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Connection Between Unimodular Interaction Constraint for Decentralized Controllers and Small Gain Bounds *

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Abstract

Decentralized controllers for a stable plant can be designed as if the plant were decoupled provided that the individual controller's Youla parameter satisfies an norm bound imposed by the off diagonal plant operators. This bound is derived from the unimodular interaction constraint associated with the set of stabilizing decentralized controllers. One result from such a bound is the quantification of the notion of weak coupling, used as a condition for the nonsingular perturbation design approach to decentralized control. Using a robust stability framework, the above bound can be related to a small gain bound. Specifically, in the two channel case, any Youla parameters which satisfy the small gain bound will satisfy the above bound however the converse is not true making this bound less conservative than small gain bound. However in a fundamental sense either framework (robust stability or unimodular interaction constraint) produce the same stability constraint for the set of stable plants. Therefore, either point of view can be used to derive these bounds. These results are extended to the multi-channel case.

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1 Introduction

A decentralized control system is a control structure where restrictions are made on the information available to the feedback channels. Specifically, a general decentralized control structure imposes a partitioning and pairing of the system inputs and outputs. This constrains the controller to be block diagonal thereby providing a individual controller for each channel of the partitioned system. Recently Gundes and Deoser [1] have provided a parameterization of the class of all stabilizing decentralized compensators. For the two channel case the parameterization involves four parameter matrices. These parameters are not free parameters, they are required to satisfy a unimodular constraint. A different method for incorporating stable factorization in the design of the decentralized control system [2] involves the use of Youla parameterization of the individual subsystems. The stable factors of the subsystem plant operator are used to parameterize the class of all stabilizing controllers for this subsystem in a manner identical to that used with centralized design [3]. The Youla parameter selected for each subsystem is done in a sequential fashion where the interconnection operators constrain the choice of parameter as each loop is closed. In this way the set of selected Youla parameters for the decentralized control system will provide closed loop stability for the overall system. In this paper we take the Gundes and Desoer [1] parameterization and recast it in the more familiar Youla parameterization form for the set of stable plant operators. However, unlike [2] sequential loop closing and sequential selection of Youla parameters will not be employed since use of the Gundes and Desoer [1] unimodular constraint for system closed loop stability will be exploited. From the unimodular constraint imposed on the parameters for the class of all stabilizing decentralized compensators a norm bound will be derived which constrains the Youla parameters of the individual subsystems in terms of the plant off diagonal operators (i.e. the interaction operators of the plant not accounted for in stabilization of the individual subsystems). The bound serves as an interaction measure and provides a upper threshold which when met by the set of subsystem Youla parameters provides a stability guarantee for the overall closed loop system. Section 2 derives these results for the two channel case. The interaction measure in the form of a norm bound effectively quantifies the notion of weak coupling which is a condition for nonsingular perturbation design of decentralized control [4] [5]. These issues will be elaborated on in section 3. Section 4 compares the bound derived for the two channel case with a small gain bound derived by placing the problem in a robust stability type framework. It will be shown that any pair of Youla parameters which satisfy this small gain bound will also satisfy the bound derived from the unimodular constraint of section 2 but the converse isn't true thereby making the aforementioned bound less conservative than the small gain bound in the two channel case. Finally, in section 5 the multi-channel case is considered. The two channel case is not generalized directly. The bound derived for the multi-channel case from the unimodular constraint is shown to be also directly derivable from the small gain theorem.

1.1 Notation

H	principle ideal domain, designates ring of proper stable rational functions
$J \subset H$	is the group of units of H
$m(H)$	set of matrices with elements in H
$\|F\|$	determinant of F
unimodular	$F \in m(H)$ is unimodular iff $\|F\| \in J$

2 Bounding Compensator Parameters

Figure 1 illustrates the two channel decentralized control problem. From [1] the parameterization of all stabilizing decentralized compensators for the two channel case with stable plant takes the following form

Lemma 1 (see [1, p. 124, Thm 4.3.5] for proof) Given that $P \in m(H)$, where $m(H)$ corresponds to the matrix ring of stable systems, then $C_d = diag(C_1, C_2)$ is a decentralized stabilizing compensator for

