ANALYTIC SURGERY
AND
ANALYTIC TORSION

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Abstract

Let (M, h) be a compact manifold in which H is an embedded hypersurface which separates M into two pieces M_+ and M_-. If h is a metric on M and x is a defining function for H consider the family of metrics

$$g_\epsilon = \frac{dx^2}{x^2 + \epsilon^2} + h$$

where $\epsilon > 0$ is a parameter. The limiting metric, g_0, is an exact b-metric on the disjoint union $\overline{M} = M_+ \cup M_-$, i.e. it gives $M_\pm$ asymptotically cylindrical ends with cross-section H. We investigate the behaviour of the analytic torsion of the Laplacian on forms with values in a flat bundle, with respect to the family of metrics g_ϵ. We find a surgery formula for the analytic torsion in terms of the 'b'-analytic torsion on $M_\pm$. By comparing this to the surgery formula for Reidemeister torsion, we obtain a new proof of the Cheeger-Müller theorem asserting the equality of analytic and Reidemeister torsion for closed manifolds, and compute the difference between b-analytic and Reidemeister torsion on manifolds with cylindrical ends. We also present a glueing formula for the eta invariant of the Dirac operator on an odd dimensional spin manifold M. This generalizes a result of Mazzeo and Melrose, who obtained a similar glueing formula under the assumption that the induced Dirac operator $\eth_H$ on H is invertible. In both cases there is an 'extra' term in the glueing formula coming from the long time asymptotics of the heat kernel. The term can be expressed in terms of a one dimensional Laplacian associated to the null space of the Laplacian on $\overline{M}$. This operator is determined by scattering data on $\overline{M}$ at zero energy, and controls the leading behaviour of small eigenvalues as $\epsilon \downarrow 0$.

Thesis Supervisor: Richard Melrose
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To my parents, Jenny and Cleve

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This work is dedicated to my parents in appreciation for their constant love and encouragement.

DECLARATION

Some of this thesis is joint work. The single and double logarithmic spaces were developed jointly with Richard Melrose and Rafe Mazzeo. Much of chapters 2, 3 and 4 is due in part to Richard Melrose, and several other ideas in the thesis were influenced or suggested by him. It is intended to publish chapters one through nine in a joint paper with Mazzeo and Melrose. Otherwise, except where noted in the text, the work described here is my own.

CONTENTS

Chapter 1. Introduction

1.1. Analytic surgery.

In this thesis, we continue the study of analytic surgery initiated in [15]. By "analytic surgery" we mean a singular deformation of a Riemannian metric on a closed manifold M that models the cutting of M along a hypersurface H (possibly disconnected), forming a manifold with boundary $\overline{M}$ ("surgery"). For simplicity we assume that H separates M; thus $\overline{M}$ is the disjoint union of two manifolds with boundary $M_{\pm}$. We consider a specific deformation which degenerates to a complete metric on $\overline{M}$ of the form $dx^2/x^2 + h$, where x is a boundary defining function for H (that is, $x \geq 0$, $H = \{x = 0\}$ and $dx \neq 0$ on H) and h is a smooth metric on M. This form of metric on a manifold with boundary, called an "exact b-metric" and studied in some detail in [17], gives $\overline{M}$ asymptotically cylindrical ends, with $\log x$ approximately arc length along the end. Specifically, we consider a family of the form

$$g_\epsilon = \frac{dx^2}{x^2 + \epsilon^2} + h;$$

this is a smooth metric on M for every $\epsilon > 0$, which develops a long neck of length $2 \log 1/\epsilon + O(1) \to \infty$ as $\epsilon \to 0$ and whose singular limit is manifestly an exact b-metric on $\overline{M}$.

Similar deformations, usually phrased in terms of a family of manifolds which have a long cylindrical neck across H with length $l \to \infty$, have been studied by several authors. There are two main reasons for interest in this procedure. One is to understand the behaviour of geometric or topological invariants such as the index of a Dirac operator, eta invariant or analytic torsion under surgery, as in [4], [6], [10], and [8]. The other is to analyse the behaviour of the spectrum of operators (such as the Laplacian) under the transition from closed manifold to complete manifold. In Mazzeo and Melrose's paper [15] and the present thesis, both questions are investigated. Of course these two problems are closely related. In this thesis glueing formulae for the eta invariant and analytic torsion under surgery are presented, but these are obtained by studying the full resolvent family of generalized Laplacians under surgery, including the analysis of accumulation of eigenvalues at the bottom of the continuous spectrum of Δ_0.

Closely related are the papers of McDonald [16] and Seeley and Singer [28], who studied metric degeneration to incomplete conic metrics, and Ji [13], who studied degeneration of Riemann surfaces to surfaces with hyperbolic cusps. It should be remarked that the approach of McDonald inspired [15] and the present work.

There are two motivations for the choice of a "cylindrical ends" metric for $\overline{M}$. One is that Atiyah, Patodi and Singer obtained their well-known global boundary condition for the Dirac operator on a manifold with boundary in [1] heuristically by considering a cylindrical end attached to the boundary. The other is that Richard Melrose has presented a detailed analysis of the Laplacian associated to an exact b-metric in [17], as well as a proof of the APS index theorem in the 'b' context. It would also be interesting to study metric degeneration to other types of complete metrics, such as metrics with asymptotically hyperbolic or Euclidean ends.

1.2. Eta invariant and analytic torsion.

The eta invariant was introduced by Atiyah-Patodi-Singer in [1] as the boundary term in the index formula for the Dirac operator on a manifold with boundary. (For a discussion of Dirac operators see [17], [2] or [14].) It is a spectral invariant, given by the analytic continuation of the eta function $\eta(s) = \sum_{\lambda\neq 0 \in \text{spec}\,\eth} \operatorname{sgn}\lambda\,|\lambda|^{-s}$ to $s=0$. Alternatively, it has a formula in terms of the heat kernel:

$$\eta(\eth) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{\frac{1}{2}}\operatorname{Tr}\left(\eth e^{-t\eth^2}\right)\frac{dt}{t}; \tag{1.1}$$

this is the formula we will exploit here. In order to generalize index formulae, it is of interest to extend the definition of eta to operators which have continuous spectrum. For manifolds with boundary and a b-metric, the heat kernel is no longer trace class but the b-eta invariant was defined in [17] as in (1.1) with Tr replaced by the 'b-Trace' (see section 2.3); it is a natural regularization of the integral. This thesis is intended to illustrate the relation between these two definitions, and to throw light on the index theorem for a manifold with corners carrying a b-metric.

The analytic torsion T is an invariant of a flat bundle E over a Riemannian manifold M introduced by Ray and Singer. It is defined by formal analogy with a formula for Reidemeister torsion, or R-torsion, denoted τ, on a simplicial complex. It is given by

$$\log T(M^n, g, E) = \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}q\log\det\Delta'_q \equiv \frac{1}{2}\sum_{q=0}^{n}(-1)^q q\left(\frac{d}{ds}\zeta_q\right)(0),$$

where ζ_q is the zeta function for Δ'_q, the Laplacian on q-forms with values in E, projected off the zero eigenspace:

$$\zeta_q(s) = \sum_{\substack{\lambda\in\text{spec}\,\Delta_q\\ \lambda\neq 0}} \lambda^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^s \operatorname{Tr} e^{-t\Delta'_q}\frac{dt}{t},\ \operatorname{Re} s > \frac{n}{2}. \tag{1.2}$$

Since for finite rank operators $\zeta'(0) = -\sum\log\lambda$, this is a natural regularization of the log determinant of an operator. Ray and Singer showed that $T(M,g)$ has the same formal properties as R-torsion and conjectured that these two torsions are equal. This was proved several years later by Cheeger and Müller independently in [7] and [22]. In recent years, several more proofs and generalizations of this result have appeared. Vishik [29], [30] has established the relationship between analytic torsion, defined using classical boundary conditions, and R-torsion on manifolds with corners. Burghelea-Friedlander-Kappeler [5] obtained a new proof using Witten's deformation of the de Rham complex via a Morse function. Müller in [23] extended the result to bundles E assuming only that the determinant bundle det E

is flat. Bismut-Zhang [3] proved a further generalization when the bundle E is not necessarily flat; the difference $\log(T/\tau)$ is given by the integral of a local 'anomaly'. It is interesting to compare Vishik's formula for T/τ on a manifold with boundary with Corollary 1.7; in both cases the ratio depends on the Euler characteristic of the boundary.

As with the eta invariant, one would like to extend the definition of the log determinant to operators which have continuous spectrum. We do this for the Laplacian on forms, as in [17, chapter 9], by defining the b-ζ function, replacing Tr with b-Tr in (1.2). Then the b-analytic torsion is defined using these regularized log determinants. In this thesis, we study the questions: how is the b-analytic torsion related to the torsion on a closed manifold? How is b-analytic torsion related to Reidemeister torsion? Most of the proofs of the equality of T and τ are indirect: one establishes that T and τ have the same glueing formula under surgery and uses this to compare the two torsions of an arbitrary manifold to the sphere S^n for which the result is known. Here we derive a glueing formula for T under *analytic* surgery, which by comparing to the analogous formula for R-torsion, enables us to compute the ratio of R-torsion and b-analytic torsion on manifolds with boundary.

1.3. Statement of Results.

Let us first recall the main results from Mazzeo and Melrose's paper [15], henceforth referred to as 'Mazzeo-Melrose'. There the same situation, for the eta invariant, was studied, under the assumption that the induced Dirac operator on H is invertible. A "surgery double space" and "surgery heat space" were introduced as spaces to carry the Schwartz kernels of $(\Delta - \lambda^2)^{-1}$ and $e^{-t\Delta_\epsilon}$ uniformly as $\epsilon \to 0$. They are blown up versions of $M^2 \times [0, \epsilon_0]_\epsilon$ and $M^2 \times [0, \infty]_t \times [0, \epsilon_0]_\epsilon$ which resolve the singularities of the space of vector fields $\mathcal{V}_s$ associated with the family g_ϵ. The resolvent and heat kernel were shown to be polyhomogeneous conormal half densities on these spaces and their leading asymptotics (model operators) at $\epsilon = 0$ were identified. It was shown that the projector Π_ϵ onto eigenfunctions with eigenvalue going to zero with ϵ is a finite rank, smoothing operator. They showed:

THEOREM. *(Mazzeo-Melrose) Let $\tilde{\eta}(\epsilon)$ be the signature of Π_ϵ. If the induced Dirac operator on H is invertible, then the eta invariant of $\eth_\epsilon$ satisfies*

$$\eta(\eth_\epsilon) - \tilde{\eta}(\epsilon) = \eta_b(\eth_{M_+}) + \eta_b(\eth_{M_-}) + \epsilon r_1(\epsilon) + \epsilon \log \epsilon\, r_2(\epsilon),$$

with $r_i \in \mathcal{C}^\infty([0, \epsilon_0]_\epsilon)$.

Douglas and Wojciechowski obtain a similar result in [10].

Unfortunately the assumption on invertibility of the operator at H excludes many interesting cases, including the surgery limit of analytic torsion when $H^*(H, E)$ does not vanish. A principal goal of this thesis is to extend to machinery of Mazzeo-Melrose to deal with the case when the operator on H has null space. To do this the constructions in Mazzeo-Melrose must be modified. One reason for this is that when Δ_H has null space the heat kernel no longer has uniform exponential decay, as it does (up to finite rank) in [15]. One must understand the leading behaviour of the heat kernel as $t \to \infty$ to calculate the integrals (1.1) and (1.2), which amounts to understanding the leading behaviour of the resolvent as $\lambda \to 0$. This means, in contrast with Mazzeo-Melrose, that one must understand the resolvent down to the bottom of the continuous spectrum of Δ_0, and in particular, the leading behaviour of small eigenvalues, that is, those going to zero with ϵ. To cope with this we perform further blowups on the spaces of Mazzeo-Melrose, to resolve singularities in the kernel that form as we approach $\lambda = \epsilon = 0$. We introduce the "logarithmic double space" X^2_{Ls} and "logarithmic heat space" X^2_{LHs}; the names are due to the application of "logarithmic blowup" (see section 2.5) to each face of the surgery double space. Then we get analogues of the results above. Introduce the function $\operatorname{ias}\epsilon = 1/\sinh^{-1}(1/\epsilon)$ ("inverse arc-sinh"), which is the reciprocal of the growth rate of the volume of M. It goes to zero with ϵ, but only logarithmically. For the resolvent we scale λ by writing $\lambda = (\operatorname{ias}\epsilon)z$ to capture the behaviour of small eigenvalues. Thus when $\epsilon = 0$, $\lambda = 0$ for all z.

THEOREM 1.1. *The resolvent* $\left(\Delta - (\operatorname{ias}\epsilon)^2 z^2\right)^{-1}$ *is a meromorphic family of conormal half densities on* $X^2_{\mathrm{Ls}} \times \mathbb{C}$, $(\operatorname{ias}\epsilon)^{-1}\times$ *smooth up to the boundary of* X^2_{Ls}. *The poles* $z(\epsilon)$ *satisfy* $\lim_{\epsilon\to 0} z(\epsilon) = 0$ *or* z_j, *where* $\{z_j^2\}$ *are the eigenvalues of a one dimensional Laplacian* $\mathrm{RN}(\Delta)$ *on* $[-1,1]$ *with boundary conditions determined by scattering data on* $\overline{M}$.

We prove this in chapter 6. Indeed we extend the result in chapter 7 to the full resolvent which includes the resolvent away from the spectrum at $\epsilon = 0$ as in Mazzeo-Melrose. From this we get the heat kernel via a contour integral. With Π_ϵ now denoting the (finite rank) projection onto eigenfunctions with eigenvalue $\lambda^2(\epsilon) = o((\operatorname{ias}\epsilon)^2)$, we have

THEOREM 1.2. *On* X^2_{LHs} *the heat kernel projected off zero modes,* $e^{-t\Delta_\epsilon} - \Pi_\epsilon$ *is* $t^{-n/2}$ *times a smooth half-density for* t *near zero and is smooth up to* $t = \infty$. Π_ϵ *itself is smooth except possibly up to* $t = \infty$.

In both cases we know the top terms at each boundary face. In principle one can calculate the Taylor series at every face at $\epsilon = 0$ to arbitrary order.

From this we read off the behaviour of the eta invariant.

THEOREM 1.3. *Let* $\eta_{\mathrm{fd}}(\epsilon)$ *be the signature of* Π_ϵ. *Then*

$$\eta(\eth_\epsilon) - \eta_{\mathrm{fd}}(\epsilon) = \eta_b(\eth_{M_+}) + \eta_b(\eth_{M_-}) + \eta(\mathrm{RN}(\eth)) + (\operatorname{ias}\epsilon)r(\operatorname{ias}\epsilon),$$

where r *is smooth.*

Thus we get, in comparison with the case when $\eth_H$ is invertible, an extra contribution $\eta(\mathrm{RN}(\eth))$ coming from the small eigenvalues.

For analytic torsion, we measure torsion relative to a fixed set of cohomology classes. If μ^i is an orthonormal basis of the surgery Hodge cohomology group $H^i_{s\text{-}\mathrm{Ho}}(M)$ defined in chapter 10, then we get

THEOREM 1.4. *$T(M,\mu^i)$ is given by*

$$\log T(M,\mu^i) = \log {}^bT(M_+,g_0) + \log {}^bT(M_-,g_0) + \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1} q \log\det \mathrm{RN}(\Delta_q).$$

Comparing to the surgery formula for R-torsion, we get

THEOREM 1.5. *The difference $\log T - \log\tau$ obeys the surgery formula*

$$\log\frac{T(M,g_\epsilon)}{\tau(M,g_\epsilon)} = \log\frac{{}^bT(M_+,g_0)}{\tau(M_+,g_0)} + \log\frac{{}^bT(M_-,g_0)}{\tau(M_-,g_0)} + \frac{1}{2}\chi_E(H)\log 2.$$

Applying Cheeger's argument from [7], we obtain

COROLLARY 1.6. *(Cheeger-Müller Theorem) For a closed manifold with flat unitary bundle E and metric g,*

$$T(M,g) = \tau(M,g).$$

COROLLARY 1.7. *For a manifold with boundary N, with flat unitary bundle E and exact b-metric g, we have*

$${}^bT(N,g) = 2^{-\chi_E(\partial N)/4}\tau(N,g).$$

1.4. Outline of the proof.

We may take as a starting point for the considerations in this thesis the fact that the eigenfunctions whose eigenvalues go to zero under surgery are *not* smooth on the "single space" X_s of Mazzeo-Melrose; indeed they are not continuous. One can easily see this by looking at the example of surgery on an interval, or a circle. Then the eigenfunctions of Δ_ϵ look like $e^{2\pi i k r/L_\epsilon}$, where $r = \sinh^{-1}(x/\epsilon)$ is arclength and $2L_\epsilon$ = length of g_ϵ. Then on the surgery space $X_s = [M\times[0,\epsilon_0]_\epsilon; H\times\{0\}]$ of Mazzeo-Melrose, this eigenfunction is equal to 1 on the surgery front face B_{ss} and $(-1)^k$ on the b-boundary B_{bb}. The oscillations disappear into the corner $B_{ss}\cap B_{bb}$. To rectify this situation we replace, in chapter 3, the single space by a new space, X_{Ls}, on which the scaled distance r/L_ϵ is a smooth function. X_{Ls} is a blown up version of X_s involving the operation of logarithmic blowup described in section 2.5. We then modify, in a methodical way, all constructions in Mazzeo-Melrose to reflect this change. The Lie Algebra $\mathcal{V}_s$ of Mazzeo-Melrose is lifted to $\mathcal{V}_{Ls}$ on X_{Ls}; properties of $\mathcal{V}_{Ls}$, including its normal operators, are discussed in chapter 3.

We microlocalize the Lie Algebra $\mathcal{V}_{Ls}$ according to the general principles set forth in [20]. This means we need the following things. First, we need a double space X^2_{Ls} to carry Schwartz kernels of "logarithmic surgery pseudodifferential operators", or Ls-ψdos, with diagonal submanifold Δ_{Ls} such that kernels of $\mathcal{V}_{Ls}$-differential operators are given precisely by all distributions on X^2_{Ls} supported on Δ_{Ls} with polynomial symbols. By replacing polynomial symbols with arbitrary classical symbols, we obtain the "small calculus". Second, the double space should have a natural map down to the single space X_{Ls} which is a b-fibration (see chapter 2) so that kernels can act on distributions on X_{Ls}. Third, there should be a triple space X^3_{Ls} with a *b*-fibration down to X^2_{Ls} so that composition of Ls-ψdos can be defined. Finally, the normal operators on $\mathcal{V}_{\mathrm{Ls}}$ should extend to "model operator maps" on Ls-ψdos given by restriction of the kernel to faces at $\epsilon = 0$. Geometric lemmas to help construct these spaces are given in the second half of chapter 2 and the surgery pseudodifferential calculus is set up in chapter 4.

In view of the Pushforward theorem of [19], discussed here in section 2.3, the fact that we have *b*-fibrations $X^3_{\mathrm{Ls}} \to X^2_{\mathrm{Ls}} \to X_{\mathrm{Ls}}$ means that we can work in the class of polyhomogeneous conormal functions throughout. This has the virtue of almost eliminating estimates from the thesis, as we easily read off the decay rate of functions from the index set specifying its (poly)homogeneities. This comes at the cost of fairly complicated spaces and geometric machinery but we are able to obtain detailed information about the resolvent, heat kernel and small eigenvalues with this method. Because of the logarithmic blowups we are able to work with "natural" index sets (see section 4.7) and show that our final objects — resolvent, heat kernel, eta invariant and analytic torsion — are smooth in the blow-up coordinates.

In chapter 5 we analyse the model problem coming from the reduced normal operator in chapter 3. This is a "new" model operator, not appearing in Mazzeo-Melrose, and Proposition 3.8 indicates that the eigenvalues of this model control the leading behaviour of small eigenvalues of Δ_ϵ. We show that this model is a nice half density on the double space; this indicates that X^2_{Ls} is the "correct" space to use, and we use the model heavily in chapter 6 in the construction for the general resolvent.

In chapters 6 and 7 we make a start on the problem of understanding $\left(\Delta - \lambda^2\right)^{-1}$ when λ approaches the spectrum. We construct the resolvent near the bottom of the continuous spectrum, 0. We blow up at $\lambda = 0$, introducing the rescaled parameter $z = \lambda \sinh^{-1}(1/\epsilon)$ which captures the scaling of small eigenvalues. To construct the parametrix we need to solve not only the symbol at the diagonal singularity but also solve a finite number of model problems at each boundary face. Compatibility conditions at the intersections of faces give boundary conditions for these model operators, which enable them to be solved uniquely; the interaction between the models on various faces and of different orders is fairly complicated.

In principle, once we have the resolvent we can construct any function of the Laplacian by functional calculus. We construct the heat kernel in this way in chapter 8. More precisely, we obtain it by performing the contour integral

$$e^{-t\Delta_\epsilon} = \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda^2}\left(\Delta - \lambda^2\right)^{-1} 2\lambda d\lambda. \tag{1.3}$$

Then we obtain the eta invariant and analytic torsion by performing the integrals (1.1) and (1.2). This integral is really a pushforward, since the integrand lives on a blown-up version of its space of parameters. We construct spaces so that the integral becomes a pushforward under a *b*-fibration. This allows us to conclude that the result is polyhomogeneous, and an extra argument shows that it is actually smooth. We compute the leading terms of the heat kernel at $t = \infty$.

In chapters 9, 10 and 11 we then make three applications of this machinery. The first is a surgery formula for the eta invariant, Theorem 1.3. The second, in chapter 10, is a Hodge version of the Mayer-Vietoris sequence in cohomology for the triple (M, M_+, M_-); we also obtain a spectral gap result between the eigenforms corresponding to cohomology on M and the other eigenforms. In chapter 11 we compute the surgery limit of analytic torsion and R-torsion (using results of chapter 10) and compare them, obtaining theorems 4 and 5. The first corollary follow then follows from Cheeger's well known approach [7] to proving the equality of analytic and R-torsion, and the second corollary follows immediately from the first.

Chapter 2. Manifolds with corners, blowups and *b*-fibrations

Here we present material on the geometry of manifolds with corners that we need in this thesis. We will assume familiarity with [19], but we also recall a fair amount of this material in the first three subchapters, although sometimes with a different presentation. We then go on to describe two "new" blowup operations which we use heavily in the sequel, logarithmic blowup and total boundary blowup.

2.1. Manifolds with corners.

We refer to [19] for a discussion of manifolds with corners, *b*-tangent space, *b*-maps, and *b*-fibrations. Let us recall here that the set of boundary hypersurfaces of a manifold with corners X is denoted $M_1(X)$, and the set of proper faces, that is, all faces excluding X itself, is denoted $M'(X)$. A boundary defining function ρ for a boundary hypersurface H of a manifold with corners Y is a smooth nonnegative function on Y such that $\rho \equiv 0$ on H, $d\rho \neq 0$ on H. A smooth map between manifolds with corners $f\colon X \to Y$ is an (interior) *b*-map if for every boundary defining function ρ_H for Y,

$$f^*\rho_H = a \cdot \prod_{G \in M_1(X)} \rho_G^{e(G,H)} \tag{2.1}$$

for some nonzero smooth function a and (uniquely determined) collection of natural numbers $e_f(G,H)$, called the boundary exponents of f.

We will be especially concerned with special *b*-maps called *b*-fibrations; the map f above is a *b*-fibration if the map f_* on the *b*-tangent bundle is surjective on each fibre, and the image of each boundary hypersurface in X is either Y or one boundary hypersurface $H \subset Y$. (This definition is different from, but equivalent to, the definition given in [19].) *b*-fibrations have good mapping properties on $M'(X)$; the image of any face $F \in M'(X)$ is a face in $M'(Y)$, and $f_{\restriction F}$ is a *b*-fibration onto its image.

p-submanifolds There are various possible definitions of a submanifold of a manifold with corners; we will use a very strong definition. A subset $S \subset Y$ is a *p*-submanifold if locally, in some coordinate system $x'_1, \dots x'_{k'}, y'_1, \dots y'_{n'-k'}$, with $x'_i \geq 0$, $y'_j \in (-\delta, \delta)$, S is given by the vanishing of some of them: $S = \{x'_{i_1} = \dots x'_{i_{l'}} = y'_{j_1} = \dots = y'_{j_{m'}} = 0\}$. Then, if S is connected, S has a tubular neighbourhood which is a bundle over S, the fibre being a neighbourhood of $0 \in \mathbb{R}_+^{k'-l'} \times \mathbb{R}^{n'-k'-m'}$. We say S is an interior *p*-submanifold if $l' = 0$ in the definition above. If f is as above and S is an interior *p*-submanifold, then $f^{-1}S$ is a *p*-submanifold; if S is not interior, then in general $f^{-1}S$ is a union of *p*-submanifolds of X. This is clear in local coordinates. Choose coordinates in Y such that S has the form above; then it is possible to choose coordinates $x_1, \dots x_k, y_1, \dots y_{n-k}$, with $x_i \geq 0$, $y_j \in (-\delta, \delta)$ locally in X so that locally f has the form

$$f(x_1, \dots x_k, y_1, \dots y_{n-k}) = \Big(\prod_{r \in I_1} x_r, \cdots \prod_{r \in I_{k'}} x_r, y_1, \dots y_{n'-k'} \Big).$$

Then $f^{-1}S$ is a union of p-submanifolds $\{x_{r_1} = \cdots = x_{r_{k'}} = y_{j_1} = \ldots y_{j_{m'}} = 0\}$, with $r_i \in I_i$.

Degrees and density bundles We will find it convenient to use the notion of the "degree" of a boundary hypersurface H of a manifold with corners X. This is simply an assignment of an integer, $d(H)$, to H. Informally this is supposed to represent the order of growth of densities allowed at H. If each boundary hypersurface of X has been assigned a degree, the degree density bundle is defined by

$$\Omega_D(X) = \prod_{H \in M_1(X)} \rho_H^{-d(H)} \Omega_b(X) = \prod_{H \in M_1(X)} \rho_H^{-d(H)-1} \Omega(X),$$

where ρ_H denotes a boundary defining function for H. Observe that b-densities have the pleasant property that $d\rho_H/\rho_H$ is a canonical factor at H, so dividing by $|d\rho_H/\rho_H|$ gives a canonical restriction $C^\infty(\Omega_b(X)) \to C^\infty(\Omega_b(X))$. With general D-densities, restriction defined by division by $|d\rho_H/\rho_H^{d(H)+1}|$ depends on the choice of boundary defining function. However, in this thesis we will have a canonical total boundary defining function (product over $M_1(X)$ of boundary defining functions) R for many of our spaces. In this case division by $|R^{-d(H)} d\rho_H/\rho_H|$ gives a canonical restriction $C^\infty(\Omega_D(X)) \to C^\infty(\Omega_D(H))$, where the degrees of boundary hypersurfaces $H \cap K$ ($K \in M_1(X)$) of H are defined by $d(H \cap K) = d(K) - d(H)$.

2.2. Blowups.

If $S \subset Y$ is a p-submanifold , the blowup of Y at S, denoted $[Y; S]$, is a manifold with corners , given as a point set by $(Y \setminus S) \cup (\mathrm{SN}^+ S)$ and with C^∞ structure the unique minimal structure such that functions on Y lifted to S, *and* polar coordinates at S, are smooth. ($\mathrm{SN}^+ S$ is the inward-pointing spherical normal bundle to S.) There is a unique smooth map $[Y; S] \to Y$ extending the identity on $Y \setminus S$, called the blowdown map. The lift of a p-submanifold $T \subset Y$ to $[Y; S]$ is defined if (i) $T \subset S$, in which case the lift is defined to be the inverse image of T under the blowdown map or (ii) $T \setminus S$ is dense in T, in which case the lift is defined as the closure of $T \setminus S$ in $[Y; S]$.

We will often perform sequences of several blowups to create new spaces in this thesis, and it will be important to know when one can exchange the order of blowup. Here we present two easy results of this nature.

LEMMA 2.1. *If S, T are p-submanifolds of Y and either (i) S and T are transverse or (ii) $T \subset S$ then $[Y; S; T] = [Y; T; S]$.*

PROOF: The proof of (i) is immediate because there NS and NT are independent and so the blowups occur in two disjoint sets of variables.

To prove (ii), it is sufficient to consider the case $T = \{0\}$, $S = \mathbb{R}^l_+ \times \mathbb{R}^m$, $Y = \mathbb{R}^k_+ \times \mathbb{R}^n$, $k \geq l$, $n \geq m$. Define $R_T^2 = \sum_{i=1}^k x_i^2 + \sum_{j=1}^n y_j^2$ and $R_S = \sum_{i=l+1}^k x_i^2 + \sum_{j=m+1}^n y_j^2$. In both $[Y; S; T]$ and $[Y; T; S]$ the lift of T and S have boundary defining functions (the lift of) R_T and R_S respectively, and a superset of coordinates on the lift of T is given by $x_i/(x_i + R_T)$, $y_j/(y_j + R_T)$ for $i, j \geq 1$ and on the lift of S by $x_i/(x_i + R_S)$, $y_j/(y_j + R_S)$ for $i \geq l+1, j \geq m+1$ on both spaces. This means that the identity map on $Y \setminus S$ extends to a smooth map $[Y;S;T] \leftrightarrow [Y;T;S]$ in both directions. Each extension is therefore a canonical diffeomorphism. ∎

2.3. Operations on conormal functions.

The principal function spaces used in this thesis are spaces of polyhomogeneous conormal functions, conormal either at a boundary hypersurface, as described in [19], or an interior p-submanifold (eg, the diagonal). For such spaces we have the Pullback and Pushforward theorems, allowing us to pull back and integrate whilst preserving polyhomogeneity. These theorems are discussed in [19]; here we recast them in the language of D-densities and also discuss the interior p-submanifold case. Let $f: X \to Y$ be a b-fibration between manifolds with corners with degrees and let $\operatorname{ex}(G) = d(G) - d(H)$ for $G \in M_1(X)$ if $f(G) = H \in M_1(Y)$, $\operatorname{ex}(G) = d(G)$ if $f(G) = Y$.

THEOREM 2.2. *(Pullback theorem) f induces a pullback map on functions*

$$f^*: \mathcal{A}^{\mathcal{J}}_{\mathrm{phg}}(Y) \to \mathcal{A}^{f^{\#}\mathcal{J}}_{\mathrm{phg}}(X)$$

where $f^{\#}(\mathcal{J})(G) = 0$ if $f(G) = Y$, $f^{\#}(\mathcal{J})(G) = \{(e(G,H)z + q, k) \mid (z,k) \in \mathcal{J}(H), q \in \mathbb{N}\}$ if $f(G) = H$.

THEOREM 2.3. *(Pushforward theorem) If $\operatorname{Re}\mathcal{K}(G) > d(G)$ for all G such that $f(G) = Y$, then the pushforward by f, that is, integration, of smooth compactly supported densities extends to a map*

$$f_{\#}: \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X;\Omega_D(X)) \to \mathcal{A}^{f_{\#}(\mathcal{K}-\mathrm{ex})}_{\mathrm{phg}}(Y;\Omega_D(Y))$$

where $f_{\#}(\mathcal{K})(H) = \{\ (z,k) \mid \exists G_1 \dots G_k$ mapping to H and $p_1 \dots p_k$ such that $(z/e(G_i,H), p_1) \in \mathcal{K}(G_i)$ and $p = p_1 + \dots + p_k + (k-1)\}$.

The coefficients of the pullback f^*h at a boundary defining function G in the first theorem are given by pullbacks of restrictions of h to $f(G)$. Under pushforward, if the inverse image of $H \in M_1(Y)$ is just one boundary hypersurface G then the coefficients of f_*u are the pushforwards of the restrictions of u to G under $f_{\restriction G}$, using consistent boundary defining functions on X and Y to restrict. If the inverse image of H is more than one boundary hypersurface of X then it is a messy business to specify in general how the coefficients of u and f_*u are related. Let us give a example to illustrate a fairly simple case, and which will be sufficient for the computations in this thesis.

EXAMPLE 2.4. Let $\mathcal{K}$ be the $\mathcal{C}^\infty$ index family for X, let $f: X \to Y$ be a b-fibration and suppose $H \in M_1(Y)$ and exactly two boundary hypersurfaces G_1, G_2 map to H, with $e_f(G_i, H) = 1$ and $\operatorname{ex}(G_i) = 0$. Let $u \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X;\Omega_D(X))$. The index set for f_*u at H is $\{(n,0),(n,1) \mid n \in \mathbb{N}\}$. Let ρ be a boundary defining function for the interior of H and r_1, r_2 boundary defining functions for G_i such that $f^*\rho = r_1 r_2$

near int $G_1 \cap G_2$. Use these boundary defining functions to restrict densities to these boundaries. Then

$$u \sim \Big(\log\rho\, a_{0,1} + a_{0,0} + O(\rho\log\rho)\Big)\Big|\frac{d\rho}{\rho^{d(H)+1}}\Big|$$

where $a_{0,1} = (f_{\restriction G_1\cap G_2})_*(u_{\restriction G_1\cap G_2})$ and, with $\psi(x)$ a cutoff function near 0, equal to 1 near 0 and vanishing for $x > 1$,

$$a_{0,0} = \lim_{\delta\to 0}\big[(f_{\restriction G_1})_*((1-\psi)(\tfrac{r_2}{\delta})\ u_{\restriction G_1}) + (f_{\restriction G_2})_*((1-\psi)(\tfrac{r_1}{\delta})\ u_{\restriction G_2}) + 2\log\delta\cdot a_{0,1}\big].$$

These pushforwards are regularized integrals; the pushforward of $u_{\restriction G_i}$ does not exist because u is not integrable up to the boundary.

More generally, if $f\colon X \to Y$ is a *b*-fibration, G is a boundary hypersurface of X with boundary defining function ρ, and u is a *b*-density on X which is integrable at all boundary hypersurfaces except possibly G, then the limit

$$\lim_{\delta\downarrow 0} f_\#\big((1-\chi_{[0,\delta]})(\rho)\,u\big) - \log\delta(f_{\restriction G})_\#(u_{\restriction G})$$

exists and is denoted $b\text{-}\int u$, or $b_\rho\text{-}\int u$ if the dependence on ρ needs emphasizing. The regularized integral depends on ρ through the section $d\rho$ of N^*G; we have

$$b_{\rho_1}\text{-}\int u - b_{\rho_2}\text{-}\int u = (f_{\restriction G})_\#(u_{\restriction G}\cdot\log(\frac{d\rho_1}{d\rho_2})).$$

If the integral is formally computing the trace of an operator, then the regularized integral is denoted $b\text{-Tr}_{d\rho}$. In the case of the regularized integral defining the *b*-eta invariant on $\eth_{M_\pm}$, the integrand at the boundary vanishes pointwise after taking the pointwise trace, and so the *b*-eta invariant is independent of the choice of boundary defining function. For the *b*-zeta function, the integrand at the boundary is constant in t, after taking pointwise trace and summing in q, and this implies that the *b*-zeta function itself is also independent of the choice of boundary defining function. Hence both these quantities are completely well defined.

The pushforward theorem also holds if we allow our densities on X to have interior singularities along a *p*-submanifold S transverse to all boundary hypersurfaces of X and to f. We denote such a space $I^m\mathcal{A}^{\mathcal{K}}_{\text{phg}}(X;\Omega_D(X);S)$ if the conormal order at S is m. To see why this is true note first that by a standard result about wavefront sets (see [12] for example) f_*u is smooth in the interior of Y. At the boundary, consider the proof of the pushforward theorem in [19]. This involves killing off terms in the asymptotic expansion of u at boundary hypersurfaces of X using test differential operators $\mathcal{B}(\mathcal{K},\mathfrak{s})$. As S is transverse to all $H \in M_1(X)$ any normal vector field r_H is f-related to a normal vector field r_G tangent to S. Then applying $\mathcal{B}(\mathcal{K},\mathfrak{s})$ kills top terms in the asymptotic expansion at G while preserving the order of conormality at S. This shows the image space is the same as in Theorem 2.3.

