

# Minimization of the Maximum Peak-to-Peak Gain: The General Multiblock Problem \*

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## Abstract

This paper presents a comprehensive study of the general  $\ell_1$ -optimal multiblock problem, as well as a new linear programming algorithm for computing suboptimal controllers. By formulating the interpolation conditions in a concise and natural way, the general theory is developed in simpler terms and with a minimum number of assumptions. In addition, further insight is gained on the structure of the optimal solution, and different classes of multiblock problems are distinguished. This leads to conceptually attractive, iterative method for finding approximate solutions with the following properties: 1) approximates multiblock problems with one-block problems by delay augmentation, 2) unifies the treatment of zero and rank interpolation conditions through robust computations, 3) provides upper and lower bounds of the optimal objective function by solving one finite dimensional linear program at each iteration, 4) for a class of problems, it generates suboptimal controllers that achieve the upper bound without order inflation, 5) both bounds as well as the solution converge to the optimal, 6) it does not require the existence of polynomial feasible solutions, and 7) gives information about the support structure of the optimal solution.

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## Notation

Let  $X$  be a real normed vector space, then  $X^*$  denotes the dual space of  $X$  containing all bounded linear functionals on  $X$ .

$\ell_1$  Space of absolutely summable sequences supported on the non-negative integers. If  $x \in \ell_1$  then  $\|x\|_1 = \sum_{k=0}^{\infty} |x(k)| < \infty$ .

$\ell_1^{p \times q}$  Space of  $p \times q$  matrices with entries in  $\ell_1$ . If  $M = (m_{ij}) \in \ell_1^{p \times q}$ , then  $\|M\|_1 := \max_{1 \leq i \leq p} \sum_{j=1}^q \|m_{ij}\|_1$ .

$\ell_\infty$  Space of all bounded sequences of real numbers supported on the non-negative integers. If  $x \in \ell_\infty$  then  $\|x\|_\infty := \sup_k |x(k)| < \infty$ .

$\ell_\infty^{p \times q}$  Space of  $p \times q$  matrices with entries in  $\ell_\infty$ . If  $M = (m_{ij}) \in \ell_\infty^{p \times q}$ , then  $\|M\|_\infty := \sum_{i=1}^p \max_{1 \leq j \leq q} \|m_{ij}\|_\infty$ . Note that  $\ell_\infty^{p \times q} = (\ell_1^{p \times q})^*$ .

$c_0^{p \times q}$  Subspace of  $\ell_\infty^{p \times q}$  consisting of all elements whose entries decay to zero, i.e.,  $\lim_{k \rightarrow \infty} m_{ij}(k) = 0$  for all  $\{ij\}$ . Note that  $(c_0^{p \times q})^* = \ell_1^{q \times p}$ .

$\lambda$  Complex variable representing the unit delay. Given  $M \in \ell_1^{p \times q}$ , define  $\hat{M}(\lambda) := \sum_{k=0}^{\infty} M(k)\lambda^k$  as the  $\lambda$ -transform of  $M$ .

$\mathcal{D}$  The open unit disk.

$P_k$  The truncation operator on sequences. Hence if  $x = \{x(i)\}_{i=0}^{\infty}$  is any sequence, then  $P_k x = \{x(0), x(1), \dots, x(k), 0, \dots\}$ .

$S_k$  Right shift by  $k$  positions. If  $x = \{x(i)\}_{i=0}^{\infty}$  is any sequence and  $k$  is a nonnegative integer, then  $S_k x = \{\overbrace{0, \dots, 0}^k, x(0), x(1), \dots\}$ .

Given a matrix  $M$ ,  $(M)_i$  will denote its  $i$ th row and  $(M)^j$  its  $j$ th column.

## 1 Introduction

Design specifications for practical control problems are often most naturally expressed in terms of time-domain bounds on the amplitude of signals (exogenous disturbances and regulated outputs). This observation has led to the introduction of a new optimization problem in the context of control system design. In [37] Vidyasagar formulated the  $\ell_1$ -optimal control problem. In contrast with the  $\mathcal{H}_\infty$  problem, the  $\ell_1$ -optimal design has as objective the minimization of the maximum peak-to-peak gain of a closed-loop system that is driven by bounded amplitude disturbances.

In 1987-88 Dahleh and Pearson introduced some basic results on the theory of  $\ell_1$  optimization. In [9] the solution to the  $\ell_1$ -optimal control problem was presented for the special case of square (i.e., one-block) systems. Then, in [11] Dahleh et al. presented the central ideas

for the solution of non-square (i.e., multiblock) problems, including a method to compute approximate suboptimal solutions iteratively. Such method is based on the solution of a linear program representing a truncated version of the original problem. Similar results extending these ideas to the continuous-time domain were introduced by the same authors in [10], as well as a solution to the fixed input optimization problem [12].

These results brought considerable attention to the problem of  $\ell_1$  optimization. In [29] a general treatment of the multiblock case was presented, where the optimal solution is shown to exist under some assumptions. Independently in [6] and [33] a method was introduced to compute lower bounds on the optimal norm, by solving a complementary linear program. A direct linear programming formulation (in the primal space) was presented in [30]. Also, [34] introduced a nice account of some convergence properties and pointed to interesting deficiencies in the theory. In [17, 18] the full state-feedback problem was addressed.

On the area of robustness, considerable advancement was made too. In [13], the necessity of the small gain theorem in the  $\ell_1$  context was analyzed. Also, [24] presented necessary and sufficient conditions for robust performance and robust stability under structured time-varying perturbations. It turns out that such conditions are relatively easy to compute making the theory more attractive from the point of view of applications. Other related work can be found in [8, 6, 3, 19, 14, 32].

The present investigation is motivated by the lack of a solid understanding of the general  $\ell_1$  multiblock problem. While various aspects of the theory are well understood, the structure of the optimal solution in the general multiblock case is not. As a result, solution methods which are based on a straightforward truncation of the full problem, suffer from significant deficiencies. Most important, they generate a sequence of suboptimal controllers of increasing order, and miss the structure of the (possibly low order) optimal controller. This issue was pointed out quite nicely in [33] where exact solutions of low order were computed. From a practical point of view, such truncation method translates into high order controllers even for the simplest multiblock problems. At the same time, it requires the existence of feasible closed-loop maps with finite pulse response, a condition that many control problems lack.

In this paper we present a comprehensive treatment of the general  $\ell_1$ -optimal multiblock problem. Contributions are made in the general theory as well as in the approximate methods of solution. With regard to the problem formulation, a more compact and natural way of characterizing the interpolation conditions of the general multiblock problem is presented. It has the advantage of simplifying many of the proofs and avoiding unnecessary assumptions (compared to previous work [29, 34]). We also present a new solution method for the general multiblock problem with the following characteristics:

- 1) Approximates multiblock problems with one-block problems by delay augmentation, thus allowing to exploit the characteristics of the optimal solutions of such problems.
- 2) Applies results from matrix theory [21] in the computation of interpolation conditions.
- 3) With each approximation (requiring the solution of only one linear program), the method provides upper and lower bounds of the optimal norm.
- 4) Under mild assumptions, both bounds converge to the optimal value of the norm.
- 5) With each approximation the method generates a feasible (i.e., stabilizing) controller that achieves the upper bound.
- 6) For a special class of multiblock problems the solutions are exact.

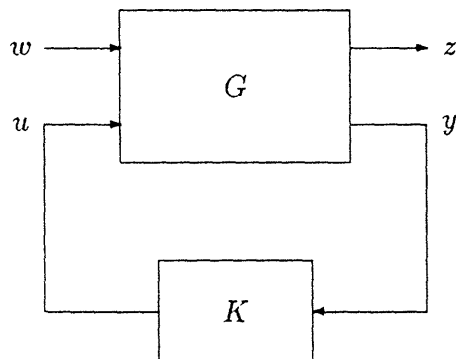


Figure 1: The Standard Problem

- 7) For a larger class of multiblock problems the sequence of suboptimal controllers does not suffer from order inflation.

Also, a result is presented relating the support characteristics of the optimal and approximate solution of multiblock problems, followed by a stronger conjecture. These results are complemented by a broad range of numerical examples, including a case study where the  $\ell_1$  and  $\mathcal{H}_\infty$  solution to the pitch axis control of the X29 aircraft are compared.

The paper is organized as follows: in section 2 the general  $\ell_1$ -optimal control problem is defined. The new interpolation conditions are presented in section 3 as well as computational procedures. This is followed by an existence result with minimum assumptions in section 4. Next, we establish the equivalence between  $\ell_1$  optimization and infinite dimensional linear programming in section 5. Section 6 contains the solution to one-block problems. The results in this section are an extension of those in [29]. Section 7 presents (approximate) methods of solution to multiblock problems. In particular, the delay augmentation method is introduced along with its convergence properties. Illustrations and examples are contained in section 8. In sections 9 and 10, we present a few results and observations (including a conjecture) on the support characteristics of these approximate solutions. Finally, we treat the X29 synthesis problem in section 11 followed by the conclusions in section 12.

## 2 Problem Formulation

The setup corresponds to the standard disturbance rejection problem formulated as a linear fractional transformation from the disturbance input to the regulated output, with the controller in the lower loop (see Figure 1). In particular, we consider the discrete time case, with the inputs and outputs being sequences of vectors. The problem is represented via an LTI finite-dimensional operator,  $G$ , that maps the disturbance vector  $w$  of dimension  $n_w$ , and the control vector  $u$  of dimension  $n_u$ , to the regulated output vector  $z$  of dimension  $n_z$ , and the

measurement vector  $y$  of dimension  $n_y$ . Thus, with the appropriate partitioning,

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \quad (1)$$

The controller action is represented by the operator  $K$  that maps the measurement sequence to the control sequence, i.e.,  $u = Ky$ . The closed-loop map from the disturbance to the regulated output, denoted  $\Phi$ , is given by:

$$\Phi = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \quad (2)$$

The  $\ell_1$ -optimal control problem can be stated as follows: among all internally stabilizing controllers, find the one that minimizes the maximum peak-to-peak gain of  $\Phi$  operating on the space of bounded disturbances with unit norm. That is,

$$\mu^\circ := \inf_{K \text{ stab.}} \sup_{\max_{1 \leq i \leq n_w} \|w_i\|_\infty = 1} \left( \max_{1 \leq k \leq n_z} \|(\Phi w)_k\|_\infty \right) = \inf_{K \text{ stab.}} \|\Phi\|_1 \quad (3)$$

In the above we have used the fact that the induced norm of an operator mapping bounded sequences in  $\mathbb{R}^{n_w}$  to bounded sequences in  $\mathbb{R}^{n_z}$  is given by the  $\ell_1^{n_z \times n_w}$  norm.

It is well known that a simpler description of the set of all (internally) stable closed-loop maps is obtained via a parameterization of all stabilizing controllers [38]. Such parameterization provides an affine expression, mapping an operator space to the set of all internally stable closed-loop maps:

$$\Phi = H - UQV \quad (4)$$

where  $H \in \ell_1^{n_z \times n_w}$ ,  $U \in \ell_1^{n_z \times n_u}$  and  $V \in \ell_1^{n_y \times n_w}$  are functions of the problem data (i.e., the operator  $G$ ), and  $Q$  is a free parameter in  $\ell_1^{n_u \times n_y}$  (i.e., stable). Furthermore, if  $G$  is LTI and finite dimensional, so are  $H$ ,  $U$  and  $V$ . Then, for any  $Q \in \ell_1^{n_u \times n_y}$ , a controller can be computed that achieves the corresponding closed-loop map,  $\Phi$ .

Consequently, the  $\ell_1$  problem can be redefined as a minimum distance problem in  $\ell_1^{n_z \times n_w}$ :

$$\mu^\circ := \inf_{R \in \mathcal{S}} \|H - R\|_1 = \inf_{\Phi - H \in \mathcal{S}} \|\Phi\|_1 \quad (5)$$

where

$$\mathcal{S} := \{R \in \ell_1^{n_z \times n_w} \mid R = UQV \text{ for some } Q \in \ell_1^{n_u \times n_y}\} \quad (6)$$

The subspace  $\mathcal{S}$  contains the set of feasible  $R$ 's. Also, from duality theory [26], problem (5) can be posed in the dual space of  $\ell_1^{n_z \times n_w}$ , that is,  $\ell_\infty^{n_z \times n_w}$  as the following maximization problem:

$$\mu^\circ = \max_{\substack{G \in \mathcal{S}^\perp \\ \|G\|_\infty \leq 1}} \langle H, G \rangle \quad (7)$$

where  $\langle H, G \rangle$  is the value of the bounded linear functional  $G$  at the point  $H$ :

$$\langle H, G \rangle = \sum_{i=1}^{n_z} \sum_{j=1}^{n_w} \sum_{k=0}^{\infty} g_{ij}(k) h_{ij}(k)$$

and  $\mathcal{S}^\perp$  is the right annihilator of  $\mathcal{S}$ :

$$\mathcal{S}^\perp = \{G \in \ell_\infty^{n_z \times n_w} \mid \langle R, G \rangle = 0 \forall R \in \mathcal{S}\}$$

Furthermore, if a solution to (5) exists, say  $\Phi^\circ$ , then it is aligned with every solution  $G^\circ$  to (7), that is  $\langle \Phi^\circ, G^\circ \rangle = \|\Phi^\circ\|_1 \|G^\circ\|_\infty$ . This implies that  $\Phi^\circ$  and  $G^\circ$  must satisfy the following alignment conditions:

- i) if  $|g_{ij}^o(t)| < \max_{1 \leq j \leq n_w} \|g_{ij}^o\|_\infty$ , then  $\phi_{ij}^o(t) = 0$
- ii)  $\phi_{ij}^o(t)g_{ij}^o(t) \geq 0$
- iii) let  $I = \{i \in [1, 2, \dots, n_z] \mid (G^o)_i \equiv 0\}$ , then  $\|(\Phi^o)_i\|_1 = \mu^o$  for all  $i$  not in  $I$
- iv) for all  $i \in I$ ,  $(\Phi^o)_i$  can be anything such that  $\|(\Phi^o)_i\|_1 \leq \mu^o$

The next section studies the solvability of the equation  $R = UQV$  for  $Q$  in  $\ell_1^{n_u \times n_y}$ .

### 3 Interpolation Conditions

Here we take some of the ideas in [11] and [29], and present a natural and compact description of the interpolation conditions for the most general MIMO case.

The notion of interpolation conditions can be viewed in at least two ways: as algebraic conditions on the matrix  $\hat{R}(\lambda)$  so that it belongs to the range of  $\hat{U}\hat{Q}\hat{V}$ , or as conditions on the nullspace of the operator  $R$ . Here we are going to exploit the algebraic notion although, for the purpose of computations, we view the interpolation conditions as a nullspace matching problem.

In the sequel it will be assumed, without loss of generality, that  $\hat{U}(\lambda)$  has full column rank (i.e., rank of  $n_u$  for almost all  $\lambda$ ) and  $\hat{V}(\lambda)$  has full row rank (i.e., rank of  $n_y$  for almost all  $\lambda$ ). Violation of these assumptions implies that there are redundancies in the controls and/or the measurements which can be easily removed.

First, a simple but useful result from complex variable theory is presented, where  $(\cdot)^{(k)}(\lambda_0)$  denotes the  $k$ th order derivative with respect to  $\lambda$ , evaluated at  $\lambda_0$ :

**Lemma 3.1** *Given a function  $f(\cdot)$  of the complex variable  $\lambda$  analytic in  $\mathcal{D}$ , then  $(f)^{(k)}(\lambda_0) = 0$  for  $k = 0, 1, \dots, (\sigma - 1)$  for  $\lambda_0 \in \mathcal{D}$  if and only if  $f(\lambda) = (\lambda - \lambda_0)^\sigma g(\lambda)$  where  $g(\cdot)$  is analytic in  $\mathcal{D}$ .*

Next, consider Smith-McMillan decompositions of the rational matrices  $\hat{U}$  and  $\hat{V}$ . (Note: to simplify notation, the complex variable argument will be omitted in most expressions.)

$$\hat{U} = \hat{L}_U \hat{M}_U \hat{R}_U \quad (8)$$

$$\hat{V} = \hat{L}_V \hat{M}_V \hat{R}_V \quad (9)$$

where  $\hat{L}_U, \hat{R}_U, \hat{L}_V$  and  $\hat{R}_V$  are (polynomial) unimodular matrices. Under the rank assumptions on  $\hat{U}$  and  $\hat{V}$ , the rational matrices  $\hat{M}_U$  and  $\hat{M}_V$  have the following diagonal structure:

$$\hat{M}_U = \begin{pmatrix} \frac{\epsilon_1}{\psi_1} & & & & \\ & \ddots & & & \\ & & \frac{\epsilon_{n_u}}{\psi_{n_u}} & & \\ 0 & \cdots & 0 & & \\ \vdots & \ddots & \vdots & & \\ 0 & \cdots & 0 & & \end{pmatrix} \quad (10)$$

$$\hat{M}_V = \begin{pmatrix} \frac{\epsilon'_1}{\psi'_1} & & 0 & \cdots & 0 \\ & \ddots & \vdots & \ddots & \vdots \\ & & \frac{\epsilon'_{n_y}}{\psi'_{n_y}} & 0 & \cdots & 0 \end{pmatrix} \quad (11)$$

Let  $\lambda_0$  be a zero of  $\hat{U}(\lambda)$ . Let  $\sigma_{U_i}(\lambda_0)$  denote the multiplicity of  $\lambda_0$  as a root of  $\epsilon_i(\lambda)$ , then  $\{\sigma_{U_i}(\lambda_0)\}_{i=1}^{n_u}$  defines a non-decreasing sequence of non-negative integers. For a given  $i \in \{1, 2, \dots, n_u\}$ ,  $\sigma_{U_i}(\lambda_0)$  is known as the algebraic multiplicity of  $\lambda_0$ . The total number of indices  $i$  for which  $\sigma_{U_i}(\lambda_0)$  is strictly positive is known as the geometric multiplicity of  $\lambda_0$ . Similarly, define  $\{\sigma_{V_j}(\lambda_0)\}_{j=1}^{n_y}$  for  $\hat{V}(\lambda)$ .

Let  $\Lambda_{UV}$  denote the set of zeros of  $\hat{U}$  and  $\hat{V}$  in  $\bar{\mathcal{D}}$ . In order to prove the interpolation theorem (i.e., apply the results of Lemma 3.1) we need the following assumption:

**Assumption 1:**  $\Lambda_{UV} \subset \mathcal{D}$ .

Consider the unimodular matrices in Equation (8). Since their inverses are polynomial, one can define the following polynomial row and column vectors:

$$\begin{aligned} \hat{\alpha}_i(\lambda) &= (\hat{L}_U^{-1})_i(\lambda) & i &= 1, 2, \dots, n_z \\ \hat{\beta}_j(\lambda) &= (\hat{R}_V^{-1})^j(\lambda) & j &= 1, 2, \dots, n_w \end{aligned} \quad (12)$$

Now we are ready to present the main interpolation theorem. These conditions are different from those in [29] and do not require coprime factorizations.