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad \text{iff} \quad C_d = \begin{bmatrix} (Q_{11} - Q_1 P_{11})^{-1} Q_1 & 0 \\ 0 & (Q_{22} - Q_2 P_{22})^{-1} Q_2 \end{bmatrix} \tag{1}$$

$$\text{for some} \quad Q_{11}, Q_{22}, Q_1, Q_2 \in m(H) \quad \text{such that} \quad \mathcal{Q} = \begin{bmatrix} Q_{11} & Q_1 P_{12} \\ Q_2 P_{21} & Q_{22} \end{bmatrix} \quad \text{is unimodular} \tag{2}$$

If the plant P is initially decoupled the interaction constraint (eq. 2) reduces to Q_{11} and Q_{22} being unimodular. To prove this the following lemma will be useful.

Lemma 2 (see [6, p. 393, Fact B.1.26] for proof) $F \in m(H)$ (where $m(H)$ corresponds to the matrix ring of proper stable systems) is unimodular iff $|F|$ is a unit in H (where H corresponds to the ring of proper stable transfer functions).

For a decoupled plant, $P_{12} = 0$ and $P_{21} = 0$, the interaction constraint reduces as follows

$$\mathcal{Q} = diag(Q_{11}, Q_{22}) \quad \text{is unimodular} \quad \Leftrightarrow \quad |\mathcal{Q}| \quad \text{is a unit} \quad \text{(by Lemma 2)}$$

Since $|\mathcal{Q}| = |Q_{11}|\,|Q_{22}|$ and Q_{11}, Q_{22} are elements of $m(H)$ then $|Q_{11}|, |Q_{22}|$ must be units in H which by Lemma 2 implies Q_{11} and Q_{22} are unimodular. The lack of coupling in the plant will

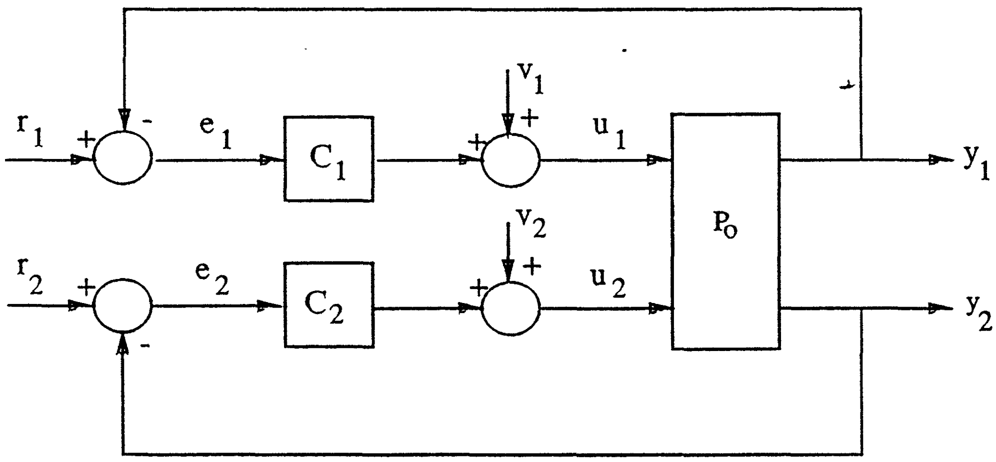

Figure 1: The Two Channel Decentralized Control Problem

allow reformulating the individual compensator parameterizations in eq. (1) to the one parameter Youla form [3]. The observation that Q_{11} and Q_{22} must become unimodular as the coupling vanishes facilitates this reformulation. Specifically, for $C_d = diag(C_1, C_2)$ both compensators can be rewritten as

$$\begin{aligned} C_i &= (Q_{ii} - Q_i P_{ii})^{-1} Q_i \qquad \text{for } i = 1 \text{ or } 2 \\ &= \left(Q_{ii}(I - Q_{ii}^{-1} Q_i P_{ii})\right)^{-1} Q_i \\ &= \left(I - Q_{ii}^{-1} Q_i P_{ii}\right)^{-1} Q_{ii}^{-1} Q_i \\ &= \left(I - \tilde{Q}_i P_{ii}\right)^{-1} \tilde{Q}_i \\ \text{where} \quad \tilde{Q}_i &= Q_{ii}^{-1} Q_i \end{aligned} \tag{3}$$

Since Q_{ii} is unimodular, $Q_{ii}^{-1} Q_i$ is an element in $m(H)$ and therefore $\tilde{Q}_i$ is also an element in $m(H)$. This then places eq. (3) in the Youla parameterization form for the case of stable plant operators P_{11} and P_{22}.

