

MODELS OF NON-STEADY-STATE ECONOMIC GROWTH
AND A DYNAMIC MODEL OF THE FIRM

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ABSTRACT

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Submitted to the Department of Economics on July 26, 1971 in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

The paper investigates the behavior of a growing economy for cases in which the steady-state conditions are not fulfilled. The first chapter, which deals with one-sector models in which the steady-state conditions are not met, investigates how the economy behaves as the aggregate effective capital-labor ratio for the economy tends to zero or infinity. Similarly, Chapter 2 investigates two-sector models in which the steady-state conditions are not fulfilled either because there are different rates of Harrod neutral technical progress in each sector, or because some capital-augmenting technical progress is present in the investment sector. It is found that these non-steady-state models parallel the steady-state growth paths in that the rates of growth of the variables tend (in most cases) to constant limits. However, differences arise between the non-steady-state models and the steady-state model when factor shares and the marginal product of capital are considered. Finally, each of these chapters investigates how factor-augmenting technical progress should be allocated within the economy, and considers under what circumstances the steady-state path is found to be optimal.

In Chapter 3 the results of the first two chapters are briefly summarized, and then the behavior of the non-steady-state economy is compared and contrasted to the characteristics of an economy in which the steady-state conditions are met. Though there is some similarity between these cases, it is found that these non-steady-state economies cannot replicate some of the major characteristics of the steady-state path. Since it is seen that the occurrence of a steady-state is quite unlikely, and since the non-steady-state economy does not generate all the accepted characteristics of a growing economy, it must be concluded that there is a basic dilemma facing the branch of economic theory that attempts to replicate the stylized facts of economic growth.

The final chapter approaches the topic of growth from a different perspective by investigating how an isolated firm in a growing economy decides what growth rate and initial size to choose. Subsequently, the chapter considers how changes in technical progress or in cost parameters affect the decisions made by this isolated firm.

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Chapter 1. The Bertrand-Vanek Model of Disequilibrium Growth -
A Synopsis and Extensions

I. Introduction

The modern theory of economic growth has progressed a long way since the razor-age instability that characterized the so-called Harrod-Domar model [16, 21]. Professor Solow's classic 1956 article [45], which showed how, by allowing smooth substitutability in the production function one could replace the instability of the Harrod-Domar model with the stability now characteristic of the neo-classical growth models, opened the flood-gates for a seemingly endless stream of papers modifying and extending the basic one-sector model. These extensions included the introduction of labor-augmenting technical progress into the one-sector world, so that the model could explain the increasing output per worker that seemed characteristic of the real world [48]. Further modifications were pursued by allowing the smooth substitutability of the Solow model to be replaced by putty-clay or clay-clay models in which the capital-labor ratio was (at least ex post) a technologically given datum [5, 24]. These models showed that, assuming that Harrod neutral technical progress was embodied in new machinery, the stability of the one-sector Solow model prevailed, and all the fundamental results of this model held for the vintage models.

Other extensions of the one-sector model included attempts to explain the occurrence of this technical progress by resorting to the

notion of learning-by-doing [2], as well as models that explored how these one-sector models should be "controlled" in order to maximize society's welfare [34, 37, 38]. While the one-sector model was being extended, two-sector models arose. These models, for the most part, emulated the convenient stable behavior that the one-sector models possessed [57, 58]. Though it is true that stability is not guaranteed in these two-sector models, various conditions were developed [7, 57, 58] that provided for their stability. Consequently, the stability of the model being assumed, extensions were made by allowing for the presence of very special cases of factor-augmenting technical progress [15, 54], and papers were produced that explored optimal behavior in these models [43, 51].

Needless to say, a complete review of the growth literature would be both exhaustive and unnecessary (for the most recent, and in our opinion, best coverage of the various growth models, see Burmeister and Dobell [9]). Yet, even with a cursory glimpse of the literature, one is struck by one concept that runs through all these models - the notion that a steady-state¹ must exist. Consequently, all the extensions

¹Broadly speaking, a steady-state may be defined as that state of the world in which the effective capital-labor ratio tends to a positive, finite limit. As a result of the constancy of the effective capital-labor ratio, the output-factor elasticities and the marginal product of capital all tend to positive, finite limits, and consequently output and capital grow at the same constant rate. Also, output per person is either constant in this steady-state, or else it grows at a constant rate if some labor-augmenting technical progress is present. In addition, for the two-sector models, the effective capital-labor ratio in each sector is constant, as is the proportion of labor allocated to each sector. While these results seem to comply well with reality, the conditions needed for a steady-state to occur are quite stringent ones.

of the one- and two-sector models are made within this constraint.

This is not to say that the economists producing these models were unaware of the possibility of non-steady-state models - rather, they chose to expand the steady-state models rather than to investigate the "darker" side of the growth model that would occur if the steady-state conditions were not fulfilled. The works of Kennedy [27], Samuelson [42], and more recently, Chang [11] show the implicit concern of economists about the possibility of non-steady-state models. However, rather than investigate what would happen if the steady-state did not occur, these models concentrated on developing mechanisms that would assure the occurrence of the steady-state path. As is well-known, these models assumed that a transformation curve between capital- and labor-augmenting technical progress existed, and they attempted to show under what conditions society (through individual entrepreneurial decision-making) would choose the steady-state path. Though these models are both interesting and informative, it is not clear to us that either a transformation curve as postulated exists, or that entrepreneurs behave as is assumed by these models.

Where, then, do all these new growth models leave us? Surprisingly, they are not very far removed from the instability of the original Harrod-Domar model. The steady-state condition for a one-sector model (under the usual assumption that the aggregate production function exhibits constant returns to scale) is a rather singular one indeed - there can be no capital-augmenting technical progress (or else the aggregate production function must be Cobb-Douglas). If any capital-augmenting technical progress occurs, there is no steady-state, and the

effective capital-labor ratio rushes away to infinity (for constant returns to scale and positive capital-augmenting technical progress). The steady-state conditions for a two-sector model are even more stringent - there can be no capital-augmenting technical progress in the investment sector, and Harrod neutral technical progress must occur at the same rate in each sector (there can be Hicks neutral technical progress in the consumption sector). If Cobb-Douglas production functions occur in either sector, these conditions can be weakened; if both production functions are Cobb-Douglas, then a steady-state will occur.

The singularity of the conditions needed for a steady-state (barring some guiding hand, as in the Kennedy-Chang models) is quite apparent, and it was to this problem that Professor Vanek turned his attention. In two earlier papers, Vanek [60, 61] considered the behavior of a one-sector growth model, assuming that capital-augmenting technological change did occur (though he maintained the assumption of constant returns to scale). In a subsequent paper, Bertrand and Vanek [4] further extended this model by allowing the aggregate production function to assume any (constant) degree of homogeneity.

The purpose of this paper is to extend the basic one-sector Bertrand-Vanek model and to consider two-sector models in which the steady-state conditions are not fulfilled. As we shall see, the fundamental problem in the one-sector model (when the steady-state conditions are not fulfilled) is that the effective capital-labor ratio, instead of tending to some finite limit, rushes away to either zero or infinity. Thus, for example, if constant returns to scale prevails, the

presence of any capital-augmenting technical progress causes society to produce machinery faster than its effective labor force is growing (unless the savings rate declines at a very special rate over time), so that continual capital-deepening occurs. Consequently, the effective capital-labor ratio (in this case) tends to infinity, and what happens to the economy depends upon how effectively the new capital can be used. Therefore, as one would expect, the aggregate elasticity of substitution, since it indicates how effectively society can absorb this new capital, plays a fundamental role in determining how this economy will behave.

In the two-sector model, our troubles are twofold. First of all, if capital-augmenting technical progress occurs in the investment sector, a problem equivalent to the one-sector case occurs in that society is producing capital faster than its effective labor force is growing, so that continual capital-deepening occurs. Consequently, the aggregate effective capital-labor ratio tends to infinity, and the elasticities of substitution in each sector (for reasons already explained) become important in determining the asymptotic behavior of the economy. Secondly, however, the two-sector model has an additional problem that is unique to it (compared to the one-sector model), since it entails continual reallocation of factors between the sectors. This problem arises if Harrod technical progress occurs at different rates in the two sectors (assuming that all technical progress is factor-augmenting, and is classified as Harrod and [or] Hicks neutral technical progress), so that, under competitive pricing (or efficient allocation of resources), a continual shifting of factors between the sectors occurs even if the aggregate effective capital-labor ratio were to remain constant.

Consequently, as we have already observed, the likelihood of a steady-state solution is quite small. On the other hand, as we shall see in Chapter 3, although the asymptotic equilibrium does meet some of the stylized facts of a growing economy, it fails to satisfactorily explain either the distribution of income within the society or the motivation for continuing investment. This seeming paradox should be kept in mind when reading the three chapters that deal with aggregate growth models, for it necessitates, in our opinion, the disaggregation of the growth model and a closer inspection of the micro-economy that is implicitly embedded within this aggregate model. Our fourth and final chapter attempts to take a small step in this direction by considering how a single, isolated firm determines both its optimal size and growth rate.

Our basic approach in this thesis will be to ask two separate questions. First, we shall inquire how an aggregate economy would behave if the steady-state conditions were not met, and how this economy would differ (at an empirically observable level) from a steady-state economy. Secondly, we investigate the different question that asks how a central planner should allocate factor-augmenting technical progress (to maximize certain criteria), assuming that a trade-off exists between various types of factor-augmenting technical progress (à la Kennedy). Our main interest in this latter question is to ascertain under what conditions a central planner (or the invisible hand, as in the cases of Kennedy-Chang) would choose to place the economy in a steady-state path.

In this first chapter we shall present the basic Bertrand-Vanek model, we shall consider the steady-state possibilities that they suggest, and we shall extend their analysis by considering the asymptotic

growth rates of the variables if a steady-state does not occur. In addition to considering how factor-augmenting technical progress should be allocated within this economy, we shall present a model that, due to the relationship between the "degree of homogeneity" of the production function and the effective capital-labor ratio, causes a steady-state (in a special sense of the word) to exist in the long-run.

Our second chapter investigates the asymptotic behavior of a two-sector model, assuming that the steady-state conditions are not fulfilled. Since, as explained earlier, the two-sector model faces two distinct problems, our approach is to consider each of these problems separately. First, we consider how the one-sector analogue of continual capital-deepening effects the two-sector model. Subsequently, we temporarily assume away the problem of capital-deepening, and we ask instead what would happen if Harrod neutral technical progress occurred at different rates in the two sectors. Finally, we combine these two separate problems and show how any combinations of factor-augmenting technical progress can be analyzed. In doing this we exhibit the asymptotic growth rates for all variables in this two-sector model, and we consider how a central planner should allocate various types of factor-augmenting technical progress.

Our third chapter briefly summarizes the results of our first two chapters and then proceeds to detail the ways in which the asymptotic equilibrium differs (at an observable level) from the steady-state path. Though we present some suggestions that might enable the asymptotic equilibrium to duplicate the stylized facts of growth, persistent doubt remains as to the likelihood of a steady-state solution

or as to the ability of the asymptotic equilibrium to duplicate these stylized facts. Consequently, we suggest that further investigation into the microeconomic characteristics of the economy, particularly in so far as factor-pricing and investment decisions are concerned, is essential.

Our last chapter then attempts a small step in this direction by analyzing an isolated firm in a growing economy. In this partial equilibrium model, which is based upon a paper by Professor Solow [49], we investigate how various types of technical progress (and changes in factor costs) effect the decisions made by the firm.

Let us now turn our attention to the basic Bertrand-Vanek model.

II. The Basic Bertrand-Vanek Model

As previously indicated, most one-sector growth models have clung to the assumption that there is no capital-augmenting technological progress and that the production function is homogeneous of degree one everywhere. The Bertrand-Vanek model, which we plan to discuss in this section (and which is more general than Vanek's two earlier papers), relaxes both of these assumptions, though it does assume that the degree of homogeneity of the production function is a constant (but not necessarily equal to one). With their model, they show that the existence of a steady-state is a rather singular affair, and thus that growth theory (in this respect) has not come as far as one might at first presume from the razor-edge instability of the Harrod-Domar model.

In this chapter we plan to follow the basic Bertrand-Vanek

model, elaborating on it in certain areas. For example, we extend somewhat their discussion of the singular cases in which a steady-state can exist. Also, we discuss in more detail the asymptotic growth rates of the variables, and we also consider what would happen if the degree of homogeneity of the production function were not constant.

Let us now outline the basic Bertrand-Vanek model:

a) All technological progress is assumed to be factor-augmenting, and the production function is assumed to have a constant degree of homogeneity.

$$1) \quad Q = F(Ke^{bt}, Le^{at}) = (Le^{at})^h f(u) \quad ; \quad h = \text{degree of homogeneity}$$

$$u \equiv (Ke^{bt}) / (Le^{at}) \equiv (z)/(x) \quad ; \quad (\dot{L}/L) = n$$

b) Factors are paid proportionally to their marginal product.²

$$2) \quad W = (\partial Q / \partial L) / h \quad ; \quad R = (\partial Q / \partial K) / h \quad ;$$

$$\phi_n \equiv [(\partial Q / \partial L)(L/Q)] \quad ; \quad \phi_k \equiv [(\partial Q / \partial K)(K/Q)] \quad ; \quad \phi_k + \phi_n = h \quad ;$$

$$\phi_k, \phi_n \geq 0$$

c) Capitalists and Workers save at constant rates ($s_k \geq s_n \geq 0$):

²If $h=1$, this obviously represents competitive pricing. However, for $h \neq 1$, it is more difficult to rationalize this pricing assumption. One possible explanation for this assumption, based on the presence of externalities that account for the non-constant returns to scale, is presented in a paper by John Chipman, "External Economies of Scale and Competitive Equilibrium," Quarterly Journal of Economics, August 1970, pages 347-385.

$$3) \quad S = sQ = [(s_k \phi_k)/h + (s_n \phi_n)/h]Q ; \quad s_k \geq s \geq s_n ;$$

S is the gross savings of the community

d) Capital depreciates at a constant rate c:

$$4) \quad \dot{K} = \dot{K} + cK ; \quad (\dot{K}/K) = [(sQ)/K] - c$$

Equations 1) - 4) are the basic ingredients of the Bertrand-Vanek model. Consider the rate of change of the effective capital-labor ratio:

$$5) \quad (\dot{u}/u) = (\dot{K}/K) + b - a - n = \{[se^{(a+n)ht} f(u)]/K + b - a - n - c\} \\ = \{[se^{[(a+n)h-(a+n-b)]t} f(u)]/u\} + b - a - n - c$$

A steady-state path implies that we can find a solution such that $[(\dot{u}/u) = 0]$, or else that we can redefine the effective capital-labor ratio so that, in those new units, the system will approach a constant, finite, non-zero effective capital-labor ratio. Two possibilities immediately occur (Bertrand and Vanek, pages 750-1):

- i) Cobb-Douglas production function - all technological change reduces to the special case of labor-augmenting technical progress
- ii)³ The parameters are such that: $(a+n)h = (a+n-b)$

³For $h=1$, this reduces to the standard neoclassical steady-state condition that there be no capital-augmenting technical progress ($b=0$); otherwise, it implies: $[b = (1-h)(a+n)]$. Essentially this says that a steady-state can occur only if the rate of capital-augmenting technical progress is just enough (and no more) to offset the decrease in the effective capital-labor ratio that would occur as the economy grows (due to decreasing returns to scale). For increasing returns to scale, the interpretation is comparable, except that the capital-augmenting technical progress must be negative (technical regression).

After we discuss the Bertrand-Vanek model in which there is no steady-state, we shall briefly consider these steady-state possibilities. At that time we shall find that, while these constitute necessary conditions for a steady-state, they are not sufficient. However, let us first present the basic Bertrand-Vanek model before we discuss this problem in more detail.

Following their analysis (their equations 8, 11, 12, 13, and 14):

$$6) \quad \dot{k} \equiv (\dot{K}/K) = (sQ)/K - c$$

$$7) \quad (\dot{k}/k) = [(k+c)/k][(a+n)h - k + (k+b-a-n)T]$$

$$8) \quad T \equiv \phi_k + E\phi_n [(\sigma-1)/(\sigma h)] \quad ; \quad E \equiv [(ds/d\phi_k)(\phi_k/s)] \quad ;$$

$$1 \geq E \geq 0 \quad ; \quad s_k \geq s_n \geq 0$$

$$9) \quad \bar{k} = [(a+n)h - (a+n-b)T]/(1-T) \quad ; \quad T \neq 1 \quad ;$$

\bar{k} is the value(s) of k such that $(\dot{k}/k) = 0$.

$$10) \quad (\dot{k}/k) = [(k+c)/k](1-T)(\bar{k}-k)$$

Bertrand and Vanek's procedure is to consider changes in the rate of growth of capital and to study the locus of k (call it \bar{k}) such that $[\dot{k}=0]$. Equation 10) indicates that if $T < 1$ (and $k > -c$, as it must be for $s > 0$ and non-zero, finite u), then whenever $k < \bar{k}$, k increases; and whenever $k > \bar{k}$, then k decreases. Therefore, the \bar{k} locus is, for $T < 1$, similar to an asymptote; and if \bar{k} tends to a constant value as $u \rightarrow \infty$, then k will tend to \bar{k} (assuming that $\dot{u} > 0$). If $T > 1$, then k will diverge from the \bar{k} locus.

These five equations are really the essence of the Bertrand-

Vanek analysis; the rest of the task is merely to see what type of behavior these equations imply. As we have seen, \bar{k} is important in determining the behavior of k ; and \bar{k} , in turn, depends only upon T (and the various parameters). From the definition of T , we can place limits on its potential values:

$$11) \quad \text{Min}[0, (\sigma-1)/\sigma] \leq T \leq \text{Max}[h, (\sigma-1)/\sigma]$$

Since whether $T \geq 1$ or $T < 1$ is obviously important, it follows that the value of the parameter h is quite important (that is, it is important whether there is increasing, constant, or decreasing returns to scale). However, we have seen that it is also critical whether:

$$12) \quad h > \frac{[(a+n-b)/(a+n)]}{\sigma} \quad . \quad \text{Therefore, let us write:}$$

$$13) \quad h^* \equiv [(a+n-b)/(a+n)] \quad ; \quad h \equiv h^* + \delta \quad . \quad \text{Then we find:}$$

$$14) \quad \bar{k} = \{(a+n-b) + \{[(a+n)\delta]/(1-T)\}\}$$

Clearly, we can not say much more about \bar{k} or T unless we are willing to make some assumption about the production function or the savings assumption.⁴

⁴Bertrand and Vanek briefly discuss this in a footnote on pages 748-9. They state that T is monotonic in u if: $h > E[(\sigma-1)/\sigma]$; this, we believe, is not necessarily true. They apparently fail to consider the changes in E caused by changes in u in arriving at the above condition. Secondly, they state that k (the actual rate of growth of capital) is monotonic in T - we are sure that they meant to say that \bar{k} , the so-called "stationary locus", is monotonic in T . It is clearly possible for k to first increase, then decrease - or vice versa. We shall discuss the time path of k later in this chapter.

$$15) \quad (d\bar{k}/dT) = \{[(a+n)\delta]/(1-T)^2\} \geq 0 \text{ as } \delta \geq 0.$$

Therefore, \bar{k} is monotonic in T (though it is discontinuous at $T=1$).

Considering T , if $s_k = s_n$ (and hence, $E=0$), we find:

$$16) \quad T = \phi_k ; \quad (dT/du) \geq 0 \text{ as } \sigma \geq 1.$$

Consequently, T is monotonic in u if $s_k = s_n$ and if the elasticity of substitution is everywhere bounded from unity.

If $s_k > s_n \geq 0$, the problem is more complicated. For simplicity, assume that the production function is a C.E.S. function. From the definition of T and E we can demonstrate the following:

$$17) \quad T = \phi_k + E\phi_n [(\sigma-1)/(\sigma h)] ; \quad E = [(s_k - s_n)\phi_k] / [(s_k - s_n)\phi_k + hs_n] ;$$

$$\phi_n = (h - \phi_k).$$

If $s_n = 0$, then $E = 1$, and therefore:

$$18) \quad T = \phi_k \{1 - [(\sigma-1)/(\sigma h)]\} + [(\sigma-1)/\sigma] ,$$

which is monotonic in ϕ_k (or constant if $h = [(\sigma-1)/\sigma]$); note that this result differs from the Bertrand-Vanek condition for the monotonicity of T - see footnote 4). Therefore, T varies between h and $[(\sigma-1)/\sigma]$ in this case.

Finally, if $s_k > s_n > 0$, we can show:

$$19) \quad T = \phi_k + \{(h - \phi_k)[(\sigma-1)/(\sigma h)][(s_k - s_n)\phi_k] / [(s_k - s_n)\phi_k + hs_n]\}$$

If we take the derivative of this expression we find that T is monotonic in ϕ_k unless:

$$20) \quad s_k > s_n > 0 \quad \underline{\text{and}} \quad h < [(s_k - s_n)/s_k][|\sigma - 1|/\sigma] \quad (\rightarrow h < 1)$$

which differs from the Bertrand-Vanek criteria (page 749) unless

$$\sigma > 1, \text{ and } \phi_k = h.$$

Since \bar{k} is monotonic in T , and ϕ_k is monotonic in u (if σ is bounded from one or else the function is a C.E.S. function), it follows that \bar{k} is monotonic in u unless condition 20) is fulfilled. However, if $h > 1$, \bar{k} is not continuous in u - there is a point of discontinuity at $T = 1$ (it is possible that $h > 1$ and that $T < 1$ everywhere, provided that σ is not everywhere bounded from one).

If the production function is of the C.E.S. variety, and if equation 20) holds, then there will be a single interior extreme point in \bar{k} as a function of u .

Armed with this knowledge, we only need to know the behavior of \bar{k} as $u \rightarrow 0$ or as $u \rightarrow \infty$ in order to complete the graph of \bar{k} . Since we have assumed a C.E.S. function (or else $s_k = s_n$ and that σ is bounded from one), $\phi_k \rightarrow 0$ or $\phi_k \rightarrow h$ as $u \rightarrow 0$ or as $u \rightarrow \infty$. Thus:

$$21) \quad \phi_k \rightarrow 0 \text{ implies } \bar{k} \rightarrow (a+n)h \text{ if } s_n \neq 0;$$

$$\phi_k \rightarrow 0 \text{ implies } \bar{k} \rightarrow [(a+n-b) + \sigma\{(a+n)h - (a+n-b)\}] \text{ if } s_n = 0$$

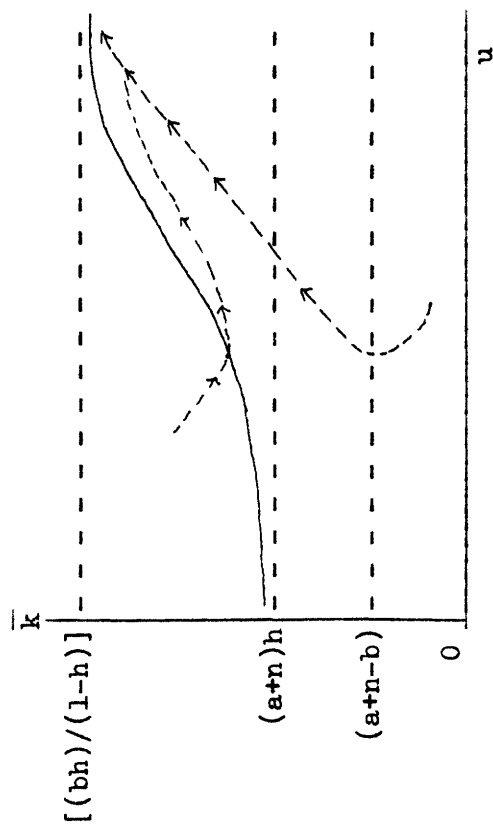
$$22) \quad \phi_k \rightarrow h \text{ implies } \bar{k} \rightarrow [(bh)/(1-h)]$$

These values hold regardless of the nature of the function, and they determine the asymptotic behavior of \bar{k} , assuming that σ is bounded from one as $u \rightarrow 0$ and as $u \rightarrow \infty$.

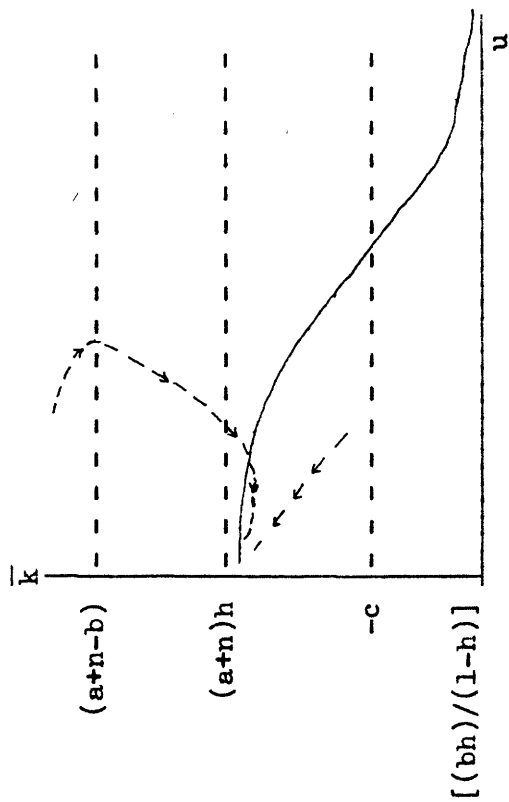
Figures I-III exhibit the path of \bar{k} , assuming that either $s_k = s_n$ and σ bounded from one, or else that the function is a C.E.S.

FIGURE I - Path of \bar{k} as a Function of u for $h < 1$

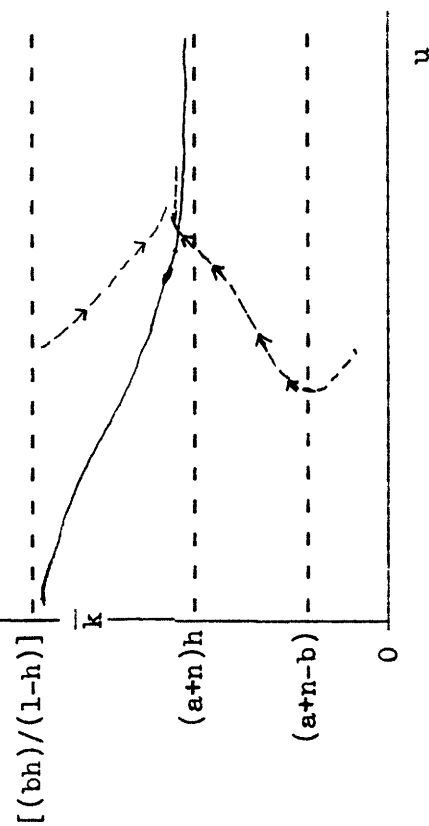
i) $\delta > 0$ ($+b > 0$)
 $\sigma > 1$



iii) $\delta < 0$, $\sigma > 1$



ii) $\delta > 0$ ($+b > 0$)
 $\sigma < 1$



iv) $\delta < 0$, $\sigma < 1$

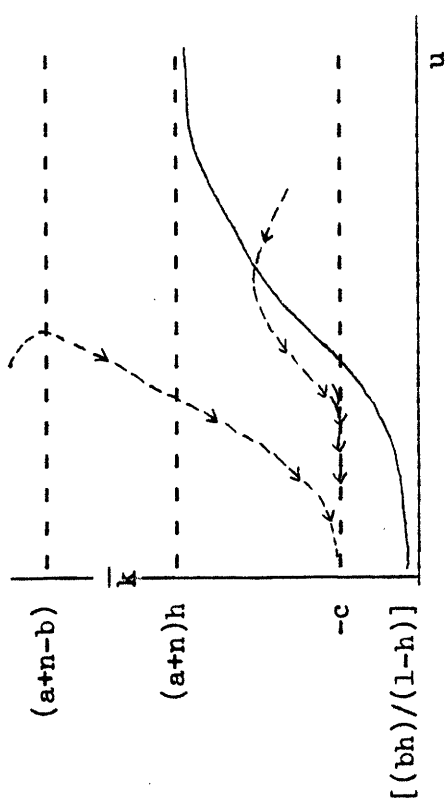


FIGURE II - Path of \bar{k} as a Function of u for $h = 1$

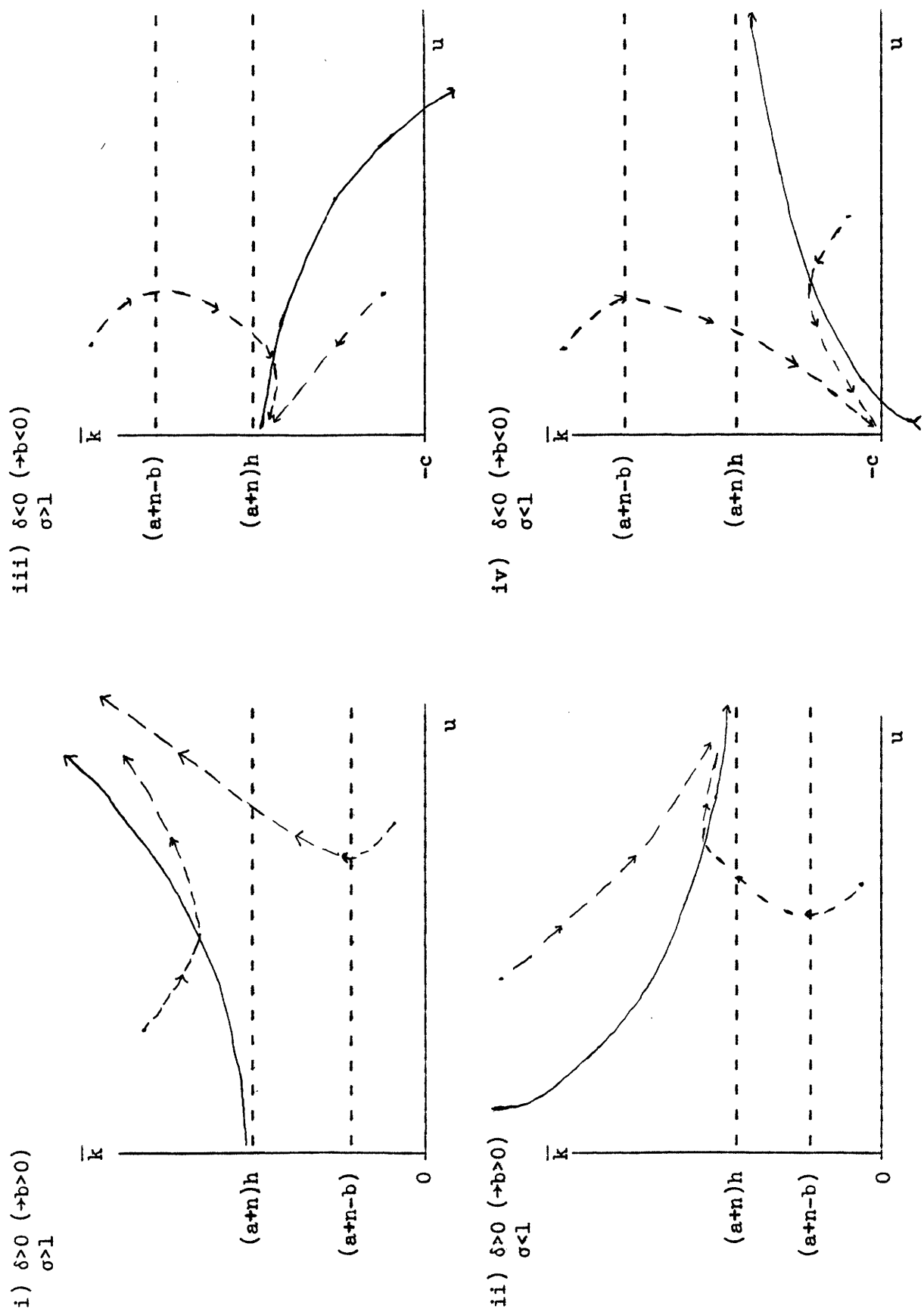
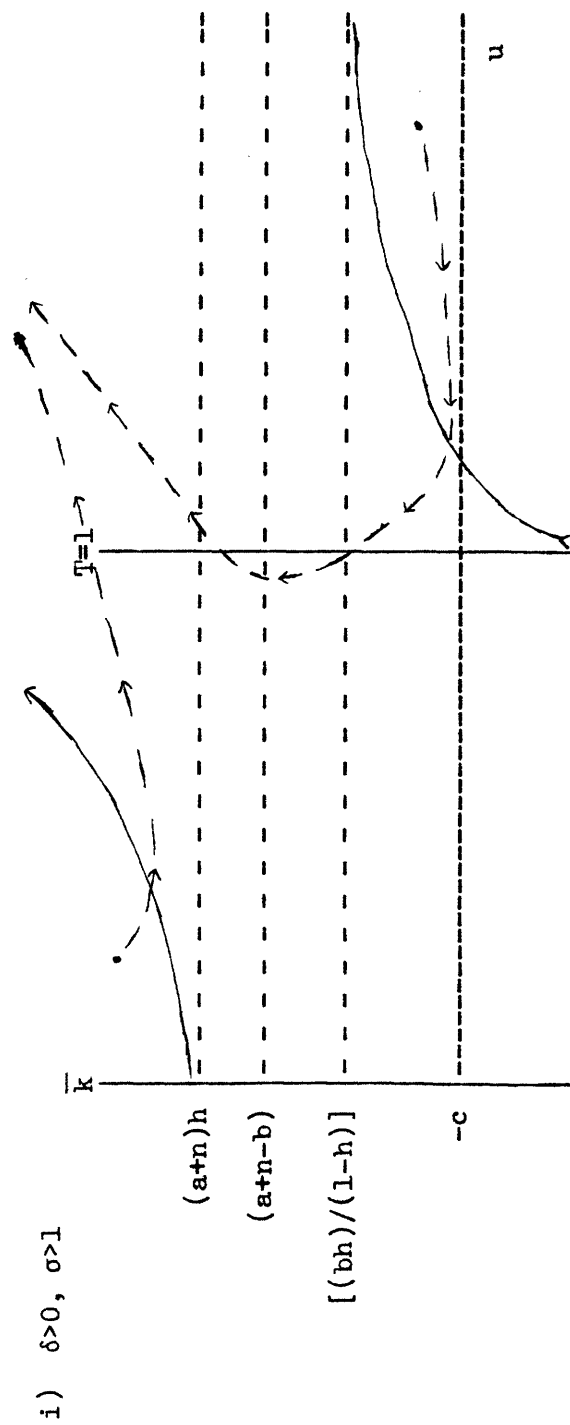


FIGURE III - Path of \bar{k} as a Function of u for $h > 1$



ii) $\delta > 0, \sigma < 1$

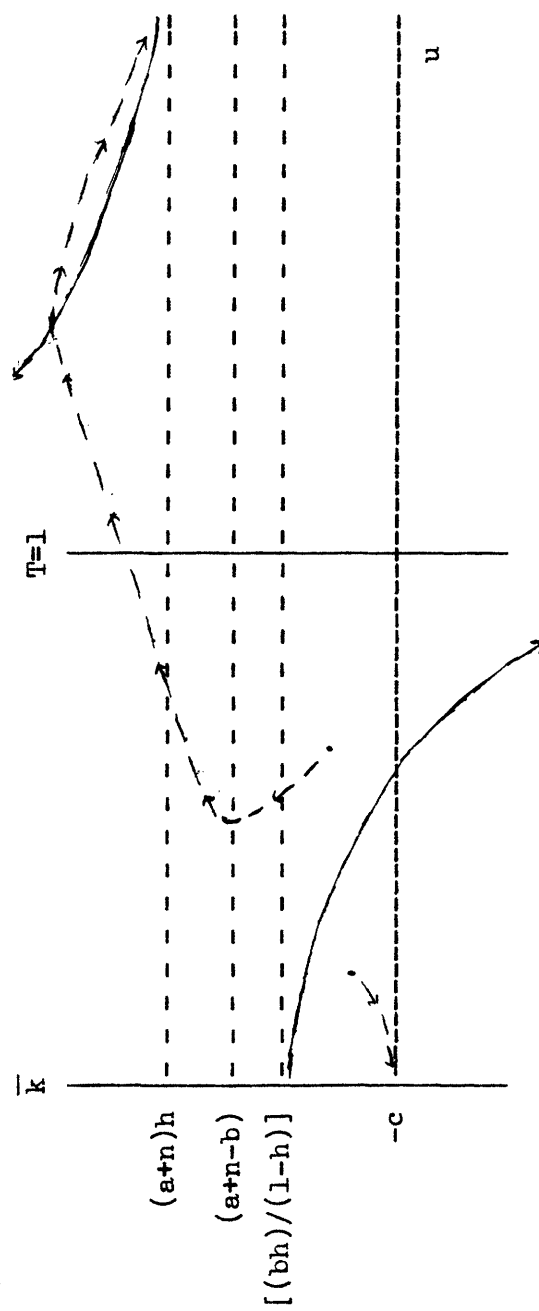
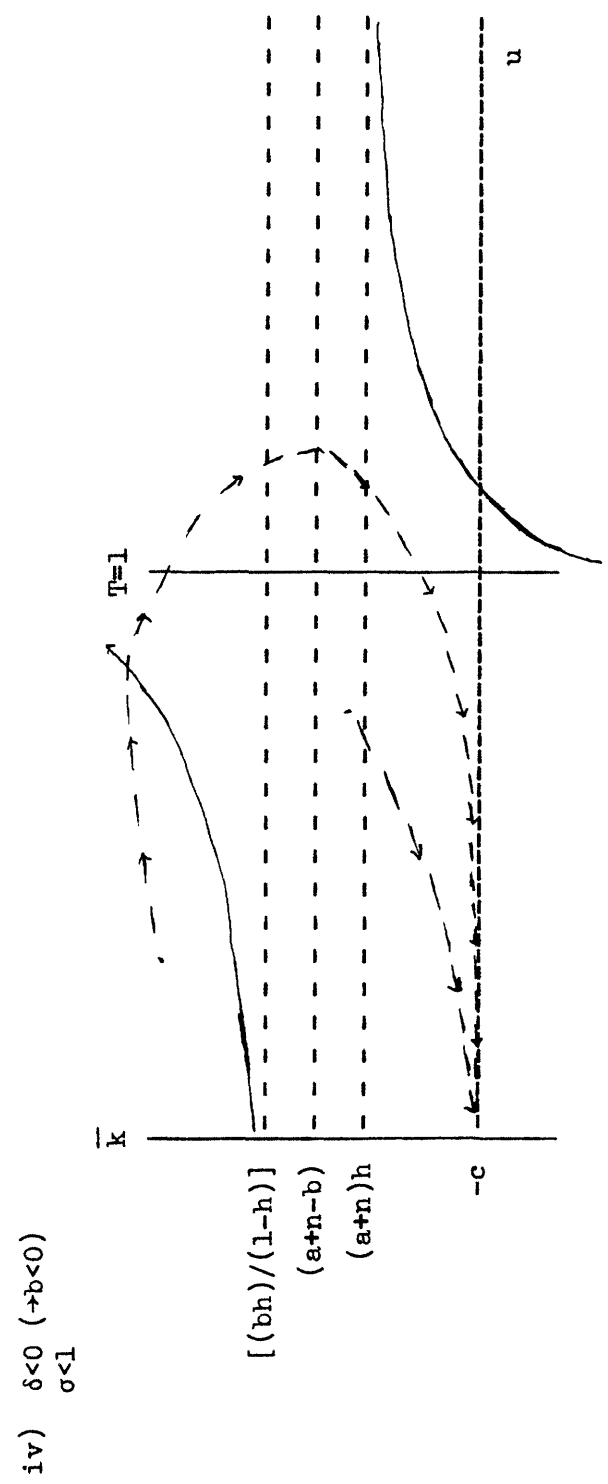
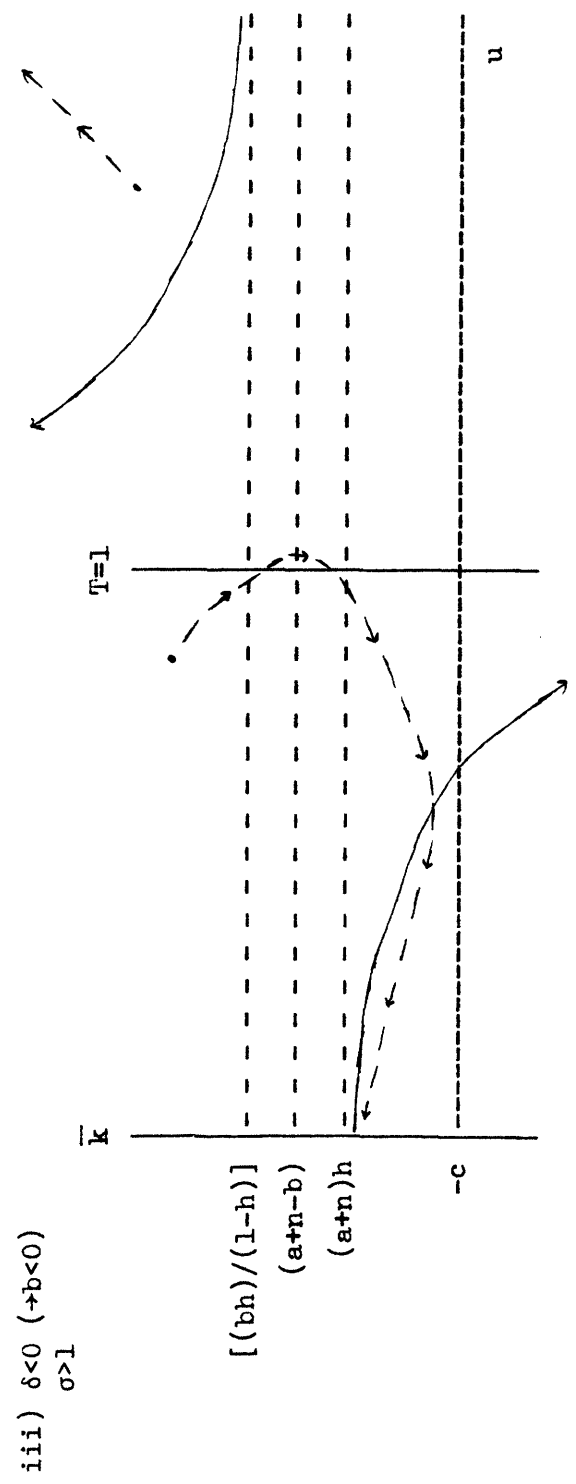


FIGURE III - Continued



function, and that condition 20) is not fulfilled.⁵ If the function is of the C.E.S. variety, but condition 20) is valid, then the curves will look basically the same except that there will be one interior extreme point; the boundary values will be the same. (In the figures, the dotted lines represent the path of (k,u) for given initial conditions.)

From equation 10), given the \bar{k} curves, we can depict the path of k in the (k,u) plane:

$$10) \quad \dot{k}/k = [(k+c)/k](1-T)(\bar{k}-k)$$

As previously noted, whenever $k < \bar{k}$, if $T < 1$, then k will increase; and when $k > \bar{k}$, $T < 1$, k decreases. Thus, we can see that for $h \leq 1$, whenever $\delta > 0$ (implies $h > [(a+n-b)/(a+n)]$, $u \rightarrow \infty$, and for $\delta < 0$, $u \rightarrow 0$. Similarly, the growth rate k approaches the "stationary rate" \bar{k} as $u \rightarrow 0$ or as $u \rightarrow \infty$, providing that \bar{k} is finite. If \bar{k} is infinite ($h=1$, $\sigma > 1$), k tends to infinity. For $h=1$, $\sigma < 1$, $\delta < 0$, \bar{k} tends to minus infinity, and k , the asymptotic growth rate, tends to its lower bound, $[-c]$. Thus, in these cases the asymptotic growth rate is independent of initial conditions, and depends only upon the various parameters of the problem. We note that in all cases the asymptotic growth rate is larger for $\sigma > 1$ than for $\sigma < 1$, given the values of the other parameters.

When we consider the case $h > 1$, the result is slightly

⁵If $s_k > s_n = 0$, then the asymptotic value of \bar{k} as $\phi_k \rightarrow 0$ is $\{(a+n-b) + \sigma[(a+n)h - (a+n-b)]\}$, instead of just $[(a+n)h]$, as we have seen in equation 21). Otherwise, the diagrams can represent that case as well.

different - the asymptotic growth rate of the system may depend upon the initial conditions.⁶ For example, we see that for $\delta > 0$, $h > 1$, $\sigma > 1$, then $k \rightarrow \infty$; however, if $\delta > 0$, $h > 1$, $\sigma < 1$, the asymptotic growth rate tends to $-c$ (as $u \rightarrow 0$) or to $(a+n)h$ (as $u \rightarrow \infty$), depending upon the initial conditions (see Figure III). It would seem that $u \rightarrow \infty$ is the more likely result, though the other result is possible (providing that $[(bh)/(1-h)] > -c$).

Similarly, if $\delta < 0$, $h > 1$ (implies $b < 0$), for $\sigma \leq 1$ there is a unique asymptotic growth rate to which the system tends $[-c]$. However, if $\sigma > 1$, it is possible that either $k \rightarrow \infty$ or $k \rightarrow (a+n)h$ [$u \rightarrow \infty$ in the former case, and $u \rightarrow 0$ in the latter case].

Since, given the initial effective capital-labor ratio, s , the aggregate gross savings rate, determines the initial rate of growth $[k(0)]$, it is possible in these two cases [$\delta > 0$, $h > 1$, $\sigma < 1$ or $\delta < 0$, $h > 1$, $\sigma > 1$] that a larger savings rate could lead to a larger asymptotic growth rate - contrary to the normal neoclassical result. This relation, though, is a step function - there might exist a critical savings rate [given $u(0)$] such that below that critical savings rate the system would tend to the smaller growth rate, whereas for larger savings rates the system would tend to the larger growth rate. Of course, for initial values of u , the savings rate might not be sufficient to alter the growth rate of the system (asymptotically).

In summary, if the function is a C.E.S. function, or if $s_k = s_n$

⁶ Bertrand and Vanek noted this possibility (page 746) for the case $h > 1$, $\delta > 0$. However, they did not elaborate on the elasticity condition, and they did not consider the case $h > 1$, $\delta < 0$, $\sigma > 1$ (because they assume $b \geq 0$).

(and σ is bounded from one), then there is a unique asymptotic growth rate to which the system tends for $h \leq 1$; and this growth rate will, in general, depend upon all the parameters of the model. For $h > 1$, it is possible that initial conditions may effect the asymptotic growth rate. These asymptotic growth rates are finite except for $h \geq 1, \sigma > 1, \delta > 0$ and perhaps $h > 1, \sigma > 1, \delta < 0$. Also, if $\delta > 0, u$, the effective capital-labor ratio, tends to infinity except perhaps in the case $\delta > 0, h > 1, \sigma < 1$. Similarly, for $\delta < 0, u \rightarrow 0$, except perhaps if $\delta < 0, h > 1, \sigma > 1$.

A. Asymptotic Growth Rates - General Case

Now that we have completed our study of the special case discussed above, we can turn our attention to the more general case in which no restrictions are placed upon the production function (or the savings parameters).

As we have seen from equation 14):

$$14) \quad \bar{k} = (a+n-b) + [(a+n)\delta]/(1-T) \quad ; \quad \delta = h - [(a+n-b)/(a+n)]$$

As long as $T < 1$, \bar{k} must always either be greater or less than $(a+n-b)$. But we know that $T < 1$ if $h < 1$; therefore, for $h < 1$, we know that \bar{k} always lies above or below $(a+n-b)$ [we are excluding the case $\delta = 0$].

Similarly, from equation 10):

$$10) \quad (\dot{k}/k) = [(k+c)/k](1-T)(\bar{k}-k)$$

As long as $T < 1$, then k increases when below \bar{k} , and decreases when above \bar{k} . Suppose $\delta > 0$; then $\bar{k} > (a+n-b)$. Whenever $k < (a+n-b)$,

it increases (though u decreases); since the growth rate (\dot{k}/k) is positive (if $k < 0$, it is still true that $\dot{k} > 0$), k must increase and eventually reach and surpass $(a+n-b)$. When this occurs, u begins to increase; but since $\bar{k} > (a+n-b)$, it follows that k will remain above $(a+n-b)$, and thus $u \rightarrow \infty$. If \bar{k} approaches a limit as $u \rightarrow \infty$ (as it will if σ is bounded from, or tends to, one), then k will approach \bar{k} . Should \bar{k} fluctuate between two limits, then k similarly will fluctuate between those limits.

Similarly we can show that if $\delta < 0$, $h < 1$, then eventually $u \rightarrow 0$, and k approaches the limiting value of \bar{k} as $u \rightarrow 0$, or else it fluctuates between the limits of \bar{k} should σ fluctuate between being greater than and less than one.

In summary, when $h < 1$, $\delta > 0$, then $u \rightarrow \infty$, and k approaches the asymptotic value of \bar{k} (determined by ϕ_k) as $u \rightarrow \infty$; and when $h < 1$, $\delta < 0$, then $u \rightarrow 0$, and k tends to the asymptotic value of \bar{k} as $u \rightarrow 0$. There is, of course, no necessity that the path of k be monotonic.

If $h = 1$, the situation is much the same. It is now possible that $T = 1$, but since we expect the output-labor elasticity to be positive for non-zero, finite values of u ($T=1$ implies $\phi_k=1$, $\phi_n=0$ for $h=1$), T can only be one asymptotically. Therefore, \bar{k} is again always either larger or smaller than $(a+n-b)$. The only real difference between this case and the previous case is that the limit of \bar{k} need not be finite. Thus, if $\sigma > 1$ as $u \rightarrow \infty$, then the limit of \bar{k} is unbounded and, for $\delta > 0$ (and $\sigma > 1$), k tends to infinity since, from equation 7) with $h=1$:

$$7) \quad (\dot{k}/k) = [(k+c)/k][(a+n-k)(1-T) + bT] \quad ,$$

and for finite k , $(\dot{k}/k) \rightarrow [(k+c)/k]b$ as $T \rightarrow 1$. Thus, in this case, $k \rightarrow \infty$ as $u \rightarrow \infty$, for $h = 1$, $\delta > 0$, $\sigma > 1$.⁷

Similarly, for $h = 1$, $\delta < 0$, $\sigma < 1$, $\bar{k} \rightarrow -\infty$ as $u \rightarrow 0$. We have already seen that for $\delta < 0$, $u \rightarrow 0$; however, there is a lower bound on the rate of growth of capital (due to depreciation) equal to $-c$. Thus, in this case, $k \rightarrow -c$.

For $h > 1$, however, the story is slightly more complicated. In general, it is now possible for T to exceed one, and thus we can not say that $\bar{k} > (a+n-b)$ when $\delta > 0$ and that $\bar{k} < (a+n-b)$ when $\delta < 0$. As an example, suppose that $\delta < 0$ for $h > 1$ (this implies $b < 0$). Then:

⁷Though Bertrand and Vanek do not explicitly discuss the growth rate of K , they find it implausible that (\dot{K}/K) should tend to infinity since (pages 747-748): "The capital-labor ratio in efficiency units will be increasing so that it could be expected that ϕ_k would eventually decrease, leading to a T less than unity (that is, this would necessarily happen if the marginal product of capital eventually became zero with increasing $[x/z]$)." This statement is wrong on three counts:

- i) If $\sigma > 1$, then ϕ_k will increase, not decrease, as the effective capital-labor ratio increases.
- ii) Obviously, it is possible for the marginal product of capital to tend to zero and for ϕ_k to remain greater than zero. If u (the effective capital-labor ratio) tends to infinity and if $\sigma > 1$, then the marginal product of capital may tend to zero, but $\phi_k \rightarrow h > 1$.
- iii) Even if u tends to infinity, the MPK may not tend to zero (even if the Inada conditions hold) since:

$$\text{MPK} = e^{\delta t} f'(u)$$

(For $s_n = 0$, the MPK tends to a positive constant $[\sigma < 1]$, though $\phi_k \rightarrow 0$. For more information on the asymptotic behavior of the MPK, see Chapter 3).

