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**Unslotted Aloha in High Speed Directional Bus Networks\***

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## UNSLOTTED ALOHA IN HIGH SPEED BIDIRECTIONAL BUS NETWORKS

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**ABSTRACT\*** We study the performance of the unslotted ALOHA multiple access protocol in a high speed bidirectional bus network. For point-to-point communications, its maximum throughput is known to be  $1/(2e)$ , and independent of the ratio of end-to-end propagation delay to packet transmission time. For broadcast communications, we show that, if the offered load density is uniform along the bus, the maximum throughput achievable by a station depends on its location along the bus. To achieve a uniform throughput density, the offered load density has to vary along the bus. We derive the optimal profile for the offered load density. In any case, the maximum throughput degrades with the ratio of end-to-end propagation delay to packet transmission time.

transmission must be received by all stations, and the success of a transmission requires that the space-time utilization of different packets is mutually exclusive.

In the conventional analysis of the unslotted ALOHA protocol in bus networks, the spatial properties are not taken into consideration. Recent work on random access in bus networks has increasingly focused on space-time properties [4], [5], [6]. Gonsalves and Tobagi have conducted a simulation study of the effects of station locations on the broadcast performance of Ethernet type bus networks using the CSMA/CD protocol [7]. They observed that, with stations uniformly distributed along the bus, those near the center of the bus obtain better performance than those near the ends. Such behavior is due to the fact that transmissions from stations close to the ends of the bus propagate longer on the bus, and are more vulnerable to collisions than those from stations close to the center of the bus. We confirm the above behavior analytically for unslotted ALOHA.

## 1 Introduction

Unslotted ALOHA is a well known random access protocol [1], [2]. Its spatial properties were first studied by Abramson, who analyzed the spatial densities of throughput and offered load in a packet radio broadcasting network with capture [3]. He derived a differential equation relating the throughput and the offered load densities. He provided solutions to two special cases: constant offered load density and constant throughput density. In this paper, we study the space-time properties of the unslotted ALOHA protocol in a high speed bidirectional passive bus network, where transmissions are in the form of packets of constant length. We follow an approach similar to that in [3].

High speed bus networks have received a lot of attention lately [8], [9]. The speed of a bus network is often specified by the parameter  $a$ , which denotes the end-to-end propagation delay normalized with respect to the average packet transmission time. Many random access strategies for high speed unidirectional bus networks have been studied by Maxemchuk, who shows that, in unidirectional bus networks, slotted protocols are always more efficient than unslotted protocols [10]. It is shown in [11] that this is not the case for point-to-point communications in bidirectional bus networks.

We distinguish between point-to-point and broadcast communications. For point-to-point communications, each transmission is designated for only one station, and the success of a transmission depends on its being free of collision only where and when the reception is intended. For broadcast communications, each

## 2 Unslotted ALOHA System

We first review the unslotted ALOHA protocol in a bidirectional bus network, of length  $D=1$ , which is assumed to have perfectly absorbing terminations at both ends. Let positions on the bus be specified with respect to the center of the bus, so that any position  $x$  must fall within the range  $[-1/2, 1/2]$ . In this paper, all functions of distance are defined for this range. Without loss of generality, we assume that all packet transmissions are of unit length.

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Each transmitting station receives an immediate error-free feedback about the success or failure of its transmitted packets. If a transmission fails, the packet is transmitted again after some random delay, independent of past transmissions and other stations. We summarize below the unslotted ALOHA model used in this paper.

- Asynchronous transmissions;
- Immediate feedback from receiver specifying success or failure;
- Offered traffic including retransmissions is approximately a Poisson process;
- Statistical equilibrium;
- For broadcast communications, each transmission must be received successfully by all stations.

### 3 Vulnerable Regions

In any contention-based multiple access system, every transmission is vulnerable to collisions. If there were no propagation delay, then the vulnerability of a transmission may be characterized by the time interval over which any other packet transmitted could cause a collision. During this time interval, which is known as the vulnerable period, the given transmission is vulnerable everywhere on the bus. In a bidirectional bus network with propagation delay, the vulnerable periods do not adequately characterize the vulnerability of transmissions because they are location dependent. We need to consider vulnerable regions in space and time, instead of vulnerable periods.