When the plant is coupled (i.e. $P_{12}, P_{21} \neq 0$) the above parameterization can be extended by accounting for the effect of the cross coupling on the $\tilde{Q}_1, \tilde{Q}_2$ terms. This effect will be accounted for in terms of a norm bound on the $\tilde{Q}_1, \tilde{Q}_2$ parameters. The following induced operator norm will be used

$$\|P\| = \sup_{\omega \in \Re} \bar{\sigma}(P(i\omega)) \tag{4}$$

Before deriving the bound on the Youla parameters the following lemma will prove useful.

Lemma 3 (see [6, p. 22, Lemma 2.2.19] for proof) For $R \in m(H)$ if $\|R\| < 1$ then $|I - R|$ is a unit in H.

To derive the bound we begin with the interaction constraint from eq. (2).

$$\mathcal{Q} = \begin{bmatrix} Q_{11} & Q_1 P_{12} \\ Q_2 P_{21} & Q_{22} \end{bmatrix} \quad \text{is unimodular} \tag{5}$$

Invoking lemma 2 and using the well known Schur determinantal formula [7], constraint 5 becomes

$$|\mathcal{Q}| = |Q_{11}| \left| Q_{22} - (Q_2 P_{21})(Q_{11}^{-1})(Q_1 P_{12}) \right| \quad \text{is a unit} \tag{6}$$

$$= |Q_{11}| \, |Q_{22}| \left| I - Q_{22}^{-1} Q_2 P_{21} Q_{11}^{-1} Q_1 P_{12} \right| \tag{7}$$

Requiring Q_{11} and Q_{22} to be unimodular ensures parameterization given by eq. (3). In addition, since Q_{11}, Q_{22} are elements of $m(H)$, $|\mathcal{Q}|$ will be a unit if and only if $\left| I - Q_{22}^{-1} Q_2 P_{21} Q_{11}^{-1} Q_1 P_{12} \right|$ is a unit. Substituting $\tilde{Q}_1$ for $Q_{11}^{-1} Q_1$ and $\tilde{Q}_2$ for $Q_{22}^{-1} Q_2$ this constraint becomes

$$|\mathcal{Q}| \quad \text{is a unit} \quad \Leftrightarrow \quad \left| I - \tilde{Q}_2 P_{21} \tilde{Q}_1 P_{12} \right| \quad \text{is a unit} \tag{8}$$

Note that $\tilde{Q}_2 P_{21} \tilde{Q}_1 P_{12}$ is a element of $m(H)$ and invoking lemma 3 means $\left| I - \tilde{Q}_2 P_{21} \tilde{Q}_1 P_{12} \right|$ will be a unit if $||\tilde{Q}_2 P_{21} \tilde{Q}_1 P_{12}|| < 1$. Use of the submultiplicative property of induce operator norms gives

$$||\tilde{Q}_2 P_{21} \tilde{Q}_1 P_{12}|| \leq ||\tilde{Q}_2|| ||P_{21}|| ||\tilde{Q}_1|| ||P_{12}|| \tag{9}$$

Therefore, forcing $||\tilde{Q}_2|| ||P_{21}|| ||\tilde{Q}_1|| ||P_{12}|| < 1$ ensures $|Q|$ is a unit and provides the following bound on the design parameters in terms of the off diagonal plant operators

$$||\tilde{Q}_2|| ||P_{21}|| ||\tilde{Q}_1|| ||P_{12}|| \quad < \quad 1 \tag{10}$$

$$||\tilde{Q}_2|| ||\tilde{Q}_1|| \quad < \quad \frac{1}{||P_{21}|| ||P_{12}||} \tag{11}$$

Thus the controller parameterization of eq. (3) will provide for closed loop stability if the above bound, eq. (11), is satisfied.