Consequently, they consider the case in which $k [= (\dot{K}/K)] \rightarrow \infty$ as an exceptional one; if they are speaking "empirically", we can hardly disagree. However, as an a priori possibility, it is just as likely as the case in which $\phi_k \rightarrow 0$ and $K \rightarrow (a+n)h$.

$$14) \quad \bar{k} = (a+n-b) + [(a+n)\delta]/(1-T)$$

$$10) \quad (\dot{k}/k) = [(k+c)/k](1-T)(\bar{k}-k)$$

Consider the minimum value of \bar{k} such that $\bar{k} > (a+n-b)$:

$$23) \quad \text{Min } \bar{k} \text{ such that } \bar{k} > (a+n-b), \delta < 0, = [(bh)/(1-h)] > (a+n-b)$$

If $k(0) < [(bh)/(1-h)]$, then k must decrease since, for $T > 1$, $k(0) < \bar{k}$

and $\dot{k} < 0$, while for $T < 1$, $k(0) > \bar{k}$, and again $\dot{k} < 0$.⁸ Thus, once

$k < [(bh)/(1-h)]$, it must decrease, eventually fall below $(a+n-b)$, and

hence $u \rightarrow 0$. As $u \rightarrow 0$, the asymptotic growth rate tends to $\text{Max}[\lim_{u \rightarrow 0}(\bar{k}), -c]$,

or else it fluctuates between the appropriate limits should σ (and ϕ_k)

fluctuate. However, if $k(0) > [(bh)/(1-h)]$, it is possible that $k > \bar{k}$,

$T > 1$; in this case, k would increase without bound, and $u \rightarrow \infty$. Note,

however, that $T > 1$ as $u \rightarrow \infty$ implies $\sigma \geq 1$, so this case cannot occur if

$\sigma < 1$ as $u \rightarrow \infty$. (However, it may occur if σ fluctuates between being

greater and less than one; the larger the intervals in u for which $\sigma < 1$,

the more likely k is to fall, and hence stay, below $[(bh)/(1-h)]$.)

Similarly, for $\delta > 0$, $h > 1$, it is now possible for $\bar{k} < (a+n-b)$.

This time we are interested in the maximum value of \bar{k} such that

$$\bar{k} < (a+n-b) :$$

$$24) \quad \text{Max } \bar{k} \text{ such that } \bar{k} < (a+n-b), \delta > 0, = [(bh)/(1-h)] < (a+n-b)$$

As in the other case, if $k(0) > [(bh)/(1-h)]$ and $k(0) < \bar{k}$,

$T < 1$, then k increases; and if $k(0) > \bar{k}$, $T > 1$, again k increases.

Thus, once $k > [(bh)/(1-h)]$, it must become larger than $(a+n-b)$, and

hence $u \rightarrow \infty$. However, if $k(0) < [(bh)/(1-h)]$, and $T > 1$, then k decreases,

⁸If $T=1$, from equation 7) we can see:

$$(\dot{k}/k) = [(k+c)/k][(a+n)(h-1) + b] < 0 \quad \text{since } h < [(a+n-b)/(a+n)] .$$

approaching $(-c)$ as its lower limit. For $T > 1$, as $u \rightarrow 0$, this implies $\sigma < 1$; thus this case cannot occur if $\sigma \geq 1$ as $u \rightarrow 0$.

In summary, if $\delta > 0$, then $u \rightarrow \infty$ except possibly for the case $\delta > 0, h > 1, \sigma < 1$ as $u \rightarrow 0$. Similarly, if $\delta < 0$, then $u \rightarrow 0$ with the possible exception of the case $\delta < 0, h > 1, \sigma > 1$ as $u \rightarrow \infty$. The asymptotic growth rate is determined by the value of \bar{k} as u tends to infinity or zero, depending upon which case we are considering.

As we have seen, there is no difference between the general case and the case in which the production function is a C.E.S. function when we are considering the asymptotic behavior of the system. However, should we allow σ to fluctuate between being asymptotically greater and less than one, a difference would emerge. Finally, we see that there is no need for the time path of k to be monotonous. Also, in our two "perverse" cases it is possible, as explained earlier, for the savings rate to alter the asymptotic growth rate of the system (though only two growth rates are possible if σ does not fluctuate between being greater and less than one).

This completes our review of the basic Bertrand-Vanek model (and our modifications of it). Before considering the two possible steady-state cases, let us now consider the growth rates of the other variables.

B. Asymptotic Growth Rates of Other Variables

Given the growth rate of capital, we are able to calculate the growth rates of the other variables from their definitions:

$$25) \quad u = [(Ke^{bt}) / (Le^{at})] ; \quad (\dot{u}/u) = (k+b-a-n) ; \quad k = (\dot{K}/K)$$

- 26) $Q = e^{(a+n)ht} f(u) ; (\dot{Q}/Q) = (a+n)\phi_n + (k+b)\phi_k$
- 27) $(C/L) = (1-s)(Q/L) ; (\dot{C}/C) - (\dot{L}/L) = (\dot{Q}/Q) - n$ since $\dot{s} \rightarrow 0$ asymptotically, and we assume $s_k < 1$ (hence, $s < 1$).
- 28) $W = [(\partial Q/\partial L)/h] = e^{[(a+n)h-n]t} [hf(u) - uf'(u)] ;$
 $(\dot{W}/W) = (a+n)h - n + \phi_k(u/u)[1 - (\sigma-1)/(\sigma h)]$
- 29) $R = [(\partial Q/\partial K)/h] = [e^{\lambda t} f'(u)]/h , \lambda \equiv [(a+n)h - (a+n-b)] ;$
 $(\dot{R}/R) = \lambda + [\phi_k\{(h-1)/h\} - \{\phi_n/(\sigma h)\}](u/u)$

In some cases (for example, $h=1, \delta>0, \sigma>1$) we are faced with an expression for (\dot{R}/R) involving: $[0^\infty]$. In these cases we can use l'Hôpital's rule to evaluate the expression. In most other cases, the results are quite straightforward. Table I summarizes the growth rates for the above variables. In some cases, the asymptotic growth rates depend upon which savings assumption is used - these cases are so indicated in the Table. Also, for $\delta < 0$, it is possible that \bar{k} would tend to some finite value, which might be either greater or less than $[-c]$ - the rate of depreciation of capital. Naturally, capital cannot decrease at a rate faster than $[-c]$ (barring direct consumption or disposal of capital) - again, these cases are so indicated in the Table.

From Table I we can study the asymptotic behavior of the various variables. As an example, suppose we are interested in per capita consumption. From Table I we can readily see that whenever $\delta>0$, $a>b$ is sufficient to guarantee that per capita consumption is always increasing (assuming $a>0$). However, for decreasing returns to scale it is possible that per capita consumption actually declines over time if capital-augmenting technical progress occurs more rapidly than labor-

TABLE I - Asymptotic Values and Growth Rates of Variables

Cases	Asymptotic Values				Asymptotic Growth Rates		
	σ	u	ϕ_k	ϕ_n	u	K	
$\delta = [h - \frac{(a+n-b)}{(a+n)}] > 0$	Limit as $u \rightarrow \infty$				$\lambda \equiv [(a+n)h - (a+n-b)]$		
1) $h < 1$ [$\rightarrow b > 0$]	> 1	∞	h	0	$[\lambda / (1-h)]$	$[(bh) / (1-h)]$	
	$\rightarrow 1$	∞	$\phi_k^* < h$	$h - \phi_k^*$	$[\lambda / (1 - \phi_k^*)]$	$\frac{[(a+n)(h - \phi_k^*) + b\phi_k^*]}{(1 - \phi_k^*)}$	
$s_n > 0$	< 1	∞	0	h	λ	$(a+n)h$	
$s_n = 0$	< 1	∞	0	h	$\lambda\sigma$	$(a+n-b) + \lambda\sigma$	
2) $h = 1$ [$\rightarrow b > 0$]	> 1	∞	1	0	∞	∞	
	$\rightarrow 1$	∞	$\phi_k^* < 1$	$1 - \phi_k^*$	$[b / (1 - \phi_k^*)]$	$[(a+n) + \frac{(b\phi_k^*)}{(1 - \phi_k^*)}]$	
$s_n > 0$	< 1	∞	0	1	b	$(a+n)$	
$s_n = 0$	< 1	∞	0	1	σb	$[(a+n-b) + \sigma b]$	
3) $h > 1$	> 1	∞	h	0	∞	∞	
	$\rightarrow 1$	∞	$\phi_k^* > 1$	$h - \phi_k^*$	∞	∞	
	$\rightarrow 1$	∞	1	$(h-1)$	∞	∞	
	$\rightarrow 1$	∞	$\phi_k^* < 1$	$h - \phi_k^*$	$[\lambda / (1 - \phi_k^*)]$	$\frac{[(a+n)h - (a+n-b)\phi_k^*]}{(1 - \phi_k^*)}$	
$s_n > 0$	< 1	∞	0	h	λ	$(a+n)h$	
$s_n = 0$	< 1	∞	0	h	$\lambda\sigma$	$[(a+n-b) + \lambda\sigma]$	
<u>PERVERSE CASE</u>	$u \rightarrow 0$	AND $k(0) < [(bh) / (1-h)]$					
	< 1	0	h	0	$(b-c-a-n)$	$-c$	
	$\rightarrow 1$	0	$\phi_k^* > 1$	$h - \phi_k^*$	$(b-c-a-n)$	$-c$	

TABLE I - Continued

Cases	Q	Asymptotic Growth Rates		
		(C/L)	W = $[(\partial Q/\partial L)/h]$	R = $[(\partial Q/\partial K)/h]$
$\delta > 0$				
<u>Growth Rates as $u \rightarrow \infty$</u>				
1) $h < 1$ [$\rightarrow b > 0$]				
$\sigma > 1$	$[bh/(1-h)]$	$\lambda/(1-h) + (a-b)$	$\lambda/[(1-h)\sigma] + (a-b)$	0
$\sigma > 1, \phi_k^* < h$	Same as K	$\lambda/(1-\phi_k^*) + (a-b)$	Same as (C/L)	0
$\sigma < 1, s_n > 0$	$(a+n)h$	$\lambda + (a-b)$	$\lambda + (a-b)$	$[(\sigma-1)/\sigma]\lambda$
$\sigma < 1, s_n = 0$	$(a+n)h$	$\lambda + (a-b)$	$\lambda + (a-b)$	0
2) $h = 1$ [$\rightarrow b > 0$]				
$\sigma > 1$	∞	∞	∞	b
$\sigma > 1, \phi_k^* < 1$	Same as K	$a + [b\phi_k^*/(1-\phi_k^*)]$	Same as (C/L)	0
$\sigma < 1, s_n > 0$	$(a+n)$	a	a	$[(\sigma-1)/\sigma]b$
$\sigma < 1, s_n = 0$	$(a+n)$	a	a	0
3) $h > 1$				
$\sigma > 1$	∞	∞	∞	∞
$\sigma > 1, \phi_k^* > 1$	∞	∞	∞	∞
$\sigma > 1, \phi_k^* \rightarrow 1$	∞	∞	∞	λ
$\sigma > 1, \phi_k^* < 1$	Same as K	$\lambda/(1-\phi_k^*) + (a-b)$	Same as (C/L)	0
$\sigma < 1, s_n > 0$	$(a+n)h$	$(h-1)n + a$	$(h-1)n + a$	$[(\sigma-1)/\sigma]\lambda$
$\sigma < 1, s_n = 0$	$(a+n)h$	$(h-1)n + a$	$(h-1)n + a$	0
<u>PERVERSE CASE</u>				
$u \rightarrow 0$ AND $k(0) < [(bh)/(1-h)]$				
$\sigma < 1$	$h(b-c) < 0$	$h(b-c) - n$	$[(\sigma-1)/\sigma][h(b-c) - n - (b-c-a-n)]$	$[bh+c(1-h)] < 0$
$\sigma > 1, \phi_k^* > 1$	$[(a+n)h + \phi_k^*(b-c-a-n)]$	$[(a+n)h - n + \phi_k^*(b-c-a-n)]$	Same as (C/L)	$[(a+n)h + c + \phi_k^*(b-c-a-n)]$

TABLE I - Continued

Cases	Asymptotic Values				Asymptotic Growth Rates	
	σ	u	ϕ_k	ϕ_n	u	K
$\delta < 0$	<u>Limit as $u \rightarrow 0$</u>					
1) $h < 1$:						
$s_n > 0$	>1	0	0	h	λ	$(a+n)h$
$s_n = 0$	>1	0	0	h	$\lambda\sigma$	$\dagger[(a+n-b) + \lambda\sigma]$
	$\rightarrow 1$	0	ϕ_k^*	$h - \phi_k^*$	$[\lambda/(1-\phi_k^*)]$	$\dagger \frac{[(a+n)h - (a+n-b)\phi_k^*]}{(1-\phi_k^*)}$
	<1	0	h	0	$[\lambda/(1-h)]$	$\dagger[(bh)/(1-h)]$
2) $h=1$ [$\rightarrow b < 0$]						
$s_n > 0$	>1	0	0	1	b	$(a+n)$
$s_n = 0$	>1	0	0	1	σb	$\dagger[(a+n-b) + \sigma b]$
	$\rightarrow 1$	0	$\phi_k^* < 1$	$1 - \phi_k^*$	$[b/(1-\phi_k^*)]$	$\dagger[(a+n) + (b\phi_k^*)/(1-\phi_k^*)]$
	<1	0	1	0	$(b-c-a-n)$	$-c$
3) $h > 1$ [$\rightarrow b < 0$]						
$s_n > 0$	>1	0	0	h	λ	$(a+n)h$
$s_n = 0$	>1	0	0	h	$\lambda\sigma$	$\dagger[(a+n-b) + \lambda\sigma]$
	$\rightarrow 1$	0	$\phi_k^* \geq 1$	$h - \phi_k^*$	$(b-c-a-n)$	$-c$
	$\rightarrow 1$	0	$\phi_k^* < 1$	$h - \phi_k^*$	$[\lambda/(1-\phi_k^*)]$	$\dagger \frac{[(a+n)h - (a+n-b)\phi_k^*]}{(1-\phi_k^*)}$
	<1	0	h	0	$(b-c-a-n)$	$-c$
<u>PERVERSE CASE</u>		$u \rightarrow \infty$	AND	$k(0) > [(bh)/(1-h)]$		
	>1	∞	h	0	∞	∞
	$\rightarrow 1$	∞	$\phi_k^* > 1$	$h - \phi_k^*$	∞	∞

† In these cases it is possible that the given rates of growth for K are less than $[-c]$; in that case, $[-c]$ becomes the growth rate for K , and the other growth rates (for other variables) are correspondingly modified.

TABLE I - Continued

Cases	<u>Asymptotic Growth Rates</u>			
	<u>Q</u>	<u>(C/L)</u>	<u>W = [(∂Q/∂L)/h]</u>	<u>R = [(∂Q/∂K)/h]</u>
$\delta < 0$	<u>Growth Rates as $u \rightarrow 0$</u>			
1) $h < 1$				
$\sigma > 1, s_n > 0$	$(a+n)h$	$(a+n)h - n$	$(a+n)h - n$	$[(\sigma-1)/\sigma]\lambda$
$\sigma > 1, s_n = 0$	$(a+n)h$	$(a+n)h - n$	$(a+n)h - n$	0
$\sigma \rightarrow 1, \phi_k^* < h$	Same as K	$\lambda/(1-\phi_k^*) + (a-b)$	Same as (C/L)	0
$\sigma < 1$	$[bh/(1-h)]$	$[(b+n)h-n]/(1-h)$	$\lambda/[(1-h)\sigma] + (a-b)$	0
2) $h=1$ [$\rightarrow b < 0$]				
$\sigma > 1, s_n > 0$	$(a+n)$	a	a	$[(\sigma-1)/\sigma]b$
$\sigma > 1, s_n = 0$	$(a+n)$	a	a	0
$\sigma \rightarrow 1, \phi_k^* < 1$	Same as K	$a + [b\phi_k^*/(1-\phi_k^*)]$	Same as (C/L)	0
$\sigma < 1$	$(b-c)$	$(b-c-n)$	$\frac{[(b-c-n)-a(1-\sigma)]}{(\sigma)}$	b
3) $h > 1$ [$\rightarrow b < 0$]				
$\sigma > 1, s_n > 0$	$(a+n)h$	$(h-1)n + a$	$(h-1)n + a$	$[(\sigma-1)/\sigma]\lambda$
$\sigma > 1, s_n = 0$	$(a+n)h$	$(h-1)n + a$	$(h-1)n + a$	0
$\sigma \rightarrow 1, \phi_k^* \geq 1$	$[(a+n)h + \phi_k^*(b-c-a-n)]$	$[(a+n)h - n + \phi_k^*(b-c-a-n)]$	Same as (C/L)	$[(a+n)h + c + \phi_k^*(b-c-a-n)]$
$\sigma \rightarrow 1, \phi_k^* < 1$	Same as K	$\lambda/(1-\phi_k^*) + (a-b)$	Same as (C/L)	0
$\sigma < 1$	$h(b-c)$	$h(b-c) - n$	$[h(b-c) - n - \{(\sigma-1)/\sigma\}(b-c-a-n)]$	$[bh + c(1-h)]$
<u>PERVERSE CASE</u>				
	$u \rightarrow \infty$	AND	$k(0) > [(bh)/(1-h)]$	
$\sigma > 1$	∞	∞	∞	∞
$\sigma \rightarrow 1, \phi_k^* > 1$	∞	∞	∞	∞

augmenting technical progress.

Similarly, for $\delta < 0$, $a \leq b$ suffices to guarantee that per capita consumption is declining in the long run. However, for $a > b$ it is possible (though not necessary) that per capita consumption would be increasing over time. Thus, the value of δ alone does not suffice to determine how per capita consumption behaves; rather, we need to consider all the parameters, and specifically whether capital- or labor-augmenting technological progress is occurring at the faster rate. (In this discussion we have ignored the two "perverse" cases mentioned earlier in this chapter).

C. Steady-State Possibilities

Before considering how technical progress should be allocated within this economy in order to maximize the discounted stream of consumption (and welfare), let us briefly consider the two steady-state possibilities mentioned by Bertrand-Vanek (pages 750-751):

- i) $\sigma = 1$ everywhere (Cobb-Douglas production function)
- ii) $[(a+n)h = (a+n-b)]$

As we shall see, either of these conditions is merely a necessary, but not sufficient, condition for the existence and stability of the steady-state path.

Consider case i) first:

$$30) \quad Q = (K e^{bt})^{\phi_k} (L e^{at})^{(h-\phi_k)} \quad ; \quad (\phi_k \text{ constant})$$

Using Vanek's pricing assumption:

$$31) \quad s = [s_k \phi_k / h] + [s_n \phi_n / h] \equiv s^* \quad ; \quad s^* \text{ is a constant, and defined}$$

to be the average gross savings rate. If $\phi_k \neq 1$, we can write:

$$32) \quad Q = K^{\phi_k} [L_0 \exp\{((a+n)h - (a+n-b)\phi_k)t / (1-\phi_k)\}]^{(1-\phi_k)} ; \quad \phi_k \neq 1$$

$$= K^{\phi_k} (L_0 A)^{(1-\phi_k)} ; \quad L = L_0 e^{nt}$$

Define $w = [K/L_0 A]$; then:

$$33) \quad (\dot{w}/w) = s*Q/K - c - [(a+n)h - (a+n-b)\phi_k] / (1-\phi_k)$$

$$= s*w^{(\phi_k-1)} - c - [(a+n)h - (a+n-b)\phi_k] / (1-\phi_k)$$

If $\phi_k < 1$ (or $\phi_k < h < 1$), a unique stable equilibrium to equation 33) exists. However, if $\phi_k > 1$, then no equilibrium exists if:

$$34) \quad -c + [(a+n)(h-\phi_k) + b\phi_k] / (\phi_k-1) \geq 0 ;$$

If $c=0$ and $b \geq 0$, then the expression in equation 34) will be positive, and no steady-state equilibrium exists, despite the fact that the production function is Cobb-Douglas. Otherwise, there exists a unique w^* such that:

$$34') \quad \text{At } w^*, \quad (\dot{w}/w) = 0; \quad \text{and } (\dot{w}/w) \geq 0 \text{ as } w \geq w^*$$

Therefore, for $\phi_k > 1$ either no equilibrium exists, or else a unique unstable equilibrium exists. Finally, if $\phi_k = 1$, we find:

$$35) \quad Q = K[L_0]^{(h-1)} e^{\delta t} ; \quad \delta = [(a+n)h - (a+n-b)]$$

$$\dot{K} = s*K[L_0]^{(h-1)} e^{\delta t} ,$$

and no steady-state exists (in which the MPK and Q/K are constant) unless $\delta = 0$, which is really just case ii). (Even if $\delta = 0$, in general, the effective capital-labor ratio will tend to zero or infinity. However, the MPK and Q/K and the shares of each factor - under Vanek's pricing assumption - will tend to constant values) Thus, in the case of Cobb-Douglas production functions it must be that $\phi_k < 1$ or $\delta=0$ (the latter being

our second special case) for a steady-state to exist and to be stable.

Consider now the second case:

$$\text{ii) } [(a+n)h = (a+n-b)] \rightarrow \delta = 0$$

This case represents the other steady-state possibility proposed by Bertrand-Vanek. Consider (\dot{u}/u) :

$$36) \quad (\dot{u}/u) = s^*f(u)/u + b - a - n - c ; [s^* = (s_k \phi_k/h) + (s_n \phi_n/h)]$$

For a steady-state to exist it must be true that:⁹

$$37) \quad b < (a+n+c)$$

Since $\delta = 0$, equation 36), which shows the rate of growth of the effective capital-labor ratio, does not explicitly depend on time, and so it would appear that we are in the traditional neoclassical world. However, we must remember that for $b \neq 0$, the production function does not exhibit constant returns to scale. From equation 36) we find:

$$38) \quad \frac{d(\dot{u}/u)}{du} = [(s_k - s_n)f''/h] + [s_n f(\phi_k - 1)/u^2]$$

If $\phi_k \leq 1$ everywhere, then this expression is never positive (assuming $s_k \geq s_n$ and $f'' < 0$; note that for $h > 1$, it is possible that $f'' > 0$, especially for a C.E.S. function). Therefore, if the Inada conditions hold, then a unique, stable steady-state exists.¹⁰

⁹This condition must be fulfilled since $h = [(a+n-b)/(a+n)] > 0$. Therefore, $h > 0$ implies $(a+n-b) > 0$, and thus $[b < (a+n+c)]$ if $c \geq 0$.

¹⁰The Inada conditions are overly strong - it suffices that:
 $s_k f'(0) > (a+n+c-b)$ and $\lim_{u \rightarrow \infty} [f(u)/u] < (a+n+c-b)$.

However, suppose $h > 1$ ($b < 0$), so that it is possible that $\phi_k > 1$.

In that case:

$$39) \quad \frac{d(\dot{u}/u)}{du} > 0 \quad \text{for } \phi_k > 1. \quad (\text{for simplicity we assume that } s_k = s_n).$$

Several possibilities now arise:

- a) If $[sf(u)/u] > (a+n+c-b)$ for all values of u , then $u \rightarrow \infty$, and no steady-state equilibrium exists.
- b) A unique unstable equilibrium exists if $\phi_k > 1$ everywhere and $\lim_{u \rightarrow 0} [sf(u)/u] < (a+n+c-b)$.
- c) Many equilibria (stable and/or unstable) may exist.

For example, suppose that $s_k = s_n$, and that the production function is a C.E.S. function. Then:

$$40) \quad \frac{d(f/u)}{du} = (f/u^2)(\phi_k - 1), \quad \text{and therefore } (f/u) \text{ has only one}$$

interior extreme point, which is a maximum (minimum) for $\sigma < 1$ ($\sigma > 1$), and it occurs at u^* such that $\phi_k(u^*) = 1$. In this case, there are three possibilities:

- i) No equilibrium exists and $u \rightarrow \infty$ ($u \rightarrow 0$) for $\sigma > 1$ ($\sigma < 1$).
- ii) There are two values of u such that $(\dot{u}/u) = 0$, the first value of u being a stable (unstable) equilibrium, the second an unstable (stable) equilibrium for $\sigma > 1$ ($\sigma < 1$).
- iii) One equilibrium occurs at the tangency between (sf/u) and the line $(a+n+c-b)$. This tangency occurs at u^* such that $\phi_k(u^*) = 1$, and it is stable (unstable) for $u < u^*$ and unstable (stable) for $u > u^*$ if $\sigma > 1$ ($\sigma < 1$).

Thus, the Bertrand-Vanek statement that a steady-state will exist

if either $\sigma=1$ or $[(a+n)h = (a+n-b)]$ proves to be a necessary, but not sufficient, condition for the existence and stability of the steady-state path. Particularly, if $h>1$ (as seems possible; however, $b<0$ does not seem too plausible), then no steady-state will result in case i) if $\phi_k \geq 1$; and in case ii), for $h>1$, it is possible that there are none, one, or several equilibria, some of which may well be unstable.

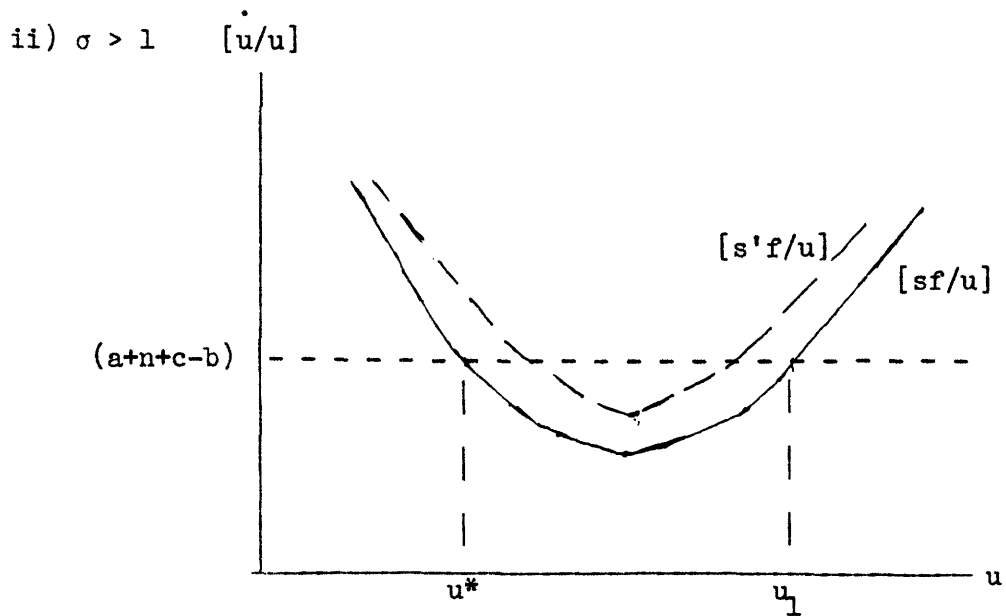
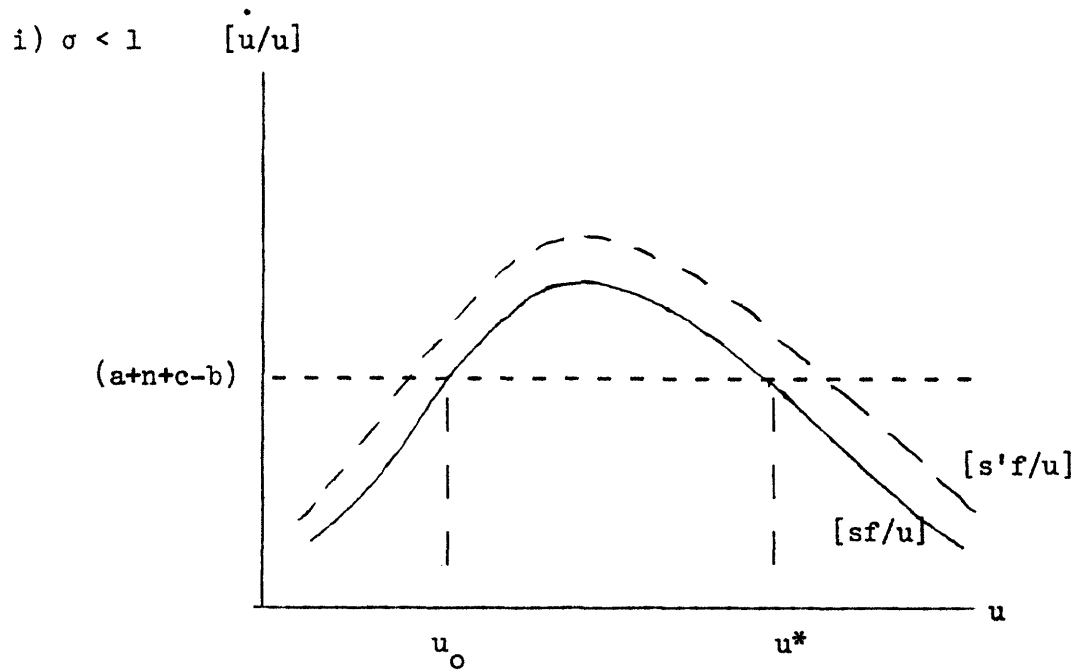
D. The Savings Rate and the Bertrand-Vanek Steady-State

In the previous section we have seen that, even if the Bertrand-Vanek conditions for a steady-state are fulfilled, there may not be a steady-state equilibrium (or it may be unstable). For the case $h<1$, $\delta=0$, a steady-state will occur and the savings rate serves to determine the effective capital-labor ratio, as in the traditional neoclassical growth models.

However, if $h>1$ (and $\delta=0$), the savings rate may play an even more influential role. We have seen in the prior section that in this case several possible roots of $[(\dot{u}/u) = 0]$ may occur, some of which are stable, others unstable. Particularly, if the function is a C.E.S. function, there may be two roots to the $(\dot{u}/u) = 0$ equation. For $\sigma < 1$, given s , if u is initially "too small", then u will tend to zero; otherwise, it will tend to its steady-state value. Conversely, for $\sigma > 1$, if $u(0)$ is "sufficiently large", then u will tend to infinity; otherwise it will tend to its steady-state value.

How does a change in the savings rate affect this system? In order to answer this question, consider Figure 4 (page 44). If we consider the case $\sigma < 1$, we see that if $u(0) < u_0$, then $u \rightarrow 0$; if $u(0) > u_0$, then $u \rightarrow u^*$;

FIGURE IV - Effect of the Savings Rate on the Bertrand-Vanek
Steady-State Growth Model - Assuming Increasing
Returns to Scale



and $u(0) = u_0$ corresponds to the unstable equilibrium. As s increases, u_0 decreases and u^* increases. If the initial $u [u(0)]$ exceeds u_0 , then the only effect of an increase in the savings rate is to increase the effective capital-labor ratio to which the system tends. However, if $u(0) < u_0$, it is possible that an increase in s may lower u_0 sufficiently so that the economy may tend to the stable root (instead of the effective capital-labor ratio tending to zero). That is, for given $u(0)$ there may exist a \tilde{s} (it is possible that u tends to zero for all $s \leq 1$) such that:

$$41) \quad u \rightarrow \begin{cases} u^* \\ u_0 \\ 0 \end{cases} \quad \text{as } s \begin{matrix} > \\ < \end{matrix} \tilde{s}$$

Obviously, the role of the savings rate is potentially more important in this model than in the normal steady-state model.

Similarly for $\sigma > 1$, there exists a u_1 such that:

$$42) \quad u \rightarrow \begin{cases} \infty \\ u_1 \\ u^* \end{cases} \quad \text{as } u(0) \begin{matrix} > \\ < \end{matrix} u_1$$

As we can see from Figure IV, an increase in s decreases u_1 and increases u^* . Consequently, it is possible that, for given $u(0)$, there may exist a \hat{s} such that:

$$43) \quad u \rightarrow \begin{cases} \infty \\ u_1 \\ u^* \end{cases} \quad \text{as } s \begin{matrix} > \\ < \end{matrix} \hat{s}$$

Therefore, assuming the Bertrand-Vanek steady-state exists, an increase in the savings rate, in addition to increasing the stable steady-state root, also increases the probability (for given initial

conditions) that, for $\sigma < 1$, the system will converge to that locally stable root; whereas for $\sigma > 1$, an increase in the savings rate increases the probability that $u \rightarrow \infty$. Consequently, a slight increase in the savings rate may prove more rewarding in this system than in the conventional steady-state models.^{11,12}

Now that we have considered the effects of the savings rate on this steady-state model, let us investigate how technological progress should be allocated between capital and labor in order to maximize society's welfare.

III. Kennedy-Von Weizsäcker Revisited

A. Maximizing the Asymptotic Rate of Growth of Consumption

As has been done by others for the special case of constant returns to scale, we can pose the following question:

"If a planner faces a transformation curve relating the rate of capital-augmenting technical progress to the rate of labor-augmenting technical progress, how should he allocate technological progress within this society?"

Specifically, assume the following transformation curve exists:

¹¹Note that, if desired, the increase in the savings rate need only be temporary, until such time as $u(T)$ is "sufficiently large" to either approach u^* ($\sigma < 1$) or to tend to infinity ($\sigma > 1$). Once this point is reached, the savings rate could be decreased again, if that were deemed desirable.

¹²If $h > 1$, $b \geq 0$, then $u \rightarrow \infty$ (barring the perverse case). However, if $h < 1$, it is possible that successive increases in n (for given rates of technical progress) may change the economy from an explosive one ($\delta > 0$, $u \rightarrow \infty$), to a steady-state economy ($\delta = 0$), to a decaying economy ($\delta < 0$, $u \rightarrow 0$).

$$44) \quad b = \ell(a) ; \quad \ell' , \quad \ell'' < 0 ; \quad \ell(A) = 0 \quad \text{and} \quad \ell(0) = B.$$

In equation 44), a represents the rate of labor-augmenting technical progress and b the rate of capital-augmenting technical progress.¹³

Before attempting to maximize the discounted stream of consumption, let us attempt to answer a slightly easier question - what should the planner do if he desires to maximize the asymptotic rate of growth of per capita consumption?¹⁴ First, consider the case $h < 1$.

Define:

$$45) \quad h^* = [(a+n-b)/(a+n)]$$

As we have seen, if $h < h^*$ ($\delta < 0$), then $u \rightarrow 0$; if $h > h^*$, $u \rightarrow \infty$; and if $h = h^*$, a steady-state exists, is unique and is stable (for $b \geq 0$). Given h , the degree of homogeneity of the production function, there exists an \hat{a} , $b(\hat{a})$ such that:

$$46) \quad h > h^* \quad \text{as} \quad a < \hat{a}. \quad ^{15} \quad \text{Also, define:}$$

¹³Though this is not necessary, we shall suppose for simplicity that neither a nor b can be negative. However, this assumption can readily be relaxed. The assumption that there can be no "technical regression" is implicitly adopted by Bertrand-Vanek; Kennedy, however, does permit negative rates of factor-augmenting technical "progress".

¹⁴If $h > 1$ ($b \geq 0$), $\sigma \geq 1$, then the asymptotic rate of growth of consumption may not exist - it will be unbounded. However, if k ($=[\dot{K}/K]$) tends to infinity, for $\sigma > 1$, then $\phi_k \rightarrow h$ and $T \rightarrow h$. Therefore:

$$(\dot{k}/k) \rightarrow [(k+c)/k][k(h-1) + b]$$

In this case, in order to maximize the rate of growth of the rate of growth of consumption (which is still unbounded for $h > 1$) all technical progress should be capital-augmenting.

¹⁵For $h < 1$, $a=A$, $b=0$ guarantees that $h < h^*$. However, it is possible that if $B < n$, then $h < h^*$ - that is, if $h < [(n-B)/n]$, then $h < h^*$, $u \rightarrow 0$ and no steady-state is possible. We shall ignore this possibility; the proper behavior in this case is readily ascertainable from examining Table I.

$$47) \quad \lambda = [(a+n)h - (a+n-b)] ; \quad \lambda(a) \begin{matrix} \geq \\ \leq \end{matrix} 0 \quad \text{as } a \begin{matrix} < \\ > \end{matrix} \hat{a}$$

From Table I we can see that:

$$48) \quad \frac{[d(C/L)]/[C/L]}{dt} = [\lambda/(1-\phi_k)] + (a-b) \quad \text{for } h < 1, \phi_k = [(uf')/f]$$

In equation 48), λ depends only upon a (given the values of the other parameters), whereas ϕ_k depends upon σ , the elasticity of substitution, as well as on the behavior of u (and hence on $[a-\hat{a}]$).

Therefore, we find:

$$49) \quad \begin{aligned} \phi_k &\rightarrow h \quad \text{if } a > \hat{a}, \sigma < 1 \quad \text{or} \quad a < \hat{a}, \sigma > 1 \\ \phi_k &\rightarrow c^*, \quad 0 < c^* < h \quad \text{if } a = \hat{a}, \quad \text{or if } \sigma > 1, u \rightarrow 0, a > \hat{a}, \quad \text{or } \sigma > 1, u \rightarrow \infty, a < \hat{a} \\ \phi_k &\rightarrow 0 \quad \text{if } a > \hat{a}, \sigma > 1 \quad \text{or} \quad a < \hat{a}, \sigma < 1 \end{aligned}$$

Returning to equation 48), and letting $Z = [(\dot{C}/C) - n]$, we find:

$$50) \quad (dZ/da) = [(h-\phi_k) + b'(a)\phi_k]/(1-\phi_k) ; \quad b'(a) = (db/da)$$

From equation 50) we can determine how to allocate technical progress in order to maximize the asymptotic rate of growth of Z . Thus, suppose $\sigma < 1$ as $u \rightarrow 0$ and as $u \rightarrow \infty$:

$$51) \quad \begin{aligned} \sigma < 1: \quad a > \hat{a} \quad \text{implies } [\phi_k \rightarrow h] \quad \text{and thus } (dZ/da) \rightarrow [(b'\phi_k)/(1-\phi_k)] < 0 \\ \quad \quad \quad a < \hat{a} \quad \text{implies } [\phi_k \rightarrow 0] \quad \text{and thus } (dZ/da) \rightarrow [h/(1-\phi_k)] > 0 \end{aligned}$$

From equation 51) we can see that if $\sigma < 1$ as $u \rightarrow 0$, it does not pay to increase a above \hat{a} ; similarly, if $\sigma < 1$ as $u \rightarrow \infty$, it does not pay to decrease a below \hat{a} . Thus, in this case we find that we do best to choose $a = \hat{a}$, and hence to choose the steady-state solution.

If $\sigma > 1$, on the other hand, it always pays to move away from \hat{a} ;

in fact, if $\sigma > 1$ as $u \rightarrow 0$ and as $u \rightarrow \infty$, then the point $a = \hat{a}$ is a minimum, and we must compare the two boundary solutions ($a = A, b = 0$ or $a = 0, b = B$) to see which gives the larger rate of growth of per capita consumption.

If $\sigma = 1$ everywhere then, as previously discussed, a steady-state exists (we are assuming $h < 1$) and the rate of growth of consumption is maximized when:

$$(db/da) = -[(h - \phi_k)/\phi_k] ; \phi_k \text{ a constant}$$

Finally, we need to consider the case in which $\sigma > 1$ as $u \rightarrow 0$ or as $u \rightarrow \infty$. Let $\phi_k \rightarrow c^*$ as $u \rightarrow 0$, and let $\phi_k \rightarrow \tilde{c}$ as $u \rightarrow \infty$. The growth rate of per capita consumption is:

$$52) \quad \frac{d(C/L)}{dt} = [\lambda/(1 - \phi_k)] + (a - b)$$

For a given ϕ_k (c^* or \tilde{c}) there is a unique (a, b) which maximizes this expression, determined by:

$$53) \quad a^*, b^* \text{ such that } b'(a^*) = -[(h - c^*)/c^*] ; \text{ or} \\ \tilde{a}, \tilde{b} \text{ such that } b'(\tilde{a}) = -[(h - \tilde{c})/\tilde{c}] .$$

Consider the following expression:

$$54) \quad M = \underset{\{a\}}{\text{Max}} \{ [\lambda/(1 - \phi_k)] + (a - b) \} = M(\phi_k)$$

Thus, M is a function of ϕ_k alone. Consider how M changes as ϕ_k changes:

$$55) \quad (dM/d\phi_k) = \{ [(b - a)/(1 - \phi_k)] + [M/(1 - \phi_k)] + \\ [(\phi_k (db/da) + (h - \phi_k))(da/d\phi_k)] / (1 - \phi_k) \}$$

Since M is maximized over a , we can find from equations 53) and 55):

$$56) \quad (dM/d\phi_k) = [(b-a) + M]/(1-\phi_k) = [\lambda/(1-\phi_k)^2]$$

Thus, an extremum occurs at $\lambda = 0$. For the second derivative we find:

$$57) \quad (d^2M/d\phi_k^2) = \{[(b'-1)(da/d\phi_k) + (dM/d\phi_k)]/(1-\phi_k) + [(b-a) + M]/(1-\phi_k)^2\} = [(b'-1)(da/d\phi_k)]/(1-\phi_k)$$

since $(dM/d\phi_k) = 0$ at an extremum.

However, from equation 53) it is clear that:

$$58) \quad (da/d\phi_k) < 0 \quad ; \quad (b'-1) < 0$$

In other words, the larger is the asymptotic value of the output-capital elasticity, the more technical progress that should be allocated to capital-augmenting technology. Therefore:

$$59) \quad M = \underset{a}{\text{Max}}[\{\lambda/(1-\phi_k)\} + (a-b)] \quad ; \quad \text{at } \phi_k = \hat{\phi}_k, a = \hat{a} \text{ and}$$

M is a minimum; $\phi_k > \hat{\phi}_k$ implies $a < \hat{a}$, $(dM/d\phi_k) > 0$;

$\phi_k < \hat{\phi}_k$ implies $a > \hat{a}$, $(dM/d\phi_k) < 0$

It is clear that no interior maximum exists for M (though it is possible that $a=0$ for $\hat{\phi}_k < \phi_k < h$, and $a=A, b=0$ for $\hat{\phi}_k > \phi_k > 0$); boundary maxima occur at $\phi_k = 0$ and at $\phi_k = h$.

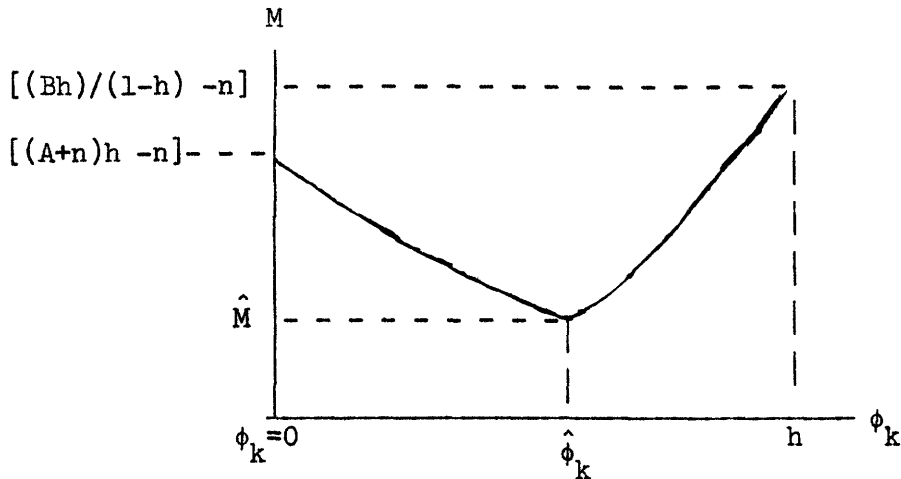
$$60) \quad M(0) = (A+n)h - n \quad ; \quad M(h) = [(Bh)/(1-h)] - n$$

$$M(0) \underset{<}{\underset{>}{\geq}} M(h) \quad \text{as} \quad h \underset{>}{\underset{<}{\leq}} [(A+n-B)/(A+n)] \quad ; \quad h < 1$$

Figure V (page 51) exhibits the behavior of M . Note also that $M(\hat{\phi}_k) \underset{<}{\underset{>}{\geq}} 0$ as $\hat{a} \underset{>}{\underset{<}{\geq}} \hat{b}$.

Suppose, as an example, that $\sigma \rightarrow 1$ as $u \rightarrow 0$, and that $\sigma > 1$ as $u \rightarrow \infty$.

FIGURE V - Graph of the Maximum Rate of Growth of Consumption as a Function of the Output-Capital Elasticity



Let $\phi_k \rightarrow c^*$ as $u \rightarrow 0$:

- 61) $c^* > \hat{\phi}_k$ implies $a^* < \hat{a}$ (and $u \rightarrow \infty$). Hence, choose $a = \hat{a}$
 $c^* = \hat{\phi}_k$ implies $a^* = \hat{a}$ (steady-state). Hence, choose $a = \hat{a}$
 $c^* < \hat{\phi}_k$ implies $a^* > \hat{a}$ (and $u \rightarrow 0$). Hence, choose $a = a^*$

That is, if c^* is greater than $\hat{\phi}_k$, the corresponding a^* would be such that $u \rightarrow \infty$, contradicting the assumption that $u \rightarrow 0$. Therefore, the best that we can do in this case is to choose the steady-state case, $a = \hat{a}$. However, for $c^* < \hat{\phi}_k$ we can do better by letting $a = a^*$ (and $u \rightarrow 0$).

We have already seen what we should do if $\sigma > 1$ as $u \rightarrow \infty$:

- 62) Choose: $a=0, b=B$; $\frac{d(C/L)}{dt} / [C/L] = [(Bh)/(1-h) - n] = M(h)$

Therefore, if $c^* \geq \hat{\phi}_k$, and $\sigma > 1$ as $u \rightarrow 0$, but $\sigma > 1$ as $u \rightarrow \infty$, we would do best by letting $a=0, b=B$ since $M(h) > M(\hat{\phi}_k)$. If $c^* < \hat{\phi}_k$,

then $M(0) > M(c^*) > M(\hat{\phi}_k)$; if $M(0) \leq M(h)$, again we do better by choosing $a=0$, $b=B$. However, if $M(0) > M(h)$, then there exists a ϕ_k^* such that:

$$63) \quad M(\phi_k^*) = M(h) ; \quad c^* < \phi_k^* < \hat{\phi}_k \quad \text{implies} \quad M(c^*) > M(h)$$

$$\phi_k^* < c^* < \hat{\phi}_k \quad \text{implies} \quad M(c^*) < M(h)$$

Consequently, if $c^* > \phi_k^*$, again we do better by letting $a=0$, $b=B$; however, if $c^* < \phi_k^*$, then we will obtain the larger growth rate by choosing $a=a^*$, $b=b^*$ (and $u \rightarrow 0$).

We could consider the other cases in exactly the same way; these results are summarized in Table II. It is obvious that quite a bit of information is needed to make the proper choice (especially when $\sigma \rightarrow 1$).

If $h=1$, then a steady-state is possible only for $b=0$. Otherwise, (assuming b is non-negative) $h > [(a+n-b)/(a+n)]$, and $u \rightarrow \infty$. Therefore, we need not worry about the asymptotic value of σ as $u \rightarrow 0$. In this case it is rather easy to decide how to allocate technical progress. Once again, Table II summarizes these results.

Finally, for the case of increasing returns to scale, a steady-state is not possible under the assumption that b is non-negative unless $\sigma \equiv 1$ - and we have discussed this case earlier. For $\sigma \neq 1$, it must be true that $u \rightarrow \infty$ (excluding the perverse case), and we can readily decide how to allocate technical progress by looking at the growth rates in Table I.

Table II summarizes the decision rules under the criterion of maximizing the asymptotic rate of growth of per capita consumption,¹⁶

¹⁶Or, if the rate of growth is unbounded, we maximize the asymptotic rate of growth of the rate of growth of consumption.

TABLE II - Allocating Factor-Augmenting Technical Progress

- 1) Define \hat{a}, \hat{b} such that: $h = [(\hat{a}+n-\hat{b})/(\hat{a}+n)]$; $b'(\hat{a}) = -[(h-\hat{\phi}_k)/\hat{\phi}_k]$
- 2) Define a^*, \tilde{a} such that: $b'(a^*) = -(h-c^*)/c^*$; $b'(\tilde{a}) = -(h-\tilde{c})/\tilde{c}$

Limit σ as

<u>$u \rightarrow 0$</u>	<u>$u \rightarrow \infty$</u>	<u>Decision Rule</u>
I) <u>$h < 1$</u>	I) <u>$h < 1$</u>	
$\sigma > 1$	$\sigma > 1$	$a=A, b=0$ or $a=0, b=B$ as $h \lesseqgtr [(A+n-B)/(A+n)]$ (If equality, then either $a=A, b=0$ or $a=0, b=B$).
$\sigma > 1$	$\sigma \rightarrow 1$	$\phi_k \rightarrow \tilde{c}$ as $u \rightarrow \infty$: a) $\tilde{c} \leq \hat{\phi}_k \rightarrow a=A, b=0$ b) $\tilde{c} > \hat{\phi}_k, h \leq [(A+n-B)/(A+n)] \rightarrow a=A, b=0$ c) $\tilde{c} > \hat{\phi}_k, h > [(A+n-B)/(A+n)]$, then there exists a $\phi_k^* > \hat{\phi}_k$ such that: i) $\tilde{c} < \phi_k^* < h \rightarrow a=A, b=0$ ii) $h > \tilde{c} > \phi_k^* \rightarrow a=\tilde{a}, b=\tilde{b}$ iii) $\tilde{c} = \phi_k^* \rightarrow$ choose either i) or ii).
$\sigma > 1$	$\sigma < 1$	$a=A, b=0$
$\sigma \rightarrow 1$	$\sigma > 1$	$\phi_k \rightarrow c^*$ as $u \rightarrow 0$: a) $c^* \geq \hat{\phi}_k \rightarrow a=0, b=B$ b) $c^* < \hat{\phi}_k, h \geq [(A+n-B)/(A+n)] \rightarrow a=0, b=B$ c) $c^* < \hat{\phi}_k, h < [(A+n-B)/(A+n)]$, then there exists a ϕ_k^* such that: i) $c^* > \phi_k^* \rightarrow a=0, b=B$ ii) $c^* < \phi_k^* \rightarrow a=a^*, b=b^*$ iii) $c^* = \phi_k^* \rightarrow$ choose either i) or ii)

TABLE II - Continued

Limit σ as				Decision Rule
$u \rightarrow 0$	$u \rightarrow \infty$			
$\sigma > 1$	$\sigma > 1$	$\phi_k \rightarrow c^*$ as $u \rightarrow 0$	and	$\phi_k \rightarrow \tilde{c}$ as $u \rightarrow \infty$.
		a)	$c^* \geq \hat{\phi}_k$, $\tilde{c} \leq \hat{\phi}_k$	$\rightarrow a = \hat{a}, b = \hat{b}$ (Steady-State)
		b)	$c^* \geq \hat{\phi}_k$, $\tilde{c} > \hat{\phi}_k$	$\rightarrow a = \tilde{a}, b = \tilde{b}$
		c)	$c^* < \hat{\phi}_k$, $\tilde{c} \leq \hat{\phi}_k$	$\rightarrow a = a^*, b = b^*$
		d)	$c^* < \hat{\phi}_k$, $\tilde{c} > \hat{\phi}_k$, then either $a = a^*$ or $a = \tilde{a}$ as:
			$\frac{[a^*(h-c^*) + b^*c^* + n(h-1)]}{(1-c^*)} > \frac{[\tilde{a}(h-\tilde{c}) + \tilde{b}\tilde{c} + n(h-1)]}{(1-\tilde{c})}$	
			(If equality, either $a = a^*$ or $a = \tilde{a}$).	
$\sigma > 1$	$\sigma < 1$	$\phi_k \rightarrow c^*$ as $u \rightarrow 0$		
		a)	$c^* < \hat{\phi}_k$	$\rightarrow a = a^*, b = b^*$
		b)	$c^* \geq \hat{\phi}_k$	$\rightarrow a = \hat{a}, b = \hat{b}$ (Steady-State)
$\sigma < 1$	$\sigma > 1$	$a = 0, b = B$		
$\sigma < 1$	$\sigma > 1$	$\phi_k \rightarrow \tilde{c}$ as $u \rightarrow \infty$		
		a)	$\tilde{c} > \hat{\phi}_k$	$\rightarrow a = \tilde{a}, b = \tilde{b}$
		b)	$\tilde{c} \leq \hat{\phi}_k$	$\rightarrow a = \hat{a}, b = \hat{b}$ (Steady-State)
$\sigma < 1$	$\sigma < 1$	$a = \hat{a}, b = \hat{b}$		(Steady-State)
II) <u>$h=1$</u>	II) <u>$h=1$</u>			
NOT	$\sigma > 1$	$a = 0, b = B$		
	$\sigma > 1$	$\phi_k \rightarrow \tilde{c}$ as $u \rightarrow \infty$		
RELE-		a)	$\tilde{c} > \hat{\phi}_k$	$\rightarrow a = \tilde{a}, b = \tilde{b}$
VANT		b)	$\tilde{c} \leq \hat{\phi}_k$	$\rightarrow a = A, b = 0$ (Steady-State)
	$\sigma < 1$	$a = A, b = 0$		(Steady-State)

TABLE II - Continued

Limit σ as

$u \rightarrow \infty$	Decision Rule
III) $h > 1$	III) $h > 1$
$\sigma > 1$	$a=0, b=B$
$\sigma \rightarrow 1$	$\phi_k \rightarrow \tilde{c}$ as $u \rightarrow \infty$
	a) $\tilde{c} \geq 1 \rightarrow a=\tilde{a}, b=\tilde{b}$ (but $k \rightarrow \infty$)
	b) $\tilde{c} < 1 \rightarrow a=\tilde{a}, b=\tilde{b}$
$\sigma < 1$	$a=A, b=0$

assuming that $a, b \geq 0$. As we can see from the above table, a steady-state will be chosen if $h \leq 1$ and $\sigma < 1$ (as $u \rightarrow 0$ and as $u \rightarrow \infty$, where relevant). For $h < 1$, a steady-state solution is not desirable if $\sigma > 1$ either as $u \rightarrow 0$ or as $u \rightarrow \infty$; it may, but need not, be chosen if $\sigma \rightarrow 1$ as $u \rightarrow 0$ or as $u \rightarrow \infty$. Clearly, for the case of increasing returns to scale, a steady-state solution is not possible under our assumptions. If constant returns to scale prevails, and if the elasticity of substitution tends to one as the effective capital-labor ratio tends to infinity, then a steady-state is possible only if the (a,b) transformation curve is not vertical at the axis ($a=A, b=0$) - that is, only if $\hat{\phi}_k > 0$.