#### 3.1 Conventional Analysis

The conventional analysis of unslotted ALOHA protocol for broadcast communications is based on the assumption of a single receiver, so that a transmission is successful only if there are no other transmissions within a vulnerable period of  $2(1+a)$ . Let  $G$  be the constant offered traffic rate, in packets per second, including retransmissions. Then, the probability of success is

$$P_a = e^{-2(1+a)G} \quad (1)$$

The throughput is given by

$$S_a = G e^{-2(1+a)G} \quad (2)$$

whose maximum with respect to  $G$  is

$$S_a^* = \left(\frac{1}{1+a}\right) \frac{1}{2e} \quad (3)$$

Note that the maximum throughput degrades with  $a$ .

#### 3.2 Space-Time Analysis

A vulnerable region associated with a transmission is the space-time region over which any other packet arriving at the network could cause a collision. The size of the vulnerable regions is a limiting factor on the performance of a contention-based protocol. In general, for a given protocol, the larger the size of the vulnerable regions, the smaller is the probability of success of each transmission.

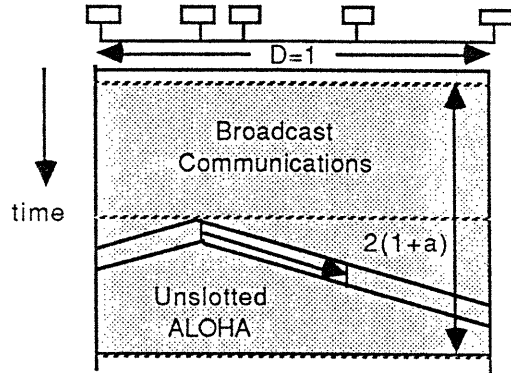


Fig. 1: Conventional Vulnerable Region for Broadcast Communications

In the conventional analysis, as we have reviewed above, the vulnerable region for broadcast communications in unslotted ALOHA is implicitly assumed to be as shown in Figure 1. The length of this vulnerable region is  $2(1+a)$  units of packet transmission time. This time interval is chosen for the worst case in which the end-stations communicate with each other.

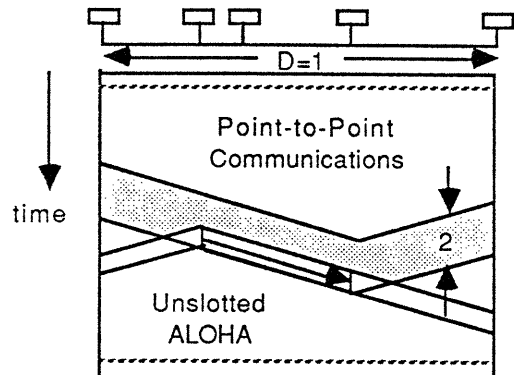


Fig. 2: Conventional Vulnerable Region for Point-to-Point Communications

In unslotted ALOHA, the vulnerable region for point-to-point communications is smaller than that for broadcast communications. As shown in Figure 2, the space-time area of a point-to-point vulnerable region is always equal to 2, regardless of the value of  $a$ . It is

well known in the literature that the maximum throughput of unslotted ALOHA for point-to-point communications is  $e^{-2G}$ .