3 Remarks

The following is a set of remarks which provide interpretation and checks on the bound of eq. (11).

Remark 1 As $||P_{12}|| \to 0$ and $||P_{21}|| \to 0$, effectively the restrictions on the parameters $\tilde{Q}_1$ and $\tilde{Q}_2$ disappear. That is the upper bound of eq. (11) becomes virtually infinite and the set of Youla parameters expands to encompass the entire matrix ring of proper stable systems $m(H)$. This is the expected result and quantifies the notion of weak coupling. Specifically, the bound of eq. (11) specifies an upper bound on the Youla parameters in terms of the off diagonal operators P_{12} and P_{21}. The expectation is that as the cross coupling in the plant becomes small (i.e. $||P_{12}|| \to \varepsilon$ and $||P_{21}|| \to \varepsilon$ where $\varepsilon \ll 1$) stabilization of the overall system is not compromised by simply ensuring that the individual compensators for P_{11} and P_{22} provide stabilization for these individual loops. And as this coupling goes to zero the expectation is that the set of individual stabilizing compensators for P_{11} and P_{22} grows to encompass the entire set of all stabilizing compensators for P_{11} and P_{22} (i.e. the parameterization in eq. (3)). In effect this is exactly what the bound of eq. (11) provides and precisely how the set of stabilizing compensators grow to encompass the entire set, is quantified by the upper bound placed on the Youla parameters in terms of P_{12} and P_{21}.

Remark 2 It is expected for a block triangular plant (i.e. $\|P_{12}\| = 0$ or $\|P_{21}\| = 0$) that no restriction should exist on the individual stabilizing controllers that can be applied to P_{11} and P_{22}. That is stabilization of the overall system is once again not compromised by simply stabilizing the individual subsystems, P_{11} and P_{22}. The bound of eq. (11) satisfies this condition. For example as $\|P_{12}\|$ goes to zero the bound on the Youla parameters $\tilde{Q}_1$ and $\tilde{Q}_2$ disappears. An interesting aspect of this is that the bound in the face of weak triangular coupling behaves similarly to case of weak coupling discussed in remark 1. That is as $\|P_{12}\| \rightarrow \varepsilon$ (or $\|P_{21}\| \rightarrow \varepsilon$) for $\varepsilon \ll 1$ there exist a rather large set of stabilizing compensators for P_{11} and P_{22} which also do not destabilize the overall system. As $\varepsilon \rightarrow 0$ this set grows to encompass the entire set of all stabilizing compensators for P_{11} and P_{22}.

4 Relation to Bound Derived From Small Gain Theorem

The problem can be approached from a robust stability point of view where the decoupled plant is treated as the nominal plant and the off diagonal plant operators, P_{12} and P_{21}, become an additive perturbation. Using the parameterization of eq. (3) we seek the constraints placed on the Youla parameters by the Small Gain Theorem. It will be shown that any Youla parameters which satisfy

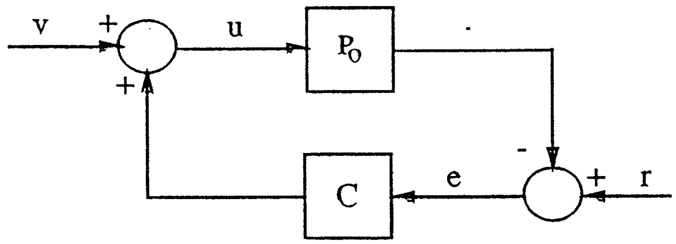

Figure 2: Centralized Two Block Problem

this small gain constraint will also satisfy the bound of eq. (11). This is reassuring in the sense that the bound of eq. (11) is derived via small gain arguments (see lemma 3). It is extended beyond the small gain bound only as a consequence of the existence of a simple determinantal formula (eq. 7) for the two channel case which allows separation of the Youla parameters from the off diagonal plant operators. As will be seen in the multi-channel case (section 5), when using the same line of reasoning as in section 2, the absence of a similar simple determinantal formula results in a bound from the multi-channel unimodular constraint which is identical to a small gain bound derived using only the robust stability framework of this section.