**Theorem 3.1** *Given  $R \in \ell_1^{n_z \times n_w}$ , there exists  $Q \in \ell_1^{n_u \times n_y}$  such that  $R = UQV$  if and only if for all  $\lambda_0 \in \Lambda_{UV} \subset \mathcal{D}$  the following conditions are satisfied:*

$$\begin{aligned} i) & \ (\hat{\alpha}_i \hat{R} \hat{\beta}_j)^{(k)}(\lambda_0) = 0 \text{ for } \begin{cases} i = 1, \dots, n_u \\ j = 1, \dots, n_y \\ k = 0, \dots, \sigma_{U_i}(\lambda_0) + \sigma_{V_j}(\lambda_0) - 1 \end{cases} \\ ii) & \ \begin{cases} (\hat{\alpha}_i \hat{R})(\lambda) \equiv 0 & \text{for } i = n_u + 1, \dots, n_z \\ (\hat{R} \hat{\beta}_j)(\lambda) \equiv 0 & \text{for } j = n_y + 1, \dots, n_w \end{cases} \end{aligned}$$

**Proof** Consider the following factorization of  $\hat{M}_U$  and  $\hat{M}_V$  (where  $\mathbf{0}$  denotes a block of zeros of appropriate dimensions):

$$\hat{M}_U =: \begin{pmatrix} \hat{\mathcal{E}}_U \hat{\Psi}_U^{-1} \\ \mathbf{0} \end{pmatrix} \quad ; \quad \hat{M}_V =: \begin{pmatrix} \hat{\Psi}_V^{-1} \hat{\mathcal{E}}_V & \mathbf{0} \end{pmatrix}$$

where  $\hat{\mathcal{E}}_U$  and  $\hat{\mathcal{E}}_V$  retain the zeros in  $\Lambda_{UV}$  while  $\hat{\Psi}_U$  and  $\hat{\Psi}_V$  capture the stable (i.e., minimum-phase) zeros of  $\hat{U}$  and  $\hat{V}$  along with their (stable) poles. Thus, both  $\hat{\Psi}_U$  and  $\hat{\Psi}_V$  are invertible in  $\ell_1$ . Then,

$$\hat{R} = \hat{L}_U \begin{pmatrix} \hat{\mathcal{E}}_U \hat{Q} \hat{\mathcal{E}}_V & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \hat{R}_V$$

where  $\hat{Q} := \hat{\Psi}_U^{-1} \hat{R}_U \hat{Q} \hat{L}_V \hat{\Psi}_V^{-1}$ . Clearly,  $\hat{Q} \in \ell_1^{n_u \times n_y}$  if and only if  $Q \in \ell_1^{n_u \times n_y}$ . Next, define the following partitions of  $\hat{L}_U$  and  $\hat{R}_V$ :

$$\hat{L}_U = \begin{pmatrix} \hat{L}_{U,1} & \hat{L}_{U,2} \end{pmatrix} \quad ; \quad \hat{R}_V = \begin{pmatrix} \hat{R}_{V,1} \\ \hat{R}_{V,2} \end{pmatrix} \quad (13)$$

where  $\hat{L}_{U,1}$  has  $n_u$  columns and  $\hat{R}_{V,1}$  has  $n_y$  rows. Then, given  $R \in \ell_1^{n_z \times n_w}$ ,

$$\begin{aligned} \exists Q \in \ell_1^{n_u \times n_y} \text{ such that } R &= UQV \\ \Updownarrow \\ \exists \bar{Q} \in \ell_1^{n_u \times n_y} \text{ such that } R &= L_{U,1} \mathcal{E}_U \bar{Q} \mathcal{E}_V R_{V,1} \end{aligned}$$

Necessity of condition *i*) follows immediately. Take any  $i \in \{1, \dots, n_u\}$  and  $j \in \{1, \dots, n_y\}$ , then

$$(\hat{\alpha}_i \hat{R} \hat{\beta}_j)(\lambda) = \prod_{\lambda_0 \in \Lambda_{UV}} (\lambda - \lambda_0)^{\sigma_{V_i}(\lambda_0)} \hat{q}_{ij}(\lambda) \prod_{\lambda_0 \in \Lambda_{UV}} (\lambda - \lambda_0)^{\sigma_{V_j}(\lambda_0)}$$

which implies condition *i*) by Lemma 3.1 and the fact that  $\hat{q}_{ij}$  is in  $\ell_1$ .

Necessity of condition *ii*) results from the following: take any  $i \in \{n_u + 1, \dots, n_z\}$  and  $j \in \{n_y + 1, \dots, n_w\}$ , then  $(\hat{\alpha}_i \hat{R})(\lambda) \equiv 0$  and  $(\hat{R} \hat{\beta}_j)(\lambda) \equiv 0$  since  $(\hat{\alpha}_i \hat{L}_{U,1})(\lambda) \equiv 0$  and  $(\hat{R}_{V,1} \hat{\beta}_j)(\lambda) \equiv 0$ .

To show that conditions *i*) and *ii*) are sufficient we proceed by backwards construction: by Lemma 3.1,

$$i) \implies \begin{pmatrix} \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_{n_u} \end{pmatrix} \hat{R} \begin{pmatrix} \hat{\beta}_1 & \dots & \hat{\beta}_{n_y} \end{pmatrix} = \hat{\mathcal{E}}_U \hat{W} \hat{\mathcal{E}}_V$$

for some  $W \in \ell_1^{n_u \times n_y}$  since  $R \in \ell_1^{n_z \times n_w}$ . Moreover,

$$ii) \implies \begin{pmatrix} \hat{\alpha}_{n_u+1} \\ \vdots \\ \hat{\alpha}_{n_z} \end{pmatrix} \hat{R} \equiv \mathbf{0} \quad \text{and} \quad \hat{R} \begin{pmatrix} \hat{\beta}_{n_y+1} & \dots & \hat{\beta}_{n_w} \end{pmatrix} \equiv \mathbf{0}$$

Therefore, combining these equations into one,

$$\hat{L}_U^{-1} \hat{R} \hat{R}_V^{-1} = \begin{pmatrix} \hat{\mathcal{E}}_U \hat{W} \hat{\mathcal{E}}_V & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

which implies that  $W = \hat{Q}$  is the solution. ■

In words, Theorem 3.1 provides a set of algebraic conditions which are necessary and sufficient for  $R$  to be feasible (i.e., equivalent to  $UQV$  for some stable  $Q$ ). The conditions in *i*) make sure that the left and right unstable zero structure of the composition  $UQV$  is preserved while the conditions in *ii*) impose the correct (normal) rank conditions on  $\hat{R}$ . In fact, it is possible to view the collection of  $\hat{\alpha}_i$ 's and  $\hat{\beta}_j$ 's for  $i > n_u$  and  $j > n_y$ , as two polynomial basis (not necessarily of minimal degree) for the left and right nullspaces of  $\hat{R}(\lambda)$  (see [23]). By virtue of the Smith-McMillan decomposition these sets of polynomial vectors are linearly independent (over the field of rational functions) so they generate a minimal set of constraints on  $\hat{R}$  (Note: the four-block case has some redundancy which can be eliminated apriori, see [18] for a detailed discussion).

In the sequel, we will refer to the conditions in *i*) as the *zero interpolation conditions*, and to the conditions in *ii*) as the *rank interpolation conditions*. Rank interpolation conditions are also known by the names of relations [11] and convolution conditions [33, 34].

Problems of the form (4) have been traditionally classified in the  $\mathcal{H}_\infty$  and  $\mathcal{H}_2$  literature according to the dimensions of the different signal spaces involved. Here we adopt the same classification:

- *One-Block Problems*: When  $n_w = n_y$  and  $n_z = n_u$ . These are also known as good rank or square problems.
- *Two-Block Column Problems*: When  $n_w = n_y$  and  $n_z > n_u$ .



- *Two-Block Row Problems:* When  $n_w > n_y$  and  $n_z = n_u$ .
- *Four-Block Problems:* When  $n_w > n_y$  and  $n_z > n_u$ .

A problem is labeled multiblock when it is not one-block. Multiblock problems are also known as bad rank problems [11, 29].

Clearly, one-block problems only require zero interpolation conditions and have no rank interpolation conditions, while two-block row (column) problems require right (left) rank interpolation conditions, and four-block problems require both left and right rank interpolation conditions.

### 3.1 Computation of Interpolation Conditions

The problem of finding the Smith-McMillan decomposition of rational matrices is at the heart of the interpolation problem. This decomposition has been studied thoroughly due to its strong connections with several important notions in system theory (e.g. multivariable zeros and poles), although mostly from an algebraic point of view [23]. The standard algebraic algorithm to compute such objects is based on the Euclidean division algorithm, known to be numerically sensitive. Nevertheless, there has been some effort in this direction, for example, by using symbolic methods from computer algebra on polynomial matrices [4]. However, it is generally desirable to have algorithms based on the state-space representation of systems, that are more easily implemented on digital computers.

Here we present an alternative approach to the problem of finding the zero interpolation conditions of a square rational matrix. Such approach avoids the explicit computation of the Smith-McMillan decomposition. Furthermore, it is computationally attractive since it is based on finding the nullspaces of certain Toeplitz-like matrices which are formed directly from the state-space representation of the system.

Although multiblock problems require rank interpolation conditions, we will show that those problem can be posed in such a way that only zero interpolations need to be considered.

In Theorem 3.1 we have shown how the internal stability of the closed-loop system is assured if the zero structure of the left unstable zeros of  $\hat{U}$  and the right unstable zeros of  $\hat{V}$  is preserved in  $\hat{R}$ . Such structure is characterized by the zero frequency, its algebraic and geometric multiplicity, and its directional properties as given by the corresponding polynomial vector  $\hat{\alpha}_i$  or  $\hat{\beta}_j$ . Despite its numerical problems, the Smith-McMillan decomposition provides the most natural way of characterizing the zero and pole structure of a rational matrix. To circumvent the formal Smith-McMillan decomposition of  $\hat{U}(\lambda)$  and  $\hat{V}(\lambda)$ , it is necessary to find an alternative set of conditions that unequivocally defines the zero structure of a rational matrix. Such a set is presented in this section.

The theory of zeros of MIMO systems has been studied extensively, both from an algebraic and state-space perspective [28, 16, 31]. It is well known that a zero of a square system given in state-space form  $[A, B, C, D]$ , is characterized by the solution of a generalized eigenvalue problem of the form [28]:

$$\begin{pmatrix} A - z_0 I & B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = 0$$

where  $z_0 := \lambda_0^{-1}$ ,  $x_0$  is known as the state zero direction and  $u_0$  is known as the zero input direction. However, the numerical stability of such eigenvalue problem deteriorates quickly when there are zeros with algebraic multiplicity greater than one. Indeed, such difficulty is

equivalent to finding the Jordan decomposition of a defective matrix (i.e., a non-diagonalizable matrix) which is known to be a hard numerical problem [22].

Although it is difficult to obtain the full zero structure directly from the state-space description of a system, the location or frequency of the zeros can be reliably computed [20]. In the sequel, we will assume that the locations of the unstable zeros of the rational (square) matrices  $\hat{U}(\lambda)$  and  $\hat{V}(\lambda)$  are available.

Following, we introduce a useful definition along with some notation.

**Definition 3.1** *Given a rational matrix  $\hat{H}(\lambda)$  analytic at  $\lambda_0$  and a positive integer  $\sigma$ ; define the following block-lower-triangular Toeplitz matrix:*

$$T_{\lambda_0, \sigma}(\hat{H}) = \begin{pmatrix} H_0 & 0 & 0 & \cdots & 0 \\ H_1 & H_0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ H_{\sigma-1} & H_{\sigma-2} & H_{\sigma-3} & \cdots & H_0 \end{pmatrix} \quad (14)$$

where the  $H_i$ 's are given by the Taylor expansion of  $\hat{H}(\lambda)$  at  $\lambda_0$ , that is,

$$\hat{H}(\lambda) = H_0 + (\lambda - \lambda_0)H_1 + (\lambda - \lambda_0)^2H_2 + (\lambda - \lambda_0)^3H_3 + \cdots$$

and  $H_i = \frac{1}{i!}(\hat{H})^{(i)}(\lambda_0)$ .

A numerically stable method was proposed in [36] to find the structural indices associated with poles and zeros of a stable rational matrix  $\hat{H}$ , by looking at the rank of  $T_{\lambda_0, \sigma}(\hat{H})$  as  $\sigma$  increases. Such approach, however, does not provide the directional information necessary to construct the interpolation conditions. Here we present an extension of the ideas in [36] by looking at the structure of the nullspace of  $T_{\lambda_0, \sigma}(\hat{H})$  for increasing values of  $\sigma$ . Such approach has strong connections with the general interpolation theory of rational matrix functions [1, 2]. In particular, it exploits the analyticity of the matrices  $\hat{U}$  and  $\hat{V}$  in the disk.

The following definition establishes some terminology [1].

**Definition 3.2** *Given an  $m \times n$  (real) rational matrix  $\hat{H}(\lambda)$  analytic at  $\lambda_0$ , a right null chain of order  $\sigma$  at  $\lambda_0$  is an ordered set of column vectors in  $\mathbb{R}^n$ ,  $\{x_1, x_2, \dots, x_\sigma\}$ , such that  $x_1 \neq 0$  and*

$$T_{\lambda_0, \sigma}(\hat{H}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\sigma \end{pmatrix} = \mathbf{0}$$

*Similarly, a left null chain of order  $\sigma$  at  $\lambda_0$  is an ordered set of row vectors in  $\mathbb{R}^m$ ,  $\{y_1, y_2, \dots, y_\sigma\}$ , such that  $y_1 \neq 0$  and*

$$T_{\lambda_0, \sigma}(\hat{H}^T) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_\sigma \end{pmatrix} = \mathbf{0}$$

The next Theorem shows that, if  $\hat{H}$  is square, the existence of a right (left) null chain of order  $\sigma$  at  $\lambda_0$  is equivalent to the existence of a zero at  $\lambda_0$  of algebraic multiplicity  $\sigma$ . It is an extension of Theorem 1.12 in [21]. Later, we will establish a complete equivalence between the structure of a zero and the null chains associated with that zero.

**Theorem 3.2** *A full rank,  $n \times n$ , rational matrix  $\hat{H}(\lambda)$ , analytic at  $\lambda_0$ , has a zero at  $\lambda_0$  of geometric multiplicity  $l$  and a sequence of structural indices equal to, at least,  $\sigma_{n-l+1}, \dots, \sigma_n$  ( $\sigma_1 = \dots = \sigma_{n-l} = 0$ ) if and only if the following conditions hold*

1. *There exist  $l$  polynomial vectors,  $\hat{u}_1, \dots, \hat{u}_l$ , such that*

$$(\hat{H}\hat{u}_j)^{(k)}(\lambda_0) = 0 \quad \text{for } k = 0, \dots, \sigma_{n-l+j} - 1 \quad \forall j = 1, \dots, l$$

2. *The set of vectors  $\{\hat{u}_1(\lambda_0), \dots, \hat{u}_l(\lambda_0)\}$  is linearly independent and*

$$\text{span}\{\hat{u}_1(\lambda_0), \dots, \hat{u}_l(\lambda_0)\} = \mathcal{N}[\hat{H}(\lambda_0)]$$

**Proof** Necessity follows directly from the Smith-McMillan decomposition of  $\hat{H}(\lambda)$ :

$$\hat{H}(\lambda) = \hat{L}(\lambda)\hat{M}(\lambda)\hat{R}(\lambda)$$

Say that the  $j$ th entry ( $j \geq n-l+1$ ) on the diagonal of  $\hat{M}$  has a factor  $(\lambda - \lambda_0)^{\sigma_j}$ . Then, pick  $\hat{u}_{j-n+l}$  to be the  $j$ th column of  $\hat{R}^{-1}$ . With this choice

$$\hat{H}\hat{u}_{j-n+l} = \hat{H}(\hat{R}^{-1})^j = (\lambda - \lambda_0)^{\sigma_j} \hat{L}\hat{p}_{j-n+l} \quad \forall j = n-l+1, \dots, n$$

where  $\hat{p}_{j-n+l}(\lambda)$  is a rational vector analytic at  $\lambda_0$ . Clearly, this implies that  $(\hat{H}\hat{u}_{j-n+l})^{(k)}(\lambda_0) = 0$  for  $k = 0, \dots, \sigma_j - 1$ , and further the set  $\{\hat{u}_1(\lambda_0), \dots, \hat{u}_l(\lambda_0)\}$  is linearly independent since  $\hat{R}$  is unimodular and spans the null space of  $\hat{H}(\lambda_0)$ .

