}}{w_0} w_0 e^{\alpha} \frac{1-(w_0 e^{\alpha})^N}{1-w_0 e^{\alpha}} \\ &= \frac{w_0^N - e^{-\alpha N}}{w_0 - e^{-\alpha}}\end{aligned} \tag{D.13}$$

$$\sum_{s=1}^{N} w_0^{N-s} e^{-\alpha(N-s)} = \sum_{s'=N-1}^{0} (w_0 e^{-\alpha})^{s'} = \frac{1-w_0^N e^{-\alpha N}}{1-w_0 e^{-\alpha}} \tag{D.14}$$

Since for $M > 1$, both exponential terms in propagator pick a minus sign

$$e^{-\alpha(\dots)} \to (\pm e^{-\alpha})^{(\dots)}, \qquad w_0^{(\dots)} \to (\pm w_0)^{(\dots)} \tag{D.15}$$

overall formulas remain the same (with $w_0 \to |w_0|$) .

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