The Plant P can be decomposed in the following manner

$$P = P_0 + \Delta = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} + \begin{bmatrix} 0 & P_{12} \\ P_{21} & 0 \end{bmatrix} \tag{12}$$

Gundes and Desoer [1] formulation of the two channel decentralized control problem (see figure 1) is in the form of the two block problem where the controller is constrained to be block diagonal (see figure 2). The parameterization of eq. (3), where $C = C_d = diag(C_1, C_2)$ guarantees internal stability for the closed loop map of the two block problem illustrated in figure 2. That is for the

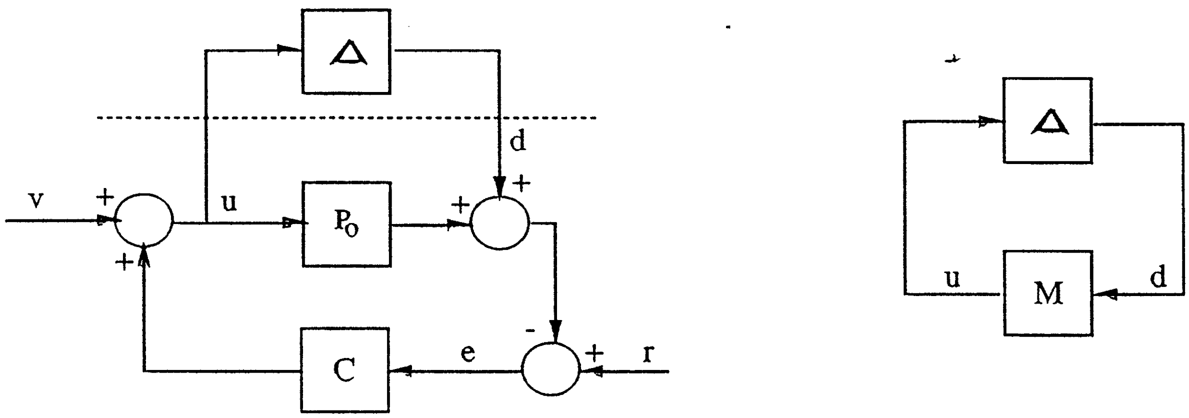

Figure 3: Transformation to Small Gain Loop

closed loop map

$$H(C,P_0): \quad \begin{bmatrix} r \\ v \end{bmatrix} \longrightarrow \begin{bmatrix} e \\ u \end{bmatrix} \quad \text{where} \quad H(C,P_0) = \begin{bmatrix} (I+P_0C)^{-1} & -(I+P_0C)^{-1}P_0 \\ (I+CP_0)^{-1}C & (I+CP_0)^{-1} \end{bmatrix} \tag{13}$$

all transfer functions which are elements of $H(C,P)$ are in $m(H)$ (i.e. they are stable). By applying the additive perturbation to the two block problem and performing the linear fractional transformation indicated in figure 3 the closed loop system is now in a form where the Small Gain Theorem can be applied directly. Note that the operator M is defined as

$$M: \quad d \longrightarrow u \qquad \text{where} \qquad M = -(I+CP_0)^{-1}C \tag{14}$$

and $M \in m(H)$ by internal stability, also $\Delta \in m(H)$ since $P \in m(H)$. Because M and Δ are both stable the Small Gain Theorem [8] provides that the closed loop remains stable as long as

$$||M\Delta|| < 1 \tag{15}$$

Substituting $C = diag(C_1, C_2)$ and $P_0 = diag(P_{11}, P_{22})$, M becomes

$$M = -(I + CP_0)^{-1}C = \begin{bmatrix} -(I + C_1P_{11})^{-1}C_1 & 0 \\ 0 & -(I + C_2P_{22})^{-1}C_2 \end{bmatrix} \tag{16}$$