The proof of sufficiency is not as straightforward. Let  $\hat{z}_j := \hat{H}\hat{u}_j \quad j = 1, \dots, l$  and define the following auxiliary rational vectors:

$$\hat{y}_j(\lambda) := (\hat{L}^{-1}\hat{z}_j)(\lambda), \quad \hat{v}_j(\lambda) := (\hat{R}\hat{u}_j)(\lambda) \quad j = 1, \dots, l$$

Then, we have that  $\hat{y}_j(\lambda) = \hat{M}(\lambda)\hat{v}_j(\lambda)$ . Note that  $\hat{u}_1(\lambda_0) \cdots \hat{u}_l(\lambda_0)$  are linearly independent if and only if  $\hat{v}_1(\lambda_0) \cdots \hat{v}_l(\lambda_0)$  are linearly independent since  $\hat{R}$  is unimodular. Further, since multiplication by a unimodular matrix preserves the zero structure, this direction of the proof can be restated as follows:

$\exists \hat{v}_j(\lambda)$  such that  $\hat{v}_1(\lambda_0) \cdots \hat{v}_l(\lambda_0)$  are linearly independent

$$\text{and } \hat{y}_j^{(k)}(\lambda_0) = 0, k = 0, \dots, \sigma_{n-l+j}-1$$

$\Downarrow$

$\exists (\lambda - \lambda_0)^{\sigma_{n-l+j}}$  in the  $n-l+j$  diagonal entry of  $\hat{M}(\lambda)$

Now, it follows from above that

$$\hat{y}_j(\lambda) = (\lambda - \lambda_0)^{\sigma_{n-l+j}} \hat{p}_j(\lambda)$$

Let  $\hat{e}_j(\lambda)$ ,  $j = 1, \dots, n$  be the diagonal entries of the matrix  $\hat{M}$ . It immediately follows that

$$\begin{pmatrix} \hat{e}_1 & & \\ & \ddots & \\ & & \hat{e}_n \end{pmatrix} (\hat{v}_1(\lambda) \cdots \hat{v}_l(\lambda)) = (\hat{p}_1(\lambda) \cdots \hat{p}_l(\lambda)) \begin{pmatrix} (\lambda - \lambda_0)^{\sigma_{n-l+1}} & & \\ & \ddots & \\ & & (\lambda - \lambda_0)^{\sigma_n} \end{pmatrix} \quad (15)$$

First, we show that the matrix  $(\hat{v}_1(\lambda_0) \cdots \hat{v}_l(\lambda_0))$  has the structure

$$\begin{pmatrix} 0 \\ \hat{V}(\lambda_0) \end{pmatrix} \quad (16)$$

The top zero block results from the fact that the matrix  $\hat{M}(\lambda_0)$  has a null space of dimension  $l$  (otherwise there will be more linearly independent vectors than  $l$ ), hence  $\hat{e}_1, \dots, \hat{e}_{n-l}$  do not have zeros at  $\lambda_0$ . From Equation 15, it follows that for all  $\lambda$

$$\begin{pmatrix} \epsilon_{n-l+1} & & \\ & \ddots & \\ & & \epsilon_n \end{pmatrix} \hat{V} = \hat{P} \begin{pmatrix} (\lambda - \lambda_0)^{\sigma_{n-l+1}} & & \\ & \ddots & \\ & & (\lambda - \lambda_0)^{\sigma_n} \end{pmatrix}$$

where the matrices  $\hat{V}$  and  $\hat{P}$  are obtained from the decompositions

$$(\hat{v}_1(\lambda) \ \dots \ \hat{v}_l(\lambda)) = \begin{pmatrix} \hat{V} \\ \hat{V} \end{pmatrix}$$

and

$$(\hat{p}_1(\lambda) \ \dots \ \hat{p}_l(\lambda)) = \begin{pmatrix} \hat{P} \\ \hat{P} \end{pmatrix}$$

Let

$$\hat{E} = \begin{pmatrix} \epsilon_{n-l+1} & & \\ & \ddots & \\ & & \epsilon_n \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} (\lambda - \lambda_0)^{\sigma_{n-l+1}} & & \\ & \ddots & \\ & & (\lambda - \lambda_0)^{\sigma_n} \end{pmatrix}$$

Then, from Equation 16, it is clear that  $\hat{V}(\lambda_0)$  has full rank. Let  $\hat{R}_1, \hat{R}_2$  be unimodular matrices such that

$$\hat{V} \hat{R}_1 = \hat{L} \text{ where } \hat{L} \text{ is lower triangular}$$

and

$$\hat{R}_2 \hat{P} = \hat{U} \text{ where } \hat{U} \text{ is upper triangular}$$

From this, Equation (17) can be factored as follows

$$\hat{E} \hat{L} = \hat{R}_2^{-1} \hat{U} \hat{D} \hat{R}_1$$

Clearly, the matrix  $\hat{E} \hat{L}$  has the same zero structure as the matrix  $\hat{U} \hat{D}$ . By direct computation of the Smith matrix of  $\hat{U} \hat{D}$ , it follows that  $(\lambda - \lambda_0)^{\sigma_{n-l+j}}$  is a factor of  $j$ th diagonal element. Since  $\hat{L}$  has full rank at  $\lambda_0$ , it follows that  $(\lambda - \lambda_0)^{\sigma_{n-l+j}}$  is a factor of  $\epsilon_{n-l+j}$ . This completes the proof.  $\blacksquare$

Note that a similar result holds for left zeros simply by replacing  $\hat{H}$  with  $\hat{H}^T$ . The following corollary restates the result of Theorem 3.2 in terms of null chains.

**Corollary 3.1** *A full rank, square, rational matrix  $\hat{H}(\lambda)$  analytic at  $\lambda_0$ , has a right (left) zero at  $\lambda_0$  of (at least) algebraic multiplicity  $\sigma$  if and only if there exists a right (left) null chain of order  $\sigma$  at  $\lambda_0$ .*

**Proof** Both directions of the proof follow immediately by equating

$$\hat{u}(\lambda) = x_1 + (\lambda - \lambda_0)x_2 + \cdots + (\lambda - \lambda_0)^{\sigma-1}x_\sigma$$

■

Note that if  $\hat{H}$  has a right zero of geometric multiplicity greater than one, say  $l$ , then there are  $l$  different right null chains (not necessarily of the same order), such that the span of the  $x_1$ 's equals the nullspace of  $\hat{H}(\lambda_0)$ . Let  $x^i$  ( $y^i$ ) denote the  $i$ th right (left) null chain of order  $\sigma_i$ , then the following definition applies [1]:

**Definition 3.3** *A canonical set of right null chains of  $\hat{H}(\lambda)$  at  $\lambda_0$  is an ordered set of right null chains, i.e.,  $x^i = (x_1^i \dots x_{\sigma_i}^i)$  for  $i = 1, \dots, l$ , such that*

- i)  $\{x_1^1, x_1^2, \dots, x_1^l\}$  are linearly independent,
- ii)  $\text{span}\{x_1^1, x_1^2, \dots, x_1^l\} = \mathcal{N}[\hat{H}(\lambda_0)]$ , and
- iii)  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_l$ .

*A canonical set of left null chains is defined similarly.*

Next, we show that the zero interpolation conditions of Theorem 3.1 can be stated in terms of the canonical set of right null chains of  $\hat{V}$  and the canonical set of left null chains of  $\hat{U}$  at each  $\lambda_0 \in \Lambda_{UV}$ . For that we need to introduce an extension of the above definition.

**Definition 3.4** *An extended set of right null chains of a full rank  $n \times n$  rational matrix  $\hat{H}(\lambda)$  at  $\lambda_0$ , is a canonical set of right null chains augmented with  $n - l$  vectors in  $\mathbb{R}^n$ , i.e.,  $\{x_1^{l+1}, \dots, x_1^n\}$ , such that the span of  $\{x_1^1, x_1^2, \dots, x_1^n\}$  is equal to  $\mathbb{R}^n$ . The order associated with these added chains is zero.*

From the above definition, if a square rational matrix has no zeros at  $\lambda_0$ , then the corresponding canonical set of null chains is empty and the extended set is a basis for  $\mathbb{R}^n$ , e.g., the columns of an  $n \times n$  identity matrix.

Next, we apply the above results and definitions to the zero interpolation conditions of a one-block problem. In the context of Theorem 3.1 we have the following equivalence: for  $j = 1, \dots, n_y$  and  $k = 0, \dots, \sigma_{V_j} - 1$ ,

$$(\hat{V}\hat{\beta}_j)^{(k)}(\lambda_0) = 0 \iff T_{\lambda_0, \sigma_{V_j}}(\hat{V})x^{n_y-j+1} = 0$$

where  $x^i$  is an extended set of right null chains for  $\hat{V}$  at  $\lambda_0$ . The sequence of  $x^i$ 's has to be reversed in the above equation due to the fact that  $\sigma_{V_j}$  is a non-decreasing sequence of algebraic multiplicities while an extended set of null chains is defined with the opposite ordering. Note that if  $\sigma_{V_j} = 0$  then both conditions are satisfied trivially (i.e., there are no conditions). Similarly, for  $i = 1, \dots, n_u$  and  $k = 0, \dots, \sigma_{U_i} - 1$ ,

$$(\hat{\alpha}_i\hat{U})^{(k)}(\lambda_0) = 0 \iff T_{\lambda_0, \sigma_{U_i}}(\hat{U}^T)y^{n_u-i+1} = 0$$

In other words, the extended set of left and right null chains are locally (i.e., for each  $\lambda_0$ ) equivalent to the polynomial vectors  $\hat{\alpha}_i$ 's and  $\hat{\beta}_j$ 's. Having made this observation, we are ready to present an alternative set of zero interpolation conditions.

Given an element of an extended set of right null chains at  $\lambda_0$ ,  $x^j$ , of order  $\sigma_j$ , define the following polynomial vector:

$$\hat{x}_{\lambda_0}^j(\lambda) := x_1^j + (\lambda - \lambda_0)x_2^j + \dots + (\lambda - \lambda_0)^{\sigma_j-1}x_{\sigma_j}^j$$

if  $\sigma_j > 0$ , and  $\hat{x}_{\lambda_0}^j(\lambda) := x_1^j$  if  $\sigma_j = 0$ . Similarly, define  $\hat{y}_{\lambda_0}^i(\lambda)$  for an element of an extended set of left null chains,  $y^i$ , of order  $\sigma_i$ . With this notation we have the following corollary.

**Corollary 3.2** *Given a one-block problem, the zero interpolation conditions of Theorem 3.1 are equivalent to the following: for all  $\lambda_0 \in \Lambda_{UV}$ ,*

$$(\hat{y}_{\lambda_0}^i \hat{R} \hat{x}_{\lambda_0}^j)^{(k)}(\lambda_0) = 0 \text{ for } \begin{cases} i = 1, \dots, n_u \\ j = 1, \dots, n_y \\ k = 0, \dots, \sigma_{U_i}(\lambda_0) + \sigma_{V_j}(\lambda_0) - 1 \end{cases}$$

where  $y^i$  and  $x^j$  are elements of the extended sets of left and right null chains of  $\hat{U}$  and  $\hat{V}$  respectively, and  $\sigma_{U_i}$  and  $\sigma_{V_j}$  are the corresponding orders (i.e., algebraic multiplicities).

**Proof** Follows directly from Theorems 3.1 and 3.2, and from the above definitions. ■

### 3.2 Computation of Null Chains

This subsection discusses a simple algorithm to compute the extended set of null chains at  $\lambda_0$  of a full rank square rational matrix analytic at  $\lambda_0$ . Let  $\hat{H}(\lambda)$  denote an  $n \times n$  rational matrix and assume that  $\lambda_0$  is given, then the algorithm is based on the computation of a basis for the nullspace of  $T_{\lambda_0, \sigma}(\hat{H})$  for increasing values of  $\sigma$ .

Consider the construction of an extended set of right null chains. By Definition 3.2, given some positive integer  $\sigma$ , any vector in the kernel of  $T_{\lambda_0, \sigma}(\hat{H})$  such that  $x_1 \neq 0$  is a potential member of the set. Let  $B_\sigma$  denote a matrix whose columns form a basis for the right nullspace of  $T_{\lambda_0, \sigma}(\hat{H})$ , then the following algorithm generates an extended set of right null chains:

**Step 1:** Compute  $B_\sigma$  for  $\sigma = 1, 2, \dots$  until the top  $n$  rows are filled with zeros (no more null chains can be extracted at this point). Then the maximum order of any chain,  $\sigma_1$ , is given by the current value of the counter ( $\sigma$ ) minus one. Note that, by Corollary 3.1, this iteration process is guaranteed to stop since the rational matrix  $\hat{H}$  is finite dimensional (i.e., its zeros have finite algebraic multiplicity).

**Step 2:** Let  $b_i$  for  $i = 1, \dots, r$  denote each column of  $B_{\sigma_1}$ . Reduce the dimension of the  $b_i$ 's by removing all sets of  $n$  contiguous zeros at the top of each vector. The result is a collection of  $r$  vectors (possibly of different dimensions) such that the top  $n$  entries of each one define a non-zero vector in  $\mathbb{R}^n$ . (Note that at least one will have dimension  $n\sigma_1$ .)

**Step 3:** Sort the resulting vectors in decreasing order of dimension. Let  $l$  be the rank of the  $n \times r$  matrix that results from collecting the first  $n$  rows of each vector. Then, select the first  $l$  vectors such that the reduced matrix that results from collecting the first  $n$  rows of each vector has rank  $l$ . Such collection forms a canonical set of right null chains.

**Step 4:** Extend the set by augmenting the collection with  $n - l$  vectors such that the set of  $n$  vectors formed with the first  $n$  rows define a basis in  $\mathbb{R}^n$ .

If the system  $\hat{H}(\lambda)$  is given in state-space form, say  $[A, B, C, D]$ , then the Toeplitz matrices  $T_{\lambda_0, \sigma}(\hat{H})$  can be easily computed using the following equation (see Definition 3.1):

$$H_k = \begin{cases} \lambda_0 C (I - \lambda_0 A)^{-1} B + D & \text{for } k = 0 \\ C (I - \lambda_0 A)^{-k-1} A^{k-1} B & \text{for } k = 1, 2, \dots \end{cases}$$

Note that  $(I - \lambda_0 A)^{-1}$  always exists since  $\lambda_0$  is in the unit disk and  $\hat{H}$  is stable (i.e., analytic in the closed unit disk). A word of warning is necessary, however, when  $\lambda_0$  is close to the unit

circle and  $A$  has a stable eigenvalue that is also close to the unit circle and next to  $\lambda_0$ . Such cases may give rise to numerical difficulties. Besides this fact, the rest of the algorithm only involves the computation of nullspaces that can be done efficiently through the well known  $QR$  or singular value decompositions [22].

### 3.3 A Simple Example

In order to illustrate the workings of the algorithm introduced in the previous section, a simple example is presented. Let  $\hat{H}(\lambda)$  be a  $3 \times 3$  polynomial matrix given by:

$$\hat{H}(\lambda) = \begin{pmatrix} (\lambda - 0.5)^2 & \lambda(\lambda + 2)(\lambda - 0.5) & 0 \\ (\lambda - 0.5)^3 & \lambda(\lambda - 0.5) & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}$$

We have chosen a polynomial matrix just to make the example tractable without the aid of a computer. Let us construct an extended set of right null chains for the zero at  $\lambda_0 = 0.5$ . According to step one, we compute the nullspace of  $T_{\lambda_0, \sigma}(\hat{H})$  for  $\sigma = 1, 2, \dots$ . In particular, for  $\sigma = 3$  we have:

$$T_{0.5, 3}(\hat{H}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & .25 & 0 & 0 & 0 \\ 1 & 1.5 & 0 & 0 & .5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & .25 \end{pmatrix} ; B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly, the first three rows of  $B_3$  are zero so we stop increasing  $\sigma$ . Then, the maximum algebraic multiplicity of  $\lambda_0 = 0.5$  is two, i.e.,  $\sigma_1 = 2$ . Next (step 2), reduce each column of  $B_3$  by eliminating the leading blocks of zeros to get:

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; b_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then (step 3), reorder the set of vectors in decreasing dimension, i.e.,  $\{b_3, b_1, b_2\}$ , and compute the rank of the matrix formed with the first three rows:

$$l = \text{rank} \left[ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right] = 2$$

Then, the canonical set of right null chains is given by  $\{x^1, x^2\}$  where

$$x^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad x^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

with their corresponding orders (i.e., algebraic multiplicity) being  $\sigma_1 = 2$  and  $\sigma_2 = 1$ . This indicates that the geometric multiplicity of  $\lambda_0$  is two. Finally (step 4), to get an extended set of right null chains we augment the collection with  $x^3 = (0 \ 0 \ 1)^T$  having order  $\sigma_3 = 0$  (by definition).

## 4 Duality and Existence

With Theorem 3.1 we have established a compact algebraic characterization of the set  $\mathcal{S}$ . Next, we need to interpret these results in the context of Equation (7), which calls for the identification of the subspace of  $\ell_\infty^{n_z \times n_w}$  which annihilates  $\mathcal{S}$ .

Following the approach in [11] and [29], we write the zero interpolation conditions as functionals acting on  $R$ . Indeed, for all  $(i, j, k)$  in the ranges established in Theorem 3.1, for  $l = 0, 1, \dots$ , and all  $\lambda_0 \in \Lambda_{UV}$ , define  $RF_{ijk\lambda_0}$  and  $IF_{ijk\lambda_0}$  in  $\ell_\infty^{n_z \times n_w}$  such that

$$[RF_{ijk\lambda_0}(l)]_{qp} := \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \alpha_{iq}(s-l) \beta_{pj}(t-s) \Re[(\lambda^t)^{(k)}] \Big|_{\lambda=\lambda_0} \quad (18)$$

and

$$[IF_{ijk\lambda_0}(l)]_{qp} := \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \alpha_{iq}(s-l) \beta_{pj}(t-s) \Im[(\lambda^t)^{(k)}] \Big|_{\lambda=\lambda_0} \quad (19)$$

where  $\Re(\lambda)$  and  $\Im(\lambda)$  denote the real and imaginary part of  $\lambda$  respectively, and  $\alpha_{iq}$  denotes the  $q$ th column of  $\alpha_i$  while  $\beta_{pj}$  denotes the  $p$ th row of  $\beta_j$ . By straightforward algebra it can be shown that  $\langle R, RF_{ijk\lambda_0} \rangle = 0$  and  $\langle R, IF_{ijk\lambda_0} \rangle = 0$  if and only if  $\hat{R}$  satisfies the zero interpolation conditions of Theorem 3.1. Note that only a finite number of sequences are required, thus the subspace spanned by the sequences associated with the zero interpolations is finite dimensional. In fact, the number of functionals is given by:

$$c_z := \sum_{\lambda_0 \in \Lambda_{UV}} \sum_{i=1}^{n_y} \sum_{j=1}^{n_u} \sigma_{U_i}(\lambda_0) + \sigma_{V_j}(\lambda_0) \quad (20)$$

A note should be made on the way  $c_z$  is computed. If a given  $\lambda_0 \in \Lambda_{UV}$  is complex then  $\bar{\lambda}_0 \in \Lambda_{UV}$  too, since  $\hat{U}$  and  $\hat{V}$  are real-rational. However, for the purpose of constructing functionals, only one of each pair of complex-conjugate zeros should be considered since the other one would generate redundant functionals. But, for the purpose of counting the number of independent functionals (i.e., computing  $c_z$ ), both zeros should be included in  $\Lambda_{UV}$ , since a complex-conjugate pair of zeros generate twice as many functionals as a real zero.