2.4. Two Blowup Lemmas.

The importance of b-fibrations has been discussed in section 1.4. In modifying spaces of Mazzeo-Melrose we often face the situation where we have a b-fibration $f: X \to Y$ and we perform operations (eg blowups) on X and Y, obtaining new spaces $\hat{X}$, $\hat{Y}$. We would like f to lift to a b-fibration $\hat{f}: \hat{X} \to \hat{Y}$. The first and second lemmas below show when one can regain a b-fibration when a submanifold is blown up in Y or X respectively. These results are due to Richard Melrose; I thank him for allowing them to appear here.

LEMMA 2.5. *Let $f: X \longrightarrow Y$ be a b-fibration between compact manifolds with corners and suppose that $T \subset Y$ is a closed p-submanifold such that for each boundary hypersurface $H \subset Y$ intersecting T, and each $G \in M_1(X)$, either $e_f(G,H) = 0$ or $e_f(G,H) = 1$. Then, with $\mathcal{S}$ the minimal collection of p-submanifolds of X into which the lift of T under f decomposes, f extends from the complement of $f^{-1}(T)$ to a b-fibration*

$$f_T: [X, \mathcal{S}] \longrightarrow [Y, T] \tag{2.2}$$

for any order of blow up of the elements of $\mathcal{S}$.

PROOF: The result is local in nature, so we may restrict attention neighbourhoods of $q \in T$ and $p \in X$ such that $f(p) = q$. Choose coordinates $x'_1, \dots, x'_{k'}, y'_1, \dots, y'_{n'-k'}$ near q, such that $q = (0, \dots, 0)$, the x'_i are nonnegative, the y'_j take values in $(-\epsilon, \epsilon)$ and in terms of which $T = \{x'_1 = \cdots = x'_l = y'_1 = \cdots = y'_m = 0\}$. Because of the assumption on the boundary exponents of f, it is possible to choose coordinates $x_1, \dots, x_k, y_1, \dots, y_{n-k}$ near $p \in X$ so that

$$\begin{gathered} f^* x'_i = \prod_{r \in I_i} x_r, \ 1 \le i \le l \text{ and } f^* y'_j = y_j, \ 1 \le j \le n' - k' \\ \text{with the } I_i \subset \{1 \dots k\} \text{ nonempty and disjoint.} \end{gathered} \tag{2.3}$$

Since f is a b-fibration, necessarily $k' \ge k$ and $n' - k' \le n - k$.

In these coordinates

$$f^{-1}(T) = \left\{ \prod_{r \in I_i} x_r = 0, \ 1 \le i \le l \text{ and }, \ y_j = 0, 1 \le j \le m \right\}.$$

Thus an element of $\mathcal{S}$, the collection of p-submanifolds into which $f^{-1}(T)$ decomposes, is determined by the choice of an index from each of the I_i. Choice of an ordering $S_1, \dots, S_N$ of the elements of $\mathcal{S}$ gives

$$S_k = \{x_{k_1} = \cdots = x_{k_l} = y_1 = \cdots = y_m = 0\}$$

with $k_i \in I_i$. Thus N is the product over i of the number of elements in I_i and for each k with $1 \le k \le N$, k_i is the unique element of I_i such that x_{k_i} vanishes on S_k.

Consider the action of blowing up S_1. This replaces S_1 by its inward pointing spherical normal bundle. The function $R_1 = x_{1_1} + \cdots + x_{1_l} + (y_1^2 + \cdots + y_m^2)^{1/2}$ defines the new boundary hypersurface so introduced. Consider the functions

$$x_i^{(1)} = \begin{cases} \frac{x_i}{R_1} \text{ if } i = 1_j \text{ for some } j \\ x_i \text{ otherwise} \end{cases}$$
$$y_j^{(1)} = \begin{cases} \frac{y_j}{R_1} \text{ if } 1 \leq i \leq m \\ y_j \text{ otherwise} \end{cases}$$

Observe that $x_{1_1}^{(1)} + \cdots + x_{1_l}^{(1)} + ((y_1^{(1)})^2 + (y_m^{(1)})^2)^{1/2} = 1$ and that $dx_{1_i}^{(1)} \neq 0$ unless $x_{1_i}^{(1)} = 1$ and similarly, $dy_j^{(1)} \neq 0$ unless $y_j^{(1)} = \pm 1$ if $1 \leq j \leq m$. Away from the front face of $[X; S_1]$ nothing has changed and near each point of this new boundary hypersurface R_1 and some $n-1$ of the n functions $x_i^{(1)}$, $y_j^{(1)}$ form a coordinate system, the one function excluded being non-zero. The lifts to $[X, S_1]$ of the submanifolds S_k, $k \geq 2$ are therefore given by the vanishing of the functions $x_{k_1}^{(1)}, \ldots, x_{k_l}^{(1)}, y_1^{(1)}, \ldots, y_m^{(1)}$, which being zero must be amongst the coordinates at each point of the lifted submanifold.

Thus, after the first blowup the combinatorial arrangement is as before, with one less submanifold S_k. We can therefore proceed to blow up $S_2, S_3, \ldots, S_N$ and define successive functions

$$R_k = x_{(k_1)}^{(k-1)} + \cdots + x_{k_l}^{(k-1)} + \left((y_1^2)^2 + \cdots + (y_m)^2\right)^2$$

$$x_i^{(k)} = \begin{cases} \frac{x_i}{R_k} \text{ if } i = k_j \text{ for some } j \\ x_i \text{ otherwise} \end{cases}$$
$$y_j^{(1)} = \begin{cases} \frac{y_j}{R_1} \text{ if } 1 \leq i \leq m \\ y_j \text{ otherwise.} \end{cases}$$

Then the R_k for $k = 1, \ldots, l$ are defining functions for the blown up surfaces.

Consider the map

$$f_T'[X, \mathcal{S}] \longrightarrow Y,\ f_T' = \beta \circ f \tag{2.4}$$

where $\beta_X \colon [X, \mathcal{S}] \longrightarrow X$ is the total blowdown map. The coordinates pull back to be of the form

$$(f_T')^* x_i' = \prod_{r \in I_i} \beta_X^* x_r = \prod_{r \in I_i} [x_r^{(N)} \prod_{k \text{ s.t. } r = k_i} R_k] = R_1 \ldots R_N \prod_{r \in I_i} x_r^{(N)}$$
$$(f_T')^* y_j' = R_1 R_2 \cdots R_N y_j^N.$$

Thus, $R' = x_1' + \cdots + x_l' + (y'^2_1 + \cdots + y'^2_m)^{1/2}$ lifts to

$$R_1 \cdots R_N \left(\sum_{i=1}^{l} \left(\prod_{r \in I_i} x_r^{(N)} \right) + \left(\sum_{j=1}^{m} (y_j^{(N)})^2 \right)^{1/2} \right).$$

The right factor does not vanish. Indeed, for it to vanish one would have a point where some $x^{(N)}_{r_i}$, $r_i \in I_i$ for each i and each $y^{(N)}_j$ vanished. But the choice $\{r_i\}$ corresponds to some submanifold S_k and after S_k is blown up, $\sum x^{(k)}_{r_i} + \big(\sum y^{(k)2}_j\big)^{1/2} = 1$. Thus

$$(f'_T)^* R' = a \cdot R_1 \dots R_N,$$
$$(f'_T)^* \frac{x'_i}{R'} = a_i \cdot \prod_{r \in I_i} x^{(N)}_r \text{ and}$$
$$(f'_T)^* \frac{y'_j}{R'} = a'_j \cdot y^{(N)}_j$$

where a, a_i and a'_j are smooth positive functions. This shows that the map (2.4) lifts to a map (2.2) which is a b-fibration. ∎

Most of the b-fibrations, $f\colon X \longrightarrow Y$, we consider below have an additional property, namely

$$f^*\Big(\prod_{H \in M_1(Y)} \rho_H\Big) = \prod_{G \in M_1(X)} \rho_G. \tag{2.5}$$

If $\rho_Y \in \mathcal{C}^\infty(Y)$ is a 'total boundary defining function' in the sense that it is the product of defining functions for all the boundary hypersurfaces of Y, then (2.5) requires $f^*\rho_Y$ to be a total boundary defining function for X. In terms of the boundary exponents this amounts to requiring that $e_f(G,H) = 0$ or 1 for each $G \in M_1(X)$ and $H \in M_1(Y)$ and that for each $G \in M_1(X)$ there exists precisely one $H \in M_1(Y)$ with $e_f(G,H) = 1$. The assumption that f is a b-fibration means that there can be at most one such H for each G.

DEFINITION 2.6. We say that a b-fibration is *simple* if it satisfies (2.5).

Following the proof of Lemma 2.5 we have also shown:

COROLLARY 2.7. *Under the conditions of Lemma 5, if f is simple then so is f_T in* (2.2).

As preamble to the next lemma, we define the relative b-tangent space of a p-submanifold. For a p-submanifold, S, of a manifold with corners, X, the (relative) b-tangent space ${}^bT_p(S,X) \subset {}^bT_pX$ at $p \in S$ is the linear space of values at p of those elements of $\mathcal{V}_b(X)$ which are tangent to S. Its dimension is $\dim S + k$ where k is the codimension of the smallest boundary face, $\mathrm{Fa}(S)$, containing S (so $k = 0$ if S is an interior p-submanifold). These spaces form a bundle ${}^bT(S,X)$ over S and the quotient by ${}^bN\,\mathrm{Fa}(S)$, the b-normal space to $\mathrm{Fa}(S)$, is canonically isomorphic to the (intrinsic) b-tangent bundle to S:

$${}^bT(S,X)\,/\,{}^bN_S\,\mathrm{Fa}(S) \cong {}^bTS. \tag{2.6}$$

LEMMA 2.8. *Let $f\colon X \longrightarrow Y$ be a b-fibration of compact manifolds with corners and suppose that $S \subset X$ is a closed p-submanifold to which f is b-transversal, in the sense that*

$$\mathrm{null}({}^bf_* \restriction {}^bT_pX) + {}^bT_p(S,X) = {}^bT_pX \;\forall\, p \in S, \tag{2.7}$$

and such that $f(S)$ is not contained in any boundary face of Y of codimension 2. Then the composition of the blowdown map $\beta\colon [X,S] \longrightarrow X$ with f is a b-fibration

$$f'\colon [X,S] \longrightarrow Y. \tag{2.8}$$

PROOF: The b-tangent map of the blowdown map β is onto ${}^bT(S,X)$ at each point of $\beta^{-1}S$. The b-transversality condition implies that composition with ${}^bf_*$ maps onto ${}^bTX$, so f' is a b-submersion. The condition on $f(S)$ means that $f(S)$ is a boundary face of codimension 1 or 0, so that f' is actually a b-fibration.

In view of (2.6) the condition of b-transversality in (2.7) is equivalent to the b-transversality of $f \restriction \mathrm{Fa}(S)$, as a b-fibration onto $f(\mathrm{Fa}(S))$, to S as a submanifold of $\mathrm{Fa}(S)$. This can also be restated as the condition that f restricts to S to a b-fibration onto $f(S)$, which is often simple to check.

2.5. Logarithmic blow up.

To handle the logarithmic behaviour of the surgery problem, when the boundary operator is not invertible, we introduce a new form of blow up. If X is a compact manifold with corners and $\rho_H \in \mathcal{C}^\infty(X)$ is a defining function for one of the boundary hypersurfaces, H, then define

$$\begin{gathered} \mathcal{C}^\infty([X,H]_{\log}) = \{g(\operatorname{ilg}\rho_H, f_1, \dots, f_p); g \in \mathcal{C}^\infty(\mathbb{R}^{p+1}), f_i \in \mathcal{C}^\infty(X)\} \\ \text{where } \operatorname{ilg}\rho_H = \frac{1}{\log\frac{1}{\rho_H}}. \end{gathered} \tag{2.9}$$

This is a new $\mathcal{C}^\infty$ structure on X, albeit diffeomorphic to the original one. In fact this $\mathcal{C}^\infty$ structure is independent of the choice of defining function ρ_H, and so defines $[X,H]_{\log}$. Since ρ_H is a $\mathcal{C}^\infty$ function of $\operatorname{ilg}\rho_H$, namely

$$\rho_H = \exp\left(-\frac{1}{\operatorname{ilg}\rho_H}\right), \tag{2.10}$$

the identity map on X is smooth as a map $\beta_{\log}\colon [X,H]_{\log} \longrightarrow X$. Clearly the operations of logarithmic blow up of two or more hypersurfaces commute. This allows us to define unambiguously the 'total logarithmic blow-up' $X_{\log}$ of X by blowing up each of the boundary hypersurfaces.

Perhaps surprisingly, an appropriate combination of the non-algebraic notion of logarithmic blowup with certain (ordinary) blow ups behaves well with respect to certain b-fibrations. To illustrate this we give a simple example.

EXAMPLE 2.9. Consider the b-fibration $g\colon X \longrightarrow Y$ where $X = [0,\infty)^2$, $Y = [0,\infty)$ and $g(x_1,x_2) = x_1x_2$. In terms of the boundary defining functions $r = \operatorname{ilg} x$, $\rho_1 = \operatorname{ilg} x_1$ and $\rho_2 = \operatorname{ilg} x_2$, we have, in the interior of X

$$g^*r = g^*\frac{1}{\log\frac{1}{x}} = \frac{1}{\log\frac{1}{x_1x_2}} = \frac{1}{\log\frac{1}{x_1} + \log\frac{1}{x_2}} = \frac{\rho_1\rho_2}{\rho_1+\rho_2}.$$

Thus g does not lift to a smooth map from $X_{\log}$ to $Y_{\log}$. If we further blow up X by defining $\hat{X} = [X_{\log}; (0,0)]$ then boundary defining functions for $\hat{X}$ are $\hat\rho_1 = \frac{\rho_1}{\rho_1+\rho_2}$, $\hat\rho_2 = \frac{\rho_2}{\rho_1+\rho_2}$, and $\hat\rho_3 = \rho_1 + \rho_2$ for the new face. Thus $g^*r = \hat\rho_1\hat\rho_2\hat\rho_3$, so it follows that g lifts to a b-fibration $\hat{g}\colon \hat{X} \longrightarrow Y_{\log}$.

We generalise this result in the next lemma. Before it is stated we need to introduce the notion of "total boundary blow up".

2.6. Total boundary blow up.

The total boundary blow up, X_{tb}, of a compact manifold with corners, X, is defined by blowing up (in the radial sense) all the boundary faces, in increasing order of dimension. Blowing up all faces of dimension $\leq k$ separates the lifts of the faces of dimension k, so there are no ambiguities of order in this definition. The boundary hypersurfaces of X_{tb} are parametrized by $M'(X)$, the set of proper boundary faces of X. In the next lemma we consider the effect of the combined operations of logarithmic blowup and total boundary blowup on simple b-fibrations. This result is applied in subsequent chapters to the surgery spaces of Mazzeo-Melrose to obtain new "logarithmic surgery spaces".

LEMMA 2.10. *Let $f\colon X \longrightarrow Y$ be a simple b-fibration of compact manifolds with corners. Then f lifts from the interior to a simple b-fibration $(f_{\log})_{\text{tb}} : (X_{\log})_{\text{tb}} \longrightarrow (Y_{\log})_{\text{tb}}$.*

PROOF: Again this is a local result. Using [19], chapter 2 we can assume that f takes the form in local coordinates:

$$(x_1, \dots x_k, y_1, \dots y_{n-k}) \longrightarrow \Big(\prod_{i\in I_1} x_i, \dots, \prod_{i \in I_k} x_i, y_1, \dots y_{n'-k'}\Big).$$

The condition (2.5) implies that the I_i form a partition of $\{1, \dots, k\}$. If $g\colon X' \longrightarrow Y'$ is a fibration of manifolds without boundary then the result holds for $f \times g$ if it holds for f. Thus the factors of $\mathbb{R}^{n-k}$ in the domain and $\mathbb{R}^{n'-k'}$ in the range can be dropped and it suffices to prove the result for maps of the form

$$(x_1, \dots x_k) \longrightarrow \Big(\prod_{i\in I_1} x_i, \dots, \prod_{i\in I_k} x_i\Big). \tag{2.11}$$

The composite of two simple b-fibrations is again a simple b-fibration and, the same operations being applied in domain and range, the result holds for the composite if it holds for the factors. The map (2.11) decomposes into the composite of simple b-fibrations of the form

$$\begin{gathered} f\colon \mathbb{R}^k_+ \longrightarrow \mathbb{R}^{k-1}_+ \\ (x_1, \dots x_k) \longmapsto (x_1, \dots, x_{k-2}, x_{k-1}x_k) \end{gathered} \tag{2.12}$$

with appropriate permutation of the coordinates, so we only need to prove the lemma for maps, f, of the form (2.12).

With $X = \mathbb{R}^k_+$ and $Y = \mathbb{R}^{k-1}_+$ let f be as in (2.12). Using the result of example 9 in the previous section, we see that f lifts to a b-fibration $\tilde{f}\colon [X_{\log}; K] \longrightarrow Y_{\log}$ where K is the lift to the logarithmic space of $\{x_{k-1} = x_k = 0\}$. Denote by $\tilde{K}$ the new boundary hypersurface produced by the blowup of K. To lift to $(X_{\log})_{\text{tb}}$, we use Lemma 2.5. Thus $(Y_{\log})_{\text{tb}}$ is produced from $Y_{\log}$ by blowing up all codimension l hypersurfaces for $l = (k-1)\dots 1$ successively. Denote the lift to $(X_{\log})_{\text{tb}}$ of hypersurface $x_i = 0$ by H_i, and denote by $\mathcal{H}_l$ the sequence of blowups $\mathcal{H}_{l,1}$ followed by $\mathcal{H}_{l,2}$ followed by $\mathcal{H}_{l,3}$ where

$$\begin{aligned}\mathcal{H}_{l,1} &= \text{all } l\text{-fold intersections of } H_1 \dots H_{k-2}\\ \mathcal{H}_{l,2} &= \text{all } l\text{-fold intersections of } H_1 \dots H_{k-2} \text{ and } \tilde{K} \text{ involving } \tilde{K}\\ \mathcal{H}_{l,3} &= \text{all } l\text{-fold intersections of } H_1 \dots H_k \text{ involving}\\ &\quad\ \text{exactly one of } \{H_{k-1}, H_k\}.\end{aligned}$$

Then by Lemma 2.5, $\tilde{f}$ lifts to a map

$$\hat{X} = [X_{\log};\ K;\ \mathcal{H}_{k-1}, \dots, \mathcal{H}_2] \longrightarrow \hat{Y}.$$

Claim: $\hat{X} = (X_{\log})_{\text{tb}}$. We show this by proving inductively that

$$\begin{aligned}\hat{X} =&[X_{\log};\ \text{all faces of codimension } \geq l+1;\\ &\mathcal{H}_{l,1};\ \mathcal{H}_{l,3};\ K;\ \mathcal{H}_{l-1} \dots \mathcal{H}_2].\end{aligned} \tag{2.13}$$

For $l = k$ this space is $\hat{X}$, for $l = 2$ it is $(X_{\log})_{\text{tb}}$. Assume that the statement is true for some l, $2 < l \leq k$. To show that it is true for $l-1$ we will use Lemma 2.1 and the following separation result:

> If all faces of $X = \mathbb{R}^k_+$ of codimension $\geq m$ have been blown up (in increasing order of dimension) then the lifts of $H_{\sigma(1)} \cap \cdots \cap H_{\sigma(p+r)}$ and $H_{\sigma(p)} \cap \cdots \cap H_{\sigma(p+s)}$ (where σ is a permutation and $s \geq r$) are disjoint if $p + s \geq m$. (This is true because they are separated when $H_{\sigma(1)} \cap \cdots \cap H_{\sigma(p+s)}$ is blown up.)

We now commute the K blowup past the $\mathcal{H}_{l-1} = \{\mathcal{H}_{l-1,1}, \mathcal{H}_{l-1,2}, \mathcal{H}_{l-1,3}\}$ blowups. By the result above, in $[X_{\log};$ all faces of codimension $\geq l+1]$ K is disjoint from all faces in $\mathcal{H}_{l-1,1}$ so the K blowup commutes with the $\mathcal{H}_{l-1,1}$ blowups. Again by this result, any two faces in $\mathcal{H}_{l-1,2}$ are disjoint, and they are all contained in K, so by Lemma 2.1 we may do the $\mathcal{H}_{l-1,2}$ blowups first. They are then, by the result, disjoint from the $\mathcal{H}_{l-1,1}$ faces, so can be commuted past these too. When we do this, they yield with the $\mathcal{H}_{l,1}$ and $\mathcal{H}_{l,3}$ blowups all the codimension l faces, so we get

$$\hat{X} = [X_{\log};\ \text{all faces of codimension } \geq l;\ \mathcal{H}_{l-1,1};\ K;\ \mathcal{H}_{l-1,3};\ \mathcal{H}_{l-2} \dots \mathcal{H}_2].$$

By the result again K is disjoint from all $\mathcal{H}_{l-1,3}$ faces so we obtain (2.13) for $l-1$. This completes the induction, so we have shown that $\tilde{f}$ lifts to $(f_{\log})_{\text{tb}} : (X_{\log})_{\text{tb}} \longrightarrow (Y_{\log})_{\text{tb}}$. Finally, both the result of the example and Lemma 2.5 preserve (2.5) so $(f_{\log})_{\text{tb}}$ is simple. ∎

Let us make some definitions concerning spaces of the form $(Z_{\log})_{\mathrm{tb}}$ for some manifold with corners Z. We define the degree $d(H)$ (see section 2.1) of a hypersurface H of $(Z_{\log})_{\mathrm{tb}}$ to be the codimension of the face of Z of which it is the blowup. The reason we introduce this notion is because of the following results on lifting densities. Define the cusp density bundle $\Omega_c(X)$ to be $\prod_{H\in M_1(X)} \rho_H^{-1}\Omega_b(X)$; in other words, the density bundle where all degrees are equal to 1. Then we have

LEMMA 2.11.

$$\beta_{\log}^*\Omega_b(X) = \Omega_c(X_{\log}) \tag{2.14}$$

$$\beta_{\mathrm{tb}}^*\Omega_c(X_{\log}) = \Omega_D((X_{\log})_{\mathrm{tb}}) \tag{2.15}$$

PROOF: (2.14) follows because $d\rho/\rho = d(\mathrm{ilg}\,\rho)/(\mathrm{ilg}\,\rho)^2$. To prove (2.15), let $F = H_1 \cap \cdots \cap H_k$ be a face of codimension k in $X_{\log}$. Denoting the boundary defining function for β_{tb}^*F by r_F, we have

$$\beta_{\mathrm{tb}}^*\rho_H = \prod_{F\subset H} r_F.$$

Under blowup of a boundary face, the b-density bundle lifts to the b-density bundle. Therefore,

$$\begin{aligned}\beta_{\mathrm{tb}}^* \prod_{H\in M_1(X)} \rho_H^{-1}\Omega_b(X_{\log}) &= \prod_{H\in M_1(X)} \prod_{F\subset H} r_F^{-1}\Omega_b((X_{\log})_{\mathrm{tb}}) \\ &= \prod_{F\in M'(X)} r_F^{-\mathrm{codim}\,F}\,\Omega_b((X_{\log})_{\mathrm{tb}}).\blacksquare\end{aligned}$$

If $S \subset M$ is a p-submanifold, define $[(Z_{\log})_{\mathrm{tb}} \times M; \partial(Z_{\log})_{\mathrm{tb}} \times S]$ to be the space $(Z_{\log})_{\mathrm{tb}} \times M$ with submanifolds $H \times S$, for all $H \in M_1\big((Z_{\log})_{\mathrm{tb}}\big)$ blown up in order of decreasing degree; there are no ordering ambiguities because all hypersurfaces of $(Z_{\log})_{\mathrm{tb}}$ of a given degree are disjoint.

LEMMA 2.12. *Let $f\colon X \longrightarrow Y$ be a simple b-fibration and $S \subset M$ be a p-submanifold then the map $(f_{\log})_{\mathrm{tb}} \times \mathrm{Id}\colon (X_{\log})_{\mathrm{tb}} \times M \longrightarrow (Y_{\log})_{\mathrm{tb}} \times M$ lifts to a simple b-fibration*

$$[(X_{\log})_{\mathrm{tb}} \times M; \partial X \times S] \longrightarrow [(Y_{\log})_{\mathrm{tb}} \times M; \partial Y \times S].$$

PROOF: We argue as in the proof of Lemma 2.10 to reduce the proof to the case where f is of the form (2.12). We apply Lemma 2.5 to the $\partial Y \times S$ blowups to lift the b-fibration.

Consider the inverse images of all the submanifolds $H \times S$ for all hypersurfaces H of Y of fixed degree d. Each such submanifold has as inverse image a union of submanifolds $G \times S$, which are disjoint if they correspond to different H. Since $(f_{\log})_{\mathrm{tb}}$ is a b-fibration and $\dim\,(X_{\log})_{\mathrm{tb}} = \dim\,(Y_{\log})_{\mathrm{tb}} + 1$, we have $d(G) = d(H)$ or $d(H) + 1$. Hence in $(X_{\log})_{\mathrm{tb}} \times M$ we can choose the order of blowup so that the $G \times S$ are also blown up in order of decreasing $d(G)$. Condition (2.5) for $(f_{\log})_{\mathrm{tb}}$ means that as H runs through all hypersurfaces of $(Y_{\log})_{\mathrm{tb}}$, G runs through all hypersurfaces of $(X_{\log})_{\mathrm{tb}}$, so we obtain precisely $[(X_{\log})_{\mathrm{tb}} \times M; \partial X \times S]$ as the new domain. ∎

Chapter 3. The Single Space

In section 1.4 we noted that the surgery spaces of Mazzeo-Melrose will not suffice to construct the resolvent of Δ when the boundary Dirac operator is not invertible. Instead, consideration of the eigenfunctions on an interval under surgery suggested that the resolvent will be a smooth (conormal) function of y, y' and the "rescaled distance"

$$\frac{\text{arclength}}{\text{total length}} = \frac{\operatorname{arcsinh}\frac{x}{\epsilon}}{2\operatorname{arcsinh}\frac{1}{\epsilon}} \longrightarrow \frac{\operatorname{ilg}\epsilon}{\operatorname{ilg}\frac{\epsilon}{x}} \text{ as } \frac{\epsilon}{x} \to 0.$$

This suggests working on a space on which these functions are smooth. We therefore make the following definition.

3.1. Definition.

We define the 'logarithmic' single surgery space by:

$$X_{\mathrm{Ls}} = \Big((X_s)_{\log}\Big)_{\mathrm{tb}}. \tag{3.1}$$

Here the single surgery space defined in [16] and Mazzeo-Melrose is obtained by blow up of H at $\epsilon = 0$:

$$X_s = [M \times [0,\epsilon_0]; H \times \{0\}]. \tag{3.2}$$

By Lemma 2.10, the b-fibration $X_s \longrightarrow [0,\epsilon_0]$ lifts to a b-fibration $X_{\mathrm{Ls}} \longrightarrow [0, \operatorname{ilg}\epsilon_0]_{\operatorname{ilg}\epsilon}$. Therefore $\operatorname{ilg}\epsilon$ is a smooth function on X_{Ls}, vanishing to first order on all boundary faces (at $\epsilon = 0$). We will write X^0_{Ls}, the "zero space", for $[0, \operatorname{ilg}\epsilon_0]_{\operatorname{ilg}\epsilon}$ below. The space X_{Ls} has four types of boundary hypersurfaces. The lift of the boundary $\epsilon = 0$ will be denoted $B_0(X_{\mathrm{Ls}})$; it is the surgery boundary. The lift of the surgery front face will be denoted $B_1(X_{\mathrm{Ls}})$ and again called the surgery front face. The new hypersurfaces constructed in the last, total boundary, blow up in (3.1) will be called the logarithmic surgery faces and denoted $B_2(X_{\mathrm{Ls}})$. There is also a 'trivial' boundary hypersurface at $\epsilon = \epsilon_0$. Both $B_0(X_{\mathrm{Ls}})$ and $B_2(X_{\mathrm{Ls}})$ have two components; these will be denoted $B_{\pm 0}(X_{\mathrm{Ls}})$, $B_{\pm 2}(X_{\mathrm{Ls}})$ with the sign corresponding to the local orientation of H.

The diffeomorphism types of these boundary hypersurfaces are easily identified. Clearly

$$B_0(X_{\mathrm{Ls}}) \cong \overline{M}_{\log} \tag{3.3}$$

is just the manifold with boundary, $\overline{M}$, obtained by cutting M along H with its boundary blown up logarithmically. The front face of X_s is the radial compactification of the normal bundle to H in M; this compactification is denoted $\overline{H}$. Lifted to $(X_s)_{\log}$ this becomes $\overline{H}_{\log}$. The final blow up does not change the structure of this face so

$$B_1(X_{\mathrm{Ls}}) \cong \overline{H}_{\log}. \tag{3.4}$$

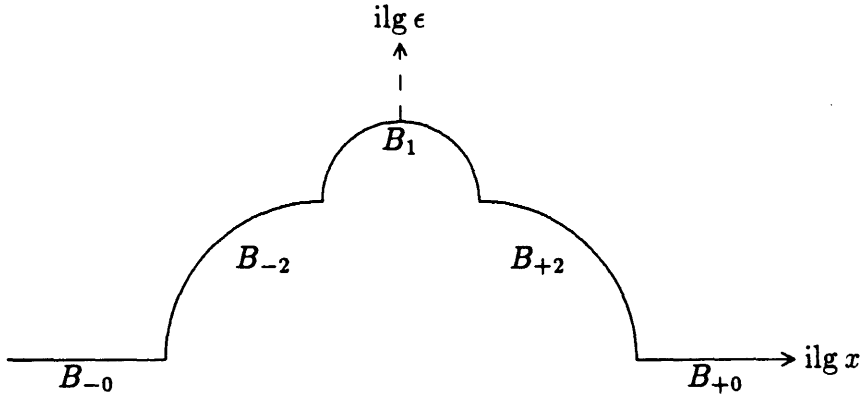


Figure 1. Boundary faces of X_{Ls}.

The essentially new faces introduced by the passage from X_s to X_{Ls} are the radial compactifications of the normal bundles to the corners of $(X_s)_{\log}$. These are interval bundles over H. The two functions $\operatorname{ilg} x$ and $\operatorname{ilg}(\epsilon/x)$ can be taken as defining functions for the corner, in the closure of the region $\epsilon < \frac{1}{2}x < \frac{1}{4}$. Thus the limiting value

$$s = \begin{cases} \lim\limits_{\operatorname{ilg} x, \operatorname{ilg}(\epsilon/x) \downarrow 0} \dfrac{\operatorname{ilg} x}{\operatorname{ilg} x + \operatorname{ilg}(\epsilon/x)} = \dfrac{\operatorname{ilg} \epsilon}{\operatorname{ilg}(\epsilon/x)} & x > 0 \\ \lim\limits_{\operatorname{ilg} -x, \operatorname{ilg}(-\epsilon/x) \downarrow 0} \dfrac{-\operatorname{ilg}(-x)}{\operatorname{ilg}(-x) + \operatorname{ilg}(-\epsilon/x)} = -\dfrac{\operatorname{ilg} \epsilon}{\operatorname{ilg}(-\epsilon/x)} & x < 0 \end{cases} \tag{3.5}$$

is a global variable along the fibres. If x is replaced by another defining function for H, $x' = a(x,y)x$ with $a > 0$, then

$$\operatorname{ilg} x' = \operatorname{ilg} x \frac{1}{1 - \operatorname{ilg} x \cdot \log a}, \quad \operatorname{ilg}(\epsilon/x') = \operatorname{ilg}(\epsilon/x) \frac{1}{1 + \operatorname{ilg}(\epsilon/x) \cdot \log a}. \tag{3.6}$$

Thus the limiting value of s in (3.5) is unchanged. It follows that $B_2(X_{\mathrm{Ls}})$ is a canonically trivial bundle over H. Occasionally we will use the notation $\widetilde{H}$ for $\partial \overline{M} = H \cup H$, the disjoint union of two copies of H, and write $\widetilde{H} \times [0,1]_s$ for $H \times ([-1,0]_s \cup [0,1]_s)$.

3.2. Densities.

The Riemannian density of g_ϵ is of the form $\nu_g = (x^2 + \epsilon^2)^{-\frac{1}{2}} \nu$ where $0 < \nu \in \mathcal{C}^\infty([0,\epsilon_0]; \Omega(M))$. We shall adopt a slightly different normalization of densities from that used in Mazzeo-Melrose and consider the (trivial) bundle, Ω_X, over $M \times [0,\epsilon_0]_\epsilon$ which has a generating section $\nu_g \otimes |d\epsilon|/\epsilon$. This bundle is just the density bundle over $M \times (0, \epsilon_0]$ and lifts to X_s to

$$(\beta[X, \{0\} \times H])^* \Omega_X \equiv \Omega_b(X_s). \tag{3.7}$$

Notice that if the extra factor of ϵ^{-1} is omitted, as it is in the normalization of Mazzeo-Melrose, then one simply gets the density bundle in (3.7) instead of the b-density bundle.

The advantage of the extra factor of ϵ^{-1} is that Ω_X lifts to be simple on X_{Ls}. By Lemma 2.11, we have

$$\beta_{\log\text{-}\mathrm{tb}}\Omega_b(X_s) = \Omega_D(X_{\mathrm{Ls}}) = \rho_0^{-2}\rho_2^{-3}\rho_1^{-2}\Omega(X_{\mathrm{Ls}}). \tag{3.8}$$

3.3. Lift of $\mathcal{V}_s(X)$.

In Mazzeo-Melrose it is shown that the Laplacian associated to a surgery metric lifts to X_s to an element of $\mathrm{Diff}_s^2(X_s)$, which is the enveloping algebra of $\mathcal{V}_s(X_s)$, the Lie subalgebra of $\mathcal{V}_b(X_s)$ consisting of those vector fields which are tangent to the fibres of ϵ = constant. $\mathcal{V}_s$ may also be described as the vector fields on X_s tangent to ϵ = constant of finite length with respect to g_ϵ. We need to consider the lift of $\mathcal{V}_s(X_s)$ to X_{Ls}. As noted in section 3.1 we know that $\mathrm{ilg}\,\epsilon$ lifts to a $\mathcal{C}^\infty$ function on X_{Ls} which is a total boundary defining function for all the boundary hypersurfaces above $\epsilon = 0$. Consider the Lie algebra

$$\begin{aligned}\mathcal{V}_{\mathrm{Ls}}(X_{\mathrm{Ls}}) = \big\{V \in \mathcal{V}_b(X_{\mathrm{Ls}}); V\,\mathrm{ilg}\,\epsilon = 0 \text{ and}\\ V \text{ is tangent to the fibres of } B_2(X_{\mathrm{Ls}}) \text{ over } [-1,1]\big\}.\end{aligned} \tag{3.9}$$

As with $\mathcal{V}_s$, $\mathcal{V}_{\mathrm{Ls}}$ is the space of $\mathcal{C}^\infty$ sections of a vector bundle. To describe this bundle directly on X_{Ls}, let ${}^bTX'_{\mathrm{Ls}} \subset {}^bTX_{\mathrm{Ls}}$ be the subbundle of codimension one given by

$${}^bTX'_{\mathrm{Ls}} = \mathrm{null}\,{}^b\,\mathrm{ilg}\,\epsilon\colon {}^bTX_{\mathrm{Ls}} \to {}^bTX^0_{\mathrm{Ls}}.$$

Since the map $\mathrm{ilg}\,\epsilon\colon X_{\mathrm{Ls}} \to X^0_{\mathrm{Ls}}$ is a b-fibration, $\mathrm{null}({}^b\,\mathrm{ilg}\,\epsilon)$ has codimension one at every point. Let $F \subset {}^bTX'_{\mathrm{Ls}\restriction B_2}$ be the subbundle tangent to the fibration $B_2 \to [-1,1]$. Then $\mathcal{V}_{\mathrm{Ls}}$ is the space of smooth sections of ${}^bTX'_{\mathrm{Ls}}$ that lie in F at B_2. As explained in [17], chapter 8, this means that there is a bundle ${}^{\mathrm{Ls}}TX_{\mathrm{Ls}} = {}^F({}^bTX'_{\mathrm{Ls}})$ (in the notation of [17]) such that $\mathcal{V}_{\mathrm{Ls}}$ is precisely the space of smooth sections of ${}^{\mathrm{Ls}}TX_{\mathrm{Ls}}$. The metric lifts to a non-degenerate fibre metric on ${}^{\mathrm{Ls}}TX_{\mathrm{Ls}}$.