Substituting in the Youla parameterization from eq. (3) each term in M reduces as follows

$$\begin{aligned} -(I + C_iP_{ii})^{-1}C_i &= -(I + (I - \tilde{Q}_iP_{ii})^{-1}\tilde{Q}_iP_{ii})^{-1}(I - \tilde{Q}_iP_{ii})^{-1}\tilde{Q}_i \qquad \text{for } i = 1 \text{ or } 2 \\ &= -\left((I - \tilde{Q}_iP_{ii})(I + (I - \tilde{Q}_iP_{ii})^{-1}\tilde{Q}_iP_{ii})\right)^{-1}\tilde{Q}_i \\ &= -\left((I - \tilde{Q}_iP_{ii}) + \tilde{Q}_iP_{ii}\right)^{-1}\tilde{Q}_i \\ &= -\tilde{Q}_i \end{aligned} \tag{17}$$

And $M\Delta$ becomes

$$M\Delta = \begin{bmatrix} -\tilde{Q}_1 & 0 \\ 0 & -\tilde{Q}_2 \end{bmatrix} \begin{bmatrix} 0 & P_{12} \\ P_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\tilde{Q}_1P_{12} \\ -\tilde{Q}_2P_{21} & 0 \end{bmatrix} \tag{18}$$

To find the constraint on the $\tilde{Q}_1$ and $\tilde{Q}_2$ the following lemma will prove useful

Lemma 4

$$\text{For} \quad H = \begin{bmatrix} 0 & H_{12} \\ H_{21} & 0 \end{bmatrix} \qquad \|H\| < 1 \quad \text{iff} \quad \|H_{12}\| < 1 \quad \text{and} \quad \|H_{21}\| < 1 \tag{19}$$

Proof

$$\begin{aligned} \|H\| &= \sup_{\omega \in \Re} \bar{\sigma}(H(i\omega)) \\ &= \sup_{\omega \in \Re} [\lambda_{\max}(H^*(i\omega)H(i\omega))]^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\omega\in\Re}\left[\lambda_{\max}\left(\begin{bmatrix} 0 & H_{12} \\ H_{21} & 0 \end{bmatrix}^* \begin{bmatrix} 0 & H_{12} \\ H_{21} & 0 \end{bmatrix}\right)\right]^{1/2} \\
&= \sup_{\omega\in\Re}\left[\lambda_{\max}\begin{pmatrix} H_{21}^*H_{21} & 0 \\ 0 & H_{12}^*H_{12} \end{pmatrix}\right]^{1/2} \\
&= \sup_{\omega\in\Re}\left[\max\left[\lambda_{\max}(H_{21}^*H_{21})^{1/2}, \lambda_{\max}(H_{12}^*H_{12})^{1/2}\right]\right] \\
&= \max\left[\sup_{\omega\in\Re}\left[\lambda_{\max}(H_{21}^*H_{21})^{1/2}\right], \sup_{\omega\in\Re}\left[\lambda_{\max}(H_{12}^*H_{12})^{1/2}\right]\right] \\
&= \max\left[\sup_{\omega\in\Re}\bar{\sigma}(H_{21}(i\omega)), \sup_{\omega\in\Re}\bar{\sigma}(H_{12}(i\omega))\right] \\
&= \max\left[||H_{21}||, ||H_{12}||\right]
\end{aligned}
$$

Therefore $\quad ||H|| < 1 \iff ||H_{12}|| < 1 \quad \text{and} \quad ||H_{21}|| < 1$ □

Thus to find the constraints on $\tilde{Q}_1$ and $\tilde{Q}_2$ we invoke lemma 4. That is $||M\Delta|| < 1$ iff $||\tilde{Q}_1 P_{12}|| < 1$ and $||\tilde{Q}_2 P_{21}|| < 1$. This then leads to the following constraint on the Youla parameters due to the Small Gain Theorem

$$
||\tilde{Q}_1|| < \frac{1}{||P_{12}||} \qquad \text{and} \qquad ||\tilde{Q}_2|| < \frac{1}{||P_{21}||} \tag{20}
$$

From the above bound we can derive the bound in equation 11 as follows

$$
\begin{aligned}
||\tilde{Q}_2|| &< \frac{1}{||P_{21}||} \\
||\tilde{Q}_2|| ||\tilde{Q}_1|| &< \frac{||\tilde{Q}_1||}{||P_{21}||} \\
&< \frac{1}{||P_{21}|| ||P_{12}||}
\end{aligned} \tag{21}
$$

This then says that any Youla parameters which satisfy the Small Gain bound of eq. (20) also satisfies the bound found earlier in section 2 given by eq. (11). However, the converse is not true.