Next, we look at the rank interpolation conditions (i.e., conditions in *ii*). Again, these algebraic conditions can be viewed as convolution of sequences. For  $i = n_u + 1, \dots, n_z$  and



$q = 1, \dots, n_w$ , define the following sequence of  $n_z \times n_w$  matrices:

$$X_{\alpha_i qt}(l) := \begin{pmatrix} \vdots & & \vdots & \overbrace{\alpha_i^T(t-l)}^{\text{qth column}} & \vdots & \vdots \\ 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \end{pmatrix} \quad (21)$$

where  $t, l \in \mathbf{Z}_+$ . Similarly, for  $j = n_y + 1, \dots, n_w$  and  $p = 1, \dots, n_z$ , define

$$X_{\beta_j pt}(l) := \begin{pmatrix} \cdots & 0 & \cdots \\ & \vdots & \\ \cdots & 0 & \cdots \\ \cdots & \beta_j^T(t-l) & \cdots \\ \cdots & 0 & \cdots \\ & \vdots & \\ \cdots & 0 & \cdots \end{pmatrix} \text{ } p\text{th row} \quad (22)$$

Then,  $\langle R, X_{\alpha_i qt} \rangle = 0$  and  $\langle R, X_{\beta_j pt} \rangle = 0$  for  $t = 0, 1, \dots$  if and only if  $\hat{R}$  satisfies the rank interpolation conditions of Theorem 3.1. Note that, in contrast with the zero interpolation sequences, the linear span of the  $X_{\alpha_i qt}$ 's and  $X_{\beta_j pt}$ 's is infinite dimensional since for every  $(i, q, p)$ ,  $t$  can take infinite values (i.e.,  $t \in \mathbf{Z}_+$ ).

The next theorem gives a sufficient condition for the existence of an optimal solution to (5). The proof is omitted since the arguments involved are essentially the same as those in [11, 29].

**Theorem 4.1** *If every  $\lambda_0 \in \Lambda_{UV}$  is strictly inside the unit disk, then there exists  $R^\circ \in \mathcal{S}$  such that*

$$\mu^\circ = \|H - R^\circ\|_1 = \inf_{R \in \mathcal{S}} \|H - R\|_1$$

Note, however, that the above result is more general than that in [29], where it is assumed that  $\hat{U}$  and  $\hat{V}$  have square partitions with no zeros on the unit circle. Such extra assumption was avoided by determining the full set of interpolation conditions directly from the Smith-McMillan decomposition of  $\hat{U}$  and  $\hat{V}$ .

## 5 $\ell_1$ Optimization and Linear Programming

This section will establish the equivalence between the primal-dual pair of optimization problems (5)-(7) and a primal-dual pair of infinite dimensional linear programs.

By definition,  $\mathcal{S}^\perp \subset \ell_\infty^{n_z \times n_w}$  is the linear span of the sequences (18), (19), (21) and (22), and  $G$  is any element in that subspace with infinity norm not greater than one. That is,

$$G \in \text{span}\{RF_{ijk\lambda_0}, IF_{ijk\lambda_0}, X_{\alpha_i qt}, X_{\beta_j pt}\} \quad (23)$$

with the appropriate index ranges.

In order to bring (5) and (7) into a standard linear programming form, it is convenient to redefine the notation, the purpose being to express both the objective and the feasible subspace in (infinite) matrix form. This is possible since the constraints that specify the feasible

subspace  $\mathcal{S}$  are no more and no less than an infinite collection of linear functionals annihilating the sequence  $R$ , which can be expressed as an infinite collection of equality constraints on the elements of the sequence  $\Phi$ .

To bring the primal objective function  $\|\Phi\|_1$  into linear form and avoid the non-linearity built into the one norm, we use a standard change of variables from linear programming: let  $\Phi = \Phi^+ - \Phi^-$ , where  $\Phi^+$  and  $\Phi^-$  are sequences of  $n_z \times n_w$  matrices with non-negative entries. That is, with a slight abuse of notation,  $\Phi^+ \geq 0$  and  $\Phi^- \geq 0$ . Then, the  $\ell_1$  norm of  $\Phi$  takes the form  $\max_i \sum_{j=1}^{n_w} \sum_{t=0}^{\infty} (\phi_{ij}^+(t) + \phi_{ij}^-(t))$  which is linear in  $(\Phi^+, \Phi^-)$ . This expression holds only if, for any  $(i, j, t)$ , either  $\phi_{ij}^+(t)$  or  $\phi_{ij}^-(t)$  is zero, which is a guaranteed property of the optimal solution. Indeed, if a feasible solution is such that  $\phi_{ij}^+(t)$  and  $\phi_{ij}^-(t)$  are strictly positive, then reducing both variables by  $\min(\phi_{ij}^+(t), \phi_{ij}^-(t))$  reduces the value of the cost and does not violate feasibility since the difference remains the same, and further, one of the two variables becomes zero. Therefore, the optimal solution will always be such that either  $\phi_{ij}^+(t)$  or  $\phi_{ij}^-(t)$  is zero. Note that this transformation doubles the number of variables representing the closed-loop response.

Consequently, the primal problem (5) can be restated as follows:

$$\begin{aligned} \mu^\circ &= \inf_{\mu, \Phi^+, \Phi^-} \mu \\ \text{subject to} & \\ & \sum_{j=1}^{n_w} \sum_{t=0}^{\infty} (\phi_{ij}^+(t) + \phi_{ij}^-(t)) \leq \mu \text{ for } i = 1, \dots, n_z \\ & \Phi - H \in \mathcal{S} \end{aligned} \quad (24)$$

Next, we shift attention to the linear constraints representing the feasible set. From the previous discussion it is clear that a given  $\Phi$  is feasible (i.e., there exists a stable  $Q$  such that  $\Phi = H - UQV$ ) if and only if

$$\begin{cases} \langle \Phi, RF_{ijk\lambda_0} \rangle = \langle H, RF_{ijk\lambda_0} \rangle \\ \langle \Phi, IF_{ijk\lambda_0} \rangle = \langle H, IF_{ijk\lambda_0} \rangle \end{cases} \text{ for } \begin{cases} \lambda_0 \in \Lambda_{UV} \\ i = 1, \dots, n_u \\ j = 1, \dots, n_y \\ k = 0, \dots, \sigma_{U_i}(\lambda_0) + \sigma_{V_j}(\lambda_0) - 1 \end{cases} \quad (25)$$

and

$$\begin{cases} \langle \Phi, X_{\alpha,qt} \rangle = \langle H, X_{\alpha,qt} \rangle \\ \langle \Phi, X_{\beta,pt} \rangle = \langle H, X_{\beta,pt} \rangle \end{cases} \text{ for } \begin{cases} i = n_u + 1, \dots, n_z \\ j = n_y + 1, \dots, n_w \\ q = 1, \dots, n_w \\ p = 1, \dots, n_z \\ t = 0, 1, 2, \dots \end{cases} \quad (26)$$

Each of these equations can be viewed as a linear equality constraint on the sequence  $\Phi$ .

At this point it is convenient to drop the tensor notation used so far and introduce a more compact, computer-ready matrix notation. Let  $M_{ij}$  denote an infinite matrix mapping  $\ell_1$  to  $\mathbb{R}^{c_z}$ , formed by collecting those coefficients of the zero interpolation functionals that act on the sequence  $\phi_{ij}$ . Similarly, define  $\bar{M}_{ij}$  to be an infinite matrix mapping  $\ell_1$  to  $\ell_1$ , formed by collecting those coefficients of the rank interpolation functionals that act on  $\phi_{ij}$ . With this notation, the set of feasible closed loop maps is characterized by the following set of equality constraints:

$$\sum_{i=1}^{n_z} \sum_{j=1}^{n_w} M_{ij} \phi_{ij} = \sum_{i=1}^{n_z} \sum_{j=1}^{n_w} M_{ij} h_{ij} =: b_1 \in \mathbb{R}^{c_z} \quad (27)$$

$$\sum_{i=1}^{n_z} \sum_{j=1}^{n_w} \bar{M}_{ij} \phi_{ij} = \sum_{i=1}^{n_z} \sum_{j=1}^{n_w} \bar{M}_{ij} h_{ij} =: b_2 \in \ell_1 \quad (28)$$

Therefore, the primal optimization problem (5) is equivalent to the following infinite dimensional linear program:

$$\begin{aligned} \mu^\circ &:= \min_{\mu, \xi, \phi_{ij}^+, \phi_{ij}^-} \mu \\ \text{subject to} & \\ \xi(i) + \sum_{j=1}^{n_w} \sum_{t=0}^{\infty} \phi_{ij}^+(t) + \phi_{ij}^-(t) &= \mu \quad \text{for } i = 1, \dots, n_z \\ \sum_{i=1}^{n_z} \sum_{j=1}^{n_w} M_{ij} (\phi_{ij}^+ - \phi_{ij}^-) &= b_1 \\ \sum_{i=1}^{n_z} \sum_{j=1}^{n_w} \bar{M}_{ij} (\phi_{ij}^+ - \phi_{ij}^-) &= b_2 \\ \xi, \phi_{ij}^+, \phi_{ij}^- &\geq 0 \end{aligned} \quad (29)$$

where  $\xi \in \mathbb{R}^{n_z}$  is a positive vector of slack variables. Note that the above linear program is infinite dimensional in the number of variables (i.e., dimension of any  $\phi_{ij}$ ) and the number of constraints (i.e., dimension of  $b_2$ ).

In order to complete this discussion, it remains to show that problem (7) is also equivalent to a linear programming problem. In fact, it can be shown that such problem corresponds to the standard dual formulation of problem (29). To illustrate this fact, we will simply write the dual form of (29) and compare it to (7). Let  $\gamma \in \ell_\infty$  denote the sequence of dual variables. To get more insight into the dual problem, let us partition  $\gamma$  according to the natural partitioning of the set of equality constraints. That is, let  $\gamma =: (-\gamma_0 \ \gamma_1 \ \gamma_2)^T$ , where  $\gamma_0 \in \mathbb{R}^{n_z}$ ,  $\gamma_1 \in \mathbb{R}^{c_z}$  and  $\gamma_2 \in \ell_\infty$  (it is convenient to have the sign of  $\gamma_0$  changed). Then, the standard dual linear program of (29) is given by:

$$\begin{aligned} \mu^\circ &= \max_{\gamma_0, \gamma_1, \gamma_2} \langle b_1, \gamma_1 \rangle + \langle b_2, \gamma_2 \rangle \\ \text{subject to} & \\ \gamma_0 &\geq 0, \quad \sum_{i=1}^{n_z} \gamma_0(i) \leq 1 \\ -\gamma_0(i) &\leq (M_{ij}^T \gamma_1 + \bar{M}_{ij}^T \gamma_2)(k) \leq \gamma_0(i) \quad \text{for } \begin{cases} i = 1, \dots, n_z \\ j = 1, \dots, n_w \\ k = 0, 1, \dots \end{cases} \end{aligned} \quad (30)$$

If one compares the above linear program with problem (7), the following relationships become apparent: 1)  $\gamma_1$  and  $\gamma_2$  are nothing but the coefficients that combine the linear functionals associated with the zero interpolation conditions and the rank interpolation conditions respectively to obtain  $G$ ; 2) the objective function results from expanding  $\langle H, G \rangle$  when  $G$  is expressed as a linear combination of the elements in the generator of  $\mathcal{S}^\perp$  with coefficients  $(\gamma_1, \gamma_2)$ ; and 3) the set of inequality constraints is equivalent to  $\|G\|_\infty \leq 1$ , while the second line of inequalities bounds  $G$  componentwise, the first line bounds the matrix  $\infty$ -norm of  $G$  by one.

## 6 One-block Problems

One-block problems have a very specific interpolation structure, namely no rank interpolation conditions. From a primal formulation point of view (see Equation (29)), this simplifies the

problem significantly by bringing the number of equality constraints down to a finite value, namely  $c_z + n_z$ . There remains, however, an infinite number of variables represented by the  $\phi_{ij}$ 's in  $\ell_1$ . Nevertheless, it has been shown by looking at the structure of the dual problem, that the underlying problem is finite dimensional [9]. Indeed, the dual formulation has an infinite number of inequality constraints but retains a finite number of variables:

$$\begin{aligned} \mu^\circ &= \max_{\gamma_0, \gamma_1} \langle b_1, \gamma_1 \rangle \\ \text{subject to} \\ \gamma_0 &\geq 0; \quad \sum_{i=1}^{n_z} \gamma_0(i) \leq 1 \\ -\gamma_0(i) &\leq (M_{ij}^T \gamma_1)(k) \leq \gamma_0(i) \quad \text{for} \quad \begin{cases} i = 1, \dots, n_z \\ j = 1, \dots, n_w \\ k = 0, 1, \dots \end{cases} \end{aligned} \quad (31)$$

Recall that  $M_{ij}^T$  is the matrix representation of an operator mapping  $\mathbb{R}^{c_z}$  to  $\ell_\infty$ . However, with Assumption 1 holding, the actual range of  $M_{ij}^T$  is in  $c_0$  since each of the columns of  $M_{ij}^T$  is in  $c_0$  and there are only finitely many of them. This is exploited in the following lemma from [34]:

**Lemma 6.1** *Let  $M$  be a full column rank infinite matrix mapping  $\mathbb{R}^n$  to  $c_0$ . Then there exists a positive integer  $N$  such that*

$$\|(I - P_N)Mx\|_\infty < \|P_N Mx\|_\infty$$

for all non-zero  $x \in \mathbb{R}^n$ .

Note, in particular, that  $N$  is independent of  $x$  and is only a function of  $M$ .

In other words, given a matrix mapping a finite dimensional space to  $c_0$ , it is always possible to bound the index at which the infinity norm of any sequence in the range is achieved.

The following theorem extends a result from [9] by exploiting this structure.

**Theorem 6.1** *The exact solution of a one-block  $\ell_1$ -optimal control problem is given by the following finite dimensional (dual) linear program,*

$$\begin{aligned} \mu^\circ &= \max_{\gamma_0, \gamma_1} \langle b_1, \gamma_1 \rangle \\ \text{subject to} \\ \gamma_0 &\geq 0, \quad \sum_{i=1}^{n_z} \gamma_0(i) \leq 1 \\ -\gamma_0(i) &\leq (M_{ij}^T \gamma_1)(k) \leq \gamma_0(i) \quad \text{for} \quad \begin{cases} i = 1, \dots, n_z \\ j = 1, \dots, n_w \\ k = 0, \dots, N_{ij} < \infty \end{cases} \end{aligned} \quad (32)$$

**Proof** Form matrices  $M_{ij}^T$  as defined before. Assume they have full column rank (if not reduce the number of columns). Apply Lemma 6.1 to each  $M_{ij}^T$  and let  $N_{ij}$  denote the corresponding index bound. Then, we claim that for every feasible solution of problem (31) all inequalities of the form  $|(M_{ij}^T \gamma_1)(k)| \leq \gamma_0(i)$  for  $k > N_{ij}$  are inactive constraints (i.e., the inequality is strict) and they can be ignored in the solution. Indeed, by Lemma 6.1, if there is an active constraint for  $k > N_{ij}$ , then there must have been a violation of a constraint for some  $k < N_{ij}$  since the  $\ell_\infty$  norm of the sequence  $M_{ij}^T \gamma_1$  is attained before  $N_{ij}$  and is always

bounded by  $\gamma_0(i)$ . ■

This fact has an immediate and important implication on the primal linear programming formulation of one-block problems. Due to the alignment conditions, if a dual optimal solution is such that all inequality constraints are inactive for  $k > N$ , then the primal optimal solution is such that it vanishes for  $k > N$ .

**Corollary 6.1** *For any one-block problem, the  $\ell_1$ -optimal closed-loop response,  $\Phi^\circ$ , has finite support (i.e., finite pulse response). Furthermore, each entry  $\phi_{ij}$  has support no greater than  $N_{ij}$ .*

Note that the  $N_{ij}$ 's provide a priori bounds on the lengths of the optimal  $\phi_{ij}$ 's. Moreover, these bounds are independent of  $H$  and only depend on the zero interpolation structure of the problem.

We conclude this section with an interesting property of most one-block problems, regarding the  $\ell_1$ -norm of each row of the optimal solution.

**Corollary 6.2** *Given a one-block problem, if for some  $i \in \{1, \dots, n_z\}$  and  $j \in \{1, \dots, n_w\}$  the matrix  $M_{ij}^T$  has full column rank, then  $\|(\Phi^\circ)_i\|_1 = \mu^\circ$ .*

**Proof** Assume  $\|(\Phi^\circ)_i\|_1 < \mu^\circ$ , then  $\xi(i) > 0$ . By the alignment conditions, this implies that  $\gamma_0(i) = 0$ , and in view of Equation (32) and the rank condition on  $M_{ij}^T$ ,  $\gamma_1$  must be zero. But this implies that  $\mu^\circ = 0$  which is a contradiction. ■

It should be noted that there are some pathological cases where the rank condition on  $M_{ij}^T$  is violated. For instance, if the given one-block problem is in fact a combination of two or more totally decoupled sub-problems, then some  $M_{ij}^T$ 's will have entire columns of zeros. In most cases, however, the solution is such that the norm of each row of  $\Phi^\circ$  is equal to  $\mu^\circ$ . It is interesting to point out the analogy between this aspect of the  $\ell_1$ -optimal solution of one-block problems, and the equivalent in  $\mathcal{H}_\infty$  optimization. In the first one, the same "gain" is achieved at all outputs while in the second one the same "gain" is achieved at all frequencies (i.e., inner solution). These are direct consequences of the corresponding norm definitions. Furthermore, the analogy extends to the multiblock case in the sense that this property does not hold in general.

## 7 Multiblock Problems

The exact solution of the one-block problem rests on the fact that the primal linear programming formulation has only finitely many equality constraints (or, equivalently, the dual formulation has finitely many variables). The multiblock problem, however, is characterized by a primal and dual formulation with an infinite number of variables and constraints. So, in principle, one can attempt to get approximate solutions by an appropriate truncation of the original problem.

There are basically two approximation methods reported in the literature. The first one, known as the *finitely many variables* (FMV) approximation, was originally introduced in [11] and further developed in [29, 34]. It results from constraining the support of the closed-loop response  $\Phi$ , thus providing a suboptimal finitely supported feasible solution to the problem. In the second approach, known as the *finitely many equations* (FME) approximation [6, 33],

only finitely many equality constraints are retained in the primal formulation of the problem, the solution of which is superoptimal but infeasible. Its value is complementary to the first approach in the sense that it generates lower bounds of the optimal norm,  $\mu^\circ$ .

The next two subsections give a more detailed description of these methods along with their main characteristics. They do not contain new results.