${}^{\mathrm{Ls}}TX_{\mathrm{Ls}}$ is supposed to be the 'correct' replacement for the usual tangent bundle TX_{Ls} in surgery geometry, in the approach outlined in [20]. We define surgery form bundles, surgery Clifford bundles, surgery frame bundles and surgery spinor bundles using ${}^{\mathrm{Ls}}TX_{\mathrm{Ls}}$. Thus the surgery cotangent bundle, ${}^{\mathrm{Ls}}T^*X_{\mathrm{Ls}}$ is defined to be the dual of ${}^{\mathrm{Ls}}TX_{\mathrm{Ls}}$. The surgery form bundle ${}^{\mathrm{Ls}}\Lambda^*X_{\mathrm{Ls}}$ is the exterior bundle of the surgery cotangent bundle. The surgery Clifford bundle ${}^{\mathrm{Ls}}\mathbb{C}\mathrm{l}\,X_{\mathrm{Ls}}$ is the fibrewise Clifford algebra of the surgery cotangent bundle with respect to the fibre metric g_ϵ (defined on ${}^{\mathrm{Ls}}T^*X_{\mathrm{Ls}}$ by duality). If M^n is spin, then the bundle of orthonormal frames of ${}^{\mathrm{Ls}}T^*X_{\mathrm{Ls}}$ lifts to a $\mathrm{Spin}(n)$ bundle $\mathrm{Spin}(X_{\mathrm{Ls}})$, which reduces over B_2 to have structure group $\mathrm{Spin}(n-1)$. The surgery spinor bundle is the associated bundle

$$S(X_{\mathrm{Ls}}) = \mathrm{Spin}(X_{\mathrm{Ls}}) \times_{\mathrm{Spin}(n)} S,$$

where S is the (irreducible) representation of $\mathrm{Spin}(n)$ on $\mathbb{C}^{2^k}$ (where $n = 2k+1$). Over B_2, $S(X_{\mathrm{Ls}})$ has a natural splitting $S(X_{\mathrm{Ls}}) = S^+(H) \oplus S^-(H)$ given by the ± 1

eigenspaces of the surgery Clifford element $dx/(\sqrt{x^2+\epsilon^2})$; restricted to each leaf $\{s = \text{constant}\}$ of B_2 they are the plus and minus spinor spaces associated with $\eth_H$.

We define the space of Ls-differential operators, $\mathrm{Diff}_{\mathrm{Ls}} X_{\mathrm{Ls}}$, to be the set of all differential operators which are sums of products of vector fields in $\mathcal{V}_{\mathrm{Ls}}$ and smooth bundle maps (in other words, the enveloping algebra of $\mathcal{V}_{\mathrm{Ls}}$ tensored with bundle maps). Then we have:

LEMMA 3.1. *The operators $\eth_\epsilon$, $d+\delta$ and Δ_ϵ, where δ is the adjoint of d with respect to g_ϵ, are* Ls*-differential operators on* ${}^{\mathrm{Ls}}\mathrm{Cl}\, X_{\mathrm{Ls}}$ *and* ${}^{\mathrm{Ls}}\Lambda^* X_{\mathrm{Ls}}$ *respectively.*

PROOF: We may write $\eth_\epsilon = \sum \mathrm{cl}(e_i)\nabla_i$, for an orthonormal frame e_i of the surgery cotangent bundle. The operator ∇_i is in $\mathcal{V}_{\mathrm{Ls}}$ and $\mathrm{cl}(e_i)$ is a smooth bundle map on ${}^{\mathrm{Ls}}\mathrm{Cl}\, X_{\mathrm{Ls}}$. Similarly, we may write $d = \sum (e_i\wedge)\nabla_i$ and $\delta = \sum \nabla_i(e_i\lrcorner)$, and $(e_i\wedge)$ and $e_i\lrcorner$ are smooth bundle maps on ${}^{\mathrm{Ls}}\Lambda^* X_{\mathrm{Ls}}$. Hence, $\Delta_\epsilon = (d+\delta)^2$ is also a Ls-differential operator. ∎

The next result identifies $\mathcal{V}_{\mathrm{Ls}}$ in terms of the Lie Algebra $\mathcal{V}_s$ of Mazzeo-Melrose:

LEMMA 3.2. *The Lie algebra $\mathcal{V}_s(X_s)$ lifts to X_{Ls} to span, over $\mathcal{C}^\infty(X_{\mathrm{Ls}})$, the boundary-fibration structure $\mathcal{V}_{\mathrm{Ls}}(X)$ given by* (3.9).

PROOF: Away from the blown up submanifolds this is obviously so. Moreover it is certainly local over open sets of M so it suffices to consider the product case $M = H \times (-1,1)$ where H is just an open set in Euclidean space. The blow-ups preserve the product structure so it suffices to consider the case that H is a point. Thus we only need consider the lift of the vector field

$$V_0 = (x^2+\epsilon^2)^{\frac{1}{2}}\frac{\partial}{\partial x}. \tag{3.10}$$

Near the corner of X_s the projective coordinates $k = \epsilon/x$ and x are valid and in terms of these

$$V_0 = (1+k^2)^{\frac{1}{2}}\left(x\frac{\partial}{\partial x} - k\frac{\partial}{\partial k}\right).$$

Under the logarithmic blow up of both boundary hypersurfaces this in turn lifts to

$$V_0 = \left(1+\exp(-\frac{2}{\kappa})\right)^{\frac{1}{2}}\left(\xi^2\frac{\partial}{\partial \xi} - \kappa^2\frac{\partial}{\partial \kappa}\right)$$

where $\xi = \mathrm{ilg}\, x$ and $\kappa = \mathrm{ilg}\, k$. Finally under the radial blow up of $\xi = \kappa = 0$ this becomes

$$V_0 = \left(1+\exp(-\frac{2(1-\rho_0)}{\rho_2})\right)^{\frac{1}{2}}\left(\rho_2^2\frac{\partial}{\partial \rho_2} - \rho_0\rho_2\frac{\partial}{\partial \rho_0}\right) \tag{3.11}$$

or

$$V_0 = \left(1+\exp(-\frac{2}{\rho_2})\right)^{\frac{1}{2}}\left(\rho_2^2\frac{\partial}{\partial \rho_2} - \rho_1\rho_2\frac{\partial}{\partial \rho_1}\right) \tag{3.12}$$

in terms of the coordinates $\rho_2 = \xi$ and $\rho_0 = \kappa/(\kappa+\xi) = (1-s)$ or $\rho_2 = \kappa$ and $\rho_1 = \xi/(\kappa+\xi) = s$ which together cover the new face. It is now easy to check that V_0 spans the algebra $\mathcal{V}_{\mathrm{Ls}}(X)$ in this case, so proving the result in general. ∎

This Lemma also shows that ${}^{\mathrm{Ls}}TX_{\mathrm{Ls}} \equiv (\beta_{\log-\mathrm{tb}})^{*s}TX_s$, where ${}^sTX_s$ is the surgery tangent space of Mazzeo-Melrose.

Consider the normal operators for the structure $\mathcal{V}_{\mathrm{Ls}}(X)$. At the level of the Lie algebra these correspond to freezing the coefficients of $\mathcal{V}_{\mathrm{Ls}}(X)$ at the various integral submanifolds. Each level surface of ϵ on which it is non-zero forms such an integral submanifold and $\mathcal{V}_{\mathrm{Ls}}(X)$ restricts to the space of all smooth vector fields on these surfaces. The more interesting integral submanifolds are those lying over $\epsilon = 0$. There are three essentially different cases. Consider first the restriction algebras. Let us define the notation $\mathcal{V}_f(Y)$ for the subalgebra of $\mathcal{V}_b(Y)$ consisting of those vector fields tangent to the leaves of a b-fibration $f\colon Y \longrightarrow Y'$ and $\mathcal{V}_c(X)$ for the cusp algebra, determined by a choice of boundary defining function ρ on a compact manifold X with boundary, Y:

$$\mathcal{V}_c(Y) = \mathcal{C}^\infty(X)\text{-span}\left\{\rho^2\frac{\partial}{\partial\rho}, \frac{\partial}{\partial y_i}\right\}$$

LEMMA 3.3. *Restriction to the three types of boundary hypersurfaces above $\epsilon = 0$ gives surjective Lie algebra homomorphisms*

$$R_0 = N_0\colon \mathcal{V}_{\mathrm{Ls}}(X) \longrightarrow \mathcal{V}_c(\overline{M}_{\log}) \tag{3.13}$$

$$R_2\colon \mathcal{V}_{\mathrm{Ls}}(X) \longrightarrow \mathcal{V}_f(\widetilde{H} \times [0,1]) \tag{3.14}$$

$$R_1 = N_1\colon \mathcal{V}_{\mathrm{Ls}}(X) \longrightarrow \mathcal{V}_c(\overline{H}_{\log}) \tag{3.15}$$

where $f\colon \widetilde{H} \times [0,1] \longrightarrow [0,1]$ is the projection.

PROOF: As in the proof of Lemma 3.2 it suffices to consider the special case where M is an interval. Then (3.13) just arises as the restriction of V_0 in (3.11) to $s = 0$, giving $\xi^2\partial/\partial\xi$, which, with the vector fields on H, generates the cusp algebra with distinguished boundary defining function ξ. The other two restriction maps can be analyzed in the same way, with (3.14) arising from the fact that V_0 in (3.12) vanishes at $\rho_2 = 0$. ∎

The notation N_0 and N_1 in (3.13) and (3.15) is justified by the fact that the null spaces of these maps are precisely the Lie ideals $\mathcal{I} \cdot \mathcal{V}_{\mathrm{Ls}}(X)$ where $\mathcal{I}$ is the ideal of functions defining the boundary hypersurface in question; thus these maps are indeed the normal operators in the sense of [18] and the range spaces can be identified with $\mathcal{C}^\infty(H; {}^{\mathrm{Ls}}TX_{\mathrm{Ls}})$ for the corresponding boundary hypersurface in $M_1(X_{\mathrm{Ls}})$.

This is not the case for restriction to $B_2(X_{\mathrm{Ls}})$. Rather the null space of the restriction map

$$\mathcal{C}^\infty\left(B_2(X_{\mathrm{Ls}}); {}^{\mathrm{Ls}}TX_{\mathrm{Ls}}\right) \longrightarrow \mathcal{V}_f(\widetilde{H} \times [0,1]) \tag{3.16}$$

is a one-dimensional Lie algebra over $B_2(X_{\mathrm{Ls}})$. This line bundle is important since it generates the reduced normal operator, the properties of which largely determine the behaviour of the small eigenvalues of the Dirac operator and Laplacian. In fact

LEMMA 3.4. *If* $V \in \mathcal{V}_{\mathrm{Ls}}(X)$ *has* $R_2(V) = 0$ *then* $V/\operatorname{ilg}\epsilon$ *is smooth up to the interior of* $B_2(X_{\mathrm{Ls}})$ *and, by projecting off any term in* $\mathcal{V}_f(\widetilde{H} \times [0,1])$, *defines a vector field* $\mathrm{RN}(V) = \mathrm{rn}(V)D_s$, *where* $\mathrm{rn}(V) \in \mathcal{C}^\infty(B_2(X_{\mathrm{Ls}}))$, *on the fibres of* $B_2(X_{\mathrm{Ls}})$ *over* $\widetilde{H}$ *which is smooth up to the boundary (but not necessarily vanishing there) and is such that* $\mathrm{rn}(V) = 0$ *if and only* $V \in \mathcal{I}(B_2(X_{\mathrm{Ls}})) \cdot \mathcal{V}_{\mathrm{Ls}}(X)$.

PROOF: Again this is just a matter of examining the behaviour of the vector field V_0, in (3.11). Using the coordinates of (3.11), in the interior of $B_2(X_{\mathrm{Ls}})$ $\operatorname{ilg}\epsilon = \rho_1\rho_2$ so the restriction of $(\operatorname{ilg}\epsilon)^{-1}V_0$ to $\xi = 0$ is just

$$\mathrm{RN}(V_0) = \frac{\partial}{\partial s}. \tag{3.17}$$

This shows it to be smooth and non-vanishing up to both boundaries of B_2. ∎

Thus, together, the two maps R_2 and RN capture the full normal operator at $B_2(X_{\mathrm{Ls}})$.

Let us now calculate quite explicitly the form of the Dirac operator and the Laplacian in local coordinates near the boundary hypersurfaces of X_{Ls}. If we restrict to the interior of a given face, we may use $\operatorname{ilg}\epsilon$ as a boundary defining function; this is a good choice because $[\eth, \operatorname{ilg}\epsilon] = [\Delta, \operatorname{ilg}\epsilon] = 0$.

LEMMA 3.5. *We have the following expressions in coordinates for the Dirac operator and the Laplacian near the interior of the boundary hypersurfaces of* X_{Ls}*:*

(i) Near interior B_1, *using coordinates* y, $r = \sinh^{-1}(x/\epsilon)$ *and* $\operatorname{ilg}\epsilon$, *we have*

$$\begin{aligned}\eth_\epsilon &= \gamma(-i\nabla_r) + \eth_H + v\cdot Q,\\ \Delta_\epsilon &= -(\nabla_r)^2 + \Delta_H + v\cdot Q'\end{aligned} \tag{3.18}$$

where v *is a function vanishing to infinite order at this face and* Q, Q' *are Ls-differential operators of order at most one, respectively two.*

(ii) Near interior B_2, *using coordinates* y, $s = \operatorname{ilg}\epsilon\cdot\sinh^{-1}(x/\epsilon)$ *and* $\operatorname{ilg}\epsilon$, *we have*

$$\begin{aligned}\eth_\epsilon &= \operatorname{ilg}\epsilon\cdot\gamma(-i\nabla_s) + \eth_H + v\cdot Q,\\ \Delta_\epsilon &= -(\operatorname{ilg}\epsilon)^2(\nabla_s)^2 + \Delta_H + v\cdot Q'\end{aligned} \tag{3.19}$$

with v, Q, Q' *as above.*

(iii) Near interior B_0, *and close to* B_2, *using coordinates* y, $\xi = \operatorname{ilg}x$ *and* $\operatorname{ilg}\epsilon$, *we have*

$$\begin{aligned}\eth_\epsilon &= \gamma(-i\xi^2\nabla_\xi) + \eth_H + v\cdot Q,\\ \Delta_\epsilon &= -(\xi^2\nabla_\xi)^2 + \Delta_H + v\cdot Q'\end{aligned} \tag{3.20}$$

with v *a function vanishing to infinite order at* B_2 *and* Q, Q' *as above. Near interior* B_0, *using the lift of coordinates on* M *and* $\operatorname{ilg}\epsilon$, *we have*

$$\begin{aligned}\eth_\epsilon &= \eth_0 + v\cdot Q,\\ \Delta_\epsilon &= \Delta_0 + v\cdot Q'\end{aligned} \tag{3.21}$$

with v *vanishing to infinite order at* B_0 *and* Q, Q' *are as before.*

PROOF: If h in the definition of the surgery metric is a product metric in $|x| < \delta$ for some defining function x for H, then (3.18) – (3.20) follow easily from (3.10) – (3.12). So write

$$\begin{aligned} h &= h_{ij}(o,y)dy^i dy^j + x \cdot h'_{ij}(x,y)dy^i dy^j + h''_j(x,y)dx dy_j + h'''(x,y)dx^2 \\ &= h_{ij}(o,y)dy^i dy^j + x \cdot h'_{ij}(x,y)dy^i dy^j \\ &\quad + \sqrt{x^2+\epsilon^2}\, h''_j(x,y)\frac{dx}{\sqrt{x^2+\epsilon^2}}dy_j + (x^2+\epsilon^2)h'''(x,y)\frac{dx^2}{(x^2+\epsilon^2)} \end{aligned}$$

where the second time we wrote the metric in terms of the smooth surgery form $dx/\sqrt{x^2+\epsilon^2}$. The lifts to X_{Ls} of x, $\sqrt{x^2+\epsilon^2}$ and $(x^2+\epsilon^2)$ vanish to infinite order at B_1 and B_2 so only contribute to $\eth$ or Δ a term of the form $v \cdot Q$ or $v \cdot Q'$. The first term is a product metric in a collar neighbourhood of H so gives the principal terms. Hence (3.18) – (3.20) are established. (3.21) follows because in the interior of B_0, the surgery metric $g_\epsilon = g_0 + \epsilon \cdot g'$ and ϵ vanishes rapidly on B_0. ∎

3.4. Models.

The Lie algebra homomorphisms in (3.13) – (3.15) extend to homomorphism of the enveloping algebras and define three of the four (closely related) 'model problems' we need to discuss in order to invert the original operator.

PROPOSITION 3.6. *For the Dirac operator (Laplacian) of the metric g_ϵ the normal operator $R_0(\eth_\epsilon)$ $(R_0(\Delta))$ is the lift to $\overline{M}_{\log}$ of the Dirac operator (Laplacian), $\eth_0$ (Δ_0), of $\overline{M}$ of the exact b-metric g_0; the normal operator $R_1(\eth_\epsilon)$ $(R_1(\Delta))$ is the lift to $\overline{H}_{\log}$ of the indicial operator of $\eth_0$ (Δ_0) as an $\mathbb{R}^+$-invariant operator in $\overline{H}$, the normal bundle to H in X; and the restriction operator $R_2(\eth_\epsilon)$ $(R_0(\Delta))$ is $\eth_H$ (Δ_H), acting on the leaves of $B_2(X_{\mathrm{Ls}})$.*

Thus the first three model problems are just the Laplacian on $\overline{M}$ and its indicial operator (in two guises). The fourth model problem arises from the reduced normal operator of Lemma 3.4.

PROPOSITION 3.7. *Let Π be the orthogonal projection on the null space of $\eth_H$ and let s be the coordinate in (3.5). If $u \in C^\infty(B_2(X_{\mathrm{Ls}}))$ is in the range of Π and $\tilde{u}$ is any smooth extension of u to X_{Ls}, then*

$$\mathrm{RN}(\eth)u = \Pi\left((\mathrm{ilg}\,\epsilon)^{-1}\eth\tilde{u} \restriction B_2(X_{\mathrm{Ls}})\right) = \gamma D_s u, \tag{3.22}$$

independent of the choice of extension. Similarly, let $\bar{\Pi}$ be the orthogonal projection on the null space of Δ_H, and let ρ_2 be a boundary defining function for B_2. If $u \in C^\infty(B_2(X_{\mathrm{Ls}}))$ is in the range of $\bar{\Pi}$ and $\tilde{u}$ is an extension of u to $C^\infty(X_{\mathrm{Ls}})$ such that $\Delta_H(\nabla_{\partial_{\rho_2}}\tilde{u}) = 0$, then $\Delta\tilde{u}$ vanishes to second order at B_2 and

$$\mathrm{RN}(\Delta)u = \bar{\Pi}\left((\mathrm{ilg}\,\epsilon)^{-2}\Delta\tilde{u} \restriction B_2(X_{\mathrm{Ls}})\right) = D_s^2 u, \tag{3.23}$$

independent of the choice of extension.

PROOF: In the interior of B_2, use s, y and $\operatorname{ilg}\epsilon$ as coordinates. Then, writing $\tilde{u}$ as a Taylor series in $\operatorname{ilg}\epsilon$ off B_2 and using the coordinate representations of $\eth$ and Δ in (3.19), the proposition follows.

The fourth model operator is therefore the ordinary differential operator γD_s (D_s^2) acting on smooth functions on $[-1,0]\cup[0,1]$ with values in the null space of $\eth_H$ (Δ_H). In fact this operator comes with boundary conditions which turn it into an unbounded self-adjoint operator on the interval $[-1,1]$. To discuss these we need to recall, from [17], some properties of the 'extended L^2 null space' of an elliptic operator. Here the cases of the Dirac operator and the Laplacian are somewhat different, so we discuss them separately. Let us treat the Laplacian first, as it is the more complicated case. Consider the null space of Δ_0 on the weighted L^2 space $x^{-\delta}L_b^2(M_\pm)$, for δ small. Each element v of this null space has an asymptotic expansion

(3.24)

$$\Delta_0 v = 0,\ v\in x^{-\delta}L_b^2(M_\pm) \Longrightarrow v = v_1\log x + v_0 + v' \text{ near } H,\ \Delta_H(v_i)=0,\ v'\in L_b^2.$$

These generalized boundary values define a map into $\operatorname{null}(\Delta_H)\oplus\operatorname{null}(\Delta_H)$; the range will be written $\Lambda_\pm$. It is necessarily a Lagrangian subspace of $\operatorname{null}(\Delta_H)\oplus\operatorname{null}(\Delta_H)$, so in particular has dimension equal to that of $\operatorname{null}(\Delta_H)$. For $\delta>0$ small enough this gives a short exact sequence

(3.25)

$$0\hookrightarrow\{v\in x^{\delta}L_b^2(M_\pm);\Delta_0 v=0\}\hookrightarrow\{v\in x^{-\delta}L_b^2(M_\pm);\Delta_0 v=0\}\longrightarrow\Lambda_\pm\longrightarrow 0$$

This Lagrangian subspace defines the boundary condition for the reduced normal operator. From these Lagrangian spaces we define two subspaces of $\operatorname{null}(\Delta_H)$:

$$\begin{aligned}\Lambda_\pm^D &= \{u'\in\operatorname{null}(\Delta_H);(u',0)\in\Lambda\}\\ \Lambda^N &= \{u''\in\operatorname{null}(\Delta_H);\exists\,(u',u'')\in\Lambda\}\end{aligned}\tag{3.26}$$

Clearly the sum of the dimensions of $\Lambda_\pm^D$ and $\Lambda_\pm^N$, is equal, for either sign, to the dimension of $\operatorname{null}(\Delta_H)$. It was also shown in [17] that $\Lambda_\pm^D$ and $\Lambda_\pm^N$ are the ±1 eigenspaces of the scattering matrix associated to $\Delta_{M_\pm}$ at $\lambda=0$, so the boundary conditions are determined by the scattering matrix.

To understand the boundary conditions associated to the fourth model operator, we prove a lemma concerning approximate small eigenfunctions u, by which we will mean $u\in\mathcal{C}^\infty(X_{\mathrm{Ls}};\Omega_D^{\frac12}X_{\mathrm{Ls}})$ such that $(\Delta-(\operatorname{ilg}\epsilon)^2z^2)u\in\rho_0^2\rho_1^2\rho_2^3\mathcal{C}^\infty(X_{\mathrm{Ls}};\Omega_D^{\frac12}X_{\mathrm{Ls}})$. "Small" refers to the fact that the eigenvalue goes to zero with $\operatorname{ilg}\epsilon$. We will see in chapter 6 that such u are indeed a good approximation to a surgery eigenfunction. For such u we have by Propostion 7, $\Delta_H\ (u\restriction B_2)=0$, so $u\restriction B_2$ can be regarded as a $\operatorname{null}(\Delta_H)$-valued function $\bar{u}$ on $[-1,0]_s\cup[0,1]_s$.

PROPOSITION 3.8. *Suppose* $u\in\mathcal{C}^\infty(X_{\mathrm{Ls}};\Omega_D^{\frac12}X_{\mathrm{Ls}})$ *satisfies*

$$(\Delta-(\operatorname{ilg}\epsilon)^2z^2)u\in\rho_0^2\rho_1^2\rho_2^3\mathcal{C}^\infty(X_{\mathrm{Ls}};\Omega_D^{\frac12}X_{\mathrm{Ls}}).\tag{3.27}$$

Then $\bar{u}$ *is* $\mathcal{C}^\infty$ *on* $[-1,1]_s$, *with*

$$(D_s^2 - z^2)\bar{u} = 0, \tag{3.28}$$

$$\text{and} \quad \begin{array}{l} \bar{u} \restriction \{s=-1\} \in \Lambda_-^D, \ D_s\bar{u} \restriction \{s=-1\} \in \Lambda_-^N, \\ \bar{u} \restriction \{s=+1\} \in \Lambda_+^D, \ D_s\bar{u} \restriction \{s=+1\} \in \Lambda_+^N. \end{array} \tag{3.29}$$

Conversely, if $\bar{u}$ *satisfies* (3.28) *and* (3.29) *then there is an extension* u *satisfying* (3.27).

Remark: This Proposition is due to Richard Melrose.

PROOF: We first show that $\bar{u}$ and $\partial_s\bar{u}$ match across $s=0$. Using the local coordinates $\operatorname{ilg}\epsilon$, $r = \sinh^{-1}(x/\epsilon)$ and y near the interior of B_1, we have by (3.18) $(\Delta - (\operatorname{ilg}\epsilon)^2 z^2) = -(\frac{\partial}{\partial r})^2 + \Delta_H$ plus terms of second order. On B_1, $u \restriction B_1$ is bounded, and $(\Delta_H - (\frac{\partial}{\partial r}^2))(u \restriction B_1) = 0$, so $u \restriction B_1$ must be constant in r; hence the values of $\bar{u} \restriction B_2$ match across $s=0$. We also have $(\Delta_H - (\frac{\partial}{\partial r})^2)((\partial_{\operatorname{ilg}\epsilon}u) \restriction B_1) = 0$, and $(\partial_{\operatorname{ilg}\epsilon}u) \restriction B_1 = r\cdot(\partial_{\rho_1}u) \in r\cdot\mathcal{C}^\infty(B_1)$, so $\partial_{\operatorname{ilg}\epsilon}u$ is linear in r: $((\partial_{\operatorname{ilg}\epsilon}u) \restriction B_1) = \alpha + \beta r$. (NB: Here we are taking the derivative $\partial_{\operatorname{ilg}\epsilon}$ keeping r and y fixed, not lifting $\partial_{\operatorname{ilg}\epsilon}$ from X^0_{Ls}.) Hence

$$\frac{\partial u}{\partial s} \restriction B_{+2} \cap B_1 = \beta = \frac{\partial u}{\partial s} \restriction B_{-2} \cap B_1,$$

so also $\partial_s u$ matches across $s=0$.

A similar argument on B_2 and B_0 shows (3.29). Since $u \restriction B_0 \in \operatorname{null}(\Delta_{\tilde{M}})$ and is bounded, the boundary value $u \restriction B_0 \cap B_2 \in \Lambda_\pm^D$. In the interior of B_0 near B_2 we can use coordinates $\xi = \operatorname{ilg} x$, $\operatorname{ilg}\epsilon$ and y. Then by (3.20) $(\Delta - (\operatorname{ilg}\epsilon)^2z^2) = -(\xi^2\frac{\partial}{\partial\xi})^2 + \Delta_{\tilde{M}}$ up to terms of second order. We therefore have $\Delta_{\tilde{M}}(\partial_{\operatorname{ilg}\epsilon}u \restriction B_0) = 0$, and $(\partial_{\operatorname{ilg}\epsilon}u) \restriction B_0 = \xi^{-1}(\partial_s u) \in \xi^{-1}\mathcal{C}^\infty(B_0)$,

$$\frac{\partial u}{\partial s} \restriction B_{\pm 2} \cap B_0 = \xi \cdot \frac{\partial u}{\partial \operatorname{ilg}\epsilon} \restriction B_0 \cap B_2 \in \Lambda_\pm^N.$$

This establishes (3.29). Finally, we show the smoothness of $\bar{u}$. Near the interior of B_2, using s, $\operatorname{ilg}\epsilon$ and y we have, by (3.19) $(\Delta - (\operatorname{ilg}\epsilon)^2z^2) = \operatorname{ilg}\epsilon^2\cdot(D_s^2 - z^2) + \Delta_H$ up to terms of infinite order. Since $(\Delta - (\operatorname{ilg}\epsilon)^2z^2)u$ vanishes to third order on B_2,

$$(D_s^2 - z^2)\bar{u} = \Delta_H((\frac{\partial}{\partial \operatorname{ilg}\epsilon})^2 u \restriction B_2).$$

The left hand side is in $\operatorname{null}(\Delta_H)$ and the right hand side is orthogonal to the null space, so each must vanish. This gives (3.28); and then since $\bar{u}$ satisfies a second order o.d.e. and $\bar{u}$ and its derivative match at $s=0$, $\bar{u}$ extends smoothly across $s=0$. This yields the necessity of (3.28) and (3.29).

To prove the sufficiency, we reverse the argument. The argument yields the (unique) first and second terms in the Taylor series off B_0 and B_1, which are compatible with the given $\bar{u}$ and are killed by the normal operators $R_0(\Delta)$, $R_1(\Delta)$.

These are also compatible with taking $\partial_{\rho_2} u \restriction B_2 = 0$. Then, by propositions 6 and 7, u satisfies (3.27). ∎

The argument to establish the boundary conditions for the Dirac operator is similar, but less complicated since it is a first order operator. In this case, we have as with Δ that if $\delta > 0$ is small enough then each element of the null space of $\eth_0$ on the weighted L^2 space $x^{-\delta} L^2_b(M_\pm)$ has a decomposition near the boundary

$$\eth_0 v = 0,\ v \in x^{-\delta} L^2_b(M_\pm) \Longrightarrow v(x,y) = v_0(y) + v'(x,y),\ \eth_H(v_0) = 0,\ v' \in x^{\delta} L^2_b(M_\pm).$$

The boundary value v_0 defines a map into null$(\eth_H)$; let us call the range $\Lambda_{\eth,\pm}$. It is a Lagrangian subspace of null$(\eth_{\tilde{H}})$ with respect to the symplectic structure on null$(\eth_{\tilde{H}})$ given by the operator $\gamma = \mathrm{cl}(dx/(\sqrt{x^2+\epsilon^2})$. Then we have for the Dirac operator a result analogous to Proposition 3.8:

PROPOSITION 3.9. *Suppose $u \in \mathcal{C}^\infty(X_{\mathrm{Ls}})$ satisfies $(\eth_\epsilon - (\mathrm{ilg}\,\epsilon)z)u = 0$. Then $\overline{u}$ is $\mathcal{C}^\infty$ on $[-1,1]_s$, with*

$$(\gamma D_s - z)\overline{u} = 0, \tag{3.30}$$

$$\text{and} \quad \begin{aligned} &\overline{u} \restriction \{s = -1\} \in \Lambda_{\eth,-} \\ &\overline{u} \restriction \{s = +1\} \in \Lambda_{\eth,+} \end{aligned} \tag{3.31}$$

The proof proceeds strictly analogously to the first part of the proof of Proposition 3.8, so it is omitted.

Chapter 4. The double space and the pseudodifferential calculus

4.1. Preliminary remarks.

In the previous chapter, Ls-differential operators were defined. In this chapter, we set up a calculus of pseudodifferential operators adapted to 'surgery geometry', in the expectation that the generalized inverses of elliptic Ls-differential operators will lie in the calculus. It is not intended to construct a 'full' calculus here; rather, to construct just enough machinery for the problem at hand, which is to construct the resolvent of Δ. Indeed, the resolvent will turn out to be a very special element of the calculus.

The strategy of the construction is:

(i) Construct a parametrix G such that

$$(\Delta - \lambda)\, G = \mathrm{Id} - R, \tag{4.1}$$

where R is a "small" remainder (residual operator).

(ii) Invert $\mathrm{Id} - R$ using the Neumann series:

$$(\mathrm{Id} - R)^{-1} = \mathrm{Id} + R + R^2 + \cdots = \mathrm{Id} + S$$

(iii) We then have

$$(\Delta - \lambda) = G \cdot (\mathrm{Id} + S)\,.$$

To carry out this program, we need to identify a space of residual operators which "iterate away", that is, such that the Neumann series makes sense, is summable and sums to an element of our calculus. Hence we need to define the powers R^j and understand the decay properties of these powers. We also need to understand the compositions $\Delta \cdot G$ (easy since Δ is a differential operator) and $G \cdot S$. In the rest of this chapter we set up this machinery.

4.2. Logarithmic Double space.

By analogy with the logarithmic single space, define

$$X^2_{\mathrm{Ls}} = \left((X^2_s)_{\log} \right)_{\mathrm{tb}}\,. \tag{4.2}$$

Recall that the surgery double space of Mazzeo-Melrose is

$$X^2_s = \left[X^2 \times [0, \epsilon_0]; H^2 \times \{0\}; X \times H \times \{0\}; H \times X \times \{0\} \right]. \tag{4.3}$$

By Lemma 2.10, the b-fibrations $\pi^2_{s,L}, \pi^2_{s,R} \colon X^2_s \longrightarrow X_s$ lift to b-fibrations $\pi^2_{\mathrm{Ls},L}, \pi^2_{\mathrm{Ls},R} \colon$ $X^2_{\mathrm{Ls}} \longrightarrow X_{\mathrm{Ls}}$, such that the following diagram commutes:

$$\begin{array}{ccccc}
X_{\mathrm{Ls}} & \xleftarrow{\pi^2_{\mathrm{Ls},L}} & X^2_{\mathrm{Ls}} & \xrightarrow{\pi^2_{\mathrm{Ls},R}} & X_{\mathrm{Ls}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
X_s & \xleftarrow{\pi^2_{s,L}} & X^2_s & \xrightarrow{\pi^2_{s,R}} & X_s .
\end{array}$$

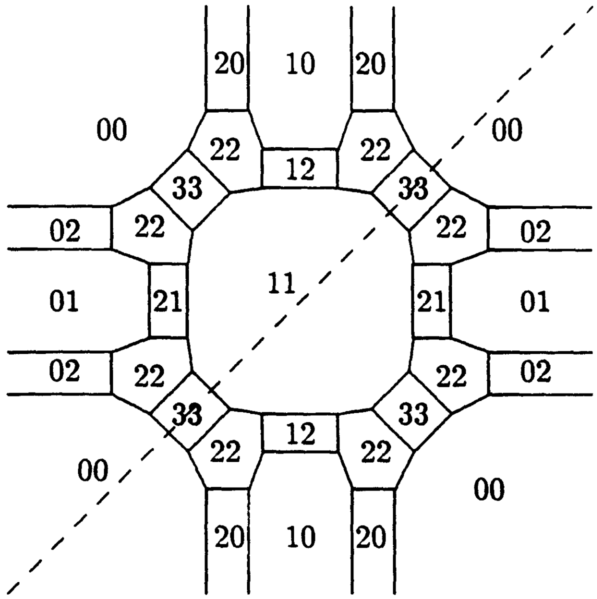


Figure 2. Representation of the boundary hypersurfaces of X^2_{Ls}.

The lifted diagonal $\Delta_{\mathrm{Ls}} \subset X^2_{\mathrm{Ls}}$ is a closed p-submanifold which is canonically diffeomorphic to X_{Ls}; indeed, both projections $\pi^2_{s,L}, \pi^2_{s,R}$ are diffeomorphisms from Δ_{Ls} to X_{Ls}. The structure algebra $\mathcal{V}_{\mathrm{Ls}}(X)$ lifts from either factor to be transversal to Δ_{Ls}.