In general, then, under the assumptions of this model, and the criterion for allocating technical progress that we have adopted, it is possible that we may seek a steady-state solution. Whether or not such a result is likely depends upon one's belief about the values of the

relevant parameters. Let us now turn our attention to a more important problem - that of maximizing the discounted flow of per capita consumption.

B. Optimal Technical Change à la Nordhaus[32]

In the previous section we have considered how technical progress should be allocated in order to maximize the asymptotic rate of growth of consumption, and we have discussed the circumstances that are likely to lead the planner to choose a steady-state solution. In this section we shall present and extend Prof. Nordhaus' model which demonstrates how technical progress should be allocated between labor-augmenting and capital-augmenting technological change. Since his paper considers only the case of constant returns to scale, we must expand his model in order to permit the production function to assume any (constant) degree of homogeneity.

Not surprisingly, the results (of the model) are not greatly changed by this modification. Naturally, in order to handle the case of increasing returns to scale (and in order to permit a steady-state solution), we must permit negative rates of capital-augmenting technical progress. In fact, the permissible range of negative rates of capital-augmenting technical progress must be unbounded if we are to "permit" the existence of a steady-state solution.

As Prof. Nordhaus points out, his paper (and hence our modification of it) shows that the steady-state is optimal only if the system starts with the "proper" initial conditions. He has not shown (and neither can we) that the steady-state is optimal for arbitrary

initial conditions (assuming the elasticity of substitution is less than one). Since this problem is not readily modified, it must be considered a serious handicap of the analysis. Similarly, we shall see that when the degree of homogeneity of the production function exceeds one, a high rate of time preference is needed to guarantee convergence of the integral and optimality of the steady-state solution. This factor also raises questions about the usefulness of the following analysis when increasing returns to scale are assumed.

The objective of Nordhaus' model (and of ours) is to maximize the discounted stream of per capita consumption. Since our model is essentially identical to his, we shall not repeat all of his equations, but instead we shall list only those equations for which our model differs from that of Prof. Nordhaus. The model is a simple one-sector model with capital- and labor-augmenting technical progress. Adopting his notation, we write:

$$64) \quad Y = F(\lambda K, \mu L) = (\mu L)^h f(x) \quad ; \quad x \equiv [(\lambda K)/(\mu L)]$$

$$65) \quad (\dot{\lambda}/\lambda) = g(\dot{\mu}/\mu) \equiv g(\beta)$$

$$66) \quad \dot{K} = sY - \delta K \quad ; \quad k \equiv (K/L) \quad ; \quad \dot{k} = s\mu^h L^{h(h-1)} f(x) - (\delta+n)k$$

These three equations represent the basic ones of the model; what follows is the Hamiltonian, and the equations obtained from seeking to optimize the Hamiltonian.

$$67) \quad H = e^{-\rho t} [(1-s)\mu^h L^{h(h-1)} f(x) + p_1 \{s\mu^h L^{h(h-1)} f(x) - (\delta+n)k\} + p_2 e^{mt} g(\beta)\lambda + p_3 e^{rt} \beta\mu]$$

In the above equation, k , λ , and μ are the state variables, s and β the

control variables, and the p_i 's the conjugate variables.¹⁷

From the Hamiltonian (Nordhaus' equation 12) we obtain the behavioral equations (his equations 13-17); as always, our formulation reduces to his for the case of constant returns to scale ($h=1$).

$$68) \quad \dot{p}_1 = (\rho + \delta + n)p_1 - v[\mu^{(h-1)}L^{(h-1)}\lambda]f'(x)$$

$$69) \quad \dot{p}_2 = [\rho - m - g(\beta)]p_2 - v[\mu^h L^{(h-1)}\lambda^{-1}]e^{-mt}xf'(x)$$

$$70) \quad \dot{p}_3 = (\rho - r - \beta)p_3 - ve^{-rt}\mu^{(h-1)}L^{(h-1)}[hf - xf']$$

$$71) \quad v = (1 - s + sp_1)$$

$$72) \quad s(t) \text{ maximizes } (1 - s + sp_1) \text{ implies } v = \max(1, p_1)$$

$$73) \quad (\partial H / \partial \beta) = p_2 e^{mt} g'(\beta)\lambda + p_3 e^{rt} \mu = 0; \quad p_i \geq 0, \quad g'' < 0 \rightarrow (\partial^2 H / \partial \beta^2) \leq 0$$

$$74) \quad \lim_{t \rightarrow \infty} [e^{-\rho t} p_1(t)] = \lim_{t \rightarrow \infty} [e^{(m-\rho)t} p_2(t)] = \lim_{t \rightarrow \infty} [e^{(r-\rho)t} p_3(t)] = 0$$

The last equation represents the transversality condition; as is well-known, it is not a necessary condition (in the case of infinite time). Also, of course, the initial conditions must be satisfied.

Following Nordhaus, we seek a stationary solution to the above equations. Letting p_1 be constant and $(\dot{x}/x) = 0$, we find:

$$75) \quad (h-1)(\beta+n) + g(\beta) = 0 \text{ determines } \hat{\beta}, \quad g(\hat{\beta}) \text{ such that:}$$

$$h = [(\hat{\beta} + n - g(\hat{\beta})) / (\hat{\beta} + n)]$$

This solution for $\hat{\beta}$ is unique, given g' , $g'' < 0$, and corresponds to our earlier results. Clearly, if $h > 1$, $g(\hat{\beta}) < 0$; thus, in order to

¹⁷Actually, $p_2 e^{mt}$ and $p_3 e^{rt}$ are the original conjugate variables; the values of m and of r are to be determined.

allow a steady-state solution in the case of increasing returns to scale, it is necessary to extend the transformation curve into negative values of capital-augmenting technical change. From equations 69) and 70) we obtain the stationary values of p_2 and p_3 :

$$76) \quad \mu = \mu^* e^{\hat{\beta}t} \quad ; \quad \lambda = \lambda^* e^{g(\hat{\beta})t} \quad ; \quad L = L_0 e^{nt}$$

$$77) \quad p_2^* = [v(\mu^*)^h (L_0)^{(h-1)} (\lambda^*)^{-1} x f'] / [\rho - \hat{\beta} + g(\hat{\beta})] \quad ; \quad m = [\hat{\beta} - 2g(\hat{\beta})]$$

$$78) \quad p_3^* = [v(\mu^* L_0)^{(h-1)} (h f - x f')] / [\rho - \hat{\beta} + g(\hat{\beta})] \quad ; \quad r = -g(\hat{\beta})$$

These equations are identical to those found by Nordhaus, when $h=1$ and $\hat{\beta} = h$ (the latter h , in his notation, does not represent the degree of homogeneity of the production function), $g(\hat{\beta}) = 0$. Clearly, the non-negativity of p_2^* , p_3^* requires:

$$79) \quad \rho > [\hat{\beta} - g(\hat{\beta})] \quad . \quad \text{From equation 73):}$$

$$80) \quad g'(\hat{\beta}) = -[(1-\alpha)/\alpha] \quad \text{or} \quad \alpha = [1/(1-g'(\hat{\beta}))] \quad ; \quad \alpha \equiv [(\partial Q/\partial K)(K/Q)]/h$$

Equation 80) uniquely determines x^* if σ is everywhere bounded from one.

Again following Nordhaus, we can determine the values of the other parameters. If $p_1 < 1$, $s=0$, $\dot{x} < 0$; and for $p_1 > 1$, $s=1$, and consumption is zero (which must be a minimum if the integral converges). Therefore, for a stationary-solution (which is optimal) we must have that $p_1=1$, and therefore from equation 68):

$$81) \quad (\mu^*)^{(h-1)} \lambda^* = [(\rho + \delta + n) / \{(L_0)^{(h-1)} f'(x^*)\}]$$

Equation 81) can be used to determine λ^* or μ^* ; note that one of them is still undetermined (or both are undetermined, but mutually constrained). For $\dot{x}=0$, using the above results, we find:

$$82) \quad s^* = h\alpha(x^*)\{[\delta+n+\beta-g(\beta)]/[\rho+\delta+n]\} \quad ; \quad \alpha \equiv [(xf')/(hf)]$$

Equations 75) - 82) are the basic equations of the model (corresponding to Nordhaus' 23 - 29) that determine the stationary state values that satisfy all the conditions imposed upon it. Note that not everything is determined - for $h=1$, μ^* and L_0 are free to be determined by initial conditions. For $h \neq 1$, one constraint is imposed upon three of the parameters (μ^* , λ^* , L_0).¹⁸

Before following Nordhaus further and showing that this solution is at least locally optimal for $\sigma < 1$, let us ask what constraints must be placed on ρ , the rate of time preference, in order to permit the optimality of this solution.

First, consider the equation for s^* ; we know that it must be true that the savings rate is less than one. Consider:

$$83) \quad Z \equiv h\alpha = \{(h/[1-g'(\hat{\beta})])\} = \{[\hat{\beta}+n-g(\hat{\beta})]/[(\hat{\beta}+n)(1-g')]\}$$

Therefore, Z depends only upon $\hat{\beta}$, which depends upon h (and $[d\hat{\beta}/dh] > 0$).

$$84) \quad (dZ/d\hat{\beta}) = [(1-Z)/(\hat{\beta}+n)] + [(Zg'')/(1-g')]$$

For $h \leq 1$, $Z < 1$; since Z is continuous in β (the derivative exists everywhere, $\beta > -n$; there may, of course, be an upper bound on β), $Z > 1$ implies $Z = 1$ for some β . But, at $Z = 1$:

$$85) \quad (dZ/d\hat{\beta}) < 0 \quad \text{at } Z = 1$$

¹⁸Earlier in this chapter we saw that for $h > 1$, even if $h = [(a+n-b)/(a+n)]$, there might be no steady-state solution. Equation 81), and a sufficiently large value of ρ (so that $s^* < 1$) eliminate this potential problem.

This implies that Z approaches one from above (since $[d\hat{\beta}/dh] > 0$), contradicting the fact that $Z < 1$ for $h \leq 1$. Thus:

$$86) \quad h\alpha < 1 \quad \text{and} \quad \rho > [\hat{\beta} - g(\hat{\beta})] \rightarrow s^* < 1$$

Similarly, to guarantee the non-negativity of p_2^* and p_3^* we also need $\rho > [\hat{\beta} - g(\hat{\beta})]$. In the steady-state:

$$87) \quad (\dot{C}/C) = \hat{\beta} + nh; \quad \text{therefore,} \quad [(\dot{C}/C) - n] = \hat{\beta} + n(h-1)$$

Convergence of the integral thus requires:

$$88) \quad \rho > [\hat{\beta} + n(h-1)] = \hat{\beta} + n[(-\hat{g})/(\hat{\beta}+n)] = \hat{\beta} - [ng(\hat{\beta})/(\hat{\beta}+n)]$$

For $\hat{g} < 0$ ($h > 1$), equation 86) is a stronger restriction on ρ ; for $\hat{g} > 0$ (and $\hat{\beta} > 0$), equation 88) is the stronger restriction. Finally, for $\hat{g} > 0$, $-n < \hat{\beta} < 0$, equation 86) is again the stronger restriction (but neither is important since $\rho > 0$ will suffice in this case).

Finally, if the transversality conditions are to be satisfied, we need:

$$89) \quad \rho > \text{Max}\{[\hat{\beta} - 2g(\hat{\beta})], -g(\hat{\beta})\}$$

For $h > 1$ ($\hat{g} < 0$), $\rho > [\hat{\beta} - 2g(\hat{\beta})]$ becomes the strongest condition (though it does not necessarily have to be fulfilled). Thus, assuming the transversality conditions must be satisfied, we must place the following restrictions on ρ to guarantee the optimality of the stationary state solution:

$$90) \quad h > 1 \rightarrow \rho > [\hat{\beta} - 2g(\hat{\beta})]; \\ h = 1 \rightarrow \rho > \hat{\beta} \quad \text{where} \quad g(\hat{\beta}) = 0;$$

90) $h < 1$ (and $\hat{\beta}, \hat{g} \geq 0$) $\rightarrow \rho > [(\hat{\beta} - (\hat{g}n))/(\hat{\beta}+n)]$ - this is for
convergence of the integral

$h < 1$ (and $\hat{g} > 0, 0 > \hat{\beta} > -n$) $\rightarrow \rho > -g$

Thus, which condition is the strongest depends upon the value of h ; in general, the restrictions on ρ are not unreasonable for $h \leq 1$. However, for $h > 1$, the necessary value of ρ may be very large indeed, depending upon the value of h and the shape of the transformation curve. In general, for $h > 1$, the convergence of the integral (and the feasibility of the stationary solution) becomes quite suspect indeed.

Let us now return to the question of the optimality of the stationary solution; we shall assume that ρ is sufficiently large to fulfill all the conditions discussed above. The method Nordhaus follows is to linearize the transformation curve (between capital- and labor-augmenting technical change) around the stationary solution and to convert the problem into one which has only one state variable and two control variables.

Since the process would be identical to what Nordhaus has already done, with the exception of allowing for the fact that the degree of homogeneity is not necessarily equal to one, we shall not bother to redo his analysis. Suffice it to say that $\sigma < 1$ is again a necessary condition for a maximum.

Nordhaus shows that the Hamiltonian is concave in k , the state variable, when it is maximized over the control variables - and this suffices to guarantee the local optimality of the solution. For our problem, the work is not quite so simple, but the result is the same. We find:

91) $H^* = \text{Max}[H]$ (control) is concave if and only if $[h - (A+1)] < 0$

where $A \equiv [-g'(\hat{\beta})] > 0$

But we have already shown:

92) $h\alpha(x^*) = [h/(1-g')] < 1 \rightarrow [h - (1-g')] \equiv [h - (A+1)] < 0$

Thus, Nordhaus' solution is equally valid for $h \neq 1$ (except that technical progress is no longer only labor-augmenting) - that is, the stationary solution is optimal if:

- a) $\sigma < 1$
- b) ρ is "sufficiently" large
- c) The initial conditions coincide with the stationary optimal solution.

Condition c) is a very strong one - it tells us nothing about behavior away from the "optimal" solution. Similarly, condition b) can prove quite strong (and quite myopic) for $h > 1$.

Before leaving this section, let us make one further observation. We have already seen that when we considered maximizing the asymptotic rate of growth of consumption, the stationary solution was best for $h \leq 1$, and $\sigma < 1$ as $x \rightarrow 0$ and as $x \rightarrow \infty$. From Table I we can also readily see that for $h > 1$, the stationary solution is also best under this criterion (assuming negative rates of capital-augmenting technical change are permissible) if $\sigma < 1$ as $x \rightarrow 0$ and as $x \rightarrow \infty$. However, as noted earlier in this chapter, it is possible that no steady-state solution occurs in this case. That is:

94) $Q = F(Ke^{gt}, Le^{\beta t})$; if $h = [(\hat{\beta}+n-\hat{g})/(\hat{\beta}+n)] > 1$, then it is possible that: $(\dot{x}/x) = [s(f/x) - (\hat{\beta}+n-\hat{g})] < 0$ for all $x \geq 0$, $s \leq 1$

Obviously, in this case one could not move directly to the stationary solution. However, if the planner could control the allocation of technical progress, he could originally allocate more technological change to capital-augmenting technical progress in order to effectively change the initial conditions for the steady-state problem. As an example:

$$\begin{aligned}
 95) \quad t \leq \tau; \quad g = g^*, \quad \beta^* = 0; \quad \text{therefore, } \lambda = e^{g^*t}, \quad \mu = \mu_0 = 1, \quad t \leq \tau \\
 \dot{x}/x = \{se^{[(h-1)n+g^*]t} f(x)\}/x - (n+\delta-g^*) \\
 t > \tau; \quad \lambda = (e^{g^*\tau})(e^{\hat{g}[t-\tau]}); \quad \mu = e^{\hat{\beta}[t-\tau]}; \quad h = [(\hat{\beta}+n-\hat{g})/(\hat{\beta}+n)] \\
 \dot{x}/x = \{[se^{g^*\tau} f(x)]/[e^{n(1-h)\tau} x]\} - [\hat{\beta}+n+\delta-\hat{g}] \geq 0 \\
 \text{for some } x.
 \end{aligned}$$

In other words, by originally allocating more capital-augmenting technical progress than would be allocated in the steady-state, the planner can guarantee the existence of a steady-state solution. After the initial adjustment period, the economy could then be placed back into its stationary path. Of course, the determination of the optimal τ , of the rates of technical progress during the adjustment period, and of the savings rate (if it is under the planner's control) is precisely the job of the Pontryagin problem. Nevertheless, this section should serve to clarify our earlier remarks and should point out that a steady-state solution exists (under the above assumptions) if the planner has control over the allocation of technological progress (and if negative rates of factor-augmenting technical change are feasible). If not, it is possible (for increasing returns to scale) that no stationary solution exists, even if we can find $\hat{\beta}$, $g(\hat{\beta})$ such that $h = [(\hat{\beta}+n-\hat{g})/(\hat{\beta}+n)]$.

IV. Factor Shares

In this chapter we have seen that if the steady-state conditions are not fulfilled the economy will approach an asymptotic equilibrium in which the growth rates tend to constant limits (if they are finite), assuming that σ is bounded from, or tends to, one. Since this result corresponds to one of the characteristics of the steady-state model, it would be interesting to determine how factor shares behave in this asymptotic equilibrium. However, the determination of factor shares obviously depends upon the pricing assumption that is made - and it is this subject that we shall discuss in this section and, in more detail, in Chapter 3.

One of the principal assumptions of the Bertrand-Vanek analysis is that factors are paid proportionally to their marginal (value) products, with the constant of proportionality being the reciprocal of the degree of homogeneity of the production function. As a consequence of this definition, the factor shares are proportional to the output-factor elasticity for the respective factors (again, the constant of proportionality is the reciprocal of the degree of homogeneity). This definition for factor-pricing, plus Euler's theorem, guarantee that factor shares will add up to one, in consonance with the economic interpretation of these definitions.

It is well-known that when constant returns to scale prevails, the above factor-pricing definition can be justified by certain reasonable assumptions on economic behavior (known as perfect competition). However, if $h \neq 1$, these assumptions prove more dubious, and we can not readily fall back on plausible economic behavior to confirm

or lend credence to this factor-pricing definition.¹⁹

The reason why the factor-pricing assumptions are important is obvious - in an asymptotic "equilibrium", in which the effective capital-labor ratio tends to infinity (or zero), if the elasticity of substitution of the production function is bounded from one, the output-factor elasticity for one of the factors must tend to zero. If we assume the "pseudo" competitive pricing mechanism outlined above, then one of the factor shares must tend to zero, an occurrence that does not appear to be consistent with reality.²⁰

However, as is clear, when the degree of homogeneity is not equal to one, this assumption for factor-pricing no longer seems particularly plausible. If we are interested in constant factor shares, we can directly make this our assumption - by assuming that in the "real" world labor is paid in proportion to its average, not marginal, product (presumably this type of behavior could arise due to market imperfections, such as collective-bargaining). If labor receives a constant fraction of its average product, then constant (and non-zero) factor shares follow by definition. Alternatively, it is possible to show that if monopolistic practices exist, even if $\phi_k \rightarrow 0$, capitalist's share will approach some positive constant limit, due to monopolistic profits. For

¹⁹See footnote 2, page 19, for one possible explanation of this factor pricing-assumption.

²⁰It may not seem reasonable that the output-factor elasticity (for some factor) tends to zero - however, the question is really what is the empirical manifestation of this result. For example, $\phi_k \rightarrow 0$ does not necessarily imply that the MPK tends to zero (if $s_n = 0$, $\sigma < 1$, $\phi_k \rightarrow 0$, but the MPK tends to some positive limit as $u \rightarrow \infty$). If the output-factor elasticities are not directly tied to factor shares, then we have no direct observation of their values.

more discussion on this subject, see Chapter 3 of this thesis.

Certainly there is no theoretical reason (at least, none proven above) to adopt this postulate of average product pricing - but neither does there appear to be a reason for postulating "pseudo-competitive" factor pricing, as Bertrand-Vanek do. However, if $h \leq 1$, it is perhaps more plausible to assume that one factor (probably labor) is paid its full marginal value product, while the other factor (capital) is paid as a residual.²¹ If this were so, then it is possible that both factors would have non-zero and asymptotically constant shares, even if the effective capital-labor ratio tends to zero or infinity. As an example, suppose that labor is paid its full marginal value product, and that $h \leq 1$. We find in that case:

$$\begin{array}{ll}
 96) \quad 1 \geq h > [(a+n-b)/(a+n)] ; \sigma > 1 & \text{Share Labor} \rightarrow 0 \\
 & \text{Share Capital} \rightarrow 1 \\
 & \sigma < 1 \quad \text{Share Labor} \rightarrow h \\
 & \text{Share Capital} \rightarrow (1-h) \geq 0 \\
 & h < [(a+n-b)/(a+n)] \leq 1 ; \sigma > 1 \quad \text{Share Labor} \rightarrow h \\
 & \text{Share Capital} \rightarrow (1-h) \geq 0 \\
 & \sigma < 1 \quad \text{Share Labor} \rightarrow 0 \\
 & \text{Share Capital} \rightarrow 1
 \end{array}$$

Although this definition does not guarantee non-zero factor shares, it does admit of that possibility. And it is, we believe, a more plausible assumption than the Bertrand-Vanek factor-pricing assumption.

²¹If increasing returns to scale prevails, this definition might lead to a situation in which one factor received more than the total output. Hence, the assumption $h \leq 1$. For $h > 1$, it seems likely that oligopolistic situations would arise - for more on this problem, see Chapter 3 of this thesis.

The adoption of either the average-product pricing assumption or the marginal-product pricing assumption in only one market would not alter the basic results of the Bertrand-Vanek growth model. Under the assumption of average-product pricing the factor shares are constant and hence so is the aggregate savings rate (assuming s_k, s_n are constants) - thus, this case is equivalent to the Bertrand-Vanek case in which $s_k = s_n$. Under the assumption of marginal-product pricing in only one factor market (the labor market), the aggregate savings propensity is always positive (assuming $s_k \geq s_n \geq 0, s_k > 0, h < 1$) and depends upon only the output-capital elasticity.

$$97) \quad W = (\partial Q / \partial L) ; \quad s = [s_n(h - \phi_k) + s_k(\phi_k + (1-h))] > 0 ; \quad s_k \geq s_n$$

$$E \equiv [(ds/d\phi_k)(\phi_k/s)] = \{[(s_k - s_n)\phi_k] / [(s_k - s_n)\phi_k + s_k(1-h) + s_n h]\}$$

Therefore, $0 \leq E \leq h < 1$

Since s , the aggregate savings rate, is always positive, and since E , the elasticity of s with respect to ϕ_k , is never greater than one, the Bertrand-Vanek model is essentially unaltered by this alternative pricing assumption. Furthermore, since $s > 0$ ($s_k \geq s_n, s_k > 0$) always, this assumption is asymptotically equivalent to the Bertrand-Vanek model with $s_n > 0$.²²

In summary, since there is (in general) no steady-state, asymptotically the output-factor elasticity of one of the factors will tend to zero if the elasticity of substitution is bounded from one. We

²²In Chapter 3 we shall show that the basic Bertrand-Vanek model is virtually unchanged by various factor-pricing assumptions.

have seen that the Bertrand-Vanek pricing assumption implies that one of the factor shares tends to zero, a result in conflict with reality. However, we have also seen that if constant returns to scale does not occur there is not a strong theoretical justification for adopting the Bertrand-Vanek pricing assumption. Consequently, we have considered two alternative assumptions that might yield non-zero factor shares for each factor. In general, it is possible to assume some combination of these two assumptions:

$$98) \quad W = a_1(Q/L) + a_2(\partial Q/\partial L) \quad ; \quad a_1, a_2 > 0$$

$$[WL/Q] = a_1 + a_2(h - \phi_k) \quad ; \quad 0 < (a_1 + a_2h) < 1$$

In this way we could guarantee that factor shares would be non-zero.

To determine what factor-pricing assumption is most plausible, it is necessary to study the microeconomic behavior of the economy - it certainly does not suffice to study the aggregate production function. However, this constitutes a different direction than that which we choose to follow - our major purpose in the preceeding discussion was simply to illustrate that it is not a necessary (or even logical) consequence of the Bertrand-Vanek model that one of the factor shares must tend to zero.

V. Variable Degree of Homogeneity of the Production Function

In this chapter we have seen that the presence of capital-augmenting technical progress makes a steady-state solution impossible unless the degree of homogeneity of the production function (assumed to

be constant) is equal to a very particular value.²³ Since there is no reason to believe that this singular case should occur, it would seem that the a priori probability of a steady-state solution, assuming that the rates of technical progress are exogenous, is virtually zero. However, we shall show in this section that if the "degree of homogeneity" of the production function is a decreasing function of the effective capital-labor ratio, then the economy will tend to a constant effective capital-labor ratio (barring perverse cases), and that this long-run equilibrium will possess most of the characteristics of a normal steady-state equilibrium. Let us now investigate why this is so.

In a recent article Koji Okuguchi [17] has shown, for a slightly more general production function, that a steady-state will exist only for very special values of the parameters. That is, Okuguchi assumes:

$$99) \quad Q = F(Ke^{\mu t}, Le^{\gamma t}) = (e^{\gamma t} L)^{[a/b]} f(x) ; \quad x \equiv [(Ke^{\mu t}) / (Le^{\gamma t})^{1/b}] ; \\ (\dot{L}/L) = n$$

For $b=1$, $a=h$ (h , the degree of homogeneity in the standard case), this becomes the production function considered in this chapter. Okuguchi shows that, for this production function, a steady-state can exist only if:

$$100) \quad [(1-a)/b] = [\mu/(n+\gamma)] ,$$

²³We have seen that if the production function is Cobb-Douglas a steady-state may exist. Also, we have seen that if the savings rate declines at just the proper rate (if $\delta > 0$), then a steady-state will occur. Neither of these possibilities seems particularly likely to us.

which is analogous to the Bertrand-Vanek condition when $b=1$, $a=h$.

Though this production function is (possibly) different from the one assumed by Bertrand-Vanek, we see that again a steady-state will not normally occur. The question then seems to be - is there some ignored mechanism that promotes the steady-state, or should we abandon the notion of a steady-state?

We have already discussed one possible mechanism - the notion that a trade-off exists between the types of technical progress. If technical progress is then allocated optimally a steady-state will be chosen under certain (fairly plausible) conditions. Nevertheless, it is not very apparent why technical progress would be so allocated in a "free enterprise" economy - there is no microeconomic explanation of this behavior, especially when we dispense with the assumption of constant returns to scale.

Another possibility for promoting the steady-state is that the production function need not be homogeneous, but rather that the degree of homogeneity depends upon the effective capital-labor ratio.²⁴ Thus:

$$101) \quad Q = F(Ke^{bt}, Le^{at}) = (Le^{at})^{h(x)} f(x) ; \quad x = [(Ke^{bt}) / (Le^{at})]$$

This production function, while not a general one, includes the homogeneous production function as a special case. Specifically, suppose h decreases as x increases (as will soon be apparent, this is not an innocent assumption, but rather it is a critical one). In this

²⁴This assumption presents the possibility (likelihood) that the marginal product of one of the factors might be negative, when that factor is varied alone.

case, assuming $h(0) > [(a+n-b)/(a+n)]$ and $h(\infty) < [(a+n-b)/(a+n)]$, there exists a unique x^* such that:

$$102) \quad h(x) \begin{cases} > \\ < \end{cases} [(a+n-b)/(a+n)] \quad \text{as} \quad x \begin{cases} < \\ > \end{cases} x^*$$

For simplicity, assume workers and capitalists save at the same (constant) rate:

$$103) \quad k \equiv (\dot{x}/x) = \{[se^{\lambda t} f(x)]/x\} - [(a+n-b)+c]; \quad \lambda \equiv [(a+n)h - (a+n-b)]$$

Depending upon the values of the parameters, it may be that:

$$104) \quad k(x^*) \begin{cases} > \\ < \end{cases} 0 \quad \text{as} \quad \{[sf(x^*)/x^*] - (a+n-b+c)\} \begin{cases} > \\ < \end{cases} 0$$

Thus, even under this assumption, it is unlikely that x^* is an equilibrium in the sense that if we started at $x(0) = x^*$, we would probably not remain there for all time. However, suppose we follow our earlier procedure and consider how (\dot{x}/x) changes over time:

$$105) \quad \dot{k} = (k+a+n-b+c)\{[(a+n)h-(a+n-b)] + k[(\phi_k-1) + (a+n)h\eta]\}$$

$$\text{where}^{25} \quad \phi_k = (xf'/f), \quad \eta = [dh/dx][x/h] < 0; \quad h \geq \phi_k \geq 0$$

Let us now investigate what happens to x (the effective capital-labor ratio) asymptotically. In order to do this, we shall divide the analysis into two parts:

$$i) \quad x(0) < x^* \quad \rightarrow \quad h > [(a+n-b)/(a+n)]$$

²⁵Note that ϕ_k is not the output-capital elasticity of the entire production function since it does not include the changes in h due to changes in x . ϕ_k corresponds to the output-capital elasticity, assuming h is constant. Thus $\phi_k > 0$ does not imply that the $MPK > 0$.

$$\text{ii) } x(0) > x^* \rightarrow h < [(a+n-b)/(a+n)]$$

where $x(0)$ is the initial value of x . Consider the first case:

$$\text{i) } x(0) < x^*$$

Again, there are several cases that need to be considered.

First, let us assume that $k(0) > 0$ ($k(0)$ is the initial rate of growth of x - naturally, it depends upon $x(0)$, as well as on all the parameters); if this is so, then x will initially increase. However, from 105) we can see that once $k > 0$ ($x < x^*$), it must remain positive since, at $k = 0$:

$$105) \quad \dot{k} = (k+a+n-b+c)[(a+n)h-(a+n-b)] > 0 \quad \text{at } k=0 \quad \text{for } x < x^*. \quad 26$$

Thus, if initially $k > 0$, it must remain positive (for $x < x^*$), and hence $x \rightarrow x^*$.

Next, suppose $k(0) < 0$ - it is necessary to further subdivide this case into two parts:

$$\text{a) } \phi_k \leq 1$$

$$\text{b) } \phi_k > 1. \quad 27$$

Suppose $\phi_k \leq 1$ - then, from 105) for $k < 0$, it is clear that $\dot{k} > 0$ for $x < x^*$ ($\eta < 0$). Consequently, k increases (though x will initially decrease); assuming ϕ_k remains less than or equal to one ($\sigma > 1$), k remains positive, and hence k must eventually reach zero. But, as we

²⁶For $s > 0$, x finite, from 103) we can see that $(k+a+n-b+c) > 0$.

²⁷Since $\phi_k \leq h$, it follows that for x near x^* , $h < 1$ (assuming $b > 0$). Therefore, $\phi_k > 1$ is possible only for x "sufficiently small" (and less than x^*), and it implies that $\sigma < 1$. We note that this case ($\sigma < 1$, $h > 1$) corresponds to one of the perverse cases discussed earlier in this chapter.

have already seen, $\dot{k} > 0$ at $k=0$ ($x < x^*$); therefore, k becomes positive, x increases, and we can return to the analysis of our previous paragraph.

Finally, suppose $\phi_k > 1$ initially (x small, $\sigma < 1$). From 105), for $k < 0$:

$$105) \quad \dot{k} = (k+a+n-b+c)\{[(a+n)h-(a+n-b)] + k(a+n)hnt + k(\phi_k-1)\}$$

Thus, it is possible that $\dot{k} < 0$ for $\phi_k > 1$. If this occurs, k decreases (towards its lower bound, $-c$), as does x (towards zero).

However, if η , the elasticity of the degree of homogeneity with respect to x , is bounded below zero ($\eta \leq \epsilon < 0$), then eventually \dot{k} must become positive.²⁸ Once this occurs, k increases, and it tends towards zero; but, as before, $\dot{k} > 0$ at $k=0$ ($x < x^*$), so k must become positive, remain positive, and consequently x tends towards x^* . Once again, therefore, we can return to our prior analysis.

On the other hand, if $\eta \rightarrow 0$ as $x \rightarrow 0$, it is possible that $\dot{k} < 0$ for all time, and consequently $k \rightarrow (-c)$ and $x \rightarrow 0$. Obviously, this case corresponds to the perverse one discussed earlier in this chapter since, if $\eta \equiv 0$, then for $h > 1$, $\sigma < 1$, it is possible that $x \rightarrow 0$ (instead

²⁸ $\dot{k} > 0$ if $[(a+n)h-(a+n-b) + k(a+n)hnt + k(\phi_k-1)] > 0$. For $x < x^*$, $[(a+n)h-(a+n-b)] > 0$, and therefore $\dot{k} > 0$ if:

$$[(a+n)hnt + (\phi_k-1)] < [(a+n)hnt + (h-1)] < 0 \quad (\text{for } k < 0).$$

But this last expression will be negative if:

$$t > [(1-h)/\{(a+n)h\eta\}] = [(h-1)/\{(a+n)h|\eta|\}] \rightarrow \dot{k} > 0$$

If $|\eta| \geq \epsilon > 0$, it follows that such a t must exist, and eventually \dot{k} must become positive.

of $x \rightarrow \infty$), as discussed earlier.

Therefore, barring this perverse case, we can see that if $x(0) < x^*$, eventually $k [= (\dot{x}/x)] > 0$, and x must eventually increase. Furthermore, for $x < x^*$, $\dot{k} > 0$ at $k=0$, and thus x must tend to x^* . Similarly, it can be shown that if $x(0) > x^*$, then eventually x must decrease and approach x^* .²⁹

What happens as $x \rightarrow x^*$? Again, there are several possibilities, depending upon whether $k(x^*) \begin{matrix} > \\ < \end{matrix} 0$. We have seen that (barring the perverse case), for $x(0) < x^*$, eventually $k(x) > 0$, and $x \rightarrow x^*$. If $k(x^*) > 0$, it follows that x not only reaches x^* , but eventually it will exceed x^* . By continuity, for t sufficiently large, if $k(x^*) > 0$, there exists an \tilde{x} such that:

$$k(x) \begin{matrix} > \\ < \end{matrix} 0 \quad \text{as} \quad x \begin{matrix} < \\ > \end{matrix} \tilde{x}$$

Furthermore, as $t \rightarrow \infty$, it is clear that $\tilde{x} \rightarrow x^*$ ($\eta \neq 0$). Therefore, x must tend to \tilde{x} , which in turn tends to x^* , and thus x tends to x^* asymptotically (barring the perverse case).

Similarly, if $k(x^*) < 0$, for t large enough (so that nt "dominates" $[\phi_k - 1]$), there exists an $\tilde{x} < x^*$ such that $k(x) \begin{matrix} > \\ < \end{matrix} 0$ as $x \begin{matrix} < \\ > \end{matrix} \tilde{x}$, and again (after sufficient time has elapsed) $x \rightarrow \tilde{x} \rightarrow x^*$, so that x converges to x^* (again, barring the perverse case). Finally,

²⁹If $b \geq 0$, then $h < 1$ for $x > x^*$, so from 105), $x > x^*$ implies $\dot{k} < 0$ for $k > 0$ (since $\phi_k \leq h < 1$) and no perverse case is possible. However, if $b < 0$ and $h[x(\infty)] > 1$ and $|\eta| \rightarrow 0$, then a perverse case may arise for $\sigma > 1$ ($\phi_k > 1$), so that $x \rightarrow \infty$. Again, this case is equivalent to one of the perverse cases ($\delta < 0$, $h > 1$, $\sigma > 1$) discussed earlier in this chapter.

if by chance $k(x^*) = 0$, then \tilde{x} coincides with x^* for all time. In this case, if $x(0) = x^*$, x remains at x^* (barring any unforeseen shocks to the system), and if $x(0) < x^*$ [$x(0) > x^*$], then $x \rightarrow x^*$ and never $x > x^*$ [$x < x^*$].

In summary, if h decreases monotonically as x increases, then eventually x must approach the finite, non-zero effective capital-labor ratio (x^*) such that: $h(x^*) = [(a+n-b)/(a+n)]$, assuming that η [$= (dh/dx)(x/h)$] is bounded below zero. If $x(0) < x^*$, then $x \leq x^*$ always, if $k(x^*) \leq 0$, whereas if $k(x^*) > 0$ then, for large t , $x \geq x^*$. Similarly, if $x(0) > x^*$, then $x \geq x^*$ always if $k(x^*) \geq 0$; whereas if $k(x^*) < 0$, eventually $x \leq x^*$. Needless to say, the path of x need not be monotonic.

Therefore, we have shown that if the "degree of homogeneity" of the production function depends upon the effective capital-labor ratio in the manner defined above, then the economy will approach a finite, non-zero effective capital-labor ratio. Furthermore, all the growth rates will approach constant, finite rates as $x \rightarrow \tilde{x}$ ($\rightarrow x^*$). For example:

$$106) \quad Q = e^{(a+n)ht} f(x) \text{ implies:}$$

$$(\dot{Q}/Q) = (a+n)h[1 + \eta(\dot{x}/x)t] + \phi_k(\dot{x}/x)$$

From 105), as $\dot{x} \rightarrow 0$ (as it must, since $x \rightarrow x^*$, x^* constant), we find:

$$107) \quad k = (\dot{x}/x) \rightarrow \{[(a+n)h - (a+n-b)] / [(1-\phi_k) - ahnt]\} \rightarrow 0 \text{ as } x \rightarrow x^*$$

Therefore:

$$108) \quad kt = (\dot{x}/x)t \rightarrow \{[(a+n)h - (a+n-b)] / [(1-\phi_k)/t - ahn]\} \rightarrow 0 \text{ as } x \rightarrow x^*$$

Hence, $[\phi_k(\dot{x}/x)]$ and $[t(\dot{x}/x)]$ tend to zero as t tends to infinity and x tends to x^* , and consequently we find asymptotically:

$$109) \quad (\dot{Q}/Q) \rightarrow (a+n)h \quad ; \quad [(\dot{C}/C) - n] \rightarrow [(a+n)h-n] = (a-b)$$

Therefore, in this asymptotic steady-state, consumption per capita will be increasing, remaining constant, or decreasing as the rate of labor-augmenting technical progress exceeds, equals, or is less than the rate of capital-augmenting technical progress. Similarly, we could readily exhibit the growth rates of the other variables. Also, since $k \rightarrow 0$ and $h \rightarrow h^*$ (the "degree of homogeneity" will be asymptotically constant), the marginal product of each factor will be positive, and consequently this model will (asymptotically) exhibit almost all of the "normal" steady-state properties (the output-factor elasticities are constant, as is the marginal product of capital, and so forth).³⁰

Clearly, if $\eta > 0$, then either this equilibrium is unstable (if $k(x^*) = 0$), or else no equilibrium exists, and the effective capital-labor ratio will tend to either zero or infinity. Needless to say, we have not demonstrated why h should behave in the "desired" manner. However, it does not seem unreasonable to us to assume that h is not constant everywhere, especially if some factor of production (other than capital or labor) is fixed or else varies exogenously; just how the degree of homogeneity varies with the effective capital-labor

³⁰Note, however, that the effective capital-labor ratio to which the system tends, and hence the long-run value of the marginal product of capital, is independent of the savings rate, unlike the normal steady-state result. In this model, the savings rate only effects how quickly the economy tends to its steady-state equilibrium (barring the perverse cases).

ratio is another question.

VI. Conclusion

In this chapter we have discussed the Bertrand-Vanek model, in which factor-augmenting technical progress can occur for each factor, and in which the degree of homogeneity of the production function is a constant, though not necessarily equal to one. The basic conclusion from this model is that a steady-state is very unlikely to occur, and thus we must be content with considering the asymptotic growth path. In this "asymptotic equilibrium" the elasticity of output with respect to one of the factors must tend to zero if the elasticity of substitution of the production function is bounded from one, and hence so must the share of that factor tend to zero, if we adopt the Bertrand-Vanek factor-pricing assumption. However, for the case of non-constant returns to scale there appears to be no logical reason to choose between the various possible pricing definitions (at least at the macroeconomic level), and thus we could define (assume) a pricing scheme in which both factor shares would remain non-zero.

Also, if we postulate a Kennedy-Von Weizsäcker model in which there exists a trade-off between capital- and labor-augmenting technical progress, we find that for $\sigma < 1$ a steady-state is a (or may be a) desirable property that the economy should seek to obtain. However, this approach presupposes the existence of a "central planner" (or some invisible hand); in order to discuss this problem for a free enterprise economy, it would be necessary to better define the microeconomic properties of that system. When we do not have constant returns to scale in the production function this task can become quite

complicated (and indeterminate).

Finally, we have exhibited a special model in which a steady-state will probably be achieved. If the "degree of homogeneity" of the production function is a monotonically declining function of the effective capital-labor ratio, then (excluding two perverse cases) the system will tend to a finite, non-zero effective capital-labor ratio, and the growth rates for the variable will be (asymptotically) constant and finite. The ultimate equilibrium effective capital-labor ratio is independent of the savings propensity, but it does depend upon the parameters of the problem, including the rates of capital- and labor-augmenting technical progress, the rate of growth of population, and the relationship between the "degree of homogeneity" of the production function and the effective capital-labor ratio.

We now leave our study of the one-sector model with persistent doubts as to the likelihood of the occurrence of a steady-state equilibrium. Let us now turn our attention to two-sector models of economic growth.

Chapter 2. Two-Sector Models of Unbalanced Growth

I. Introduction

In the past chapter we have followed Prof. Vanek's approach and have investigated the asymptotic behavior of variables when Hicks neutral (or any capital-augmenting) technical progress is present in a one-sector model. In this chapter we plan to extend the analysis by considering two related problems:

- 1) Hicks neutral technical progress in the investment sector
- 2) Harrod neutral technical progress in only one sector

It is well-known that if there is Hicks technical progress in the consumption good sector then a steady-state does indeed exist (if Harrod neutral technical progress occurs at the same rate in each sector); the problem of Hicks neutral technical progress in the investment sector is like (though more difficult than) the problem studied in the first chapter. On the other hand, the problem of Harrod neutral technical progress in only one sector (or at different rates in the two sectors) is a different problem since it necessitates factor reallocation between the two sectors, even if the aggregate capital-labor ratio (or effective capital-labor ratio) were held fixed. At the end of this chapter we shall show how, by using the analysis from Parts I and II, any type of factor-augmenting technical progress can be treated.

Before turning to the analysis, let us make a few remarks. Just as in the one-sector model, where the existence of a steady-state hinged upon a very particular degree of homogeneity of the production function, given the other parameters, the corresponding constraints for the two-

sector model seem just as unreasonable. That is, assuming both production functions are homogeneous of degree one, then, for a steady-state to exist it must be true that Hicks technical progress, if it occurs at all, is restricted to the consumption goods sector; and that Harrod neutral technical progress, if present at all, must occur at the same rate in each sector. To us these assumptions seem quite strong and unwarranted (until some mechanism can be shown to exist that causes this pattern of technical progress). Therefore, we consider it quite important to examine what happens to the economy if the steady-state conditions are not fulfilled, and to consider how this economy would differ from the textbook steady-state economy.

As we shall see, the non-steady-state economy will tend to a state in which the growth rates of the variables (if finite) approach constant limits, assuming that the elasticities of substitution are bounded from, or tend to, one. However, this asymptotic equilibrium deviates from the traditional steady-state results in several respects, not the least of which concern factor shares in each sector (and for the economy as a whole), and the asymptotic value of the marginal product of capital. These differences of the asymptotic equilibrium from the steady-state results, and the implausibility of the steady-state path itself, seem to pose a difficult dilemma for modern growth theory. We shall have more to say on this topic in Chapter 3.

Let us now turn to our analysis of the two-sector model. In this first section we shall investigate the problem of Hicks neutral technical progress in only the investment sector, a problem that is quite similar to the one-sector model studied in Chapter 1.

II. Hicks Technical Progress in the Investment Sector

In this section we shall attempt to determine what happens to the economy if Hicks neutral technical progress occurs only in the investment sector. To investigate this problem we shall use a traditional two-sector model (such as Uzawa, [58], [59]), and we shall employ the approach Vanek used in studying the one-sector model [4]. As we shall see, our results in this case do not differ greatly from those found in the one-sector model.

Specifically, we assume that there are two sectors - the consumption good sector (C), and the investment good sector (M). The production functions are assumed to be homogeneous of degree one in capital and labor, and Hicks technical progress is assumed to occur at rate A in sector M (whether or not it occurs in the consumption good sector is immaterial). Thus, using traditional notation:

- 1) $C = F_c(K_c, L_c) = L_c f_c(k_c) = L\gamma f_c(k_c)$
- 2) $M = e^{At} F_m(K_m, L_m) = L_m e^{At} f_m(k_m) = L(1-\gamma)e^{At} f_m(k_m)$
- 3) $\gamma \equiv (L_c/L)$, $(1-\gamma) \equiv (L_m/L)$; $k_i \equiv (K_i/L_i)$; $\gamma = [(k-k_m)/(k_c-k_m)]$

Since the rewards to factors must be equated in each sector (under competition), it is found that the capital-labor ratio in each sector depends only upon the wage-rental ratio (ω):

$$4) \quad \omega = [(f_c - k_c f'_c)/f'_c] = [(f_m - k_m f'_m)/f'_m]; \quad f'_i > 0, \quad f''_i < 0.$$

The Inada conditions are assumed to hold (when possible; for a C.E.S. function, $\sigma \neq 1$, not all four conditions can be satisfied). Also, so that we need not worry about uniqueness of equilibrium, the capital-intensity

condition is assumed:

$$5) \quad k_c \geq k_m \quad \text{for all } \omega.$$

As is well-known, it would suffice to postulate that the elasticity of substitution in the consumption good sector (σ_c) is greater than one [17]; however, there are times when we wish to explore the case in which $\sigma_c < 1$ - hence we assume that the factor-intensity condition holds (as ω becomes large, this implies that $\sigma_c \geq \sigma_m$).

Furthermore, we assume that the consumption good is the numeraire, so that P , the price level, is the price of the investment good in terms of the consumption good. Since the return on capital must be the same in each sector, we find:

$$6) \quad P = [(e^{-At} f'_c)/f'_m] ; \quad P \equiv P_m ; \quad P_c \equiv 1$$

When it is not ambiguous we shall omit the independent variable - thus, $[f'_c(k_c)] \equiv f'_c$.

Finally, to complete the model we need a savings assumption. It is possible to follow Vanek and assume that capitalists and workers have different savings behavior; or, alternatively, we could assume that they both save at the same rate. As we have seen in the one-sector model, there is no asymptotic difference in these cases as long as workers do some savings. Therefore, we shall investigate two cases:

$$a) \quad s_k = s_n = s$$

$$b) \quad s_k > s_n = 0$$

We omit the case $s_k > s_n > 0$; the reason that this case is unimportant is that, clearly: $s_k \geq s \geq s_n$ (where s is the average savings rate for

the economy), and as the capital-labor ratio tends to infinity, s will approach some positive limit (given that the elasticity of substitution is bounded from, or tends to, one). Consequently, this case is asymptotically equivalent to case a). On the other hand, if case b) pertains, then s may approach zero, and since the essential problem in this model is that physical investment (for a constant s) tends to increase forever, allowing s to approach zero may afford some new behavior.

We could, if we wished to complicate the model, assume that the owners of capital in the consumption goods sector have a different savings propensity than the owners of capital in the investment sector (for example, Stiglitz [52]). However, since in our model capital is completely malleable, we really see no justification for this complicating assumption.

The savings rate allows us to complete our model, and determines our basic equations - the market equilibrium equation and the capital-accumulation equation. Under the two savings assumptions (for the rest of this part of the paper, a subscript a denotes that $s_k = s_n = s$, whereas a subscript b denotes that $s_k > s_n = 0$; naturally, we assume that savings equals investment) we have the following relationships (for simplicity, we assume that there is no depreciation):

$$7a) \quad S = s(C + PM) = PM = PK$$

$$7b) \quad S = \hat{s}(C + PM) = PM = PK \quad ; \quad \hat{s} = s_k \text{ (share capital)} = s_k [k / (\omega + k)]$$

Using this equation, we find our market equilibrium equation, which must hold at all times:

$$8a) \quad s\gamma(\omega+k_c) - (1-s)(1-\gamma)(\omega+k_m) = 0$$

$$8b) \quad s_k k \gamma(\omega+k_c) - (\omega+k-s_k k)(1-\gamma)(\omega+k_m) = 0$$

These equations define ω as a function of k (if they are monotonic - $k_c \geq k_m$ suffices); note that time does not appear explicitly in either equation.

Assuming that investment equals savings, and that there is no depreciation, we find the following equations:

$$9a) \quad (\dot{K}/K) = \{[(1-\gamma)e^{At} f_m] / k\}$$

$$9b) \quad (\dot{K}/K) = \{[(1-\gamma)e^{At} f_m] / k\}$$

These are the capital-accumulation curves; since s does not appear explicitly, they appear to be the same. However, since s affects the relationship between ω and k , they are not, in fact, the same curves.