This is seen by considering the following example. If $||P_{12}|| \to 0$ the bound on $\tilde{Q}_1$ and $\tilde{Q}_2$ from eq. (11) disappears but as can be seen from eq. (20) the Small Gain bound still constrains $\tilde{Q}_2$ when $||P_{21}|| \neq 0$. Hence arbitrary parameters $\tilde{Q}_1$ and $\tilde{Q}_2$ which satisfy the bound given by eq. (11) may not satisfy the bound imposed by the Small Gain condition given by eq. (20). This illustrates that the bound of eq. (11) encompasses a larger set of Youla parameters which will stabilize the closed loop system then is given by the small gain bound eq. (20). Note however that in a fundamental sense these two seemingly different frameworks (unimodular constraint v.s. stability robustness) give precisely the same conditions for stability and hence result in the same bound. This is seen as follows, stability of the closed loop involving the stable operators Δ and M in figure 3 is guaranteed as long as $|I - M\Delta|$ is a unit. Substituting in the matrix values for $M\Delta$ from eq. (18) results in precisely the constraint of eq. (8) which is derived from the unimodular constraint of eq. (5).

5 Generalization to Multi-Channels

From [1] the unimodular constraint for the m channel case where $P \in m(H)$ is

$$\mathcal{Q} = \begin{bmatrix} Q_{11} & Q_1P_{12} & Q_1P_{13} & \cdots & Q_1P_{1m} \\ Q_2P_{21} & Q_{22} & Q_2P_{23} & \cdots & Q_2P_{2m} \\ Q_3P_{31} & Q_3P_{32} & Q_{33} & \cdots & Q_3P_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ Q_mP_{m1} & Q_mP_{m2} & Q_mP_{m3} & \cdots & Q_{mm} \end{bmatrix} \tag{22}$$

Directly generalizing the method in section 2 for finding an interaction measure in the form of a norm bound on the Youla parameters for the individual subsystem compensators would require finding a determinantal formula for the following matrix

$$\tilde{Q} = \begin{bmatrix} I & \tilde{Q}_1P_{12} & \tilde{Q}_1P_{13} & \cdots & \tilde{Q}_1P_{1m} \\ \tilde{Q}_2P_{21} & I & \tilde{Q}_2P_{23} & \cdots & \tilde{Q}_2P_{2m} \\ \tilde{Q}_3P_{31} & \tilde{Q}_3P_{32} & I & \cdots & \tilde{Q}_3P_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ \tilde{Q}_mP_{m1} & \tilde{Q}_mP_{m2} & \tilde{Q}_mP_{m3} & \cdots & I \end{bmatrix} \tag{23}$$

And deriving a norm which would allow separation of the $\tilde{Q}_i$'s and P_{ij}'s in the form of an inequality which provides that $|\tilde{Q}|$ is a unit (see section 2 eq. (5) through eq. (11)). The complexity of determinantal formula for the m channel case precludes this approach. Another approach which generalizes the intent of the bound in eq.(11) for the multi-channel case and takes advantage of the equivalence of the stability constraint in both the robust stability framework and unimodular interaction setting (as noted at the end of section 4) is as follows. Rewriting eq. (23) we obtain

$$\tilde{Q} = \begin{bmatrix} I & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} + \begin{bmatrix} \tilde{Q}_1 & & & \\ & \tilde{Q}_2 & & \\ & & \ddots & \\ & & & \tilde{Q}_m \end{bmatrix} \begin{bmatrix} 0 & P_{12} & \cdots & P_{1m} \\ P_{21} & 0 & \cdots & P_{2m} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \cdots & 0 \end{bmatrix} \tag{24}$$