For broadcast communications in unslotted ALOHA, the actual vulnerable region for a transmission is shown in Figure 3. Let  $V_a(x)$  be the space-time area of this vulnerable region. It is easy to verify that

$$V_a(x) = 2 + a/2 + 2ax^2 \quad (4)$$

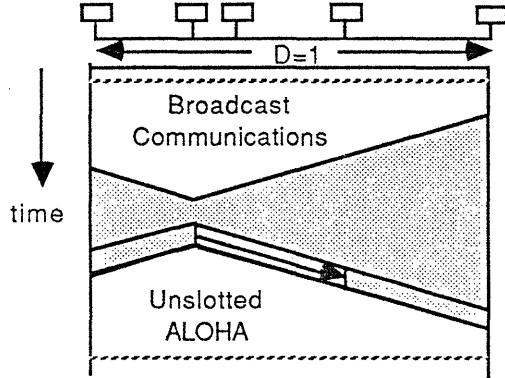


Fig. 3: Actual Vulnerable Region for Broadcast Communications

As shown in Figure 4,  $V_a(x)$  is symmetric about, and minimized at,  $x=0$ . Hence, we could expect the throughput performance to be a function of  $x$ , and is largest in the middle of the bus.

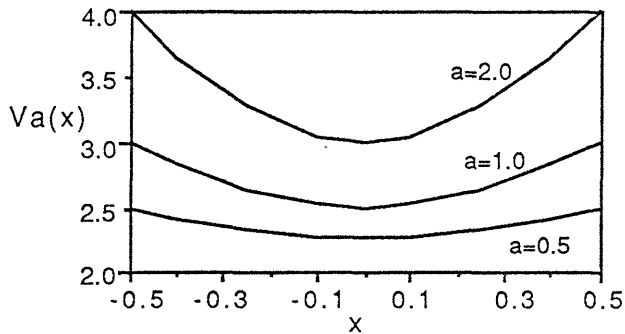


Fig. 4: Area of Vulnerable Region

Since  $V_a(x)$  increases with  $a$ , and is less than  $(2+a) \leq 2(1+a)$ , the broadcast throughput of the unslotted ALOHA protocol indeed degrades as  $a$  increases, but more slowly than that under the conventional assumption.

#### 4 Maximum Throughput

In this section, we evaluate the broadcast throughput of unslotted ALOHA. We show that, if the offered load density is uniform along the bus, the maximum

throughput depends on the location along the bus. To achieve a uniform throughput density, the offered load density has to vary along the bus. In any case, the maximum throughput degrades with the ratio of end-to-end propagation delay to packet transmission time.

**Theorem:** Consider unslotted ALOHA in a bidirectional bus network. Let the position of a given source station be represented by  $x \in [-1/2, 1/2]$ . Let  $g(x)$  be the offered traffic rate density at location  $x$ , in packets per second. The throughput density at location  $x$  for broadcast communications is

$$S_a(x) = g(x) P_a(x) \quad (5)$$

where  $P_a(x)$  is the spatial density of the probability of success.  $S_a(x)$  is the solution to the following differential equation.

$$S_a'(x)g(x) = S_a(x) \left\{ g'(x) - g(x)h_a(x) \right\} \quad (6)$$

where

$$h_a(x) = 2a \left\{ \begin{array}{l} x \quad 1/2 \\ \int_{-1/2}^x g(z) dz - \int_x^{1/2} g(z) dz \end{array} \right\} \quad (7)$$

and  $f'(x)$  denotes the derivative of a function,  $f(x)$ , with respect to  $x$ .

**Proof:** Let  $k_a(x,z)$  be the temporal length of the vulnerable region at location  $z$  when the transmission originates at location  $x$ . For broadcast communications, as shown in Figure 3,

$$k_a(x,z) = 2(1+|z-x|) \quad (8)$$

The spatial density for the probability of success is

$$P_a(x) = \exp \left\{ - \int_{-1/2}^{1/2} k_a(x,z) g(z) dz \right\} \quad (9)$$

Taking the derivative of (5), multiplying each side by  $g(x)$ , and using (9), we obtain

$$S_a'(x)g(x) - g'(x)S_a(x) = g(x)S_a(x) \int_{-1/2}^{1/2} k_a'(x,z)g(z) dz \quad (10)$$

It is easy to verify that

$$k_a'(x,z) = \begin{cases} -2a & \text{if } x < z \\ 0 & \text{if } x = z \\ +2a & \text{if } x > z \end{cases} \quad (11)$$

It follows that (6) holds with  $h_a(x)$  defined below.

$$\begin{aligned} h_a(x) &= \int_{-1/2}^{1/2} k_a'(x,z) g(z) dz \\ &= 2a \left\{ \begin{array}{l} x \quad 1/2 \\ \int_{-1/2}^x g(z) dz - \int_x^{1/2} g(z) dz \end{array} \right\} \end{aligned} \quad (12)$$

Q.E.D.