Following Vanek's analysis, we define:

$$X \equiv (\dot{K}/K) ; \quad [(k/k) = X - n] ; \quad \text{we seek } \bar{X} \text{ such that } (\dot{X}/X) = 0.$$

In general, \bar{X} will depend upon k . \bar{X} is what Vanek refers to as a "quasi-asymptote" for the rate of growth of capital.

$$10) \quad (\dot{X}/X) = \left\{ \left[\frac{d(1-\gamma)}{dt} \right] \left[\frac{1}{(1-\gamma)} \right] + A + \alpha_m \left(\frac{\dot{k}_m}{k_m} \right) - (X-n) \right\}$$

where α_m is the competitive share of capital in the investment sector.

From equation 8a) or 8b), we can determine ω as a function of k , and hence we can calculate the total derivatives needed in 10):

$$11) \quad \left(\frac{\dot{k}_m}{k_m} \right) = \left[\left(\frac{dk_m}{d\omega} \right) \left(\frac{\omega}{k_m} \right) \right] \left[\left(\frac{d\omega}{dk} \right) \left(\frac{k}{\omega} \right) \right] \left(\frac{k}{k} \right) = (\sigma_m / \sigma) (X-n)$$

$\sigma \equiv \left[\left(\frac{dk}{d\omega} \right) \left(\frac{\omega}{k} \right) \right]$, is the aggregate elasticity of

substitution for the economy as a whole.

Similarly, using the definition of $(1-\gamma)$, we find its total derivative:

$$11') \quad \left\{ \frac{d(1-\gamma)}{dt} \right\} / (1-\gamma) = \left\{ \left[\gamma \sigma_c k_c + (1-\gamma) \sigma_m k_m \right] (1/\sigma) - k \right\} (X-n) / (k_c - k)$$

From equations 10) through 11') we determine (\dot{X}/X) :

$$12) \quad (\dot{X}/X) = (TX - Tn + A) \quad \text{where:}$$

$$13) \quad T = \left[\sigma_c k_c \gamma + \sigma_m (1-\gamma) \{ k_m (1-\alpha_m) + k_c \alpha_m \} - k_c \sigma \right] / [\sigma (k_c - k)]$$

Defining \bar{X} to be the locus such that $(\dot{X}/X) = 0$, we find:

$$14) \quad \bar{X} = n - (A/T) \quad ; \quad T \neq 0$$

(If $A=0$, $\bar{X}=n$, the normal steady-state case). Substituting back into equation 12):

$$15) \quad (\dot{X}/X) = -T(\bar{X}-X)$$

If $T < 0$, then whenever $\bar{X} > X$, $(\dot{X}/X) > 0$, and X increases; if $\bar{X} < X$, $(\dot{X}/X) < 0$, and X will decrease. Thus, whenever $T < 0$, the \bar{X} locus is an asymptote in the sense that, whenever $X > \bar{X}$, X decreases, and whenever $X < \bar{X}$, X increases. If $T > 0$ (as we shall see, this is not possible for $k_c \geq k_m$ or $\sigma_c \geq 1$), this relationship is reversed, and X moves away from the \bar{X} locus. Thus, the sign of T is critical. In order to find T , we must first calculate σ ; since the values of T and σ depend upon the savings assumption, we shall get different results for our two cases.

$$16a) \quad \sigma_a = \frac{[(1-s)s\omega(k_c - k_m)^2 + \sigma_c k_c (\omega + k_m)^2 (1-s) + \sigma_m s k_m (\omega + k_c)^2]}{[s(k_c - k_m) + (\omega + k_m)][k_c (\omega + k_m) - s\omega(k_c - k_m)]}$$

$$16b) \quad \sigma_b = \frac{[\omega s_k (k_c - k_m) + \sigma_c (\omega + k_m) (\omega + k_m - s_k k_m) + \sigma_m k_m s_k (\omega + k_c)]}{(\omega + k_m) [(\omega + k_m) + s_k (k_c - k_m)]}$$

Clearly, $\sigma_a > 0$ everywhere; this need not be true for σ_b ($\sigma_b < 0$ implies that the market equilibrium curve "bends back" in the (k, ω) plane - that is, to each k , there may be more than one ω satisfying the market equilibrium conditions. In this case we can not express ω as a unique function of k). However, clearly $\sigma_b > 0$ if (see Drandakis [17] or Burmeister [7]):

$$i) \quad k_c \geq k_m \quad \text{or} \quad ii) \quad \sigma_c + \sigma_m \geq 1$$

(Since, as we shall see, for $T_a < 0$ it suffices that $k_c \geq k_m$ or $\sigma_c \geq 1$, we adopt the factor-intensity hypothesis so that we are free to study different values of the elasticities). Note that if $\sigma_c = \sigma_m = 1$, then $\sigma = 1$ in both cases; otherwise, the value of σ will depend upon ω .

With this information, we can now determine the value of T ; observe that T will appear as a function of ω , but since, from the market equilibrium equation, ω depends upon k , T depends upon k .

$$17a) \quad T_a = \frac{\{-\omega[(1-s)k_c(\omega+k_m)(k_c-k_m) + (1-s)k_c(\omega+k_m)^2\sigma_c + \sigma_m s k_m (\omega+k_c)^2]\}}{\{(\omega+k_c)(\omega+k_m)[k_c(\omega+k_m) - s\omega(k_c-k_m)]\}\sigma_a}$$

$$17b) \quad T_b = [(-\omega)/\{(\omega+k_m)\sigma_b\}]$$

Since the sign of T_a is related to the condition that a unique equilibrium exists ($A=0$), $T_a < 0$ whenever:

$$i) \quad k_c \geq k_m \quad \text{or} \quad ii) \quad \sigma_c \geq 1$$

Clearly, $T_b < 0$ whenever $\sigma_b > 0$; and we have already seen under what

conditions σ_b is positive (note that T_b may be negative while T_a is positive).

Assuming the factor-intensity condition holds, we have:

$$T_a, T_b \leq 0 \text{ always, and hence } \bar{X} = [n - (A/T)] \geq n \text{ (} A > 0 \text{)}.$$

A. Special Cases

Before considering the general case, we shall consider three special cases:

$$\text{i) } k_c = k_m \quad (\rightarrow \sigma_c = \sigma_m ; \sigma_c \neq 1)$$

$$\text{ii) } k_c = k_m \quad \text{and} \quad \sigma_c = \sigma_m = 1$$

$$\text{iii) } k_c \neq k_m, \quad \text{but} \quad \sigma_c = \sigma_m = 1$$

Case i) is comparable to the normal one-sector model except that technological change occurs only for the investment good, so that the price of this good will decline over time. On the other hand, cases ii) and iii), in which both production functions are Cobb-Douglas, are very special cases since steady-states will occur in each of these two cases (for any type of factor-augmenting technical change). Let us now briefly consider these special cases.

$$\text{i) } k_c = k_m \quad (\text{implies } \sigma_c = \sigma_m)$$

From 17a), by substituting in for σ_a , we find that whenever $k_c = k_m$ (and $\sigma_c = \sigma_m$):

$$18a) \quad T_a = [(-\omega)/(\omega+k)] \quad ; \quad \text{similarly,}$$

$$18b) \quad T_b = [(-\omega)/\{\sigma_c(\omega+k)\}]$$

since $k = k_c = k_m$ and $\sigma = \sigma_c = \sigma_m$. For T_a we find:

$$19a) \quad (dT_a/d\omega) \begin{matrix} > \\ < \end{matrix} 0 \quad \text{as} \quad \sigma \begin{matrix} > \\ < \end{matrix} 1$$

The result for T_b is not readily ascertainable since, in general, it will depend upon the third derivative of the production function. If we assume that σ_c (and σ_m) is a constant, then the condition for T_b is identical to 19a):

$$19b) \quad (dT_b/d\omega) \begin{matrix} > \\ < \end{matrix} 0 \quad \text{as} \quad \sigma \begin{matrix} > \\ < \end{matrix} 1 \quad \text{for} \quad \sigma_c, \sigma_m \text{ constant}$$

However, we are more interested in the asymptotic behavior of T_a, T_b (given that they are negative elsewhere). Clearly, under competitive pricing:

$$20a) \quad T_a = -(1-\alpha) ; \quad 20b) \quad T_b = -[(1-\alpha)/\sigma_c] , \quad \text{where } (1-\alpha)$$

is the aggregate share of labor in the economy. Since both T_a and T_b are negative, but finite, everywhere it follows that $\bar{X} > n$ everywhere (and hence k tends to infinity). Therefore:

$$21a) \quad \sigma > 1 \rightarrow [\bar{X} \rightarrow \infty]; \quad \sigma < 1 \rightarrow [\bar{X} \rightarrow (n+A)]; \quad \sigma \rightarrow 1 \rightarrow [\bar{X} \rightarrow \{n + A/(1-\alpha)\}]$$

$$21b) \quad \sigma > 1 \rightarrow [\bar{X} \rightarrow \infty]; \quad \sigma < 1 \rightarrow [\bar{X} \rightarrow (n+A\sigma)]; \quad \sigma \rightarrow 1 \rightarrow [\bar{X} \rightarrow \{n + A/(1-\alpha)\}]$$

(α is the asymptotically constant share of capital as $\sigma \rightarrow 1$).

Thus, for $\sigma > 1$, X will grow without bound, whereas for $\sigma \leq 1$, X has a finite limit to which it tends. For $\sigma < 1$, the asymptotic growth rate is larger under the first savings assumption, while for $\sigma \rightarrow 1$, the two growth rates are the same.

Note that this result corresponds to our one-sector result that, for $h = 1$, X grows without bound whenever $\sigma > 1$, whereas for $\sigma \leq 1$, it approaches a finite limit.

$$\text{ii) } k_c = k_m ; \sigma_c = \sigma_m = 1$$

The case $\sigma = 1$ is a rather special case since $\bar{X} = [n + A/(1-\alpha)]$ is constant for all k , and consequently a steady-state equilibrium exists, rather than just an "asymptotic equilibrium". Thus:

$$22) \quad M = e^{At} K_m^\alpha L_m^{1-\alpha} = K_m^\alpha [L_m e^{\{At/(1-\alpha)\}}]^{1-\alpha}$$

Note that the growth rate corresponds to the rate of growth of population plus the rate of Harrod neutral technical progress in M. Technical progress takes place only in M; however, for sector C we can write ($k_c = k_m$):

$$23) \quad C = K_c^\alpha L_c^{1-\alpha} = e^{-At} K_c^\alpha [L_c e^{\{At/(1-\alpha)\}}]^{1-\alpha}$$

Therefore, we can envision the situation as one in which Harrod neutral technical progress takes place at the same rate in each sector, as well as Hicks neutral technical progress at a negative rate in sector C (and hence a steady-state exists).

Therefore, in this special case a steady-state exists, and in that steady-state:

$$24) \quad \bar{k} = [K/(L e^{\{A/(1-\alpha)\}t})] \quad \text{is constant}$$

$$25) \quad (\dot{K}/K) = n + [A/(1-\alpha)] ; \quad (\dot{C}/C) = n + [(\alpha A)/(1-\alpha)]$$

$$\text{iii) } k_c \neq k_m ; \sigma_c = \sigma_m = 1$$

$$26) \quad C = K_c^d L_c^{1-d} ; \quad M = e^{At} K_m^f L_m^{1-f} ; \quad d \neq f$$

Once again, we can envision the technical progress in M as being Harrod

neutral, and in C as being a mix of Harrod neutral technical progress at the same rate as in M, plus a negative rate of Hicks neutral technical progress in C:

$$27) \quad M = K_m^f [L_m e^{[(At)/(1-f)]}] (1-f)$$

$$28) \quad C = [e^{-A[(1-d)/(1-f)]t}] K_c^d [L_c e^{[(At)/(1-f)]}] (1-d)$$

A steady-state exists in which $\bar{k} = [K/(L e^{[A/(1-f)]t})]$ approaches a constant value, so that:

$$29) \quad (\dot{K}/K) = n + [A/(1-f)] ; \quad (\dot{k}/k) = (\dot{\omega}/\omega) = [A/(1-f)]$$

$$30) \quad (\dot{C}/C) = n + [(dA)/(1-f)]$$

Therefore, if $\sigma_c = \sigma_m = 1$, there is a steady-state equilibrium; and if $k_c = k_m$, $\sigma_m = \sigma_c \leq 1$, then there is a finite asymptotic rate of growth for the economy, whereas if $\sigma_m > 1$, the rate of growth is unbounded. So far, we are not far from our one-sector world. Let us now turn to our more general case.

B. General Case - $k_c \geq k_m$

Even in our more general approach we shall maintain the factor-intensity hypothesis to guarantee the non-positiveness of T_a and T_b . Because of the complexity of the expressions for T_a and T_b it is not possible to exhibit their path (and hence the path of \bar{X}) for all k (without assumptions on third derivatives of the production function). However, since $T_a, T_b \leq 0$ everywhere, we have:

$$31) \quad \bar{X}_i = n - (A/T_i) \geq n \quad (T_i \neq 0); \quad i = a, b; \quad \text{and since:}$$

$$32) \quad (\dot{X}/X) = -T(\bar{X}-X) \quad , \quad (\dot{k}/k) = (X-n),$$

k tends to infinity. Therefore, we can be content with considering the asymptotic values of T_a and T_b .

In order to calculate the asymptotic values of T_a and T_b we first must calculate the asymptotic values of σ_a and σ_b . Though we are principally interested in the case $k_c \geq k_m$, $\omega \rightarrow \infty$, we also calculated the limits for $k_c < k_m$ and as $\omega \rightarrow 0$.

From Table I (on the following page) we can see that asymptotically $\sigma_b \geq 0$ in all cases, even if $k_c < k_m$ or $[\sigma_c + \sigma_m] < 1$.

Since:

$$17b) \quad T_b = -[\omega/(\omega+k_m)][1/\sigma_b] \quad ,$$

we know that asymptotically T_b must be non-positive in all cases.

Also, from the Table we can see that as $\omega \rightarrow \infty$, then $\sigma_a \geq \sigma_b$ in all cases; whereas, as $\omega \rightarrow 0$, $\sigma_b \geq \sigma_a$ in all cases.

Given σ_b , it is quite clear that the asymptotic behavior of T_b depends only upon σ_m . If $\sigma_m > 1$, then as $k \rightarrow \infty$, $T_b \rightarrow 0$; if $\sigma_m < 1$, $T_b \rightarrow [-1/\sigma_b]$. Finally, if $\sigma_m \rightarrow 1$, then $T_b \rightarrow -[(1-\alpha_m^*)/\sigma_b]$, where $(1-\alpha_m^*)$ is the asymptotically constant share of labor in sector M.

Correspondingly, given Table I, we can calculate the value of T_b as $\omega \rightarrow 0$ and as $\omega \rightarrow \infty$. These results appear in Table II. (It should be noted that as $\omega \rightarrow \infty$, for $k_c \geq k_m$, it must be true that $\sigma_c \geq \sigma_m$; whereas, as $\omega \rightarrow 0$, for $k_c \geq k_m$, $\sigma_c \leq \sigma_m$. Hence, from Table I we can see that $\sigma_b \rightarrow 1$ as $\sigma_m \rightarrow 1$, given $k_c \geq k_m$).

Since, given $k_c \geq k_m$ and that k tends to infinity, it follows that \bar{X}_b tends to:

TABLE I - Asymptotic Value of Aggregate Elasticity of Substitution

$\sigma_c > \sigma_m$	Value as $\omega \rightarrow \infty$		Value as $\omega \rightarrow 0$	
	σ_a	σ_b	σ_a	σ_b
1) $\sigma_c = \sigma_m$	σ_c	σ_c	σ_c	σ_c
2) $\sigma_c > \sigma_m$	$k_c > k_m$		$k_c < k_m$	
	a) $\sigma_c > \sigma_m > 1$	σ_m σ_m	σ_m σ_c	
	b) $\sigma_c > \sigma_m = 1$	1 1	1 σ_c	
	c) $\sigma_c > 1, \sigma_m < 1$	1 1	1 σ_c	
	d) $\sigma_c = 1, \sigma_m < 1$	1 1	1 1	
	e) $\sigma_m < \sigma_c < 1$	σ_c σ_c	σ_c σ_c	
3) $\sigma_c < \sigma_m$	$k_c < k_m$		$k_c > k_m$	
	a) $\sigma_m > \sigma_c > 1$	σ_c σ_c	σ_c σ_c	
	b) $\sigma_m > \sigma_c = 1$	1 1	1 1	
	c) $\sigma_m > 1, \sigma_c < 1$	1 σ_c	1 1	
	d) $\sigma_m = 1, \sigma_c < 1$	1 σ_c	1 1	
	e) $\sigma_c < \sigma_m < 1$	σ_m σ_c	σ_m σ_m	

TABLE II - Asymptotic Values of T_b

	Value as $\omega \rightarrow \infty$	Value as $\omega \rightarrow 0$
$\sigma_m > 1$	0	$-[1/\sigma_b]$
$\sigma_m = 1$	$-[(1-\alpha)/\sigma_b]$	$-[(1-\alpha)/\sigma_b]$
$\sigma_m < 1$	$-[1/\sigma_b]$	0

$$\begin{aligned}
 33) \quad \sigma_m > 1 & \quad \bar{X}_b \rightarrow \infty \\
 \sigma_m \rightarrow 1 & \quad \bar{X}_b \rightarrow [n + \{A/(1-\alpha)\}] \\
 \sigma_m < 1 & \quad \bar{X}_b \rightarrow [n + A\{\min(\sigma_c, 1)\}]
 \end{aligned}$$

Thus, for $\sigma_m \leq 1$, (k/k) asymptotically approaches a finite rate of growth, whereas for $\sigma_m > 1$, (k/k) will grow without bound. σ_c is not very important in determining the asymptotic rate of growth except that, for $\sigma_m < 1$ and $\sigma_c < 1$, the larger σ_c , the quicker is the asymptotic rate of growth. As in the traditional result, the savings rate has no influence on the (asymptotic) rate of growth.

Returning to the case $s = s_k = s_n$, we see that our task is not quite so simple. In Table III (on the following page) we present the asymptotic values of T_a , indicating, where applicable, the determinates of the sign (as $T_a \rightarrow 0$, we indicate whether it approaches plus zero or minus zero - as always, the assumption that $k_c \geq k_m$ suffices to guarantee the non-positiveness of T_a).

From the Table we see that, in most cases, T_a is negative (or approaches zero from negative values); however, in some cases, T_a approaches zero from positive values. If we maintain that $k_c \geq k_m$, then we see that $T_a \leq 0$; and, in fact, $T_a = 0$ whenever $\sigma_m > 1$, and $T_a < 0$ whenever $\sigma_m \leq 1$. Thus, once again, the elasticity of substitution of the investment good sector is critical. Using Table III we have:

$$\begin{aligned}
 34) \quad \omega \rightarrow \infty: \sigma_m > 1 & \rightarrow \bar{X}_a \rightarrow \infty; \quad \sigma_m \rightarrow 1 \rightarrow \bar{X}_a \rightarrow [n + \{A/(1-\alpha_m)\}]; \\
 \sigma_m < 1 & \rightarrow \bar{X}_a \rightarrow [n + A]
 \end{aligned}$$

For $\sigma_m \rightarrow 1$ we see that labor's share enters the expression for the asymptotic growth rate due to the (asymptotic) equivalence of

TABLE III - Asymptotic Value of T_a

σ_c, σ_m	Value as $\omega \rightarrow \infty$	Value as $\omega \rightarrow 0$
1) $\sigma_c = \sigma_m$		
a) $\sigma_c = \sigma_m > 1$	-0	-1
b) $\sigma_c = \sigma_m = 1$	$-(1-\alpha_m)$	$-(1-\alpha_m)$
c) $\sigma_c = \sigma_m < 1$	-1	± 0 as $[\sigma_c + (k_c/k_m)-1] \lesseqgtr 0$
2) $\sigma_c > \sigma_m$	$k_c > k_m$	$k_c < k_m$
a) $\sigma_c > \sigma_m > 1$	-0	-1
b) $\sigma_c > \sigma_m = 1$	$-(1-\alpha_m)$	$-(1-\alpha_m)$
c) $\sigma_c > 1, \sigma_m < 1$	-1	-0
d) $\sigma_c = 1, \sigma_m < 1$	-1	-0
e) $\sigma_m < \sigma_c < 1$	-1	+0
3) $\sigma_c < \sigma_m$	$k_c < k_m$	$k_c > k_m$
a) $\sigma_m > \sigma_c > 1$	-0	-1
b) $\sigma_m > \sigma_c = 1$	-0	-1
c) $\sigma_m > 1, \sigma_c < 1$	± 0 as $[\sigma_m + \sigma_c] \gtrless 2$	-1
d) $\sigma_m = 1, \sigma_c < 1$	$-(1-\alpha_m)$	$-(1-\alpha_m)$
e) $\sigma_c < \sigma_m < 1$	-1	-0

α_m denotes the (asymptotically) constant share of capital (under perfect competition) in sector M.

Harrod and Hicks neutral technical progress. Whenever $\sigma_m > 1$, (\dot{k}/k) grows without bound, whereas for $\sigma_m < 1$, (\dot{k}/k) approaches the growth rate A . Notice that for $\sigma_m \leq 1$, $\sigma_c \geq 1$, the two savings assumptions yield the same asymptotic growth rate. However, for $\sigma_m < 1$ and $\sigma_c < 1$, the asymptotic growth rate is larger for the economy in which both workers and capitalists save. And that growth rate is independent of the savings rate.

In summary, given the capital-intensity condition (or the appropriate constraints on the sectoral elasticities of substitution) and that both the production functions are homogeneous of degree one, the economy will approach a finite asymptotic rate of growth only if the elasticity of substitution in the investment sector is not asymptotically larger than one. The elasticity of the consumption good sector is unimportant in determining the growth rate when both workers and capitalists save at the same rate; however, if only capitalists save, then the elasticity of substitution in the consumption sector can effect the growth rate when $\sigma_m < 1$ and $\sigma_c < 1$.

Let us now see what happens if we relax the factor-intensity assumption.

C. Relaxing the Factor Intensity Assumption

Our analysis so far has been predicated upon the assumption that $k_c \geq k_m$. This assumption has simplified the analysis by guaranteeing the monotonicity of the market equilibrium curve and the "stability" of the "equilibrium" capital-labor ratio (for a given instant of time). As we shall see, the assumption is not necessary in the case in which everybody saves at the same rate, so that, even

without the factor-intensity assumption, the system will behave as described above. However, if workers do no savings, the analysis is more complicated since it becomes necessary to have information on the third derivatives of the production functions (for $s_k > s_n = 0$, there is no guarantee of causality of the system unless $k_c \geq k_m$; see Burmeister and Dobell [9]). Let us now consider our growth model when the factor-intensity assumption is relaxed.

Consider first the second savings assumption - if $k_c < k_m$ (and $[\sigma_c + \sigma_m] < 1$), it is possible that $T_b > 0$ ($\sigma_b < 0$). But $\sigma_b < 0$ implies that the market equilibrium curve is not monotonic and ω can not be expressed as a function of k . Instead, we must treat ω as the independent variable:

$$22) \quad (\dot{k}/k) = (\sigma_b)(\dot{\omega}/\omega)$$

In order to follow our earlier analysis we must take the derivative of this equation - and this, in turn, requires knowledge of third derivatives of production functions. Since we are reluctant to make such assumptions (even if the functions are C.E.S., the analysis is quite complicated), we shall ignore this problem, and examine the case in which everybody saves at the same rate.

Employing our first savings assumption, we know that the market equilibrium curve is monotonic. Returning to equation 12):

$$12) \quad (\dot{X}/X) = T(X-n) + A$$

We have already seen that, if $T < 0$, k tends to infinity, and $X > n$. Suppose that $T > 0$; if $X \geq n$, then k increases, as does X , and

k tends to infinity. If T remains positive asymptotically, it will tend to zero, and hence X tends to infinity. If T becomes negative, then what happens to X depends on T 's asymptotic value, as previously discussed. Once $X \geq n$, it can never become less than n , and therefore only the asymptotic value of T (as $k \rightarrow \infty$) matters.

Suppose $X < n$ initially: k will decrease, but what happens to X (\dot{X}/X) depends on T as well as X itself (if $T < 0$, $(\dot{X}/X) > A > 0$; however, if $T > 0$ it is possible that $(\dot{X}/X) < 0$). If $(\dot{X}/X) > 0$, then X must eventually reach n , in which case we return to the analysis of the previous paragraph. Therefore, suppose $(\dot{X}/X) < 0$; clearly, $X=0$ is a lower bound on X (we are assuming no depreciation or consumption of capital). As $k \rightarrow 0$, either T becomes negative, or it tends to zero. Therefore, there exists a k^* such that, for $k < k^*$, $(\dot{X}/X) > 0$ ($k < k^*$, $T_a < [A/n]$). Though k may continue to decrease for a while, X must increase, eventually reaching n . When $X=n$, k ceases falling, and then begins to increase (as $X > n$), and once again we can return to the analysis of our previous paragraph. Therefore, as long as T cannot remain asymptotically greater than zero (and it cannot), k must eventually tend to infinity. As k tends to infinity, X tends to infinity if T approaches zero; otherwise, it approaches a finite rate of growth.

In summary, relaxing the factor-intensity condition (for $s_k = s_n$) does not affect the asymptotic behavior of the system, though it may affect the time path (consequently, it appears possible for k to first become arbitrarily small, and later to become arbitrarily large). As long as $\sigma_m \leq 1$, X approaches some finite growth rate; if $\sigma_m > 1$, X approaches infinity, and if σ_m fluctuates between being greater and less than one, X will fluctuate between its limits.

Therefore, in the constant savings case we are prepared to drop the assumption that $k_c \geq k_m$.

D. Growth Rates of Other Variables

As in the one-sector case, we are interested in the growth rates of all variables, not just in the growth rate of capital. In this case, in which Hicks neutral technical progress occurs in sector M (and there is no labor-augmenting technical progress) our results are quite straightforward, and are quite similar to the one-sector model. Later, when we consider different rates of Harrod neutral technical progress in each sector, as well as Hicks neutral technical progress in the investment sector, we shall see that the results become more complicated, and the similarity with the one-sector model disappears.

For the moment, consider the growth rates of the other variables:

$$35) \quad \dot{M} = \dot{K} = XK; \text{ therefore, } (\dot{M}/M) = X + (\dot{X}/X)$$

$$36) \quad (\dot{\omega}/\omega) = [(X-n)/\sigma]$$

$$37) \quad (\dot{P}/P) = -A + (\alpha_c - \alpha_m)(\dot{\omega}/\omega)$$

$$38) \quad (\dot{R}/R) = -(1-\alpha_c)(\dot{\omega}/\omega)$$

$$39) \quad (\dot{C}/C) = n + (\dot{R}/R) + \max[(\dot{\omega}/\omega), (\dot{k}/k)]$$

$$40) \quad (\dot{W}/W) = (\dot{\omega}/\omega) + (\dot{R}/R) = \alpha_c (\dot{\omega}/\omega)$$

Obviously, the values of these growth rates depends upon σ_m ; some also depend upon the value of σ_c , as well as which savings

assumption is employed. For $\sigma_m \leq 1$,¹ the analysis is straightforward and consists of merely substituting the asymptotic values of X and α_i into the relations given above. However, for $\sigma_m > 1$, the process can be more complicated, and sometimes entails using l'Hôpital's rule. For example, if $\sigma_m > 1$ and $\sigma_c < 1$, we find for (\dot{C}/C) :

$$39') \quad (\dot{C}/C) = n - (1-\alpha_c)(\dot{\omega}/\omega) + (\dot{\omega}/\omega) = n + \alpha_c (\dot{\omega}/\omega)$$

Clearly, $(\dot{\omega}/\omega)$ tends to infinity and α_c tends to zero; in order to calculate the limit, we can employ l'Hôpital's rule² (or else look at the time derivative of the expression).

In Table IV (page 101) we present the values of these asymptotic growth rates, as calculated from equations 35) through 40). These values are given as depending upon the asymptotic behavior of σ_c and σ_m ; should σ_m fluctuate between (for example) being greater and less than one, then the growth rates should fluctuate between their respective limits.

Several comments about the Table are in order. First of all, whenever $\sigma_c < 1$, the per capita consumption is not increasing in the

¹ $\sigma_i < 1$ means that asymptotically σ_i is bounded below unity; a comparable interpretation holds for $\sigma_i > 1$. For $\sigma_i = 1$, this means that σ_i asymptotically approaches one, and α_i is the asymptotically constant share of capital in sector i - assumed to be neither zero nor one.

² When calculating these limits we needed to assume either:

i) σ_c, σ_m were non-zero and finite, or :

ii) σ_c, σ_m changed sufficiently slowly - for example, it would suffice if the elasticity of σ_i with respect to ω tends to zero.

Neither of these conditions seems terribly unreasonable to us.

TABLE IV - Asymptotic Growth Rates of Variables
with Hicks Technical Progress in M

	$\sigma_m > 1$			$\sigma_m \rightarrow 1$			$\sigma_m < 1$		
	$\sigma_c > 1$	$\sigma_c \rightarrow 1$	$\sigma_c < 1$	$\sigma_c > 1$	$\sigma_c \rightarrow 1$	$\sigma_c < 1$	$\sigma_c > 1$	$\sigma_c \rightarrow 1$	$\sigma_c < 1$
$s_k = s_n$									
M	∞	∞	∞	$n+A^*$	$(n+A^*)^\dagger$	$n+A^*$	$n+A$	$n+A$	$n+A$
C	∞	∞	n	$n+A^*$	$n+\alpha_c A^*$	n	$n+A$	$n+\alpha_c A$	n
P	$-A$	$-\infty$	$-\infty$	0	$-A^*(1-\alpha_c)$	$-A^*$	0	$-A(1-\alpha_c)$	$-A$
W	∞	∞	0	A^*	$\alpha_c A^*$	0	A	$\alpha_c A$	0
R	0	$-\infty$	$-\infty$	0	$-A^*(1-\alpha_c)$	$-A^*$	0	$-A(1-\alpha_c)$	$\frac{-A}{\max(\sigma_c, \sigma_m)}$
ω	∞	∞	∞	A^*	A^*	A^*	A	A	$\frac{A}{\max(\sigma_c, \sigma_m)}$
K	∞	∞	∞	$n+A^*$	$n+A^*$	$n+A^*$	$n+A$	$n+A$	$n+A$
k	∞	∞	∞	A^*	A^*	A^*	A	A	A
<hr/>									
$s_k > s_n = 0$									
M	∞	∞	∞	$n+A^*$	$n+A^*$	$n+A^*\sigma_c$	$n+A$	$n+A$	$n+A\sigma_c$
C	∞	∞	n	$n+A^*$	$n+A^*\alpha_c$	n	$n+A$	$n+A\alpha_c$	n
P	$-A$	$-\infty$	$-\infty$	0	$-A^*(1-\alpha_c)$	$-A^*$	0	$-A(1-\alpha_c)$	$-A$
W	∞	∞	0	A^*	$\alpha_c A^*$	0	A	$\alpha_c A$	0
R	0	$-\infty$	$-\infty$	0	$-A^*(1-\alpha_c)$	$-A^*$	0	$-A(1-\alpha_c)$	$-A$
ω	∞	∞	∞	A^*	A^*	A^*	A	A	A
K	∞	∞	∞	$n+A^*$	$n+A^*$	$n+A^*\sigma_c$	$n+A$	$n+A$	$n+A\sigma_c$
k	∞	∞	∞	A^*	A^*	$A^*\sigma_c$	A	A	$A\sigma_c$

† In the above Table, $A^* = [A/(1-\alpha_m)]$

α_i represents the asymptotic share of capital in sector i ; $1 > \alpha_i > 0$

asymptotic growth path. Since, for $\sigma_c < 1$, $\alpha_c \rightarrow 0$, this means that the capital-deepening does not serve to increase consumption - all the benefits of technical progress serve merely to lead to greater outputs of machinery. Also, it can readily be seen that asymptotically the rate of growth of consumption is independent of the savings rate, and that per capita consumption grows at the same rate as the wage rate (when finite). Finally, we note that when $\sigma_c > 1$, $\sigma_m \leq 1$, the rate of growth of the price of the investment good tends to zero (though P tends to infinity), despite the presence of technical progress in M . This occurs because sector C is better situated to utilize the ever-increasing stock of capital.

When we consider the case $\sigma_m > 1$, we can see that most growth rates become infinite (positively or negatively). We know:

$(\dot{X}/X) = A + (X-n)T$; But $T \rightarrow 0$, and $XT \rightarrow 0$ (shown by using l'Hôpital's rule). Therefore:

$$41) \quad (\dot{X}/X) \rightarrow A \quad \text{as} \quad t \rightarrow \infty, \quad \sigma_m > 1$$

Consequently, the capital stock (and the capital-labor ratio) grows at an ever-increasing rate - as Nordhaus [32, page 61] says: "Robots are making robots at an ever increasing rate."

In summary, if σ_m is bounded below unity, or asymptotically approaches unity, the system approaches the growth rates shown in the Table, which also depend on the asymptotic value of σ_c . If σ_m is bounded above unity, then most of the growth rates tend to infinity; and if σ_m fluctuates between these values, so will the respective growth rates.

Before considering the case of Harrod neutral technical progress in only one sector, let us examine one more question: If a planner has a choice between allocating technical progress to sector M or sector C, where should he allocate it?

E. Allocating Hicks Technical Progress Between the Investment and Consumption Sectors

So far we have seen that, if $\sigma_m \leq 1$ asymptotically, the economy will approach a path in which the growth rates of the variables tend to constant limits. Obviously, this case is similar to the one-sector model studied in Chapter 1, and, as in that chapter, we can ask how a planner should allocate technical progress within the economy. (Though we do not explicitly deal with the problem of allocating research funds, this exercise can be considered as indicating how research funds should be allocated within the economy). Our principal interest in this problem is to determine under what conditions a steady-state path is likely to be chosen by the planner.

Therefore, let us suppose we have the choice of either:

- i) Hicks neutral technical progress in sector C at rate A and none in sector M, or
- ii) Hicks neutral technical progress in sector M at rate A and none in sector C.

Also, suppose our goal is to maximize the steady-state (or asymptotic) rate of growth of consumption. We would like to know, given this objective, where to allocate technical progress. We know that if Hicks neutral technical progress occurs only in sector C, then a steady-state exists and C grows at rate $(n+A)$. From Table IV we can find the asymptotic rate of growth of C; comparing these, we show in Table V

TABLE V - Allocating Hicks Technical Progress Between Sectors

	All in One Sector			Continuous Trade-Off		
	$\sigma_m > 1$	$\sigma_m \rightarrow 1$	$\sigma_m < 1$	$\sigma_m > 1$	$\sigma_m \rightarrow 1$	$\sigma_m < 1$
$\sigma_c > 1$	M	M	Either	M	Both	Both
$\sigma_c \rightarrow 1$	M	M or C as $[\alpha_c + \alpha_m] > 1$ < 1	C	M	Both	Both
$\sigma_c < 1$	C	C	C	C	C	C

how one should allocate technical progress (under the above criterion).

From Table V we see that it never pays to allocate technical progress to sector M when $\sigma_c < 1$ - the consumption sector simply cannot take advantage of the continual capital-deepening. On the other hand, when $\sigma_c > 1$, you should always allocate the technical progress to M (you are indifferent if $\sigma_c > 1$, $\sigma_m < 1$) - the indirect route of increasing consumption through capital-deepening is more effective. Finally, if $\sigma_c \rightarrow 1$, we need to consider more carefully the value of σ_m .

Alternatively, suppose we postulate the existence of a trade-off frontier between technical progress in the two sectors. Let:

- a) ℓ - the rate of Hicks technical progress in C; and
- b) g - the rate of Hicks technical progress in M

$$42) \quad g = g(\ell); \quad g', g'' < 0; \quad g(0) = A, \quad g(\lambda) = 0; \quad g, \ell \geq 0. \quad \text{Thus:}$$

$$43) \quad (\dot{c}/c) = \ell + [(\dot{c}/c)\{g\}] \equiv n(g) + \ell; \quad c \equiv (C/L)$$

In the above equation, $n(g)$ [the rate of growth of per capita

consumption (asymptotically) due to Hicks technical progress in M] is to be determined from Table IV. From Table IV we find that:

If $\sigma_c < 1$, $n(g) = 0$, and (\dot{c}/c) is maximized for $\ell = \lambda$, $g = 0$

If $\sigma_c \geq 1$, $\sigma_m \leq 1$, $n(g) = [(g\alpha_c)/(1-\alpha_m)]$

Maximizing equation 43) with respect to ℓ yields:

$$44) \quad 1 + [(g'\alpha_c)/(1-\alpha_m)] = 0; \quad \text{or} \quad g' = -[(1-\alpha_m)/\alpha_c]; \quad g'' < 0$$

$$\alpha_c > 0, \quad \alpha_m < 1$$

Thus, in this case the optimal solution is to allocate technical progress until the slope of the transformation curve is just equal to the negative of the ratio of the share of labor in M to the share of capital in C (for $\alpha_c = \alpha_m$, this is equivalent to the one-sector Kennedy condition [27]). In general, you should allocate some technical progress to each sector, though a corner solution is possible.

Finally, for $\sigma_m > 1$, $\sigma_c \geq 1$, consumption grows at an ever-increasing (and asymptotically infinite) rate if there is any Hicks technical progress in M. As shown earlier:

$$45) \quad (\dot{X}/X) \rightarrow g; \quad (\dot{c}/c) \rightarrow \ell + \alpha_c (\dot{\omega}/\omega), \quad \alpha_c > 0$$

From 45) it is clear that maximization of (\dot{c}/c) entails a corner solution - with $\ell = 0$, $g = A$. In this case it pays to allocate all technical progress to sector M.

Therefore, only if $\sigma_c \geq 1$, and $\sigma_m \leq 1$ will it pay to allocate some technical progress to each sector (under the given criterion).

Consider equation 44):

$$44) \quad g' = -[(1-\alpha_m)/\alpha_c]; \quad g'' < 0$$

We see that the larger α_c (or α_m), ceteris paribus, the greater the portion of technical progress that should be allocated to M. That is, the greater the elasticity of output with respect to capital, in either sector, the larger the share of technical progress that we should allocate to M. Naturally, the converse is also true. Finally, as α_m and α_c approach one, the greater is the possibility of a corner solution in which all technical progress is allocated to M; and as α_c and α_m approach zero, the more likely is a corner solution with all technical progress allocated to C.

This completes our study of the effects of Hicks neutral technical progress in the investment sector. We have seen that when $\alpha_m \leq 1$, there exist finite asymptotic growth rates to which the system tends. Also, we have discussed the problem of allocating factor-augmenting technological change within this economy. Let us now attempt to develop a model in which Harrod neutral technical progress occurs in only one sector.

III. Harrod Neutral Technical Progress in the Consumption Sector

In the previous section we have studied the problem of what would happen if there were Hicks neutral technical progress in M. In this part we plan to study a different problem - that in which there is Harrod neutral technical progress in only one sector. Specifically, we start by assuming that the technical progress occurs only in sector C; later we show how this can be generalized to cover the case in which it occurs in sector M, or in both sectors, though at different rates. The problem we are considering in this section arises not

because capital tends to grow too quickly, but rather because, even for a fixed ω , the ratio of the marginal-productivities in at least one sector will be shifting, and hence factors will be continually reallocated within the economy. Consequently, this problem is fundamentally different from either the one-sector model or the problem just studied.

As we shall see from the ensuing analysis, the aggregate (effective) capital-labor ratio for the economy as a whole will tend to a constant, finite limit, as will the (effective) wage-rental ratio and the capital-labor ratio (in efficiency units) in the investment sector. However, due to the presence of different rates of Harrod neutral technical progress in the two sectors, the effective capital-labor ratio in the consumption sector will tend to either zero or infinity. Consequently, this model will differ from the normal steady-state model principally in terms of the fraction of labor allocated to each sector and in terms of factor shares in the consumption sector.

To see this, let us assume:

$$46) \quad M = F_m(K_m, L_m) \quad ; \quad C = F_c(K_c, L_c e^{dt}) \quad ; \quad d \text{ may be positive or negative.}$$

First of all, it is clear that if $\sigma_c = 1$, then we are done - the technical progress in C is equivalent to Hicks neutral technical progress, and a steady-state exists in that case. Thus, assume $\sigma_c \neq 1$. Furthermore, we shall assume that the elasticity of substitution in C is bounded from one.

In pursuing the analysis, we employ the normal two-sector

model. Below we outline some of our basic relations:

$$47) \quad k_i = (K_i/L_i) \quad ; \quad x = [k_c e^{-dt}]$$

$$48) \quad M = L_m f_m(k_m) \quad ; \quad C = L_c e^{dt} f_c(x)$$

$$49) \quad \omega = \{[e^{dt}(f_c - x f'_c)]/f'_c\} = \{(f'_m - k'_m f'_m)/f'_m\}$$

$$50) \quad P_c = 1; \quad P_m \equiv P = (f'_c/f'_m)$$

$$51) \quad \gamma = (L_c/L) = [(k - k_m)/(k_c - k_m)] \quad ; \quad (1-\gamma) = [(k_c - k)/(k_c - k_m)]$$

It is clear that k_m depends upon ω ; however, k_c (and x) depends upon t as well as ω . Thus, even if ω is held constant, other variables change. Since the presence of Harrod neutral technical progress in only one sector involves this continual shifting in x , it is clear that there must be a continual reallocation of resources ($\sigma_c \neq 1$). Consequently, an "equilibrium" can only occur asymptotically.

Below we illustrate how some of the variables shift over time, assuming that the wage-rental ratio is held constant:

$$52) \quad [(\partial x/\partial t)(1/x)] = -d\sigma_c \quad ; \quad [(\partial k_c/\partial t)(1/k_c)] = d(1-\sigma_c)$$

$$53) \quad [(\partial P/\partial t)(1/P)] = d(1-\alpha_c)$$

$$54) \quad [(\partial \gamma/\partial t)(1/\gamma)] = -\{[k_c d(1-\sigma_c)]/(k_c - k_m)\}$$

$$55) \quad [(\partial \{1-\gamma\}/\partial t)(1/\{1-\gamma\})] = \{[\gamma k_c d(1-\sigma_c)]/[(1-\gamma)(k_c - k_m)]\}$$

From the above equations it is obvious that the sign of $[d(1-\sigma_c)]$ is important:

$$56) \quad d(1-\sigma_c) > 0 \quad \text{if} \quad \text{a) } d > 0, \sigma_c < 1 \quad \text{or} \quad \text{b) } d < 0, \sigma_c > 1$$

$$57) \quad d(1-\sigma_c) < 0 \quad \text{if} \quad \text{a) } d < 0, \sigma_c < 1 \quad \text{or} \quad \text{b) } d > 0, \sigma_c > 1$$

Finally, adopting the simple savings assumption that everyone saves at the same rate, we can derive our two basic equations - the capital-accumulation equation (58) and the market equilibrium curve (59).

$$58) \quad \dot{k} = (1-\gamma)f'_m - nk = 0 \text{ defines } L(\omega, k, t) = 0$$

$$59) \quad (1-s)(1-\gamma)(\omega+k_m) - s\gamma(\omega+k_c) = 0 \text{ defines } H(\omega, k, t) = 0$$

We know from the traditional two-sector models that for these two curves to have a unique (stable) intersection it suffices to have either $k_c \geq k_m$ or $\sigma_c \geq 1$. From equations 56) and 57) we have three possible cases:

- i) Whenever $\sigma_c > 1$, a unique intersection exists.
- ii) For $\sigma_c < 1$, $d > 0$, k_c increases over time (for fixed ω), so that (after sufficient time has elapsed) there will be a unique intersection.
- iii) However, for $d < 0$ and $\sigma_c < 1$, neither condition is satisfied and we can not be certain that a unique equilibrium exists in this case.

In this latter case it can be shown that as $k_c \rightarrow 0$, $\sigma_m > 0$ suffices to guarantee that an intersection of these two curves exists and is unique. Thus, in the first two cases we can be sure of unique intersections, whereas in the third case a unique asymptotic equilibrium exists.

The question we now seek to answer is: how do these curves shift over time and how does their intersection shift? To answer these questions we must examine the partial derivatives of the implicit functions defined by 58) and 59).

$$60) \quad (\partial L/\partial t) = (1-\gamma)f_m[\gamma k_c d(1-\sigma_c)]/[(1-\gamma)(k_c-k_m)]$$

Assuming sufficient time has elapsed, $k_c > k_m$ as $d(1-\sigma_c) > 0$. Thus:

$$60') \quad (\partial L/\partial t) > 0 \text{ in all cases (for } t > t^* \text{ so that the 'proper'}$$

factor-intensity relationships have been established). Similarly:

$$61) \quad (\partial L/\partial k) = -[(1-\gamma)f_m k_c]/[k(k_c-k)] < 0 \text{ as } d(1-\sigma_c) > 0, \text{ and}$$

$$62) \quad (\partial L/\partial \omega) = \frac{\{[(1-\gamma)f_m]/\omega\}[k_c \sigma_c \gamma(\omega+k_m) + k_m \sigma_m(\omega+k_c)(1-\gamma)]}{[(\omega+k_m)(k_c-k)]}$$

$$(\partial L/\partial \omega) > 0 \text{ as } d(1-\sigma_c) > 0$$

Using the implicit function theorem we find:

$$63) \quad (\partial k/\partial t)_{L=0} = k \gamma d(1-\sigma_c) > 0 \text{ as } d(1-\sigma_c) > 0$$

$$64) \quad (\partial \omega/\partial t)_{L=0} = -\{[\omega(\omega+k_m)k_c \gamma d(1-\sigma_c)]/[k_c \sigma_c \gamma(\omega+k_m) + k_m \sigma_m(1-\gamma)(\omega+k_c)]\}$$

$$(\partial \omega/\partial t)_{L=0} < 0 \text{ as } d(1-\sigma_c) > 0$$

Therefore, the sign of $d(1-\sigma_c)$ is critical in determining how the curve shifts. Figure I depicts the behavior of this curve over time. Next, consider what happens to the market equilibrium curve:

$$65) \quad (\partial H/\partial t) = \{[(1-s)(1-\gamma)(\omega+k_m)d(1-\sigma_c)k_c(\omega+k)]/[(k_c-k)(\omega+k_c)]\} > 0$$

$$66) \quad (\partial H/\partial k) = -\{[(1-s)(1-\gamma)(\omega+k_m)]/[(k_c-k)\gamma]\} < 0 \text{ as } d(1-\sigma_c) > 0$$

$$67) \quad (\partial E/\partial \omega) = \frac{(1-s)[\omega(k-k_m)(k_c-k) + (\omega+k)\{\sigma_c k_c(\omega+k_m)\gamma + \sigma_m k_m(1-\gamma)(\omega+k_c)\}]}{[(\omega+k_c)\omega(k-k_m)]}$$

$$67') \quad (\partial H / \partial \omega) \gtrless 0 \quad \text{as} \quad d(1 - \sigma_c) \gtrless 0$$

(Note that for both the market equilibrium curve and the capital-accumulation curve that $[dk/d\omega] > 0$).

Using the above equations, we can show how the variables shift:

$$68) \quad (\partial k / \partial t)_{H=0} = \{ [k_c(\omega+k)\gamma d(1-\sigma_c)] / (\omega+k_c) \} \gtrless 0 \quad \text{as} \quad d(1-\sigma_c) \gtrless 0$$

$$69) \quad (\partial \omega / \partial t)_{H=0} = \frac{-[\omega(\omega+k_m)(\omega+k)k_c\gamma d(1-\sigma_c)]}{[\omega(k-k_m)(k_c-k) + (\omega+k)\{\sigma_c k_c \gamma(\omega+k_m) + \sigma_m k_m(1-\gamma)(\omega+k_c)\}]}$$

$$(\partial \omega / \partial t)_{H=0} \gtrless 0 \quad \text{as} \quad d(1-\sigma_c) \gtrless 0$$

Figure II (page 112) depicts how the market equilibrium curve shifts over time.

So far, we are not able to determine what happens to the intersection of these two curves - in order to do this we must compare the magnitudes of the two shifts.³ Thus, if we call (k^*, ω^*) the (unique) intersection of these two curves at time t , then:

$$70) \quad \dot{\omega}^* \gtrless 0 \quad \text{as} \quad |[(\partial k / \partial t)(1/k)]_{L=0}| \gtrless |[(\partial k / \partial t)(1/k)]_{H=0}|,$$

where the derivatives are evaluated at (k^*, ω^*) . Or, for simplicity, let:

$$71) \quad \eta_{ca} = [(\partial k / \partial t)(1/k)]_{L=0} ; \quad \eta_{me} = [(\partial k / \partial t)(1/k)]_{H=0} ; \quad \text{then:}$$

$$72) \quad \dot{\omega}^* \gtrless 0 \quad \text{as} \quad [|\eta_{ca}| / |\eta_{me}|] \gtrless 1.$$

³We have seen for $d < 0$, $\sigma_c < 1$, it is possible that several intersections may occur. In this case, we assume t is sufficiently large (k_c sufficiently small) to guarantee that only one intersection occurs.

Figure I - Shifting of the Capital-Accumulation Curve
Due to Harrod Technical Progress Only in Sector C

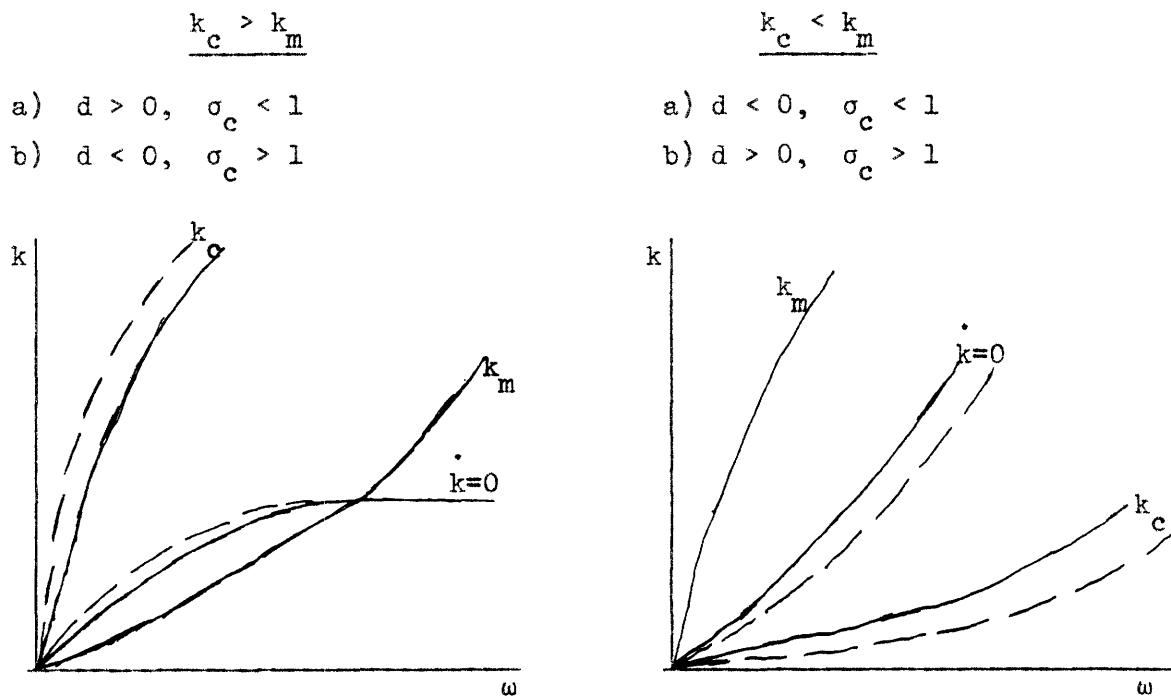
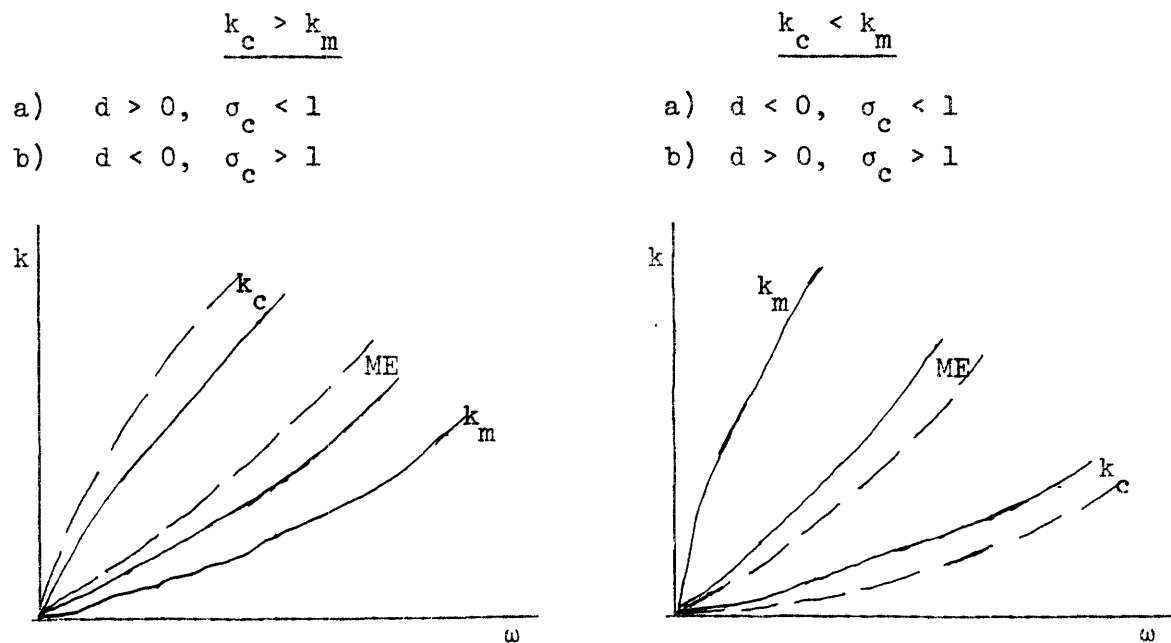


Figure II - Shifting of the Market Equilibrium Curve
Due to Harrod Technical Progress Only in Sector C



But, from equations 63) and 68) we have:

$$73) \quad [n_{ca}/n_{me}] = [k(\omega+k_c)]/[k_c(\omega+k)] \lesssim 1 \quad \text{as } k \lesssim k_c \quad (\text{as } d[1-\sigma_c] \gtrsim 0).$$

Thus, we find:

$$74) \quad \dot{\omega}^* \lesssim 0 \quad \text{as } d(1-\sigma_c) \gtrsim 0$$

In other words, if $d(1-\sigma_c) < 0$, then $k_c < k_m$; in this case, the $k = 0$ curve intersects the market equilibrium curve from below, and since the capital-accumulation curve shifts more than the market equilibrium curve, ω^* must increase; the reverse holds for $d(1-\sigma_c) > 0$.