We shall label the new boundary hypersurfaces of X^2_s as $B_{11}(X^2_s)$, arising from the first blowup of (4.3), $B_{01}(X^2_s)$ arising from the second blowup and $B_{10}(X^2_s)$ arising from the third blowup. The lift of the boundary at $\epsilon = 0$ will be denoted $B_{00}(X^2_s)$; the boundary over $\epsilon = \epsilon_0$ will generally be igonored. We use the same notation for the boundary hypersurfaces of $(X_s)_{\log}$ and for the boundary hypersurfaces of X^2_{Ls} to which they lift. The remaining boundary hypersurfaces arise from the blow up of both codimension three and codimension two faces in the final step of the definition (4.2). The boundary hypersurfaces arising from codimension three faces will be denoted $B_{22}(X^2_{\mathrm{Ls}})$, those arising from the codimension two faces will be denoted $B_{21}(X^2_{\mathrm{Ls}})$, $B_{12}(X^2_{\mathrm{Ls}})$, $B_{33}(X^2_{\mathrm{Ls}})$, $B_{20}(X^2_{\mathrm{Ls}})$ and $B_{02}(X^2_{\mathrm{Ls}})$ according as the face blown up comes from $B_{11}(X^2_s) \cap B_{01}(X^2_s)$, $B_{11}(X^2_s) \cap B_{10}(X^2_s)$, $B_{11}(X^2_s) \cap B_{00}(X^2_s)$, $B_{10}(X^2_s) \cap B_{00}(X^2_s)$ and $B_{01}(X^2_s) \cap B_{00}(X^2_s)$. As usual the connectedness properties of these manifolds depend on the orientation and separation properties of H.

The structure of these faces is given below. Recall that $B_0(X_{\mathrm{Ls}}) = \overline{M}_{\log}$, $B_1(X_{\mathrm{Ls}}) = \overline{H}_{\log}$.

PROPOSITION 4.1. *There are canonical diffeomorphisms:*

$$B_{00}\left(X^2_{\mathrm{Ls}}\right) \equiv \left((\overline{M}^2_{\mathrm{tb}})_{\log}\right)_{\mathrm{tb}}, \tag{4.4}$$

$$B_{10}\left(X^2_{\mathrm{Ls}}\right) \equiv (B_1 \times B_0)_{\mathrm{tb}}, \tag{4.5}$$

$$B_{01}\left(X^2_{\mathrm{Ls}}\right) \equiv (B_0 \times B_1)_{\mathrm{tb}}, \tag{4.6}$$

$$B_{11}\left(X^2_{\mathrm{Ls}}\right) \equiv \left((\overline{H}^2_{\mathrm{tb}})_{\log}\right)_{\mathrm{tb}}, \tag{4.7}$$

$$B_{22}\left(X^2_{\mathrm{Ls}}\right) \equiv [(B^2_2)_{\mathrm{tb}}; \Delta_{\mathrm{fib}}, \Delta^{-1}_{\mathrm{fib}}], \tag{4.8}$$

$$B_{21}\left(X^2_{\mathrm{Ls}}\right) \equiv B_2 \times B_1, \tag{4.9}$$

$$B_{12}\left(X^2_{\mathrm{Ls}}\right) \equiv B_1 \times B_2, \tag{4.10}$$

$$B_{20}\left(X^2_{\mathrm{Ls}}\right) \equiv B_2 \times B_0, \tag{4.11}$$

$$B_{02}\left(X^2_{\mathrm{Ls}}\right) \equiv B_0 \times B_2, \tag{4.12}$$

$$B_{33}\left(X^2_{\mathrm{Ls}}\right) \equiv (B_2)^2. \tag{4.13}$$

In (4.8) Δ_{fib} $(\Delta^{-1}_{\mathrm{fib}})$ *is the fibre diagonal (anti-diagonal) of* B^2_2, *for the fibration over* $[0,1]$, *lifted to* $(B^2_2)_{\mathrm{tb}}$.

PROOF: The maps $\pi^2_{\mathrm{Ls},L}$ and $\pi^2_{\mathrm{Ls},R}$ restricted to B_{mn} map to B_m and B_n respectively, for $(mn) \neq (33)$. In fact $(\pi^2_{\mathrm{Ls},L}, \pi^2_{\mathrm{Ls},R})$ lifts from the interior of B_{mn} to map diffeomorphically to the spaces indicated. B_{33} is canonically diffeomorphic to $\widetilde{H}^2 \times [0,1]_t \times [0,1]_{\log} \equiv \widetilde{H}^2 \times [0,1]^2$ so (4.13) follows. ∎

4.3. Densities.

Lifting the canonical isomorphism

$$\pi_L{}^*\, \Omega_b(M \times [0,\epsilon_0]) \otimes \pi_R{}^*\Omega_b(M \times [0,\epsilon_0]) \equiv \Omega_b\left(M^2 \times [0,\epsilon_0]\right) \otimes \Omega_b([0,\epsilon_0])$$

to the (original) surgery spaces one obtains a canonical isomorphism

$$\pi^{2\,*}_{s,L}\, \Omega_b(X_s) \otimes \pi^{2\,*}_{s,R}\, \Omega_b(X_s) \equiv \Omega_b\left(X^2_s\right) \otimes \Omega_b([0,\epsilon_0]). \tag{4.14}$$

By the above remarks, the lift of these density bundles to the logarithmic surgery spaces gives the canonical isomorphism

$$\pi^{2\,*}_{\mathrm{Ls},L}\, \Omega_D X_{\mathrm{Ls}} \otimes \pi^{2\,*}_{\mathrm{Ls},R}\, \Omega_D X_{\mathrm{Ls}} \equiv \Omega_D X^2_{\mathrm{Ls}} \otimes \Omega_c\left(X^0_{\mathrm{Ls}}\right). \tag{4.15}$$

The square root of this density bundle equation,

$$\pi^{2\,*}_{\mathrm{Ls},L}\, \Omega^{\frac12}_D X_{\mathrm{Ls}} \otimes \pi^{2\,*}_{\mathrm{Ls},R}\, \Omega^{\frac12}_D X_{\mathrm{Ls}} \equiv \Omega^{\frac12}_D X^2_{\mathrm{Ls}} \otimes \Omega^{\frac12}_c\left(X^0_{\mathrm{Ls}}\right), \tag{4.16}$$

will be used in Subsection 5.

4.4. Logarithmic Surgery Pseudodifferential Operators.

The purpose of the logarithmic double space is to carry the kernels of logarithmic surgery pseudodifferential operators (Ls-ψdos for short). These will be defined directly in terms of the properties of their kernels. Let E and F be vector bundles over M, and $\tilde{E}$, $\tilde{F}$ their lifts via $\pi^2_{\mathrm{Ls},L}, \pi^2_{\mathrm{Ls},R}$ respectively to X^2_{Ls}. An Ls-ψdo will be defined to be a distributional section of $\mathrm{Hom}(\tilde{E},\tilde{F}) \otimes \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}$, conormal to Δ_{Ls} with specified behaviour at the boundary of X^2_{Ls}. Before giving the definition, the kernels of some simple operators are studied.

EXAMPLE 4.2. The kernel of the identity operator is, in original coordinates (x, y) near $H \subset M$,

$$\mathrm{Id} = \delta_x(x') \cdot \delta_y(y') \cdot |dx \cdot dx' \cdot dy \cdot dy'|^{\frac{1}{2}} .$$

To study this on X^2_{Ls}, it is convenient to multiply by the formal factor $|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}|^{\frac{1}{2}}$, which makes it a half density on X^2_{Ls}. Near $B_{00} \cap B_{33}$, using coordinates x'/x, $\rho_{33} = \mathrm{ilg}\, x$, and $\rho_{00} = \mathrm{ilg}\,\epsilon/\rho_{33}$, we have

$$\mathrm{Id} \cdot \left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{\frac{1}{2}} = \delta\left(\frac{x'}{x} - 1\right) \cdot \delta_y(y') \cdot \left|\frac{d(x'/x)}{x'/x} \cdot \frac{d\rho_{33}}{\rho_{33}^3} \cdot \frac{d\rho_{00}}{\rho_{00}^2} \cdot dy \cdot dy'\right|^{\frac{1}{2}} ,$$

and near $B_{33} \cap B_{11}$, with $\rho_{33} = \mathrm{ilg}(\epsilon/x)$ and $\rho_{11} = \mathrm{ilg}\,\epsilon/\rho_{33}$,

$$\mathrm{Id} \cdot \left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{\frac{1}{2}} = \delta\left(\frac{x'}{x} - 1\right) \cdot \delta_y(y') \cdot \left|\frac{d(x'/x)}{x'/x} \cdot \frac{d\rho_{33}}{\rho_{33}^3} \cdot \frac{d\rho_{11}}{\rho_{11}^2} \cdot dy \cdot dy'\right|^{\frac{1}{2}} .$$

We see that the kernel of the identity is, in these regions, a distribution conormal to Δ_{Ls}, with values in $\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}$, with elliptic symbol $\equiv 1$; this is true globally.

For another example, consider the differential operator V_0 of (3.10). We have $V_0 = (1 + \epsilon^2/x^2)^{\frac{1}{2}} x \frac{\partial}{\partial x}$. In the above two regions, the kernel of V_0 is given by

$$V_0 \cdot \left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{\frac{1}{2}} = (1 + \frac{\epsilon^2}{x^2})^{\frac{1}{2}} \delta'\left(\frac{x'}{x} - 1\right) \cdot \delta_y(y') \cdot \left|\frac{d(x'/x)}{x'/x} \cdot \frac{d\rho_{33}}{\rho_{33}^3} \cdot \frac{d\rho_{00}}{\rho_{00}^2} \cdot dy \cdot dy'\right|^{\frac{1}{2}}$$

and

$$V_0 \cdot \left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{\frac{1}{2}} = (1 + \frac{\epsilon^2}{x^2})^{\frac{1}{2}} \delta'\left(\frac{x'}{x} - 1\right) \cdot \delta_y(y') \cdot \left|\frac{d(x'/x)}{x'/x} \cdot \frac{d\rho_{33}}{\rho_{33}^3} \cdot \frac{d\rho_{11}}{\rho_{11}^2} \cdot dy \cdot dy'\right|^{\frac{1}{2}} .$$

If there are no y factors present (ie the one dimensional case), this again has an elliptic symbol, namely, ξ. Indeed, since $\mathcal{V}_{\mathrm{Ls}}$ lifts from the left factor of X^2_{Ls} to be transversal to Δ_{Ls}, Ls-differential operators are given precisely by kernels supported on the diagonal with arbitrary polynomial symbols.

We define the set of Ls-ψdos of order k and index family $\mathcal{K}$ to be the sum of two pieces, the small calculus of order k and the boundary terms of index family $\mathcal{K}$:
(4.17)

$$\Psi^{k,\mathcal{K}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}},E,F) = \Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}},E,F)+\Psi^{-\infty,\mathcal{K}}_{\mathrm{bdy}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}},E,F).$$

When the bundles E and F are trivial, or are understood then we will simplify the notation to $\Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$. The small calculus is the "microlocalization"of Ls-differential operators:
(4.18)

$$\Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}},E,F) = \Big\{K \in I^{k-1/4}_{\mathrm{class}}(X^2_{\mathrm{Ls}};\Delta_{\mathrm{Ls}};\mathrm{Hom}(E,F)\otimes\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}});$$
$$K \text{ vanishes rapidly at all boundaries disjoint from } \Delta_{\mathrm{Ls}}\Big\}.$$

This notation means that K is classical conormal to Δ_{Ls} of order $k-1/4$, with values in the bundle $\mathrm{Hom}(E,F)\otimes\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}$. The "$-1/4$" in the order comes from the the extra ϵ dimension; this space of kernels corresponds to order k operators in the usual sense. The boundary piece is smooth in the interior of X^2_{Ls}, hence the superscript $-\infty$, and is polyhomogeneous at all boundaries, with index set $\mathcal{K}(B_{\mathrm{gf}})$ at B_{gf} (here, the index family $\mathcal{K}$ is by definition an assignment of an index set to each boundary hypersurface of X^2_{Ls}):

$$\Psi^{-\infty,\mathcal{K}}_{\mathrm{bdy}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}},E,F) = \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}\otimes\mathrm{Hom}(E,F)). \tag{4.19}$$

The definition of the residual space of operators will be deferred until composition has been discussed.

4.5. Action on Distributions.

In general the action of $A \in \Psi^k(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}},E,F)$ on $u \in C^{-\infty}(X_{\mathrm{Ls}};E)$ is defined by a generalization of the formula "$(Au)(x) = \int A(x,y)u(y)dy$":

$$Au\cdot\mu = \pi^2_{\mathrm{Ls},L\,*}\left(A\cdot\pi^{2\,*}_{\mathrm{Ls},L}\mu\cdot\pi^{2\,*}_{\mathrm{Ls},R}u\cdot\left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{-\frac{1}{2}}\right). \tag{4.20}$$

To explain this: the pushforward is defined naturally on densities. To make $A\cdot\pi^{2\,*}_{\mathrm{Ls},R}u$ a density on X^2_{Ls}, by (4.16) it is necessary to pull up from the left factor an arbitrary (non-vanishing) half-density μ and then cancel off one of the formal $\left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{\frac{1}{2}}$ factors. Then, by (4.15), we have a density, which by letting the $\mathrm{Hom}(E,F)$ part of A act on u can be regarded as F-valued. Pushing forward yields a F-valued density on X_{Ls}, which when divided by μ is a half-density independent of μ. The Pushforward theorem, Theorem 2.3, implies that $\Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}},E,E)$ maps $\mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X_{\mathrm{Ls}};E)$ to itself and that $\Psi^{-\infty,\mathcal{K}}_{\mathrm{bdy}}(X^2_{\mathrm{Ls}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}},E,F)$ maps $\mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X_{\mathrm{Ls}};E)$ to $\mathcal{A}^{\mathcal{G}}_{\mathrm{phg}}(X_{\mathrm{Ls}};F)$ for suitable $\mathcal{K},\mathcal{F},\mathcal{G}$. As these results are not essential here, the details of proofs are omitted.

4.6. The Triple Space.

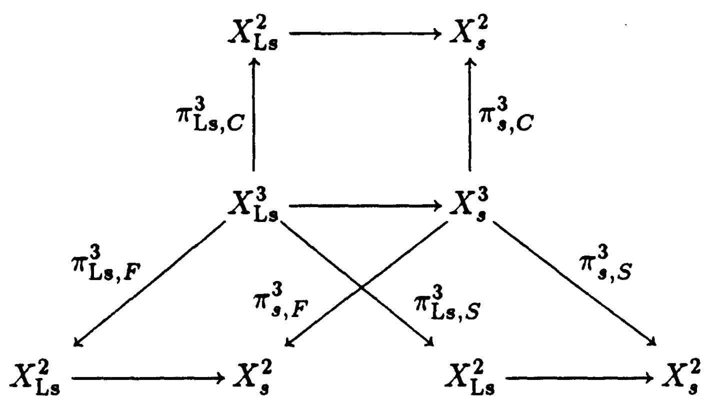
The purpose of the triple space is to define the composition of Ls-ψdos. Composition of integral kernels requires three sets of variables, one of which is integrated out. To integrate in the context of Ls-ψdos we need a "triple space" with a b-fibration down to the double space; by the Pushforward theorem this will preserve polyhomogeneity at the boundary.

We define the triple logarithmic space exactly analogously to the single and double logarithmic spaces. It is obtained by further blowup from the triple space X_s^3 considered in Mazzeo-Melrose and [16]:

$$X_{\mathrm{Ls}}^3 = \left((X_s^3)_{\log}\right)_{\mathrm{tb}}. \tag{4.21}$$

Another application of Lemma 2.10 implies that we have a commutative diagram:

(4.22)



where the maps $\pi^3_{\mathrm{Ls},F}, \pi^3_{\mathrm{Ls},C}, \pi^3_{\mathrm{Ls},S}$ are b-fibrations.

We lift the b-density bundle of X_s^3 to X^3_{Ls} as with the double space in (4.15):

$$\Omega_D X^3_{\mathrm{Ls}} = \pi^*_{\log\,\mathrm{tb}} \Omega_b X_s^3$$

and get, as before, a canonical isomorphism

$$\pi^{3\,*}_{\mathrm{Ls},F}\Omega_D X^2_{\mathrm{Ls}} \otimes \pi^{3\,*}_{\mathrm{Ls},C}\Omega_D X^2_{\mathrm{Ls}} \otimes \pi^{3\,*}_{\mathrm{Ls},S}\Omega_D X^2_{\mathrm{Ls}} \equiv \Omega_D X^3_{\mathrm{Ls}} \otimes \Omega_D X^3_{\mathrm{Ls}} \otimes \pi^*\Omega_c X^0_{\mathrm{Ls}}.$$

The square root of this density bundle equation

$$\pi^{3\,*}_{\mathrm{Ls},F}\Omega_D^{\frac12} X^2_{\mathrm{Ls}} \otimes \pi^{3\,*}_{\mathrm{Ls},C}\Omega_D^{\frac12} X^2_{\mathrm{Ls}} \otimes \pi^{3\,*}_{\mathrm{Ls},S}\Omega_D^{\frac12} X^2_{\mathrm{Ls}} \equiv \Omega_D X^3_{\mathrm{Ls}} \otimes \pi^*\Omega_c^{\frac12} X^0_{\mathrm{Ls}} \tag{4.23}$$

is used below in the composition formula.

4.7. Compositon and the Residual Space.

The composition of two Ls-ψdos is in general defined by a formula similar to (4.20) for the action of a distribution. It generalizes the formula "$(AB)(x,y) = \int A(x,z)B(z,y)dz$":

$$AB \cdot \nu = \pi^3_{\mathrm{Ls},C\,*}\left(\pi^{3\,*}_{\mathrm{Ls},C}\nu \cdot \pi^{3\,*}_{\mathrm{Ls},F}A \cdot \pi^{3\,*}_{\mathrm{Ls},S}B \cdot \left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{-\frac12}\right). \tag{4.24}$$

Here, ν is a nonvanishing Ls-half density on X^2_{Ls}. The term in the large parentheses is by (4.23) a density on X^3_{Ls} so it can be pushed forward to X^2_{Ls}.

We wish to establish the composition results listed in section 4.1. One of these, $\Delta \cdot G$, $G \in \Psi^{-2,0}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ is easy to understand, since composition on the left by a Ls-differential operator has an alternative description in terms of lifting Ls-vector fields by $\pi^2_{\mathrm{Ls},L}\colon X^2_{\mathrm{Ls}} \longrightarrow X_{\mathrm{Ls}}$, and letting them act on G. It is then immediate that composition with Δ maps $\Psi^{-2,\mathcal{E}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}) \longrightarrow \Psi^{0,\mathcal{E}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$.

A second requirement was to find a "residual space" of operators that iterate away. We shall define as our class of residual operators, $\Psi^{-\infty}_{\mathrm{res}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$, a space of conormal Ls-half densities R such that, if we regard R as a family of operators $R(\mathrm{ilg}\,\epsilon)$ parametrized by $\mathrm{ilg}\,\epsilon$, and multiplied by our formal factor $\left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{1/2}$, we have an estimate of the form $\|R\|_{\mathrm{HS}}(\mathrm{ilg}\,\epsilon) \to 0$, where HS denotes Hilbert-Schmidt norm. This is a convenient norm with which to work; a Hillbert-Schmidt operator is compact and the norm is easy to calculate from the kernel of the operator, it is just the L^2 norm of the kernel:

$$\|R\|^2_{\mathrm{HS}}(\mathrm{ilg}\,\epsilon) = \left\| R \left| \frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2} \right|^{-\frac{1}{2}} \right\|^2_{L^2(M,g_\epsilon)} = \pi_*(|R|^2) \left| \frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2} \right|^{-1} (\mathrm{ilg}\,\epsilon)$$

where $\pi_*\colon X^2_{\mathrm{Ls}} \longrightarrow X^0_{\mathrm{Ls}}$ is integration along the leaves $\mathrm{ilg}\,\epsilon = \text{constant}$.

We now calculate a multiweight $\mathfrak{t}$ such that $\mathcal{A}^{\mathfrak{t}}_{-}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ has this property. The notation used here is that of [19]. Note that if $R \in \mathcal{A}^{\mathfrak{t}}_{-}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$, then

$$\begin{aligned} |R|^2 &\in \mathcal{A}^{2\mathfrak{t}}_{-}(X^2_{\mathrm{Ls}}; \Omega_D X^2_{\mathrm{Ls}}) \\ &\in \mathcal{A}^{2\mathfrak{t}-\mathfrak{d}}_{-}(X^2_{\mathrm{Ls}}; \Omega_b) \end{aligned}$$

where $\mathfrak{d}$ is the degree multiweight. Therefore, with π as above,

$$\begin{aligned} &\pi_*(|R|^2) \in \mathcal{A}^{\pi_\#(2\mathfrak{t}-\mathfrak{d})}_{-}(X^0_{\mathrm{Ls}}; \Omega_b) \\ \Rightarrow\ &\pi_*(|R|^2) \left| \frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2} \right|^{-1} \in \mathcal{A}^{\pi_\#(2\mathfrak{t}-\mathfrak{d})+1}_{-}(X^0_{\mathrm{Ls}}; \Omega) \end{aligned}$$

and $\pi_\#(2\mathfrak{t} - \mathfrak{d}) + 1$ will be a positive multiweight provided $2\mathfrak{t} - \mathfrak{d} + 1 \geq 0$. Hence for any $\mathfrak{t} \geq (1/2)(\mathfrak{d} - 1)$ we have a suitable residual space $\mathcal{A}^{\mathfrak{t}}_{-}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$.

Observe that the polyhomogeneous space

$$\Psi^{-\infty,\mathcal{E}}_{\mathrm{bdy}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}) \subset \Psi^{-\infty}_{\mathrm{res}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}) \text{ if } \mathcal{E}(G) > \frac{d(G)-1}{2}.$$

For the purpose of constructing the resolvent of Δ, we can identify a smaller, polyhomogeneous residual space which is closed under composition and taking Neumann series. We will want to stick with operators that have only integral powers in their index sets, since our parametrix will be constructed as a (finite) Taylor series. However, for our residual space to be closed under composition, we will have to allow logarithmic terms as well, that is, $(n, k) \in \mathcal{E}(H)$ with $k > 0$. In the case of our error $\mathrm{Id} - \Delta \cdot G$ we will show after the event that all logarithmic terms vanish but this is certainly special to the problem at hand.

DEFINITION 4.3. An index set E is *natural* if all its powers are natural numbers, that is, $(z,k) \in E \Rightarrow z \in \mathbb{N}$. An index family is natural if all its index sets are natural.

Note that if f is a simple b-fibration, then the operators $f^{\#}$, $f_{\#}$ of Theorems 2.2 and 2.3 preserve naturality. Let $\mathcal{E}$ be the $\mathcal{C}^\infty$ (and hence natural) index family

$$\begin{aligned}\mathcal{E}(H) &= \{(n,0) \mid n \geq 1\} \text{ if degree}(H) = 1\\ &= \{(n,0) \mid n \geq 2\} \text{ otherwise.}\end{aligned}$$

Then $\Psi_{\text{bdy}}^{-\infty,\mathcal{E}}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}}) \subset (\text{ilg}\,\epsilon)^{1-\tau} \cdot \Psi_{\text{res}}^{-\infty}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})\ \forall\tau > 0$, which implies

$$\Psi_{\text{bdy}}^{-\infty,\mathcal{E}}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})^k \subset (\text{ilg}\,\epsilon)^{k-\tau} \cdot \Psi_{\text{res}}^{-\infty}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})\ \forall\tau > 0. \tag{4.25}$$

Moreover, by the Pushforward theorem (4.25) is another polyhomogeneous space:

$$\Psi_{\text{bdy}}^{-\infty,\mathcal{E}}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})^k = \Psi_{\text{bdy}}^{-\infty,\mathcal{E}_k}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})$$

for some index family $\mathcal{E}_k$. $\mathcal{E}_k$ is a natural index family, since $\Omega_D X^2_{\text{Ls}}$ is given by $\Omega_b X^2_{\text{Ls}}$ times integral powers of boundary defining functions, and the operators $f_{\#}, f^{\#}$ yield only integral powers, because the maps $\pi^3_{\text{Ls},F}$, $\pi^3_{\text{Ls},C}$, $\pi^3_{\text{Ls},S}$ are simple b-fibrations. And by (4.25), Inf Re $\mathcal{E}_k \geq k$. Hence there is a natural index set $\mathcal{F} = \cup_k \mathcal{E}_k$ such that $\Psi_{\text{bdy}}^{-\infty,\mathcal{F}}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}}) \subset \Psi_{\text{res}}^{-\infty}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})$ is closed under composition. Let us call it the parametrix-residual space, $\Psi_{\text{par-res}}^{-\infty}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})$.

4.8. Symbol Map.

In the ordinary pseudodifferential operator calculus, we use the symbol map as a tool to invert elliptic operators (see [12], chapter 18). In the logarithmic surgery calculus, we also have a symbol map on the small calculus defined exactly analogously. The symbol of $A \in \Psi^k_{\text{small}}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})$ is the Fourier transform of A transverse to Δ_{Ls}; this Fourier transform is a classical symbol by definition of $\Psi^k_{\text{small}}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})$. To make the density factors work out, we should divide A by the formal half density factor $|\frac{d(\text{ilg}\,\epsilon)}{(\text{ilg}\,\epsilon)^2}|^{1/2}$, multiply by the half-density $\exp^{-i(\xi\log(x'/x)+\eta\cdot(y'-y))}|\frac{dx'}{x'}dy'd\xi d\eta|^{1/2}$ and integrate over x' and y', obtaining a half-density. Thus $\sigma(A)(p,\xi,\eta)|\frac{dx}{x}dyd\xi d\eta|^{1/2}$ is given by

$$\int e^{-i(\xi\log(x'/x)+\eta\cdot(y'-y))}A(p,\frac{x'}{x},y'-y)\psi(|y-y'|+|\log\frac{x'}{x}|)|\frac{dx'}{x'}dy'||\frac{dx}{x}dyd\xi d\eta|^{\frac{1}{2}}.$$

Here $p \in \Delta_{\text{Ls}}$, ψ is a cutoff function and we use $\log(x'/x)$, $y'-y$ for coordinates transverse to Δ_{Ls}. The half-density on the left hand side of this equation is a canonical factor on $N^*\Delta_{\text{Ls}}$, so can be cancelled; the symbol then is a function on $N^*\Delta_{\text{Ls}}$, which is polyhomogeneous on the fibres. It is of course the same as the lift of the symbol defined in Mazzeo-Melrose on Δ_s, lifted to Δ_{Ls}. The principal symbol $\sigma_k(A)$ is the degree k part; as usual it is invariant of coordinate changes in x and y. There is a surjective quantization map from symbols of order k to $\Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ given by inverse Fourier transform of the symbol in the (ξ, η) directions. Hence the symbol map may be used to solve operator equations $P \cdot G = \mathrm{Id}$, $P \in \Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ elliptic, up to $\Psi^{-\infty}_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ errors. These errors unfortunately are not uniformly compact in ϵ, since elements of $\Psi^{-\infty}_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ do not vanish at the boundaries B_{00}, B_{11}, B_{33}. To construct parametrices for elliptic operators one also needs to solve model problems at the boundary.

LEMMA 4.4. *Δ_ϵ is an elliptic Ls-ψdo.*

PROOF: Away from B_1 and B_2 this is true because the metric is non-degenerate. At B_1 and B_2 we have, up to a symbol of at most second order vanishing to infinite order (in Taylor series) at H:

$$\sigma_2(\Delta)(\xi, \eta) = |\xi|^2 + |\eta|^2. \blacksquare$$

4.9. Model Operators.

We will define model operators for $A \in \Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ and for $A \in \Psi^0_{\mathrm{bdy}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$, where 0 denotes the $\mathcal{C}^\infty$-index family. For a "full" calculus we would also want to define them for general $\Psi^{\mathcal{K}}_{\mathrm{bdy}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$, but for this thesis that is not necessary.

The model operator N_{mn} at face B_{mn} is in principle just the restriction of A to B_{mn}. To make density factors work out, refer back to Proposition 4.1. Equip $\overline{M}_{\log}$ and $\overline{H}_{\log}$ with the cusp density bundles $\Omega_c(\overline{M}_{\log})$, $\Omega_c(\overline{H}_{\log})$, and oH with the usual density bundle $\Omega(\mathrm{oH})$. Equip their products with the product density bundles, lift to the B_{mn}, and denote the resulting density bundles $\Omega_{\mathrm{Ls}} B_{mn}$. Then at B_{mn} if we divide by the canonical factor $\left|d\rho_{mn}/(\rho_{mn}(\mathrm{ilg}\,\epsilon)^d)\right|^{1/2}$ where d is the degree of B_{mn}, we get a restriction map $N_{mn}\colon \mathcal{C}^\infty(\Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}) \longrightarrow \mathcal{C}^\infty(\Omega^{1/2}_{\mathrm{Ls}} B_{mn})$.

These model operators have different characters. N_{00} and N_{11} map $\Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ to $\Psi^k_{c,\mathrm{small}}(\overline{M})$ and $\Psi^k_{c,\mathrm{small}}(\overline{H})$, the "cusp" pseudodifferential operators on $\overline{M}$ and $\overline{H}$, which microlocalize the cusp algebra defined above Lemma 3.3. We do not need any facts about cusp pseudodifferential operators beyond the fact that b-pseudodifferential operators lift to cusp pseudodifferential operators under logarithmic blowup. These two model operators are the lifts to X^2_{Ls} of the normal homomorphisms in Mazzeo-Melrose.

The model operator N_{33} maps $\Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ to a family of translation invariant operators in $\Psi^k_{c,\mathrm{small}}(\overline{H})$ parametrized smoothly by s, the coordinate in (3.5) along $\Delta_{\mathrm{Ls}} \cap B_{33}$. This is because B_{33} is the lift of the $B_0 \cap B_1$ to X^2_{Ls}, so the lifts of $\mathcal{V}_{\mathrm{Ls}}$ from both the left and right are tangent to the leaves s = constant ($= s'$), and so we get an "indicial operator" (see [17] or [15]) on each leaf. For an operator B lifted from $\Psi^k_{\mathrm{small}} X^2_s$, $N_{33}(B)$ will be the indicial operator of B on each leaf $s =$ constant, but in general the $N_{33}(A)$ will depend on s. There is a consistency condition between the symbol map and each of the models N_{00}, N_{11}, N_{33}: the restriction of the symbol to each of these faces equals the symbol of the model operator.

The other models $N_{01}, N_{10}, N_{12}, N_{21}, N_{02}, N_{20}$ and N_{22} map $\Psi^{-\infty,0}_{\mathrm{bdy}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ to $\mathcal{C}^\infty(N_{mn}; \Omega^{1/2}_{\mathrm{Ls}} B_{mn})$, and $\Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ to zero, since kernels in $\Psi^k_{\mathrm{small}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ vanish to infinite order at all these boundaries.

To construct the resolvent of Δ we will need to know the model operators of $\Delta \cdot G$.

LEMMA 4.5. *In the notation of Lemma 3.3 we have*

$$N_{mn}(\Delta \cdot G) = R_m(\Delta) \cdot N_{mn}(G), \quad (mn) \neq (33); \tag{4.26}$$

$$N_{33}(\Delta \cdot G) = I(\Delta) \cdot N_{33}(G). \tag{4.27}$$

PROOF: Lemma 4.1 identifies the various boundary hypersurfaces $B_{mn} \neq B_{33}$ as blown up products of boundary hypersurfaces of X_{Ls}. Since Δ acts on G by products of $\mathcal{V}_{\mathrm{Ls}}$-vector fields lifted from the left, which are tangent to the boundary, (4.26) follows. For N_{33}, $\mathcal{V}_{\mathrm{Ls}}$ lifts from the left to be tangent to the leaves s = constant, so Δ acts by $I(\Delta)$ on each leaf. ∎

Finally, we discuss the reduced normal operator $\mathrm{RN}(\Delta)$. The following is a double space version of Lemma 3.7.

LEMMA 4.6. *Suppose* $N_{22}(\Delta G) = 0$, *that is,* $\Delta_H(N_{22}(G)) = 0$. *Let* ρ_{22} *be a boundary defining function for* B_{22} *and let* s, s' *be the coordinates on* X^2_{Ls} *defined by* (3.5). *If* $\Delta_H(\nabla_{\partial \rho_{22}})G = 0$ *then* ΔG *vanishes to second order at* B_{22} *and*

$$N_{22}\left(\frac{\Delta G}{(\mathrm{ilg}\,\epsilon)^2}\right) = D^2_s(N_{22}(G)). \tag{4.28}$$

The result follows from Lemma 3.7.

4.10. Neumann Series for Residual Operators.

If $R \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ then we know from the discussion of section 4.7 that $R^j \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$, $\|R^j\|_{\mathrm{HS}} = \mathcal{O}((\mathrm{ilg}\,\epsilon)^{\frac{j}{2}-\tau})$ $\forall \tau$. Therefore, $\sum_{j=0}^\infty R^j$ converges to $\mathrm{Id} + S$ in Hilbert-Schmidt norm (for example), where S is Hilbert-Schmidt; in fact, $\|S\|_{\mathrm{HS}} = O((\mathrm{ilg}\,\epsilon)^{1/2-\tau})$ $\forall \tau$. This however says nothing about the smoothness of S. The next lemma states that the sum of the Neumann series actually stays in the parametrix-resolvent space.

LEMMA 4.7. *Let* $R \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$, *and let* $\mathrm{Id} + S = \sum_{j=0}^\infty R^j$ *as above. Then* $S \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$.

PROOF: First we show $S \in \mathcal{A}^{\tau}_{-}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$ for all $\tau < 1/2$. To do this, we must show

$$\mathcal{V}_b(X^2_{\rm Ls})^k \cdot S \subset \operatorname{ilg}\epsilon^{\tau} L^2(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) \text{ for all } k,\ \tau < \frac{1}{2}.$$

We know that R^j is conormal; in fact,

$$R^j \in \mathcal{A}^{\frac{j}{2}-\delta}_{-}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) \text{ for all } \delta > 0.$$

Hence, if $V_1, \dots, V_k \in \mathcal{V}_b(X^2_{\rm Ls})$, then

$$V_1 \dots V_k R^j \in \mathcal{A}^{\frac{j}{2}-\delta}_{-}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) \text{ for all } k,\ \delta > 0.$$

It follows that $V_1 \dots V_k R^j$ converges in Hilbert-Schmidt norm, for ϵ small. Since V_i is a closed operator on the space of continuous maps from $[0, \operatorname{ilg}\epsilon_0]$ to Hilbert-Schmidt operators on M, it follows that $\sum_j V_1 \dots V_k R^j = V_1 \dots V_k S$, so $V_1 \dots V_k S \in \mathcal{A}^{\tau}_{-}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$ for all $\tau < 1/2$, as claimed.

To complete the proof, consider the identity

$$S = R + R^2 + \dots + R^{2j} + R^j S R^j. \tag{4.29}$$

The right hand side is in $\Psi^{-\infty}_{\text{par-res}}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) + \mathcal{A}^{j-\delta}_{-}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$. Thus $S \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) + \mathcal{A}^{j-\delta}_{-}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$ for all j, which implies $S \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$. ∎

4.11. Composition of small calculus with residual calculus.

In this section we analyse products like $G \cdot S$ of section 4.1.

LEMMA 4.8. *The composition formulae*

$$\Psi^{k,0}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) \cdot \Psi^{-\infty}_{\text{par-res}}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) \subset (\operatorname{ilg}\epsilon)\Psi^{-\infty,\mathcal{F}}_{\text{bdy}}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$$
$$\Psi^{-\infty}_{\text{par-res}}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) \cdot \Psi^{k,0}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls}) \subset (\operatorname{ilg}\epsilon)\Psi^{-\infty,\mathcal{F}}_{\text{bdy}}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$$

hold, where $\mathcal{F}$ is a natural index family.