## 7.1 The FMV Approximation Method

Let  $N$  be the order of approximation or support of  $\Phi$ , then the FMV primal formulation is given by the following linear program:

$$\begin{aligned}
\bar{\nu}_N &:= \min_{\mu, \xi, \phi_{ij}^+, \phi_{ij}^-} \mu \\
\text{subject to} & \\
\xi(i) + \sum_{j=1}^{n_w} \sum_{k=0}^N \phi_{ij}^+(k) + \phi_{ij}^-(k) &= \mu \text{ for } i = 1, \dots, n_z \\
\sum_{i=1}^{n_z} \sum_{j=1}^{n_w} M_{ij}(\phi_{ij}^+ - \phi_{ij}^-) &= b_1 \\
\sum_{i=1}^{n_z} \sum_{j=1}^{n_w} \bar{M}_{ij}(\phi_{ij}^+ - \phi_{ij}^-) &= b_2 \\
\phi_{ij}^+(k) = \phi_{ij}^-(k) &= 0 \text{ for } k > N \\
\xi, \phi_{ij}^+, \phi_{ij}^- &\geq 0
\end{aligned} \tag{33}$$

Note that without the constraints  $\phi_{ij}^+(k) = \phi_{ij}^-(k) = 0$  for  $k > N$ , Equation (33) is equivalent to the full (un-truncated) optimization problem. Clearly, the added constraints will make  $\bar{\nu}_N \geq \mu^\circ$  in general. It is yet unclear, however, if the resulting problem is finite dimensional or not, since we still carry an infinite number of constraints. A closer look at the matrices  $\bar{M}_{ij}$  will answer this question.

Recall that these matrices represent the rank interpolation conditions (albeit some specific reordering) of the form (see Theorem 3.1):

$$\begin{pmatrix} \alpha_{n_u+1} \\ \vdots \\ \alpha_{n_z} \end{pmatrix} * \Phi = \begin{pmatrix} \alpha_{n_u+1} \\ \vdots \\ \alpha_{n_z} \end{pmatrix} * H$$

and

$$\Phi * \begin{pmatrix} \beta_{n_y+1} & \cdots & \beta_{n_w} \end{pmatrix} = H * \begin{pmatrix} \beta_{n_y+1} & \cdots & \beta_{n_w} \end{pmatrix}$$

where the results from the right-hand-side convolutions are collected in the infinite vector  $b_2$ . The matrix representation of the convolution of the  $\alpha_i$ 's and  $\beta_j$ 's on the different entries of  $\Phi$ , say  $\phi_{ij}$ , is precisely given by  $\bar{M}_{ij}$ . Therefore, such infinite matrices will have a band structure inherited from the fact that the  $\hat{\alpha}_i(\lambda)$ 's and  $\hat{\beta}_j(\lambda)$ 's are polynomials.

In view of this particular structure, forcing  $\phi_{ij}(k) = 0$  for  $k > N$  will make the product  $(\bar{M}_{ij}\phi_{ij})(k)$  eventually vanish for  $k > N + \text{constant}$ , where the constant depends on the order of the polynomials  $\hat{\alpha}(\lambda)$ 's and  $\hat{\beta}(\lambda)$ 's. If, however, the infinite vector  $b_2$  is not zero at that point, then the equality constraints will be violated for any  $\Phi$ , implying that the added constraints have transformed the feasible set into an empty set and that the linear program has no solution. Furthermore, this will always be the case if  $b_2$  has infinite support, no matter how large  $N$

is chosen to be. This leads to the following theorem and corollary (equivalent results can be found in [29]).

**Theorem 7.1** *Given a multiblock problem, there exists a finitely supported feasible solution,  $\Phi$ , if and only if  $\alpha_i * H$  and  $H * \beta_j$  are finitely supported for  $i = n_u + 1, \dots, n_z$  and  $j = n_y + 1, \dots, n_w$ .*

**Corollary 7.1** *Given a positive integer  $N$ , the FMV problem (33) has a non-empty feasible set and therefore a solution, if and only if  $(\alpha_i * H)(k) = 0$  and  $(H * \beta_j)(k) = 0$  for  $k > N + \text{constant}$ ,  $i = n_u + 1, \dots, n_z$  and  $j = n_y + 1, \dots, n_w$ , where the constant depends on the order of  $\hat{\alpha}_i$  and  $\hat{\beta}_j$ .*

It is clear from the above results that there is a class of multiblock problems for which the FMV method fails regardless of the order of approximation  $N$ . Also, given any multiblock problem, there is in general a lower bound for  $N$  under which the FMV method also fails. A way to avoid this difficulty is to approximate  $H$  arbitrarily close with a finitely supported sequence (e.g.,  $P_k H$ ). Such approach, however, has the effect of increasing the order of the suboptimal solution and therefore the order of the controller that achieves it.

Without overlooking these limitations, we are going to assume for the rest of this subsection that the problems at hand allow polynomial feasible solutions and that  $N$  is large enough to capture at least one of such solutions.

Under these assumptions, it is clear that all but finitely many constraints in (33) are satisfied trivially, so that the problem is in effect a finite dimensional linear program. The next theorem shows that it has nice convergence properties [11].

**Theorem 7.2** *In the FMV method,  $\bar{v}_N \rightarrow \mu^\circ$  as  $N \rightarrow \infty$ .*

Besides the necessary assumptions regarding the existence of polynomial feasible solutions, the FMV approximation method suffers from two other significant drawbacks: 1) Although it provides an upper bound for  $\mu^\circ$  and a feasible solution that achieves it, it gives no information about how far away from optimal the solution is, and 2) the compensators obtained with this method suffer from order inflation (i.e., the order of the controller increases with  $N$ ). These aspects of the solutions will be illustrated through an example at the end of this section.

## 7.2 The FME Approximation Method

The first drawback was solved independently in [6] and [33] by introducing a second optimization problem, the FME approximation method. Such method further exploits the structure of the matrices  $\bar{M}_{ij}$  to get lower bounds on  $\mu^\circ$ . The name stems from the fact that only finitely many equality constraints associated with the rank interpolation conditions are included in the optimization problem. The rest are simply ignored. Therefore, the solution obtained will in general fail to satisfy those constraints that were left out, rendering it infeasible to the un-truncated problem. A formal statement of the FME approximation problem (in its primal

form) is as follows:

$$\begin{aligned}
\underline{\nu}_N &:= \min_{\mu, \xi, \phi_{ij}^+, \phi_{ij}^-} \mu \\
\text{subject to} & \\
&\xi(i) + \sum_{j=1}^{n_w} \sum_{k=0}^{\infty} \phi_{ij}^+(k) + \phi_{ij}^-(k) = \mu \text{ for } i = 1, \dots, n_z \\
&\sum_{i=1}^{n_z} \sum_{j=1}^{n_w} M_{ij}(\phi_{ij}^+ - \phi_{ij}^-) = b_1 \\
&\left( \sum_{i=1}^{n_z} \sum_{j=1}^{n_w} \bar{M}_{ij}(\phi_{ij}^+ - \phi_{ij}^-) \right) (k) = b_2(k) \text{ for } k = 0, \dots, N-1 \\
&\xi, \phi_{ij}^+, \phi_{ij}^- \geq 0
\end{aligned} \tag{34}$$

This truncation scheme transforms the original problem into one with a finite number of constraints but still an infinite number of variables. An argument similar to the one used for the one-block problem shows that the above infinite dimensional linear program is indeed equivalent to a finite dimensional one. Let  $\bar{M}_{ij,N}$  denote the truncated  $\bar{M}_{ij}$  (i.e., the first  $N$  rows of it). Since  $\bar{M}_{ij,N}$  has only a finite number of rows, then the combined matrix

$$\begin{pmatrix} M_{ij}^T & \bar{M}_{ij,N}^T \end{pmatrix}$$

maps a finite dimensional space to  $\ell_\infty$ . Moreover, due to the band structure of  $\bar{M}_{ij}$ , all the columns of the combined matrix are in  $c_0$  and thus the range is in  $c_0$ . Therefore, by Lemma 6.1 and Theorem 6.1, the FME problem is equivalent to a finite dimensional linear program whose solution has finite support.

The sequence of linear programs in (34) are such that the number of constraints increases with  $N$ . Therefore,  $\underline{\nu}_N$  forms a non-decreasing sequence bounded from above by  $\mu^\circ$ . The next theorem shows that it actually converges to  $\mu^\circ$  [34].

**Theorem 7.3** *In the FME method,  $\underline{\nu}_N \rightarrow \mu^\circ$  as  $N \rightarrow \infty$ .*

Based on these convergence properties, a multiblock problem can be solved iteratively to any degree of approximation by solving two finite dimensional linear programs, corresponding to the FMV and FME truncation schemes, at each iteration. The stopping criteria is based on the upper and lower bounds provided in each iteration. This holds only if there exists finitely supported feasible solutions to the problem.

### 7.3 Delay Augmentation Method

Following, a new method is presented by the name of delay augmentation (DA). This method provides a conceptually attractive and computationally efficient way of solving general multi-block problems, with the added benefit of not requiring assumptions on the existence of polynomial feasible solutions and with the capacity of generating suboptimal controllers without order inflation.

The main idea is very simple:

1. augment  $U$  and  $V$  with pure delays (i.e., right shifts) such that the augmented problem is one-block,



2. apply all the machinery developed for one-block problems to the augmented system,
3. reduce it back to the original system and compute the controller.

In more precise terms, partition the original system as follows:

$$\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} Q \begin{pmatrix} V_1 & V_2 \end{pmatrix} \quad (35)$$

where  $U_1 \in \ell_1^{n_u \times n_u}$  and  $V_1 \in \ell_1^{n_y \times n_y}$ . Then, augment  $U$  and  $V$  with  $N$ th order shifts and augment the free parameter  $Q$  accordingly:

$$\begin{pmatrix} \Phi_{11,N} & \Phi_{12,N} \\ \Phi_{21,N} & \Phi_{22,N} \end{pmatrix} := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} - \begin{pmatrix} U_1 & 0 \\ U_2 & S_N \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} V_1 & V_2 \\ 0 & S_N \end{pmatrix} \quad (36)$$

or, equivalently,

$$\Phi_N := H - U_N Q_N V_N =: H - R_N \quad (37)$$

where  $U_N$ ,  $Q_N$  and  $V_N$  have the obvious definitions. Clearly, problem (37) is of the one-block class since  $U_N \in \ell_1^{n_z \times n_z}$  and  $V_N \in \ell_1^{n_w \times n_w}$ . By expanding Equation (36) we have

$$\Phi_N = H - U Q_{11} V - S_N \tilde{R}_N \quad (38)$$

and

$$\tilde{R}_N := \begin{pmatrix} 0 & U_1 Q_{12} \\ Q_{21} V_1 & Q_{21} V_2 + U_2 Q_{12} + S_N Q_{22} \end{pmatrix}$$

where the fact that these are all time invariant operators has been used. With this notation we are ready to define the delay augmentation problem of order  $N$  as the following optimization problem:

$$\underline{\mu}_N := \inf_{Q_N \in \ell_1^{n_z \times n_w}} \|H - U_N Q_N V_N\|_1 \quad (39)$$

It follows from the above definition that  $\underline{\mu}_N$  is a lower bound for  $\mu^\circ$  since

$$\underline{\mu}_N \leq \inf_{\substack{Q_{11} \in \ell_1^{n_u \times n_y} \\ Q_{12} = Q_{21} = Q_{22} = 0}} \|H - U_N Q_N V_N\|_1 = \inf_{Q_{11} \in \ell_1^{n_u \times n_y}} \|H - U Q_{11} V\|_1 = \mu^\circ$$

In other words, the extra degree of freedom in the free parameter  $Q_N$  (as compared to  $Q$ ) makes the construction of superoptimal solutions possible. Such solutions, however, are clearly infeasible to the un-augmented problem. Also, it is interesting to note that the extra parameters (namely  $Q_{12}$ ,  $Q_{21}$  and  $Q_{22}$ ) have no effect on the solution  $\Phi_N(k)$  for  $k < N$  due to the presence of the shift operator in Equation (38). And even more interesting, the term  $\Phi_{11}$  is not affected at all by the added parameters (note the block of zeros in  $\tilde{R}_N$ ). This observation will let us construct a suboptimal feasible solution directly from the solution of (39).

Given some positive integer  $N$ , let

$$\underline{\mu}_N = \|\Phi_N^\circ\|_1 = \|H - U Q_{11}^\circ V - S_N \tilde{R}_N^\circ\|_1$$

then, clearly

$$\mu^\circ = \inf_{Q \in \ell_1^{n_u \times n_y}} \|H - U Q V\|_1 \leq \|H - U Q_{11}^\circ V\|_1 =: \bar{\mu}_N \quad (40)$$

Or, equivalently, the solution obtained by making the extra free parameters zero after solving (39) is feasible and suboptimal to the un-augmented problem. The following lemma summarizes these results.

**Lemma 7.1** *Given a positive integer  $N$  and definitions (39) and (40), then*

$$\underline{\mu}_N \leq \mu^\circ \leq \bar{\mu}_N$$

where  $\bar{\mu}_N$  is achieved with  $Q_{11}^\circ$ .

Before addressing the convergence properties of this method, a word on existence is in order. Recall that existence is assured if there are no zero interpolations on the boundary of the unit disk. Now, it may happen that a multiblock problem that satisfies this condition augments into a one-block problem that does not. Indeed, notice that the left zeros of  $\hat{U}_N$  are given by the left zeros of  $\hat{U}_1$  plus a multiple zero at the origin (due to the block of delays,  $\lambda^N I$ , resulting from the  $\lambda$ -transform of  $S_N$ ). Clearly, the left zeros of  $\hat{U}$  are also left zeros of  $\hat{U}_1$ . However,  $\hat{U}_1$  may have more zeros, possibly on the boundary of the disk. For example, let

$$\hat{U}_N = \begin{pmatrix} (\lambda - 1) & 0 \\ (\lambda - 0.5) & \lambda^N \end{pmatrix}$$

At  $\lambda = 1$  the above matrix loses rank, indicating the existence of a zero at the boundary of the unit disk. However, reordering the outputs before augmenting with delays avoids this difficulty:

$$\hat{U}_N = \begin{pmatrix} (\lambda - 0.5) & 0 \\ (\lambda - 1) & \lambda^N \end{pmatrix}$$

Note that the original  $\hat{U}$  has no left zeros since the rows are coprime.

The same applies to the right zeros of  $\hat{V}$ . In many instances this situation may be reversed by a proper reordering of inputs and outputs, such that the resulting  $\hat{U}_1$  and  $\hat{V}_1$  have no zeros on the boundary respectively. In any case, this limitation has little practical implications since it is always possible to find approximate rational solutions to (39) that are arbitrarily close to  $\underline{\mu}_N$ . In view of this, we will make the following simplifying assumption:

**Assumption 2:**  $\hat{U}_1(\lambda)$  and  $\hat{V}_1(\lambda)$  have no zeros on the unit circle.

Note that under this assumption the results of Theorem 3.1 are applicable. Furthermore, in the analysis that follows we will be able to exploit the existence of optimal solutions for any  $N$  and thus avoid the epsilon-delta arguments that would result from rational approximations.

By definition, problem (39) is equivalent to the following primal-dual pair:

$$\underline{\mu}_N = \min_{R_N \in \mathcal{S}_N} \|H - R_N\|_1 = \sup_{\substack{G_N \in \mathcal{S}_N^\perp \\ \|G_N\|_\infty \leq 1}} \langle H, G_N \rangle \quad (41)$$

It is easy to see that, as  $N$  increases, the subspace  $\mathcal{S}_N$  gets smaller and such that

$$\mathcal{S}_N \supseteq \mathcal{S}_{N+1} \supseteq \cdots \supseteq \mathcal{S} \quad (42)$$

since the only change in the interpolation structure is due to a higher multiplicity of the zero at the origin. Therefore,  $\underline{\mu}_N$  forms a non-decreasing sequence, bounded from above by  $\mu^\circ$ .

The next theorem states an interesting convergence result.

**Theorem 7.4** *Given the sequence  $\Phi_N^\circ$ , there exists a subsequence that converges weak\* to some  $\Phi^\circ$ . If the optimal solution is unique then the whole sequence converges weak\* to it.*

**Proof** Clearly  $\Phi_N^o$  forms a bounded sequence in  $\ell_1^{n_z \times n_w}$ , then there exists a *weak\**-convergent subsequence  $\Phi_{N_s}^o$  by the Banach-Alaoglu theorem. Let  $\Phi^{w^*}$  denote such limit point. As mentioned before,  $\Phi_N^o$  is infeasible to the original (un-augmented) problem. However, we will show that  $\Phi^{w^*}$  is in fact feasible. From Equation (38), after taking the *weak\** limit, we have:

$$\Phi^{w^*} = H - (UQ_{11}^o V)^{w^*} - (S_N, \bar{R}_{N_s}^o)^{w^*} = H - U(Q_{11}^o)^{w^*} V$$

where the superscript  $w^*$  denotes *weak\** limit. The last term drops since  $\bar{R}_{N_s}^o$  is uniformly bounded in  $N$ . For if  $\{\bar{R}_{N_s}^o\}$  were unbounded, then  $\{Q_{11}^o\}$  would necessarily be unbounded to keep  $\underline{\mu}_N$  bounded. But this contradicts the fact that  $\underline{\mu}_N$  is larger than  $\|H_{11} - U_1 Q_{11,N}^o V_1\|_1$ . Therefore,  $\Phi^{w^*}$  is feasible. To show that  $\Phi^{w^*}$  is actually an optimal solution, we need to view  $\Phi_N^o$  as a bounded linear operator from  $c_0^{n_z \times n_w}$  to  $\mathbb{R}$  (i.e., bounded linear functional on  $c_0^{n_z \times n_w}$ ) with strong operator limit  $\Phi^{w^*}$ . In such context we have the following inequality (see [25], page 269):

$$\|\Phi^{w^*}\|_1 \leq \liminf_{j \rightarrow \infty} \|\Phi_{N_s}^o\|_1 \leq \|\Phi^o\|_1$$

Therefore, since  $\Phi^{w^*}$  is feasible, all inequalities above are in fact equalities and  $\Phi^{w^*} = \Phi^o$ .

Finally, if the solution is unique then the whole sequence converges to  $\Phi^o$  *weak\**. ■

The last claim in the above lemma simply reflects the fact that if there are several optimal solutions,  $\Phi^o$ , then a sequence of DA problems can be such that  $\Phi_N^o$  (in the limit) “jumps” from one optimal solution to the other therefore not converging as a whole. Then, a subsequence that “keeps track” of a single optimal solution will converge *weak\** to it. This technicality is unnecessary when the optimal solution is unique.

An immediate corollary to Theorem 7.4 is the following:

**Corollary 7.2** *The sequence of lower bounds,  $\underline{\mu}_N$ , converge to  $\mu^o$  as  $N \rightarrow \infty$ .*

Next, we focus on the convergence properties of the dual sequence  $G_N$ . In the context of Equation (41) we state the following Theorem. (Note that  $G_N^o$  as well as  $G^o$  may not be unique.)