In order to see what happens to k^* , we must consider the relative shifts of the two curves in the ω direction.

$$75) \quad \delta_{ca} \equiv [(\partial\omega/\partial t)(1/\omega)]_{L=0} ; \quad \delta_{me} \equiv [(\partial\omega/\partial t)(1/\omega)]_{H=0}$$

$$76) \quad \dot{k}^* \gtrsim 0 \quad \text{as } |\delta_{ca}| \gtrsim |\delta_{me}| ;$$

the derivatives are evaluated at (k^*, ω^*) .

From equations 64) and 69) we find:

$$77) \quad [\delta_{ca}/\delta_{me}] = \frac{[\omega(k-k_m)(k_c-k) + (\omega+k)\{\sigma_c k_c \gamma(\omega+k_m) + \sigma_m k_m (1-\gamma)(\omega+k_c)\}]}{(\omega+k)[\sigma_c k_c \gamma(\omega+k_m) + \sigma_m k_m (1-\gamma)(\omega+k_c)]}$$

Therefore:

$$77') \quad [\delta_{ca}/\delta_{me}] \geq 1 \quad (\text{equality only if } k=k_m \text{ or } k=k_c)$$

Since there can not be specialization ($1 > s > 0$), $[\delta_{ca}/\delta_{me}] > 1$ (except asymptotically as $k_c \rightarrow \infty$ or $k_c \rightarrow 0$), and thus k^* increases over time. Table VI (page 114) summarizes how (ω^*, k^*) change over time,

TABLE VI - Changes in Equilibrium Values Due to Harrod Technical Progress in C

	$\sigma_c > 1$	$\sigma_c < 1$
$d > 0$	$\dot{k}^* > 0, \dot{\omega}^* > 0$	$\dot{k}^* > 0, \dot{\omega}^* < 0$
$d < 0$	$\dot{k}^* > 0, \dot{\omega}^* < 0$	$\dot{k}^* > 0, \dot{\omega}^* > 0$

assuming the uniqueness of the intersection of the two curves.

We have now determined how the "equilibrium" values shift over time - k^* always increases, whereas ω^* may increase or decrease, depending upon the value of $d(1-\sigma_c)$. Clearly, for $k_c > k_m$, k^* cannot increase forever - an upper bound is established by:

$$78) \quad \dot{k} = 0, \quad k_m \leq k, \quad \text{for } \gamma=0, \quad f(\hat{k}^*) - n\hat{k}^* = 0; \quad k^* \leq \hat{k}^*$$

Since k_c continually shifts over time, our problem is intrinsically an asymptotic one. Let us now attempt to determine what does occur asymptotically.

A. Asymptotic Behavior

So far in this section we have seen how the "equilibrium" k and ω shift over time. However, it is clear that no steady-state occurs in this case ($\sigma_c \neq 1$), and therefore our main interest concerns the asymptotic values of the variables. As we shall see, both k and ω will approach finite limits as $t \rightarrow \infty$. Let us now see how these limits are determined.

For any finite (ω, k) , k_c must eventually tend to zero or infinity as time becomes large. Thus, as $t \rightarrow \infty$:

$$79) \quad d(1-\sigma_c) > 0 \rightarrow [k_c \rightarrow \infty]; \quad d(1-\sigma_c) < 0 \rightarrow [k_c \rightarrow 0]; \quad [\infty > \omega > 0]$$

$$80) \quad \text{But, } k_c \rightarrow \infty \text{ implies: } (1-\gamma) = [(k_c - k)/(k_c - k_m)] \rightarrow 1; \quad [\infty > (\omega, k) > 0]$$

$$81) \quad \text{And, } k_c \rightarrow 0 \text{ implies: } (1-\gamma) \rightarrow [k/k_m] \leq 1$$

If $d(1-\sigma_c) > 0$ we can show:

$$82) \quad \dot{k} = 0 \text{ tends to: } [f'_m - nk] = 0$$

$$83) \quad (1-s)(1-\gamma)(\omega+k_m) - s\gamma(\omega+k_c) = 0 \text{ tends to:}$$

$$83') \quad (\omega+k_m) - s(\omega+k) = 0$$

Equations 82) and 83') are the asymptotic relationships as $k_c \rightarrow \infty$; clearly, from 83'), $k_m < k$ unless $s = 1$. Since $k_c > k_m$, there exists a unique intersection between these two curves, and it is towards this intersection that the economy tends. The value of $[\hat{k}, \hat{\omega}]$ (the intersection of these two asymptotic curves) depends only upon s , n , and the production function in industry M - C influences this equilibrium only insofar as it determines the sign of $d(1-\sigma_c)$. From equations 82) and 83') the value of $\hat{\omega}$ is determined by:

$$84) \quad [\omega/(\omega+k_m)] - [(1/s) - (f'_m/n)] = 0 \quad \psi(\omega) = 0$$

Equation 84) has a unique solution ($\hat{\omega}$) since:

$$85) \quad \psi(0) = \infty, \quad \psi(\infty) < 0, \quad \text{and} \quad \psi' < 0 \quad \text{when} \quad \psi = 0.$$

Thus, if k_c tends to infinity, there exists a unique asymptotic equilibrium. As the economy approaches this equilibrium,

k^* increases, and ω^* decreases.

Next, suppose that $d(1-\sigma_c) < 0$, so that $k_c \rightarrow 0$. As already stated, for $\sigma_c < 1$, we can not be sure that a unique equilibrium exists for all time. However, as $k_c \rightarrow 0$, it suffices that $\sigma_m > 0$ in order for a unique intersection to exist between the two curves. As $k_c \rightarrow 0$, our two equations approach the following limits:

$$86) \quad \dot{k} = 0 \text{ tends to } [(k/k_m)(f_m - nk_m)] = 0 \text{ as } k_c \rightarrow 0.$$

$$87) \quad (1-s)(1-\gamma)(\omega+k_m) - s\gamma(\omega+k_c) = 0 \text{ tends to}$$

$$87') \quad k[(\omega+k_m) - sk_m] - s\omega k_m = 0 \text{ as } k_c \rightarrow 0.$$

As we can see from equation 86), there exists a unique $\hat{\omega}$ such that $\dot{k} = 0$ (independent of k); and $[\partial k / \partial \omega] < 0$ (at $k=0$). Since 87), the market equilibrium curve, is upward sloping, a unique (and stable) equilibrium exists. Hence, $\hat{\omega}$ is determined from equation 86), and \hat{k} from equation 87'), by using the value of $\hat{\omega}$ obtained from equation 86).

In summary, the presence of Harrod neutral technical progress (at a positive or negative rate) in sector C leads to an asymptotic equilibrium in which both k and ω approach finite limits. If $d(1-\sigma_c)$ is positive, then the "equilibrium" value of ω (as defined by the intersection of the $\dot{k}=0$ curve and the market equilibrium curve) decreases over time, approaching its (non-zero) asymptotic limit, whereas k increases over time. If $d(1-\sigma_c)$ is negative, then the equilibrium values of ω and k increase (if a unique intersection exists) over time, approaching their finite asymptotic limits. Finally, if $\sigma_c=1$, then Harrod neutral technical progress is equivalent to Hicks

neutral technical progress, and a steady-state exists.⁴

B. Asymptotic Growth Rates

Now that we have seen what happens to ω and k asymptotically, we would like to determine how the other variables behave. Since ω and k tend to constant limits, it follows that k_m also tends to some constant limit. Consider the following variables:

$$88) \quad M = L(1-\gamma)f_m$$

$$89) \quad C = e^{\int \gamma dt} L \gamma f_c$$

$$90) \quad P = [f'_c/f'_m]$$

$$91) \quad \gamma$$

$$92) \quad (1-\gamma)$$

Equations 53) - 55) tell us how P , γ and $(1-\gamma)$ change for constant ω . Again, we need to divide our analysis into two parts: $d(1-\sigma_c) > 0$ and $d(1-\sigma_c) < 0$. Also, we need to consider the case in which $\sigma_c \rightarrow 1$ asymptotically. As already stated, if $\sigma_c = 1$ everywhere, a steady-state exists.

$$i) \quad d(1-\sigma_c) > 0 \quad \text{and} \quad k_c \rightarrow \infty, \quad (1-\gamma) \rightarrow 1 \quad \text{and} \quad \alpha_c \rightarrow 1$$

$$90') \quad (\dot{P}/P) \rightarrow d(1-\alpha_c) \rightarrow 0$$

⁴Earlier we assumed $\sigma_c < 1$ or $\sigma_c > 1$ everywhere; since, as $t \rightarrow \infty$, $x \rightarrow 0$ ($d > 0$) or $x \rightarrow \infty$ ($d < 0$), and since σ_c depends only on x , it suffices that σ_c be bounded from unity asymptotically. That is, the same asymptotic result holds true if $\sigma_c < 1$ [or $\sigma_c > 1$] only asymptotically, though it may fluctuate between the two for intermediate values of x .

$$91') \quad (\dot{\gamma}/\gamma) \rightarrow -d(1-\sigma_c) < 0$$

$$92') \quad \frac{[d(1-\gamma)]}{dt} / (1-\gamma) \rightarrow \{[\gamma d(1-\sigma_c)] / (1-\gamma)\} \rightarrow 0$$

$$89') \quad (\dot{C}/C) \rightarrow n + d + (\dot{\gamma}/\gamma) + \alpha_c (\dot{x}/x) \rightarrow n$$

$$88') \quad (\dot{M}/M) \rightarrow n$$

$$\text{ii) } d(1-\sigma_c) < 0 \text{ and } k_c \rightarrow 0, (1-\gamma) \rightarrow [k/k_m], \alpha_c \rightarrow 0$$

$$90'') \quad (\dot{P}/P) \rightarrow d$$

$$91'') \quad (\dot{\gamma}/\gamma) \rightarrow 0$$

$$92'') \quad \frac{[d(1-\gamma)]}{dt} / (1-\gamma) \rightarrow 0$$

$$89'') \quad (\dot{C}/C) \rightarrow n + d + (\dot{\gamma}/\gamma) + \alpha_c (\dot{x}/x) \rightarrow [n + d]$$

$$88'') \quad (\dot{M}/M) \rightarrow n$$

Finally, it is possible that $\sigma_c \rightarrow 1$ asymptotically; in this case k_c may tend to zero or infinity, but α_c approaches some non-zero value.

$$90''') \quad (\dot{P}/P) \rightarrow d(1-\alpha_c)$$

$$91''') \quad (\dot{\gamma}/\gamma) \rightarrow 0$$

$$92''') \quad \frac{[d(1-\gamma)]}{dt} / (1-\gamma) \rightarrow 0$$

$$89''') \quad (\dot{C}/C) \rightarrow [n + d(1-\alpha_c)]$$

$$88''') \quad (\dot{M}/M) \rightarrow n$$

Thus, if $k_c \rightarrow \infty$ (and σ_c bounded from one) there is no continual growth in per capita consumption or the price level. The reason for this is, if $d > 0$, then $\sigma_c < 1$ - the resources are continually shifted away from C, offsetting the gains from technical progress. On

the other hand, if $d < 0$ (technical regression) then x tends to infinity and the output-capital elasticity tends to one - and this is enough to offset the negative effect on consumption due to the technical regression (and the shifting of resources out of C).

Therefore, if $k_c \rightarrow \infty$, then the effect of technical progress is transitory (σ_c bounded from one) in the sense that the asymptotic growth rates of C, M, and P are the same as if there were no technical progress at all. Finally, consider the asymptotic values of P and C:

$$93) \quad C = [(1-s)/s]PM \quad ; \quad P = [f'_c(x)]/[f'_m(k_m)]$$

If the Inada conditions hold, then P and (C/L) tend to infinity for $d > 0$ (since $x \rightarrow 0$, $\sigma_c < 1$), whereas P and (C/L) tend to zero for $d < 0$. Thus, even though the growth rates of P and (C/L) tend to zero, their asymptotic values tend to zero or infinity. For $d > 0$, there is a bonus of greater consumption due to the technical progress, whereas for $d < 0$, there is a decrease in consumption. However, if the Inada conditions are not fulfilled (and not all four can hold for a C.E.S. function unless $\sigma=1$), then P and (C/L) both have finite, non-zero limits (as M always does in this case).

However, if k_c tends to zero ($d > 0$, $\sigma_c > 1$ or $d < 0$, $\sigma_c < 1$), then the fruits of technical progress show up in the growth rates of the variables. For example, if $d > 0$, $\sigma_c > 1$, then the continually decreasing effective capital-labor ratio in C (x) eventually ceases to be important since the output-capital elasticity tends to zero, and a point is reached when no more resources are transferred out of (or into) C, so that the share of labor employed in C remains constant. Once this stage is reached, the effect of technical progress dominates

TABLE VII - Asymptotic Growth Rates of Variables with Harrod Neutral Technical Progress Occurring Only in Sector C

Variable	$\sigma_c > 1$	$d > 0$		$d < 0$	
		$\sigma_c > 1$	$\sigma_c < 1$	$\sigma_c > 1$	$\sigma_c < 1$
M	n	n	n	n	n
C	$[n+d(1-\alpha_c)]$	$[n+d]$	n	n	$[n+d]$
P	$d(1-\alpha_c)$	d	0	0	d
γ	0	0	$-d(1-\sigma_c)$	$-d(1-\sigma_c)$	0
$(1-\gamma)$	0	0	0	0	0
W	$d(1-\alpha_c)$	d	0	0	d
R	$d(1-\alpha_c)$	d	0	0	d
k	0	0	0	0	0
ω	0	0	0	0	0

and C will increase over time (for $d > 0$). Table VII summarizes these results.

C. Extending the Analysis - Harrod Technical Progress Only in the Investment Sector

Suppose that Harrod neutral technical progress occurs at rate g in M and not at all in C:

$$M = F_m(K_m, L_m e^{gt}) \quad ; \quad C = F_c(K_c, L_c)$$

This can be rewritten to indicate that Harrod neutral technical progress occurs at rate g in both sectors, and at rate $d (= -g)$ in sector C:

$$M = F_m(K_m, L_m e^{gt}) \quad ; \quad C = F_c(K_c, [L_c e^{gt}] e^{dt}) \quad ; \quad d = -g$$

As we know, when Harrod neutral technical progress occurs at the same rate in each sector, a steady-state exists, and the appropriate variables, when expressed in efficiency units, are constant. Therefore, we can use our previous analysis to study this case, except that we now interpret our variables as being expressed in efficiency units. For example, in this case $\bar{\omega}$ and \bar{k} (the efficiency variables) would both approach positive, finite values, though ω and k would each grow asymptotically at rate g . Also, consumption per capita would grow at rate g plus the rate determined from Table VII with $[d = -g]$. Consequently, (C/L) would grow at rate g if $\sigma_c > 1$, and it would not (asymptotically) grow if $\sigma_c < 1$. Thus, we are prepared to handle the occurrence of Harrod neutral technical progress in M alone, or at different rates in the two sectors.

Similarly, if capital-augmenting technical progress also occurs in C, we can treat this as Hicks neutral technical progress in C plus some negative rate of Harrod neutral technical progress in C. Again, our analysis would be identical to that just completed, except that consumption, the price level, the wage rate and the rental rate (in numeraire units) would all grow at the additional rate of Hicks neutral technical progress in C, as well as at the rates determined from Table VII.

Finally, if Hicks neutral technical progress occurs in M (it can also occur in C), and Harrod neutral technical progress occurs at the same rate in each sector, then the analysis would be identical to that performed in Part I, except that the variables would be expressed in efficiency units.

Therefore, the only case we can not yet treat is if Hicks

neutral technical progress occurs in M and if Harrod neutral technical progress occurs at different rates in the two sectors. Note, however, that if $\sigma_m = 1$, then we can always treat the problem as though Harrod neutral technical progress occurs at the same rate in each sector (or else as though there were no capital-augmenting technical progress in M), and thus our earlier analysis can cope with this case. Similarly, if $\sigma_c = 1$, we can arrange it so that Harrod neutral technical progress occurs at the same rate in each sector. Therefore, if:

$$C = F_c(e^{gt}K_c, e^{dt}L_c) ; \quad M = F_m(e^{at}K_m, e^{bt}L_m) ,$$

then the only case we can not yet handle is:

$$i) \quad a \neq 0, \text{ and } (b-a) \neq (d-g) \text{ and } \sigma_c, \sigma_m \neq 1$$

The reason our previous analysis can not handle this case is that, in treating the case of Harrod neutral technical progress in only one sector we assumed that (ω, k) remained finite - and this will not be true if some Hicks technical progress occurs in M. To treat this case we must return to (and modify) our analysis of Part I.

Before considering this case, let us briefly summarize our findings on the asymptotic growth rates in various cases:

$$ii) \quad a \neq 0, \sigma_c, \sigma_m \neq 1, \text{ but } (b-a) = (d-g)$$

When the variables are expressed in efficiency units, they will asymptotically grow at the rates determined in Part I, with Hicks technical progress in M at rate a . Naturally, these rates depend upon the elasticities of substitution in each sector. To derive the growth rates of the original variables, one need only add the growth rates that occur in a normal two-sector model with Harrod neutral technical progress at rate $(b-a)$ in each sector.

iii) $a = 0, b \neq (d-g)$

If $\sigma_c = 1$, a steady-state exists with Harrod neutral technical progress at rate b in each sector, and Hicks neutral technical progress at rate $[(d-b)(1-\alpha_c) + g\alpha_c]$ in sector C. If $\sigma_c \neq 1$, then Harrod neutral technical progress occurs at rate b in both sectors, and at rate $(d-g-b)$ in sector C, as well as Hicks neutral technical progress in sector C at rate g .

iv) σ_c or $\sigma_m = 1$

As explained previously, any problem can be converted into a type ii) or type iii) problem.

v) $a = 0$ and $b = (d-g)$

This represents the normal steady-state case with Harrod neutral technical progress at rate b in both sectors and Hicks neutral technical progress at rate g in C.

vi) $a = 0$ and $\sigma_m = 1$

This case is essentially the same as case iii).

vii) $a = 0$ and $\sigma_c = 1$

In this case a normal steady-state occurs since it can be converted into a type v) problem due to the equivalence of Hicks and Harrod technical progress in C.

viii) $a \neq 0, (b-a) = (d-g),$ and $\sigma_m = 1$

The Hicks neutral technical progress in M can be converted into Harrod neutral technical progress, and thus this case is equivalent to case iii).

ix) $a \neq 0, (b-a) = (d-g),$ and $\sigma_c = 1$

This case is essentially the same as case ii).

x) $a = 0, b = (d-g), \sigma_c, \sigma_m = 1$

This is the normal steady-state case, and the restrictions on the elasticities of substitution are unnecessary.

Thus, the only case we cannot yet handle is case i) - let us now turn our attention to this case.

IV. Hicks Technical Progress in the Investment Sector and Harrod Technical Progress in the Consumption Sector

So far, we have considered how Hicks technical progress in sector M or Harrod technical progress in sector C will affect the two-sector growth model. Specifically, we have found that in the former case the capital-labor ratio tends to infinity, but the growth rates approach finite limits for $\sigma_m \leq 1$ (and the results are comparable to the one-sector model. In the latter case, which can only occur in multi-sector models, the capital-labor ratio and wage-rental ratio for the economy tend to finite limits, and the principal difference between this case and the normal steady-state case is that the output-capital elasticity (and capital's share under competitive pricing) tends to zero or one in the consumption sector ($\sigma_c \neq 1$). In this part we shall consider both cases simultaneously, and we shall find that, though the analysis is comparable to that for Part I, our results will be more complicated. For example, the growth rates of the system depend not only on σ_c and σ_m and the rates of technical progress, but they also depend on whether Hicks neutral technical progress in M occurs at a faster rate than Harrod neutral technical progress in C. Let us now investigate why this is so.

For simplicity, we assume that only Hicks neutral technical progress occurs in M and only Harrod neutral technical progress occurs in C; it is quite clear from our previous discussions that all other cases can be readily incorporated into this case.

Using our previous notation:

$$i) \quad a = b, \quad \text{and} \quad g = 0, \quad d \neq 0; \quad \sigma_c, \quad \sigma_m \neq 1$$

For simplicity, we adopt the simple savings assumption: $s_k = s_n$.

Our two basic equations, as discussed in Parts II and III, are:

$$94) \quad \dot{X} \equiv (\dot{K}/K) = [(1-\gamma)e^{at}f'_m]/k$$

$$95) \quad (1-s)(1-\gamma)(\omega+k_m) - s\gamma(\omega+k_c) = 0 \quad \text{defines:} \quad H(\omega, k, t) = 0$$

We must remember that from equation 95) ω is defined as a function of t as well as of k . Similarly, k_c (it appears in equation 94) in $[1-\gamma]$) depends on time as well as on ω .

We follow our analysis of Part II, page 85, and find:

$$96) \quad (\dot{X}/X) = \left\{ \frac{d(1-\gamma)}{dt} / (1-\gamma) \right\} + a + \alpha_m (\dot{k}_m/k_m) - (X-n)$$

$$97) \quad (\dot{k}_m/k_m) = \{[\sigma_m(X-n)]/\sigma\}, \quad \text{where} \quad \sigma = [(dk/d\omega)(\omega/k)], \quad \text{depends} \\ \text{on } t \text{ as well as on } k.$$

$$98) \quad \frac{d(1-\gamma)/dt}{(1-\gamma)} = \frac{(X-n)[(1/\sigma)\{\gamma\sigma_c k_c + (1-\gamma)\sigma_m k_m\} - k] + k_c \gamma d(1-\sigma_c)}{(k_c - k)}$$

Thus, comparable to equation 12, page 86:

$$99) \quad (\dot{X}/X) = (TX - Tn + A) + \{[k_c \gamma d(1-\sigma_c)]/(k_c - k)\}$$

where T and σ are defined to be the same as in part II, equations 16a) and 17a), except that they implicitly depend upon time.

As earlier, σ is always positive; we have seen that T is negative if either: i) $k_c \geq k_m$ or ii) $\sigma_c \geq 1$

In part III we showed that if ω tends to a finite limit, then eventually $[d(1-\sigma_c)/(k_c - k)] > 0$. Unfortunately, this is not

necessarily true if ω grows without bound, and so we cannot be sure of the positiveness of the last term in 99).

From the previous two parts it is clear that both k and ω tend to infinity; therefore, we must be content to consider the asymptotic growth path. We proceed as in Part II, considering what happens to the growth rates for various values of σ_c and σ_m .

Table VIII presents the asymptotic growth rates; several comments about the Table are in order. First of all, it no longer suffices (in some cases) to state whether σ_c is greater than, equal to, or less than one - in some cases we must further subdivide these intervals. Secondly, when we write that σ_c tends to one (or σ_m tends to one), we assume that it asymptotically approaches one - if σ_c or σ_m were unity throughout, then our prior analysis would suffice, as previously explained. Thirdly, in calculating the limits we assumed (as in Part II) that σ_c and σ_m were bounded from zero or infinity - or else approaches those values "sufficiently slowly" (see footnote 2, page 100).

Though it is true, as we can see from the Table, that k , ω , and k_m always tend to infinity, the same is not true of x (the effective capital-labor ratio in C). In fact, what happens to x depends upon the relationship between A and d (the rates of Hicks technical progress in M and Harrod technical progress in C, respectively), and, in some cases, σ_m . For example, from the Table we see that for $\sigma_c \geq 1$, $\sigma_m \rightarrow 1$, then:

$$\text{sign}(\dot{x}/x) = \text{sign}[A - d(1 - \alpha_m)]$$

When the expression in brackets is positive, x tends to infinity; when it is negative, x tends to zero. If $A = d(1 - \alpha_m)$, then the asymptotic value of x depends upon the initial conditions, the behavior of σ_m as $\omega \rightarrow \infty$, and so forth. In general, x can tend to either zero or infinity in

TABLE VIII - Asymptotic Values and Growth Rates with Hicks Neutral Technical Progress in M at Rate A and Harrod Neutral Technical Progress in C at Rate d

Values σ_c, σ_m	Asymptotic Values			Asymptotic Growth Rates			
	σ	$x \neq k_c e^{-dt}$	α_c	α_m	K or M	k	R
<u>$\sigma_m > 1$:</u>							
$\sigma_c > 1$	$\text{Min}[\sigma_c, \sigma_m]$	∞	1	1	∞	∞	0
$\sigma_c \rightarrow 1$	1	∞	α_c^*	1	∞	∞	$-\infty$
$\sigma_c < 1$	1	∞	0	1	∞	∞	$-\infty$
<u>$\sigma_m > 1$:</u>							
$\sigma_c > 1$							
$A > d(1-\alpha_m)$	1	∞	1	α_m^*	$n + [A/(1-\alpha_m)]$	$[A/(1-\alpha_m)]$	0
$A = d(1-\alpha_m)$	1	0 or ∞	0 or 1	α_m^*	$n + [A/(1-\alpha_m)]$	$[A/(1-\alpha_m)]$	0
$A < d(1-\alpha_m)$	1	0	0	α_m^*	$n + [A/(1-\alpha_m)]$	$[A/(1-\alpha_m)]$	$[d - \{A/(1-\alpha_m)\}]$
<u>$\sigma_c \rightarrow 1$:</u>							
$A > d(1-\alpha_m)$	1	∞	α_c^*	α_m^*	$n + [A/(1-\alpha_m)]$	$[A/(1-\alpha_m)]$	$[(1-\alpha_c)(d - \{A/(1-\alpha_m)\})]$
$A = d(1-\alpha_m)$	1	0 or ∞	α_c^*	α_m^*	$n + [A/(1-\alpha_m)]$	$[A/(1-\alpha_m)]$	0
$A < d(1-\alpha_m)$	1	0	α_c^*	α_m^*	$n + [A/(1-\alpha_m)]$	$[A/(1-\alpha_m)]$	$[(1-\alpha_c)(d - \{A/(1-\alpha_m)\})]$

Values σ_c, σ_m	σ	$x = k e^{-dt}$	α_c	α_m	K or M	k	R
<u>$\sigma_m \rightarrow 1$:</u>							
<u>$\sigma_c < 1$</u>							
$A > d(1 - \alpha_m)$	1	∞	0	α_m^*	$n + [A/(1 - \alpha_m)]$	$[A/(1 - \alpha_m)]$	$[d - \{A/(1 - \alpha_m)\}]$
$A = d(1 - \alpha_m)$	1	0 or ∞	0 or 1	α_m^*	$n + [A/(1 - \alpha_m)]$	$[A/(1 - \alpha_m)]$	0
$A < d(1 - \alpha_m)$	1	0	1	α_m^*	$n + [A/(1 - \alpha_m)]$	$[A/(1 - \alpha_m)]$	0
<u>$\sigma_m < 1$:</u>							
<u>$\sigma_c > 1$</u>							
$A > d$	1	∞	1	0	$[n + A]$	A	0
$A = d$	1	∞	1	0	$[n + A]$	A	0
$A < d$	†	0	0	0	$[n + A]$	A	$\text{Max}[0, d - (A/\sigma_m), (d - A)/\sigma_c]$
<u>$\sigma_c \rightarrow 1$</u>							
$A > d$	1	∞	α_c^*	0	$[n + A]$	A	$[(d - A)(1 - \alpha_c)]$
$A = d$	1	∞	α_c^*	0	$[n + A]$	A	0
$A < d$	1	0	α_c^*	0	$[n + A]$	A	$[(d - A)(1 - \alpha_c)]$
<u>$\sigma_c < 1$</u>							
$A > d$	††	∞	0	0	$[n + A]$	A	$\text{Max}[(d - A)/\sigma_c, d - (A/\sigma_m)]$
$A = d$	1	∞	0	0	$[n + A]$	A	0
$A < d$	1	0	1	0	$[n + A]$	A	0

† $A \geq \sigma_m d + \sigma = \sigma_m$; $A < \sigma_m d$, $\sigma = \sigma_c$ or σ_m as $\sigma_c \leq \{[(A - d)\sigma_m]/(A - d\sigma)\}$; (for equality, $\sigma \in [\sigma_c, \sigma_m]$)

†† $\sigma = \sigma_m$ or σ_c as $\sigma_c \leq \{[(A - d)\sigma_m]/(A - d\sigma)\}$; (for equality, $\sigma \in [\sigma_c, \sigma_m]$)

TABLE VIII - Continued

Asymptotic Growth Rates

Values σ_c, σ_m	W and (C/L)	ω	P	X
<u>$\sigma_m > 1$:</u>				
$\sigma_c > 1$	∞	∞	$-A$	∞
$\sigma_c \rightarrow 1$	∞	∞	$-\infty$	∞
$\sigma_c < 1$	d	∞	$-\infty$	∞
<u>$\sigma_m \rightarrow 1$:</u>				
$\sigma_c > 1$	$\text{Max}[d, A/(1-\alpha_m)]$	$[A/(1-\alpha_m)]$	$\text{Max}[0, d - \{A/(1-\alpha_m)\}]$	$\frac{\sigma_c [A-d(1-\alpha_m)]}{(1-\alpha_m)}$
$\sigma_c \rightarrow 1$	$(1-\alpha_c)d + \frac{\alpha_c A}{(1-\alpha_m)}$	$[A/(1-\alpha_m)]$	$(1-\alpha_c)[d - \{A/(1-\alpha_m)\}]$	$\frac{\sigma_c [A-d(1-\alpha_m)]}{(1-\alpha_m)}$
$\sigma_c < 1$	$\text{Min}[d, A/(1-\alpha_m)]$	$[A/(1-\alpha_m)]$	$\text{Min}[0, d - \{A/(1-\alpha_m)\}]$	$\frac{\sigma_c [A-d(1-\alpha_m)]}{(1-\alpha_m)}$
<u>$\sigma_m < 1$:</u>				
$\sigma_c > 1$	$\text{Max}[A, d]$	$\text{Min}[A/\sigma_m, \text{Max}\{A, [A-d(1-\sigma_c)]/\sigma_c\}]$	$\text{Max}[0, (d-A)]$	$\text{Max}[\sigma_c(A-d), \text{Min}\{(A-d), [\sigma_c(A-d\sigma_m)/\sigma_m]\}]$ $(A-d)$
$\sigma_c \rightarrow 1$	$\alpha_c A + d(1-\alpha_c)$	A	$[(d-A)(1-\alpha_c)]$	
$\sigma_c < 1$	$\text{Min}[A, d]$	$\text{Min}[A/\sigma_m, \text{Max}\{A, [A-d(1-\sigma_c)]/\sigma_c\}]$	$\text{Min}[0, (d-A)]$	$\text{Max}[\sigma_c(A-d), \text{Min}\{(A-d), [\sigma_c(A-d\sigma_m)/\sigma_m]\}]$

this case. There are other examples in which this problem arises.

If we compare Table VIII to Table IV of Part II, we see that whenever $d=0$ (hence $A>d$), the two tables are identical, as expected. However, when $d\neq 0$, the growth rates of R , W , ω , C , x , and P will, in general, depend upon d (as well as A), whereas the growth rates of K , k , and M depend only upon A and the values of σ_c and σ_m .

As we can readily see from the Table, the growth rates of K , k , ω , and M are always positive. The growth rates of (C/L) and W are also positive for $A, d > 0$; however, if $\sigma_c \leq 1$ and $d < 0$, then they may be negative. On the other hand, the growth rates of P and R may be negative, zero, or positive, depending upon the elasticities of substitution and the relationship between A and d (the growth rates of P and R can be positive only if $d > A > 0$). Therefore, if $d > A$ (and not both $\sigma_c, \sigma_m > 1$), then the growth rates of P and R will be non-negative - the technical progress in C "outweighs" the technical progress in M in this case.

Finally, given Table VIII we can handle the general case in which both types of factor-augmenting technical progress occur in each sector:

$$M = F_m(K_m e^{at}, L_m e^{bt}) = e^{at} F_m(K_m, L_m e^{(b-a)t})$$

$$C = F_c(K_c e^{gt}, L_c e^{dt}) = e^{gt} F_c(K_c, L_c e^{(b-a)t} e^{d^*t}); \quad d^* = (d+a)-(b+g)$$

Thus, the general case⁵ is equivalent to:

⁵The assumption that technical progress is only factor-augmenting is a restriction in itself; thus, this represents the general case, given that restriction.

TABLE IX - Asymptotic Growth Rates - General Case

<u>Variables</u>	<u>Asymptotic Growth Rates</u>
K, k, M, and ω	$[\rho(a, d^*) + (b-a)]$
R and P	$[\rho(a, d^*) + g]$
W and (C/L)	$[\rho(a, d^*) + (b+g-a)]$

$\rho(a, d^*)$ is the rate of growth for the appropriate variable obtained from Table VIII when Hicks technical progress occurs at rate a in M and Harrod technical progress occurs at rate d^* in C, with $d^* = [(d+a) - (b+g)]$.

-
- i) Hicks technical progress in C at rate g
 - ii) Harrod technical progress in both sectors at rate $(b-a)$
 - iii) Hicks technical progress in M at rate a , and
 - iv) Harrod technical progress in C at rate d^*

We already know how to handle parts i) and ii) since they yield steady-state solutions; furthermore, iii) and iv) are the cases we have just studied. Therefore, the growth rates in this general case will be the sum of the growth rates from the respective parts. Table IX summarizes the growth rates for each of the variables.

V. Factor Shares

In the previous discussion, we have determined the asymptotic growth rates of the variables - as we have seen, all the variables will approach some asymptotically constant (if finite) growth rate, the magnitude of which depends upon many of the parameters of the

problem. However, in general, the effective capital-labor ratio in each sector will tend to either zero or infinity, and so the factor shares (under competitive pricing) will tend to zero or one unless the elasticity of substitution in that sector tends to one. For those who are firm believers in marginal cost pricing, this result will hardly seem plausible. However, it should be noted that while the factor shares in each sector may tend to zero or one, for the economy as a whole they may approach some intermediate value. Thus, from the market equilibrium equation we can write:

$$100) \quad sY = s(C + PM) = PM ; \text{ or, } C = [(1-s)/s]PM$$

Clearly, the share of capital for the economy is the weighted average of the capital-share for each sector:

$$101) \quad \alpha = [(RK)/(RK + WL)] = [R(K_c + K_m)/(C + PM)] = [(1-s)\alpha_c + s\alpha_m]$$

Thus, unless both α_c and α_m tend to zero (or both tend to one), α will tend to some intermediate value.

Therefore, while the factor shares in each sector will, in general, tend to zero or one, it is quite possible that factor shares for the economy will be non-zero for both factors. From Table VIII we can determine the factor shares in each sector, and given s , we can subsequently determine factor shares for the economy as a whole. We shall have more to say on this topic and other related issues in Chapter 3.

VI. Maximizing the Asymptotic Rate of Growth of Consumption

In Part II we discussed the problem of allocating technical progress (a surrogate for research funds) in order to maximize the

asymptotic rate of growth of consumption, given that technical progress had to be Hicks neutral. In general, we found (Table V) that, unless $\sigma_c < 1$, the planner should allocate at least some technical progress to the investment sector, so that a steady-state would not occur. We can now generalize this discussion to include all types of factor-augmenting technical progress.

A. Allocating Labor-Augmenting Technical Progress Between Sectors

As our first example, let us assume that technical progress is only of the labor-augmenting variety, so that:

$$M = F_m(K_m, L_m e^{ht}) ; \quad C = F_c(K_c, L_c e^{gt}) ; \quad h = h(g) ; \quad h, g \geq 0.$$

$$g = \lambda, h(\lambda) = 0 ; \quad g = 0, h(0) = A ; \quad h', h'' < 0$$

Naturally, $h(g)$ represents the trade-off between technical progress in the two sectors; though it may seem reasonable for this frontier to be symmetrical, we do not impose this a priori restriction.

If Harrod neutral technical progress occurs at rate h in M and at rate g in C , this is equivalent to it occurring at rate h in both sectors and at rate $d [= (g-h)]$ in sector C . Thus:

$$102) \quad (\dot{C}/C) = [h + \delta(d)] , \quad \text{where } \delta(d) \text{ is the rate of growth of}$$

consumption when Harrod neutral technical progress occurs only in C .

Since we have already studied this problem in Part III, we can immediately write:

$$103) \quad d(1-\sigma_c) > 0 \rightarrow (\dot{C}/C) = (n+h); \quad d(1-\sigma_c) < 0 \rightarrow (\dot{C}/C) = (n+g)$$

$$d = 0 \rightarrow (\dot{C}/C) = (n+g) ; \quad \sigma_c \rightarrow 1 \rightarrow (\dot{C}/C) = [n + \alpha_c h + (1-\alpha_c)g]$$

If $\sigma_c = 1$ everywhere, then a steady-state exists, and C grows at the same rate as the asymptotic growth rate for the case in which $\sigma_c \rightarrow 1$. Given equation 103), we can make our decision about how to allocate technical progress, given the (asymptotic) value of σ_c :

$$\begin{aligned}
 104) \quad \sigma_c > 1: \quad g > h &\rightarrow (\dot{C}/C) = n + g; \quad g = \lambda, h = 0 \\
 \sigma_c > 1: \quad g < h &\rightarrow (\dot{C}/C) = n + h; \quad g = 0, h = A \\
 \sigma_c < 1: \quad (\dot{C}/C) &= n + \text{Min}(h, g) = n + \hat{g}; \quad h(\hat{g}) = \hat{g}, d = 0 \\
 \sigma_c \rightarrow 1: \quad (\dot{C}/C) &= n + \alpha_c h^* + (1 - \alpha_c)g^*; \quad h'(g^*) = -[(1 - \alpha_c)/\alpha_c]; \quad h'' < 0
 \end{aligned}$$

We can see that for $\sigma_c < 1$, one should allocate technical progress equally between the two sectors, so that a steady-state would exist. If $\sigma_c = 1$, one should (barring a corner solution) allocate some technical progress to each sector, but a steady-state would exist, as stated previously.

If asymptotically $\sigma_c \rightarrow 1$, then, in general, some technical progress should be allocated to each sector, but a steady-state will probably not occur. Finally, if $\sigma_c > 1$, then one should allocate all technical progress to only one sector, that sector depending upon the shape of the transformation curve. Thus, as $\lambda \gtrless A$, the planner should allocate all technical progress to C, to either sector, or to M. Therefore, in the case $\lambda = A$, the planner is indifferent (under this criterion) as to where technical progress is to be allocated, except that it all must go to one sector.

So far we have overlooked one factor - if $d [= (g-h)] > 0$, then x (the effective capital-labor ratio in C) tends to zero, whereas if $d < 0$, x tends to infinity. Clearly, the asymptotic value of σ_c as $x \rightarrow 0$ may be different from the value of σ_c as $x \rightarrow \infty$. Thus, the planner may have to

base his decision on the sign of d , as well as the asymptotic value of σ_c .

For example, suppose $\sigma_c < 1$ as $x \rightarrow 0$, and $\sigma_c > 1$ as $x \rightarrow \infty$. From 104), it is quite clear that the growth rate for $\sigma_c > 1$ is greater than that for $\sigma_c < 1$. Therefore, the planner must choose $x \rightarrow \infty$ (thus, $d < 0$). This implies that he should choose $g=0, h=A$ even if $A < \lambda$.

Similarly, suppose $\sigma_c \rightarrow 1$ as $x \rightarrow 0$, and $\sigma_c > 1$ (or $\sigma_c < 1$) as $x \rightarrow \infty$. If $\sigma_c \rightarrow 1$:

$$105) \quad (\dot{C}/C) = n + \alpha_c h^* + (1-\alpha_c)g^* \quad ; \quad h'(g^*) = -[(1-\alpha_c)/\alpha_c]$$

Thus, h^* and g^* now depend only upon α_c . Define:

$$106) \quad \theta = [\alpha_c h^* + (1-\alpha_c)g^*] \quad ; \quad h'(g^*) = -[(1-\alpha_c)/\alpha_c]$$

Then it can readily be shown that:⁶

⁶All of the following results are derivable from the definition of θ :
 $\theta = \alpha_c h^* + (1-\alpha_c)g^*$, where $h'(g^*) = -[(1-\alpha_c)/\alpha_c]$; $(dg^*/d\alpha_c) < 0$

Thus:

$$i) \quad (d\theta/d\alpha_c) = (h^* - g^*) + [\alpha_c (dh^*/dg^*) + (1-\alpha_c)](dg^*/d\alpha_c) \quad ;$$

Therefore, $(d\theta/d\alpha_c) = [(h^* - g^*) + 0] = (h^* - g^*)$, and

$$ii) \quad (d^2\theta/d\alpha_c^2) = [(dh^*/dg^*) - 1](dg^*/d\alpha_c) = \\ -[1 + \{(1-\alpha_c)/\alpha_c\}](dg^*/d\alpha_c) = -(1/\alpha_c)(dg^*/d\alpha_c) > 0$$

Therefore, θ is a minimum for $\hat{\alpha}_c$ such that $h^* = g^*$; there are no interior maxima, just boundary maxima.

$$iii) \quad \theta(0) = \lambda, \quad \theta(1) = A$$

By continuity, if $A \neq \lambda$, (assume $A > \lambda$), then there exists $\tilde{\alpha}_c$ such that:

$$iv) \quad A > \lambda, \quad \alpha_c > \tilde{\alpha}_c, \quad \theta(\alpha_c) > \lambda$$

Therefore, depending on the slope of the transformation curve, the planner may choose either:

$$h^* > g^* > 0 \quad \text{or} \quad h = A^*, g = 0 \quad (\text{for } \alpha_c > \tilde{\alpha}_c).$$

A comparable result holds for $A < \lambda$.

- 107) $\text{Max}(A, \lambda) \geq \theta \geq \hat{g}$ where $h(\hat{g}) = \hat{g}$
- 108) There exists a unique $\hat{\alpha}_c$ such that: $\theta(\hat{\alpha}_c) = \hat{g}$; $\alpha_c \neq \hat{\alpha}_c$, $\theta > \hat{g}$
- 109) $A = \lambda$ implies $\theta(\alpha_c) < A$ if $\alpha_c(1-\alpha_c) \neq 0$
- 110) If $A > \lambda$, there exists an $\tilde{\alpha}_c$ such that: $\alpha_c > \tilde{\alpha}_c \rightarrow \theta(\alpha_c) > \lambda$
- 111) If $A < \lambda$, there exists an α_c^\dagger such that $\alpha_c < \alpha_c^\dagger \rightarrow \theta(\alpha_c) > A$

Given the above properties of θ , we can decide how to allocate technical progress should $\sigma_c \rightarrow 1$. For example, if $\sigma_c \rightarrow 1$ as $x \rightarrow 0$, and $\sigma_c < 1$ as $x \rightarrow \infty$, we find that, given $\alpha_c(0)$ as $x \rightarrow 0$:

- 112) If $\alpha_c(0) < \hat{\alpha}_c$ ($\hat{\alpha}_c$ implies $h^* = g^*$), then $g^* > h^* \geq 0$ (g^* , h^* determined from $h'(g^*) = -[(1-\alpha_c(0))/\alpha_c(0)]$), and $g^* > h^*$ implies $x \rightarrow 0$. Therefore, the maximum occurs at g^* , h^* . However, if:
- 113) $\alpha_c(0) > \hat{\alpha}_c$, then $h^* > g^* \geq 0$, which implies $x \rightarrow \infty$, and hence $\sigma_c < 1$.⁷ Thus, in this case we find:
- 114) To maximize (\dot{C}/C) choose: $g^* = \hat{g} = h(\hat{g})$ if $\alpha_c(0) \geq \hat{\alpha}_c$, and choose: $g^* > h^* \geq 0$ [where $h'(g^*) = -\{(1-\alpha_c(0))/\alpha_c(0)\}$] if $\alpha_c(0) < \hat{\alpha}_c$

⁷For $\alpha_c > \hat{\alpha}_c$, $h^* > g^*$ implies $x \rightarrow \infty$. Given $\sigma_c \rightarrow 1$ as $x \rightarrow 0$, for fixed α_c , it follows that we seek:

i) $\text{Max}_{\{g\}}[\theta] = [\alpha_c h + (1-\alpha_c)g]$ such that $g \geq h$. But since θ

is concave in g (given α_c), and since $\alpha_c > \hat{\alpha}_c$, this implies $g^* = \hat{g} = h(\hat{g})$, the same solution as for $\sigma_c < 1$, and hence a steady-state.

Therefore, it is not enough to consider the asymptotic values of σ_c , but we also need information about the shape of the transformation curve (its slope at $h(\hat{g}) = \hat{g}$), and the asymptotic value(s) of α_c .

Of course, we must follow the same procedures in all cases in which $\sigma_c \rightarrow 1$ as either $x \rightarrow 0$ or as $x \rightarrow \infty$ (or both). Thus, if $\sigma_c \rightarrow 1$ as $x \rightarrow 0$ and as $x \rightarrow \infty$, we **must** consider both the values of $\alpha_c(0)$ and of $\alpha_c(\infty)$.

Table X (on page 138) summarizes the results for all possible cases.

B. Allocating Technical Progress Within the Consumption Sector

Alternatively, we might suppose that the planner has to decide how to allocate factor-augmenting technical progress within C (somewhat analogous to the one-sector Kennedy model).

$$115) \quad M = F_m(K_m, L_m) \quad ; \quad C = F_c(K_c e^{ht}, L_c e^{gt}) = e^{ht} F_c(K_c, L_c e^{(g-h)t})$$

Letting $d = (g-h)$, we have:

$$116) \quad (\dot{C}/C) \rightarrow [h + \delta(d)], \quad \text{where } \delta(d) \text{ is the rate of growth of}$$

consumption that results from Harrod neutral technical progress in C alone. However, it is obvious that this problem is the same as the one just discussed. That is, under the criterion of maximizing the asymptotic rate of growth of consumption, the problem is comparable if the choice is either:

- i) Between Harrod technical progress in C or in M
- ii) Between varying amounts of factor-augmenting technical progress within C.

In either case, the planner should reach the same decision as to how to best allocate technical progress (in so far as the occurrence of a

TABLE X - Allocating Labor-Augmenting Technical Progress
Between Sectors

$$\text{Either } C = F_c(K_c, L_c e^{gt}), \quad M = F_m(K_m, L_m e^{ht})$$

$$\text{Or } C = F_c(K_c e^{ht}, L_c e^{gt}), \quad M = F_m(K_m, L_m)$$

$h = h(g); \quad g, h \geq 0; \quad h(0) = A; \quad h(\lambda) = 0; \quad h', h'' < 0$

x is the effective capital-labor ratio in C.