PROOF: Let $A \in \Psi^{k,0}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$ and $B \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\rm Ls};\Omega^{\frac12}_D X^2_{\rm Ls})$. First we note that the lift of $\Delta_{\rm Ls} \subset X^2_{\rm Ls}$ under $\pi^3_{{\rm Ls},F}$ or $\pi^3_{{\rm Ls},S}$ is transverse to $\pi^3_{{\rm Ls},C}$. Standard results about wavefront sets show that the singularities of $\pi^{3\,*}_{{\rm Ls},F}A$ conormal to $\pi^{3\,*}_{{\rm Ls},F}\Delta_{\rm Ls}$ are wiped out by $\pi^3_{{\rm Ls},C\,*}$, so the kernel of the composition is smooth in the interior. We need to analyse the behaviour at the boundary.

Write ρ_i for the product of boundary defining functions over all boundary hypersurfaces of degree i in $X^3_{\rm Ls}$, and r_i for the same product in $X^2_{\rm Ls}$. Observe that $\pi^3_{{\rm Ls},F}$, $\pi^3_{{\rm Ls},C}$, and $\pi^3_{{\rm Ls},S}$ are simple b-fibrations with the following 'degree property':

if $H \in M_1(X^3_{\mathrm{Ls}})$ is mapped to $G \in M_1(X^2_{\mathrm{Ls}})$, then $d(H) = d(G)$ or $d(G)+1$. Hence we have

$$\pi^{3}_{\mathrm{Ls},F}{}^{*}A \in I^{k-1/4}_{\mathrm{class}}(X^3_{\mathrm{Ls}}; \pi^{3}_{\mathrm{Ls},F}{}^{*}\Omega^{\frac{1}{2}}_{D}X^2_{\mathrm{Ls}}; \pi^{3}_{\mathrm{Ls},F}{}^{*}(\Delta_{\mathrm{Ls}})),$$
$$\pi^{3}_{\mathrm{Ls},S}{}^{*}B \in \rho_1\rho_2\rho_3^2\rho_4^2\mathcal{A}^{\mathcal{E}'}_{-}(X^3_{\mathrm{Ls}}; \pi^{3}_{\mathrm{Ls},S}{}^{*}\Omega^{\frac{1}{2}}_{D}X^2_{\mathrm{Ls}})$$

where $\mathcal{E}'$ is natural. Hence by the the degree property and Lemma 2.11,

(4.30)
$$\pi^{3}_{\mathrm{Ls},C}{}^{*}\nu\cdot\pi^{3}_{\mathrm{Ls},F}{}^{*}A\cdot\pi^{3}_{\mathrm{Ls},S}{}^{*}B\cdot\Big|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\Big|^{-\frac{1}{2}} \in \rho_1\rho_2\rho_3^2\rho_4^2 I^{k-1/4}_{\mathrm{class}}\mathcal{A}^{\mathcal{E}'}_{-}(X^3_{\mathrm{Ls}}; \Omega_D X^3_{\mathrm{Ls}}; \pi^{3}_{\mathrm{Ls},F}{}^{*}(\Delta_{\mathrm{Ls}})),$$

where we use the notation of section 2.3. Using the degree property and the Pushforward theorem we have

$$\begin{aligned} AB\cdot\nu &= \pi^3_{\mathrm{Ls},C\,*}\left(\pi^{3}_{\mathrm{Ls},C}{}^{*}\nu\cdot\pi^{3}_{\mathrm{Ls},F}{}^{*}A\cdot\pi^{3}_{\mathrm{Ls},S}{}^{*}B\cdot\Big|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\Big|^{-\frac{1}{2}}\right) \\ &\in r_1^0 r_2^1 r_3^1\mathcal{A}^{\pi^3_{\mathrm{Ls},C\#}\mathcal{E}'}_{-}(X^2_{\mathrm{Ls}}; \Omega_D X^2_{\mathrm{Ls}}). \end{aligned} \tag{4.31}$$

Since $\pi^3_{\mathrm{Ls},C}$ is a simple b-fibration, $\pi^3_{\mathrm{Ls},C\,\#}\mathcal{E}' \equiv \mathcal{F}$ is natural. In fact by further analysing the stretched projections from X^3_{Ls} to X^2_{Ls} we can show

$$AB\cdot\nu \in r_1^1 r_2^1 r_3^1\mathcal{A}^{\mathcal{F}}_{-}(X^2_{\mathrm{Ls}}; \Omega_D X^2_{\mathrm{Ls}}). \tag{4.32}$$

Consider a boundary hypersurface G of X^2_{Ls} of degree one, and suppose $H \in M_1(X^3_{\mathrm{Ls}})$ is mapped by $\pi^3_{\mathrm{Ls},C}$. Then $d(H) = 1$ or 2. If $d(H) = 1$, then the above argument shows that this yields the power r_1^1 at G. If $d(H) = 2$, then we use the following fact about how hypersurfaces in X^3_{Ls} map under the three stretched projections to X^2_{Ls}: every $H \in M_1(X^3_{\mathrm{Ls}})$ of degree two projects to a degree one hypersurface under at most one stretched projection. Therefore, H must map to a degree 2 hypersurface under both $\pi^3_{\mathrm{Ls},F}$ and $\pi^3_{\mathrm{Ls},S}$. This implies that the left hand side of (4.30) has top power 2 at H. This then produces top power r_1^1 at G. So we have established (4.32). The same arguments goes through for composition in the other order. Since $r_1r_2r_3 = (\mathrm{ilg}\,\epsilon)$ up to a smooth nonzero function, the lemma is proved. ∎

Chapter 5. One dimensional surgery resolvent

We return to the reduced normal operator of section 3.4. This model operator does not appear in Mazzeo-Melrose and is precisely the new ingredient that allows us to construct the surgery parametrix down to $\lambda = 0$.

5.1. Scaling property.

Let $\overline{K}(s,s',z)|ds\cdot ds'|^{1/2}$ be the kernel of $D_s^2 - z^2$ on the interval $[-1,\ 1]_s$, acting on functions $\overline{u}$ taking values in the vector space $V = \operatorname{null}(\Delta_H)$, with boundary conditions as in (3.29). To illustrate the scaling property of this operator, let $K\cdot|dg_\epsilon dg'_\epsilon|^{1/2}$ be the kernel of $\left(\Delta - (\operatorname{ias}\epsilon)^2 z^2\right)^{-1}$ on V-valued functions defined on $[-1,1]_x$ with surgery metric $g_\epsilon = dx^2/(x^2+\epsilon^2)$ and with boundary conditions as above. Then arclength is

$$r = \int_0^x \frac{d\tilde{x}}{\sqrt{\tilde{x}^2+\epsilon^2}} = \sinh^{-1}\frac{x}{\epsilon}$$

so the length of the interval with respect to g_ϵ is $2\sinh^{-1}(1/\epsilon) \equiv 2L_\epsilon$.

Define the function $\operatorname{ias} a \equiv 1/\sinh^{-1} a$, so that $\operatorname{ias}\epsilon = L_\epsilon^{-1}$. Since $\operatorname{ias}\epsilon = \operatorname{ilg} 2\epsilon + O(\operatorname{ilg}\epsilon^\infty) = \operatorname{ilg}\epsilon(1+(\log 2)\operatorname{ilg}\epsilon)^{-1}$, $\operatorname{ilg}\epsilon$ and $\operatorname{ias}\epsilon$ are C^∞ functions of each other, however as the above equation shows, $\operatorname{ias}\epsilon$ is a more natural function to use than $\operatorname{ilg}\epsilon$ in this setting. Let s = rescaled arclength = r/L_ϵ. So $s\in[-1,1]\,\forall\epsilon$. We have $\Delta_\epsilon = D_r^2 = (\operatorname{ias}\epsilon)^2 D_s^2$, so therefore

$$\begin{aligned}(\operatorname{ias}\epsilon)^2\left(D_s^2 - z^2\right)K = \operatorname{Id} &= \delta(r-r')|dg_\epsilon dg'_\epsilon|^{1/2}\\ &= (\operatorname{ias}\epsilon)\delta(s-s')|dg_\epsilon dg'_\epsilon|^{1/2}.\end{aligned}$$

Thus, using coordinates s, s', K scales in ϵ as $(\operatorname{ias}\epsilon)^{-1}$:

$$K(s,s',\epsilon,z) = (\operatorname{ias}\epsilon)^{-1}\overline{K}(s,s',z). \tag{5.1}$$

Let us now multiply K by the formal density factor $\left|\frac{d(\operatorname{ilg}\epsilon)}{(\operatorname{ilg}\epsilon)^2}\right|$ and lift to the logarithmic double space $X^2_{\mathrm{Ls}}([-1,1]_x)$.

LEMMA 5.1. *The lift of K to $X^2_{\mathrm{Ls}}\times\mathbb{C}_z$ is a D-density meromorphic in z, conormal to Δ_{Ls} and $(\operatorname{ias}\epsilon)^{-1}\times$ smooth up to all boundary hypersurfaces. In other words, K is a meromorphic family of Ls-ψdos with index family $C^\infty\cup\{(-1,0)\}$.*

PROOF: All these assertions follow easily from (5.1). Meromorphy in z follows from meromorphy of the resolvent $\overline{K} = (\Delta - z^2)^{-1}$. To show that it lifts to be conormal to Δ_{Ls}, observe that $\overline{K}(s,s',z) = -1/2|s-s'| + \operatorname{smooth}(s,s',z)$. Since s is a smooth function on X_{Ls}, $(\operatorname{ias}\epsilon)^{-1}\cdot\operatorname{smooth}(s,s',z)$ lifts to be $(\operatorname{ias}\epsilon)^{-1}\times$ smooth on X^2_{Ls}. We need only show that $(\operatorname{ias}\epsilon)^{-1}1/2|s-s'|$ lifts to be conormal on X^2_{Ls}. This is clear away from $\epsilon=0$, so we need only check this in Taylor series at $\epsilon = 0$. Near the interior of B_{11}, $(\operatorname{ias}\epsilon)^{-1}|s-s'| = |\sinh^{-1}(x/\epsilon) - \sinh^{-1}(x'/\epsilon)|$. Since x/ϵ, x'/ϵ are coordinates on the interior of B_{11}, this is conormal to $\Delta_{\mathrm{Ls}} = \{x/\epsilon = x'/\epsilon\}$. Near B_{33} and B_{00},

$$(\mathrm{ias}\,\epsilon)^{-1}(s-s') = \big(\log(x/\epsilon) - \log 2 - \log(x'/\epsilon) + \log 2 + O(\rho_{33}^{\infty}\rho_{00}^{\infty})\big)$$

so $(\mathrm{ias}\,\epsilon)^{-1}|s-s'| = |\log(x/x')| + O(\rho_{33}^{\infty}\rho_{00}^{\infty})$ which is conormal to $\Delta_{\mathrm{Ls}} = \{\log(x/x') = 0\}$. ∎

5.2. Scattering Matrix.

Let us calculate the scattering matrix, as defined in [17], chapter 6, for the two one dimensional Laplacians $N_0(\Delta) = \Delta_0$ on $M_\pm = [-1,0]_x$ and $[0,1]_x$. On $[0,\pm 1]_x$, $\Delta_0 = -(xD_x)^2$ is a b-Laplacian near $x=0$, and looks like $-D_x^2$ at $x = \pm 1$, with mixed Dirichlet-Neumann boundary conditions as above. The scattering solutions of $(\Delta_0 - z^2)u = 0$ are given by

$$\left\{v(x^{iz} + x^{-iz}), w(x^{iz} - x^{-iz}) \mid v \in \Lambda_\pm^D,\ w \in \Lambda_\pm^N\right\}.$$

The scattering matrix can be read off from this:

$$S(z) = \mathrm{proj}_{\Lambda_\pm^D} - \mathrm{proj}_{\Lambda_\pm^N},$$

independent of z. We also read off that the smooth and logarithmically growing null spaces of Δ_0 are precisely $\Lambda_\pm^D$ and $\Lambda_\pm^N$. It follows that the reduced normal operator of $\Delta_\epsilon - (\mathrm{ias}\,\epsilon)^2 z^2$ on $[-1,1]_x$ with boundary conditions as above is the operator $D_s^2 - z^2$ with the same boundary conditions. In this case, therefore, the reduced normal operator "reproduces" the original operator.

5.3. Properties at the boundary.

The boundary conditions associated with Δ imply that

$$\overline{K}(\pm 1, s', z) \in \mathrm{Hom}(V, \Lambda_\pm^D)\,,$$
$$\partial_s \overline{K}(\pm 1, s', z) \in \mathrm{Hom}(V, \Lambda_\pm^N).$$

Because Δ is self-adjoint, we also have

$$\overline{K}(s, \pm 1, z) \in \mathrm{Hom}(\Lambda_\pm^D, V)\,,$$
$$\partial_{s'} \overline{K}(s, \pm 1, z) \in \mathrm{Hom}(\Lambda_\pm^N, V).$$

Therefore, we have $\overline{K}(\pm 1, \pm 1, z) \in \mathrm{Hom}(\Lambda_\pm^D, \Lambda_\pm^D)$ and $\overline{K}(\pm 1, \mp 1, z) \in \mathrm{Hom}(\Lambda_\pm^D, \Lambda_\mp^D)$. We also have, since $(\Delta - z^2)\cdot \overline{K} = \delta(s-s')$,

$$\lim_{s'\downarrow s} \partial_s \overline{K}(s,s',z) = \mathrm{Id} + \lim_{s'\uparrow s} \partial_s \overline{K}(s,s',z).$$

Hence

$$\begin{aligned}\mathrm{Hom}(\Lambda_+^N, V) \ni \lim_{s\to 1} \partial_{s'} \overline{K}(s,1,z) &= \lim_{s'\to 1} \left(\partial_s \overline{K}(1,s',\overline{z})\right)^* \\ &= -\mathrm{Id} + \lim_{s'\to 1} \left(\partial_s \overline{K}(s,1,\overline{z})\right)^*\end{aligned}$$

and $\lim_{s'\to 1} \left(\partial_s \overline{K}(s,1,\overline{z})\right)^* \in \mathrm{Hom}(V, \Lambda_+^D)$. It follows that

$$\begin{aligned}\partial_{s'}\overline{K}(1-0,1,z) &= -\operatorname{proj}_{\Lambda_+^N} + A(z) \\ \partial_{s}\overline{K}(1-0,1,z) &= \quad \operatorname{proj}_{\Lambda_+^D} + A^*(\overline{z}) \\ \partial_{s'}\overline{K}(1,1-0,z) &= \quad \operatorname{proj}_{\Lambda_+^D} + A(z) \\ \partial_{s}\overline{K}(1,1-0,z) &= -\operatorname{proj}_{\Lambda_+^N} + A^*(\overline{z}).\end{aligned} \qquad \text{where } A(z) \in \mathrm{Hom}(\Lambda_+^N, \Lambda_+^D) \tag{5.2}$$

Similar results hold at $s = s' = -1$.

5.4. Eigenvalues.

The spectrum of $\mathrm{RN}(\Delta)$ is, by standard elliptic theory, a discrete sequence $0 \le z_0^2 < z_1^2 < \cdots$ with each z_j^2 of finite multiplicity. The kernel $\overline{K}(s,s',z)$ of $\left(\mathrm{RN}(\Delta) - z^2\right)^{-1}$ is meromorphic in z with poles only at the $\pm z_j$. These poles are simple for all $z_j \neq 0$ with residue equal to $(2z_j)^{-1}$ times the projection onto the jth eigenspace and has a double pole at 0 if $z_0 = 0$, with residue zero and coefficient of z^{-2} equal to the projection onto the null space. This different behaviour at 0 is just the result of using z^2 rather than z as the spectral parameter.

We next discuss the eigenvalues of the reduced normal Dirac operator, since these contribute to our formula for the limit of the eta invariant. Recall from chapter 3 that the reduced normal operator of $\eth_\epsilon$ at B_2 is γD_s acting on $\mathrm{null}(\eth_H)$-valued functions $\overline{u}$ on the interval $[-1,1]$, where $\gamma \equiv \mathrm{cl}(dx/\sqrt{x^2+\epsilon^2})$ is the matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ with respect to the splitting of the spinor bundle $S = S^+ \oplus S^-$ at B_2. This model operator has boundary conditions $\overline{u} \restriction \pm 1 \in \Lambda_{\eth,\pm}$, where $\Lambda_{\eth,\pm}$ are the spaces of $\mathcal{C}^\infty$ solutions to $\eth_{M_\pm} = 0$.

Notice that if $\overline{u}$ is an eigenfunction of γD_s, with these boundary conditions, with eigenvalue z then $\overline{u} = Ae^{izs} + Be^{-izs}$, where $\gamma A = A$ and $\gamma B = -B$. Then $Ae^{i(z+k\pi)s} + Be^{-i(z+k\pi)s}$ also satisfies the boundary conditions, and so is an eigenfunction of γD_s with eigenvalue $z + k\pi$. Hence the eigenvalues of γD_s are periodic with period π; there are $\dim \mathrm{null}(\eth_M)$ eigenvalues in the interval $[0,\pi)$.

5.5. Heat kernel and large $|z|$ asymptotics of $\overline{K}$.

One can write down an explicit formaula for the heat kernel for the heat kernel $e^{-t\,\mathrm{RN}(\Delta)}$ using the reflection principle. This is a convenient way to obtain the large $|z|$ asymptotics of $\overline{K}(s,s',z)$ and to compute the determinant and eta invariant of $\mathrm{RN}(\Delta)$ and $\mathrm{RN}(\eth)$ respectively.

When reflecting $\phi \in V_3$ at $s = \pm 1$, one should take $+\phi$ if $\phi \in \Lambda_\pm^D$ and $-\phi$ if $\phi \in \Lambda_\pm^N$; that is, in general one should take $S_\pm \phi$, where $S_\pm = \operatorname{proj} \Lambda_\pm^D - \operatorname{proj} \Lambda_\pm^N$

is the scattering matrix for $\Delta_{M_\pm}$. There are four 'series' of reflections, originating from ϕ at s': they are

$$\begin{aligned} (S_+S_-)^k S_+\phi &\text{ at } && 4k+2-s', \\ (S_-S_+)^k S_-\phi &\text{ at } && -4k-2-s', \\ (S_+S_-)^k \phi &\text{ at } && 4k+s', \\ (S_-S_+)^k \phi &\text{ at } && -4k+s'. \end{aligned} \tag{5.3}$$

Thus $e^{-t\,\mathrm{RN}(\Delta)}(s,s')$ is given by

$$\begin{aligned} \frac{1}{\sqrt{4\pi t}}\Big\{ e^{-|s-s'|^2/4t}\,\mathrm{Id} + \sum_{k=0}^{\infty}\Big[e^{-|4k+s'-s|^2/4t}(S_+S_-)^k + \\ e^{-|-4k+s'-s|^2/4t}(S_-S_+)^k + e^{-|4k+2-s-s'|^2/4t}(S_+S_-)^k S_+ + \\ e^{-|-4k-2-s-s'|^2/4t}(S_-S_+)^k S_- \Big]\Big\}. \end{aligned} \tag{5.4}$$

From this one can use the transform

$$(\Delta - z^2)^{-1} = \int_0^\infty e^{-t\Delta} e^{tz^2}\,dt \tag{5.5}$$

to derive the asymptotics of the model resolvent $\overline{K}$ as $|z| \to \infty$, $\operatorname{Im} z < 0$. Choose a contour of integration for (5.5) so that $\operatorname{Re} t > 0$, $\operatorname{Re} tz^2 < 0$. Consider the transform applied to a term

$$\frac{1}{\sqrt{4\pi t}} e^{-|A\pm s-s'|^2/4t} \cdot B$$

of (5.4). If $|A \pm s - s'|$ is bounded away from zero near (s,s') then

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-|A\pm s-s'|^2/4t} e^{tz^2}\,dt$$

is rapidly decreasing in $|z|$, uniformly in any sector $-\pi < \arg z < -\delta$. Hence, for (s,s') away from $(1,1)$ and $(-1,-1)$ the only term that contributes to (polynomial) asymptotics in z is the first, $1/\sqrt{4\pi t} e^{-|s-s'|^2/4t}\,\mathrm{Id}$ and for (s,s') near $(\pm 1, \pm 1)$, the only terms that contribute are

$$\frac{1}{\sqrt{4\pi t}}\left(e^{-|s-s'|^2/4t}\,\mathrm{Id} + e^{-|\pm 2-s-s'|^2/4t} S_\pm\right).$$

Thus, performing the integral (5.5), we get, in any sector as above, for (s,s') away from the two points $(-1,-1)$ and $(1,1)$ we have

$$\overline{K}(s,s',z) = \frac{e^{-iz|s-s'|}}{2iz}\,\mathrm{Id} + O\left(e^{-C/|z|}\right). \tag{5.6}$$

and near the corners $(\pm 1, \pm 1)$, we have

$$\begin{aligned}\overline{K}(s,s',z) &= \frac{1}{2iz}\left(e^{-iz|s-s'|} + e^{-iz|\pm 2-s-s'|}\right)\operatorname{proj}\Lambda_\pm^D \\ &+\frac{1}{2iz}\left(e^{-iz|s-s'|} - e^{-iz|\pm 2-s-s'|}\right)\operatorname{proj}\Lambda_\pm^N + O\left(e^{-C/|z|}\right).\end{aligned} \tag{5.7}$$

Comparing these formulae with (5.2), we see that $A(z)$ is exponentially decreasing as $|z| \to \infty$ in this sector.

5.6. Determinant and eta invariant.

From the expression (5.4) we can explicitly calculate the determinant of $\mathrm{RN}(\Delta)$ and the eta invariant of $\mathrm{RN}(\eth)$ in terms of the subspaces $\Lambda_\pm^D$, $\Lambda_\pm^N$ which determine them. First we treat the determinant of $\mathrm{RN}(\Delta)$. This will be a crucial result for relating analytic torsion to R-torsion. To state this, first decompose the vector space V into an orthogonal direct sum $V = V_1 \oplus V_2 \oplus V_3$, where

$$\begin{aligned}V_1 &= \Lambda_+^D \cap \Lambda_-^D \oplus \Lambda_+^N \cap \Lambda_-^N \\ V_2 &= \Lambda_+^D \cap \Lambda_-^N \oplus \Lambda_-^D \cap \Lambda_+^N \\ V_3 &= V \ominus (V_1 \oplus V_2).\end{aligned}$$

Write $\Lambda_\pm^{D,r}$, $\Lambda_\pm^{N,r}$ for the 'reduced' subspaces $\Lambda_\pm^D \cap V_3$, $\Lambda_\pm^N \cap V_3$, two pairs of orthogonal complements in V_3 all of which intersect only in $\{0\}$, and write $S_\pm^r$ for the reduced scattering matrix $S_\pm(0)_{\restriction V_3}$ at $\lambda = 0$. The Laplacian Δ splits into a direct sum $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ with Δ_i acting on sections of V_i, and so $\log\det\Delta = \log\det\Delta_1 + \log\det\Delta_2 + \log\det\Delta_3$.

PROPOSITION 5.2. *We have*

$$\log\det\Delta_1 = 2\dim V_1 \log 2 \tag{5.8}$$

$$\log\det\Delta_2 = \dim V_2 \log 2 \tag{5.9}$$

$$\log\det\Delta_3 = \log\det(\mathrm{Id} - S_+^r S_-^r). \tag{5.10}$$

PROOF: The operator Δ_1 has nonzero eigenvalues $(\frac{\pi k}{2})^2$, $k \geq 1$, with multiplicity $\dim V_1$ and the operator Δ_2 has eigenvalues $(\frac{\pi(k-1/2)}{2})^2$, $k \geq 1$, with multiplicity $\dim V_2$. The log determinant of these two operators can be calculated from the values of the Riemann zeta function and its derivative at $s = 0$; we obtain (5.8) and (5.9). To compute (5.10), we use the definition of the zeta function, (1.2), and the expression (5.4) above for V_3, that is, with $S_\pm$ replaced by $S_\pm^r$. Since $\dim\operatorname{null}\Delta_3 = \{0\}$, we have

$$-\log\det\Delta_3 = \int_0^\infty \frac{dt}{t}\int_{-1}^1 \left(\operatorname{tr} e^{-t\Delta_3}(s,s) - \operatorname{tr}\frac{\mathrm{Id}}{\sqrt{4\pi t}}\right) ds.$$

Observe that $\dim \Lambda_{\pm}^{D,r} = \dim \Lambda_{\pm}^{N,r} = 1/2 \dim V_3$. For if not, then the sum of the dimensions of some two of these subspaces would be bigger than $\dim V_3$ and then they would have to intersect nontrivially. Hence $\operatorname{tr} S_{\pm}^r = 0$, and by repeatedly cyclically permuting and using $(S_{\pm}^r)^2 = \mathrm{Id}$, we have $\operatorname{tr}(S_+^r S_-^r)^k S_+^r = \operatorname{tr}(S_-^r S_+^r)^k S_-^r = 0$ for all k. So the trace of the heat kernel on V_3 is, at $s = s'$,

$$\operatorname{tr} e^{-t\Delta_3} = \frac{1}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} e^{-|4k|^2/4t} \big(\operatorname{tr}(S_+^r S_-^r)^k + \operatorname{tr}(S_-^r S_+^r)^k \big).$$

Next we use the formula (from tables)

$$\int_0^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-A^2/4t} \frac{dt}{t} = \frac{1}{A}.$$

Hence, performing the s and t integrals gives, for each k

$$-2\Big(\sum_{k=1}^{\infty} \frac{\operatorname{Tr}(S_+^r S_-^r)^k}{4k} + \frac{\operatorname{Tr}(S_-^r S_+^r)^k}{4k} \Big). \tag{5.11}$$

Since each pair of spaces $\Lambda_{\pm}^{D,r}, \Lambda_{\pm}^{N,r}$ only intersects at zero, therefore $\|S_+^r S_-^r\|$ and $\|S_-^r S_+^r\|$ are less than one (operator norm). Hence the power series (5.11) converges to

$$\begin{aligned} &\frac{1}{2}\Big(\operatorname{tr}\log(\mathrm{Id} - S_+^r S_-^r) + \operatorname{tr}\log(\mathrm{Id} - S_-^r S_+^r) \Big) \\ &= \frac{1}{2}\Big(\log\det(\mathrm{Id} - S_+^r S_-^r) + \log\det(\mathrm{Id} - S_-^r S_+^r) \Big). \end{aligned}$$

Finally, $\det(\mathrm{Id} - S_+^r S_-^r) = \det S_+^r(S_+^r - S_-^r) = \det(S_+^r - S_-^r)S_+^r = \det(\mathrm{Id} - S_-^r S_+^r)$, so we obtain (5.10). ∎

To state the analogous result for the eta invariant of $\mathrm{RN}(\eth)$, define the 'superdeterminant' Sdet of an operator $A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(-1)} \end{pmatrix}$, diagonal with respect to γ, by

$$\operatorname{Sdet} A = \det A^{(1)} (\det A^{(-1)})^{-1}.$$

PROPOSITION 5.3. *The eta invariant of* $\mathrm{RN}(\eth)$ *is given by*

$$\eta(\mathrm{RN}(\eth)) = \frac{1}{\pi} \log \operatorname{Sdet}(\mathrm{Id} - S_+ S_-).$$

PROOF: We compute the integral (1.1) for the eta invariant, using the representation (5.4). Applying $\mathrm{RN}(\eth) = \gamma D_s$ to the term

$$\frac{1}{\sqrt{4\pi t}} e^{-|A \pm s - s'|^2/4t} \cdot B$$

(where B is a matrix) gives, at $s = s'$,

$$\frac{At^{-\frac{3}{2}}}{4\sqrt{\pi}} e^{-|A|^2/4t}(\gamma B).$$

Consider the trace $\mathrm{tr}(\gamma B)$ for B one of the matrices in (5.3). For the square of the Dirac operator, $\Lambda^D_\pm = \Lambda_{\eth,\pm}$ and $\Lambda^N_\pm = \Lambda^\perp_{\eth,\pm}$ since $\Lambda^D_\pm$ is Lagrangian with respect to γ. Hence $\gamma S_\pm = -S_\pm \gamma$. Since the trace of a product of matrices is invariant under cyclic permutation, we get $\mathrm{tr}\,\gamma(S_+S_-)^k S_+ = \mathrm{tr}\,\gamma(S_-S_+)^k S_- = 0$ and $\mathrm{tr}\,\gamma(S_+S_-)^k = -\,\mathrm{tr}\,\gamma(S_-S_+)^k$. Also, the trace of γ itself is zero. Hence,

$$\begin{aligned}
\eta(\mathrm{RN}(\eth)) &= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}}\, \mathrm{Tr}\left(\gamma D_s e^{-t\,\mathrm{RN}(\eth)^2}\right) dt \\
&= \sum_k \frac{1}{4\pi} \int_0^\infty \frac{dt}{t^2} \Big\{ \int_{-1}^1 ds\ 8k e^{-|4k|^2/4t}\, \mathrm{tr}\left(\gamma(S_+S_-)^k\right) \Big\} \\
&= \frac{1}{2\pi} \sum_k 4k\, \mathrm{tr}\left(\gamma(S_+S_-)^k\right) \int_0^\infty \frac{\partial}{\partial t}\Big(\frac{4e^{-|4k|^2/4t}}{|4k|^2}\Big) dt \int_{-1}^1 ds \\
&= \frac{1}{\pi} \sum_k \mathrm{tr}\Big(\frac{\gamma(S_+S_-)^k}{k}\Big) \\
&= \frac{1}{\pi}\, \mathrm{tr}\,\gamma \log(\mathrm{Id} - S_+S_-).
\end{aligned}$$

Since γ anticommutes with both S_+ and S_-, it commutes with S_+S_-. That is, S_+S_- is diagonal with respect to γ and so we can write the last line in this equation as

$$\begin{aligned}
\mathrm{tr}\,\gamma \log(\mathrm{Id} - S_+S_-) &= \mathrm{tr} \log(\mathrm{Id} - S_+S_-)^{(1)} - \mathrm{tr} \log(\mathrm{Id} - S_+S_-)^{(-1)} \\
&= \log \mathrm{Sdet}(\mathrm{Id} - S_+S_-). \ \blacksquare
\end{aligned}$$

Chapter 6. Resolvent with scaled spectral parameter

In this chapter we will construct the resolvent $\Delta_\epsilon - (\text{ias}\,\epsilon)^2 z^2$ of the surgery Laplacian and prove theorem 1.1. Here the spectral parameter $\lambda = (\text{ias}\,\epsilon)z$ is scaling as $\text{ias}\,\epsilon$, as in the one dimensional example in section 5.1. Hence at $\text{ias}\,\epsilon = 0$ the spectral parameter is zero for all z. We treat z as a parameter and perform the construction on X^2_{Ls}. In the next chapter we will construct the "full" resolvent on a bigger space with contains both the region $X^2_{\text{Ls}} \times \mathbb{C}_z$ and $X^2_{\text{Ls}} \times (\mathbb{C}\setminus\mathbb{R}^+)_\lambda$ and unifies the resolvent constructed in this chapter with that constructed in Mazzeo-Melrose.

We construct a parametrix away from the spectrum of $\text{RN}(\Delta)$ in sections 1 — 5, assuming that Δ_0 has no L^2 null space. Then in the following sections we derive the true resolvent from the parametrix, first away from $\text{spec}\,\text{RN}(\Delta)$, then near $\text{spec}\,\text{RN}(\Delta)$ and finally in the case where Δ_0 does have L^2 null space.

6.1. Preliminaries.

Motivated by Lemma 5.1, let us look for a two-sided parametrix G in the space $\Psi^{-2,-1}(X^2_{\text{Ls}}; \Omega^{\frac12}_D X^2_{\text{Ls}})$, where '$-1$' is short for the index set $\mathcal{C}^\infty \cup (-1,0)$. G should satisfy

$$\begin{aligned} \Delta G - \text{Id} &\in \prod \rho_{mn}^{e(B_{mn})} \mathcal{C}^\infty(X^2_{\text{Ls}}; \Omega^{\frac12}_D X^2_{\text{Ls}}) \subset \Psi^{-\infty}_{\text{par-res}}(X^2_{\text{Ls}}; \Omega^{\frac12}_D X^2_{\text{Ls}}) \\ G\Delta - \text{Id} &\in \prod \rho_{mn}^{e(B_{mn})} \mathcal{C}^\infty(X^2_{\text{Ls}}; \Omega^{\frac12}_D X^2_{\text{Ls}}) \subset \Psi^{-\infty}_{\text{par-res}}(X^2_{\text{Ls}}; \Omega^{\frac12}_D X^2_{\text{Ls}}) \end{aligned} \tag{6.1}$$

where $e(B_{mn})$ equals 1 for $(mn) = (00)$, (01), (10) and (11) and equals 2 otherwise. In the presence of L^2 null space, we will also have a finite rank part of order $(\text{ias}\,\epsilon)^{-2}$: $G \in \Psi^{-2,0}(X^2_{\text{Ls}}; \Omega^{\frac12}_D X^2_{\text{Ls}}) + \Psi^{-\infty,-2}_{\text{bdy}}(X^2_{\text{Ls}}; \Omega^{\frac12}_D X^2_{\text{Ls}})$.

Model Operator Equations at each face By Proposition 4.1, the interior of each face B_{mn}, $(mn) \neq (33)$, is canonically diffeomorphic to the interior of $B_m \times B_n$. Then, Lemma 3.5 gives us a formula for the Laplacian near the interior of each of these faces, using the coordinates y, $r = \sinh^{-1}(x/\epsilon)$ for B_1, y, $s = (\text{ias}\,\epsilon)\sinh^{-1}(x/\epsilon)$ for B_2 and y, $\xi = \text{ilg}\,x$ near B_0. For B_{33} we can use coordinates y, y', $\log(x'/x)$, s and then the Laplacian looks like

$$\Delta = I(\Delta) + v \cdot Q = -(\nabla_{\log(x'/x)})^2 + \Delta_H + v \cdot Q$$

where v vanishes to infinite order at B_{33} and Q is a Ls-ψdo of order at most two.

To take advantage of the fact that $[\Delta, \text{ias}\,\epsilon] = 0$ we write G as a Taylor series in $\text{ias}\,\epsilon$ off each face:

$$G = \sum_{j=-1}^{e(B_{mn})} (\text{ias}\,\epsilon)^j G^{(j)}_{mn} + O\big((\text{ias}\,\epsilon)^{e(B_{mn})+1}\big) \text{ near } B_{mn}.$$

Then Δ acts on this Taylor series by acting on the $G^{(j)}_{mn}$, but not the $(\text{ias}\,\epsilon)^j$. The model equations of (6.1) involving the identity operator are

$$\Delta_{\overline{M}} G^{(0)}_{00} = G^{(0)}_{00} \Delta_{\overline{M}} = \mathrm{Id}_{\overline{M}} \tag{6.2}$$

$$\Delta_{\overline{H}} G^{(0)}_{11} = G^{(0)}_{11} \Delta_{\overline{H}} = \mathrm{Id}_{\overline{H}} \tag{6.3}$$

$$\Delta_{\overline{H}} G^{(0)}_{33} = G^{(0)}_{33} \Delta_{\overline{H}} = \mathrm{Id}_{\overline{H}} . \tag{6.4}$$

The other equations for $j = -1$ or 0 are

$$R_m(\Delta) G^{(j)}_{mn} = G^{(j)}_{mn} R_n(\Delta) = 0 \tag{6.5}$$

for all (mn), $j = -1$ and all $(mn) \neq (00), (11), (33)$ for $j = 0$. The equations for $j = 1$ are

$$R_m(\Delta) G^{(1)}_{mn} = G^{(1)}_{mn} R_n(\Delta) = z^2 G^{(-1)}_{mn}. \tag{6.6}$$

Finally we have equations involving the reduced normal operator:

$$(D^2_s - z^2) G^{(-1)}_{2m} = 0 = (D^2_{s'} - z^2) G^{(-1)}_{m2}. \tag{6.7}$$

This involves only terms of order $j = -1$ because the reduced normal terms occur two terms down in the Taylor series. Hence for $j \geq 0$ the reduced normal term is of order ≥ 2 and is an error term in (6.1). This means that the models $G^{(0)}_{22}, G^{(1)}_{22}$ are free, in the sense that the only requirement is that they be smooth and $\mathrm{null}(\Delta_H) \otimes \mathrm{null}(\Delta_H)$-valued. It is possible to define such a model, compatible with all adjacent faces, if and only if the adjacent faces are $\mathrm{null}(\Delta_H) \otimes \mathrm{null}(\Delta_H)$-valued and compatible between themselves at the intersection with B_{22}. Hence in the sequel we will simply verify this condition and not write down explicit models $G^{(0)}_{22}, G^{(1)}_{22}$.