**Theorem 7.5** *Given the sequence  $G_N^o$ , there exists a subsequence that converges *weak\** in  $\ell_\infty^{n_z \times n_w}$  to an optimal solution  $G^o$ . Furthermore, if the solution  $G^o$  is unique, then the whole sequence converges *weak\** to it.*

**Proof** Clearly the sequence  $G_N^o$  is bounded by one. Then, by the Banach-Alaoglu theorem, there exists a subsequence that converges *weak\** in  $\ell_\infty^{n_z \times n_w}$ . Also, from Equation (42) we have that

$$S_N^\perp \subseteq S_{N+1}^\perp \subseteq \dots \subseteq S^\perp$$

Or, equivalently,  $G_N^o$  is feasible to the original (dual) problem for all  $N$ . Further, it can be shown that the feasible subspace  $S^\perp$  is *weak\**-closed [11, 29], then  $G_N^o$  converges *weak\** to a feasible limit point, say  $G^{w^*}$ . Therefore,

$$\underline{\mu}_{N_s} = \langle H, G_{N_s}^o \rangle \rightarrow \langle H, G^{w^*} \rangle$$

But, by Corollary 7.2,  $\underline{\mu}_{N_s} \rightarrow \mu^o$ , thus,  $\mu^o = \langle H, G^{w^*} \rangle$ . This implies that  $G^{w^*}$  is in fact an optimal dual solution,  $G^o$ , since it achieves the optimal value and is feasible.

If the solution,  $G^\circ$ , is unique then the whole sequence converges *weak\** to it.  $\blacksquare$

Next, we focus our attention on the sequence of suboptimal solutions that attain the upper bound  $\bar{\mu}_N$ . Let  $\bar{\Phi}_N := H - UQ_{11}^\circ V$ , then  $\bar{\mu}_N = \|\bar{\Phi}_N\|_1$  by definition. It is easy to see that  $\bar{\Phi}_N$  forms a bounded sequence in  $\ell_1^{n_z \times n_w}$  (if not  $\bar{\Phi}_{11,N}^\circ$  and thus  $\bar{\mu}_N$  would be unbounded). Therefore, there exists a subsequence that converges *weak\** in  $\ell_1^{n_z \times n_w}$ . Also,  $\bar{\Phi}_N$  is clearly feasible to the original problem for any  $N$ , and since  $\mathcal{S}$  is *weak\**-closed [11], then all *weak\** limit points are feasible. The question is whether or not the subsequence  $\bar{\mu}_N = \|\bar{\Phi}_N\|_1$  converges to  $\mu^\circ$  in general.

In order to give a proper answer to this question, it is useful to make the following observation first made in [33]. In Corollary 6.2 we have shown that most one-block problems have optimal solutions with all row norms equal to  $\mu^\circ$ . To illustrate why this is not the case with multiblock problems, consider the following SISO example:

$$\phi_1 = h_1 - u_1 q$$

where all operators are in  $\ell_1$  and  $\hat{u}_1(\lambda)$  has no zeros on the unit circle. Let  $\phi_1^\circ$  denote an  $\ell_1$ -optimal solution to such (one-block) problem, that is achieved with  $q^\circ$ . Next, add a new row to the problem,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} q$$

such that  $\|h_2 - u_2 q^\circ\|_1 < \mu^\circ$  (this is always possible simply by choosing a scalar weight on the second row of small enough value). Then, it is clear that an optimal solution to the new two-block column problem is still given by  $q^\circ$  and that  $\|\phi_2^\circ\|_1 < \|\phi_1^\circ\|_1 = \mu^\circ$ . In other words, the new row does not affect the optimal solution which is given by the first row alone. In contrast with a one-block problem with two outputs, a two-block problem with two outputs has to minimize both outputs with just one scalar free parameter sequence,  $q$ . The ‘‘shortage’’ of degrees of freedom is what makes this situation more common in multiblock problems.

Having noted this behavior, we can present the main theorem concerning convergence of the upper bound,  $\bar{\mu}_N$ .

**Theorem 7.6** *Given a general multiblock problem, let  $\bar{\Phi}_{N_i}^\circ$  converge *weak\** to an optimal solution  $\Phi^\circ = H - UQ^\circ V$  such that  $\|(\Phi^\circ)_i\|_1 = \mu^\circ$  for  $i \in \{1, \dots, n_u\}$ . Then,  $\bar{\Phi}_N$  converges strongly (i.e., in the norm) to  $\Phi^\circ$  as  $N \rightarrow \infty$ , and further,  $\bar{\mu}_N \rightarrow \mu^\circ$ .*

**Proof** It is a well known fact that if a sequence  $x_n \in \ell_1$  converges to  $x^{w*}$  *weak\**, and if  $\|x_n\|_1 \rightarrow \|x^{w*}\|_1$ , then  $x_n$  converges to  $x^{w*}$  strongly. However, such result is valid only for scalar and row-vector sequences in  $\ell_1$  (it is easy to think of a counter-example in the general matrix case). Therefore, we apply it to each individual row of  $\bar{\Phi}_N$ , to conclude the following:  $(\bar{\Phi}_N)_i$  converges strongly (i.e., in the norm) to  $(\Phi^\circ)_i$  for all  $i \in \{1, \dots, n_z\}$  such that  $\|(\Phi^\circ)_i\|_1 = \mu^\circ$ .

At the same time, from Assumption 2,  $\hat{U}_1$  and  $\hat{V}_1$  have full normal rank, so the map from  $Q_{11}$  to  $\bar{\Phi}_{11,N}$  is continuous with continuous inverse, that is

$$\hat{Q}_{11} = \hat{U}_1^{-1}(\hat{H}_{11} - \hat{\Phi}_{11,N})\hat{V}_1^{-1}$$

Then, using the fact that  $\|(\Phi^\circ)_i\|_1 = \mu^\circ$  for  $i \in \{1, \dots, n_u\}$ , we conclude that  $\bar{\Phi}_{11,N}^\circ$  converges strongly to  $\Phi_{11}^\circ$  which in turn implies that  $\hat{Q}_{11}^\circ$  converges strongly to  $Q^\circ$  and the result follows.  $\blacksquare$

The above theorem suggests that the construction of the feasible solution that attains the upper bound,  $\bar{\Phi}_N$ , can be viewed as an attempt to compute the *weak\** limit of the sequence  $\Phi_N^\circ$ , by “throwing away the tail” contained in the term  $S_N \bar{R}_N^\circ$ .

It should be stressed at this point, that non-pathological multiblock problems have optimal solutions where at least  $n_u$  of the  $n_z$  rows achieve the optimal norm (a natural extension of how optimal solutions of one-block problems behave). Furthermore, those rows that do not achieve the optimal norm can be left out of the optimization problem without affecting the overall solution, so eventually, the problem can be reduced. In general, however, a well posed control problem will tend to have none of its rows “redundant”, so  $\bar{\mu}_N$  usually converges to  $\mu^\circ$  without further considerations. In this context we have the following corollary valid for two-block column problems of the form:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} QV$$

**Corollary 7.3** *Given a two-block column problem, if  $\|\Phi_2^\circ\|_1 < \mu^\circ$  then  $\bar{\Phi}_N$  is the exact optimal solution for any  $N$ .*

**Proof** Follows immediately from the fact that the first block-row  $H_1 - U_1 Q_1 V$  is independent of the extra free parameter. That is,

$$\begin{aligned} \Phi_{1,N}^\circ &= H_1 - U_1 Q_1^\circ V \\ \Phi_{2,N}^\circ &= H_2 - U_2 Q_1^\circ V - S_N Q_2^\circ V \end{aligned}$$

Then, for any  $N$  we have

$$\|\Phi_{1,N}^\circ\|_1 \geq \mu^\circ \geq \underline{\mu}_N = \max(\|\Phi_{1,N}^\circ\|_1, \|\Phi_{2,N}^\circ\|_1) \geq \|\Phi_{1,N}^\circ\|_1$$

Thus, equality is attained throughout and the result follows, i.e.,  $Q_1^\circ = Q^\circ$ . ■

Theorem 7.6 and Corollary 7.3 dictate that a reordering of outputs needs to be done so that the first  $n_u$  rows of  $\Phi$  achieve the optimal norm  $\mu^\circ$ . The question is, then, how to find a priori which rows of the problem are not going to achieve the optimal norm. A brute force answer to this question is simply to solve all possible one-block problems formed by taking  $n_u$  rows out of the given  $n_z$  rows. If any solution is such that all the rows that were left out have smaller norm than the corresponding  $\mu^\circ$ , then those rows are the inactive ones and should be ordered in  $U_2$ . (In fact these rows can be removed altogether.) However, this approach may require a considerable amount of work. We will return to this difficulty later.

Two-block row problems, show a similar behavior. Indeed, such problems may have columns that are inactive in the optimization process in the sense that they can be removed without affecting the solution. Note that in the previous case, the phenomenon of inactive rows was intimately related with the fact that the  $\ell_1$  norm on matrices takes the maximum row norm, which allowed us to easily construct an example.

If the DA method is applied to a two-block row problem such that the columns associated with  $V_2$  are inactive, then again the solution  $\bar{\Phi}_N$  is exact for any  $N$ . However,  $\underline{\mu}_N$  will not give the exact optimal norm (although it will tend to it) since the extra parameter contributes in reducing the norm of  $\Phi_{12,N}^\circ$ .

Finally, let us point out that both forms of redundancy (row and column) can occur in a multiblock problem simultaneously. This discussion motivates the following definitions.

**Definition 7.1** Given a general multiblock problem, a one-block partition is defined by taking  $n_y$  inputs and  $n_u$  outputs of the full problem, such that the reduced problem corresponds to a one-block problem with full normal rank  $U$  and  $V$ .

**Definition 7.2** In a multiblock problem, a one-block partition is totally dominant (TD) if the optimal free parameter  $Q^\circ$  obtained from its solution also solves the original multiblock optimization problem.

It follows from these definitions that, if there is a TD one-block partition corresponding to the partitions  $U_1$  and  $V_1$ , then the DA method provides the exact answer for any  $N$ . The next section illustrates some of these properties.

In summary, in the DA method,  $\underline{\mu}_N$  always converges to  $\mu^\circ$ , and  $\bar{\mu}_N$  converges to  $\mu^\circ$  if the first  $n_u$  rows are active.

## 8 A Comparison of Methods

This section provides a general comparison of the approximation methods presented, based on a few simple multiblock examples. To facilitate further study, the first two selected problems are the same as those treated in other references [11, 33]. Particular attention will be paid to two aspects of the solutions: first, the support characteristics of the sequence of solutions, and second, the order of the sub-optimal controller they generate.

**EXAMPLE I:** Consider the following two-block column problem: given the SISO plant  $P$ , minimize the  $\ell_1$  norm of the weighted sensitivity and complementary sensitivity,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} W_1(1 - PK)^{-1} \\ W_2PK(1 - PK)^{-1} \end{pmatrix}$$

where

$$\hat{P}(\lambda) = \frac{\lambda(\lambda - 0.5)}{(\lambda - 0.1)(1 - 0.5\lambda)}$$

and

$$\hat{W}_1(\lambda) = \frac{0.02}{1 - 0.2\lambda} ; \quad \hat{W}_2(\lambda) = \frac{0.004\rho}{1 - 0.6\lambda}$$

Note that a variable scalar weight on  $\phi_2$ , denoted  $\rho$ , has been included. By adjusting  $\rho$ , we will be able to generate two interesting cases: case (a) where  $\phi_1$  is TD (for “small”  $\rho$ ) and case (b) where both rows are active in the optimization (for “intermediate”  $\rho$ ). The workings of Theorem 7.6 will be illustrated by reordering the outputs and forcing the TD row to be in the “wrong” place.

The results are presented in tables showing, for each  $N$ , the DA lower bound ( $\underline{\mu}_N$ ), the DA upper bound ( $\bar{\mu}_N$ ) and the FMV upper bound ( $\bar{\nu}_N$ ). The FME lower bound is omitted since it is equal to  $\underline{\mu}_N$  in this particular case. In general, however,  $\underline{\mu}_N$  converges faster than  $\underline{\nu}_N$  since the delay augmentation method generates more constraints than the FME method for any given  $N$ . These extra constraints are the ones that ensure feasibility of  $\Phi_{11,N}$ . To illustrate this point, consider the following case:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} q$$

where  $\hat{u}_1(\lambda)$  and  $\hat{u}_2(\lambda)$  are coprime. Further, assume that  $\hat{u}_1(\lambda)$  has an unstable zero at  $\lambda_0$ . Consequently, the FME method generates the following rank constraints (note there are no left zeros of  $\hat{U}$ ):

$$(\phi_1 * u_2 - \phi_2 * u_1)(k) = (h_1 * u_2 - h_2 * u_1)(k) ; k = 0, \dots, N - 1 \quad (43)$$

Now consider the DA method of order  $N$ :

$$\hat{U}_N = \begin{pmatrix} \hat{u}_1 & 0 \\ \hat{u}_2 & \lambda^N \end{pmatrix}$$

Let us construct the left zero interpolations for this  $\hat{U}_N$ . Multiplying  $\hat{U}_N$  on the left by  $(\hat{u}_2 - \hat{u}_1)$  we get  $(0 - \hat{u}_1 \lambda^N)$ . This implies that the left zeros of  $\hat{U}_N$  are given by the zeros of  $\hat{u}_1$  and a zero at the origin of multiplicity  $N$ . Further, the directional properties of such zero are captured by the vector  $(\hat{u}_2 - \hat{u}_1)$ . Therefore, the zero interpolation conditions are given by Equation (43) plus the following:

$$\hat{\phi}_1(\lambda_0) = \hat{h}_1(\lambda_0)$$

Note that this last constraint becomes redundant as  $N \rightarrow \infty$ .

In this particular numerical example, however, both lower bounds are equal due to the fact that the unstable zeros of  $\hat{u}_1(\lambda)$  are also zeros of  $\hat{u}_2(\lambda)$ .

Also included are the support characteristics of  $\Phi_N^o$  and of the FMV solution along with the order of the suboptimal controllers that achieve the corresponding upper bounds.

To describe the support characteristics we define a function,  $len(\cdot)$ , mapping  $\ell_1^{n \times m}$  to  $\mathbf{Z}_+^{n \times m}$  in the following way: given  $\Phi \in \ell_1^{n \times m}$ , then  $[len(\Phi)]_{ij}$  is a non-negative integer equal to the maximum  $k$  for which  $\phi_{ij}(k)$  is not zero, plus one. Also, we denote the order of a controller  $K$  by  $ord(K)$ .

**Case (a):** In this case let  $\rho = 1$  and keep the same ordering of outputs as above (i.e., sensitivity first). The results are shown in Table 1. Clearly the solution given by the delay augmentation method is exact since the upper and lower bounds are equal for any  $N$ . Then, in the context of Corollary 7.3, the first row corresponding to the weighted sensitivity is TD. Indeed, a simple computation shows that  $\|\phi_2^o\|_1 = 0.2040 < \|\phi_1^o\| = 0.7822$ . Note how the support of the second row of the augmented optimal solution increases with  $N$  while the first row remains constant and equal to the optimal of the un-augmented problem. Since the controller is computed from the first row only, it is also exact and constant as  $N$  increases. In contrast, the FMV solution has increasing support on both rows, thus generating a suboptimal controller of increasing order that approximates the second order optimal controller. Note that for some  $N$ 's, the FMV problem has no solution (indicated with a dash) since the feasible set is empty.

Next, consider the same problem but with the outputs reordered (i.e., the complementary sensitivity in the first row). Table 2 shows how violating the conditions of Theorem 7.6 affects the convergence of the upper bound (note that the lower bound does converge as shown in Theorem 7.4). Although the upper bound does not converge, it is interesting to note that for  $N \geq 2$  the length of  $\phi_{2,N}^o$  (i.e the weighted sensitivity) locks at a value of 3, which coincides with the length of the optimal solution. This seems to be a general characteristic of the DA method as we shall see later. At the same time, there is a clear order inflation on the suboptimal controller due to the constant increase in the length of  $\phi_{1,N}^o$ . (Note: FMV results are

$N$	DA				FMV		
	$\underline{\mu}_N$	$\bar{\mu}_N$	$len(\Phi_N^o)^T$	$ord(K)$	$\bar{\nu}_N$	$len(\Phi_N^o)^T$	$ord(K)$
1	0.78222	0.78222	(3 2)	2	-	-	-
2	0.78222	0.78222	(3 3)	2	-	-	-
3	0.78222	0.78222	(3 4)	2	1.31912	(4 4)	4
4	0.78222	0.78222	(3 5)	2	0.97459	(5 5)	5
5	0.78222	0.78222	(3 6)	2	0.87547	(6 6)	6
6	0.78222	0.78222	(3 7)	2	0.83292	(7 7)	7

Table 1: Comparison of Methods: Example I, case (a) where the first row is TD.

not included in Table 2 since such method is not affected by reordering.)

**Case (b):** Let  $\rho = 6$  and place the sensitivity back in the first row. For this weighting, both rows are active in the optimization as shown by the gradual convergence of the upper and lower bound (see Table 3). Note that, even though the controller order growth is comparable in both methods, the support characteristics are quite different. Most interesting, the length of  $\phi_{2,N}^o$  remains equal to 4 for  $N > 2$  suggesting the possibility that, by changing the order of the outputs, a low order sub-optimal controller can be computed. This is in fact the case, as shown in Table 4. (This procedure does not apply to the FMV method since the sub-optimal solutions obtained by this method are such that all entries of  $\Phi(k)$  are supported at  $k = N$ .) It is interesting how in both cases (a) and (b), a proper ordering of the outputs results in a much better approximation of the solution (exact if one row is TD) in the sense that, after some  $N$ , the sequence of sub-optimal controllers are of fixed order and asymptotically approaching the optimal one. This is not an isolated case. Many other multiblock problems for which reliable numerical approximations were computed behave in this way when solved by the DA method. In other words, given a general multiblock problem, there seems to be a one-block partition that preserves a polynomial optimal solution, and further, such support structure is eventually captured by the delay augmentation method for a large enough  $N$ . Then, a proper ordering of inputs and outputs that places the one-block partition in the first  $n_u$  rows and  $n_y$  columns of  $\Phi$  (corresponding to  $U_1$  and  $V_1$ ) will generate a sequence of sub-optimal controllers without order inflation.

These observations suggest that an efficient algorithm for computing low order sub-optimal controllers can be as follows: given a general multiblock problem,

**Step 1:** Pick a positive integer  $N$ .

**Step 2:** Solve the corresponding delay augmentation problem.

**Step 3:** Compute  $len(\Phi_N^o)$  and reorder inputs and outputs such that the set of  $n_u \times n_y$  input-output pairs of minimum length correspond to  $\Phi_{11}$ .