Limit σ_c as

$x \rightarrow 0$	$x \rightarrow \infty$	$A = \lambda$	$A > \lambda$	$A < \lambda$
$\sigma_c > 1$	$\sigma_c > 1$	$g=0, h=A (x \rightarrow \infty)$ or $g=A, h=0 (x \rightarrow 0)$	$g=0, h=A (x \rightarrow \infty)$	$g=\lambda, h=0 (x \rightarrow 0)$
$\sigma_c > 1$	$\sigma_c \rightarrow 1$	$g=\lambda, h=0 (x \rightarrow 0)$	There exists α_c^* : $\alpha_c \leq \alpha_c^*, g=\lambda, h=0$ $\alpha_c > \alpha_c^*, h^* > g^* \geq 0$ $h'(g^*) = -(1-\alpha_c)/\alpha_c$ and $x \rightarrow \infty$.	$g=\lambda, h=0 (x \rightarrow 0)$
$\sigma_c > 1$	$\sigma_c < 1$	$g=\lambda, h=0 (x \rightarrow 0)$	$g=\lambda, h=0 (x \rightarrow 0)$	$g=\lambda, h=0 (x \rightarrow 0)$
$\sigma_c \rightarrow 1$	$\sigma_c > 1$	$g=0, h=A (x \rightarrow \infty)$	$g=0, h=A (x \rightarrow \infty)$	There exists α_c^* : $\alpha_c \geq \alpha_c^*, g=0, h=A$ $\alpha_c < \alpha_c^*, g^* > h^* \geq 0,$ $h'(g^*) = -(1-\alpha_c)/\alpha_c$ and $x \rightarrow 0$.
$\sigma_c \rightarrow 1$	$\sigma_c \rightarrow 1$	In this case it is unimportant whether $A \gtrless \lambda$. The values of $\alpha_c(0)$ and of $\alpha_c(\infty)$ are critical There exists an $\hat{\alpha}_c$ such that: $h^* \gtrless g^*$ and $x \rightarrow$ steady-state 10 as $\alpha_c \gtrless \hat{\alpha}_c$		
		1) If $\alpha_c(0) = \alpha_c(\infty)$, then $h' = -(1-\alpha_c)/\alpha_c; h \neq g$ if $\alpha_c \neq \hat{\alpha}_c$		
		2) $\alpha_c(\infty) > \hat{\alpha}_c, \alpha_c(0) \geq \hat{\alpha}_c; h' = -(1-\alpha_c(\infty))/\alpha_c(\infty); x \rightarrow \infty$		
		3) $\alpha_c(\infty) \leq \hat{\alpha}_c, \alpha_c(0) < \hat{\alpha}_c; h' = -[1-\alpha_c(0)]/\alpha_c(0); x \rightarrow 0$		

TABLE X - Continued

Limit σ_c as

$x \rightarrow 0$	$x \rightarrow \infty$	$A = \lambda$	$A > \lambda$	$A < \lambda$
$\sigma_c \rightarrow 1$	$\sigma_c \rightarrow 1$	4) $\alpha_c(\infty) < \alpha_c(0) = \hat{\alpha}_c$; $g^* = h^*$; Steady-State		
(Continued)		5) $\alpha_c(0) > \alpha_c(\infty) = \hat{\alpha}_c$; $g^* = h^*$; Steady-State		
		6) $\alpha_c(\infty) < \hat{\alpha}_c$, $\alpha_c(0) > \hat{\alpha}_c$; $g^* = h^*$; Steady-State		
		7) $\alpha_c(\infty) > \hat{\alpha}_c$, $\alpha_c(0) < \hat{\alpha}_c$; let: $h'_1(g_1) = \frac{-[1-\alpha_c(\infty)]}{\alpha_c(\infty)}$		
		$h'_2(g_2) = -[1-\alpha_c(0)]/\alpha_c(0)$;		
		Let: $\delta_1 = [\alpha_c(\infty)h_1 + \{1-\alpha_c(\infty)\}g_1]$		
		$\delta_2 = [\alpha_c(0)h_2 + \{1-\alpha_c(0)\}g_2]$. Then, choose:		
		g_1 or g_2 as $\delta_1 \geq \delta_2$ ($\delta_1 = \delta_2$, indifferent between g_1 and g_2)		
$\sigma_c \rightarrow 1$	$\sigma_c < 1$	Define $\hat{\alpha}_c$ such that $h^* \geq g^*$ as $\alpha_c \geq \hat{\alpha}_c$		
		1) $\alpha_c(0) \geq \hat{\alpha}_c$; choose $g^* = h^*$; Steady-State		
		2) $\alpha_c(0) < \hat{\alpha}_c$; choose $g^* > h^* \geq 0$ ($x \rightarrow 0$), where $h'(g^*) = -\{[1-\alpha_c(0)]/\alpha_c(0)\}$		
$\sigma_c < 1$	$\sigma_c > 1$	$g=0, h=A$ ($x \rightarrow \infty$)	$g=0, h=A$ ($x \rightarrow \infty$)	$g=0, h=A$ ($x \rightarrow \infty$)
$\sigma_c < 1$	$\sigma_c \rightarrow 1$	Define $\hat{\alpha}_c$ such that $h^* \geq g^*$ as $\alpha_c \geq \hat{\alpha}_c$		
		1) $\alpha_c(\infty) \leq \hat{\alpha}_c$; choose $g^* = h^*$; Steady-State		
		2) $\alpha_c(\infty) > \hat{\alpha}_c$; choose $h^* > g^* \geq 0$ ($x \rightarrow \infty$) where: $h'(g^*) = -\{[1-\alpha_c(\infty)]/\alpha_c(\infty)\}$		
$\sigma_c < 1$	$\sigma_c < 1$	$g^* = h^* > 0$	$g^* = h^* > 0$	$g^* = h^* > 0$
		(Steady-State in all three cases)		

steady-state is concerned), and the asymptotic growth rates are the same in the two cases. Also, there can only be a steady-state if asymptotically $\sigma_c < 1$ (as $x \rightarrow 0$ and as $x \rightarrow \infty$), if $\sigma_c = 1$ everywhere, or perhaps if $\sigma_c \rightarrow 1$ (a necessary condition is that $\sigma_c \leq 1$ asymptotically as $x \rightarrow 0$ and as $x \rightarrow \infty$).

C. Allocating Capital-Augmenting Technical Progress Between Sectors

As another example, suppose that we have the following:

$$117) \quad C = F_c(K_c e^{gt}, L_c) ; \quad M = F_m(K_m e^{ht}, L_m) . \quad \text{This can be rewritten as:}$$

$$C = e^{gt} F_c(K_c, L_c e^{-ht} e^{(h-g)t}) ; \quad M = e^{ht} F_m(K_m, L_m e^{-ht})$$

$$h, g \geq 0 ; \quad h = h(g) ; \quad h', h'' < 0 ; \quad h(0) = A, \quad h(\lambda) = 0$$

Consequently, this case is equivalent to:

- i) Hicks technical progress in C at rate g
- ii) Harrod technical progress in both sectors at rate $(-h)$
- iii) Harrod technical progress in C at rate $(h-g)$
- iv) Hicks technical progress in M at rate h

From our earlier discussions we know that:

$$118) \quad (\dot{C}/C) = g + (-h) + \delta[h, (h-g)] , \quad \text{where } \delta[h, (h-g)] \text{ is the}$$

rate of growth of consumption when Hicks technical progress takes place at rate h in M and Harrod technical progress takes place at rate $(h-g)$ in C. These rates are readily ascertainable from Table VIII, page 127.

From Table VIII we can see that an important consideration in determining the growth rates is whether the rate of Hicks technical progress in M is greater than the rate of Harrod technical progress in C:

119) That is, it is important if: $h > d$ or $h > d(1-\alpha_m)$.

However, since $d = (h-g)$, and since $h, g \geq 0$, it follows that:

120) $h \geq d$ always, with equality only if $g = 0$. Also,

121) $h > d(1-\alpha_m)$ always ($\alpha_m > 0$).

Thus, many of the different possible subcases are eliminated from Table VIII, and since $h \geq d$, x (the effective capital-labor ratio in C) always tends to infinity. Therefore, unlike the previous problem, we need not worry about the asymptotic value of σ_c as $x \rightarrow 0$.

Table XI presents the decisions that the planner should make, which depend upon the values of the elasticities of substitution. From the Table we see that whenever $\sigma_c < 1$, it does not matter (under this criterion) what decision the planner makes - in any case, the asymptotic growth rate of per capita consumption will be zero. This is true simply because capital-augmenting technical progress in either sector eventually leads to capital-deepening in C, and since the elasticity of substitution in C is less than one, this capital-deepening is of little use in increasing the asymptotic growth rate of consumption.

On the other hand, if $\sigma_c \geq 1$, then σ_m is the determinant of our actions. Since $\sigma_c \geq 1$, capital-deepening in C can lead to increasing consumption levels. If $\sigma_m > 1$, the technological progress in M will lead to an ever-increasing rate of growth of capital, and therefore we should direct all our technical progress to sector M. On the other hand, if $\sigma_m < 1$, then the investment sector can not (asymptotically) benefit from the capital-deepening, so in this case we should direct all our new technology to sector C - directly increasing the level of consumption. Finally, if $\sigma_m \rightarrow 1$, then technological change should be

TABLE XI - Allocating Capital-Augmenting Technical Progress
Between Sectors

$$C = F_c(K_c e^{gt}, L_c) ; M = F_m(K_m e^{ht}, L_m) ; h = h(g), h, g \geq 0 ;$$

$$h', h'' < 0 ; h(0) = A, h(\lambda) = 0$$

σ_m	σ_c	Allocation Rule	(\dot{C}/C)
$\sigma_m > 1$	$\sigma_c > 1$	$g = 0, h = A$	∞
	$\sigma_c \rightarrow 1$	$g = 0, h = A$	∞
	$\sigma_c < 1$	Indifferent	n
$\sigma_m \rightarrow 1$	$\sigma_c > 1$	$h'(g^*) = -[(1-\alpha_m)/\alpha_m]$	$n + \{[\alpha_m h^* + (1-\alpha_m)g^*]/(1-\alpha_m)\}$
	$\sigma_c \rightarrow 1$	$h'(g^*) = -[(1-\alpha_m)/\alpha_m]$	$n + \alpha_c \{[\alpha_m h^* + (1-\alpha_m)g^*]/(1-\alpha_m)\}$
	$\sigma_c < 1$	Indifferent	n
$\sigma_m < 1$	$\sigma_c > 1$	$g = \lambda, h = 0$	$n + \lambda$
	$\sigma_c \rightarrow 1$	$g = \lambda, h = 0$	$n + \alpha_c \lambda$
	$\sigma_c < 1$	Indifferent	n

divided between the two sectors (the allocation of technological change does not depend on whether $\sigma_c > 1$ or $\sigma_c \rightarrow 1$).

In this case there can only be a steady-state if $h=0, \sigma_c=1$ everywhere, or else if both σ_c and σ_m equal one. Consequently, a steady-state is very unlikely to occur in this case.

Finally, we also see that, unlike the previous cases, the shape of the transformation curve (given its concavity) is unimportant except in the case $\sigma_m \rightarrow 1$.

D. Allocating Technical Progress Within the Investment Sector

Another possible alternative is to assume that there exists a trade-off between capital- and labor-augmenting technical progress in the investment sector, and to ask what decision the planner should make about allocating this technical progress if he seeks to maximize the asymptotic rate of growth of consumption. Thus, assume:

$$122) \quad C = F_c(K_c, L_c) ; \quad M = F_m(K_m e^{ht}, L_m e^{gt}) ; \quad h=h(g); \quad h, g \geq 0, \quad h', h'' < 0$$

Equation 122) can be rewritten as:

$$123) \quad M = e^{ht} F_m(K_m, L_m e^{(g-h)t}) ; \quad C = F_c(K_c, L_c e^{(g-h)t} e^{(h-g)t})$$

Following our previous procedure, we find:

$$124) \quad (\dot{C}/C) = (g-h) + \delta[h, (h-g)] \quad , \quad \text{where } \delta[h, (h-g)] \text{ is the rate}$$

of growth of consumption as determined from Table VIII (h is the rate of Hicks technical progress in M, (h-g) the rate of Harrod technical progress in C). However, from equation 124) it is apparent that this is equivalent to the problem that we have just investigated, so that Table XI applies equally well to this case. Once again, a steady-state will be chosen only if:

$$i) \quad \sigma_m, \sigma_c \equiv 1 \quad \text{or} \quad ii) \quad \sigma_m < 1, \sigma_c \equiv 1$$

E. Induced Technical Progress in Each Sector

As our final case we shall investigate what decision the planner should make (under the criterion of maximizing the asymptotic rate of growth of consumption) if there is a Kennedy-type trade-off in

each sector. Specifically, assume:

$$125) \quad C = F_c(K_c e^{dt}, L_c e^{gt}) \quad ; \quad M = F_m(K_m e^{bt}, L_m e^{at})$$

$$b = b(a) \quad ; \quad a, b \geq 0; \quad b', b'' < 0; \quad b(0) = B, \quad b(A) = 0$$

$$d = d(g) \quad ; \quad g, d \geq 0; \quad d', d'' < 0; \quad d(0) = D, \quad d(G) = 0$$

Professor Chang [11] has studied this model under the assumption that each firm seeks to maximize the rate of reduction of per unit costs, given constant factor prices. In his model he found that there is a locally stable steady-state equilibrium (assuming $s_k > s_n = 0$) if:

$$i) \quad k_c \geq k_m \quad \underline{\text{and}} \quad ii) \quad \sigma_c, \sigma_m < 1 \quad \text{everywhere}$$

This result is, of course, akin to Professor Samuelson's [42] one-sector model in which he showed that the "Kennedy" equilibrium was stable only if the elasticity of substitution were less than one.

In investigating this case, we are interested in determining under what conditions the planner will choose a steady-state solution. As we shall see, the most likely case for the occurrence of a steady-state solution corresponds to the Chang case, $\sigma_c < 1$, $\sigma_m < 1$ (however, unlike in Chang's model, these are only asymptotic restrictions on σ_i). However, unlike the Chang case, it is possible that a steady-state would be chosen even if $\sigma_m \geq 1$, or if $\sigma_c \rightarrow 1$. On the other hand, if $\sigma_c > 1$, then the planner will never choose a steady-state since the consumption sector can profit from the continual capital-deepening that occurs in the asymptotic equilibrium if there is capital-augmenting technical progress in either sector.

Clearly, all the information that we need to solve this problem is contained in Table VIII. We must consider each case separately (for different values of the elasticities), and determine whether the

planner should choose a steady-state solution.⁸

$$i) \quad \sigma_m > 1, \quad \sigma_c \geq 1$$

In this case, if some capital-augmenting technical progress is present in the investment sector, then the rate of growth of consumption is unbounded. Furthermore, since the rate of growth of the rate of growth of capital depends upon the rate of capital-augmenting technical progress in M, it follows that we should allocate as much capital-augmenting technical progress to sector M as is possible, and consequently a steady-state solution is not desirable in this case.

$$ii) \quad \sigma_m > 1, \quad \sigma_c < 1$$

Though the rate of growth of capital is unbounded in this case, the rate of growth of consumption is finite since the consumption sector cannot profitably avail itself of the continual capital-deepening. If any capital-augmenting technical progress occurs in M, then we find:

$$126) \quad (\dot{C}/C) = n + g, \quad \text{and maximization of 126) implies } d=0, g=G.$$

The values of (a,b) are unimportant, provided that $b > 0$. On the other hand, if no capital-augmenting technical progress occurs in M, we find from Table VII:

$$126') \quad (\dot{C}/C) = n + d + A + \text{Min}[0, (g-d-A)] ; \quad b=0, a=A.$$

Maximization of 126') yields: $g^* = (d^* + A)$, assuming that this is feasible (that is, that $G \geq A$). If we compare our results from 126) to

⁸ Chang assumes that $\sigma_c, \sigma_m < 1$ everywhere; we are interested only in the asymptotic values of σ_c and of σ_m .

those for 126'), we find that for $G > A$, it pays to have some capital-augmenting technical progress in M, and so we would choose $d=0$, $g=G$, $b > 0$ (and hence there is no steady-state). If $G = A$, then we would lose nothing (in terms of the rate of growth of consumption) by choosing the steady-state solution, whereas if $G < A$ (and $d \geq 0$), a steady-state is not possible since either capital-augmenting technical progress will occur in M, or else Harrod technical progress will occur at different rates in the two sectors.

Thus, the planner is unlikely to choose a steady-state in this case.

$$\text{iii) } \sigma_m \rightarrow 1, \sigma_c > 1$$

From Table VIII we can show:

$$127) \quad (\dot{C}/C) = n + d + (a-b) + \text{Max}\left[\left\{\frac{b}{(1-\alpha_m)}\right\}, (g-d+b-a)\right] = \\ n + \text{Max}\left[\left\{d + a + \left[\frac{(\alpha_m b)}{(1-\alpha_m)}\right]\right\}, g\right]$$

In general, the planner will not seek a steady-state solution in this case. If:

$$D + \text{Max}\left[\left\{a + \left[\frac{(\alpha_m b)}{(1-\alpha_m)}\right]\right\}, \{a\}\right] \geq D + \text{Max}(A,B) > G,$$

then the planner should choose $d=D$, $g=0$, and (a^*, b^*) such that:

$$b'(a^*) = -\left[\frac{(1-\alpha_m)}{\alpha_m}\right]$$

Obviously, there is no steady-state solution under these circumstances.

On the other hand, if:⁹

⁹This is possible only if: $D + \text{Max}(A,B) > G$, which implies that the $d(g)$ curve is skewed, and that the technical progress transformation curve is smaller in M than in C. There seems to be no reason to believe that this case should occur.

$$D + \underset{\{a\}}{\text{Max}}[a + \{(\alpha_m b)/(1-\alpha_m)\}] < G,$$

the planner should choose $d=0$, $g=G$; the values of (a,b) do not affect the asymptotic rate of growth of consumption. However, even if he chose $a=A$, $b=0$ (so that there is no capital-augmenting technical progress in M), Harrod neutral technical progress would still occur at different rates in the two sectors. Furthermore, even if we allowed b to be negative, so that $a \geq G$, there could be no steady-state solution in that case because of the presence of capital-augmenting technical progress (at a negative rate) in sector M.

Thus, the planner will not choose a steady-state in this case.

$$\text{iv) } \sigma_m \rightarrow 1, \sigma_c \rightarrow 1$$

In this case we find:

$$128) \quad (\dot{C}/C) = n + [\alpha_c d + (1-\alpha_c)g] + \alpha_c [a + \{(\alpha_m b)/(1-\alpha_m)\}]$$

Maximization of equation 128) yields: $d'(g^*) = -[(1-\alpha_c)/\alpha_c]$, and

$b'(a^*) = -[(1-\alpha_m)/\alpha_m]$. In general, $b^* \neq 0$, so that no steady-state solution will be chosen (we are not considering Cobb-Douglas functions - we have discussed this case earlier). Even if $b=0$, $a=A$ (a corner solution), there is no a priori reason to expect g^* , d^* to be such that: $[(g^* - d^*) = A]$, though it is possible that this may occur. Thus, while it is possible that the planner may seek a steady-state solution in this case, such an occurrence would be a singular result indeed.

$$\text{v) } \sigma_m \rightarrow 1, \sigma_c < 1$$

Proceeding as in other sections, we find:

$$129) \quad (\dot{C}/C) = n + d + (a-b) + \text{Min}[\{b/(1-\alpha_m)\}, (g+b-d-a)]. \quad \text{Therefore:}$$

$$129') \quad (\dot{C}/C) = n + \text{Min}[g, \{a + d + \{(\alpha_m b)/(1-\alpha_m)\}\}].$$

Letting $H = \text{Max}_{\{a\}}[a + \{(\alpha_m b)/(1-\alpha_m)\}] \geq A$, we see that maximization of

(\dot{C}/C) implies choosing (d^*, g^*) such that: $g^* = d^* + H$. (If d^*, g^* exist - for $H > G$, and $g, d \geq 0$, no such d^*, g^* exist). Therefore, if $H \leq G$, then (a^*, b^*) and (d^*, g^*) are determined. For $H < G$, $d^* > 0$, whereas for $H = G$, $d^* = 0$. In either case, however, it "pays" to maximize the expression $[a + \{(\alpha_m b)/(1-\alpha_m)\}]$, and clearly there is no reason to expect this to yield $b^* = 0$. However, it is feasible that $b^*=0$, $a^*=A$, and if this occurs, then the planner would seek a steady-state solution. This result, however, corresponds to a corner solution, and consequently a steady-state solution is quite unlikely.

On the other hand, if $H > G$, (and $d \geq 0$), it is clear from 129') that g determines the rate of growth of consumption, and hence we should choose $d=0$, $g=G$. However, since $H > G$, and since $(\dot{C}/C) = n+G$, nothing is gained by maximizing $[a + \{(\alpha_m b)/(1-\alpha_m)\}]$, provided that we choose \tilde{a}, \tilde{b} such that this expression is at least equal to G . Specifically, if $A \geq G$, then nothing is lost (asymptotically) by choosing $a=A$, $b=0$ (and $d=0$, $g=G$). Consequently, the planner could choose a steady-state solution if $A = G$; if $A > G$, the planner need not choose any capital-augmenting technical progress, but a steady-state will not occur since Harrod technical progress occurs at a quicker rate in M than in C .

In summary, while a steady-state solution might be chosen in this case, such an occurrence would be a singular result, and thus we do not expect the planner to choose a steady-state path in this case.

$$\text{vi) } \sigma_m < 1, \sigma_c > 1$$

In this case the growth rate of consumption is:

$$130) \quad (\dot{C}/C) = d + a - b + n + \text{Max}[b, (g+b-a-d)] = n + \text{Max}[g, (a+d)]$$

Since the elasticity of substitution in M is less than one, there is nothing to be gained from capital-augmenting technical progress in M. Therefore, the planner should allocate technical progress as follows:

- a) If $G > [A + D]$, then choose: $d=0, g=G$; (a,b) unimportant
- b) If $G < [A + D]$, then choose: $d=D, g=0$; $a=A, b=0$
- c) If $G = [A + D]$, then choose either case a) or case b)

Case b) would seem to be the most plausible one, and clearly it does not provide a steady-state solution. Even if case a) pertains, there will not be a steady-state since, for $b=0, a=A < G$, and hence Harrod technical progress occurs at different rates in the two sectors.

Therefore, no steady-state solution will be chosen in this case.

$$\text{vii) } \sigma_m < 1, \sigma_c \rightarrow 1$$

Proceeding as in prior sections, we find:

$$131) \quad (\dot{C}/C) = n + \alpha_c a + [\alpha_c d + (1-\alpha_c)g]. \quad \text{In order to maximize}$$

the rate of growth of consumption, we would choose: $a=A, b=0$, and (d^*, g^*) such that: $d'(g^*) = -[(1-\alpha_c)/\alpha_c]$. Note that once again there is no reason to allocate any capital-augmenting technical progress to M since the elasticity of substitution in that sector is less than one.

There can be a steady-state in this case only if: $[g^* - d^*] = A$. Since α_c is a parameter, there is certainly no reason to expect this equality to hold (if $G < A$, there can never be a steady-state), though it might occur by chance. Thus, we conclude that a steady-state

solution would be a singular occurrence in this case.

$$\text{viii) } \sigma_m < 1, \sigma_c < 1$$

This case coincides with Chang's assumptions on the elasticities of substitution, and it yields the best chance for a steady-state solution. We find:

$$132) \quad (\dot{C}/C) = n + d + a - b + \text{Min}[b, (g+b-d-a)] = n + \text{Min}[g, (a+d)]$$

From 132) it is clear that there is no reason to allocate any capital-augmenting technical progress to M - therefore we choose $a=A$, $b=0$. Furthermore, if possible, we should choose g^* , d^* such that: $g^* = [A + d^*]$. Therefore, if $G \geq A$, a unique steady-state solution exists and will be chosen by the planner in this case (if $G > A$, then there will be both Hicks and Harrod technical progress in C, whereas if $G = A$, then only Harrod technical progress will occur in each sector). If $G < A$, a steady-state solution is possible (and will be chosen) only if we permit d to be negative, and if $\text{Max}(g) \geq A$. Should we maintain our restriction that d be non-negative, then only labor-augmenting technical progress will occur in each sector, but it will occur at a faster rate in the investment sector.

From the preceding analysis we have seen that the only case in which the planner is likely to choose a steady-state solution is if the elasticity of substitution in each sector is less than one (Not surprisingly, this case corresponds to Chang's results for the stability of the steady-state solution in a laissez-faire economy). We have seen, however, that the steady-state path might be chosen if $\sigma_m \geq 1$, $\sigma_c < 1$, or $\sigma_c \rightarrow 1$, $\sigma_m \leq 1$, though these possibilities correspond to singular cases. If $\sigma_c > 1$, no steady-state solution is ever desirable because of

the ability of the consumption sector to productively employ the ever-increasing capital stock (in efficiency units).

VII. Conclusion

In this lengthy chapter we have studied two-sector models of growth that, in general, do not lead to steady-state solutions. We have determined the asymptotic growth rates of variables for these non-steady-state models, and we have presented a generalized framework that can handle any combinations of factor-augmenting technical progress in this two-sector world.

Finally, we postulated the existence of trade-offs between certain types of factor-augmenting technical progress, and we have investigated how a central planner should allocate technical progress, assuming that he was trying to maximize the asymptotic (or steady-state) rate of growth of consumption. Basically, we found that his decision would depend upon the asymptotic values of the elasticities of substitution in each sector, as well as on the nature of the transformation curve between different types of technical progress. In some cases he may deem it desirable to seek a steady-state solution, while in others he will not. While we have not considered every possible trade-off between types of factor-augmenting technical progress, we have explored five particular cases. It is quite clear that other cases could readily be treated within the context of the model used in this chapter.

This concludes our study of the growth rates of variables for one- and two-sector models in which there is no steady-state path. In the next chapter we shall compare the characteristics of this non-steady-

state path to those characteristics attributed to the one- and two-sector steady-state models. As we shall see, these two models (the steady-state and the non-steady-state models) differ principally in the values of the observed factor shares (under competitive or "pseudo-competitive" pricing) and in the motivation for investment in the non-steady-state case.

Chapter 3: A Summing Up - The Steady-State and the Asymptotic Path:
Progress and Problems

I. Introduction

The previous two chapters have dealt with the conditions needed for a steady-state to occur, and they have examined how the economy will behave if no steady-state path exists. We have found, like others before us, that the conditions needed for a steady-state to exist are quite stringent, and that there appears to be little reason to believe (a priori) that these conditions are in fact fulfilled. On the other hand, we shall see in this chapter that the asymptotic equilibrium can not, in general, duplicate all the stylized facts of growth. The failure of the asymptotic equilibrium to explain the observed relationships in a growing economy and the implausibility of a steady-state equilibrium illustrate the dilemma facing the theory of economic growth. Somehow the theory must be able to explain the observed empirical relationships without placing the types of constraints (and seemingly unreasonable ones) on the system that must hold if the steady-state equilibrium is to occur. Though we have no ready answer to this dilemma, it is instructive to consider what constraints must be placed on the economy to achieve the steady-state path, and to investigate just how the asymptotic equilibrium deviates from the basic properties of the steady-state. In addition, it is helpful to consider what assumptions (or types of behavior) would enable the asymptotic equilibrium to meet the stylized facts of growth. It is these issues that we shall consider in this chapter.

Specifically, if we consider the one-sector model studied earlier, we find:

$$1) \quad \dot{x}/x = [\{se^{\delta t} f(x)\}/x] + b - a - n - \lambda; \quad (\lambda \text{ is the rate of depreciation}).$$

$$2) \quad \delta = [(a+n)h - (a+n-b)] \quad ; \quad x = [(Ke^{bt})/(Le^{at})]$$

Obviously, if $\delta = 0$, then a steady-state solution will exist for a constant s (given the other "normal" neo-classical assumptions); otherwise, it is most unlikely that a steady-state should exist.

Specifically, if $\delta < 0$, then it is clearly impossible to maintain indefinitely a fixed value of x (since s is bounded by one), and no steady-state solution is feasible ($\sigma \neq 1$). An example of this case is if no capital-augmenting technical progress is present ($b = 0$), and if decreasing returns to scale prevails ($h < 1$). Economically, this says that as growth takes place in factor inputs, output grows at a slower rate, and so in order to maintain the same effective capital-labor ratio, we must invest a continually larger fraction of output. Eventually, all output would be invested, but this still would prove incapable of providing enough new machines to maintain the steady-state path, and hence a steady-state cannot be maintained for all time.

Alternatively, if $\delta > 0$, then it is clear that if:

$$3) \quad s = s_0 e^{-\delta t},$$

then a steady-state solution will exist, and otherwise it will not ($\sigma \neq 1$). Note, however, that this implies that the fraction of resources devoted to investment declines over time - an assumption that does not appear to be empirically validated.

We might, in order to justify this very special saving behavior,

seek some economic mechanism that would predict this declining savings rate. For example, it might be assumed that investors seek to maintain a constant marginal physical product of capital:

$$4) \quad R = (\partial Q / \partial K) = e^{\delta t} f'(x)$$

However, from 4) it is apparent that this assumption will not maintain a steady-state path ($\delta \neq 0$):

$$5) \quad \frac{d(\partial Q / \partial K)}{dt} = 0 \text{ implies } \delta = [(1 - \phi_k) / \sigma] (\dot{x} / x),$$

where ϕ_k represents the elasticity of output with respect to capital.

In general, it is very difficult to think of any compelling economic mechanism that serves to maintain the steady-state path.¹ We have also seen that the problem is even more difficult in the two-sector (or multi-sector) model since, in addition to needing the proper fraction of output to be invested, we also have to worry about the problem of continual reallocation of resources within the economy due to differing rates of technical progress in each sector.² Therefore, in the two-sector model, even if the production function of the investment good is Cobb-Douglas, there still will not normally be a steady-state because there is no guarantee that Harrod neutral technical progress occurs at

¹Some possibilities discussed earlier are a Cobb-Douglas production function, a Kennedy-Samuelson innovation frontier, or a continually declining (as x increases) "degree of homogeneity" of the production function. None of these seems to be very compelling as a priori assumptions or arguments.

²Chang [11], in a Kennedy-type model, tries to provide a mechanism to eliminate this problem. However, as in the Kennedy model, it is rather dubious that the process that Chang envisions actually occurs in a decentralized economy.

the "proper" rate in the consumption sector.

Since there does not appear to be any compelling reason to believe that the conditions for a steady-state will be fulfilled, it becomes imperative to compare the behavior of this "non-steady-state" economy to those characteristics attributed to an economy generating a steady-state path. In our previous chapters we have seen that if the elasticity of substitution is bounded from one,³ then the "non-steady-state" economy will approach a path in which the physical variables grow at constant rates (Table I, page 37, and Table VIII, page 127).

How, then, does this asymptotic path differ from the "reality" that the steady-state purports to describe? The basic properties of the steady-state growth path are:

- 1) Output and Capital grow at the same constant rate
- 2) Output/Capita grows at a constant (non-negative) rate
- 3) Effective capital-labor ratio tends to a constant, and thus:
- 4) The marginal product of capital and the output-elasticity of each factor tends to a finite, positive value
- 5) The share of each factor tends to a non-zero value
- 6) Wages grow at a constant (possibly zero) rate
- 7) A constant fraction of output is saved and invested⁴

On the other hand, the basic properties of the asymptotic path

³If $\sigma > 1$ and $h \geq 1$, $b \geq 0$ (and not equality for both h and b), then many of the asymptotic growth rates become infinite. However, since this case does not seem plausible (and most studies show $\sigma < 1$), we shall consider only cases in which the growth rates are finite. Also, σ need not be bounded from one - it may tend to one (but not fluctuate between being greater than, equal to, and/or less than one).

⁴For convenience, this is normally treated as an assumption. However, this proposition would hold true if the proportion of investment to output were an increasing function of the MPK, and hence 7) can be interpreted as a result, not an assumption, of the steady-state model.

(assuming that the growth rates are finite, that σ is bounded from, or tends to one, and that $s_k \geq s_n > 0$)⁵ are as follows:

- 1) Output and Capital grow at the same asymptotic rate
- 2) Output/Capita grows at an asymptotically constant rate
(which may be negative)
- 3) The effective capital-labor ratio tends to zero or infinity, and thus:
- 4) The output-elasticity of one factor tends to zero (if σ is bounded from one), while the marginal product of capital probably tends to zero (for more on this, see Part III of this chapter)
- 5) The share of one factor (under the Vanek pricing assumption) tends to zero if σ is bounded from one, or tends to some non-zero constant value if σ tends to one.
- 6) Wages grow at an asymptotically constant rate (which may be negative)
- 7) An asymptotically constant fraction of output is saved and invested

From the above lists we can see that the basic differences between the steady-state path and the asymptotic path are in properties 3) - 6). However, property 3) is not a directly observable magnitude, but rather the constancy of the effective capital-labor ratio is normally inferred from properties 4) to 6) of the steady-state model. Property 4) itself is not directly observable in the real world, but rather is normally construed to be reflected in the relatively constant factor shares and in the approximately constant real interest rate.

⁵If $s_n = 0$, and if the growth rates are finite, then, for $\sigma < 1$, $h > [(a+n-b)/(a+n)]$, or for $\sigma > 1$, $h < [(a+n-b)/(a+n)]$, the growth rates of Q and K will be different. Similarly, if $s_n = 0$ and if $\phi_k \rightarrow 0$, then the share of output being invested tends to zero. See Table^k I, page 37, in Chapter 1, for further details.

From an empirical point of view, then, the two models differ in their ability to explain the relatively constant real interest rate and the relatively constant non-zero factor shares. In order to discuss factor shares (proposition 5), it is necessary to make some assumptions about factor pricing. However, if we relax the assumption of constant returns to scale, then the traditional assumption of competitive pricing⁶ holds little special merit, and we are called upon to consider other models of pricing factor inputs.

Therefore, the asymptotic path appears to explain the constant growth rates of the physical variables as well as the steady-state path does. One problem encountered with the asymptotic path is when we consider the distribution of income within the society. Another problem encountered is that the MPK may tend to zero in the asymptotic path, and then we are logically compelled to explain why investment still occurs. Therefore, it is our opinion that the asymptotic growth model lacks the following parts:

- i) A mechanism for explaining factor pricing
- ii) An explanation of the causes of investment

Before we attempt to discuss these issues, let us briefly consider how a change in the factor pricing assumption would affect the

⁶Vanek's assumption that factors are paid proportionally to their marginal products (the constant of proportionality being the reciprocal of the degree of homogeneity) holds little general relevance, though Prof. Chipman [12] shows that this type of factor pricing might result if deviations from constant returns to scale were due solely to externalities. The Chipman model is a quite special one, however, for it assumes that each firm behaves as though constant returns to scale prevails, that each firm is a very small part of the market, and that externalities are due solely to total industry output. Furthermore, it seems to us that this approach avoids the essentials of non-competitive pricing.

the growth path reached in this asymptotic equilibrium.

II. Factor Pricing and Asymptotic Growth Rates

In this section we intend to show that the asymptotic behavior of the economy is independent of the factor pricing assumption.

Recalling the Vanek pricing assumption, we can write:

$$6) \quad W = [(\partial Q/\partial L)/h] \quad ; \quad R = [(\partial Q/\partial K)/h]$$

It is quite clear that if a proportional savings function is assumed, then total savings is independent of the distribution of income within the society. Suppose that the Cambridge savings assumption is adopted:⁷

$$7) \quad S = s_k(RK) + s_n(WL) \quad ; \quad S = \text{gross savings}$$

We have already investigated how the growth path behaves for the factor pricing assumption of 6) and the savings assumption in 7); in this case the system will approach constant growth rates for Q and K ($h < 1$ or $\sigma < 1$). Consider the following factor pricing assumption:

$$8) \quad W = (1-a)(\partial Q/\partial L) \quad ; \quad 1 \geq a \geq 0 \quad .$$

$$R = [(Q - WL)/K] = [(1/h)(\partial Q/\partial K) + (a + (1/h) - 1)(\partial Q/\partial L)(L/K)]$$

If $a = 0$, this reduces to the assumption that labor is paid the value of its marginal product (a most dubious assumption for $h > 1$)⁸; for

⁷Naturally, we are assuming that the government policy is designed to maintain full-employment, so that savings equals investment.

⁸If $h > 1$, it seems most probable that competition will break down in either the factor market or the product market or both. Later in this chapter we shall discuss a situation in which (for $h > 1$) competition is maintained in the factor market, but is replaced by oligopolistic behavior in the product market. If $h > 1$, and if labor is paid its marginal value product, then its total share might exceed one.

$a = [1 - (1/h)]$, this becomes the Vanek pricing assumption.⁹ From 7) and 8) we find:

$$9) \quad S = s_k(RK) + s_n(WL) = [(\partial Q/\partial L)(L/h)\{s_k - (s_k - s_n)(1-a)h\} + s_k(\partial Q/\partial K)(K/h)]. \quad \text{Let:}$$

$$10) \quad s_n^* = [s_k - (s_k - s_n)(1-a)h] ; \quad 0 \leq s_n^* \leq s_k \quad \text{if} \quad 1 \geq a \geq [(h-1)/h]$$

Upon making this substitution into 9), we arrive at the following:

$$11) \quad S = s_n^*[(\partial Q/\partial L)(L/h)] + s_k[(\partial Q/\partial K)(K/h)]. \quad 10$$

Clearly equation 11) is fundamentally identical to the basic Vanek savings equation, though the values of the parameters may differ.

However, if workers do no savings out of their income ($s_n=0$), for the Vanek pricing assumption we can show:

$$12) \quad S = \dot{K} = s_k[(\partial Q/\partial K)(K/h)] ; \quad (\dot{K}/K) = s_k[(\partial Q/\partial K)/h]$$

For $\sigma < 1$, and/or $h < 1$, (\dot{K}/K) approaches a constant, finite limit, and so must the marginal product of capital. However, under the more general pricing assumption in 8) we have:

$$13) \quad S = s_k[1 + h(a-1)][(\partial Q/\partial L)(L/h)] + s_k[(\partial Q/\partial K)(K/h)] ; \quad s_n = 0$$

Therefore, if $[1 + (a-1)h] > 0$, then $(\partial Q/\partial K) \rightarrow 0$, since the effective capital-labor ratio tends to infinity and $\sigma < 1$.

⁹As we shall briefly discuss later, this assumption can be derived by assuming perfect competition in the labor market, but monopolistic practices in the product market. In that case, $(1-a)$ is the marginal revenue (for $P=1$, the numeraire), so that a may be interpreted as the reciprocal of the elasticity of the demand curve.

¹⁰Economically, $a \geq [(h-1)/h]$ may be interpreted as placing a limit on the elasticity of demand as a function of h . If $h < 1$, then market power is unlikely, and hence $a=0$, in which case:

$$s_n^* = [s_n h + s_k(1-h)] > 0.$$

Nevertheless, it seems rather singular a case to assume $s_n = 0$; if $s_k \geq s_n > 0$, and $s_n^* > 0$, as seems plausible, then the growth rates approached by these two systems will be identical since the growth rates are independent of the savings parameters.

Thus, in the general case in which something is saved out of each type of income (and $a \geq [(h-1)/h]$), the growth path of the system is independent of the factor pricing assumption (for those cases that we have considered). This is tautologically true if everybody saves at the same rate (for all types of income).

Now that we know that the growth path of the system does not depend on the factor pricing assumption (unless $s_n = 0$; naturally, this statement is made within the limits of the previous analysis), let us briefly consider the asymptotic behavior (and values) of certain variables for the growth models studied in the previous two chapters.

III. Asymptotic Values of Variables

As has been continually emphasized, the system studied in the previous chapters will approach an "equilibrium" in which the physical quantities (Q , K) will grow at constant rates. However, we would like to know how some of the other variables behave. For example, what happens to the following variables in this asymptotic equilibrium?

- i) (Q/K) and $(\partial Q/\partial K)$.

Let us first consider the one-sector model. If we assume that everyone saves at the same rate, we know:

$$14) \quad (\dot{K}/K) = s(Q/K) ; \text{ for simplicity, assume no depreciation.}$$

Restricting our attention to those cases in which the growth rates

remain finite, we have found:¹¹

$$15) \quad \sigma < 1, \quad (\dot{K}/K) \rightarrow [(a+n)h] ;$$

$$\sigma > 1, \quad h < 1, \quad (\dot{K}/K) \rightarrow [(bh)/(1-h)]$$

$$\sigma \rightarrow 1, \quad (\dot{K}/K) \rightarrow \{[(a+n)h - (a+n-b)\phi_k^*]/(1-\phi_k^*)\} ; \quad \phi_k^* < 1. \quad ^{12}$$

Therefore, from 14), if (\dot{K}/K) approaches a positive, finite limit then (Q/K) tends to a constant, positive, finite limit. However, if $\sigma < 1$, then $\phi_k \rightarrow 0$, so that $(\partial Q/\partial K) \rightarrow 0$. If $\sigma \geq 1$, then $(\partial Q/\partial K)$ also tends to a finite limit (for $h < 1$).

Next, suppose that $s_k \geq s_n > 0$. In that case:

$$16) \quad (\dot{K}/K) = [(s_n/h)(\partial Q/\partial L)(L/Q)(Q/K) + (s_k/h)(\partial Q/\partial K)] ; \quad ^{13}$$

From our earlier table on growth rates (Table I, page 37) we know that the growth rates for 16) are equivalent to those for 14). Since $x \rightarrow \infty$, then $\phi_k \rightarrow 0$ and $\phi_n \rightarrow h$ if $\sigma < 1$. However, $\phi_k = [(\partial Q/\partial K)(K/Q)] \rightarrow 0$, implies either $(\partial Q/\partial K) \rightarrow 0$ or $(K/Q) \rightarrow 0$. However, from 16) it is clear that for finite (\dot{K}/K) , (Q/K) is finite ($s_n > 0$), so we conclude that $(\partial Q/\partial K) \rightarrow 0$. Therefore,

$$17) \quad \sigma < 1, \quad (\dot{K}/K) \rightarrow [(s_n/h)(Q/K)h] = [s_n(Q/K)] ; \quad (\partial Q/\partial K) \rightarrow 0 .$$

On the other hand, if $\sigma > 1$, then $\phi_k \rightarrow h$, $\phi_n \rightarrow 0$, and:

¹¹In the ensuing discussion we shall assume that $\delta = [(a+n)h - (a+n-b)] > 0$, so that $x \rightarrow \infty$ (barring the perverse case discussed earlier) - this assumption seems most plausible to us. Clearly, for $\delta < 0$, the discussion is quite symmetrical.

¹² ϕ_k^* is the asymptotic value of $[(\partial Q/\partial K)(K/Q)]$ as $x \rightarrow \infty$; if $\phi_k^* \rightarrow 0$, the results are comparable to those for $\sigma < 1$; if $\phi_k^* \geq 1$, the growth rates are unbounded.

¹³As explained earlier in this chapter, this can be interpreted as a more general form of the capital-accumulation curve, valid for different types of factor pricing assumptions.

$$18) \quad \sigma > 1, \quad (\dot{K}/K) \rightarrow [(s_k/h)(\partial Q/\partial K)]. \quad 14$$

Finally, if $\sigma \rightarrow 1$, $\phi_k \rightarrow \phi_k^* < 1$, we find:

$$19) \quad \sigma \rightarrow 1, \quad (\dot{K}/K) = (Q/K)[(s_n/h)(h-\phi_k^*) + \phi_k^*(s_k/h)] ,$$

and (Q/K) as well as $(\partial Q/\partial K)$ tend to finite limits.

Finally, if $s_n = 0$, we have:

$$20) \quad (\dot{K}/K) = [(s_k/h)(\partial Q/\partial K)] .$$

Since (\dot{K}/K) tends to a finite limit ($\sigma < 1$, or $h < 1$), $(\partial Q/\partial K)$ also approaches a finite, nonzero limit. If $\sigma < 1$, $(Q/K) \rightarrow \infty$ since $\phi_k \rightarrow 0$; for $\sigma \geq 1$, (Q/K) will also approach a finite, nonzero limit. The table on the following page summarizes these results under the assumption that all growth rates are finite and that $x \rightarrow \infty$.

From the table we see that there are many possibilities, depending upon the values of the parameters. However, if $\sigma < 1$ (which is the most prevalent empirical result), then $(\partial Q/\partial K) \rightarrow 0$ unless $s_n = 0$; and the assumption that $s_n = 0$ is a very strong one, since s_n can be interpreted not only as the propensity of workers to save, but it also depends upon the "degree of exploitation" of workers by capitalists, and the capitalists propensity to save (see page 160). Thus, if we assume $h \geq 1$, and that σ is bounded from unity, then in order to achieve finite growth rates we are forced to assume that $\sigma < 1$, and hence the MPK tends to zero ($s_n \neq 0$). Should the MPK tend to zero, we then are obligated to rethink our savings assumption, and to delve into the investment

¹⁴This result follows from: $[(\partial Q/\partial K)(K/Q)] \rightarrow h$; if (Q/K) is finite, then $[(\partial Q/\partial L)(L/Q)](Q/K) \rightarrow 0$; if (Q/K) is infinite, then so must be $[\partial Q/\partial K]$ (since ϕ_k tends to h), violating the assumption that (\dot{K}/K) is finite.

TABLE I - Asymptotic Values of the Average and Marginal Product of Capital

Cases	(\dot{K}/K)	(Q/K)	$(\partial Q/\partial K)$	ϕ_k
I) $\sigma < 1$				
a) $s_k \geq s_n > 0$	$[(a+n)h]$	$[(a+n)h]/s$	0	0
b) $s_k > s_n = 0$	$[(a+n)h + (1-\sigma)(a+n-b)]$	∞	$[(K/K)(h/s_k)]$	0
II) $\sigma \rightarrow 1$ ($\phi_k^* < 1$)	$\frac{[(a+n)h - (a+n-b)\phi_k^*]}{(1-\phi_k^*)}$	$[(K/K)(1/s)]$	$[(K/K)(\phi_k^*/s)]$	ϕ_k^*
III) $\sigma > 1$ ($h < 1$)	$[(bh)/(1-h)]$	$[(bh)/s(1-h)]$	$[(bh^2)/(1-h)s]$	h

In the above table, $s \equiv [s_n + (s_k - s_n)(\phi_k/h)]$

behavior of the capitalists.

In the previous chapters we have considered how the marginal product of labor (MPL) changes over time. As is apparent from Table I in Chapter 1 (page 37), the MPL increases over time if $h \geq 1$. For decreasing returns to scale, if labor-augmenting technical progress occurs at least as rapidly as capital-augmenting technical progress, then the MPL will also increase over time in that case. However, if $h < 1$, it is possible that for some values of the other parameters, the MPL will decrease over time.

It is clear that the situation is considerably more complex in the multi-sector case because of the presence of more than one good. In this case, it is possible to think of the marginal product of capital in terms of the consumption good or the investment good.¹⁵ Similarly, one must

¹⁵Since the ultimate purpose of investment (presumably) is consumption, it might be argued that only the MPK in consumption units matters. However, it is quite possible that there are several consumption goods, one

speak of the output-factor elasticities for each sector, and so on.

Assuming the same two-sector model as discussed earlier, it is possible to compute the asymptotic behavior of some of the variables. Since constant returns to scale is assumed, the competitive factor pricing mechanism is more plausible, and hence it can be argued that the output-factor elasticities can be interpreted as the competitive factor shares in each sector.¹⁶ As has already been pointed out, Harrod neutral technical progress in each sector at the same rate leaves the growth model unaltered (when variables are expressed in efficiency units). Therefore, it suffices to limit our attention to considering Hicks technical progress in M and Hicks and Harrod technical progress in C. Assuming that Hicks neutral technical progress occurs at rate \underline{a} in M and at rate \underline{g} in C, and that Harrod neutral technical progress occurs at rate \underline{d} in C (\underline{d} may be negative), we would like to find the asymptotic values of:

$$21) \quad (L_m/L) ; \alpha_c, \alpha_m, \alpha ; (\partial C/\partial K_c) , (\partial M/\partial K_m) .$$

α_i represents the share of capital in sector i (under competitive pricing), and α is the share of capital in the economy as a whole. In Table VIII, page 137, we have computed α_i , as well as (\dot{K}/K) . Using the market

¹⁵of which (at least) is produced in the same way as machines, so that the MPK in investment units also represents the MPK in terms of some consumption good. (If a constant fraction of income is spent on this new consumption good, the basic growth model is unaltered). In this case, the choice of numeraire units seems quite arbitrary.

¹⁶The assumption of constant returns to scale (CRS) is made to avoid discussing the multi-sector allocation problem. If the degree of homogeneity is the same in each sector, and if the Vanek pricing assumption is made, the subsequent analysis (for $h \neq 1$) does not differ much from that already performed. However, the Vanek pricing assumption has little justification, and the issue of factor-allocation within the economy should not be dismissed so lightly.

equilibrium equation, the marginal product equations, and the basic differential equation, we can readily derive the following:

$$22) \quad (\dot{K}/K) = (M/K) = [(L_m e^{at} f'_m)/K] = [K_m/(K\alpha_m)](e^{at} f'_m)$$

$$23) \quad [K_m/(K\alpha_m)] = [s/(1-s)][K_c/(K\alpha_c)]$$

$$24) \quad [L_m/\{L(1-\alpha_m)\}] = [s/(1-s)][L_c/\{L(1-\alpha_c)\}]$$

Since we already know (\dot{K}/K) , we can calculate $(\partial M/\partial K_m) [= (e^{at} f'_m)]$ if we can find $[K_m/(K\alpha_m)]$. Using the above equations we can then derive the asymptotic values shown in the table on the following page.

As is apparent from the Table, a great variety of behavior is possible, depending upon the values of the parameters. It is interesting to note that the marginal product of capital in machine units never becomes infinite (for finite values of $[\dot{K}/K]$); this corresponds to the one-sector results. On the other hand, the MPK in consumption units will always tend to either zero or infinity, depending upon the values of the parameters.¹⁷ It is difficult to say what constitutes the most plausible case, but it should be pointed out that if σ_c and σ_m are both bounded from unity, those cases for which $(\partial M/\partial K_m)$ is nonzero correspond to rather high shares for capital ($= [1-s]$). Also, in every case except $\sigma_c \rightarrow 1$, $\sigma_m \rightarrow 1$, the factor shares in at least one of the sectors tends to zero (or one). Thus, it cannot be claimed that this model explains the observed factor shares in the real world and the

¹⁷The ambiguity in the table in the case $\sigma_m \rightarrow 1$, $a = d(1-\alpha_m)$, stems from the inability to determine what happens to the effective capital-labor ratio in sector C. It should be noted that if $\sigma_m \equiv 1$, this case would correspond to a steady-state solution.

TABLE II - Asymptotic Values in the Two-Sector Growth Model

$$(\partial M / \partial K_m) = [n + \{a/(1-\alpha_m)\}][(1-s)\alpha_c + s\alpha_m]/s \quad ; \quad \alpha_m < 1$$

$$(L_m/L) = \{[s(1-\alpha_m)]/[s(1-\alpha_m) + (1-s)(1-\alpha_c)]\} \quad ; \quad \alpha_m \text{ or } \alpha_c < 1$$

$$R \equiv (\partial C / \partial K_c)$$

Case	(R/R)	Value R if g=0	α_c	α_m	α
<u>$\sigma_m < 1$:</u>					
<u>$\sigma_c < 1$:</u>					
a > d	$g + \text{Min}[0, \text{Max}\{[d - (a/\sigma_m)], [(d-a)/\sigma_c]\}]$	0	0	0	0
a = d	Same as above [= g]	$f'_c(\infty) = 0$	0	0	0
a < d	Same as above [= g]	$f'_c(0) = \infty$	1	0	(1-s)
<u>$\sigma_c \rightarrow 1$:</u>					
a > d	$[g + (d-a)(1-\alpha_c)]$	0	α_c	0	$(1-s)\alpha_c$
a = d	$[g + (d-a)(1-\alpha_c)] = g$	$f'_c(\infty) = 0$	α_c	0	$(1-s)\alpha_c$
a < d	$[g + (d-a)(1-\alpha_c)]$	∞	α_c	0	$(1-s)\alpha_c$
<u>$\sigma_c > 1$:</u>					
a > d	$g + \text{Max}[0, \{(d-a)/\sigma_c\}, \{d - (a/\sigma_m)\}]$	$f'_c(\infty) = 0$	1	0	(1-s)
a = d	Same as above [= g]	$f'_c(\infty) = 0$	1	0	(1-s)
a < d	Same as above	∞	0	0	0
<u>$\sigma_m \rightarrow 1$:</u>					
<u>$\sigma_c < 1$:</u>					
a > d(1- α_m)	$g + \text{Min}[0, \{d - [a/(1-\alpha_m)]\}]$	0	0	α_m	$s\alpha_m$
a = d(1- α_m)	Same as above	0 or ∞	0 or 1	α_m	$[(1-s)\alpha_c + s\alpha_m]$
a < d(1- α_m)	Same as above	$f'_c(0) = \infty$	1	α_m	$[(1-s) + s\alpha_m]$

TABLE II - Continued

Case	(R/R)	Value R if $g=0$	α_c	α_m	α
<u>$\sigma_m \rightarrow 1$:</u>					
<u>$\sigma_c \rightarrow 1$:</u>					
$a > d(1-\alpha_m)$	$g + (1-\alpha_c)[d - \{a/(1-\alpha_m)\}]$	0	α_c	α_m	$[(1-s)\alpha_c + s\alpha_m]$
$a = d(1-\alpha_m)$	$g + (1-\alpha_c)[d - \{a/(1-\alpha_m)\}]$	0 or ∞	α_c	α_m	$[(1-s)\alpha_c + s\alpha_m]$
$a < d(1-\alpha_m)$	$g + (1-\alpha_c)[d - \{a/(1-\alpha_m)\}]$	∞	α_c	α_m	$[(1-s)\alpha_c + s\alpha_m]$
<u>$\sigma_c > 1$:</u>					
$a > d(1-\alpha_m)$	$g + \text{Max}[0, \{d - [a/(1-\alpha_m)]\}]$	$f'_c(\infty) = 0$	1	α_m	$[(1-s) + s\alpha_m]$
$a = d(1-\alpha_m)$	Same as above [=g]	0 or ∞	0 or 1	α_m	$[(1-s)\alpha_c + s\alpha_m]$
$a < d(1-\alpha_m)$	Same as above	∞	0	α_m	$s\alpha_m$

apparent constancy of the real interest rate over time (presumably reflecting the marginal product of capital).

In summary, we have seen that in both the one- and two-sector models the asymptotic value of the MPK may tend to zero, though this result seems more likely to occur in the one-sector case. Also, we have seen that the asymptotic path leaves much to be desired in its attempt to explain observed factor shares. Let us now comment briefly on these problems.