We will not derive the general solution to the differential equation in the above theorem. Instead, we consider two special cases.

### Case #1: Constant Offered Load Density

Suppose that  $g(x)$  is constant, such that

$$g(x) = G \quad (13)$$

From (7), we have

$$h_a(x) = 4aGx \quad (14)$$

From (6), (13), and (14), we obtain the following differential equation.

$$S_a'(x) = -4aGxS_a(x) \quad (15)$$

Solving (15), we obtain the broadcast throughput density of the unslotted ALOHA as follows.

$$S_a(x) = G e^{-(2+a)G} \left\{ e^{-2aG(x^2-1/4)} \right\} \quad (16)$$

Note that for a given  $G$ ,  $S_a(x)$  is minimized at the ends and maximized at the center of the bus. It follows from (16) that

$$G e^{-(2+a)G} \leq S_a(x) \leq G e^{-(2+a/2)G} \quad (17)$$

The total throughput is

$$S_a(G) = \int_{-1/2}^{1/2} S_a(x) dx \quad (18)$$

Note that we have explicitly indicated in (18) the dependence of the total throughput on  $G$ .

We can write

$$S_a(G) = G e^{-(2+a/2)G} \left\{ \frac{\pi}{2aG} \right\}^{1/2} \operatorname{erf} \left( \left\{ \frac{aG}{2} \right\}^{1/2} \right) \quad (19)$$

where  $\operatorname{erf}(\cdot)$  is the following standard error function:

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-w^2} dw \quad (20)$$

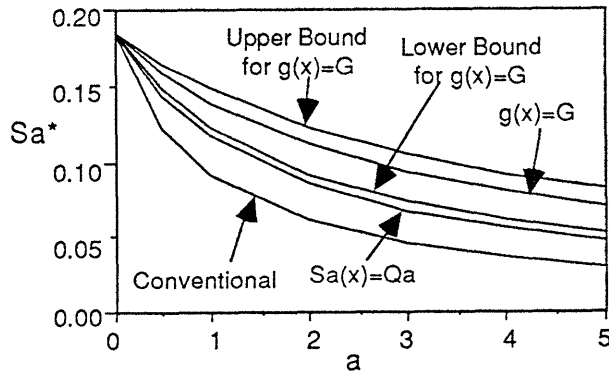


Fig. 5: Maximum Throughput

For any given  $a \geq 0$ , we can determine the maximum throughput  $S_a^*$ , defined as follows.

$$S_a^* = \max_G \{ S_a(G) \} \quad (21)$$

From (17) and (18), we have the following bounds.

$$\left( \frac{1}{1+a/2} \right) \frac{1}{2e} \leq S_a^* \leq \left( \frac{1}{1+a/4} \right) \frac{1}{2e} \quad (22)$$

In Figure 5, we show  $S_a^*$  and its bounds. We have included the result of Case #2 and that of conventional analysis for comparison.

### Case #2: Constant Throughput Density

Suppose that  $S_a(x)$  is independent of location, such that

$$S_a(x) = Q_a. \quad (23)$$

This corresponds to the interesting case where all stations have the same throughput. We assume that  $g(x)$  is symmetric about the center of the bus. Thus,

$$g(x) = g(-x) \quad (24)$$

Taking the derivative of (7) and using (24), we obtain

$$h_a'(x) = 4ag(x) \quad (25)$$

From (6) and (23), we have

$$g'(x) = h_a(x)g(x) \quad (26)$$

Taking the derivative of (26), multiplying each side by  $g(x)$ , and using (25), we obtain

$$g''(x)g(x) = 4ag^3(x) + \{g'(x)\}^2 \quad (27)$$

Solving the above differential equation, using (24), we obtain

$$g(x) = \frac{b^2}{2a \cos^2(bx)} = \frac{b^2}{2a} \sec^2(bx) \quad (28)$$

for some constant  $b$ .