$N$	$\underline{\mu}_N$	$\bar{\mu}_N$	$len(\Phi_N^o)^T$	$ord(K)$
1	0.22000	1.1602	(3 2)	2
2	0.29195	1.9939	(4 3)	4
3	0.42826	3.1464	(5 3)	5
4	0.55995	3.9859	(6 3)	6
5	0.65664	4.5189	(7 3)	7
6	0.71550	4.8077	(8 3)	8
7	0.74789	4.9504	(9 3)	9
8	0.76483	5.0171	(10 3)	10
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
15	0.78159	5.1878	(15 3)	15

Table 2: Comparison of Methods: Example I, case (a) where the second row is TD.

**Step 4:** If reordering was necessary in Step 3, solve the reordered system for the same  $N$ . Then, check the difference between the upper and lower bounds, i.e.,  $\bar{\mu}_N - \underline{\mu}_N$ . If such difference is small enough stop, otherwise increase  $N$  by one (or more) and go to Step 2.

In order to illustrate the workings of such algorithm we include a four-block example.

**EXAMPLE II:** Consider the following 2-input-2-output four-block problem where the regulated signals are the output of the plant and the control sequence (weighted with the scalar  $\rho$ ), and the input disturbances are a disturbance at the plant output with frequency weighting  $\hat{W}_1(\lambda)$  and measurement noise with frequency weighting  $\hat{W}_2(\lambda)$ . That is,

$$\Phi = \begin{pmatrix} (1 - PK)^{-1}W_1 & PK(1 - PK)^{-1}W_2 \\ \rho K(1 - PK)^{-1}W_1 & \rho K(1 - PK)^{-1}W_2 \end{pmatrix}$$

where

$$\hat{W}_1(\lambda) = \frac{0.4}{1 - 0.6\lambda} \quad ; \quad \hat{W}_2(\lambda) = \frac{1 - 0.75\lambda}{1 - 0.25\lambda}$$

$\rho = 0.1$  and  $\hat{P}(\lambda)$  is the same as in Example I. Then, the results in Table 5 are obtained by applying the above algorithm starting with  $N = 3$ . For  $N = 10$ , the sub-optimal controller is of order five and achieves a norm that is within half a percent of the optimal. (The jump in order is most likely due to convergence to another optimal solution.) In contrast, it can be shown that the FMV method has no polynomial feasible solution for any  $N$  (due to the way  $\hat{W}_1$  and  $\hat{W}_2$  enter the problem). This example shows how the delay augmentation algorithm can generate low order sub-optimal controllers even when the FMV method has no solution.

$N$	DA				FMV		
	$\underline{\mu}_N$	$\bar{\mu}_N$	$len(\Phi_N^\circ)^T$	$ord(K)$	$\bar{\nu}_N$	$len(\Phi_N^\circ)^T$	$ord(K)$
1	0.78222	1.2243	(3 2)	2	-	-	-
2	0.79333	1.2547	(4 3)	3	-	-	-
3	0.90230	1.5255	(5 4)	5	1.3191	(4 4)	3
4	0.99522	1.0389	(5 4)	5	1.0564	(5 5)	4
5	1.0015	1.0105	(6 4)	6	1.0121	(6 6)	6
6	1.0024	1.0043	(7 4)	7	1.0044	(7 7)	7
7	1.0026	1.0030	(8 4)	8	1.0030	(8 8)	8
8	1.0026	1.0027	(9 4)	9	1.0027	(9 9)	9

Table 3: Comparison of Methods: Example I, case (b) where no row is TD.

## 9 Support Structure of Optimal Solutions

Here we explore the support characteristics of the optimal solution in more general terms. The numerical examples in the previous section suggest that it may be possible to infer the support of the optimal solution by observing how the superoptimal solutions,  $\Phi_N^\circ$ , evolve as  $N$  increases. Here we make an important step in this direction by showing that such support structure is “hinted to” by the support of the sequence of superoptimal solutions.

We have already shown that, given a multiblock problem, there exists a subsequence of super-optimal dual solutions,  $G_N^\circ$ , whose *weak\** limit point,  $G^\circ$ , is feasible and optimal (Theorem 7.5). By exploiting this result in combination with the alignment conditions, we will show that the finitely supported partition of the optimal solution is eventually “captured” by the sequence of super-optimal solutions. For that purpose we need the following well known lemma.

**Lemma 9.1** *If a sequence  $G_N \in \ell_\infty^{n \times m}$  converges weak\* to  $G$ , then for any positive integer  $L < \infty$ ,  $\|P_L(G_N - G)\|_\infty \rightarrow 0$  as  $N \rightarrow \infty$ .*

Note that the above lemma implies that each individual entry of  $G_N$  also enjoys this convergence property, i.e.,  $\|P_L(g_{ij,N} - g_{ij})\|_\infty \rightarrow 0$  as  $N \rightarrow \infty$ , for all  $i = 1, \dots, n$  and  $j = 1, \dots, m$ .

Next, let us review the alignment properties of the optimal solutions. Optimality implies that each optimal solution to the primal problem must be aligned with every optimal solution to the dual problem. In particular, if an optimal dual solution,  $G^\circ$ , is such that

$$|g_{ij}^\circ(t)| < \max_{1 \leq j \leq n_w} \|g_{ij}^\circ\|_\infty \text{ for all } t > T$$

then all optimal primal solutions are such that  $\phi_{ij}^\circ(t) = 0$  for  $t > T$ . Note that, according to

$N$	$\underline{\mu}_N$	$\bar{\mu}_N$	$len(\Phi_N^o)^T$	$ord(K)$
1	0.95745	1.1602	(3 2)	2
2	0.95745	1.1602	(3 3)	2
3	0.98658	1.0586	(4 4)	3
4	0.99889	1.0157	(4 5)	3
5	1.0019	1.0053	(4 6)	3
6	1.0022	1.0031	(4 7)	3
7	1.0026	1.0027	(4 8)	3
8	1.0026	1.0026	(4 9)	3

Table 4: Comparison of Methods: Example I, case (b) with the outputs reordered.

the notation developed in section 5,  $\max_{1 \leq j \leq n_w} \|g_{ij}^o\|_\infty$  is nothing but  $\gamma_0^o(i)$ . The next theorem puts all these pieces together.

**Theorem 9.1** *Given a multiblock problem, if all optimal dual solutions are such that  $|g_{ij}^o(T)| = \gamma_0^o(i)$  for some  $T \in \mathbf{Z}_+$  and  $|g_{ij}^o(t)| < \gamma_0^o(i)$  for all  $t > T$  then, for every  $L > T$  there exists a positive integer  $N^*$  such that  $\phi_{ij,N}^o(t) = 0$  for  $T < t \leq L$  and for any  $N \geq N^*$ .*

**Proof** (Note: to simplify notation we drop sub-indices  $i, j$  and superindex ‘o’.) Given some  $L > T$ , pick  $\varepsilon > 0$  such that

$$\min_{T < t \leq L} (\gamma_0 - |g(t)|) = \varepsilon \quad (44)$$

By Lemma 9.1, for every  $L > T$  there exists  $N^*$  such that

$$\|P_L(g_N - g)\|_\infty < \frac{\varepsilon}{2} \quad (45)$$

for all  $N > N^*$ . First we prove (by contradiction) that  $|g_N(t)| < \gamma_{0,N}$  for  $T < t < L$  and for any  $N \geq N^*$ . The result then follows from the alignment conditions.

Given  $N > N^*$ , assume that  $|g_N(t_1)| = \gamma_{0,N}$  for  $T < t_1 \leq L$ . Then, by Equation (44) and (45),

$$\gamma_{0,N} - \gamma_0 \leq |g_N(t_1)| - |g(t_1)| - \varepsilon < \frac{\varepsilon}{2} - \varepsilon$$

Therefore,

$$\gamma_0 - \gamma_{0,N} > \frac{\varepsilon}{2} \quad (46)$$

Next, consider the point  $t = T$ . From Equation (45) and the fact that  $|g_N(t)| \leq \gamma_{0,N}$  in general, we have

$$\gamma_0 - \gamma_{0,N} \leq |g(T)| - |g_N(T)| < \frac{\varepsilon}{2}$$

$N$	DA				FMV		
	$\underline{\mu}_N$	$\bar{\mu}_N$	$len(\Phi_N^o)$	$ord(K)$	Comments	$\bar{\nu}_N$	$ord(K)$
3	60.453	102.34	$\begin{pmatrix} 5 & 3 \\ 5 & 3 \end{pmatrix}$	4	Reorder inputs	-	-
3	60.400	81.161	$\begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}$	2	Keep order	-	-
4	64.702	81.161	$\begin{pmatrix} 3 & 6 \\ 3 & 7 \end{pmatrix}$	2	"	-	-
5	68.284	81.161	$\begin{pmatrix} 3 & 7 \\ 5 & 9 \end{pmatrix}$	2	"	-	-
6	70.721	72.850	$\begin{pmatrix} 6 & 7 \\ 6 & 11 \end{pmatrix}$	5	"	-	-
7	70.754	71.874	$\begin{pmatrix} 6 & 8 \\ 8 & 13 \end{pmatrix}$	5	"	-	-
8	70.888	71.500	$\begin{pmatrix} 6 & 9 \\ 10 & 15 \end{pmatrix}$	5	"	-	-
9	71.040	71.615	$\begin{pmatrix} 6 & 11 \\ 10 & 17 \end{pmatrix}$	5	"	-	-
10	71.089	71.408	$\begin{pmatrix} 6 & 12 \\ 12 & 19 \end{pmatrix}$	5	"	-	-
11	71.110	71.146	$\begin{pmatrix} 12 & 13 \\ 13 & 21 \end{pmatrix}$	12	"	-	-
12	71.113	71.122	$\begin{pmatrix} 13 & 14 \\ 19 & 23 \end{pmatrix}$	14	"	-	-

Table 5: Example II: Delay Augmentation Algorithm

which contradicts Equation (46). This implies that  $\phi_N(t) = 0$  for  $T < t \leq L$  and  $N > N^*$  which is the desired result. ■

In other words, given the conditions of the theorem above, and for  $N$  large enough, there is a “gap” of zeros (between  $T$  and  $L$ ) in  $\phi_{i,N}^o(t)$  which gets wider as  $N$  increases, i.e., as  $L$  increases. However,  $T$  does not change for  $N$  large enough, giving a clue on the length of the finitely supported entries of  $\Phi^o$ . The difficulty is that we do not have an a priori estimate of how large  $N$  has to be to capture  $T$ .

It is worth pointing out that Theorem 9.1 can be applied to the FMV sequence of suboptimal solutions too, since the corresponding duals also have a *weak\** convergent subsequence [34]. However, there is an important difference in the way the DA and FMV sequence of solutions behave, which was pointed out in the previous section. Indeed, while the FMV solutions are consistently supported for  $t > L$ , the DA solutions are not. This observation was crucial in constructing low order suboptimal controller. We expand these ideas in the following section.

## 10 Observations

This section includes a few observations based on a fair amount of computational experience using the delay augmentation method and on some intuitive ideas on the problem of  $\ell_1$  optimization in general. It is by no means a formal or precise presentation. It is simply intended to give some lead into new ideas that might open the way to finding the exact solution of multiblock problems in general. In particular, a conjecture is stated, establishing a stronger connection between the support structure of the optimal solution and the DA method.

Observe the way the DA method works. It transforms a general multiblock problem into a square one, therefore generating polynomial super-optimal solutions,  $\Phi_N^o$ . Without changing the order of inputs and outputs, the sequence  $\Phi_N^o$  will increase its length as  $N$  increases. However, it was noted in previous examples that not every entry of  $\Phi_N^o$  increases its length in the same way. In fact, a closer look at the sequence  $\Phi_N^o$  suggests that the support of some of its entries stops changing after some  $N$ . This is exactly what happened in Example I, case (a) and (b), where the support of one of the entries of  $\Phi_N^o$  remained the same after some  $N$  regardless of the ordering. In Example II, the pattern also occurs but for  $N > 12$  (not shown in Table 5). Next, note that  $\Phi_{11,N}^o = \bar{\Phi}_{11,N}$  since that block of the problem is not affected by the extra free parameters. Therefore, for each  $N$ ,  $\bar{\Phi}_{11,N}$  is polynomial. Then, if those entries of  $\Phi_N^o$  that have constant support after some  $N$  are collected (by reordering) in  $\Phi_{11,N}^o$ ,  $\bar{\Phi}_{11,N}$  will have constant support. Interestingly, those entries of constant support seem to be always enough to define a one-block partition and therefore fill the necessary entries of  $\bar{\Phi}_{11,N}$ . Furthermore, many multiblock problems seem to have this property.

A multiblock problem in this class can be viewed as dominated by a one-block partition. In other words, there is an embedded one-block problem that is further constrained by the rank interpolation conditions. Such constraints, however, are not enough to change the polynomial nature of the optimal solution corresponding to that partition, although, in general, they have the effect of increasing its order. With this we extend the notion of TD one-block partitions where the added constraints due to the rank interpolation conditions were totally inactive.

**Definition 10.1** *Given a multiblock problem, a one-block partition is partially dominant (PD) if all  $\ell_1$  optimal solutions are polynomial in the entries corresponding to such partition.*

Clearly, a TD one-block partition is also PD but not vice versa. Based on this definition we state the following conjecture.

**Conjecture 10.1** *Given a multiblock problem with a PD one-block partition, there exists a positive integer  $N^*$  such that the DA solution,  $\Phi_N^o$ , for  $N > N^*$  captures the exact support of the sequences corresponding to the PD one-block partition. Furthermore, since the actual linear program splits  $\Phi_N^o$  into the difference of two positive sequences ( $\Phi_N^{o+}$  and  $\Phi_N^{o-}$ ), the sign of the non-zero entries of the exact solution corresponding to the PD partition is also captured. That is, for any pair of indices  $(i, j)$  in the PD partition, and  $N > N^*$ ,*

$$\begin{aligned} \phi_{ij}^o(k) = 0 &\iff \phi_{ij,N}^o(k) = 0 \\ \phi_{ij}^o(k) > 0 &\iff \phi_{ij,N}^o(k) > 0 \\ \phi_{ij}^o(k) < 0 &\iff \phi_{ij,N}^o(k) < 0 \end{aligned}$$

This conjecture is supported by a fair amount of numerical experiments covering the most obvious combinations (i.e., two-block row and column problems and four-block problems with different input-output dimensions). At the same time, it is consistent with Theorem 9.1 but

stronger. Indeed, the conjecture claims that the superoptimal solution will not be supported for  $t \geq L$ . This conjecture, if proven correct, has interesting consequences. To illustrate some of the ideas involved, consider the following simple two-block column problem:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} q$$

and assume, without loss of generality, that  $\hat{u}_1$  and  $\hat{u}_2$  are polynomials (this can always be obtained by polynomial factorization of  $\hat{U}$ ). Further, assume that  $(h_1 \ h_2)^T$  is a polynomial feasible solution and that the outputs are ordered such that  $\phi_1$  is PD. Then we have the following equality due to the rank interpolation conditions:

$$\hat{u}_2 \hat{\phi}_1^o - \hat{u}_1 \hat{\phi}_2^o = \hat{u}_2 \hat{h}_1 - \hat{u}_1 \hat{h}_2 \quad (47)$$

Assume that all zeros common to  $\hat{u}_1$  and  $\hat{u}_2$  have been canceled out from the above equation. Clearly, the right hand side of Equation (47) is polynomial, and furthermore, the first term on the left hand side is polynomial since we assumed that  $\phi_1$  is PD. Therefore, the second term on the left hand side must be polynomial. This implies that two situations are possible: either  $\hat{\phi}_2^o$  is polynomial or it has stable poles that are canceled by stable zeros of  $\hat{u}_1$ .

This observation has interesting implications. On one hand, there is a class of multiblock problems with polynomial optimal solutions that is characterized by the absence of stable zeros in  $\hat{u}_1$ . Such solutions can then be computed exactly by either the FMV or the DA method. On the other hand, if  $\hat{u}_1$  has stable zeros and  $\hat{\phi}_2^o$  is infinitely supported, the rate at which  $\hat{\phi}_2^o$  decays is given by a subset of the stable zeros of  $\hat{u}_1$ . This information could be used to transform the original problem into a finite dimensional one for which exact solutions are computable. This approach is currently under investigation. It should be noted that the above ideas can be easily extended to the general multiblock problem.

Finally, note that if the above conjecture is correct, the DA algorithm would automatically reorder any TD partition in  $\Phi_{11}$  and provide the exact answer, without the need to solve all possible combinations of one-block problems (see discussion after Corollary 7.3).

## 11 A Synthesis Example

In this section we apply the DA method to a specific control problem, namely, the pitch axis control of the X29 aircraft. The motivation for doing so is two-fold: first, to illustrate the use of the delay augmentation method in a more realistic problem, and second, to have a first look at the frequency domain features of an  $\ell_1$ -optimal design (albeit for one particular example). In order to give some perspective to this presentation, we will compare the characteristics of the  $\ell_1$  design with those of an  $\mathcal{H}_\infty$  optimal design.

It should be stressed, however, that this particular control problem was not chosen for the purpose of demonstrating extreme behaviors of the  $\ell_1$  and  $\mathcal{H}_\infty$  optimal solutions. Rather, it was candidly selected as an interesting control problem in general.

The X29 aircraft poses an interesting control problem due to its revolutionary forward-swept wing design. With such configuration, the center of gravity lies behind the aerodynamic center of pressure, rendering the aircraft statically unstable. Thus, a control system has to actively stabilize the aircraft during flight.

We are interested in designing a digital controller for a simple model of the pitch dynamics of the aircraft. The airplane has three types of control surfaces: canard wings, flaperons on the

main wings and strakes on the tail. In order to simplify the model, the action of these control surfaces are lumped into one equivalent actuator with first order dynamics. Similarly, the gyroscopes and accelerometers are modeled by an equivalent sensor with neglectable dynamics. Thus, the system can be approximately represented by the following continuous time SISO plant [35]:

$$\hat{P}(s) = \underbrace{\frac{(s+3)}{(s+10)(s-6)}}_{\text{airframe}} \underbrace{\frac{20}{(s+20)}}_{\text{equiv. actuator overhead}} \underbrace{\frac{(s-26)}{(s+26)}}_{\text{overhead}} \quad (48)$$

where  $s$  is the Laplace variable. The airframe factor corresponds to a simplified model of the pitch dynamics of the airplane flying at a low altitude and with an air speed of approximately 0.9 Mach. The overhead factor lumps the equivalent low frequency phase lag introduced by the dynamics that were neglected in deriving the reduced model (48). In particular, this all-pass factor is an approximate representation of the collected phase lag of the gyroscopic sensor dynamics, the actuator servo dynamics, the airframe flexible modes, and the digital implementation (i.e., pre-filter, zero order hold and computing delay) corresponding to a sampling period  $\Delta t = 1/30$  seconds.