IV. Factor Pricing and Factor Shares

As observed repeatedly in this chapter, as well as in previous chapters, the asymptotic path fails to explain the distribution of income within the society if the elasticity of substitution is bounded from one (under the Vanek pricing assumption). That this result

is inevitable becomes apparent immediately upon realizing that the effective capital-labor ratio must tend to either zero or infinity in this "asymptotic equilibrium".

Where does this leave us in attempting to explain reality in this "asymptotic" world? We have stated earlier that the steady-state, though it performs well in mapping reality, is a singular result indeed; and the asymptotic path apparently does not satisfactorily explain the income distribution in society. Clearly, something must give, and it is our feeling that the weakest link in the chain is the factor pricing assumption. If the assumption of constant returns to scale is dropped, no great justification remains for resorting to competitive pricing. Even if we accept the notion of constant returns to scale, the world we seek to explain does not, in our eyes, duplicate the competitive world that the model presumes.

For simplicity, we shall concentrate on the one-sector model. If decreasing returns to scale prevails, then it is still possible to maintain the assumption of competitive pricing.¹⁸ Assuming $h < 1$, if labor is paid its marginal value product, then the share of labor should equal $(h - \phi_k)$. For $\sigma < 1$, $\phi_k \rightarrow 0$ (assuming the effective capital-labor ratio tends to infinity), and so labor's share tends to h , leaving the residual, $(1-h)$, for capitalists. Thus, if the economy experiences decreasing returns to scale, and if $\sigma < 1$, as seems plausible, then under competitive pricing the share accruing to each factor will

¹⁸ However, it now becomes necessary to go beyond the aggregate model and to observe the size of each firm. Clearly, with non-constant returns to scale, the aggregate output is likely to depend upon industry structure and the number of firms present in each industry (assuming that it is not externalities that account for the decreasing returns to scale).

approach some positive limit.¹⁹ Though this would appear to concur with reality, it does ignore the all-important question of what determines the firm size in the micro-economy, and how are the decreasing returns to scale manifested?

If $h > 1$, as seems possible, then competitive pricing is meaningless. In this circumstance one would expect to see the economy dominated by major firms (industries), and an alternative pricing assumption is needed. As a simple alternative, it is possible to imagine that firms have market power, so that marginal revenue deviates from price, but that they behave as perfect competitors in the factor market (this latter assumption is made for simplicity). In this case, firms would hire each factor until its marginal revenue product equalled its cost. Assuming each firm had comparable production and demand conditions, the allocation of factors within this economy (for given W, R) would be identical to that for a competitive economy.

How would factor shares in such a world behave? Since each firm possesses market power, we know:

$$25) \quad W = MR(\partial Q/\partial L); \quad [(WL)/(PQ)] = (MR/P)(\partial Q/\partial L)(L/Q) = (MR/P)\phi_n$$

Even if $\phi_n (= [h - \phi_k])$ tends to $h > 1$ (and thus $\phi_k \rightarrow 0$), labor's share may well be less than one.²⁰ Letting η represent the elasticity of

¹⁹This assumes that the effective capital-labor ratio tends to infinity, which in turn assumes that there must be some capital-augmenting technical progress (since we have assumed $h < 1$).

²⁰Of course, it must be less than, or equal to, one for it to have any economic meaning (assuming free disposal). What guarantees that this will be so? Clearly, the number of firms that can profitably operate depends on h ; the larger h , ceteris paribus, the fewer the number of firms that can survive (under our assumptions), and hence the greater the disparity between MR and P must be.

demand, we have:

$$26) \quad [(WL)/(PQ)] = [(\eta-1)/\eta]\phi_n \rightarrow h[(\eta-1)/\eta] \quad \text{for } \phi_n \rightarrow h$$

Thus, depending upon the elasticity of demand, labor's share may be significantly less than one. Clearly, for the viability of business (under the above assumptions) it must be true that:

$$27) \quad h[(\eta-1)/\eta] < 1$$

This relationship tells us that the degree of increasing returns to scale places an upper bound (in absolute value) on the elasticity of demand, presumably through affecting the number of firms in the market.

It is possible to formalize the above presentation (though in a static sense) by assuming Cournot behavior in the product market, competitive behavior in the factor market, and to utilize this information to determine factor shares. If we do this, assuming there are two products, each facing unitary elastic demand (as in the growth models), and that there are N firms in each market²¹ (it is quite simple to allow a different number of firms in each market), we find:

$$28) \quad [(WL)/(PQ)] = \phi_n [(N-1)/N] \rightarrow h[(N-1)/N] \quad , \quad \text{for } \phi_n \rightarrow h$$

$$[(RK)/(PQ)] = \phi_k [(N-1)/N] \rightarrow 0 \quad \text{for } \phi_n \rightarrow h, \quad \phi_k \rightarrow 0$$

$$[(Profits)/(PQ)] = \{1 - h[(N-1)/N]\} ; = [(1/N)] \quad \text{for } h = 1.$$

Thus, it is quite plausible that, even though the output-elasticity of capital tends to zero, its observed share, when combined with

²¹As stated earlier, h places limits on the number of firms that can survive in the market. This appears clearly in the second order conditions (for the above model) for a relative profit maximum as $N \leq [(2h)/(h-1)]$, $h > 1$.

monopolistic profits, approaches a positive limit. For example, if $h = 1.2$, and $N = 3$, then:²²

$$29) \quad [(WL)/(PQ)] \rightarrow .8 \quad ; \quad [(RK)/(PQ)] + [(Profits)/(PQ)] \rightarrow .2$$

It is quite obvious that the approach pursued above is quite naive; nevertheless, it does readily illustrate a possible explanation of the relative constancy of observed factor shares within the context of the asymptotic equilibrium. What is needed is a more dynamic approach to this problem, one that explains how the market position of the firms change through time due to advertising, their individual investment decisions, and so forth. Though we have not pursued this problem further, the next chapter does deal with the problem of how firms decide what growth rates to pursue, and how changes in various parameters affect this decision.

Before concluding this chapter, there is one more fundamental problem that must be discussed. In this past section we have attempted to show that once we leave the competitive framework, the "asymptotic equilibrium" is as capable as is the steady-state equilibrium of explaining the relative constancy of factor shares. However, even if our preceding comments are valid, there still remains the more fundamental problem of the zero marginal product of capital and the incentives for investment.

²² Obviously, nothing is special about the values used here. Though the proofs of the above statements are omitted, they can readily be demonstrated by using the normal macro-demand equations and by dividing the economy into two industries, the firms in each industry pursuing Cournot behavior. For brevity, the work is not included here.

V. Investment Decisions and the Marginal Product of Capital

The most fundamental failing of the asymptotic equilibrium (in our opinion) is its inability to explain both the observed interest rate and the consequent investment decisions of the firm. In equilibrium it is assumed that investment is carried out until the marginal product of capital equals the real interest rate (barring capital gains); however, in the previous sections we have seen that, in general, the marginal product of capital does not tend to a constant, positive limit but rather normally tends to zero.²³ If this happens, the logical question is why any investment takes place, given that the interest rate is positive.

Unfortunately, we have no definite answers to this problem. One apparent answer that may suggest itself is that capital is the bearer of progress, and as such new investment provides society with new technology (vintage capital). Though the assumption of vintage capital is, in our opinion, a plausible one, it does not answer our problems. We have considered a vintage clay-clay model (with Hicks neutral technical progress) and have found that the marginal product of capital (of the newest vintage) still tends to zero, while the growth rates of the system are identical to those in the non-vintage case (for $\sigma \rightarrow 0$).²⁴ Thus, even though technology is embodied in the

²³In the one-sector case, if $\sigma \geq 1$, then the MPK tends to a positive limit (if the growth rates are finite). Also, for $s_n = 0$, the MPK tends to a positive limit. In the two-sector case, the MPK in capital-numeraire units may tend to zero or a finite limit (for finite growth rates); in terms of the consumption good as the numeraire unit, anything can happen to the MPK.

²⁴Our work on this model is omitted simply because of the identity of its results to those of the non-vintage case.

new machines, the marginal product of each additional machine tends to zero because the economic lifetime of the machines (under competitive pricing) tends to zero (for discrete time, there will be unused machines of even the newest vintage). Though we have not investigated the putty-putty or putty-clay vintage models, we feel strongly that even in these cases the MPK will tend to zero (for $\sigma < 1$).

A second, and more promising, alternative is to assume that the level of technology depends upon the amount of past investment (learning-by-doing). In this case the social MPK will remain positive even if the extra output generated directly by an additional machine (the private MPK) tends to zero. For example, suppose technology takes the following form:²⁵

$$30) \quad Q = K^\lambda F(K, Le^{at}) = K^\lambda [\bar{L}f(\bar{k})] \quad ; \quad \bar{L} = Le^{at}, \quad \bar{k} = (K/\bar{L}), \quad \lambda < 1$$

Assuming that everyone saves at the same constant rate, and that $\sigma < 1$, we can show:

$$31) \quad (\dot{K}/K) \rightarrow [(a+n)/(1-\lambda)] \quad , \quad \text{and}$$

$$32) \quad (\partial Q/\partial K) = \{[K^\lambda f(\bar{k})]/\bar{k}\}(\lambda + \phi_k) = \{(\dot{K}/K)(1/s)(\lambda + \phi_k)\}; \quad \text{therefore:}$$

$$32') \quad (\partial Q/\partial K) \rightarrow \{[\lambda(a+n)]/[s(1-\lambda)]\} > 0 \quad , \quad \sigma < 1$$

In other words, since the extent of technological efficiency depends upon the amount of investment, further investment, even if it

²⁵This form of technical progress is postulated because of its similarity to the Arrow learning-by-doing model [2]. Actually, the production function is just a special case of increasing returns to scale, with $h = (1 + \lambda)$, and $\phi_k \rightarrow \lambda$ as $\bar{K} \rightarrow \infty$.

contributes nothing directly to output, does increase output through improving technology.

There are several obvious problems that arise from the above formulation. In 32) we have found the social MPK; however, if the world is composed of small businesses, the private MPK still tends to zero, so the private motivation for investment is still lacking. If instead the world is composed of large businesses, then, even though some externalities may still occur, the private MPK will (in general) exceed zero, should their investment constitute a "sizeable" portion of total investment.²⁶

The other problem that arises is that the production function in 30) does not exhibit constant returns to scale (as pointed out by Arrow, among many others). Obviously, both factors cannot be paid their social marginal products. Under the Arrow assumptions, competition prevails and labor receives its marginal value product, leaving capital "underpaid" (giving rise to the need for, or desirability of, capital subsidies). However, it is quite clear that if this technology tends to lead to large enterprises (by not being a total externality), there is no reason to assume that competitive pricing prevails, and the private MPK may well remain positive (Also, we must then discuss non-competitive pricing, as in the last section).

Though learning-by-doing technical progress can conceivably help to explain the presence of continuing investment despite the fact

²⁶This line of inquiry leads us to ponder whether this learning-by-doing technology is passed on to all firms, or if it can be internalized. Following this line of reasoning would lead us into discussions of market structure and hence disaggregate investment decisions. We feel that this latter course is one that needs to be investigated (in a dynamic framework).

that the physical marginal product of capital tends to zero, we feel that it is fundamental (especially when the steady-state does not exist) to inquire just why investment takes place at all. For example, if instead of assuming that the value of investment is proportional to income, we assume that the investment demand for each sector is proportional to its output, we find that a steady-state will exist if Hicks neutral technical progress does not occur in at least one of the sectors.²⁷ (The normal result is that there can be no capital-augmenting technical progress in the investment sector). Though this condition is still quite strong, it is weaker than the corresponding result in the normal two-sector model.

Obviously, there are other possibilities that could be investigated. It is our opinion (because of the failure of the "asymptotic equilibrium" to fully replicate the stylized facts of growth) that it is quite essential to consider a more disaggregated model in order to determine what investment for the economy as a whole will be.

The two-sector model is considerably more complicated because of the large variety of results that may occur (as is seen from Table II of this chapter, page 167). When there are several consumption goods, we do not know which to treat as the numeraire; consequently, it is unclear (in the absence of specific utility functions) just what is

²⁷This assumes that Harrod neutral technical progress occurs at the same rate in each sector. If there are more than two sectors, then there can only be Hicks neutral technical progress in one sector if a steady-state solution is to exist (assuming that investment is as above, and that the demand for each good is unitary elastic).

meant by the interest rate or by the marginal product of capital. However, should we consider the capital good to be the numeraire unit, then in a large number of cases the MPK in terms of this good (which might also be a consumption good) is finite, but nonzero, so that in this case the "investment question" is less critical than in the one-sector model.

In summary, we have seen that our aggregate growth model presents us with quite a dilemma. On the one hand, the properties of the steady-state equilibrium seem to coincide quite well with our observations of the real world, but the chances of such a steady-state occurring seem quite small indeed. On the other hand, the asymptotic equilibrium appears to explain the relatively constant growth rates observed in the economy, but it does not readily account for the observed factor shares or the presence of continuing investment despite positive interest rates (for $\sigma < 1$). Whereas in the steady-state equilibrium the MPK tends to a positive, finite limit (so that the ex post observation of a constant fraction of output being invested is quite plausible), in the asymptotic equilibrium the presence of continuing investment as the MPK tends to zero (but interest rates remain positive) is more suspect. Obviously, then, the problem of explaining the stylized facts of a growing economy has not been settled quite so well as the literature might lead one to conclude. Accordingly, it is our feeling that more attention must be devoted to explaining the "whys" of investment, principally through a more disaggregate view of the economy. In our final chapter we hope to take a small step in that direction by considering the growth decisions made by an isolated firm in a growing economy.

Chapter 4: Technology and the Growing Firm - A Simple Model

I. Introduction

So far in this thesis we have considered aggregate models of an economy, and we have studied how various types of technological change affect the growth behavior of this economy. In this chapter we shall adopt a slightly more disaggregate view and shall investigate how different types of technical progress affect the decisions of a growing firm.

Specifically, we shall adopt the model of a firm as developed by Professor Solow [49]. In that paper Prof. Solow formulated a simple model of a growing firm in order to investigate how the (qualitative) behavior of a classical profit-oriented firm might differ from that of the more "modern" growth-oriented firm. Though we shall not be pursuing this question at any length, the model developed by Prof. Solow is readily adaptable to our needs.

Though we do not choose to pursue Prof. Solow's line of inquiry at any length, we feel that several comments about this subject are in order. It is our understanding that one reason for the "existence" of growth-oriented firms is due to the separation of ownership from control of the business. Given this presumption, it is ironic to note that what Prof. Solow calls the owner-oriented firm will always (for all types of technical progress) choose a larger growth rate than a firm that maximizes its value without any concern for the opportunity cost of the capital to the original investors. Thus, if we assume that modern technology has given rise to a class of managers (technocrats?) who run the corporation without direct control from the owners of the

corporation, it seems plausible to assume that the rewards to these technocrats will depend upon the value of the firm (stock options, stock dividends, bonuses). If these managers are interested in their own well-being, then they will choose a smaller growth rate than a firm which is identical in technology, but is run instead by (or for) its owners.

The above result is a consequence of the fact that the manager-run firm, disregarding the opportunity cost on capital, will choose a larger initial size than will the owner-oriented firm. As a result of this larger size, the manager-run firm receives a lower per unit price for output, and hence it finds growth less appealing than its (originally) smaller owner-oriented counterpart.

The question may arise as to how the manager-oriented firm can ignore the original opportunity cost of capital. An explanation for this phenomenon (as for the Williamson, Marris type firms) must of necessity rely upon some imperfection in the capital markets (perhaps due to lack of, or imperfect knowledge). Thus, the manager-oriented firm will attempt to raise more funds (than its owner-oriented counterpart would) through larger initial stock-offerings, or, in the case of a take-over of an owner-oriented firm by a manager-oriented firm, through subsidiary stock offerings. The managers can then use these additional funds to expand the size of the firm beyond that size which the owner-oriented firm would choose. However, if people subscribe to these excessive stock offerings, it must be because of some lack of knowledge on their part, since they surely must consider the opportunity cost to themselves of their own funds.

We also feel (given our "classical" bias) that the "growth-

oriented" theorists fail to fully explain why their firms behave the way they do.¹ One possible explanation for the "growth-oriented" behavior, we believe, may be a failure to distinguish (or perceive) the difference between short-run profits and long-run profits. Specifically, it may seem that, for given technological and economic conditions, the firm is growing faster than seems profitable; however, over the long-run, the faster growth rate will (ceteris paribus) allow the firm to gain a larger share of the market, and hence allow it to seize some of the larger profits that accrue to the major firm in the industry. Thus, the firm is sacrificing short-run profits to obtain more market power, and hence larger profits in the long-run. To a casual observer it may appear that the firm is growing more quickly than is optimal, but to one who is privy to all the information, this may not appear to be the case.²

¹Williamson and Marris [31, 63, and 64] argue that the size of the staff enters the managers' utility function, causing them to behave differently than a profit-maximizing firm would. Presumably, in terms of the Solow model this might mean that L , the size of the labor force, enters the utility function (as a proxy for staff). Consequently, this would effect the decisions made by the firm. Williamson also argues that direct rewards ("emoluments") to the managers enters their utility function. While this is certainly plausible, this latter assumption would not affect the operation of a manager-oriented firm (per Solow) since the bonuses come out of profits. However, if a fixed proportion of net profits must go to managers as bonuses, this would effect the operations of an otherwise owner-oriented firm since it is effectively equivalent to an increase in capital costs to the owners (since part of the dividend flow does not return to them). In order to see how an increase in capital costs affects the decisions of the firm, consult the Appendix to this chapter.

²Naturally, this type of reasoning would lead us into oligopoly and game theory. Thus, if there are two firms in an industry, each may choose a larger growth rate than would otherwise be optimal, hoping to improve their market position. However, if both firms behave in this way, no additional market power will accrue to either firm (or, at least, additional market power cannot accrue to both firms), and it will appear ex post as though each firm, and the industry, is growing faster than a profit-oriented firm should. However, it is possible that

If we accept the hypothesis that firms might choose larger growth rates to improve their market position, it is possible that, in turn, this improved market position may lead them to select even higher growth rates. Specifically, we shall see that within the context of the model developed by Prof. Solow, the more inelastic the demand curve, ceteris paribus, the larger the growth rate chosen by the firm. Since a traditional definition of monopoly power is in terms of the elasticity of demand facing the firm, it seems plausible to say that the improved market position found by the growing firm may induce it to grow even quicker. To carry this analysis much further would necessitate formulating a specific model of the market structure, market demand, and so forth - and this is something that we have not yet done.

Given our classical bias, and the above remarks, we shall assume that the firm is, in fact, a "profit-maximizer". Using this assumption, and the basic Solow model, we shall investigate how different rates of technical progress (and different price behavior) affect the growth rate chosen by the firm. But before we do this, we shall present a brief synopsis of the Solow model of the firm.

II. The Solow Model of the Firm

In this section we shall attempt to present the basic model developed by Prof. Solow, and to briefly summarize his results. The basic assumptions are:

- 1) Fixed coefficients in production prevails

² neither firm acting alone should lower its growth rate, and thus, without collusion, these profit-oriented firms will choose a larger growth rate than would seem to be optimal, and hence they will appear to be "growth-oriented" firms.

- 2) There are constant returns to scale in production
- 3) There is no technological progress, or else it must be labor-augmenting (in which case the wage rate is assumed to grow at the same rate as technical progress).³
- 4) Depreciation occurs at rate f
- 5) The price of capital, m , and the wage rate (in efficiency units) a , are constants
- 6) Demand is iso-elastic, with elasticity n , $n > 1$
- 7) In order to cause demand to increase over time, it is necessary to advertise. For demand to grow at rate g , a fraction $s(g)$ of gross revenue must be spent on advertising.⁴

Given these assumptions, the firm is assumed to choose a constant price and a constant rate of growth (and hence the initial size of the firm) in order to maximize the present value of the firm. While it may seem like an "imposition" to force the firm to choose a strategy that prevails for all time, it must be remembered that this is the essence of a steady-state model. Also, we shall show later in this chapter that, if the problem is formulated using the Maximum principle, there is a

³It is the purpose of this paper to relax this assumption. The assumption made corresponds to those made for aggregate growth models. However, this assumption seems unrealistic since:

- a) There appears to be empirical evidence of capital-augmenting technical progress being present in industry
- b) Even if only labor-augmenting technical progress occurs, it seems likely to occur at different rates in each sector, and the wage rate can not, therefore, grow at the same rate as productivity in each and every sector.

⁴It is assumed that there exist g_m, g_1 such that: $s(g_m)=0$, $s(g_1) = 1$; $s', s'' > 0$.

unique growth rate and initial size of the firm that will allow the firm to pursue the steady-state strategy.

Formulating the model mathematically, we can write:

- 1) $Q = \text{Min}(bK, bL)$
- 2) $K = K_0 e^{gt}$; $Q = bK_0 e^{gt}$
- 3) $I = \dot{K} + \text{Depreciation} = (f + g)K_0 e^{gt}$
- 4) $Q_0 = P_0^{-n}$ implies: $P_0 = Q_0^{-(1/n)}$
- 5) $T(g) = [1 - s(g)]$; $T(g)$ is the fraction of revenue not spent on advertising
- 6) Dividends $\equiv \text{Div} = [TPQ - aL - m(f+g)K_0 e^{gt}] =$
 $[Tb^\theta K_0^\theta e^{gt} - aK_0 e^{gt} - m(f+g)K_0 e^{gt}]$; $\theta = [1 - (1/n)] > 0$
- 7) $V = \int_0^\infty [(\text{Div})e^{-it} dt] = [(Tb^\theta K_0^\theta - [a + m(f+g)]K_0) / (i-g)]$.

Given the value of the firm,⁵ the management must choose the initial size (K_0) and the growth rate (g) in order to maximize V .

However, in making its decision, the firm might take no account of the initial cost of capital - that is, there might be no opportunity cost attributed to capital (This could arise if the planners had an interest in the value of the firm, but did not provide the original capital). Thus, the question arises as to what criterion should be used for maximizing the value of the firm. Prof. Solow considers two possibilities:

⁵It might be argued that, due to myopia, imperfect capital markets, and so forth, the value of the firm should be:

$$V = \int_0^\infty [U(D)e^{-it} dt] ; \text{ (where } U(D) \text{ is the utility derived}$$

from dividends). Though we have not investigated this case, it seems likely that if $U'' < 0$, this assumption would lead the firm to choose slower growth rates than for expression 7) above.

$$I) \quad (\partial V / \partial K_0) = 0, \quad (\partial V / \partial g) = 0$$

$$II) \quad (\partial V / \partial K_0) = j, \quad (\partial V / \partial g) = 0, \quad j \geq m$$

Though Professor Solow suggests that in a world of perfect capital markets it must be true that $[(\partial V / \partial K_0) = m]$, he apparently does not feel that a compelling argument can be made to guarantee this equality in the real world. However, if one deducts from the value of the firm the cost of the initial capital, we have:

$$7') \quad V^* = V - mK_0.$$

It is clear that maximization of V^* leads to the criterion: $V_k = m, V_g = 0$. Since V , the value of the firm, is computed by calculating the present discounted value of the flow of dividends, it certainly seems appropriate to deduct from this stream of dividends the initial outlays. Therefore, we feel that $V_k = m$ is the "proper" criterion, though we shall explore both cases.⁶ Prof. Solow also focuses his attention on two cases (which we shall call Criterion I and Criterion II):

$$I) \quad V_k = 0, \quad V_g = 0$$

$$II) \quad V_k = m, \quad V_g = 0$$

As indicated in the introduction, Criterion I can be interpreted as representing the manager-oriented firm, since no consideration is given to the owners' original outlays, whereas Criterion II is more suitable for what Prof. Solow calls the "owner-oriented firm".

Given the form of V , and the maximization criterion that is

⁶As we shall see, when the Maximum Principle is used, the only steady-state solution implies that $V_k = m, V_g = 0$. This is because in deciding how much capital to invest, we implicitly attribute an opportunity cost to the initial capital.

adopted, one can then proceed to calculate the optimal solution for the firm. Under the assumption that $V_k=0$, $V_g=0$, Prof. Solow proceeds to sketch each curve (both are negatively sloped in the (K_o, g) plane), and to show what a maximum solution would look like.⁷ We can briefly summarize his results as follows:

- a) If the curves never intersect, there is no interior solution, and the smallest feasible g is the optimal solution.
- b) There must be some interior relative maximum if:

$$\left[\frac{a+mf}{a+m(i+f)} \right] > \left[\frac{\theta T(0)}{T(0)+iT'(0)} \right] \geq \theta$$

Note that for θ near one ($n \rightarrow \infty$), this condition will not be fulfilled, and hence it is possible that in this case no interior solution exists.

- c) If there is more than one intersection of the curves, at least one of them must be a saddle-point. However, though Prof. Solow does not point this out, there would appear to be no guarantee of a unique interior (relative) maximum, and hence it may be necessary, in the case of several relative maxima, to compare the extreme values at each point to determine which is the global maximum.⁸

As an alternative, consider the case: $V_k = j \geq m$, $V_g = 0$.

Prof. Solow shows that the $V_k = j \geq m$ curve is either always negatively sloped; or else has at most one change in the sign of the derivative

⁷From the second order conditions it can readily be seen that:
 $\left| \left[\frac{dK_o}{dg} \right]_{(V_k=0)} \right| < \left| \left[\frac{dK_o}{dg} \right]_{(V_g=0)} \right|$ at a maximum, and otherwise the intersection is a saddle-point (no interior minimum is possible).

⁸This same problem arises when technical progress is considered, and it proves quite burdensome since we wish to determine what happens to the growth rate as a result of changes in certain parameters. We shall assume that (except for singular cases) the same root (intersection) remains the dominant one (optimal one), thus enabling us to consider only how each of the intersections shifts.

(for $j = m$, the slope is always strictly negative). Furthermore, since $V_{kk} < 0$ everywhere, it follows that the $V_k = j$ curve ($j > 0$) lies inside the $V_k = 0$ curve. From this fact we can infer:

- a) If there is an intersection of the $V_k=0$ and $V_g=0$ curves, then there must be one for the $V_k=j > 0$ and the $V_g=0$ curves, though the reverse is not necessarily true.⁹
- b) If there is a unique intersection of these two curves, the owner-oriented firm will always choose a larger growth rate than the management-oriented firm. If there are several interior solutions, then each relative maximum occurs at a higher growth rate for the owner-oriented firm. It is possible that as the $V_k=j$ curve shifts inward (as j increases) that some "roots" may be lost; however, the root corresponding to the largest growth rate can not be lost since, for $K_o = 0$, the $V_g=0$ curve must lie below the $V_k=j \geq 0$ curve.

As a third alternative (and to allow him to deal with the growth-oriented firms), Prof. Solow considers the case in which the initial size of the firm is fixed, and only the growth rate is to be chosen by the firm. Since we are not mainly interested in comparing the owner-oriented firm to the management-oriented firm, we shall not even consider this alternative. We do feel, however, that it is less plausible to assume the firm will choose a steady-state strategy when it has no choice over its original size. Thus, if the initial size is too small, it may choose a larger initial growth rate to increase

⁹Thus, for $V_k=0$, $V_g=0$, there might be a boundary solution, while for $V_k=j > 0$, $V_g=0$, there might be an interior solution.

the size of the firm.¹⁰

Since our interest lies primarily in the growth rates chosen by the firm, consider the "growth-rate determining" equations (we shall assume, as Prof. Solow implicitly does, that a unique interior maximum exists):

$$I) V_k = V_g = 0 ; [T + (i-g)T'] / T = \theta[a + m(i+f)] / [a + m(f+g)]$$

$$II) V_k = m, V_g = 0 ; T + n(i-g)T' = 0 ; (\text{or } [T + (i-g)T'] = \theta T)$$

As already noted, it is clear that the owner-oriented firm chooses a larger growth rate than the other firm. How does a change in the elasticity of demand affect the growth rate chosen by the firm?

First consider the case $V_k = m, V_g = 0$:¹¹

$$8) [T + n(i-g)T'] \equiv F(n,g) = 0 ; F_n < 0 \text{ clearly.}$$

If we consider the second order conditions for a maximum we find:

$$9) [V_{kk}V_{gg} - (V_{gk})(V_{kg})] > 0 \text{ implies } [n(i-g)T'' + (1-n)T'] < 0 ;$$

$$10) \text{ Therefore: } F_g = [(1-n)T' + n(i-g)T''] < 0 \text{ at a maximum. Thus:}$$

$$11) (dg/dn) = -[F_n/F_g] < 0.$$

That is, by considering the second order conditions, it is possible to show that the greater the elasticity of demand, the slower

¹⁰This assertion is borne out by the observation that there is a unique steady-state solution to the Pontryagin problem. Thus, there is only one set of initial conditions that will lead the firm to choose a steady-state path (Assuming the price of output is not fixed for the firm).

¹¹Prof. Solow has suggested in his paper that as $n \rightarrow \infty$, $g \rightarrow i$, if the restrictions on $T(g)$ are ignored (Solow, *op. cit.*, [49]). However, if $T', T'' < 0$, even if $T(i) > 0$, then, as n increases, g decreases, and as $n \rightarrow \infty$, $T(\hat{g}) \rightarrow 0$, not $g \rightarrow i$.

the firm will choose to grow.¹² If we identify the larger elasticity of demand with a more highly competitive industry, then it follows that the more competitive the industry, the more slowly growing the firm will be. Combined with the assumption that a firm may seek a larger growth rate in order to increase its monopoly power (and thus decrease n), it can be seen that these two processes may be reenforcing.

Though we might speculate that the optimal size of the firm will increase as the elasticity of demand increases (from the static notion that monopolies underproduce), this is not necessarily so.

Specifically:

$$12) \quad K_0 = \{[\theta b^\theta T]/[a + m(i+f)]\}^n \quad ; \quad \theta = [1 - (1/n)]$$

$$13) \quad (dK_0/dn) = (K_0/n)[\ln(bK_0) + (1/\theta) + (T'/T)(dg/d\theta)]$$

Without further information on T'' it does not appear possible to say what will happen to K_0 as n changes. Economically, it appears that if K_0 is initially sufficiently small, the increase in the price elasticity may cause price to fall (ceteris paribus), thereby causing the firm to contract its output. However, as n becomes sufficiently large, it seems likely that further increases in n are likely to elicit increases in K_0 .

Similarly, for the case $V_k=0$, $V_g=0$ it can be shown, using the second order conditions, that an increase in n will lead the firm to

¹²As usual, the possibility of multiple roots of the equation $[T + n(i-g)T'] = 0$ exists. In this case, we can say that each root associated with a maximum decreases and that each root associated with a saddle-point increases as n increases. If the chosen growth root does not change, then our conclusion stands. It seems that, except in singular cases, a change in n will not lead the firm to choose a different growth root.

choose a slower growth rate.¹³ However, since the work is rather tedious, we shall omit the proof.

This concludes our brief summary of Prof. Solow's paper. In addition to outlining his model, we have found that:

- i) The owner-oriented firm will choose a larger growth rate than the management-oriented firm.
- ii) The more competitive is the industry (as measured by the elasticity of demand facing the firm), the slower each firm will choose to grow.

In the following sections we shall consider how changes in the model affect the optimal growth rate for each type of firm. Our first analysis deals with a problem suggested by Prof. Solow himself - how do our results change if the firm is allowed to change its price at a constant rate over time?

III. The Solow Firm and Price Strategy

In this section we adopt Prof. Solow's suggestion and allow the price to change at a constant rate over time. Adopting his notation:

$$14) \quad Q = Q_0 e^{gt} \quad ; \quad P = P_0 e^{-\pi t} \quad ; \quad \text{thus, } Q = (P_0)^{-n} e^{(h+n\pi)t} \quad ,$$

$g = (h + n\pi)$, where h is the rate of increase in demand due to advertising.

$$15) \quad \text{Div}(t) = T(h)b^{\theta} K_0^{\theta} e^{(g-\pi)t} - [a + m(g+f)]K_0 e^{gt}$$

If the firm adopts a policy of falling prices, eventually dividends will become negative. If there is free disposal of its

¹³All the previous warnings regarding multiple equilibria apply here. From now on we shall assume (for simplicity) that a unique interior solution exists. If not, our previous cautions must be considered.

assets, then the firm should shut down when the dividends fall to zero. However, it seems unlikely that a firm planning to terminate its operations in the not-too-distant future would pursue a steady-state path until its demise. Thus, the framework of this model is not suited to deal with a falling price situation. Also, it is clear that a policy of price reduction is not compatible with the notion of a firm interested in its own self-preservation.¹⁴

If we assume that a firm pursues a policy of falling prices, as well as a continuous rate of growth of output until it shuts down, then its current value would be (assuming it shuts down when dividends fall to zero):

$$16) \quad V = \int_0^T [(Div)e^{-it}] dt \quad ; \quad \text{Dividends equal zero at } T \text{ such that:}$$

$$e^{-\pi T} = \{ [a + m(f+g)] / [T(h)b^\theta K_0^{\theta-1}] \} \quad . \quad \text{Thus:}$$

$$17) \quad V = \left(\frac{Tb^\theta K_0^\theta}{(i+\pi-g)} - \frac{[a+m(f+g)]K_0}{(i-g)} \right) + \left(\frac{\pi K_0 [a+m(f+g)]^{[(i+\pi-g)/\pi]}}{(i-g)(i+\pi-g)[Tb^\theta K_0^{\theta-1}]^{[(i-g)/\pi]}} \right)$$

For falling prices ($\pi > 0$), this expression exceeds the one developed by Prof. Solow because he implicitly assumes that the firm must live forever, forcing it to "pay" negative dividends. The last expression on the right hand side of equation 17) indicates how much the firm saves by closing its doors when dividends reach zero (instead of continuing to operate forever).

¹⁴ Because of oligopolistic markets, it may be that a certain asymmetry in price changes exists. That is, the firm may be able to pursue a policy of rising prices without reprisals, but its policy of falling prices may be followed by competitors, offsetting the initial advantages of the lower prices (This is, of course, the kinked-demand curve phenomenon). This problem points out, we believe, the danger of a partial-equilibrium approach.

If the firm pursues a falling price strategy, then its optimal lifetime depends upon all the decision variables of the firm. An increase in K_0 , ceteris paribus, leads to an earlier shut-down because the price falls in response to the larger output. Comparably, the quicker the firm chooses to grow, ceteris paribus, the sooner the firm will shut-down because a larger fraction of gross revenue is spent on advertising. An increase in the rate of decline in prices works in two directions - it leads to lower advertising expenditures (for the same growth rate for the firm), but it also means prices are falling more rapidly.

Since expression 17) is a rather difficult one to work with, and since falling prices do not seem appropriate in a steady-state model, we shall assume that either:

- a) Prices increase or stay the same over time, or
- b) The firm must operate forever.

If we make either assumption, we can then derive the expression for the present value of the firm (which is the same as Prof. Solow's expression):

$$18) \quad V = \{[\pi(h)b^\theta K_0^\theta]/(i+\pi-g)\} - K_0\{[a + m(f+g)]/(i-g)\}$$

As earlier, there are several criteria that may be used in determining the optimal values of (g, K_0, π) .

- I) $V_k = V_g = V_\pi = 0$
- II) $V_k = m, V_g = V_\pi = 0$

Using criterion I, and solving the resulting simultaneous system, we find:

$$19) \quad [T + n(i+\pi-g)T'] = 0$$

$$20) \quad \pi = -\{[m(i-g)^2]/[a + m(i+f)]\} < 0$$

$$21) \quad K_o = [\{\theta T b^0(i-g)\}/\{[a + m(f+g)](i+\pi-g)\}]^n$$

From these equations we can see that the management-oriented firm will choose a strategy of rising prices. How does a change in n affect g ?

$$19) \quad F(n, \pi, g) \equiv [T + n(i+\pi-g)T'] = 0$$

$$20) \quad F_n = (i-g)T' - n(i+\pi-g)\pi T'' < 0 \quad (h = [g - n\pi]; \pi < 0)$$

But, from the second order condition we can show that:¹⁵

$$21) \quad F_g = (\partial F/\partial g) + (\partial F/\partial \pi)(d\pi/dg) < 0$$

$$22) \quad \text{Therefore, } (dg/dn) < 0; \quad (d\pi/dn) = (d\pi/dg)(dg/dn) < 0$$

Thus, as in the case of the firm that holds price constant, an increase in the elasticity of demand will lead to a lower growth rate but, paradoxically, to more rapidly rising prices. It would be interesting to know whether the firm with fixed prices or the firm with rising prices (for the manager-oriented firm) chooses a larger growth rate; unfortunately, even though we might expect the constant-price firm to choose the larger growth rate, we have not been able to demonstrate this. Fortunately, when we consider our second criterion, our results are more definitive.

$$II) \quad V_k = m, \quad V_g = V_\pi = 0$$

Using this alternative criterion, and proceeding as before, we

¹⁵This result follows directly from the second order condition; for the sake of brevity, the proof is omitted.

derive:¹⁶

$$23) \quad [T + n(i+\pi-g)T'] = 0$$

$$24) \quad \pi = 0; \text{ therefore, } [T + n(i-g)T'] = 0$$

$$25) \quad K_0 = \{[\theta T b^0] / [a + m(i+f)]\}^n$$

The owner-oriented firm, when faced with the opportunity to vary prices over time, will behave in exactly the same way as his counterpart firm that is constrained to hold prices fixed over time (though he can set the initial price level). Thus, all our comments from the prior section regarding the owner-oriented firm are applicable in this case. Specifically, the growth rate chosen by the firm depends only on n and i (and the shape of $T(g)$), and an increase in either will decrease the optimal growth rate of the firm.

Also, if we compare our results for criteria I and II, we see that the owner-oriented firm chooses a smaller initial size (and hence a larger initial price), but a larger growth rate. Therefore, even though the management-oriented firm will initially provide the public with more output at a lower price, it will eventually be dominated (in total size) and undersold by the owner-oriented firm.

Now that we have considered how firms will vary their prices, if allowed to do so, let us next consider what effect technical progress will have on the decisions made by the firms. As we shall see, the analysis becomes more complicated as a result of this new assumption.

¹⁶This result coincides with the steady-state solution obtained from the Maximum Principle - see Section VII of this chapter.

IV. The Effect of Disembodied Capital-Augmenting Technical Progress on the Firm

In this section we shall consider what effect changes in technology have upon the decisions of the firm. As Prof. Solow mentions in his paper, if the technical progress is disembodied and labor-augmenting, and if the wage rate increases at the same rate as labor productivity, then the model is unchanged (except for the physical quantity of labor). However, if the wage rate changes at a different rate than does the productivity of labor (as is certainly possible in a many sector model), then the Solow model will be altered. For simplicity, we shall adopt Prof. Solow's assumption regarding the wage rate and labor productivity, and instead we shall focus our attention on the effect of disembodied capital-augmenting technical progress. In a subsequent section we shall consider how our results change if the technical progress is embodied in new machinery.

Assuming technical progress occurs at rate c , and that the firm plans to grow at rate g , we calculate:

$$26) \quad Q = bK_0 e^{gt} \quad ; \quad K = K_0 e^{(g-c)t}$$

$$27) \quad \text{Div} = T b^\theta K_0^\theta e^{gt} - a K_0 e^{gt} - m(g+f-c)K_0 e^{(g-c)t}$$

$$28) \quad V = \{ [T b^\theta K_0^\theta - a K_0] / (i-g) \} - \{ [m(f+g-c)K_0] / (i+c-g) \}$$

Note that the larger c , ceteris paribus, the less investment the firm will undertake. In fact, if c is sufficiently large, the firm may discard capital over time (if $c > [f + g_1]$, where $T(g_1) = 0$, then clearly capital will be discarded over time). However, since it seems plausible that $f > c$, we shall not worry about this possibility.

The firm's problem is to choose K_0 and g ; however, as always, there are several possible criteria. We shall explore the two considered in previous sections:

$$I) \quad V_k = V_g = 0$$

$$II) \quad V_k = m, \quad V_g = 0$$

For $V_k = 0, V_g = 0$, we can derive:

$$29) \quad V_k = \frac{\theta T b^\theta K_0^{\theta-1} - a - \{[m(i-g)(f+g-c)]/[i+c-g]\}}{(i-g)} = 0$$

$$30) \quad V_g = \frac{K_0 \{b^\theta K_0^{\theta-1} [T+(i-g)T'] - a - \{[m(i-g)^2(i+f)]/(i+c-g)^2\}}{(i-g)^2} = 0$$

From 29) and 30) we can see that an increase in the rate of technical progress, ceteris paribus, will increase the marginal values of both K_0 and of g :

$$31) \quad (\partial V_g / \partial c) > 0, \quad (\partial V_k / \partial c) > 0.$$

Furthermore, since $V_{kk} < 0$, and $[V_{gk}]_{(V_g=0)} < 0$, it follows that an increase in c shifts both the $V_k=0$ and the $V_g=0$ curves outward.

Unfortunately, it is not possible to show that either curve is monotonic for $c > 0$. For $V_k=0$ we find:

$$32) \quad [V_{kg}]_{(V_k=0)} = \{[\theta b^\theta K_0^{\theta-1} T' - m\{1 - [c(i+f)/(i+c-g)^2]\}]/(i-g)\};$$

$$32') \quad V_{kk} < 0$$

For K_0 sufficiently small (g sufficiently large), this expression [32] will be negative; however, as g becomes small, this expression might

become positive. Nor is there any guarantee that V_{kg} changes sign only once. Therefore, we can not obtain much information about the shape of the $V_k=0$ curve, except that for small K_0 it must be downward sloping.¹⁷

Let us see if we can learn any more about the shape of the $V_g=0$ curve. Clearly, for $V_g=0$ it must be true that $g \leq \hat{g}$ (where $[T(\hat{g}) + (i-\hat{g})T'(\hat{g})] = 0$); as $K_0 \rightarrow 0$, $g \rightarrow \hat{g}$. Also, as $g \rightarrow 0$, K_0 approaches some finite, positive value. Therefore, the $V_g=0$ curve must be negatively sloped at some points. Furthermore:

$$33) \quad V_{gk} < 0 \quad \text{for} \quad V_g = 0$$

$$34) \quad V_{gg} = [K_0/(i-g)^3][b^\theta K_0^{\theta-1}(i-g)^2 T'' + \{[2mc(i+f)(i-g)^2]/(i+c-g)^3\}]$$

Without further information on the shape of T , it is not possible to ascertain the value of V_{gg} ; however, as $K_0 \rightarrow 0$, $g \rightarrow \hat{g}$, and $V_{gg} < 0$ (provided that $T''(g) \leq \epsilon < 0$). Thus, for large g , the $V_g=0$ curve will be negatively sloped; elsewhere, it is not possible to determine its slope.¹⁸

When the two curves intersect, $[V_{kg}]_{(V_k=0)} = [V_{gk}]_{(V_g=0)}$, and thus the $V_k=0$ curve must be negatively sloped at any intersection of the

¹⁷For $g \geq \hat{g}$, where $[T(\hat{g}) + (i-\hat{g})T'(\hat{g})] = 0$, $[dK_0/dg]_{(V_k=0)} < 0$; also, if $i > (c+f)$, then $\frac{d[dK_0/dg]}{dc} < 0$. Since $[dK_0/dg] < 0$ for $c=0$, it follows that for $c > 0$, $i > (c+f)$, then $[dK_0/dg]_{(V_k=0)} < 0$.

However, there is no reason to assume that i is as large as is needed for this result to apply.

¹⁸If $T''=0$ for $g \leq g^*$, then $V_{gg} > 0$ in this range. Since it is likely that there will be some increase in demand over time even without advertising, it seems plausible to assume that $T''=0$ for some $g^* > 0$. Thus, the $V_g=0$ curve is likely to have positive slope for small values of g .

two curves. Furthermore, for a maximum it must be true that:

$$35) \quad V_{gg} < 0, \quad V_{kk} < 0, \quad \text{and} \quad [V_{gg} V_{kk} - (V_{gk})^2] > 0$$

Therefore, if $V_g=0$ has positive slope when the curves intersect, the intersection is a saddle-point; and if $V_k=0$ has a larger slope (in absolute value), then the intersection is again a saddle-point. Only if $V_g=0$ has a steeper (and negative) slope (dK_o/dg) when the curves cross can the intersection be a maximum.

As Prof. Solow does, we can consider how these curves behave as they approach the axes. For $V_k=0$, as $K_o \rightarrow 0$, $T \rightarrow 0$ (call the value of g such that $T = 0$, g_1); similarly, for $V_g=0$, as $K_o \rightarrow 0$, g must tend to \hat{g} (where $[T(\hat{g}) + (i-\hat{g})T'(\hat{g})] = 0$). Since $T'' < 0$, it follows that $\hat{g} < g_1$; therefore, if the two curves ever intersect, at least one of these intersections must be a maximum.¹⁹ So far, there is no guarantee that the curves ever intersect; however, if $K_g > K_o$ (where $V_k[0, K_o]=0$, $V_g[0, K_g]=0$), then there must be at least one interior solution. In other words, if:

$$36) \quad \left(\frac{[T(0) + iT'(0)]}{\theta T(0)} \right) > \left(\frac{[a + \{[mi^2(i+f)]/(i+c)^2\}]}{[a + \{[mi(f-c)]/(i+c)\}]} \right) ; \quad (f \geq c)$$

then the two curves must intersect (for $c=0$, this reduces to Prof. Solow's expression that was referred to earlier). If $K_g < K_o$, it is possible that the curves never cross (in which case the smallest

¹⁹From the fact that $\hat{g} < g_1$, it follows that the intersection of these two curves corresponding to the largest g (if there is more than one intersection) must be at least a relative maximum since the $V_g=0$ curve must have a steeper slope $[(dK_o/dg)]$ at this intersection.

feasible value of g is the optimal growth rate), or else that the curves intersect several times.

To summarize:

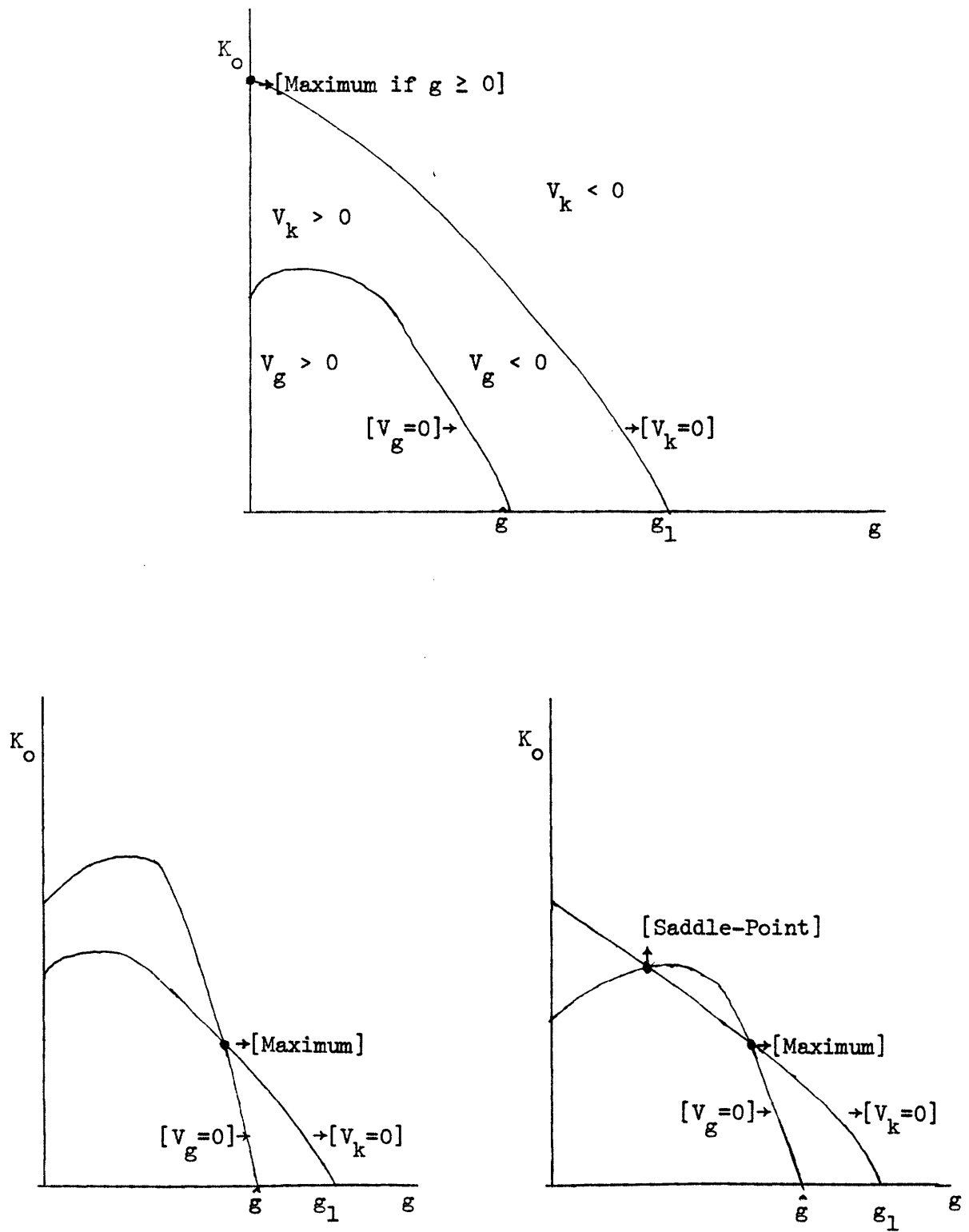
- a) There is no guarantee that the curves are monotonic.
- b) The $V_k=0$ curve has negative slope whenever the $V_g=0$ curve lies below it.
- c) Both curves are negatively sloped for "large" values of g .
- d) If the curves ever intersect, at least one of the intersections must be a relative maximum. Also, the $V_k=0$ curve must have negative slope whenever the two curves intersect.
- e) There is no guarantee of the existence or of the uniqueness of an interior maximum.

The diagrams on the following page indicate some of the possibilities of the behavior for the $V_k=0$ and the $V_g=0$ curves; naturally, there are other possibilities. Obviously, the simplicity of the Solow model disappears when we consider the possibility of capital-augmenting technical progress.

Assuming that an interior solution exists, it remains to be determined how the optimal solution (K^*, g^*) changes as c increases. Since both curves are shifted outward, it is not possible to determine by inspection how the intersection shifts (though at least one of K^* and g^* must increase). Using the fact that a maximum occurs at the intersection of the two curves (assuming the proper slopes), and that a change in c shifts both curves, we can derive (at a maximum):

$$37) \quad (dg/dc) = \frac{(-V_{gk})(V_{kc}) \left[\left\{ \frac{(V_{gc} V_{kk})}{(V_{kc} V_{gk})} \right\} - 1 \right]}{[V_{gg} V_{kk} - (V_{kg})^2]}$$

FIGURE I - Possible Behavior of the $V_k=0$ and the $V_g=0$ Curves



$$38) \quad (dK_o/dc) = \frac{(-v_{gg})(v_{kc})[1 - \{(v_{gc}v_{kg})/(v_{kc}v_{gg})\}]}{[v_{gg}v_{kk} - (v_{kg})^2]}$$

At a maximum the denominator of each expression must be positive; also, since $v_{gg} < 0$, $v_{gk} < 0$, and $v_{kc} > 0$, it follows that:

$$39) \quad \text{sign}(dg/dc) = \text{sign}[\{(v_{gc}v_{kk})/(v_{kc}v_{gk})\} - 1] ; \text{ and,}$$

$$\text{sign}(dK_o/dc) = \text{sign}[1 - \{(v_{gc}v_{kg})/(v_{kc}v_{gg})\}]$$

Thus, from the second order condition it is apparent that at least one of (dg/dc) and (dK_o/dc) must be positive.