Define  $R$  as follows.

$$R = \pi / (2b) \quad (29)$$

Note that  $g(x)$  is unbounded if

$$|x| \geq R \quad (30)$$

If  $b > \pi$ , then  $R < 1/2$ , and (9) implies that  $P_a(x) = 0$  for  $x \in [-1/2, 1/2]$ . It follows that  $S_a$  can only be zero. For a given  $b$ , [3] defines the Sisyphus Distance as the value of  $x$  with which  $g(x)$  in (28) becomes unbounded. It does not appear to have any physical meaning in this case, as  $b$  is an arbitrary parameter. In the analysis below,  $b$  is always smaller than  $\pi$ , so that  $R > 1/2$ .

To evaluate  $P_a(x)$ , we recall the following identity.

$$\int x \sec^2(bx) dx = \frac{x}{b} \tan(bx) + \frac{1}{b^2} \ln \{ \cos(bx) \} \quad (31)$$

Using (8), (9), (28), and (31), we obtain

$$P_a(x) = \exp \left\{ -b \{ \tan(b/2) \} \left( 1 + \frac{2}{a} \right) - \ln \{ \sec^2(bx) \} \right\} \quad (32)$$

From (5), (28), and (32), one obtain

$$S_a(b) = Q_a = \frac{b^2}{2a} \exp \left\{ -\frac{b}{a} \{ \tan(b/2) \} (a+2) \right\} \quad (33)$$

where  $S_a(b)$  is the total throughput as a function of  $b$ .

For any given  $a \geq 0$ , we can determine the maximum throughput,  $S_a^*$ , defined as follows.

$$S_a^* = \max_b \{ S_a(b) \} \quad (34)$$

Taking the derivative of (34) with respect to  $b$ , and setting it to zero, we obtain

$$\left( \frac{b}{2} \right)^2 \tan^2 \left( \frac{b}{2} \right) + \left( \frac{b}{2} \right) \tan \left( \frac{b}{2} \right) + \left( \frac{b}{2} \right)^2 = \left( \frac{a}{a+2} \right) \quad (35)$$

Equation (35) can be solved numerically to determine the value of  $b$  which maximizes  $S_a(b)$  in (33).

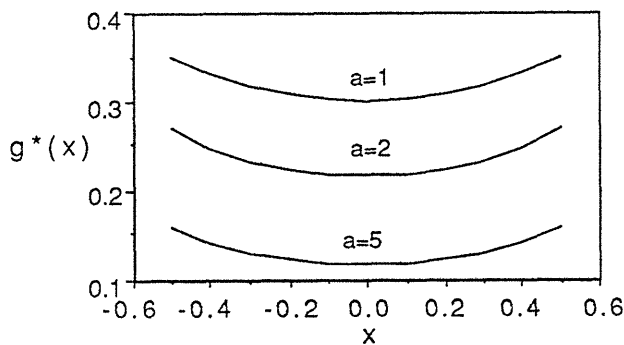


Fig. 6: Optimal Offered Load Density

We show in Figure 5 the behavior of  $S_a^*$  as a function of  $a$ . The optimal offered load density,  $g^*(x)$ , which is obtained from (28) with the optimal value of  $b$ , is shown in Figure 6. Note that  $g^*(x)$  decreases with increasing value of  $a$ . As  $g^*(x)$  is proportional to the number of retransmissions, this confirms the observation in [7].

## 5 Conclusion

From the above results, we can make the following observations. Conventional analysis of unslotted ALOHA underestimates its maximum throughput for broadcast communications. The maximum broadcast throughput is higher for the case with constant offered load density than for the case with constant throughput density.

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