Consider the following formal synthesis problem:

$$\inf_{K \text{ stab.}} \left\| \begin{array}{c} W_1 K S \\ W_2 S \end{array} \right\|_1$$

where  $S$  is the sensitivity function. Such problem requires the discrete time version of Equation (48) and two weighting transfer functions. The  $\lambda$ -domain model of the plant,  $\hat{P}(\lambda)$ , is obtained by discretizing Equation (48) assuming a zero order hold at the plant input and a synchronized sampling of the (pre-filtered) plant output. The weights are chosen as follows: let  $\tilde{W}_1$  be a scalar equal to 0.01 and let  $\tilde{W}_2(\lambda)$  be the discrete time version of the continuous time transfer function  $(s+1)/(s+0.001)$  for a sampling period  $\Delta t = 1/30$ . This choice of weights reflects a trade-off between low frequency performance and the control effort.

Note that a controller designed for the discrete-time model of a continuous-time plant completely ignores the inter-sampling behavior of the system. An optimal controller designed in this way is actually sub-optimal for the original hybrid system. This notwithstanding, we will carry out the design and comparison entirely in the discrete domain (both for  $\ell_1$  and  $\mathcal{H}_\infty$  designs), taking the discrete time plant model and weights as the starting point.

### 11.1 Computing an $\ell_1$ sub-optimal controller

With this problem set-up we are ready to apply the delay augmentation algorithm as described in Section 9. Table 6 shows the sequence of results obtained in this case, starting with  $N = 4$ . Note how the length of the response corresponding to the weighted sensitivity stops increasing after  $N = 7$ , suggesting that such row is PD. For  $N = 80$  the achieved  $\ell_1$  norm is within one percent of the optimal so we stop the iteration process. It is interesting to note how slowly the upper bound converges to the optimal. This behavior is consistent with the observations made in Section 10 regarding the rate of decay of  $\Phi$  when one row is PD. Indeed, if the first row corresponding to the weighted sensitivity is PD, then the rate of decay of the second row is dictated by the stable zeros of  $\hat{u}_1(\lambda)$ . It is easy to check that such transfer function contains two stable zeros that are close to the unit circle. Then, if the optimal second row decays slowly, the extra free parameter ( $q_2$ ) corresponding to the DA solution will be significant even for large values of  $N$ .

$N$	$\underline{\mu}_N$	$\bar{\mu}_N$	$len(\Phi_N^o)^T$	$ord(K)$	Comments
4	3.254	1256.4	(10 5)	11	Reorder outputs
5	4.024	7.619	(5 5)	6	Keep order
6	4.045	5.059	(5 6)	6	"
7	4.048	5.052	(6 7)	6	"
8	4.051	4.652	(6 8)	6	"
9	4.051	4.319	(6 9)	6	"
10	4.052	4.224	(6 10)	6	"
⋮					
20	4.053	4.196	(6 20)	6	"
⋮					
40	4.053	4.158	(6 36)	6	"
⋮					
80	4.054	4.091	(6 69)	6	"

Table 6: X29 Synthesis Problem: Delay Augmentation Algorithm

Next, we will compare the time and frequency domain characteristics of the  $\ell_1$  sub-optimal design corresponding to  $N = 80$  with an  $\mathcal{H}_\infty$  design. The comparison will be based on three different aspects of the solutions: 1) operator norms, 2) frequency response characteristics, and 3) time response characteristics.

Table 7 shows how the  $\ell_1$  and  $\mathcal{H}_\infty$  norms of the two solutions compare. As expected, the  $\mathcal{H}_\infty$  design achieves better  $\mathcal{H}_\infty$  norms while the  $\ell_1$  design achieves better  $\ell_1$  norms. A cross examination shows that both solutions are fairly good in terms of both measures. In fact, this does not come as a surprise in view of the following norm inequality [5] valid for any stable

		$\ \cdot\ _{\mathcal{H}_\infty}$	$\ \cdot\ _1$	$ord(\bar{K})$
$\mathcal{H}_\infty$ design	$\Phi$	2.4	5.2	5
	$W_1 K S$	2.0	3.3	
	$W_2 S$	2.4	5.2	
$\ell_1$ design	$\Phi$	3.8	4.1	6
	$W_1 K S$	2.8	4.1	
	$W_2 S$	3.6	4.1	

Table 7: Operator norm comparison ( $\Delta t = 1/30$ )



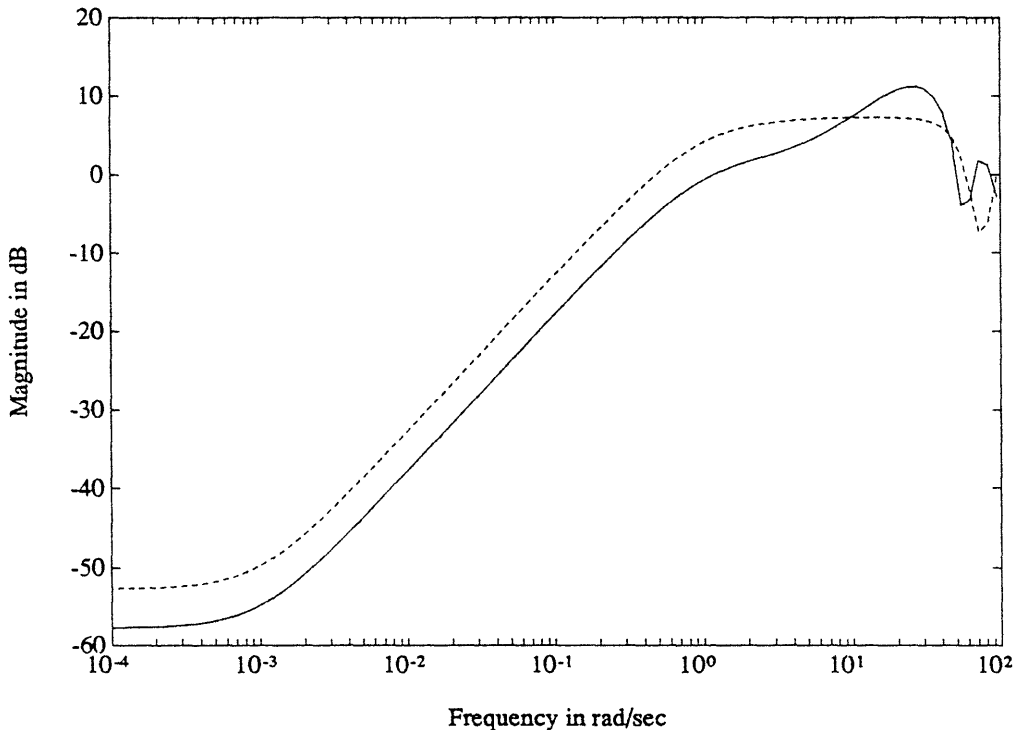


Figure 2: Frequency response of  $\hat{S}$  for  $\ell_1$  design (full line) and  $\mathcal{H}_\infty$  design (dashed line)

finite dimensional system  $H \in \ell_1^{p \times q}$ :

$$\|H\|_{\mathcal{H}_\infty} \leq \sqrt{p}\|H\|_1 \leq \sqrt{p}(2n+1)\sqrt{q}\|H\|_{\mathcal{H}_\infty}$$

where  $n$  is its McMillan degree. Thus, minimizing any of the two norms will also “push down” the other one, particularly in a low order problem as the one under consideration.

Next, let us examine the frequency domain features. Both designs have fairly similar frequency domain characteristics as shown in Figures 2 and 3. While the  $\ell_1$  design has better disturbance rejection at low and medium frequencies, it overshoots at high frequencies where the  $\mathcal{H}_\infty$  norm is achieved. In fact, Figure 3 shows that both controllers have very similar response, the only significant difference being at frequencies close to  $\pi/\Delta t$ . An interesting difference, though, is that the  $\ell_1$  design results in an unstable controller while the  $\mathcal{H}_\infty$  design does not. Finally, we compare the weighted and unweighted sensitivity step response of both designs (Figures 4 and 5). Note how the output of the plant,  $y$ , converges to zero faster in the  $\ell_1$  design than in the  $\mathcal{H}_\infty$  design (Figure 5). This is a direct result of the smaller weighted steady state error in the  $\ell_1$  design (see Figure 4) and the pole of  $\tilde{W}_2$  at 0.9999 (almost a pure integrator).

## 12 Conclusions

A complete and comprehensive study of the general  $\ell_1$ -optimal multiblock problem has been presented. It advances the understanding of these problems both from a theoretical and a

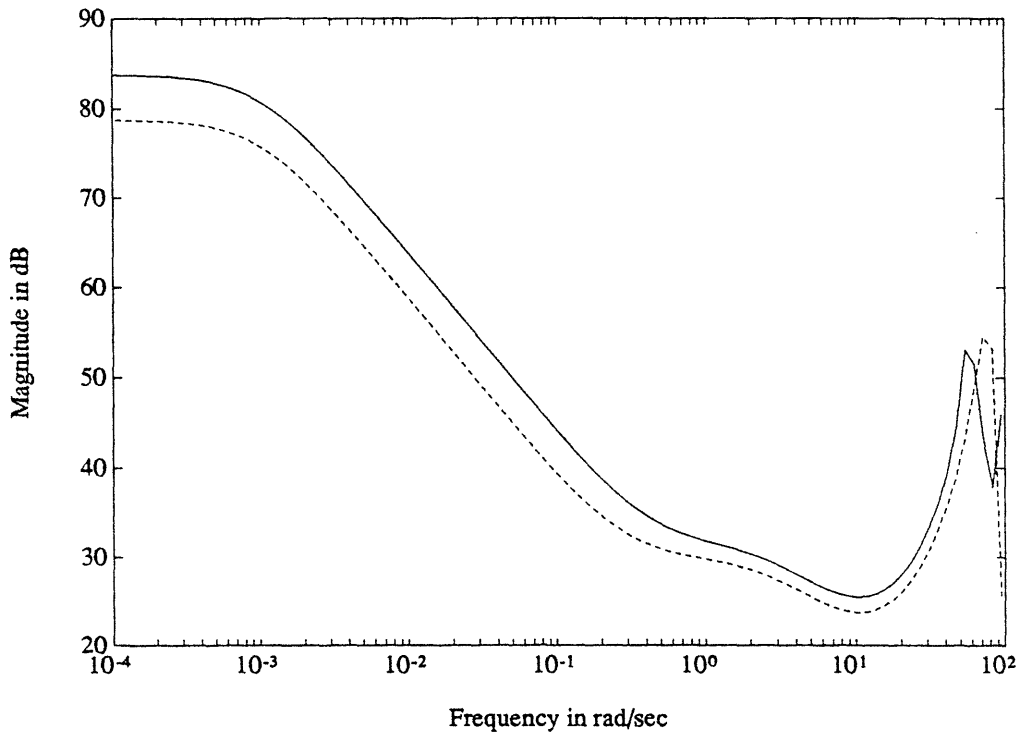


Figure 3: Frequency response of  $\hat{K}$  for  $\ell_1$  design (full line) and  $\mathcal{H}_\infty$  design (dashed line)

practical point of view.

The paper makes the following contributions:

- 1) The interpolation conditions are stated in a concise and natural way. As a result the general theory is developed in simpler terms and with a minimum number of assumptions.
- 2) Methods for computing the interpolation conditions were tied up directly to matrix theory.
- 3) Further insight was gained on the structure of the optimal solution which allowed us to distinguish between different classes of multiblock problems (i.e., problems with TD or PD one-block partitions).
- 3) A new method for computing suboptimal (or optimal in some special cases) solutions was proposed that exploits such structure. With this method, a sequence of suboptimal controllers can be computed iteratively avoiding (for a class of problems) the problem of order inflation. Each iteration requires the solution of one finite dimensional linear program, and generates upper and lower bounds of the optimal norm with the proper convergence properties. In contrast, previously known approximation schemes required the solution of two linear programs at each iteration, and generated suboptimal controllers with increasing order. In addition, the DA method unifies the treatment of zero and rank interpolations and avoids the coprime factorization of  $U$  and  $V$  (this was required in previous work [29]). Further, this approach generates a minimal set of constraints describing the feasible subspace [18].

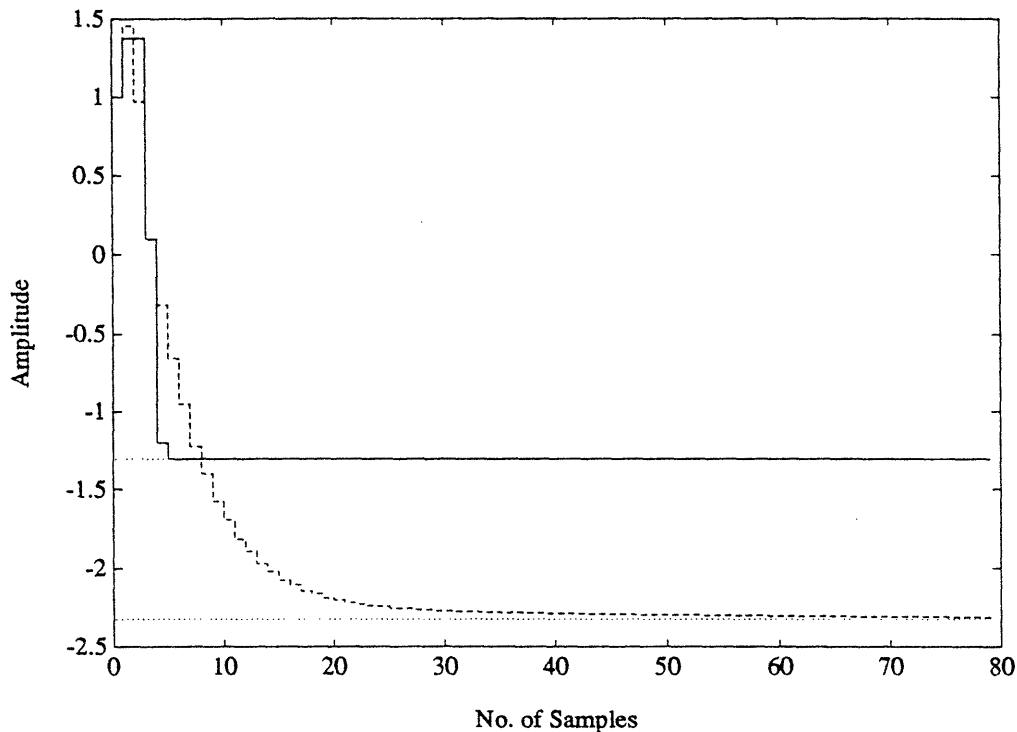


Figure 4: Weighted sensitivity step response for  $\ell_1$  design (full line) and  $\mathcal{H}_\infty$  design (dashed line)

- 4) A result was presented relating the support characteristics of the optimal and super-optimal solutions, followed by a stronger conjecture.

Several examples were worked out to illustrate the properties of the DA method. In particular, a multiblock problem corresponding to the X29 pitch axis control was solved. The operator norms and frequency domain properties of the solutions were compared with those of a standard  $\mathcal{H}_\infty$  design. Although the designs turned out to be quite similar, some differences were found at high frequencies.

As a final note, let us point out that there are still important open questions to be answered in connection with  $\ell_1$  optimization. From a theoretical point of view, stronger results regarding the support structure of the optimal solution are needed. In particular, a proof for the conjecture presented. As pointed out before, proving such conjecture could provide the insight to uncover the underlying finite dimensional structure that the general multiblock problem may have. Also, the existence in general of optimal rational solutions is an interesting open question connected to the above.

Finally, a model reduction theory in the context of  $\ell_1$  optimization would be of significant practical value. Recall that multiblock as well as one-block problems may have high order optimal controllers (depending on the interpolation data). A straightforward approach to computing lower order suboptimal controllers results from restricting the appropriate entries of  $\Phi$  to have fixed finite support. But such approach may be far from optimal. Therefore,

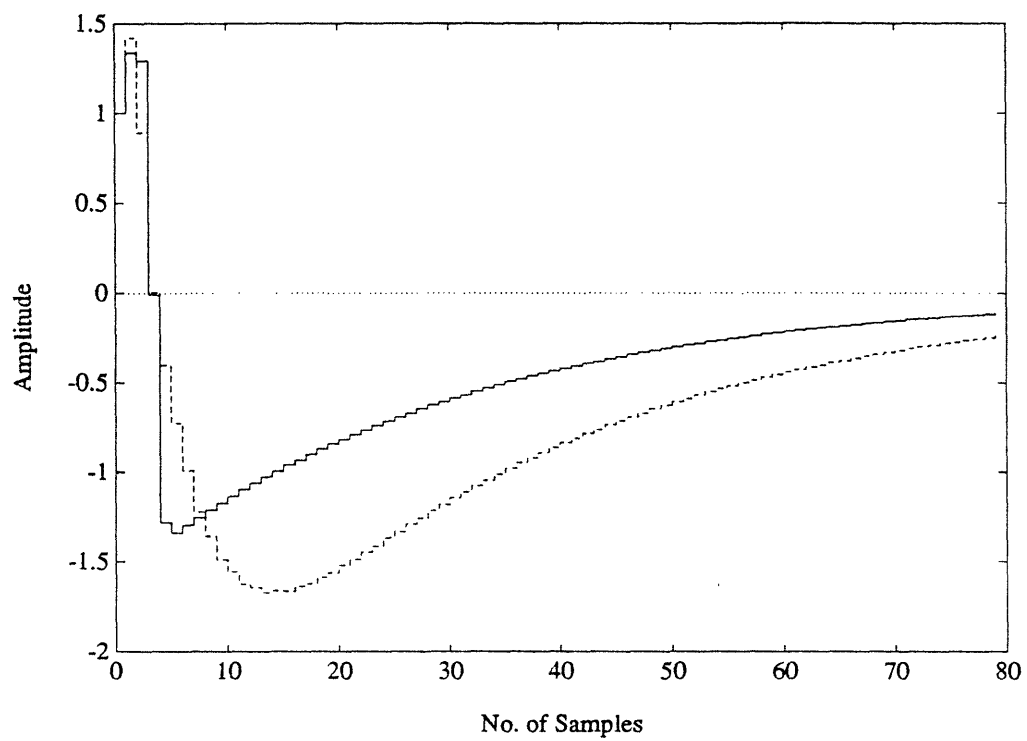


Figure 5: Sensitivity step response from  $w$  to  $y$  for  $\ell_1$  design (full line) and  $\mathcal{H}_\infty$  design (dashed line)

optimal model reduction techniques would be useful in practical design.

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