Compatibility between models on adjacent faces If the model operators are continuous functions, it is sufficient to check compatibility between models on adjacent faces in the interior of the intersection. Suppose B_{mn} and B_{pq} intersect, and suppose that ρ_{mn} and ρ_{pq} are boundary defining functions, valid in the interior of $B_{mn} \cap B_{pq}$, such that $\rho_{mn}\rho_{pq} = \mathrm{ias}\,\epsilon$. On each face it has a Taylor series

$$\sum_{i \geq -1} (\mathrm{ias}\,\epsilon)^i G^{(i)}_{mn} \text{ or } \sum_{j \geq -1} (\mathrm{ias}\,\epsilon)^j G^{(j)}_{pq},$$

and at $B_{mn} \cap B_{pq}$, $G^{(i)}_{mn}$ ($G^{(j)}_{pq}$) has a Taylor series, which we write

$$\sum_{j \geq -1} \rho_{pq}^{j-i} \big(G^{(i)}_{mn}\big)_{pq,j-i} \quad \Big(\sum_{i \geq -1} \rho_{mn}^{i-j} \big(G^{(j)}_{pq}\big)_{mn,i-j} \Big).$$

Comparing, we see

$$\big(G^{(i)}_{mn}\big)_{pq,j-i} = \big(G^{(j)}_{pq}\big)_{mn,i-j} \tag{6.8}$$

is a necessary and sufficient condition for these models to be compatible.

Next we recall some facts about *b*-Laplacians which we will need, and then define some notation which will make writing down terms in the parametrix easier.

Resolvent of a b-Laplacian Let us start by considering the model equation (6.2):

$$\Delta_{\overline{M}} G^{(0)}_{00} = G^{(0)}_{00} \Delta_{\overline{M}} = \mathrm{Id}_{\overline{M}}.$$

This equation might seem problematical, since $\Delta_{\overline{M}}$ is not invertible. Let us recall some facts from [17] some properties of the resolvent $(\Delta - \lambda^2)^{-1}$. In chapter 6 of this book, it is shown that $\Delta_{\overline{M}} - \lambda^2$ is invertible in the *b*-calculus for $\operatorname{Im}\lambda < 0$ and the inverse extends meromorphically to a neighbourhood of 0 in $\mathbb{C}_\lambda$ with a simple pole at $\lambda = 0$, the residue being projection onto the smooth null space of $\Delta_{\overline{M}}$. Thus at $\lambda = 0$, $\Delta_{\overline{M}}$ has a Laurent series

$$\Delta_{\overline{M}} = \sum_{j \geq -1} \lambda^j \operatorname{Res}^{(j)}_{\overline{M}}, \tag{6.9}$$

with $\operatorname{Res}^{(-1)}_{\overline{M}} = -i \operatorname{proj}_{\mathcal{C}^\infty \text{null}}$. Applying $(\Delta - \lambda^2)$ to (6.9) shows that $\Delta_{\overline{M}} \cdot \operatorname{Res}^{(0)}_{\overline{M}} = \mathrm{Id}$, so that $\operatorname{Res}^{(0)}_{\overline{M}}$ is an inverse to $\Delta_{\overline{M}}$ (though not of course a bounded operator on L^2_b), and for $j \geq 1$, $\Delta_{\overline{M}} \operatorname{Res}^{(j)}_{\overline{M}} = \operatorname{Res}^{(j-2)}_{\overline{M}}$. Let us note here that if $\Delta_{\overline{M}}$ has null space, then the series (6.9) starts at $\lambda = -2$, with $\operatorname{Res}^{(-2)}_{\overline{M}} = -\operatorname{proj}_{L^2 \text{null}}$, and then we have $\Delta_{\overline{M}} \cdot \operatorname{Res}^{(0)}_{\overline{M}} = \mathrm{Id} - \operatorname{proj}_{L^2 \text{null}}$. Similar results hold for $\Delta_{\overline{H}}$; here L^2 null space cannot occur.

We will also need the notion of scattering solutions and scattering matrix. For definiteness we will describe these just for M_+. Let $\{\phi_J\}$ be an orthonormal basis for V, which splits into $\{\phi_i\}$, $1 \leq i \leq \dim \Lambda^D_+$, a basis for Λ^D_+, and $\{\phi_\alpha\}$, $\dim \Lambda^D_+ + 1 \leq \alpha \leq \dim V$, a basis for Λ^N_+. We will use, in the summation convention, the form of the index – capital, small roman or small greek – to determine whether the sum is over a basis of V, Λ^D_+ or Λ^N_+. The scattering solutions $\Phi_J(\lambda)$, with λ near 0, $\operatorname{Im}\lambda < 0$, are defined by the solutions to

$$(\Delta - \lambda^2)\Phi_J(\lambda) = 0 \tag{6.10}$$

with the boundary behaviour $\Phi_J(\lambda) \sim x^{-i\lambda}\phi_J(y) + v$, with $v \in L^2_b$ near H. They have an expansion

$$\Phi_J(\lambda) = x^{-i\lambda}\phi_J(y) + x^{i\lambda} S_{JK}(\lambda)\phi_K(y) + O(x^\delta),$$

with $\delta > 0$ uniformly near $\lambda = 0$. S is the scattering matrix; it is meromorphic near $\lambda = 0$, symmetric, unitary for λ real, and satisfies $S(\lambda)S(-\lambda) = \mathrm{Id}$; from this we deduce that $S(0) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$ and $\partial_\lambda S(0)$ is block diagonal with respect to the splitting $V = \Lambda^D_+ \oplus \Lambda^N_+$.

Let us denote $1/k!(\frac{\partial}{\partial\lambda})^k\Phi_J(\lambda)_{\restriction\lambda=0}$ by $\Phi_J^{(k)}$, and $1/k!(\frac{\partial}{\partial\lambda})^k S_{JK}(\lambda)_{\restriction\lambda=0}$ by $S_{JK}^{(k)}$. By differentiating (6.10), we obtain

$$\Delta_{\overline{M}}\Phi_J^{(0)} = \Delta_{\overline{M}}\Phi_J^{(1)} = 0,$$

$$\Delta_{\overline{M}}\Phi_J^{(2)} = \Phi_J^{(0)}. \tag{6.11}$$

We have $\Phi_\alpha^{(0)} \equiv 0$, $\Phi_j^{(1)} = S_{jk}^{(1)}\Phi_k^{(0)}$, and $\Phi_j^{(0)}$ $(i\Phi_\alpha^{(1)})$ form the smooth (logarithmically growing) null space of $\Delta_{\overline{M}}$.

We need to know the top terms in the (polyhomogeneous) expansion of $\Delta_{\overline{M}} - \lambda^2$ at the faces of B_{00}, that is, those terms with exponent tending to zero as $\lambda \to 0$. At the "front face", $x = x' = 0$, corresponding to $B_{00} \cap B_{33}$, we have

(6.12)

$$(\Delta_{\overline{M}} - \lambda^2)^{-1} \sim \frac{1}{2i\lambda}\left((x'/x)^{\pm i\lambda}\phi_J(y)\phi_J(y') + x^{i\lambda}x'^{i\lambda}\phi_J(y)S_{JK}(\lambda)\phi_K(y')\right), \ \frac{x'}{x} \lessgtr 1$$

At the left boundary $x/x' = 0$, corresponding to $B_{00} \cap B_{20}$, we have

$$(\Delta_{\overline{M}} - \lambda^2)^{-1} \sim \frac{1}{2i\lambda}\left(x^{i\lambda}\phi_J(y)\Phi_J(p')\right) \tag{6.13}$$

and at the right boundary $x'/x = 0$, corresponding to $B_{00} \cap B_{02}$, we have

$$(\Delta_{\overline{M}} - \lambda^2)^{-1} \sim \frac{1}{2i\lambda}\left(x'^{i\lambda}\Phi_J(p)\phi_J(y')\right). \tag{6.14}$$

The "$\mathfrak{T}$" notation In the construction below, we will want some notation for transferring information about the one dimensional resolvent $\overline{K}$ to the space $X^2_{\mathrm{Ls}}(M)$ for general $\overline{M}$. Let us first define $\mathfrak{T}(\phi_J) = \phi_J(H)$, where by ϕ_J is meant, in the first instance, a section of the vector bundle V over $[-1,1]$ and in the second instance, a section of the appropriate bundle over the faces $B_1(X_{\mathrm{Ls}}(M))$ or $B_2(X_{\mathrm{Ls}}(M))$. On the face B_0 we define $\mathfrak{T}(\Phi_J^{(k)}[0,\pm1])$ to be zero if $\Phi_J^{(k)}[0,\pm1] = 0$ and $\Phi_J^{(k)}(M_\pm)$ otherwise. Extending $\mathfrak{T}$ to the space $\mathrm{End}\,V$ in the natural way, we can map models $\overline{K}^{(j)}_{mn}$ to models on $X^2_{\mathrm{Ls}}(M)$, for all odd j and all even j except for $(mn) = (00), (33), (11)$ (these terms are not smooth, and hence not globally valued in the span of the functions for which $\mathfrak{T}$ is defined).

6.2. Terms of order $(\mathrm{ias}\,\epsilon)^{-1}$.

Consider equations (6.5) for $j = -1$. At first sight it would appear that $G^{(00)}_{mn} \equiv 0$ is an acceptable solution, since there are no forcing terms of order $(\mathrm{ias}\,\epsilon)^{-1}$. However we have already seen that the one dimensional resolvent is of order $(\mathrm{ias}\,\epsilon)^{-1}$. In fact the general case behaves in much the same way:

Claim $G^{(-1)}_{22}$ is the resolvent of the fourth model problem, that is, the one-dimensional resolvent for $D_s^2 - z^2$ with boundary conditions (3.29), $\overline{K}(s, s', z)$, studied in chapter 5.

Proof $G_{22}^{(-1)}$ is required to satisfy (6.5) and (6.7). The first equation means we can regard $G_{22}^{(-1)}$ as a one dimensional kernel with values in $\mathrm{End}(V)$, where $V = \mathrm{null}(\Delta_H)$. Consider the boundary conditions given by compatibility conditions with adjacent faces.

On B_{12} and B_{11} the left model operator is $\Delta_{\overline{H}}$, given in local coordinates by (3.18). Since $G_{12}^{(-1)}$ and $G_{11}^{(-1)}$ are bounded, they are constant in $r = \sinh^{-1}(x/\epsilon)$, the variable across B_{12} and B_{11}, so they give trivial matching conditions across $s = 0$ for $G_{22}^{(-1)}$, as in the proof of Proposition 3.8. We similarly get trivial matching conditions across B_{21}, B_{11} at $s' = 0$ and across B_{33} at $s = s'$. Hence $G_{22}^{(-1)}$ is continuous on $[-1,1]_s \times [-1,1]_{s'}$. At $s = \pm 1$, $G_{22}^{(-1)}$ matches with $G_{02}^{(-1)}$ which is bounded and $\mathrm{null}(\Delta_{\overline{M}}) \times \mathrm{null}(\Delta_H)$-valued; hence $G_{22}^{(-1)} \restriction s = \pm 1$ takes values in $\Lambda_\pm^D \otimes V \equiv \mathrm{Hom}(V, \Lambda_\pm^D)$ and similarly $G_{22}^{(-1)} \restriction s' = \pm 1$ takes values in $V \otimes \Lambda_\pm^D \equiv \mathrm{Hom}(\Lambda_\pm^D, V)$.

In the interior the derivatives $\partial_s G_{22}^{(-1)}$, $\partial_{s'} G_{22}^{(-1)}$ match across B_{12}, B_{21}. The derivative $\partial_s G_{22}^{(-1)}$ at $s = \pm 1$ matches, as in the proof of Proposition 3.8, with the ξ^{-1} term of $G_{02}^{(0)}$ at $\xi = 0$, so $\partial_s G_{22}^{(-1)} \restriction s = \pm 1 \in \mathrm{Hom}(V, \Lambda_\pm^N)$ and similarly $\partial_{s'} G_{22}^{(-1)} \restriction s' = \pm 1 \in \mathrm{Hom}(\Lambda_\pm^N, V)$. Across B_{33}, compatibility between $G_{22}^{(-1)}$ and $G_{33}^{(0)}$ requires that at the intersection of B_{22} and B_{33}, $(\partial_{s'} - \partial_s) G_{22}^{(-1)}$ matches the $\log(x'/x)$ coefficient of $G_{33}^{(0)}$. The model equation for $G_{33}^{(0)}$ is $I(\Delta_{\overline{M}}) G_{33}^{(0)} = \mathrm{Id}_{\overline{H}}$, so $G_{33}^{(0)}$ is given by

$$G_{33}^{(0)} = \sum_{j=1}^{\infty} \frac{e^{-\sigma_j |\log(x'/x)|}}{2\sigma_j} \mathrm{proj}_{V_j} - \frac{1}{2} \big|\log(x'/x)\big| \,\mathrm{proj}_V + A(s) + B(s)\log(x'/x)$$

where V_j is the j-th nonzero eigenspace of Δ_H with eigenvalue σ_j^2, and $A(s)$, $B(s)$ are $\mathrm{End}(V)$-valued functions of s. There is a jump of $\mathrm{Id} \in \mathrm{End}(V)$ in the coefficient of $\log(x'/x)$ between the two sides of B_{33}. Hence $(\partial_{s'} - \partial_s) G_{22}^{(-1)}$ has a jump of Id across $s = s'$. These conditions on $G_{22}^{(-1)}$ uniquely determine that $G_{22}^{(-1)} = \overline{K}(s, s', z)$, the kernel of the one dimensional resolvent studied in chapter 5, so the claim is established.

Indeed, not only does this argument yield $G_{22}^{(-1)}$, it gives us all the $G_{mn}^{(-1)}$. Compatibility of $G_{mn}^{(-1)}$ with $G_{22}^{(-1)}$ requires that $G_{mn}^{(-1)} = \mathfrak{T}(K_{mn}^{(-1)})$ for all (mn). The compatibility of these terms itself follows from the compatibility on $X^2_{\mathrm{Ls}}([-1,1])$.

6.3. Terms of order $(\mathrm{ias}\,\epsilon)^0$.

To find the next term in the Taylor series at each face, we start with the faces B_{00}, B_{11} and B_{33} whose normal equations, (6.2) – (6.4) have "forcing terms" on the right hand side. Let us start with $G_{00}^{(0)}$. This we know is given by

$$G_{00}^{(0)} = \mathrm{Res}_{\overline{M}}^{(0)} + \text{terms in } \mathrm{null}(\Delta_{\overline{M}}) \otimes \mathrm{null}(\Delta_{\overline{M}}).$$

The regular part of the resolvent can be calculated from (6.12) - (6.14). They are, at the front face,

$$\begin{aligned}
&\frac{1}{2}\left(-\left|\log\frac{x'}{x}\right|\operatorname{Id}+(\log x+\log x')S_{JK}(0)\phi_J(y)\phi_K(y')-i\phi_J(y)\phi_K(y')S^{(1)}_{JK}\right)\\
(6.15)\qquad &=\left(-\log x\operatorname{proj}_{\Lambda^N_+}+\log x'\operatorname{proj}_{\Lambda^D_+}-\frac{i}{2}\phi_J(y)\phi_K(y')S^{(1)}_{JK}\right),\ \frac{x'}{x}<1\\
&=\left(\log x\operatorname{proj}_{\Lambda^D_+}-\log x'\operatorname{proj}_{\Lambda^N_+}-\frac{i}{2}\phi_J(y)\phi_K(y')S^{(1)}_{JK}\right),\ \frac{x'}{x}>1,
\end{aligned}$$

at the left boundary

$$\begin{aligned}
&\frac{1}{2}\left(\log x\phi_J(y)\Phi^{(0)}_J(p')-i\phi_J(y)\Phi^{(1)}_J\right)\\
=&\frac{1}{2}\left(\log x\phi_j(y)\Phi^{(0)}_j(p')-i\phi_\alpha(y)\Phi^{(1)}_\alpha(p')-i\phi_j(y)S^{(1)}_{jk}\Phi^{(0)}_k(p')\right),
\end{aligned}$$

and at the right boundary

$$\begin{aligned}
&\frac{1}{2}\left(\log x'\Phi^{(0)}_J(p)\phi_J(y')-i\Phi^{(0)}_J\phi_J(y')\right)\\
=&\frac{1}{2}\left(\log x'\Phi^{(0)}_j(p)\phi_j(y')-i\Phi^{(1)}_\alpha(p)\phi_\alpha(y')-iS^{(1)}_{jk}\Phi^{(0)}_k(y)\phi_j(y')\right).
\end{aligned}$$

The relation between these expressions at the front face and at the left and right boundaries is not immediately evident since there only appears $\partial_\lambda S_{jk}$, with small roman indices, in the last two. The reason is that the $\partial_\lambda S_{\alpha\beta}$ piece is contained in the $\tilde{\Phi}_\alpha$ term, which may have a part smooth up to the boundary "beneath" the principal, logarithmically increasing one. Since $\partial_\lambda S_{JK}(0)$ is block diagonal with respect to the splitting $V=\Lambda^D_+\oplus\Lambda^N_+$, the other pieces $\partial_\lambda S_{i\alpha}(0)$, $\partial_\lambda S_{\alpha i}(0)$ are zero. The null space terms are determined by compatibility with $G^{(-1)}_{22}$. Returning once more to the one dimensional operator, we have

$$\begin{aligned}
K^{(0)}_{00}&=(\text{ias}\,\epsilon)^{-1}\left(\overline{K}(s,s',z)-\overline{K}(1,1,z)\right)_{\restriction B_{00}}\\
&=(\text{ias}\,\epsilon)^{-1}\left((s-1)\partial_s\overline{K}(1,1,z)+(s'-1)\overline{K}(1,1,z)\right)_{\restriction B_{00}}\\
&=\log x\,\partial_s\overline{K}(1,1,z)+\log x'\partial_{s'}\overline{K}(1,1,z)\\
&=\log x\left(-\operatorname{proj}_{\Lambda^N_+}+A^*(\overline{z})\right)+\log x'\left(\operatorname{proj}_{\Lambda^D_+}+A(z)\right)\quad\frac{x'}{x}<1\\
&=\log x\left(\operatorname{proj}_{\Lambda^D_+}+A^*(\overline{z})\right)+\log x'\left(-\operatorname{proj}_{\Lambda^N_+}+A(z)\right)\quad\frac{x'}{x}>1
\end{aligned}$$

for x, $x'>0$; we used (5.2) in the last line. The projection terms are terms already appearing in (6.15), and the $A(z)$ terms are null space terms which, by definition, give compatibility with $K^{(-1)}_{22}$. Hence, let us take

$$\begin{aligned}
(6.16)\qquad G^{(0)}_{00}&=\operatorname{Reg}_{\tilde{M}}(0)+\mathfrak{T}\left(\log x\,A(z)+\log x'A^*(\overline{z})\right).\\
&=\operatorname{Reg}_{\tilde{M}}(0)+A_{\alpha j}(z)\Phi^{(1)}_\alpha(p)\phi_j(y')+A^*_{j\alpha}(\overline{z})\phi_j(y)\Phi^{(1)}_\alpha(p')
\end{aligned}$$

We use the same reasoning for $G_{33}^{(0)}$ and $G_{11}^{(0)}$. Using coordinates $\log(x'/x)$, s, ias ϵ near B_{33}, we have

$$
\begin{aligned}
\overline{K}_{33}^{(0)} &= (\text{ias}\,\epsilon)^{-1}\left(\overline{K}(s, s+\text{ias}\,\epsilon\log(x'/x), z) - \overline{K}(s,s,z)\right)_{\restriction B_{11}} \\
&= \log(x'/x)\left(\partial_s\overline{K}(s,s,z)\right) \\
&= -\frac{1}{2}|\log\frac{x'}{x}|\phi_J(y)\phi_J(y') + -\frac{1}{2}\log\frac{x'}{x}\left(D_{IJ}(s,z)\phi_I(y)\phi_J(y') - D^*{}_{IJ}(s,\overline{z})\phi_I(y')\phi_J(y)\right).
\end{aligned}
\tag{6.17}
$$

The $1/2\,|\log(x'/x)|$ term already appears in $\text{Res}_{\overline{H}}^{(0)}$, and the other terms are required by compatibility with $G_{22}^{(-1)}$. Referring to (6.15), we should also add in a term for compatibility with the $\partial_\lambda S$ terms in $G_{00}^{(0)}$. Hence we define

$$
\begin{aligned}
G_{33}^{(0)} = \text{Res}_{\overline{H}}^{(0)} &+ \mathfrak{I}\Big(-\frac{1}{2}\log\frac{x'}{x}\left(D_{IJ}(s,z)\phi_I(y)\phi_J(y') - D^*{}_{IJ}(s,\overline{z})\phi_I(y')\phi_J(y)\right)\Big) \\
&- \frac{i}{2}\psi(2-s-s')e^{-iz(2-s-s')}\phi_J(y)S_{JK}^{(1)}\phi_K(y').
\end{aligned}
\tag{6.18}
$$

Here ψ is a cutoff function, with support near 0. The factor $e^{-iz(2-s-s')}$ may appear somewhat mysterious; it will be needed in the next chapter when we join these models with those constructed in Mazzeo-Melrose. The $G_{11}^{(0)}$ term is similar except that the last term is not required as B_{11} is away from B_{00}:

$$
G_{11}^{(0)} = \text{Res}_{\overline{H}}^{(0)} + \frac{r-r'}{2}\mathfrak{I}\Big(D_{IJ}(0,z)\phi_I(y)\phi_J(y') - D^*{}_{IJ}(0,\overline{z})\phi_I(y')\phi_J(y)\Big).
$$

Next consider $G_{02}^{(0)}$ and $G_{20}^{(0)}$. To satisfy compatibility with $G_{22}^{(-1)}$, we must take

$$
G_{02}^{(0)} = \mathfrak{I}(\overline{K}_{02}^{(0)}) + \mathcal{C}^\infty([-1,1]_{s'}; \mathcal{C}^\infty\,\text{null}(\Delta_{\overline{M}})\otimes V^*).
$$

To make this term compatible with $G_{00}^{(0)}$, let us take

$$
G_{02}^{(0)} = \mathfrak{I}(\overline{K}_{02}^{(0)}) - \frac{i}{2}\psi(1-s')e^{-iz(1-s')}\Phi_j^{(0)}(p)S_{jk}^{(1)}\phi_k(y'). \tag{6.19}
$$

We define $G_{20}^{(0)}$ similarly. Again the exponential factor is irrelevant here but will be required in the next chapter.

Finally for the other faces (except B_{22}, as per the comment in section 1) we define

$$
G_{mn}^{(0)} = \mathfrak{I}(\overline{K}_{mn}^{(0)})\ , (mn) = 12,\ 21,\ 01,\ 10.
$$

Compatibility of these terms follows from the compatibility of the $\overline{K}_{mn}^{(j)}$.

6.4. Terms of order (ias ϵ).

These terms are only required for $e(B_{mn}) = 2$, and as outlined above, not for B_{22}. Let us start with $G^{(1)}_{02}$. To satisfy compatibility with the model equation, we must have

$$G^{(1)}_{02} = \mathfrak{T}(\overline{K}^{(1)}_{02}) + \mathcal{C}^\infty([-1,1]_{s'}; \mathrm{null}(\Delta_{\overline{M}}) \otimes V^*).$$

For compatibility with $G^{(0)}_{00}$ we may take the null space term to be zero; however let us add a term which will be required in the next chapter:

$$G^{(1)}_{02} = \mathfrak{T}(\overline{K}^{(1)}_{02}) + \frac{z}{2i}\psi(1-s')e^{-iz(1-s')}\Phi^{(2)}_\alpha(p)\phi_\alpha(y').$$

(Recall that $\Phi^{(2)}_\alpha$ has zero leading, $(\log x)^2$ term. Hence there is no compatibility required between the last term and $G^{(-1)}_{22}$ as might at first appear.) We define $G^{(1)}_{20}$ similarly. The $G^{(1)}_{33}$ term may, in the same way, be taken to be $\mathfrak{T}(\overline{K}^{(1)}_{33})$ but again we add a term required in the next chapter:

$$G^{(1)}_{33} = \mathfrak{T}(\overline{K}^{(1)}_{33}) + \frac{z}{2i}\psi(2-s-s')e^{-iz(2-s-s')}\phi_J(y)S^{(2)}_{JK}\phi_K(y'). \tag{6.20}$$

All other terms are away from B_{00} so we can safely define them by

$$G^{(1)}_{mn} = \mathfrak{T}(\overline{K}^{(1)}_{mn}), (mn) = 12,\ 21,\ 01,\ 10.$$

Again compatibility follows from compatibility of the $\overline{K}^{(j)}_{mn}$.

6.5. Compatibility with the symbol.

The total symbol, $\sigma_{\mathrm{tot}}(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2)$, is equal to $\sigma_{\mathrm{tot}}(\Delta_\epsilon) + O((\mathrm{ias}\,\epsilon)^2)$. In our chosen coordinates, $\sigma_{\mathrm{tot}}(\Delta_\epsilon)$ is constant, to infinite order at the boundary so compatibility with the symbol requires that terms $G^{(-1)}_{mm}$ and $G^{(1)}_{33}$ are smooth, and $\sigma_{\mathrm{tot}}(G^{(0)}_{mm}) = \sigma_{\mathrm{tot}}(\Delta_\epsilon)_{\restriction B_{mm}}$. This is true because the restriction of Δ_ϵ to B_{00} (B_{11}, B_{33}) is $\Delta_{\overline{M}}$ ($\Delta_{\overline{H}}$) and $G^{(0)}_{00}$ ($G^{(0)}_{11}$, $G^{(0)}_{33}$) are chosen to be, up to smoothing operators, the inverses of $\Delta_{\overline{M}}$ ($\Delta_{\overline{H}}$). It follows that one can construct a holomorphic family of Ls-ψdos $G(z)$ in $\Psi^{-2,-1}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$ restricting to the models which we have defined and solving the full symbol equations $\sigma_{\mathrm{tot}}(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2) \circ \sigma_{\mathrm{tot}} G(z) = 1$.

This completes our construction of the parametrix.

6.6. From parametrix to resolvent.

We have outlined already, in section 4.1, the process of getting the actual resolvent from our parametrix $G(z)$. We have now, for $z \in \mathbb{C} \setminus \mathbb{R}^+$, a holomorphic family $G(z)$ such that (6.1) holds. Let $-R = \Delta G - \mathrm{Id} \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$. Then by Lemma 4.7, we have

$$(\mathrm{Id} - R)^{-1} = \mathrm{Id} + S,\ S \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}).$$

Hence

$$(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2)G(z)(\mathrm{Id} + S(z)) = \mathrm{Id}$$

so by Lemma 4.8,

(6.21)

$$\left(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2\right)^{-1} = G + G\cdot S \in \Psi^{-2,-1}(X^2_{\mathrm{Ls}}; \Omega_D^{\frac{1}{2}} X^2_{\mathrm{Ls}}) + \Psi^{-\infty,\mathcal{F}}_{\mathrm{bdy}}(X^2_{\mathrm{Ls}}; \Omega_D^{\frac{1}{2}} X^2_{\mathrm{Ls}})$$

where $\mathcal{F}$ is natural (see Definition 3).

In fact, it is not hard to show that all coefficients of elements of the index family $\mathcal{F}$ with nonzero logarithmic behaviour are zero. For suppose that there is a nonzero logarithmic term $\rho^j_{mn} \log \rho^k_{mn} \cdot a$, where a is a section defined on B_{mn}, occurring at power j, with j minimal. Since there is no other terms lower in the Taylor series with logarithmic behaviour, a must be killed by the reduced normal operators of Δ:

$$R_m(\Delta)\cdot a = a \cdot R_n(\Delta) = 0.$$

But the solutions to the model problems on each face have the property (with one exception) that they are either smooth up to each boundary $B_{mn} \cap B_{m'n'}$ of the face or blow up like $1/\rho_{m'n'}$ there. The only exception is B_{33}, which has this property with respect to B_{22} but not B_{00} or B_{11}. In any case, there is some face $B_{m'n'} \neq B_{33}$ at which a is nonzero, we have, by compatibility of Taylor series, a term on $B_{m'n'}$ which behaves like $\log \rho_{mn}$. This contradicts the assertion above that solutions to models problems do not have logarithmic behaviour at the boundary. Therefore no such term exists.

It follows then that $\left(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2\right)^{-1} \in \Psi^{-2,-1}(X^2_{\mathrm{Ls}}; \Omega_D^{\frac{1}{2}} X^2_{\mathrm{Ls}})$.

6.7. Near the discrete spectrum of $\mathrm{RN}(\Delta)$.

Recall from Section 5 that $\overline{K}(s,s',z) = (D_s^2 - z^2)^{-1}$ is meromorphic in z with poles at the spectrum $0 \leq z_0^2 < z_1^2 \cdots$. We have already constructed the resolvent away from the discrete spectrum. In this section, we will construct the resolvent in a neighbourhood of one of the z_j; at first, we shall take $z_j \neq 0$.

Near z_j the kernel $\overline{K}$ has the behaviour

$$\overline{K}(s,s',z) = \frac{\mathrm{proj}\, V_j}{2z_j(z_j - z)} + \overline{K}_0(s,s',z) \tag{6.22}$$

where V_j is the jth eigenspace of $\mathrm{RN}(\Delta)$ and $\overline{K}_0$ is holomorphic near z_j. We have

$$(D_s^2 - z^2)\overline{K}_0(s,s',z) = \mathrm{Id} - \frac{z_j + z}{2z_j}\,\mathrm{proj}\, V_j \tag{6.23}$$

so $\overline{K}_0$ is an inverse up to finite rank near z_j.

Let us express this now on $[-1,1]_x$ with surgery metric $g_\epsilon = dx^2/(x^2+\epsilon^2)$. With $s = (\mathrm{ias}\,\epsilon)\sinh^{-1}(x/\epsilon)$ as in chapter 5, and defining

$$K_0(s,s',z,\mathrm{ias}\,\epsilon) = (\mathrm{ias}\,\epsilon)^{-1}\overline{K}_0(s,s',z)|ds ds' \frac{d\,\mathrm{ilg}\,\epsilon}{(\mathrm{ilg}\,\epsilon)^2}|^{\frac{1}{2}}$$

$$E(s,s',\mathrm{ias}\,\epsilon) = \frac{\mathrm{ias}\,\epsilon}{2}\,\mathrm{proj}\, V_j |ds ds' \frac{d\,\mathrm{ilg}\,\epsilon}{(\mathrm{ilg}\,\epsilon)^2}|^{\frac{1}{2}},$$

we have $(\Delta_\epsilon - (\text{ias}\,\epsilon)^2 z^2)K_0 = \text{Id} - E$ on $X^2_{\text{Ls}}([-1,1])$. Returning to our manifold M, by Proposition 3.8 we can construct from $\overline{e} \in V_j$ an approximate small eigenfunction e on X^2_{Ls} such that $(\Delta - (\text{ias}\,\epsilon)^2 z_j^2)e \in \rho_0^2\rho_1^2\rho_2^3\mathcal{C}^\infty(X_{\text{Ls}};\Omega_D^{\frac{1}{2}}X_{\text{Ls}})$. Let $\{\overline{e}_i\}_{i=1}$ be a basis of V_j, and define

$$W = \frac{\text{ias}\,\epsilon}{2}\, e_i(p)e_i(p')\Big|\frac{d(\text{ilg}\,\epsilon)}{(\text{ilg}\,\epsilon)^2}\Big|^{\frac{1}{2}} \in \Psi^{-\infty,0}_{\text{bdy}}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}}). \tag{6.24}$$

W is a uniformly finite rank operator of rank $N_j = \dim V_j$, such that

$$W^{(k)}_{mn} = \mathfrak{T}(E^{(j)}_{mn})$$

for $k \leq 1$.

The first step in the construction of our parametrix near z_j is analogous to the construction of $\overline{K}_0$, an inverse for $D_s^2 - z^2$ orthogonal to the troublesome terms e_i. Here, we define the $G^{(j)}_{mn}$ as before, but using $\overline{K}_0$ instead of $\overline{K}$. Then we have, by comparing G with K_0,

$$(\Delta - (\text{ias}\,\epsilon)^2 z^2)G(z) = \text{Id} + \frac{z_j + z}{2z_j}W - R(z) \tag{6.25}$$

with W as above and $R \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\text{Ls}};\Omega_D^{\frac{1}{2}}X^2_{\text{Ls}})$. Inverting $\text{Id} - R$ as before, we get

$$(\Delta - (\text{ias}\,\epsilon)^2 z^2)\big(G(z)(\text{Id} + S(z))\big) = \text{Id} + \frac{z_j + z}{2z_j}W(\text{Id} + S(z)) = \text{Id} - W'(z)$$

where $\text{Id} + S = (\text{Id} - R)^{-1}$. We now have an inverse up to the uniformly rank N_j error term $W' = (z_j + z)/(2z_j)W(\text{Id} + S)$. The null space of $\text{Id} - W'$ is contained in range $W = \text{span}\,\{e_i\}$. Let

$$W'(z)e_i = a_{ik}(z)e_k$$

where each $a_{ik}(z)$ is a holomorphic map from a neighbourhood of z_j to $\mathcal{A}^{\mathcal{G}}_{\text{phg}}([0,\delta])$, and $\mathcal{G}$ is natural. The function

$$q(\text{ias}\,\epsilon, z) = \det(\delta_{ik} - a_{ik}(z)).$$

is holomorphic in z and polyhomogeneous conormal in $\text{ias}\,\epsilon$ with natural index set, and such that

$$q(0,z) = \left(\frac{z - z_j}{2z_j}\right)^{N_j}. \tag{6.26}$$

Hence for small $\text{ias}\,\epsilon$, q has exactly N_j zeroes near z_j and does not vanish in some small annulus around z_j. Hence q vanishes precisely at the eigenvalues of Δ corresponding to $z_j^2 \in \text{spec RN}(\Delta)$. The inverse $\big(\text{Id} - W'\big)^{-1} = \text{Id} + F$ is therefore meromorphic in z, in the sense that $q \cdot (\mathrm{Id} + F)$ is holomorphic, and conormal in $\mathrm{ias}\,\epsilon$ with natural index set. The resolvent is

$$\left(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2\right)^{-1} = G(z)(\mathrm{Id} + S(z))(\mathrm{Id} + F) \in (\mathrm{ias}\,\epsilon)^{-1}\Psi^{-2,\mathcal{G}'}\left(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}\right)$$

and is meromorphic in the same sense as F, with $\mathcal{G}'$ natural.

As in the last section we can show that actually we can take $\mathcal{G}'$ the C^∞ index family. Indeed, we already know this is so in any region away from z_j. Also we may calculate the projector onto small eigenfunctions corresponding to z_j by the contour integral

$$\Pi_j = \frac{1}{2\pi i}\int_C \left(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2\right)^{-1} 2z_j dz$$

where C is a small circle that encloses all the zeroes of q, for small $\mathrm{ias}\,\epsilon$, and keeps away from all eigenvalues corresponding to other z_k. The value is, by the results of the previous section, in $\Psi^{-2,0}\left(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}\right)$. Moreover, the full symbol of the Laplacian is holomorphic near z_j, so the singularity at Δ_{Ls} is removed and the result is actually in $\Psi^{-\infty,0}_{\mathrm{bdy}}\left(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}}\right)$. Hence we have shown that the projector onto small eigenvalues corresponding to one z_j is smooth on the space X^2_{Ls} and that the resolvent itself, $\left(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2\right)^{-1}$, is smooth on X^2_{Ls} and meromorphic in z.