By application of lemma 3, $|\tilde{Q}|$ will be a unit if

$$\left\| \begin{bmatrix} \tilde{Q}_1 & & & \\ & \tilde{Q}_2 & & \\ & & \ddots & \\ & & & \tilde{Q}_m \end{bmatrix} \begin{bmatrix} 0 & P_{12} & \cdots & P_{1m} \\ P_{21} & 0 & \cdots & P_{2m} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \cdots & 0 \end{bmatrix} \right\| < 1 \tag{25}$$

or equivalently

$$\left\| diag(\tilde{Q}_1, \cdots, \tilde{Q}_m) \right\| < \left\| \begin{matrix} 0 & P_{12} & \cdots & P_{1m} \\ P_{21} & 0 & \cdots & P_{2m} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \cdots & 0 \end{matrix} \right\|^{-1} \tag{26}$$

or by lemma 4

$$\left\| \tilde{Q}_i \right\| < \left\| \begin{matrix} 0 & P_{12} & \cdots & P_{1m} \\ P_{21} & 0 & \cdots & P_{2m} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \cdots & 0 \end{matrix} \right\|^{-1} \quad \forall \quad i \tag{27}$$

Note however that this bound is identical to that given by the Small Gain Theorem for the multi-channel case. For the multi-channel case

$$M = \begin{bmatrix} \tilde{Q}_1 & & & \\ & \tilde{Q}_2 & & \\ & & \ddots & \\ & & & \tilde{Q}_m \end{bmatrix} \qquad \Delta = \begin{bmatrix} 0 & P_{12} & \cdots & P_{1m} \\ P_{21} & 0 & \cdots & P_{2m} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \cdots & 0 \end{bmatrix} \tag{28}$$

The Small Gain bound requirement $||M\Delta|| < 1$ is equivalent to eq.(25). Remarks from section 3 extend in an analogous fashion to the above multi-channel bound.

6 Conclusion

The set of stabilizing compensators for a decoupled, two channel, plant consists of a compensator of the form $C_d = diag(C_1, C_2)$ where the individual compensators C_1 and C_2 have a Youla parameterization. For coupled stable plants this parameterization can be extended by constraining the norm of the Youla parameters by the norm of the off diagonal plant operators P_{12} and P_{21} as was done in section 2, eq. (11). This bound was derived from the unimodular interaction constraint associated with the parameterization of stabilizing compensators found in section 2, lemma 1, eq. (2). One result from such a bound is the quantification of weak coupling with respect to stabilizing decentralized compensators. This bound provides for the recovery of the entire set of stabilizing compensators for the individual plant operators P_{11} and P_{22} as the coupling goes to zero. A relationship is made to a bound derived using the Small Gain Theorem. It is shown that Youla parameters for the decentralized controllers which satisfy this small gain bound (eq. 20) will also satisfy the bound derived via the interaction constraint (eq. 11). It is noted that fundamentally the robust stability framework setup in section 4 effectively produces the same stability constraint as the unimodular interaction constraint of section 2. This observation is then used in section 5 when extending the bound to the multi-channel case.

References

[1] Gundes, A.N. and Desoer, C.A. *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators: Lecture Notes in Control and Information Sciences, Vol. 142.* Springer-Verlag, 1990.

[2] Tan, X.-L. and Ikeda, M. Decentralized stabilization for expanding construction of large-scale systems. *IEEE Trans. AC, Vol. 35, pp. 644-651*, 1990.

[3] Francis, B. *A Course in H_∞ Control Theory: Lecture Notes in Control and Information Sciences, Vol. 88.* Springer-Verlag, 1987.

[4] Sandell Jr., N.R. Varaiya, P. Athans, M. and Safonov, M.G. Survey of decentralized control methods for large scale systems. *IEEE Trans. AC, Vol. 23, pp. 108-128*, 1978.

[5] Gajic, Z. Petkovski, D. and Shen, X. *Singularly Perturbed and Weakly Coupled Linear Control Systems: Lecture Notes in Control and Information Sciences, Vol. 140.* Springer-Verlag, 1990.

[6] Vidyasagar, M. *Control System Synthesis: A Factorization Approach.* MIT Press, 1985.

[7] Horn, R.A. and Johnson, C.R. *Matrix Analysis.* Cambridge University Press, 1985.

[8] Desoer, C.A. and Vidyasagar, M. *Feedback Systems: Input-Output Properties.* Academic Press, 1975.