It is not possible to determine the value (or sign) of (dg/dc) for all values of c ; however, for c near zero we find:

$$40) \quad \text{sign}(dg/dc) = \text{sign}[a + m(f+2g-i)]$$

Thus, for i sufficiently small ($i \leq f$ suffices), the management-oriented firm that faces disembodied capital-augmenting technical progress will choose a larger growth rate than a firm which does not possess technical progress. However, if i is rather large, it is possible that $(dg/dc) < 0$ (and thus $(dK_o/dc) > 0$).²⁰ If $(dg/dc) > 0$, it is not possible to determine the sign of (dK_o/dc) without further knowledge of T .²¹

²⁰In other words, if people are "quite myopic", they may be tempted to use the fruits of technological progress for immediate plunder rather than for long-term growth. Thus, firms facing identical cost and demand conditions (except, presumably, for interest rates), may respond differently in different societies. Particularly, in a developing country, where the rate of time preference may be high, technological change may lead to lower growth rates for the firm (and perhaps for the economy, depending upon who does the savings).

²¹Similarly, it is not possible in this case to show that the firm that has any positive level of technological progress will grow

The procedure followed to determine the sign of (dg/dc) is applicable for determining how changes in any of the parameters affect g or K_0 (In his paper, Prof. Solow, in order to compare the classical profit-maximizing firm to the growth-oriented firm, performed these comparative static operations. However, he did this under the assumption that the initial K_0 was given to the firm). Thus, for any parameter z we find:

$$41) \quad (dg/dz) = \frac{(-v_{gk})[(v_{kz})\{[(v_{gz} v_{kk})/(v_{kz} v_{gk})] - 1\}]}{[v_{gg} v_{kk} - (v_{kg})^2]}$$

$$42) \quad (dK_0/dz) = \frac{(-v_{gg})[(v_{kz})\{1 - [(v_{gz} v_{kg})/(v_{kz} v_{gg})]\}]}{[v_{gg} v_{kk} - (v_{kg})^2]}$$

Since the denominator is positive at a maximum, as is $(-v_{gk})$ and $(-v_{gg})$, the sign of each expression depends upon the term in brackets in the numerator. Therefore, we find:

- 43) $v_{kz} < 0, v_{gz} > 0$ implies $(dK_0/dz) < 0, (dg/dz) > 0$.
- $v_{kz} > 0, v_{gz} < 0$ implies $(dK_0/dz) > 0, (dg/dz) < 0$.
- $v_{kz} > 0, v_{gz} > 0$ implies at least one of $(dg/dz), (dK_0/dz)$ is positive.
- $v_{kz} < 0, v_{gz} < 0$ implies at least one of $(dg/dz), (dK_0/dz)$ is negative.

Following this procedure (and assuming $c=0$), we find:²²

²¹ faster than the firm with no technical progress. See the appendix for more details.

²² In the appendix we shall consider how any price change (or change in technology) affects the growth decisions made by the firm. This is done in the case of embodied technological progress, as well as

- 44) a) $(dK_o/di) > 0$, $(dg/di) < 0$
 b) $(dK_o/da) < 0$, $(dg/da) > 0$
 c) $(dK_o/dm) ?$, $(dg/dm) < 0$
 d) $(dK_o/dn) ?$, $(dg/dn) < 0$

As is expected (and as Prof. Solow finds for constant K_o), an increase in the price of capital goods or in the rate of discount leads the firm to choose lower growth rates. (Remember that 44) was derived by using the criterion that $V_k=0$, and by assuming that $c=0$). In the case of a change in the discount rate, since this does not represent an increase in costs per se (but rather a bias against the future), the smaller growth rate leads the firm to choose a larger initial size.²³ However, an increase in the wage rate actually results in a larger growth rate for the firm.

We have seen that for $V_k=0$, $V_g=0$, the model becomes quite complicated for $c > 0$; and we have seen that we cannot reach many definitive results, even concerning how changes in technological progress affect the growth rate chosen by the firm. If we adopt the criterion for the owner-oriented firm things become slightly more tractable. Let us now investigate this case.

$$\text{II) } V_k = m, \quad V_g = 0$$

Using this criterion we arrive at the following conditions:

²² for disembodied technological progress. Also, it is done for the owner-oriented firm as well as for the manager-oriented firm.

²³ Note the similarity of this result to that which we found for (dg/dc) - that is, if i is large enough, the firms may take profits immediately, foregoing the opportunity for growth.

$$45) \quad V_k - m = \frac{[\theta T b^\theta K_o^{\theta-1} - a - \{[m(i+f)(i-g)]/(i+c-g)\}]}{(i-g)} = 0$$

$$V_g = \frac{K_o [b^\theta K_o^{\theta-1} \{T+(i-g)T'\} - a - \{[m(i+f)(i-g)^2]/(i+c-g)^2\}]}{(i-g)^2} = 0$$

Since $V_k=m$ lies inside the $V_k=0$ curve, it follows that if an interior solution exists in the latter case, than it exists in the former case. Thus, as in the case $c=0$, the owner-oriented firm will choose a larger growth rate but smaller initial size than the manager-oriented firm.²⁴

Unfortunately, we are still not able to say that either curve is monotonic. All our previous comments regarding the shapes of the two curves (see page 198) are also valid in this case. From equation 45) we can show:

$$46) \quad T + n(i-g)T' + \left\{ \frac{[(n-1)Tmc(i-g)(i+f)]}{[a(i+c-g)^2 + m(i-g)(i+f)(i+c-g)]} \right\} = 0$$

When $c=0$, this reduces to the case considered by Solow; if $c > 0$, it follows that in equilibrium (since n is assumed to be larger than one):

$$47) \quad [T + n(i-g)T'] < 0$$

Therefore, the owner-oriented firm with capital-augmenting technical progress will (at least for small values of c) choose a larger growth rate than a comparable firm that has no technical progress.

More formally, using the technique described earlier, we find:

²⁴The usual warnings regarding multiple roots hold in this case, as in all other cases.

$$48) \quad \text{sign}(dg/dc) = \text{sign}\left\{\left(\frac{V_{gc} V_{kk}}{V_{kc} V_{gk}}\right) - 1\right\} =$$

$$\text{sign}\left\{\frac{2[a(i-g) + \{[m(i+f)(i-g)^2]/(i+c-g)\}]}{[a(i+c-g) + \{[m(i+f)(i-g)^2]/(i+c-g)\}]} - 1\right\} > 0 \quad \text{at } c=0$$

As we can readily see, for $c=0$ this expression is always positive; furthermore, for $[i > (c+g)]$, $(dg/dc) > 0$ everywhere.²⁵ Since, for convergence, we must assume $i > g$, if c is not "too large", this condition seems quite likely to be fulfilled.²⁶ Unfortunately, without further assumptions on $T(g)$, we cannot ascertain the sign of (dK_0/dc) .

Though we shall not present the actual computations, it should be noted that the "neat" solution of a constant price strategy for the owner-oriented firm disappears in the face of capital-augmenting technical progress. Even if the firm is forced to operate forever (thus absorbing some unnecessary losses in the case of falling prices), the owner-oriented firm with some technical progress will choose to pursue a falling-price strategy.²⁷

Obviously, however, this firm would do even better if it terminated its operations at some point in time, and consequently a steady-state model is hardly the proper framework for analyzing this

²⁵Note, paradoxically, that the larger i is in this case, the more certain we are that $(dg/dc) > 0$; just the opposite case holds for the management-oriented firm.

²⁶Though it is not necessarily true that $(dg/dc) > 0$ for all values of c , we show in the appendix that the owner-oriented firm with some capital-augmenting technical progress always chooses a larger growth rate than the firm that has no technical progress.

²⁷If $(\dot{P}/P) = -\pi$, assuming that the firm operates forever, we find:
 $0 < \pi = \{[m(i+f)c]/[\{[a(i+c-g)^2]/(i-g)^2\} + m(i+f)]\} < c$.

If the firm were allowed to shut-down when dividends fell to zero, it clearly could do even better.

case.²⁸ Though we do not plan to offer an alternative model at this time, it is worth observing that the presence of capital-augmenting technical progress causes "perverse" behavior for the Solow firm (which is geared to the steady-state), as it does for the neoclassical growth models.

Our next task, one which is a logical extension of this section, is to investigate how our results change if the technical progress is embodied in new capital (though it is still assumed to be capital-augmenting). Since the fruits of technical progress are bestowed only upon new capital, we should expect the firm to choose a smaller initial size, but larger growth rate, than in the case in which technical progress is disembodied. Let us now see if our intuition is substantiated.

V. Embodied Capital-Augmenting Technical Progress
and the Solow Firm

Before proceeding to the analysis, let us take a closer look at the model we have been using. The firm is assumed to start with some "chunk" of capital, and it is assumed to keep adding to this capital so that the total available stock of capital grows at a constant rate (of $[g-c]$). If we think about this, it is apparent that while a finite amount of capital of vintage $t=0$ exists, only infinitesimal amounts of

²⁸The question arises as to why the firm, instead of shutting down, does not just cut its output and raise prices. This approach, which is not acceptable in the steady-state model, indicates the limiting behavior of our assumptions. However, we feel that a more general equilibrium approach, properly representing the oligopolistic nature of the market (and thus giving rise to kinked-demand curve phenomena), would make a falling-price strategy much less likely to be chosen.

any other vintage exist ($\dot{K}(t) = K_t = (g+f-c)K_0 e^{(g-c)t}$ is the rate of flow of machines of vintage t). Though this may seem slightly strange, it causes no problems as long as all machines are used. However, if older machines are eventually discarded (due to economic obsolescence), then at some point in time a discontinuity in output will occur (when this large block of capital is discarded). As we shall see, this aspect of the model (plus the fact that the economic lifetime of capital depends upon the other variables under the firm's control) will cause us a great deal of trouble when we consider the case of embodied labor-augmenting technical progress.

Assuming technical progress is embodied and only capital-augmenting (or, if any labor-augmenting technical progress occurs, it is assumed to be disembodied, and for simplicity wages are assumed to grow at the same rate as the labor-augmenting technical progress), we have:

49) Q_v = flow of output from capital of vintage v .
 K_v = flow of capital of vintage v . Thus,
 $Q_v = bK_v e^{cv}$; K_0 = initial block of capital
 $L_v = K_v e^{cv}$

Next, assume that depreciation occurs at rate f on capital of all vintages:

50) $Q_v(t)$ = flow output at time t from capital of vintage v
 $K_v(t)$ = amount of capital of vintage v left at time t
 $L_v(t)$ = amount of labor used at time t on capital of vintage v
 $K_v(t) = K_v e^{-f(t-v)}$, $t \geq v$.
 $Q_v(t) = Q_v e^{-f(t-v)}$, $t \geq v$; and so forth.

Finally, assume the firm seeks to grow at (a constant) rate g :

$$51) \quad K_v(t) = (g+f)K_0 e^{(g-c)v} e^{-f(t-v)} = (g+f)K_0 e^{(g+f-c)v} e^{-ft} ; \quad t \geq v$$

$$Q_v(t) = b(g+f)K_0 e^{(g+f)v} e^{-ft} ; \quad \text{therefore,}$$

$$52) \quad Q(t) = \int_0^t [Q_v(t)dv] + Q_0(t) = bK_0 e^{gt} ; \quad L(t) = K_0 e^{gt} , \quad \text{and:}$$

$$Q_0(t) = bK_0 e^{-ft}$$

Therefore, the flow of dividends at time t is:

$$53) \quad \text{Div}(t) = T b^\theta K_0^\theta e^{gt} - a K_0 e^{gt} - m(g+f)K_0 e^{(g-c)t} ; \quad \text{and thus:}$$

$$54) \quad V = \int_0^\infty [\text{Div}(t)e^{-it}dt] = \left[\frac{[b^\theta K_0^\theta T - a K_0]}{(i-g)} - \frac{[m(g+f)K_0]}{(i+c-g)} \right]$$

For $c=0$ this reduces to Prof. Solow's expression; for $c > 0$, note that the value of the "vintage" firm is always less than that of the firm in which technical progress is disembodied (call it V'). That is, we know:

$$28) \quad V' = \{ [b^\theta K_0^\theta T - a K_0] / (i-g) \} - \{ [m(g+f-c)K_0] / (i+c-g) \}$$

Let us now investigate how the vintage firm will behave under each of our two criteria:

$$I) \quad V_k = V_g = 0$$

$$55) \quad V_k = \{ [[b^\theta K_0^{\theta(\theta-1)} T - a] / (i-g) \} - \{ [m(f+g)] / (i+c-g) \} = 0$$

$$56) \quad V_g = [K_0 / (i-g)^2] [b^\theta K_0^{\theta(\theta-1)} \{ T + (i-g)T' \} - a - \left[\frac{m(i+c+f)(i-g)^2}{(i+c-g)^2} \right]] = 0$$

Since $V_{kc}, V_{gc} > 0$, the curves for $c > 0$ both lie outside those for $c = 0$, and hence we can not tell by inspection what happens to K_0 and g (except that at least one of them must increase). Also, observe that the $V_k=0$ and the $V_g=0$ curves each lie inside their corresponding curves

for disembodied technical progress, and hence all that we can now conclude is that either K_0 or g (or both) is lower in the vintage case than in the disembodied case.

As in the previous section, it is not possible to show that the $V_k=0$ and the $V_g=0$ curves are negatively sloped at all points. However, all of our prior remarks regarding the slopes of the curves remain qualitatively unchanged. Thus, when the curves intersect, $V_k=0$ must have negative slope; and if the curves ever intersect, at least one of these intersections must be a relative maximum; and so forth.

Will the firm choose a larger growth rate for the case of vintage technical progress than for the non-vintage case? Since both curves shift inward (for the vintage model vis-a-vis the non-vintage model) it is not possible to tell a priori. Remembering that the $V_g=0$ curve must have a larger slope (in absolute value) at the intersection of the two curves (for it to be a relative maximum), it follows that if at the g which optimizes the non-vintage case (call it \bar{g}), the $V_k=0$ curve lies above the $V_g=0$ curve (for the vintage case), then the optimal g must decrease, and conversely. That is, if:

57) \bar{g}, \bar{K}_0 is the optimal solution for the non-vintage case; and define K_1, K_2 such that: $V_k(\bar{g}, K_1) = 0$, and $V_g(\bar{g}, K_2) = 0$ for the vintage case. Therefore, if g' is the optimal solution for the vintage case, then:

57a) $g' \begin{matrix} \geq \\ < \end{matrix} \bar{g}$ as $K_2 \begin{matrix} \geq \\ < \end{matrix} K_1$

Using the fact that the pair (\bar{g}, \bar{K}_0) is the solution for the non-

vintage case, we find:²⁹

$$58) \quad (K_1/\bar{K}_0)^{[\theta-1]} = \frac{[a + \{[m(f+g)(i-g)]/(i+c-g)\}]}{[a + [m(f+g-c)(i-g)]/(i+c-g)]} \equiv A, \text{ and:}$$

$$(K_2/\bar{K}_0)^{[\theta-1]} = \frac{[a + \{[m(i+c+f)(i-g)^2]/(i+c-g)^2\}]}{[a + [m(i+f)(i-g)^2]/(i+c-g)^2]} \equiv B. \text{ Thus:}$$

$$59) \quad (K_2/K_1) = (A/B)^n.$$

However, through simple multiplication it can readily be shown that

$A > B$. Therefore:

$$60) \quad (A/B) > 1 \text{ implies: } (K_2/K_1) > 1, \text{ and thus: } g' > \bar{g}$$

Consequently, we have confirmed our intuition: the vintage firm will choose a larger growth rate than the non-vintage firm (assuming they are otherwise identical). Since the $V_k=0$, $V_g=0$ curves shift inward (in the vintage case), the vintage firm must choose a smaller initial size, given that it chooses a larger growth rate. In other words, vintage technical progress raises the price of current machines compared to future machines (effectively), and hence it induces the firm to "start smaller and grow bigger", compared to its non-vintage counterpart. In this respect it is like a decrease in the discount rate.

²⁹For $V_k=0$, $[\theta b^\theta K^{(\theta-1)}_T - a - \{[m(f+g)(i-g)]/(i+c-g)\}] = 0$ in the vintage case, and $[\theta b^\theta K^{(\theta-1)}_T - a - \{[m(f+g-c)(i-g)]/(i+c-g)\}] = 0$ in the non-vintage case. Since this expression is evaluated at the same g (called \bar{g}), we readily obtain the ratio $(K_1/\bar{K}_0)^{[\theta-1]}$; and similarly for $V_g=0$ we can obtain $(K_2/\bar{K}_0)^{[\theta-1]}$. Finally, by looking at the ratio of these ratios, our conclusion follows.

If we next inquire how the onset of technical progress affects the firm (if the technical progress is embodied), we can derive for small values of c ($c = 0$):

$$61) \quad \text{sign}[dg/dc] = \text{sign}\left[\frac{[a+m(g+f)]}{[a+m(i+f)]}\{2 + [(i-g)/(f+g)]\} - 1\right] > 0$$

Therefore, the firm will (initially) increase its growth rate due to the onset of technical progress; unfortunately, it does not appear possible to say that $[dg/dc]$ is positive for all values of c . In other words, it is possible that for large values of c , a further increase in the rate of technical progress may cause the firm to choose a lower growth rate.³⁰ As always, the sign of $[dK_0/dc]$ is ambiguous [for $(dg/dc) > 0$] without further information on the nature of T .

Let us now see how our results change for the owner-oriented firm.

$$II) \quad V_k = m, \quad V_g = 0$$

$$62) \quad V_k - m = \frac{[\theta b^\theta K_0^{(\theta-1)} T - a - \{[m(i+c+f)(i-g)]/(i+c-g)\}]}{(i-g)} = 0$$

$$63) \quad V_g = [K_0/(i-g)^2][b^\theta K_0^{(\theta-1)}\{T+(i-g)T'\} - a - \left\{\frac{m(i+c+f)(i-g)^2}{(i+c-g)^2}\right\}] = 0$$

As in prior sections, it is easy to show that the owner-oriented firm will choose a larger growth rate than the management-oriented firm.

³⁰As in the disembodied case in which $V_k = m$, $V_g = 0$, it is possible to demonstrate that a firm which faces some positive level of capital-augmenting technical progress will choose a larger growth rate than a firm that has no technical progress. For these results, and others, consult the Appendix at the end of this chapter.

Also, since both curves ($V_k = m$, $V_g = 0$) for the vintage case lie inside their counterparts for the non-vintage case, it follows that at least either the size of the firm or the growth rate of the firm must be smaller for the vintage firm, compared to the non-vintage firm. However, proceeding exactly as we did in the management-oriented case, it can readily be shown that the vintage firm will choose a higher growth rate, and hence a smaller initial size, than its non-vintage counterpart. Naturally, the underlying economic reasons for this result are the same in this case as they were in the management-oriented case.

When we consider how technical progress affects the firm, we can readily see that, for small values of c , the growth rate increases as the rate of technical progress increases. Furthermore, for any value of c , we find:

$$64) \quad \text{sign}[dg/dc] = \text{sign}\left\{ \frac{[a + \{m(i+c+f)(i-g)\}/(i+c-g)](i-g)}{[a + \{m(i+c+f)(i-g)^2\}/(i+c-g)^2]}(i+c-g) \right\} \left[2 + \frac{(i+c-g)}{(f+g)} \right] - 1$$

For $c=0$, this expression is clearly positive. Though it is not necessarily positive for all values of the parameters, $[dg/dc]$ will be positive for all values of c if either $i > f$ or $i > (c+g)$ (for small c , this latter inequality must hold for convergence of the integral for the value of the firm).³¹ Again, it is not possible to determine how a change in the rate of technical progress affects the size of the firm (if $[dg/dc] > 0$).

³¹As for the embodied case, $V_k = 0$ and $V_g = 0$, we can show that the firm with any positive level of technical progress will choose a larger growth rate than the firm that has no technical progress. For further details, consult the Appendix to this chapter.

Though we shall not bother to exhibit the proof, it can be shown that for the vintage model, as in the non-vintage case, the larger the elasticity of demand, the slower the firm will choose to grow:

$$65) \quad [dg/dn] < 0 \quad \text{for } V_k=0, V_g=0 \quad \text{or for } V_k=m, V_g=0$$

As in prior cases, it is not possible to determine how the size of the firm is changed due to changes in the elasticity of demand without better knowledge of the values of the parameters and the shape of $T(g)$.

Finally, we can ask how our model will behave if we allow the firm to pursue an optimal price strategy, in addition to choosing its initial size and its desired growth rate. Assuming the firm must operate forever (even with falling prices), we find that, as in the disembodied case, the owner-oriented firm will choose a falling-price strategy.³² Thus, unless we can resort to the notion of a kinked demand curve (or some other oligopolistic practice), the owner-oriented firm will choose self-annihilation, and the basic assumptions of this model become quite dubious indeed.

In summary, we have found so far that:

- 1) The more elastic the demand curve, the slower the firm will choose to grow.
- 2) The vintage firm will always choose a larger growth rate and smaller initial size than its non-vintage counterpart.
- 3) For small rates of technical progress, an increase in the rate of technical progress will lead to larger growth rates for the firm (except, perhaps, in the manager-oriented, disembodied firm); for the owner-oriented

³² Assuming the firm must operate forever, we find:

$$0 < \pi = \left\{ \frac{[mc(i+c+f)]}{[m(i+c+f) + \{[a(i+c-g)^2]/(i-g)^2\}]} \right\} < c, \quad \text{where } (\dot{P}/P) = -\pi.$$

vintage firm, this may be true for any rate of technological progress (if $i > [c+g]$).

- 4) As in prior cases, the owner-oriented firm is more growth-minded than his management-oriented counterpart.
- 5) The constant price strategy of the stagnant (that is, the firm with no technical progress) owner-oriented firm disappears in the presence of technical progress.

It is nice to summarize the results we have obtained so far, since when we consider a vintage model with labor-augmenting technical progress we shall find that the problem does not seem to be tractable. Let us now see why this is so.

VI. Embodied Labor-Augmenting Technical Progress and the Solow Firm

For reasons outlined earlier, the problem with embodied labor-augmenting technical progress is much more complicated than those problems we have considered so far. If the wage rate is increasing over time at the same rate as technical progress, then it is clear that, after some period of time has elapsed, it will no longer be profitable to use machines that are older than some specific vintage. However, this economic discarding of machines (which does not arise when technical progress is either capital-augmenting or disembodied) causes us problems for two reasons:

- a) If the firm is portrayed as starting with some initial amount of capital (of vintage $t = 0$), and growing at a constant rate (so that there are only infinitesimal amounts of capital of other vintages), it is clear that, when the initial block of capital is discarded, there will

be a discontinuity in output.³³

- b) Secondly, and more fundamentally, the economic lifetime of the machines depends upon both the growth rate and the size that the firm chooses. This aspect of the problem makes it impossible (for us) to arrive at any conclusions.

Also, it is clear that a) and b) interact since the economic life of capital that is chosen determines when the original chunk of capital is discarded, and this, in turn, affects the current value of the firm.

It is possible to avoid the discontinuity problem by assuming that there are only infinitesimal amounts of each vintage. However, for the firm to reach a constant growth rate in this case it must originally grow faster than its final steady-state rate of growth (until the first machine is discarded). This, in turn, makes it difficult to consider the problem in a Solow framework in which prices are rigid (allowing them to vary merely complicates the analysis) and advertising expands demand at a constant rate. Even if simplifying assumption are made,³⁴ so that the problem can be treated in the context of a steady-state growth path, the solution remains essentially unsolvable.

³³This is not a necessary assumption - it is possible to assume that the firm starts with capital of various vintages (Though one must still postulate how the magnitude of each type of vintage capital is determined by the firm, since in this problem the firm is free to choose its own initial conditions). However, even under this assumption, the analysis proves intractable because the optimal economic lifetime of capital (for the firm) is an internal variable.

³⁴For example, we could assume that no sales are made until the steady-state growth path is reached - output would just be given away as an advertising gimmick. Alternatively, we could assume that output is sold at a price just sufficient to cover labor costs - and this would be another advertising scheme (Naturally, this assumes that the price can be set large enough to cover labor costs; however, for there to be any long-run profits, this must be possible).

For example, suppose technical progress is embodied in labor and occurs at rate d ; the wage rate is assumed to also increase at rate d . If the firm starts with an initial block of capital of vintage o , and if it seeks to grow at rate g , then we can write:

$$66) \quad K_v = (g+f)K_o e^{gv}$$

$$Q_v(t) = b(g+f)K_o e^{(g+f)v} e^{-ft}$$

If j is the age of the oldest machine in use, then we can find:

$$67) \quad Q(t) = \int_0^t [Q_v(t)dv] + Q_o e^{-ft} = bK_o e^{gt} ; \quad t \leq j$$

$$Q(t) = \int_{(t-j)}^t [Q_v(t)dv] = bK_o [1 - e^{-(g+f)j}] e^{gt} ; \quad t > j$$

Similarly, assuming technical progress is labor-augmenting and embodied:

$$68) \quad L(t) = [K_o / (g+f-d)] [(g+f)e^{(g-d)t} - de^{-ft}] ; \quad t \leq j$$

$$L(t) = [\{K_o(g+f)\} / (g+f-d)] [1 - e^{-(g+f-d)j}] e^{(g-d)t} ; \quad t > j$$

Proceeding in this way, we calculate the value of the firm:

$$69) \quad V = \int_0^j [\text{Div}(t)e^{-it} dt] + \int_j^\infty [\text{Div}(t)e^{-it} dt]$$

$$= \left\{ \left[\frac{Tb^\theta K_o^\theta}{(i-g)} \right] [1 - e^{(g-i)j}] \{1 - [1 - e^{-(g+f)j}]^\theta\} \right.$$

$$\quad \left. - \left[\frac{aK_o(i+f)}{(i-g)(i+f-d)} \right] [1 - e^{-(i+f-d)j}] \right.$$

$$\quad \left. - \left[\frac{m(g+f)K_o}{(i-g)} \right] \right\}$$

We can either treat j as a decision variable for the firm (in addition to K_o and g), or else we can assume that j is determined such that the marginal revenue product of the last worker is just equal to his cost

(which is equal to the wage rate, assuming perfect competition in the factor market):³⁵

$$70) \quad Q_v = be^{dv} L_v \text{ implies } MRP_L = [be^{dv}(\theta TP)] ; \text{ Wage Rate} = ae^{dt}$$

$$\text{Therefore, } e^{d(t-v)} = [(\theta bTP)/a] \text{ implies } e^{dj} = [(\theta bTP)/a]$$

Clearly, the larger is g , the sooner machines must be discarded since a larger fraction of total revenue is devoted to advertising. Also, the larger firm (with the smaller price) will tend to discard its older equipment sooner. Unfortunately, even if we make this a priori substitution, the problem proves intractable.

Economically, it is clear that the embodied technical progress is actually a burden to the entrepreneur, assuming that wages rise at an equivalent rate. All the benefits of the technical progress are passed on to the worker (as is also the case for disembodied labor-augmenting technical progress);³⁶ however, since the fruits of technical progress are not spread over all machines, the capitalist is thereby hurt due to the technological obsolescence of some of his machines.³⁷ Since an increase in d , ceteris paribus, decreases the economic lifetime of machines, it is to be expected that this lowers both V_k and V_g (V_{kd} ,

³⁵ Presumably, these two methods will give the same result; unfortunately, both methods prove intractable.

³⁶ This indicates that the Kennedy-Samuelson model relies crucially upon the assumption that firms decide alone what type of technological progress to choose. If there were collusive behavior (for example, pooling research funds), and if they realized the effect of their decision on wages (but could not wholly offset it with monopsony power), they may instead choose capital-augmenting technical progress, even for $\sigma < 1$.

³⁷ It is clear that the value of the firm (68) is less than the value of the firm for the case in which no technical progress occurs.

$V_{gd} < 0$). Thus, it would appear that either K_0 or g (or both) will decrease as a result of the occurrence of embodied labor-augmenting technical progress (and the consequent increase in wages).

Unfortunately, we have not been able to take the analysis much further. We have seen that, with embodied labor-augmenting technical progress, the faster growing companies will tend to discard their machines sooner than slower growing companies (other things being equal). However, due to the complexity of the analysis, we have been able to conclude little else.

As the final model we present an optimal control formulation of the Solow firm (with constant technology). We shall see that the only steady-state path satisfying the equations of motion corresponds to a case already discussed - that of the owner-oriented firm ($V_k = m$, $V_g = 0$).

VII. The Solow Firm and Optimal Planning

In the preceding sections we have investigated how the Solow firm would respond to various changes in parameters, and we have demonstrated how this firm chooses its optimal size and growth rate under two different criteria. Alternatively, it is possible to formulate this problem as an optimal control problem, and to investigate how the firm should behave, assuming that it seeks to maximize the discounted value of the flow of dividends. As we shall see, if the firm starts with a very particular initial capital stock, it will choose to behave exactly the same as does the owner-oriented firm that we studied earlier in this chapter.

Thus, suppose that we choose the basic Solow model, except that we allow prices to change over time. Letting g be the rate of

increase in demand due to advertising, we can write:

$$71) \quad (\dot{Q}/Q) = [g - n(\dot{P}/P)]$$

Assuming that there are no excess machines (that is, the initial stock of capital is sufficiently small), and that there is no technical progress:

$$72) \quad (\dot{Q}/Q) = (\dot{K}/K) \quad \text{implies} \quad \dot{K} = [(\dot{Q}/Q)(Q/b)]$$

But, due to investment, we know:

$$73) \quad m\dot{K} = s[TPQ - a(Q/b) - mf(Q/b)] ,$$

where the term in brackets is the amount of funds available for net investment after advertising expenses, workers, and depreciation have been paid.³⁸ Therefore:

$$74) \quad (\dot{Q}/Q) = [(sb)/m][TP - (a/b) - m(f/b)]$$

$$75) \quad (\dot{P}/P) = [1/n][g - s(b/m)\{TP - (a/b) - m(f/b)\}]$$

$$76) \quad \text{Dividends} = (1-s)[TP - (a/b) - m(f/b)]Q$$

Allowing λ_1 and λ_2 to represent the shadow prices, we can formulate the Hamiltonian as follows:

$$77) \quad H = \{(1-s)[TP - (a/b) - m(f/b)]Q + \lambda_1 s(b/m)[TP - (a/b) - m(f/b)]Q + \lambda_2 (P/n)[g - s(b/m)\{TP - (a/b) - m(f/b)\}]\}e^{-it}$$

The control variables are s and g ; the state variables are Q and P . For s we find:

³⁸If $[TP - (a/b) - m(f/b)]Q \leq 0$, then $s = 0$. In that case, 72) should read: $m\dot{K} = s'[TPQ - a(Q/b)] - fK$, where s' is now the gross savings rate. However, we are only interested in problems where K is small initially, so this problem need not concern us here.

$$78) \quad s \text{ maximizes } [(1-s) + s(b/m)\{\lambda_1 - [(\lambda_2^P)/(nQ)]\}];^{39}$$

$$\text{This implies: } s \begin{cases} = 0 \\ \in (0,1) \\ = 1 \end{cases} \text{ as } \{(b/m)[\lambda_1 - \{(\lambda_2^P)/(nQ)\}]\} \begin{cases} < \\ > \end{cases} 1$$

$$79) \quad \text{Therefore, } \underset{\{s\}}{\text{Max}}[(1-s) + s(b/m)\{\lambda_1 - [(\lambda_2^P)/(nQ)]\}] \equiv \gamma$$

$$\gamma = \text{Max}[1, \{(b/m)[\lambda_1 - \{(\lambda_2^P)/(nQ)\}]\}]$$

Next, choosing g to maximize 76), we find:

$$80) \quad T'(g) = [(-\lambda_2)/(n\gamma Q)]$$

Finally, there are the two Euler equations:

$$81) \quad [TP - (a/b) - m(f/b)][(1-s) + s\lambda_1(b/m)] + \dot{\lambda}_1 - i\lambda_1 = 0$$

$$82) \quad \{TQ[(1-s) + s(b/m)\{\lambda_1 - [(\lambda_2^P)/(nQ)]\}]\} \\ + (\lambda_2/n)[g - s(b/m)\{TP - (a/b) - m(f/b)\}] + \dot{\lambda}_2 - i\lambda_2 = 0$$

In addition, we also have the differential equations for P and Q themselves. To summarize, we have the following equations:

$$i) \quad \gamma = \text{Max}[1, \{(b/m)[\lambda_1 - \{(\lambda_2^P)/(nQ)\}]\}]$$

$$ii) \quad T' = [(-\lambda_2)/(n\gamma Q)]$$

$$iii) \quad (\dot{\lambda}_1/\lambda_1) = i - [TP - (a/b) - m(f/b)][\{(1-s)/\lambda_1\} + s(b/m)]$$

$$iv) \quad (\dot{\lambda}_2/\lambda_2) = i - [(TQ\gamma)/\lambda_2] - (1/n)[g - s(b/m)\{TP - (a/b) - m(f/b)\}]$$

$$v) \quad (\dot{Q}/Q) = s(b/m)[TP - (a/b) - m(f/b)]$$

$$vi) \quad (\dot{P}/P) = (1/n)[g - s(b/m)\{TP - (a/b) - m(f/b)\}]$$

We seek a stationary solution that satisfies these equations. Clearly,

³⁹If $[TP - (a/b) - m(f/b)] \leq 0$, then $s = 0$.

$s = 1$ is not a solution unless the growth rate exceeds i (in which case the integral does not converge). Similarly, $s = 0$ is not a solution if there is to be some positive growth rate.⁴⁰ Thus, for $0 < s < 1$:

$$83) \quad [\lambda_1 - \{(\lambda_2 P)/(nQ)\}] = (m/b)$$

Also, for g to be constant over time implies:

$$84) \quad (\dot{\lambda}_2/\lambda_2) = (\dot{Q}/Q) \quad (\text{for } \gamma \text{ constant})$$

Using equations 83) and 84), plus the equations of motion, we can derive:

$$85) \quad \dot{\lambda}_1 = \dot{P} = 0$$

$$86) \quad \dot{P} = 0 \rightarrow (\dot{Q}/Q) = (\dot{\lambda}_2/\lambda_2) = g = s(b/m)[TP - (a/b) - m(f/b)]$$

$$87) \quad (\dot{\lambda}_2/\lambda_2) = g \rightarrow [T + n(i-g)T'] = 0, \text{ which determines } \bar{g}^{41}$$

Using equations i) - vi), and assuming a steady-state solution, we can derive the necessary initial conditions for each of the variables (as well as their subsequent values over time):

$$88) \quad Q_0 = bK_0 = \{[(im+mf+a)n]/[b(n-1)T]\}^{-n} ; \quad Q = Q_0 e^{gt}$$

$$89) \quad P = [Q_0]^{-(1/n)}$$

$$90) \quad s = \{[g(n-1)]/[in+f+(a/m)]\} < 1 \quad (s \text{ is the net savings rate})$$

$$91) \quad \lambda_1 = \{[nm(i-g)+a+m(g+f)]/[b(n-1)(i-g)]\}$$

⁴⁰Without knowledge of $T(g)$ it is not possible to exclude such a solution. However, we shall assume that $T(g)$ is such that a solution for $g > 0$ exists.

⁴¹This is the same equation found by Prof. Solow for the owner-oriented firm; as before, it does not appear possible to guarantee the uniqueness of the solution.

$$92) \quad \lambda_2 = \{[(im+mf+a)n]/[b(n-1)T^{(n+1)/n}]\}^{-n} [e^{gt}/(i-g)]$$

These are the only values of the variables, given the values of the parameters, that will generate a steady-state solution that satisfies the equations i) - vi) (assuming that \bar{g} , as determined from 87), is unique); unfortunately, we have not been able to determine the optimal path for the enterprise if the initial capital stock differs from the optimal level as determined in 88).

Let us now return for a moment to the static Solow model:

$$93) \quad V = \{[Tb^\theta K_o^\theta - a - m(f+g)K_o]/(i-g)\} ; \quad V_k = m \quad \text{implies:}$$

$$94) \quad [\theta T b^\theta K_o^{\theta-1}] = [a + m(i+f)] ; \quad \text{and therefore, using } Q_o = bK_o :$$

$$95) \quad Q_o = \{[b(n-1)T]/[(mi+mf+a)n]\}^n ,$$

which is the same value for Q_o as we found in 88) for the control problem. Therefore, the optimal control problem implicitly attributes an opportunity cost to capital equal to its price. Consequently, the steady-state solution to the optimal control problem is equivalent to the optimal solution for the owner-oriented firm that was discussed earlier in this chapter (assuming that there is no technical progress). Finally, we can not readily determine the behavior of the firm if it starts with a capital stock different from that found in 88).

VIII. Conclusion - The Solow Model and the Steady-State

In this chapter we have explored how various forms of technical progress affect the behavior of a dynamic firm. We have discussed how changes in various parameters affect the decisions made by the firm, and we have seen how different "types" of firms respond to the same stimuli. Specifically, we have shown that:

- a) For "all" firms, the more elastic is the demand curve, the slower the firm will choose to grow.
- b) For all types of technical progress considered, the owner-oriented firm will choose a larger growth rate than the firm that disregards the opportunity cost of the initial stock of capital.
- c) The growth rate chosen by the firm will be larger (and the initial size smaller) if technical progress is embodied, rather than disembodied (and capital-augmenting).
- d) In general, the firm will not choose a constant price strategy if there is any capital-augmenting technical progress.
- e) The optimal growth rate chosen by the firm depends upon all of the parameters of the model (unless there is no technical progress and the firm is owner-oriented).
- f) For the management-oriented firm, an increase in the rate of disembodied, capital-augmenting technical progress might lead to a decline in the growth rate chosen by the firm. If the firm (country) has a very high rate of time preference, this result is likely.
- g) For the owner-oriented firm, an increase in the rate of technical progress (for small values of the rate of technical progress) will increase the growth rate chosen by the firm.
- h) If technical progress is embodied and labor-augmenting, and if wages increase at the rate of technical progress, then the quicker growing the firm is, the sooner it will discard capital as being economically obsolete.
- i) Embodied labor-augmenting technical progress, if accompanied by wages that increase at the same rate as technical progress, actually decreases the value of the firm. This, we feel, sheds some

doubt on the Kennedy-Samuelson results if it is assumed (recognized?) that decisions on types of technological progress are made by large corporations that are cognizant of their effect on wages.

There are other results that could be listed; we feel, however, that the major results have been enumerated above.

The model of the growing firm formulated by Prof. Solow, and extended in this chapter, though simplistic in style, is an important contribution because it focuses on what we consider a major factor in influencing a firm's dynamic behavior - the need to expend a significant part of its resources to ensure a growing demand for its product. However, when one attempts to extend the results of this chapter to apply to the economy as a whole, several weaknesses of the model become evident:

a) The assumption of Fixed Coefficients - though this assumption simplifies the analysis, it is not really an important one, and it could readily be dropped.

b) The Steady-State assumption - clearly, the model is aimed at providing a picture of a firm that is essentially unchanging over time, as is the steady-state economy. However, this assumption has been seen to be quite dubious when we allow the firm (that has capital-augmenting technical progress) to vary prices.

c) The "Long-Run" - in the Solow model, a firm (and, implicitly, all firms) may grow at a faster rate than the economy as a whole over the relevant planning period. This stems from the fact that the long run is "not that long". Also, the model implicitly assumes that firms' plans (in terms of expanding demand) are always fulfilled. This brings

us to our final point.

d) The Partial-Equilibrium Approach - obviously, this model is a partial equilibrium one - no concern is given to the larger wages that presumably will occur if a firm grows faster than the economy as a whole (for an extended period of time), and no attention is focused on the market structure of the economy.

The fact that the model is partial equilibrium in nature is not a criticism of the model itself, since its purpose was to analyze how different firms respond to various exogenous changes. However, if the model is to be adapted to help explain the growth behavior of an economy (and the demand for investment in that economy), it is our feeling that it must be embedded in a more general equilibrium approach. For example, the assumption that the firm faces a demand curve that is not completely elastic implicitly recognizes the existence of market power; the logical extension would seem to be to formulate a model that explicitly recognizes the oligopolistic structure of the market. In that model, growth might serve to increase market power (and hence profits) - this is in contrast to asking how the growth rate responds to an exogenous change in market structure.

Unfortunately, it is easier to talk of such a model than to construct it. Nevertheless, it is our feeling that greater understanding of the growth behavior of an economy can only be obtained by disaggregating the growth model and by attempting to explain how firms within this growing economy might behave.

IX. Appendix - Responses of the Firm's Growth Rate
to Parameter Changes

In the main part of this chapter we have shown how changes in the rate of technological progress affect the growth decisions made by the firm, for very low levels of technical progress ($c = 0$). In this Appendix we hope to show that these comparative static results can be extended. Specifically, we ask:

1) Will the firm with some technical progress always choose a larger growth rate than the firm with no technical progress (for the owner-oriented and the management-oriented firm)?

2) How does a change in the price of labor or capital affect the growth decisions made by the firm?

In addition to attempting to answer these questions for the owner-oriented and the management-oriented firms, we shall attempt to answer them for the case of embodied, as well as disembodied, technical progress.

Consider first how the presence of technological change affects the growth decisions. We have seen that the firm chooses the growth rate at which the $V_k=0$ ($V_k=m$) and $V_g=0$ curves intersect, assuming that they have the "proper" slope. For $c = 0$, these curves will (might) intersect at some growth rate - call it \bar{g} . The onset of technological change causes both of these curves to shift; if, in the presence of this new technology, the $V_k=0$ curve lies above the $V_g=0$ curve at the old growth rate \bar{g} , then the new growth rate (g') must be greater than \bar{g} , since the $V_g=0$ curve must have a greater slope (in absolute value) at the maximum.

As an example, consider the case of disembodied technical progress (it is capital-augmenting) for the management-oriented firm ($V_k=0$). In that case we have:

$$96) \quad V_k = 0 \rightarrow [\theta T(g)b^\theta K_0^{(\theta-1)}] = \{[a+m(i-g)(f+g-c)]/(i+c-g)\}$$

$$V_g = 0 \rightarrow \{[T+(i-g)T']b^\theta K_0^{(\theta-1)}\} = \{[a+m(i-g)^2(i+f)]/(i+c-g)^2\}$$

Let \bar{g} , \bar{K}_0 represent the optimal solution for the case $c=0$; let K_1 represent the value of K_0 such that $V_k(K_1, \bar{g}, c) = 0$ for $c > 0$; let K_2 be chosen such that $V_g(K_2, \bar{g}, c) = 0$ for $c > 0$. Then, if g' represents the new optimal growth rate:⁴²

$$97) \quad g' \begin{matrix} \geq \\ < \end{matrix} \bar{g} \quad \text{as} \quad K_2 \begin{matrix} \geq \\ < \end{matrix} K_1$$

From 96) we find:

$$98) \quad [K_1/\bar{K}_0]^{(\theta-1)} = \frac{[a + \{[m(i-g)(f+g-c)]/(i+c-g)\}]}{[a + m(f+g)]} ;$$

$$[K_2/\bar{K}_0]^{(\theta-1)} = \frac{[a + \{[m(i-g)^2(f+g-c)]/(i+c-g)^2\}]}{[a + m(i+f)]} ;$$

Therefore,

$$99) \quad [K_2/K_1] \begin{matrix} \geq \\ < \end{matrix} 1 \quad \text{as} \quad \frac{[a + \{[m(i-g)(f+g-c)]/(i+c-g)\}][a+m(i+f)]}{[a + \{[m(i-g)^2(i+f)]/(i+c-g)^2\}][a+m(i+f)]} \begin{matrix} \geq \\ < \end{matrix} 1$$

Without knowledge of the value of the parameters, we can not determine the value of the ratios in 99); however, if $[(f+2g) > (i+c)]$, this suffices to guarantee that the growth rate with technical progress will

⁴²As usual, the possibility of multiple roots exists. In that case, we may talk about the largest such root, which must be a relative maximum.

be larger ($g' > \bar{g}$)⁴³ (note that for $a = 0$, this is also a necessary condition).⁴⁴

However, in the case of embodied technical progress for the management-oriented firm we find that we can show that the firm with some positive rate of technical progress will always choose the larger growth rate. In this case we find:

$$100) \quad [K_2/K_1] \underset{<}{\geq} 1 \quad \text{as} \quad \left(\frac{[a + \{m(f+g)(i-g)\}/(i+c-g)] [a+m(i+f)]}{[a + \{m(i+c+f)(i-g)^2\}/(i+c-g)^2] [a+m(g+f)]} \right) \underset{<}{\geq} 1$$

By expanding 100) we can show that $[K_2/K_1] > 1$; therefore, the firm with some embodied, capital-augmenting technical progress will choose a larger growth rate than a comparable firm with no technical progress.

Using the same technique, we can readily show that the owner-oriented firm which possesses some capital-augmenting technical progress will choose a larger growth rate than a comparable firm that receives no advances in technique (this is true for both embodied and disembodied technical progress). Note that this result does not imply that an even larger rate of growth of technical progress will lead to even larger growth rates for the firm; it merely tells us that a firm

⁴³Note that the larger is i (the rate of discount), the less likely it is that the firm with technical progress will choose a larger growth rate than the firm with no technical progress.

⁴⁴Observe that if $m=0$, there is no change in the optimal growth rate due to capital-augmenting technical progress. This is true in all cases (of capital-augmenting technical progress) and is obvious, since if capital is free, it is irrelevant to the firm how much capital must be "purchased". By continuity, it seems plausible to argue that the smaller m , ceteris paribus, the less important capital-augmenting technical progress will be to the decisions made by the owner or manager. Thus, we would expect the onset of capital-augmenting technical progress to have a larger effect on decisions in an underdeveloped country, where capital costs are high relative to labor costs, than in a capital rich country.

with some (any) positive rate of capital-augmenting technical progress will choose a larger growth rate than the firm that is technologically stagnant (except, perhaps, for the manager-oriented, disembodied case).

In summary, in all cases except the disembodied, management-oriented case, we can conclude that the presence of capital-augmenting technical progress will lead the firm to choose a larger growth rate than it otherwise would if there were no technical progress. For the case of disembodied technical progress in the management-oriented firm, no definitive results can be stated without further knowledge of the relevant parameters.

Next, let us consider how a change in cost conditions affects the growth decisions made by the firm. Specifically, we would like to know how an increase in the wage rate (a) or in capital costs (m) will affect the growth rate chosen by the firm. Since an increase in a (or in m) leads to a decrease in V_k and V_g (ceteris paribus), it follows that either the optimal g or the optimal K_0 (or both) must decrease. Therefore:

$$101) \quad [dg/da] > 0 \rightarrow [dK_0/da] < 0 ; \text{ similarly for } [dg/dm].$$

Fortunately, it is relatively simple to calculate $[dg/da]$ and $[dg/dm]$ using the technique discussed earlier. If we assume (for example) an owner-oriented firm, for $a = \bar{a}$, we have:

$$102) \quad V_k(\bar{g}, \bar{K}_0, \bar{a}) = m ; \quad V_g(\bar{g}, \bar{K}_0, \bar{a}) = 0$$

at a maximum. Now let a (or m) increase to $a' > \bar{a}$, and define K_1, K_2 as follows:

$$103) \quad V_k(\bar{g}, K_1, a') = m ; \quad V_g(\bar{g}, K_2, a') = 0$$

As before, if g' represents the new growth rate chosen by the firm, then:

$$104) \quad g' \begin{matrix} > \\ < \end{matrix} \bar{g} \quad \text{as} \quad K_2 \begin{matrix} > \\ < \end{matrix} K_1$$

Clearly, the same technique can be used for changes in capital costs.

As an example, consider the case of an owner-oriented firm facing disembodied, capital-augmenting technical progress. For a maximum we have:

$$105) \quad [\theta T b^{\theta} K_0^{(\theta-1)}] = a + \{[m(i+f)(i-g)]/(i+c-g)\}$$

$$[T + (i-g)T'] b^{\theta} K_0^{(\theta-1)} = a + \{[m(i+f)(i-g)^2]/(i+c-g)^2\}$$

Using the notation defined above, at \bar{g} we find:

$$106) \quad g' \begin{matrix} > \\ < \end{matrix} \bar{g} \quad \text{as} \quad K_2 \begin{matrix} > \\ < \end{matrix} K_1, \quad \text{and} \quad K_2 \begin{matrix} > \\ < \end{matrix} K_1 \quad \text{as:}$$

$$107) \quad \frac{\left\{ \frac{[a' + \{[m'(i+f)(i-g)]/(i+c-g)\}][\bar{a} + \{[\bar{m}(i+f)(i-g)^2]/(i+c-g)^2\}]}{[\bar{a} + \{[\bar{m}(i+f)(i-g)]/(i+c-g)\}][a' + \{[m'(i+f)(i-g)^2]/(i+c-g)^2\}]} \right\}}{1} \begin{matrix} > \\ < \end{matrix}$$

Let us consider changes in capital costs only ($m' > \bar{m}$, $a' = \bar{a}$); from 107) we can readily see that:

$$108) \quad K_2 > K_1 \quad \text{and thus} \quad [dg/dm] > 0, \quad \text{for} \quad c > 0.$$

(Note that if $c = 0$, then $K_2 = K_1$, regardless of the values of a and m ; this corresponds to Prof. Solow's observation that, in this case, the growth rate of the firm depends only upon the rate of discount and the elasticity of demand, in addition to the shape of $T(g)$). Similarly, letting $a' > \bar{a}$, $m' = \bar{m}$, we find for this case:

$$109) \quad K_2 < K_1 \quad \text{and therefore} \quad [dg/da] < 0 \quad \text{for} \quad c > 0.$$

Therefore, in the case of the owner-oriented firm with disembodied capital-augmenting technical progress, an increase in capital

costs actually leads to a larger growth rate, though a smaller initial size, for the firm. This may be rationalized by realizing that, due to the technical progress, capital costs in the future (for a given growth rate) are lower than present capital costs, and thus the decrease in size as a result of the larger capital costs makes it profitable to expand the growth rate of the firm. On the other hand, technical progress, since it is only capital-augmenting, does not offset (in the future) the larger labor costs, so there is no reason to expect the firm to choose a larger growth rate as a result of increased labor costs.

We have seen in 108) and 109) above that increased labor costs have exactly the opposite effect (qualitatively) on the growth rate as does an increase in capital costs. When we consider all possible cases we find that this result still holds:

$$110) \quad \text{sign}[dg/da] = - \text{sign}[dg/dm]$$

The table on the following page summarizes the effects of changes in capital or labor costs on both the size of the firm and its growth rate for all cases.

From the table we can see that, for the owner-oriented firm, an increase in capital costs actually leads to a more rapidly growing firm, while for the management-oriented firm, no definitive results are available (though it can readily be seen that an increase in capital costs is more likely to lead to a larger growth rate for the firm with embodied technical progress than for the firm with disembodied technical progress). If i (the discount rate) is sufficiently large, then $[dg/dm] < 0$ - in this case, the high rate of time preference makes growth more costly, and thus "overcomes" the benefits of capital-

TABLE I - Growth Rate and Size Responses Due to Changes in Capital or Labor Costs

Case	Sign [dg/dm]	Sign [dk/dm]	Sign [dg/da]	Sign [dk/da]
I) $c = 0$				
a) $V_k = V_g = 0$	-	?	+	-
b) $V_k = m, V_g = 0$	0	-	0	-
II) Disembodied ($c > 0$)				
a) $V_k = V_g = 0$	$\text{sign}[c(f+g-c)-(i-g)(i+c-g)]$	- if $[dg/dm] \geq 0$? otherwise	- $\text{sign}[dg/dm]$	- if $[dg/da] \geq 0$? otherwise
b) $V_k = m, V_g = 0$	+	-	-	?
III) Embodied ($c > 0$)				
a) $V_k = V_g = 0$	$\text{sign}[c(f+g)-(i-g)(i+c-g)]$	- if $[dg/dm] \geq 0$? otherwise	- $\text{sign}[dg/dm]$	- if $[dg/da] \geq 0$? otherwise
b) $V_k = m, V_g = 0$	+	-	-	?

augmenting technical progress (in terms of lowering future capital costs). However, as is apparent from the table, $[dg/da]$ is positive in that case.

There appears to be little else that can be said about the table, so we shall let it "speak for itself".

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