If $z_0 = 0$, the argument is the same, but some of the formulae must be modified to accommodate the presence of a double pole at $z = 0$. Equation (6.22) should be replaced by

$$\overline{K}(s, s', z) = \mathrm{proj}\, V_j\left(1 + z^{-2}\right) + \overline{K}_0(s, s', z).$$

In equation (6.25), we must replace $(z_j + z)/2z_j$ by $(1 + z^2)$. Finally, equation (6.26) is replaced by

$$q(0, z) = z^{2N}.$$

6.8. In the presence of L^2 null space.

Let $\{\phi_i\}$ be an orthonormal basis for the L^2_b-null space of $\Delta_{\overline{M}}$ on $M_+ \cup M_-$ (on half-densities). Lift the ϕ_i to smooth half-densities on X_{Ls}, also denoted ϕ_i, which vanish to infinite order at B_1 and B_2. This is possible since on the space $\overline{M}$, ϕ_i decays at least as some positive power at the boundary, so on $\overline{M}_{\log}$ it vanishes to infinite order. We can then form the finite rank operator $\Pi_{L^2} = \phi_i(p)\phi_i(p')\left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{1/2}$ which restricts at B_{00} to the projector onto L^2_b null space of Δ.

We have, near $\lambda = 0$,

$$\left(\Delta - \lambda^2\right)^{-1} = \mathrm{Reg}_{\bar{M}}(0) - \lambda^{-2}\Pi_{L^2},$$

with $\mathrm{Reg}_{\bar{M}}(0)$ regular near $\lambda = 0$. Since $\lambda^{-2} = (\mathrm{ias}\,\epsilon)^{-2} z^{-2}$, we expect a term looking like $(\mathrm{ias}\,\epsilon)^{-2} z^{-2}\Pi_{L^2}$ in our resolvent. Hence, we will get a nonzero term $G^{(-2)}_{00}$ of order $(\mathrm{ias}\,\epsilon)^{-2}$; since Π_{L^2} vanishes to infinite order at all other faces this should be the only term of order $(\mathrm{ias}\,\epsilon)^{-2}$.

Define $G_{00}^{(-2)} = \Pi_{L^2} \restriction B_{00}$, and let the other $G_{mn}^{(j)}$ terms be as above. The $G_{00}^{(-2)}$ term is not a good approximation to the term in the resolvent we expect, but it is a holomorphic term which has approximate range the projection onto the expected null space, which is what we need for the construction to work. Then instead of (6.25), we get

$$(\Delta - (\text{ias}\,\epsilon)^2 z^2)G = \text{Id} - (1 + z^2)(W + \Pi_{L^2}) - R,$$

with R in the parametrix-residual space, and W the approximate projection onto small eigenvalues corresponding to $0 \in \text{RN}(\Delta)$. The $z^2\Pi_{L^2}$ term comes from $-(\text{ias}\,\epsilon)^2 z^2$ hitting $G_{00}^{(-2)}$ and $\text{Id} - \Pi_{L^2}$ is produced from Δ hitting $G_{00}^{(0)} = \text{Reg}_{\bar{M}}(0)$. Inverting $(\text{Id} - R)$, we get

$$(\Delta - (\text{ias}\,\epsilon)^2 z^2)G(\text{Id} + S) = \text{Id} - (1 + z^2)(W + \Pi_{L^2})(\text{Id} + S),$$

where $(\text{Id} + R)^{-1} = \text{Id} + S$. Then $(1 + z^2)(W + \Pi_{L^2})(\text{Id} + S) = Y'$ is a finite rank operator which we treat like W' in the last section. The range of Y' is spanned by e_i and ϕ_i. Let $\delta_{kl} + a_{kl}$ be the matrix of $\text{Id} - Y'$ relative to the above basis. Then

$$q(\text{ias}\,\epsilon, z) = \det(\delta_{kl} - a_{kl}(z))$$

is holomorphic in z and polyhomogeneous conormal in $\text{ias}\,\epsilon$ with natural index set, and such that

$$q(0, z) = z^{2N},$$

where N here is the sum of the dimensions of the L^2 null space of $\Delta_{\overline{M}}$ and the null space of $\text{RN}(\Delta)$. The inverse $\left(\text{Id} - Y'\right)^{-1} = \text{Id} + F'$ is therefore meromorphic in z, in the sense that $q \cdot (\text{Id} + F')$ is holomorphic, and conormal in $\text{ias}\,\epsilon$ with natural index set. The resolvent is

$$\left(\Delta - (\text{ias}\,\epsilon)^2 z^2\right)^{-1} = G(z)(\text{Id} + S(z))(\text{Id} + F') \in \Psi^{-2,-2}(X_{\text{Ls}}^2; \Omega_D^{\frac{1}{2}} X_{\text{Ls}}^2)$$

and is meromorphic in the same sense as F'.

The results of this section and the previous one show that the limit as $\epsilon \to 0$ of the small eigenvalues approach 0 or one of the eigenvalues of $\text{RN}(\Delta)$, as claimed in Theorem 1.1.

6.9. Very small eigenvalues.

The projectors Π_j onto the eigenspace corresponding to z_j are smooth, that is, have a complete asymptotic expansion in powers of $\text{ias}\,\epsilon$ at each boundary hypersurface of X_{Ls}. For those corresponding to $z = 0$, which will be referred to as 'very small eigenvalues', this expansion is particularly simple.

PROPOSITION 6.1. *The very small eigenvalues are rapidly vanishing in* $\text{ias}\,\epsilon$.

PROOF: Consider the models of the regular part of the resolvent at $z = 0$, which we will denote $\widetilde{\mathrm{Res}}_{mn}^{(j)}$. Then, for $j \geq 0$, these models agree with those of $\mathrm{Res}_{mn}^{(j)}$; in other words, all the models for $\left(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2\right)^{-1}$ at $z = 0$ with $j \geq 0$ are actually regular at $z = 0$. This is because the models (6.2) — (6.7) for $j \geq 0$, for $z = 0$ (and only for $z = 0$) do not link models with different j. Also the compatibility conditions between between the models with $j \leq -1$ and $j \geq 0$ do not detect the pole in $\mathrm{Res}_{mn}^{(-1)}$ and $\mathrm{Res}_{00}^{(-2)}$ at $z = 0$. For $\mathrm{Res}_{00}^{(-2)}$ this is because Π_{L^2} is rapidly decreasing at all other boundary hypersurfaces. For $\mathrm{Res}_{mn}^{(-1)}$ this is because the models is constant to infinite order at each boundary hypersurface (indeed one can lift E from the double space X_s^2 of Mazzeo-Melrose). Hence, in the argument of the previous section, we may take $G_{mn}^{(j)} = \widetilde{\mathrm{Res}}_{mn}^{(j)}$ for all j, E as in section 6.8 and then we have

$$\left(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2\right)^{-1} G = \mathrm{Id} - (1 + z^2)(W + \Pi_{L^2}) - R$$

where now $R \in \Psi^{-\infty,+\infty}\left(X_{\mathrm{Ls}}^2; \Omega_D^{\frac{1}{2}} X_{\mathrm{Ls}}^2\right)$, that is, R has a smooth kernel vanishing to infinite order at $\mathrm{ias}\,\epsilon = 0$. Therefore

$$(\mathrm{Id} - R)^{-1} = \mathrm{Id} + S, \;\; S \in \Psi^{-\infty,+\infty}\left(X_{\mathrm{Ls}}^2; \Omega_D^{\frac{1}{2}} X_{\mathrm{Ls}}^2\right).$$

Following the argument in the previous section, we get

$$q(\mathrm{ias}\,\epsilon, z) = z^{2N} + O((\mathrm{ias}\,\epsilon)^\infty),$$

and it follows that the zeroes of q, which are the very small eigenvalues, are $O((\mathrm{ias}\,\epsilon)^\infty)$. ∎

Chapter 7. Full Resolvent

7.1. Resolvent spaces.

In this chapter we will unify the resolvent $\left(\Delta - (\mathrm{ias}\,\epsilon)^2 z^2\right)^{-1}$, constructed in the last chapter, with the resolvent $(\Delta - \lambda^2)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, constructed in Mazzeo-Melrose.

At the level of parameters, $\mathrm{ias}\,\epsilon$ and z (or $\lambda = z\,\mathrm{ias}\,\epsilon$), these regions are united in the space

$$X^0_{\mathrm{LsR}} = \left[X^0_{\mathrm{Ls}} \times \overline{\mathbb{C}}_\lambda; \{0\}_{\mathrm{ilg}\,\epsilon} \times \{0\}_\lambda\right],$$

the "zero-resolvent space". We use $\overline{\mathbb{C}}$, the complex numbers compactified with a circle at infinity, to stay in the class of compact manifolds with corners. At $\mathrm{ilg}\,\epsilon = 0$ there are two boundary hypersurfaces, $B^{\mathbb{C}} = [\overline{\mathbb{C}}; 0]_\lambda$, a punctured complex plane and $B^0 = \overline{\mathbb{C}}_z$, where $z = \lambda/\mathrm{ias}\,\epsilon$, a disc (hemisphere). We use Lemma 2.12 to create single, double and triple spaces that lie above X^2_{LsR}:

$$\begin{aligned} X_{\mathrm{LsR}} &= \left[X_{\mathrm{Ls}} \times \overline{\mathbb{C}}_\lambda; \partial X_{\mathrm{Ls}} \times \{0\}_\lambda\right] \\ X^2_{\mathrm{LsR}} &= \left[X^2_{\mathrm{Ls}} \times \overline{\mathbb{C}}_\lambda; \partial X^2_{\mathrm{Ls}} \times \{0\}_\lambda\right] \\ X^3_{\mathrm{LsR}} &= \left[X^3_{\mathrm{Ls}} \times \overline{\mathbb{C}}_\lambda; \partial X^3_{\mathrm{Ls}} \times \{0\}_\lambda\right]. \end{aligned} \tag{7.1}$$

Here we have used the notation of Lemma 2.12. Then, by Lemma 2.12, we have a commutative diagram of simple b-fibrations:

$$\begin{array}{ccccccc} X^3_{\mathrm{LsR}} & \longrightarrow & X^2_{\mathrm{LsR}} & \longrightarrow & X_{\mathrm{LsR}} & \longrightarrow & X^0_{\mathrm{LsR}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X^3_{\mathrm{Ls}} & \longrightarrow & X^2_{\mathrm{Ls}} & \longrightarrow & X_{\mathrm{Ls}} & \longrightarrow & X^0_{\mathrm{Ls}} \end{array}.$$

Each boundary hypersurface B_{mn} of X^2_{Ls} lifts to two hypersurfaces in X^2_{LsR}, one a blown up version of $B_{mn} \times \overline{\mathbb{C}}_z$, which we will denote B^0_{mn} and the other a blown up version of $B_{mn} \times \overline{\mathbb{C}}_\lambda$, which will be denoted $B^{\mathbb{C}}_{mn}$. For X^2_{LsR} we define the degree of $B^0_{mn}(X^2_{\mathrm{LsR}})$ and $B^{\mathbb{C}}_{mn}(X^2_{\mathrm{LsR}})$ to be the same as that of $B_{mn}(X^2_{\mathrm{Ls}})$.Then we have:

LEMMA 7.1. *Let π_j be the map $X^j_{\mathrm{LsR}} \to X^j_{\mathrm{Ls}}$. Then there are isomorphisms*

$$\Omega_D(X^j_{\mathrm{LsR}}) \equiv \pi_j^* \Omega_D(X^j_{\mathrm{Ls}}) \otimes \left|\frac{d\lambda d\overline{\lambda}}{\lambda\overline{\lambda} + (\mathrm{ias}\,\epsilon)^2}\right|. \tag{7.2}$$

PROOF: More generally, we will show that this result holds whenever one takes a manifold with corners Y with degrees and forms $[Y \times \mathbb{C}; \mathcal{H} \times \{0\}]$, where $\mathcal{H} = \{$all boundary hypersurfaces at $\mathrm{ias}\,\epsilon = 0\}$.

The result is clear away from $\lambda = 0$, so suppose we are near $\lambda = 0$ and near a corner of codimension k in $X^j_{\mathrm{Ls}} \times \mathbb{C}$. Let $H_1, \ldots H_k$ be the boundary hypersurfaces forming the corner, let $r_1, \dots r_k$ be boundary defining functions with $r_1 \dots r_k = \text{ias}\,\epsilon$ and suppose that $H_1 \dots H_k$ are blown up in that order, creating new hypersurfaces $H_1^0 \dots H_k^0$. Then the new boundary defining functions are ρ_i^0 where these are defined inductively by

$$(\rho_i^0)^2 = r_i^2 + |\lambda_{i-1}|^2$$
$$\lambda_i = \frac{\lambda_{i-1}}{\rho_i^0}.$$

Let $\rho_i^{\mathbb{C}} = r_i/\rho_i^0$ be a boundary defining function for the lift $H_i^{\mathbb{C}}$ of $H_i \times \mathbb{C}$. Ignoring the y factors in X^j_{Ls}, which are irrelevant here, the right hand side of (7.2) is

$$\tag{7.3} \prod_{i=1}^{k} \left(\frac{dr_i}{r_i^{d(H_i)+1}} \right) \frac{d\lambda d\overline{\lambda}}{|\lambda|^2 + (r_1 \dots r_k)^2}$$
$$= \prod_{i=2}^{k} \left(\frac{dr_i}{r_i^{d(H_i)+1}} \right) \frac{1}{(\rho_1^0 \rho_1^{\mathbb{C}})^{d(H_i)+1}} \frac{dr_1 d\lambda d\overline{\lambda}}{|\lambda|^2 + (r_1 \dots r_k)^2}.$$

Writing $dr_1 d\lambda d\overline{\lambda}$ in polar coordinates around H_1, we get

$$\prod_{i=2}^{k} \left(\frac{dr_i}{r_i^{d(H_i)+1}} \right) \frac{1}{(\rho_1^0 \rho_1^{\mathbb{C}})^{d(H_i)+1}} \frac{(\rho_1^0)^2 d\rho_1^0 d\lambda_1 d\overline{\lambda}_1}{(\rho_1^0)^2 (|\lambda_1|^2 + (\rho_1^{\mathbb{C}} r_2 \dots r_k)^2)}$$
$$= \prod_{i=2}^{k} \left(\frac{dr_i}{r_i^{d(H_i)+1}} \right) \frac{d\rho_1^0}{(\rho_1^0 \rho_1^{\mathbb{C}})^{d(H_i)+1}} \frac{d\lambda_1 d\overline{\lambda}_1}{|\lambda_1|^2 + (\rho_1^{\mathbb{C}} r_2 \dots r_k)^2}.$$

This is an expression of essentially the same form as (7.3) near (the lifts of) $H_2 \times \{0\}, \dots H_k \times \{0\}$ since $\rho_1^{\mathbb{C}} = 1$ there. Hence applying this reasoning $k-2$ times gives

$$\prod_{i=1}^{k} \left(\frac{d\rho_i^0}{(\rho_i^0 \rho_i^{\mathbb{C}})^{d(H_i)+1}} \right) \frac{d\lambda_k d\overline{\lambda}_k}{|\lambda_k|^2 + (\rho_1^{\mathbb{C}} \dots \rho_k^{\mathbb{C}})^2}.$$

which is explicitly a D-density near H_k^0, since all the $\rho_i^{\mathbb{C}}$ are equal to 1 there.

We also need to check (7.2) near intersections $H_i^0 \cap H_i^{\mathbb{C}}$. To do this, we only need to blow up $H_1 \times \{0\} \dots H_i \times \{0\}$, since all other blowups lie away from $H_i^{\mathbb{C}}$. Thus we can assume $i = k$. Then we have the boundary defining functions $\rho_k^{\mathbb{C}}$ or $\tilde{\rho}_k^{\mathbb{C}} = r_k/\lambda_{k-1}$ for $H_k^{\mathbb{C}}$, ρ_1^0 or λ_{k-1} for H_k^0 and $\omega = \overline{\lambda}_{k-1}/|\lambda_{k-1}|$ as angular coordinate. We have

$$\prod_{i=1}^{k-1} \left(\frac{d\rho_i^0}{(\rho_i^0 \rho_i^{\mathbb{C}})^{d(H_i)+1}} \right) \frac{dr_k}{r_k^{d(H_k)+1}} \frac{d\lambda_{k-1} d\overline{\lambda}_{k-1}}{|\lambda_{k-1}|^2 + (\rho_1^{\mathbb{C}} \dots \rho_{k-1}^{\mathbb{C}} r_k)^2}$$
$$= \prod_{i=1}^{k-1} \left(\frac{d\rho_i^0}{(\rho_i^0 \rho_i^{\mathbb{C}})^{d(H_i)+1}} \right) \frac{\lambda_{k-1} d\tilde{\rho}_k^{\mathbb{C}}}{(\tilde{\rho}_k^{\mathbb{C}} \lambda_{k-1})^{d(H_k)+1}} \frac{d\lambda_{k-1} \, |\lambda_{k-1}| d\omega}{|\lambda_{k-1}|^2 (1 + (\rho_1^{\mathbb{C}} \dots \rho_{k-1}^{\mathbb{C}} \tilde{\rho}_k^{\mathbb{C}})^2)}$$
$$= \prod_{i=1}^{k-1} \left(\frac{d\rho_i^0}{(\rho_i^0 \rho_i^{\mathbb{C}})^{d(H_i)+1}} \right) \frac{d\tilde{\rho}_h^{\mathbb{C}} d\lambda_{k-1}}{(\tilde{\rho}_h^{\mathbb{C}} \lambda_{k-1})^{d(H_k)+1}} \frac{\overline{\omega} d\omega}{1 + (\rho_1^{\mathbb{C}} \dots \rho_{k-1}^{\mathbb{C}} \tilde{\rho}_k^{\mathbb{C}})^2}.$$

Again, this is explicitly a D-density, so we have proved the lemma. ∎

7.2. Operator Calculus.

To carry out the constructions of the previous chapter for the full resolvent space, we need the ingredients of that chapter — small pseudodifferential calculus, boundary terms, parametrix-residual space, closure of the parametrix residual space under Neumann sums, and composition formula for $\Psi^{-2,\mathcal{F}}$ with the parametrix-residual space — extended to the "full resolvent" setting. Here we can take advantage of the functorial nature of the constructions of chapter 4, and simply assert that we obtain all these things from that chapter by replacing the spaces X^j_{Ls} with X^j_{LsR}. We will denote the spaces of LsR-pseudodifferential operators by

$$\Psi^{m,\mathcal{F}}(X^2_{\mathrm{LsR}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{LsR}})$$

and the parametrix-residual space by

$$\Psi^{-\infty}_{\text{par-res}}(X^2_{\mathrm{LsR}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{LsR}}) = \Psi^{-\infty,\mathcal{F}'}_{\text{bdy}}(X^2_{\mathrm{LsR}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{LsR}}),$$

where $\mathcal{F}'$ is a natural index family; the one, in fact, which assigns to B^0_{mn} and $B^{\mathbf{C}}_{mn}$ the index set $\mathcal{F}(B_{mn})$, where $\mathcal{F}$ is the index set from section 4.7. As with our previous parametrix-residual space, it is closed under taking Neumann series; the proof follows the same lines as that of Lemma 4.7. Finally we have composition formulae just as in Lemma 4.8.

7.3. Full Parametrix.

We will look for a parametrix G in the space $\lambda^{-2}\Psi^{-2,\mathcal{F}}(X^2_{\mathrm{LsR}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{LsR}})$ such that

$$(\Delta-\lambda^2)G-\mathrm{Id} = R \in \Psi^{-\infty}_{\text{par-res}}(X^2_{\mathrm{LsR}};\Omega^{\frac{1}{2}}_D X^2_{\mathrm{LsR}})$$

away from the continuous spectrum, that is, in the set $X^0_{\mathrm{LsR}} \setminus \{\lambda \in \mathbb{R}, z = \infty\}$. This includes the region where z is finite, down to $\operatorname{ias}\epsilon = 0$, and joins the region $|z|$ large, $\delta < |\arg(z)| < \pi - \delta$ with λ small, $\delta < |\arg(\lambda)| < \pi - \delta$.

We will construct the parametrix G as before, as a finite Taylor series off the boundary hypersurfaces at $\operatorname{ias}\epsilon = 0$. We will continue to write $G^{(j)}_{mn}$ for the term at B_{mn} of order $(\operatorname{ias}\epsilon)^j$ and write $G^{(\mathbf{C})}_{mn}$ for the term at $B^{\mathbf{C}}_{mn}$ of order 0 (there will be no other nonzero terms on the faces with $\lambda \neq 0$).

The correct models for nonzero λ can be read off from the parametrix constructed in Mazzeo-Melrose. Lifting their results to X^2_{LsR}, we have

$$\begin{aligned} G^{(\mathbf{C})}_{00} &= \left(\Delta_{\overline{M}}-\lambda^2\right)^{-1} \\ G^{(\mathbf{C})}_{11} &= \left(\Delta_{\overline{H}}-\lambda^2\right)^{-1} \\ G^{(\mathbf{C})}_{33} &= \left(\Delta_{\overline{H}}-\lambda^2\right)^{-1} \end{aligned} \tag{7.4}$$

and all others are zero. Indeed, it is shown there that the error in this parametrix is conormal of positive order on the original surgery space X^2_s, and so after logarithmic blowup vanishes to infinite order at all other boundary hypersurfaces. One can also derive these models directly on the logarithmic resolvent space, but this would conceal the simplicity of the results. We have defined the $G^{(j)}_{mn}$ in the previous chapter, so as to satisfy the model operator equations and compatibility amongst themselves. We need to check that the two sets of models are compatible. As explained in chapter 6, for $G^{(0)}_{22}$ and $G^{(1)}_{22}$ we only need to check that adjacent faces are $\operatorname{null}(\Delta_{\overline{H}})\otimes\operatorname{null}(\Delta_{\overline{H}})$-valued and compatible amongst themselves; it is not necessary to write down explicit models for these terms.

Compatibility with $G^{(\mathbb{C})}_{00}$ The face $B^{\mathbb{C}}_{00}$ intersects B^0_{00}, B^0_{20}, B^0_{22}, B^0_{33} and B^0_{02}. Consider compatibility with $G^{(-1)}_{22}$. In the interior of the intersection, we may take boundary defining functions λ for B^0_{22} and $1/z$ for $B^{\mathbb{C}}_{00}$, and coordinates $\lambda\log x$, $\operatorname{ilg}(x'/x)/\operatorname{ilg} x$, y and y' along the intersection. We refer back to (6.8) for the compatibility condition. We calculate from (6.12) that $\big(G^{(\mathbb{C})}_{00}\big)_{22,-1}$ is

$$\frac{1}{2i}\left((x'/x)^{\pm i\lambda}\phi_J(y)\phi_J(y')+x^{i\lambda}x'^{i\lambda}\phi_J(y)S_{JK}(0)\phi_K(y')\right),$$

and from (5.7), $\big(G^{(-1)}_{22}\big)_{00,\mathbb{C},1}$ is given by

$$\frac{e^{-iz|s-s'|}+e^{-iz|s-(2-s')|}}{2i}\operatorname{proj}\Lambda^D_{\pm}+\frac{e^{-iz|s-s'|}-e^{-iz|s-(2-s')|}}{2i}\operatorname{proj}\Lambda^N_{\pm}.$$

These agree since $S(0)=\operatorname{proj}_{\Lambda^D_{\pm}}-\operatorname{proj}_{\Lambda^N_{\pm}}$ and $e^{-iz|s-s'|}=(x'/x)^{\pm i\lambda}$ and $e^{-iz|s-(2-s')|}=x^{i\lambda}x'^{i\lambda}$.

Next consider compatibility with $G^{(j)}_{02}$. We may use boundary defining functions λ for B^0_{02} and $1/z$ for $B^{\mathbb{C}}_{00}$, and coordinates $\lambda\log x'$, $\operatorname{ilg} x$, y and y' along the intersection. Let us do the calculation at $s=s'=1$. The calculation for the compatibility of $G^{(\mathbb{C})}_{00}$ and $G^{(-1)}_{02}$ is very similar to the above, so we omit it. To check compatibility of $G^{(\mathbb{C})}_{00}$ and $G^{(0)}_{02}$, we compute from (6.14)

$$\big(G^{\mathbb{C}(0)}_{00}\big)_{22,0}=\frac{i}{2}e^{i\lambda\log x'}\Phi^{(1)}_J(x,y)\phi_J(y'),$$

and from (5.7) and (6.19)

$$\begin{aligned}\big(G^{(0)}_{02}\big)_{00,0}&=\mathfrak{T}(\overline{K}^{(0)}_{02})+\frac{i}{2}\psi((1-s')/2)e^{-iz(1-s')}\Phi^{(1)}_j(p)S^{(1)}_{jk}\phi_k(y')\\&=\mathfrak{T}(\partial_s\overline{K}(1,1-0,z))+\frac{i}{2}e^{-iz(1-s')}\Phi^{(1)}_j(p)S^{(1)}_{jk}\phi_k(y')\\&=\frac{1}{2}e^{-iz(1-s')}\Phi^{(1)}_\alpha\phi_\alpha+\frac{i}{2}e^{-iz(1-s')}\Phi^{(1)}_j(p)S^{(1)}_{jk}\phi_k(y'),\end{aligned}$$

which agree. To check compatibility of $G^{(\mathbb{C})}_{00}$ and $G^{(1)}_{02}$, we compute

$$\big(G^{(\mathbb{C})}_{00}\big)_{22,1}=\frac{1}{2i}e^{i\lambda\log x'}\Phi^{(2)}_J(p)\phi_J(y'),$$

$$\begin{aligned}(G_{02}^{(0)})_{00,\mathbb{C},-1} &= (\mathfrak{T}(\overline{K}_{02}^{(1)}))_{00,\mathbb{C},-1} + \frac{1}{2i}e^{-iz(1-s')}\Phi_\alpha^{(2)}(p)\phi_\alpha(y') \\ &= \frac{1}{2i}e^{-iz(1-s')}\Phi_j^{(2)}(p)\phi_j(y') + \frac{1}{2i}e^{-iz(1-s')}\Phi_\alpha^{(2)}(p)\phi_\alpha(y'),\end{aligned}$$

using (5.7) again, which agrees.

The checking of $G_{00}^{(\mathbb{C})}$ and $G_{33}^{(j)}$ is very similar to the above calculations, so we omit it. Let us check compatibility of $G_{00}^{(\mathbb{C})}$ and $G_{00}^{(j)}$. Again we use λ and $1/z$ as boundary defining functions. Then

$$(G_{00}^{(\mathbb{C})})_{00,-1} = \operatorname{Res}_{\overline{M}}^{(-1)} = -i\,\operatorname{proj}_{\mathcal{C}^\infty\,\mathrm{null}}$$

as remarked after (6.9), and

$$(G_{00}^{(-1)})_{00,\mathbb{C},1} = \mathfrak{T}((\overline{K}(1,1,z))_{00,\mathbb{C},1}) = \mathfrak{T}(-i\,\operatorname{proj}\Lambda_\pm^D)$$

by (5.7), which agrees. For $j=0$ we have

$$(G_{00}^{(\mathbb{C})})_{00,0} = \operatorname{Res}_{\overline{M}}^{(0)}$$

and from (6.16)

$$(G_{00}^{(0)})_{00,\mathbb{C},0} = \operatorname{Res}_{\overline{M}}^{(0)}$$

since $A(z)$ is rapidly decreasing in z.

Compatibility with $G_{33}^{(\mathbb{C})}$ The face $B_{33}^{\mathbb{C}}$ intersects B_{22}^0 and B_{33}^0. In the interior of $B_{33}^{\mathbb{C}} \cap B_{22}^0$ we may use boundary defining functions $\lambda, 1/z$ and coordinates $\lambda\log(x'/x)$, $(s+s')/2$, y, y'. We need check compatibility only with $G_{22}^{(-1)}$.

For $G_{33}^{(\mathbb{C})}$ we have an explicit expression in terms of the eigenvalues σ_j and eigenspaces V_j of Δ_H (where $\sigma_0 = 0$ and $V_0 = V$):

$$G_{33}^{(\mathbb{C})} = (\Delta_{\overline{H}} - \lambda^2)^{-1} = \sum_{j=0}^{\infty} \frac{e^{-\sqrt{\sigma_j^2-\lambda^2}|\log(x'/x)|}}{2\sqrt{\sigma_j^2-\lambda^2}}\operatorname{proj}_{V_j}. \tag{7.5}$$

Hence

$$(G_{33}^{(\mathbb{C})})_{22,-1} = \frac{e^{-i\lambda|\log(x'/x)|}}{2i}\operatorname{proj}_V$$

and by (5.7) and (5.6)

$$(G_{22}^{(-1)})_{33,\mathbb{C},1} = \frac{e^{-iz|s-s'|}}{2i}\operatorname{proj}_V$$

which agree. (Note that, in (5.7), $e^{-iz(2-s-s')}$ and $e^{-iz(-2+s+s')}$ are rapidly vanishing at $B_{33}^{\mathbb{C}}$.)

We next check compatibility of $G_{33}^{(\mathbb{C})}$ and $G_{33}^{(j)}$. Again we use boundary defining functions $\lambda, 1/z$. Then from (7.5) we calculate

$$\begin{aligned}
(G_{33}^{(\mathbb{C})})_{33,-1} &= \frac{\mathrm{proj}_V}{2i} \\
(G_{33}^{(\mathbb{C})})_{33,0} &= \mathrm{Res}_{\overline{H}}^{(0)} \\
(G_{33}^{(\mathbb{C})})_{33,1} &= -\frac{(\log(x'/x))^2}{4i}\,\mathrm{proj}_V\,.
\end{aligned}$$

From (5.7) and (5.6), again observing that $e^{-iz(2-s-s')}$ is rapidly vanishing at $B_{33}^{\mathbb{C}}$, we see that the first one agrees with $(G_{33}^{(-1)})_{33,\mathbb{C},1}$. From (6.18), since $D(s,z)$ vanishes rapidly as $|z| \to \infty$, the second line agrees with $(G_{33}^{(0)})_{33,\mathbb{C},0}$. Finally we calculate $(G_{33}^{(1)})_{33,\mathbb{C},-1}$. Using coordinates as in (6.17), we have

$$\begin{aligned}
(G_{33}^{(1)})_{33,\mathbb{C},-1} &= \mathfrak{T}\big((\overline{K}_{33}^{(1)})_{33,\mathbb{C},-1}\big) \\
&= \frac{\log(x'/x)^2}{2}\,\partial_s^2\overline{K}(\tilde{s},\tilde{s},z) \\
&= -\frac{(\log x'/x)^2}{4i}\,z^2\overline{K}(s,s,z).
\end{aligned}$$

which checks with the third line above.

Compatibility with $G_{11}^{(\mathbb{C})}$ The checking of these compatibility conditions follows the same lines, but is considerably simpler; we leave these as an exercise for the reader.

Compatibility with the symbol We showed in the last chapter that the models $G_j^{(mn)}$ were compatible with the symbol map. On the faces $B_{mm}^{\mathbb{C}}$, the total symbol $\sigma_{\mathrm{tot}}(\Delta-\lambda^2)_{\restriction B_{mm}} = \sigma_{\mathrm{tot}}(\Delta_{\overline{M}})$ or $\sigma_{\mathrm{tot}}(\Delta_{\overline{H}})$ to infinite order after logarithmic blowup, so these models are consistent with the symbol map too. Really this is just lifting the compatibility result from Mazzeo-Melrose to this space.

It follows that one can construct a parametrix G restricting to all the given models and compatible with the symbol.

7.4. Full Parametrix to Full Resolvent.

This is done in exactly the same way as in chapter 6. We may also argue as before to show that the resolvent is smooth away from the diagonal. Let us summarize the results of chapters 6 and 7:

THEOREM 7.2. *The resolvent family* $\mathrm{Res}(\lambda) = (\Delta-\lambda^2)^{-1}$ *lifts to the resolvent double space to an element of* $\Psi^{-2,0}(X_{\mathrm{LsR}}^2;\Omega_D^{\frac12}X_{\mathrm{LsR}}^2) + \lambda^{-2}\,\Psi_{\mathrm{bdy}}^{-\infty,0}(X_{\mathrm{LsR}}^2;\Omega_D^{\frac12}X_{\mathrm{LsR}}^2)$ *away from the continuous spectrum* $\{\lambda \text{ real}, |z|=\infty\}$. *The leading terms on each* *face are:*

$$\begin{aligned}
\mathrm{Res}_{mn}^{(-1)} &= \mathfrak{T}\big((\mathrm{RN}(\Delta) - z^2)^{-1}\big) \\
\mathrm{Res}_{00}^{(-2)} &= \mathrm{proj}_{L^2} \\
\mathrm{Res}_{00}^{(\mathbb{C})} &= (\Delta_{\overline{M}} - \lambda^2)^{-1} \\
\mathrm{Res}_{33}^{(\mathbb{C})} &= (\Delta_{\overline{H}} - \lambda^2)^{-1} \\
\mathrm{Res}_{11}^{(\mathbb{C})} &= (\Delta_{\overline{H}} - \lambda^2)^{-1} \\
\textit{other}\ \mathrm{Res}_{mn}^{(\mathbb{C})} &\equiv \emptyset.
\end{aligned}$$

PROOF: The statement about smoothness follows from the argument given in section 6.6. The models $\mathrm{Res}_{mn}^{(j)}$ are given by $G_{mn}^{(j)} + (G \cdot S)_{mn}^{(j)}$. By Lemma 4.8, $G \cdot S$ has no order -1 terms up to the boundary, so the $\mathrm{Res}_{mn}^{(-1)}$ terms are given by $G_{mn}^{(-1)} = \mathfrak{T}\big((\mathrm{RN}(\Delta) - z^2)^{-1}\big)$. At $B_{mn}^{\mathbb{C}}$ the parametrix is good to infinite order so the models of the resolvent agree with those of our parametrix. ∎

Chapter 8. Heat Kernel

In this chapter we will define the logarithmic heat space and associated spaces, and prove Theorem 1.2. The heat kernel $e^{-t\Delta_\epsilon}$ is given by the operator valued contour integral

$$\frac{1}{2\pi i}\int_\Gamma e^{-t\lambda^2}(\Delta-\lambda^2)^{-1}2\lambda d\lambda \tag{8.1}$$

where $\Gamma \in \mathbb{C}$ is a contour that encloses (in a suitable sense) spec $\Delta_\epsilon \subset [0,\infty)$. In the spirit of this thesis, we will perform the integral as a pushforward under a b-fibration between appropriate spaces. Then the Pushforward theorem will tell us that the result is polyhomogeneous and will give us the top order terms explicitly at each face. In the next section we will define a big space, the "heat-resolvent space", lying above both the heat space and the resolvent space on which the integrand of (8.1) lies.

8.1. The Heat Space and the Heat-Resolvent Space.

The heat kernel for finite time $[0,T^2]$ has been treated in Mazzeo-Melrose. There are no changes that need to be made to treat the case when Δ_H has null space; the heat space X^2_{hs} is a suitable space to carry the heat kernel in general. It is in the behaviour of the heat kernel for large times that the situation is very different; in Mazzeo-Melrose the heat kernel was rapidly decreasing as $t\to\infty$ uniformly in ϵ (up to finite rank) but here this is no longer the case. In fact, the structure of the heat kernel near $t=\infty$ has the same degree of complexity as the resolvent near $\lambda=0$.

To define the heat space for large times $t\geq C^2$, let $\tau=\sqrt{t}$ and denote by $[C,\infty]_\tau$ the compactification of the interval $\{C\leq\tau<\infty\}$ with boundary defining function τ^{-1} at infinity. Then we define the logarithmic heat space to be, in the notation of Lemma 2.12,

$$X^{2,C}_{\mathrm{LHs}}=\left[X^2_{\mathrm{Ls}}\times[C,\infty]_\tau;\partial X^2_{\mathrm{Ls}}\times\{\tau=\infty\}\right].$$

Denote the lifts of $B_{mn}\times[C,\infty]_\tau$ by $B_{mn}(X^{2,C}_{\mathrm{LHs}})$ or just B_{mn} if this is unambiguous in context, and denote the result of blowing up $B_{mn}\times\{\infty\}$ by B^∞_{mn}. Let the degrees of $B_{mn}(X^{2,C}_{\mathrm{LHs}})$ and B^∞_{mn} be the same as that of B_{mn}, and define the degree of (the lift of) $X^2_{\mathrm{Ls}}\times\{\infty\}$ to be 0 and the degree of $X^2_{\mathrm{Ls}}\times\{C\}$ to be -1. This means the densities are smooth up to $\tau=C$, reflecting the fact that this is an artificial, and ignorable, boundary.

The heat-resolvent space is a space which maps down to both the heat space and the resolvent space, ie, it has both t and λ variables. It is not quite the space on which the integrand of 1 lies, since we also have to restrict to a contour of integration; this is done in the next section. To define the heat-resolvent space, let $\mathcal{B}_j$, $\mathcal{B}^\infty_j$ be the set of B_{mn} (B^∞_{mn}) for X^2_{LHs} of degree j, for $j=1\cdots3$. The heat-resolvent space is defined to be

$$X^{2,C}_{\mathrm{LsHR}}=\big[X^2_{\mathrm{LHs}}\times\overline{\mathbb{C}}_\lambda;\mathcal{B}^\infty_3\times\{0\}_\lambda;\mathcal{B}_3\times\{0\}_\lambda;\mathcal{B}^\infty_2\times\{0\}_\lambda;\mathcal{B}_2\times\{0\}_\lambda; \\ \mathcal{B}^\infty_1\times\{0\}_\lambda;\mathcal{B}_1\times\{0\}_\lambda\big]. \tag{8.2}$$

The $\mathcal{B}_3^\infty \times \{0\}$ blowup separates the $\mathcal{B}_3 \times \{0\}$ faces from the $\mathcal{B}_2^\infty \times \{0\}$ and the $\mathcal{B}_1^\infty \times \{0\}$ faces, and the $\mathcal{B}_2^\infty \times \{0\}$ blowup separates the $\mathcal{B}_2 \times \{0\}$ faces from the $\mathcal{B}_1^\infty \times \{0\}$ faces. Hence, the heat-resolvent space is also given by

$$\begin{aligned} X^{2,C}_{\mathrm{LsHR}} = \big[X^2_{\mathrm{LHs}} \times \overline{\mathbb{C}}_\lambda; \mathcal{B}_3^\infty \times \{0\}_\lambda; \mathcal{B}_2^\infty \times \{0\}_\lambda; \mathcal{B}_1^\infty \times \{0\}_\lambda;\\ \mathcal{B}_3 \times \{0\}_\lambda; \mathcal{B}_2 \times \{0\}_\lambda; \mathcal{B}_1 \times \{0\}_\lambda\big]. \end{aligned} \tag{8.3}$$

We will want both descriptions in order to lift maps to $X^{2,C}_{\mathrm{LsHR}}$. Denote the hypersurfaces of $X^{2,C}_{\mathrm{LsHR}}$ by B^0_{mn}, $B^{\mathbb{C}}_{mn}$, $B^{\infty,0}_{mn}$ or $B^{\infty,\mathbb{C}}_{mn}$ according as they are the lift of $B_{mn}(X^{2,C}_{\mathrm{LHs}}) \times \{0\}$, $B_{mn}(X^{2,C}_{\mathrm{LHs}}) \times \{\mathbb{C}\}$, $B^\infty_{mn}(X^{2,C}_{\mathrm{LHs}}) \times \{0\}$, or $B^\infty_{mn}(X^{2,C}_{\mathrm{LHs}}) \times \{\mathbb{C}\}$, respectively. Define $d(B^0_{mn}) = d(B^{\mathbb{C}}_{mn}) = d(B^{\infty,0}_{mn}) = d(B^{\infty,\mathbb{C}}_{mn}) = d(B_{mn})$.

These spaces have been defined with good mapping properties in mind. By Lemma 2.8, the *b*-fibration $X^2_{\mathrm{Ls}} \times [C,\infty]_\tau \to X^2_{\mathrm{Ls}}$ lifts to a *b*-fibration $\pi_H \colon X^{2,C}_{\mathrm{LHs}} \to X^2_{\mathrm{Ls}}$, and the *b*-fibration $X^{2,C}_{\mathrm{LHs}} \times \overline{\mathbb{C}}_\lambda \to X^{2,C}_{\mathrm{LHs}}$ lifts to a *b*-fibration $\pi_{\mathrm{heat}} \colon X^{2,C}_{\mathrm{LsHR}} \to X^{2,C}_{\mathrm{LHs}}$. By Lemma 2.5, the *b*-fibration $X^{2,C}_{\mathrm{LHs}} \times \mathbb{C} \to X^2_{\mathrm{Ls}} \times \mathbb{C}$ lifts to a *b*-fibration $\pi_{\mathrm{res}} \colon X^{2,C}_{\mathrm{LsHR}} \to X^2_{\mathrm{LsR}}$.

We then have the bundle isomorphisms:

LEMMA 8.1.

$$\Omega_D(X^{2,C}_{\mathrm{LHs}}) \equiv \pi_H^* \Omega_D(X^2_{\mathrm{Ls}}) \otimes \left|\frac{d\tau}{\tau}\right|$$

$$\Omega_D(X^{2,C}_{\mathrm{LsHR}}) \equiv \pi^* \Omega_D(X^2_{\mathrm{LsR}}) \otimes \left|\frac{d\tau}{\tau}\right|$$

$$\Omega_D(X^{2,C}_{\mathrm{LsHR}}) \equiv \pi^* \Omega_D(X^{2,C}_{\mathrm{LHs}}) \otimes \left|\frac{d\lambda d\overline{\lambda}}{\lambda\overline{\lambda} + (\mathrm{ias}\,\epsilon)^2}\right|.$$

PROOF: The first two follow easily from the fact that the *b*-density bundle lifts to the *b*-density bundle when a boundary face is blown up. The third equation follows from Lemma 7.1. ∎

8.2. Contour spaces.

The contour of integration, Γ, we choose to do the integral (8.1) is a *p*-submanifold of X^0_{LsR} of codimension one as illustrated in the figure. It should have the following properties:

(i) Γ lies in the right half plane;

(ii) For $|z| > 1$,

$$\Gamma = \text{closure}\,\{\arg z = \pm\frac{\pi}{6}\};$$

(iii) For $|z| < 2$, Γ is given by a relation $\{\operatorname{Re} z = f(\operatorname{Im} z)\}$, with f smooth, such that for $|z| > 1$ this relation is given by (ii) above and $f(0) = a$, where $a > 0$ is real and a^2 is strictly less than the smallest nonzero eigenvalue of $\mathrm{RN}(\Delta)$;

(iv) Γ is disjoint from the poles of $(\Delta - \lambda^2)^{-1}$, for ilg ϵ small.

Condition (iv) is possible and compatible with (iii) by the results of chapter 6.

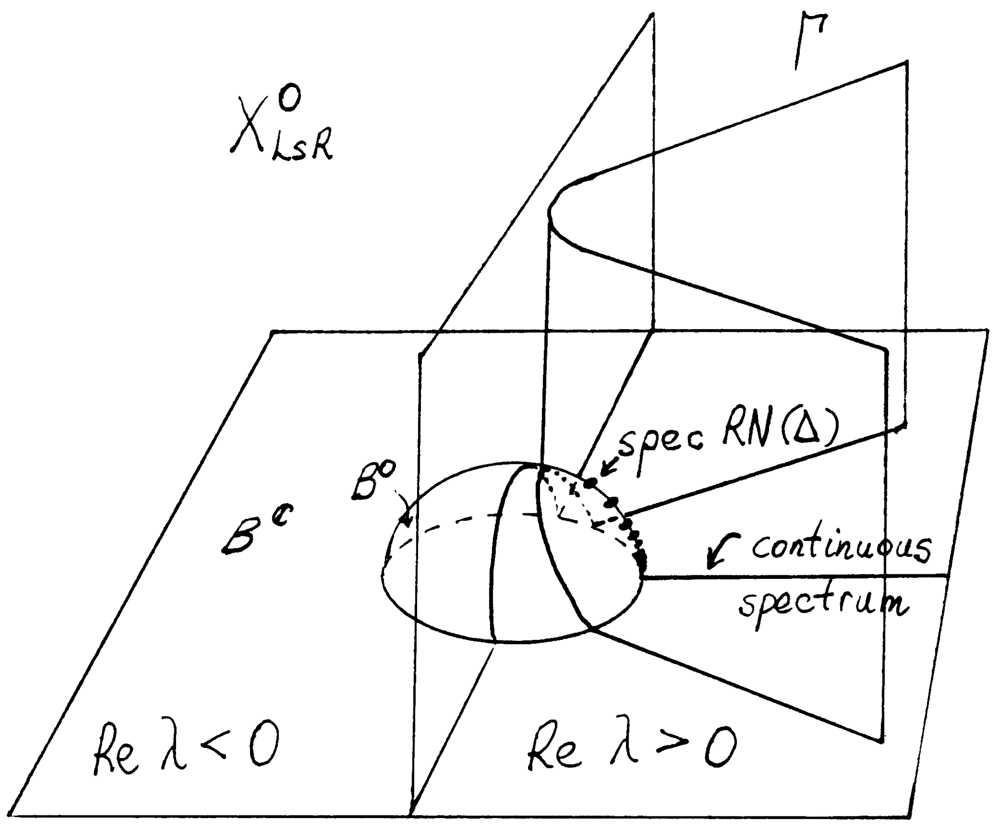


Figure 3. The contour Γ.

It follows that Γ "encloses" all of the spectrum of Δ except that corresponding to L^2 null space of $\Delta_{\overline{M}}$ and $z_0 \in$ spec RN(Δ), if $z_0 = 0$. We denote $\Gamma \cap B^0$ by $\mathcal{C}$, a contour in the right half z-plane enclosing all positive spectrum of RN(Δ), and $\Gamma \cap B^{\mathrm{C}}$ by $\gamma_+ \cup \gamma_-$, where $\gamma_\pm = \{\arg\lambda = \pm\pi/6\}$.

Since Γ is an interior p-submanifold of X^0_{LsR} of codimension one, the inverse images of Γ in X^2_{LsR} and $X^{2,C}_{\mathrm{LsHR}}$ are also interior p-submanifolds of codimension one, which we will denote X^2_{LsC} and X^2_{LsHC} respectively, the "double contour space" and the "double heat-contour space". Moreover the maps $X^2_{\mathrm{LsHC}} \to X^2_{\mathrm{LsC}} \to \Gamma$ are simple b-fibrations. Since these spaces are interior p-submanifolds of the respective resolvent space, their boundary hypersurfaces are given by their intersection with boundary hypersurfaces of the ambient space. We will use the same notation for maps between the resolvent spaces and their restrictions to the contour spaces. The map $X^2_{\mathrm{LsHC}} \to \Gamma$ will be denoted π_{par}. We also use the same notation for the boundary hypersurface of the contour spaces as for the corresponding resolvent space, and assign the same degree. With these degrees,

$$\left|\frac{d\,\mathrm{ias}\,\epsilon}{(\mathrm{ias}\,\epsilon)^2}\frac{d\lambda}{\lambda}\right|$$

is a nonzero smooth section of $\Omega_D(\Gamma)$. Similarly, we have

$$\Omega_D(X^2_{\mathrm{LsC}}) \cong \pi^*\Omega_D(X^2_{\mathrm{Ls}}) \otimes |\frac{d\lambda}{\lambda}|$$
$$\Omega_D(X^2_{\mathrm{LsHC}}) \cong \pi^*\Omega_D(X^{2,C}_{\mathrm{LHs}}) \otimes |\frac{d\lambda}{\lambda}|.$$

It will be convenient in the proof of Proposition 8.2 to have the densities written in this product form.

We need one more space before carrying out the integral (8.1). This is a space to carry the function $e^{t\lambda^2}$. Note that this is smooth on the space

$$[[T,\infty]_\tau \times \mathbb{C}; \{\tau = \infty, \lambda = 0\}]. \tag{8.4}$$

Hence by Lemma 2.1, this lifts to a smooth function on the space

$$\begin{aligned} \tilde{X}^0_{\mathrm{LsHR}} =& [X^0_{\mathrm{Ls}} \times [T,\infty]_\tau \times \mathbb{C}; \{\mathrm{ilg}\,\epsilon = 0, \tau = \infty, \lambda = 0\}; \{\tau = \infty, \lambda = 0\}; \\ & \{\mathrm{ilg}\,\epsilon = 0, \lambda = 0\}; \{\mathrm{ilg}\,\epsilon = 0, \tau = \infty\}] \\ =& [X^0_{\mathrm{Ls}} \times [T,\infty]_\tau \times \mathbb{C}; \{\mathrm{ilg}\,\epsilon = 0, \tau = \infty\}; \{\mathrm{ilg}\,\epsilon = 0, \tau = \infty, \lambda = 0\}; \\ & \{\mathrm{ilg}\,\epsilon = 0, \lambda = 0\}; \{\tau = \infty, \lambda = 0\}]. \end{aligned} \tag{8.5}$$

Indeed, by Lemmas 1, 5 and 8, this space maps down to both the space (8.4) and X^0_{LsR} with *b*-fibrations. Lifting $\Gamma \subset X^0_{\mathrm{LsR}}$ to $\tilde{X}^0_{\mathrm{LsHR}}$, we obtain the "zero-heat-contour space" $X^0_{\mathrm{LsHC}} = \pi^{-1}\Gamma$. This is an appropriate space to carry the function $e^{t\lambda^2}$. Now observe that, in fact, $\pi^{-1}\Gamma$ does not meet the face $\{\tau = \infty, \lambda = 0\}$, the last face to be blown up above, and so for the purposes of defining X^0_{LsHC} this blowup could be omitted. Let us define the "zero-heat-resolvent space" X^0_{LsHR} (no tilde) to be the space in (8.5) with this blowup omitted:

$$\begin{aligned} X^0_{\mathrm{LsHR}} = [X^0_{\mathrm{Ls}} \times [T,\infty]_\tau \times \mathbb{C}; \{\mathrm{ilg}\,\epsilon = 0, \tau = \infty\}; \{\mathrm{ilg}\,\epsilon = 0, \tau = \infty, \lambda = 0\}; \\ \{\mathrm{ilg}\,\epsilon = 0, \lambda = 0\}]. \end{aligned}$$

Then, by Lemma 2.5, and using the second description (8.3) of $X^{2,C}_{\mathrm{LsHR}}$, the map

$$X^2_{\mathrm{Ls}} \times [C,\infty]_\tau \times \mathbb{C} \longrightarrow X^0_{\mathrm{Ls}} \times [C,\infty]_\tau \times \mathbb{C}$$

lifts to a *b*-fibration

$$X^{2,C}_{\mathrm{LsHR}} \longrightarrow X^0_{\mathrm{LsHR}}$$

and so

$$X^2_{\mathrm{LsHC}} \longrightarrow X^0_{\mathrm{LsHC}}$$

is also a *b*-fibration. The point of this argument with $\tilde{X}^0_{\mathrm{LsHR}}$ and X^0_{LsHR} is to show that $e^{t\lambda^2}$ lifts to be smooth on X^2_{LsHC}, although it is *not* smooth on $X^{2,C}_{\mathrm{LsHR}}$. Finally we have the bundle isomorphism:

$$\Omega_D(X^2_{\mathrm{LsHC}}) \cong \Omega^{\frac{1}{2}}_D(X^2_{\mathrm{LsR}}) \otimes \Omega^{\frac{1}{2}}_D(X^{2,C}_{\mathrm{LHs}}) \otimes \Omega^{\frac{1}{2}}_D(X^0_{\mathrm{LsHC}}) \otimes |\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}|^{-\frac{1}{2}}.$$

8.3. Behaviour as $t \to \infty$.

Our contour Γ does not enclose the eigenvalues of Δ_ϵ corresponding to L^2 zero modes of $\Delta_{\overline{M}}$ or the value $0 \in \operatorname{spec} \mathrm{RN}(\Delta)$. Hence, the integral (8.1) will yield the heat kernel projected off these modes. We will denote this by $e^{-t\Delta_\epsilon^\perp}$. Similarly, $\mathrm{RN}(\Delta)^\perp$ ($\Delta_{\overline{M}}^\perp$) will denote these operators projected off their (L^2) zero modes. Let also Π_0 denote projection onto the 0 eigenvalue of $\mathrm{RN}(\Delta)$. In the previous chapter, we constructed spaces and maps as follows:

(8.6)
$$\begin{array}{ccccc} & & X^2_{\mathrm{LsHC}} & & \\ & \overset{\pi_{\mathrm{res}}}{\swarrow} & \downarrow{\scriptstyle \pi_{\mathrm{heat}}} & \overset{\pi_{\mathrm{par}}}{\searrow} & \\ X^2_{\mathrm{LsC}} & & X^{2,C}_{\mathrm{LHs}} & & X^0_{\mathrm{LsHC}} \end{array}$$

Explicitly, including density factors, $e^{-t\Delta_\epsilon^\perp}$ is given by
(8.7)
$$e^{-t\Delta_\epsilon^\perp} \cdot \mu^{\frac{1}{2}} = \pi_{\mathrm{heat}\,*}\left(\pi_{\mathrm{res}}^*\left(\Delta_\epsilon - \lambda^2\right)^{-1} \cdot \pi_{\mathrm{heat}}^* \mu^{\frac{1}{2}} \cdot \pi_{\mathrm{par}}^*(2\lambda^2 e^{-t\lambda^2} \nu^{\frac{1}{2}}) \cdot \left|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\right|^{-\frac{1}{2}}\right)$$

where μ is the canonical density on $X^{2,C}_{\mathrm{LHs}}$, and ν is the D-density $\left|\frac{dt}{t}\frac{d\,\mathrm{ilg}\,\epsilon}{\mathrm{ilg}\,\epsilon^2}\frac{d\lambda}{\lambda}\right|$ on X^0_{LsHC}. Since $\left(\Delta_\epsilon - \lambda^2\right)^{-1} \in \lambda^{-2}\Psi^{-2,0}(X^2_{\mathrm{LsR}}; \Omega_D^{\frac{1}{2}} X^2_{\mathrm{LsR}})$, the product in (8.7) is a smooth D-density on X^2_{LsHC}. Note that π_{heat} has the property that every boundary hypersurface in X^2_{LsHC} maps to a boundary hypersurface of the same degree, and each boundary hypersurface of $X^{2,C}_{\mathrm{LHs}}$ is the image of exactly two boundary hypersurfaces of X^2_{LsHC}, which intersect. Hence, by the pushforward theorem,

$$e^{-t\Delta_\epsilon^\perp} \in \mathcal{A}^{0\overline{\cup}0}_{\mathrm{phg}}(X^{2,C}_{\mathrm{LHs}}; \Omega_D),$$

where $0\overline{\cup}0$ is the natural index family which assigns to each boundary hypersurface the index set

$$\{(n,0),(n,1) \mid n \in \mathbb{N}\}.$$

To calculate the coefficients of these terms, we refer the reader back to the example in section 2.3. Note that the index set allows logarithmic terms at every boundary hypersurface. We show below that in fact the logarithmic terms do not appear. Using coordinates on boundary hypersurfaces as in Lemma 3.5 and t ($T = t(\mathrm{ias}\,\epsilon)^2$) on B_{mn} (B^∞_{mn}), let us write the heat kernel as a Taylor series in $\mathrm{ias}\,\epsilon$ at each face, denoting the term of order k at B by $H^{(k)}(B)$. The following proposition contains the 'long time information' of Theorem 1.2 and more:

PROPOSITION 8.2. *The heat kernel projected off null modes,* $e^{-t\Delta_\epsilon^\perp}$, *is a smooth* D*-density on the long-time heat space* $X^{2,C}_{\mathrm{LHs}}$. *The top terms at each face are given* *by:*

$$
\begin{aligned}
&H^{(1)}(B^{\infty}_{mn}) = \mathfrak{T}\big(e^{-T\,\mathrm{RN}(\Delta)^{\perp}}\big),\\
&H^{(j)}(B_{t\infty}) = \emptyset\\
(8.8)\qquad &H^{(0)}(B_{00}) = e^{-t\Delta^{\perp}_{\overline{M}}}\\
&H^{(0)}(B_{11}) = e^{-t\Delta_{\overline{H}}}\\
&H^{(0)}(B_{33}) = e^{-t\Delta_{\overline{H}}}\\
\text{other } (B_{mn}):\ &H^{(1)}(B_{mn}) = -\Pi_0.
\end{aligned}
$$

PROOF: First consider the faces B^{∞}_{mn}. The boundary hypersurfaces in X^2_{LsHC} that map to B^{∞}_{mn} are $B^{\infty,0}_{mn}$ and $B^{\infty,\mathrm{C}}_{mn}$. At $B^{\infty,\mathrm{C}}_{mn}$ the function $e^{-t\lambda^2}$, lifted to X^2_{LsHC}, vanishes to infinite order so there is no contribution to the asymptotic expansion at B^{∞}_{mn} from $B^{\infty,\mathrm{C}}_{mn}$, and moreover, this shows that there are no logarithmic terms at B^{∞}_{mn}.

At $B^{\infty,0}_{00}$ the top term in the integrand (8.7) is

$$
\begin{aligned}
&\lambda^{-2}\,\mathrm{proj}_{L^2}\,\Big|d\nu d\nu'\frac{dt}{t}\Big|e^{-Tz^2}2\lambda d\lambda\\
=&z^{-2}\,\mathrm{proj}_{L^2}\,\Big|d\nu d\nu'\frac{dt}{t}\Big|e^{-Tz^2}2zdz.
\end{aligned}
$$

This is integrated over a contour which encloses no poles of the integrand, and which vanishes rapidly as $z \to \infty$. Hence, this integral gives zero; there is no contribution to the face B^{∞}_{00} of order $(\mathrm{ias}\,\epsilon)^0$.

The next top term, at each face $B^{\infty,0}_{mn}$, is

$$
\begin{aligned}
H^{(1)}(B^{\infty}_{mn}) &= (\mathrm{ias}\,\epsilon)^{-1}\int_C (\mathrm{ias}\,\epsilon)^{-1}e^{-t\lambda^2}\mathfrak{T}\big((\Delta-(\mathrm{ias}\,\epsilon)^2z^2)^{-1}\big)2\lambda d\lambda\Big|d\nu d\nu'\frac{d\,\mathrm{ilg}\,\epsilon}{\mathrm{ilg}\,\epsilon}\frac{dt}{t}\Big|\\
&= \int_C e^{-Tz^2}\mathfrak{T}\big((\Delta-(\mathrm{ias}\,\epsilon)^2z^2)^{-1}\big)2zdz\Big|d\nu d\nu'\frac{d\,\mathrm{ilg}\,\epsilon}{\mathrm{ilg}\,\epsilon}\frac{dt}{t}\Big|\\
&= \mathfrak{T}\big(e^{T\,\mathrm{RN}(\Delta)^{\perp}}\big),
\end{aligned}
$$

since the integral encloses all points of the spectrum of $\mathrm{RN}(\Delta)$ except 0.

At $B_{t\infty}$, $e^{-t\lambda^2}$ vanishes to infinite order so $e^{-t\Delta^{\perp}_{\epsilon}} \equiv 0$ to all orders at $B_{t\infty}$.

Next consider the faces B_{mn}. The boundary hypersurfaces that map to B_{mn} are B^0_{mn} and B^{C}_{mn}. Recall that, by Theorem 7.2, the resolvent has just one term in the full Taylor series at $B^{\mathrm{C}}_{00}, B^{\mathrm{C}}_{11}, B^{\mathrm{C}}_{33}$, namely $(\Delta_{\overline{M}}-\lambda^2)^{-1}$, $(\Delta_{\overline{H}}-\lambda^2)^{-1}$ and $(\Delta_{\overline{H}}-\lambda^2)^{-1}$ respectively, and vanishes to infinite order at the other faces B^{C}_{mn}. At the faces B^0_{mn}, the top terms are, for B^0_{00}, $G^{(-2)}_{00} = z^{-2}\,\mathrm{proj}_{L^2}$, and for the other faces, $G^{(-1)}_{mn} = \mathfrak{T}\big((\Delta-(\mathrm{ias}\,\epsilon)^2z^2)^{-1}\big)$. The integrand in (8.7) also contains the factor λ^2, which increases the power at which the terms on the faces B^0_{mn} appear by two.

Therefore the only possibility for a logarithmic term is at $B_{00}(X^{2,C}_{\mathrm{LHs}})$, at power zero. This term comes from the "corner" $B^0_{00} \cap B^{\mathrm{C}}_{00}$, which consists of two copies of $B_{00}(X^{2,C}_{\mathrm{LHs}})$. The integrand is, at this corner, equal to $2\,\mathrm{proj}_{L^2} \cdot \left|d\nu d\nu' \frac{d\,\mathrm{ilg}\,\epsilon}{\mathrm{ilg}\,\epsilon}\frac{dt}{t}\right|$. Note that the direction of integration is opposite on the two copies: at one the contour is coming in from infinity, and at the other the contour is going out to infinity. Since the contour integral is a directed integral, these two contributions cancel, giving no logarithmic term on $X^{2,C}_{\mathrm{LHs}}$.

The smooth term at B_{00} is, by Example 2.4, given by the sum of the *b*-integrals on B^0_{00} and B^{C}_{00}. The contribution from B^{C}_{00} is

(8.9)

$$\lim_{\delta\downarrow 0}\left[\sum_{\pm}\int_{\gamma_\pm\cap|\lambda\geq\delta|}(\Delta-\lambda^2)^{-1}e^{-t\lambda^2}2\lambda^2\frac{d\lambda}{\lambda}-\log\delta\left\{(\Delta-\lambda^2)^{-1}e^{-t\lambda^2}2\lambda^2\right\}_{|\lambda=\delta e^{\pm i\pi/6}}\right].$$

The $\log\delta$ term can be manipulated as follows:

$$\begin{aligned}\left\{\log\delta(\Delta-\lambda^2)^{-1}e^{-t\lambda^2}2\lambda^2\right\}_{|\lambda=\delta e^{\pm i\pi/6}} &= 2\,\mathrm{proj}_{L^2}\log(\delta\cdot e^{\pm i\pi/6})+o(1)\\ &= \int_{\delta e^{\pm i\pi/6}}^{1} 2\,\mathrm{proj}_{L^2}\frac{d\lambda}{\lambda} = \int_{\delta e^{\pm i\pi/6}}^{1} 2\,\mathrm{proj}_{L^2}\,e^{-t\lambda^2}\frac{d\lambda}{\lambda}+o(1)\\ &= \int_{\delta e^{\pm i\pi/6}}^{\infty e^{\pm i\pi/6}} 2\,\mathrm{proj}_{L^2}\,e^{-t\lambda^2}\frac{d\lambda}{\lambda}.\end{aligned}$$

Putting this expression back in (8.9), we can let $\delta\to 0$, obtaining as the contribution from B^{C}_{00}

$$\sum_{\pm}\int_{\gamma_\pm}\left((\Delta-\lambda^2)^{-1}-\mathrm{proj}_{L^2}\right)e^{-t\lambda^2}2\lambda^2\frac{d\lambda}{\lambda}=e^{-t\Delta^{\perp}_{M}}.$$

From B^0_{00}, the contribution is

$$b\text{-}\int_C \mathrm{proj}_{L^2}\frac{dz}{z}=0,$$

because the integrand is an exact multiple of dz/z and is therefore completely cancelled by the *b*-regularization.

The top term at B_{11} and B_{33} is given by

$$H^{(0)}(B_{11})=H^{(0)}(B_{33})=\int_{\gamma_\pm}(\Delta-\lambda^2)^{-1}e^{-t\lambda^2}2\lambda^2\frac{d\lambda}{\lambda}=e^{-t\Delta_{\overline{H}}}.$$

Finally, we will calculate the $H^{(1)}$ term at each face B_{mn}, for $(mn)\neq(00)$, (11), (33). This comes from the top term at B^0_{mn}:

$$H^{(1)}(B_{mn})=\int_C \mathfrak{T}((\mathrm{RN}(\Delta)-z^2)^{-1})2z\,dz.$$

From (5.6) we have that, on these faces $(\mathrm{RN}(\Delta) - z^2)^{-1}$ is exponentially decreasing as $|z| \to \infty$, $\arg z = \pm\pi/6$. Hence, by dominated convergence we can calculate this integral as

$$\begin{aligned} H^{(1)}(B_{mn}) &= \lim_{T \downarrow 0} \int_C e^{-Tz^2} \mathfrak{T}((\mathrm{RN}(\Delta) - z^2)^{-1} 2z dz) \\ &= \lim_{T\downarrow 0} \mathfrak{T}(e^{-T\,\mathrm{RN}(\Delta)} - \Pi_0) \\ &= -\mathfrak{T}(\Pi_0) \end{aligned}$$

on these faces. This completes the proof of the proposition. ∎

8.4. Very small eigenvalues.

In the last section we split off from Δ_ϵ the projection onto eigenfunctions corresponding to very small eigenvalues. We will denote by Π_ϵ the projection onto these eigenfunctions, and write Π_{L^2} (Π_0) for the projection onto eigenvalues corresponding to L^2 zero modes (zero modes of $\mathrm{RN}(\Delta)$). Let us analyse the behaviour of $\Pi_\epsilon e^{-t\Pi_\epsilon}$.

PROPOSITION 8.3. *The operator $\Pi_\epsilon e^{-t\Pi_\epsilon}$ is a uniformly finite rank operator, with Schwartz kernel smooth on $X^{2,C}_{\mathrm{LHs}}$ everywhere except possibly up to $B_{t\infty}$. The top terms at each face $\neq B_{t\,\infty}$ are*

$$\begin{aligned} \left(\Pi_\epsilon e^{-t\Pi_\epsilon}\right)^{(0)}_{00} &= \Pi_{L^2} \\ \left(\Pi_\epsilon e^{-t\Pi_\epsilon}\right)^{(1)}_{mn} &= \mathfrak{T}(\Pi_0) \\ \left(\Pi_\epsilon e^{-t\Pi_\epsilon}\right)^{\infty(1)}_{mn} &= \mathfrak{T}(\Pi_0). \end{aligned}$$

PROOF: The operator $\Pi_\epsilon e^{-t\Pi_\epsilon}$ is given by a contour integral

$$\Pi_\epsilon e^{-t\Pi_\epsilon} = \frac{1}{2\pi i} \int_{|z|=c} \left(\Delta - \lambda^2\right)^{-1} e^{-t\lambda^2} 2\lambda d\lambda$$

for c small enough that the contour excludes all points of $\operatorname{spec} \mathrm{RN}(\Delta)$ apart from zero. By the results of chapter 6, the contour encloses precisely those eigenfunctions corresponding to Π_ϵ and no others, for ϵ sufficiently small. The problem with this integral is that the function $e^{-t\lambda^2}$ is growing rapidly on the (appropriate) heat-contour space. To fix this, weight both sides with a factor small enough to kill this growth:

$$e^{-tc^2(\mathrm{ias}\,\epsilon)^2} \cdot \Pi_\epsilon e^{-t\Pi_\epsilon} = \frac{1}{2\pi i} \int_{|z|=c} \left(\Delta - \lambda^2\right)^{-1} e^{-tc^2(\mathrm{ias}\,\epsilon)^2} e^{-t\lambda^2} 2\lambda d\lambda.$$

By the pushforward theorem this is smooth on $X^{2,C}_{\mathrm{LHs}}$. The added factor $e^{-tc^2(\mathrm{ias}\,\epsilon)^2}$ is smooth and nonzero on all faces except $B_{t\,\infty}$, where it vanishes rapidly. Hence $\Pi_\epsilon e^{-t\Pi_\epsilon}$ itself is smooth everywhere except possibly at $B_{t\,\infty}$. ∎

Notice that there is no reason to believe that $\Pi_\epsilon e^{-t\Pi_\epsilon}$ is in fact smooth up to this face, since in principle the small eigenvalues could cross zero infinitely often as $\epsilon \to 0$.

8.5. Full Heat Kernel.

To obtain the full heat kernel we need to discuss the heat kernel for small time $[0, C^2]$ and join with the long time heat kernel constructed above. This is a simple matter of lifting the result from Mazzeo-Melrose to our logarithmic spaces.

In Mazzeo-Melrose, the heat kernel was constructed on the space

$$X^2_{\text{hs}} = \left[X^2_s \times [0,\infty]_t; (\Delta_s \times \{t=0\})_{\text{par}}\right].$$

Here Δ_s is the diagonal p-submanifold in X^2_s and the blowup is performed parabolically, that is, with homogeneity two in t (see [17], chapter 7). For our purpose we just need this space restricted to $\{t \le C^2\}$; this affects the blowup not at all. The heat kernel for finite time lifts to the space

$$X^{2,[0,C]}_{\text{LHs}} = \left[X^2_{\text{Ls}} \times [0, C^2]_t; (\Delta_{\text{Ls}} \times \{t=0\})_{\text{par}}\right].$$

In fact, since Δ_s is transverse to the boundary of X^2_s, the logarithmic and total blowups that turn X^2_s into X^2_{Ls} commute with the diagonal blowup. Thus there is a blow-down map $X^{2,[0,C]}_{\text{LHs}} \to X^2_{\text{hs}} \cap \{t \le C^2\}$ and we lift the heat kernel by pulling back under this map. (We must also multiply the heat kernel by $1/t\epsilon$ to correct for different normalization of densities used in Mazzeo-Melrose.) Then the heat kernel lifts to be a D-half-density on $X^{2,[0,C]}_{\text{LHs}}$, with the models

$$\begin{aligned} H^{(0)}(B_{00}) &= e^{-t\Delta_{\overline{M}}} \\ H^{(0)}(B_{11}) &= e^{-t\Delta_{\overline{H}}} \\ H^{(0)}(B_{33}) &= e^{-t\Delta_{\overline{H}}} \end{aligned}$$

and all other models $\equiv 0$. The two heat spaces $X^{2,[0,C]}_{\text{LHs}}$ and $X^{2,C}_{\text{LHs}}$ are both just a product of X^2_{Ls} with a time interval (canonically) away from $t=0$ and infinity. Hence the spaces can be joined at $t = C^2$ to produce the full heat space X^2_{LHs}. Since the heat kernel is unique, it extends smoothly across $t = C^2$ to be the full heat kernel of this space. Let us summarize the properties of the full heat kernel:

THEOREM 8.4. *The heat kernel is a D-density on the heat space X^2_{Ls}, $t^{-n/2}\times$ smooth near $t = 0$, and smooth everywhere else except (possibly) at the face $B_{\text{t}\,\infty}$. The top terms at each face $\neq B_{\text{t}\,\infty}$ are given by:*

$$\begin{aligned} H^{(0)}(B^{\infty}_{00}) &= \operatorname{proj} L^2\operatorname{null}(\Delta_{\overline{M}}) \\ H^{(1)}(B^{\infty}_{mn}) &= \mathfrak{T}(e^{-T\,\text{RN}(\Delta)}),\ T = t(\text{ias}\,\epsilon)^2 \\ H^{(0)}(B_{00}) &= e^{-t\Delta_{\overline{M}}} \\ H^{(0)}(B_{11}) &= e^{-t\Delta_{\overline{H}}} \\ H^{(0)}(B_{33}) &= e^{-t\Delta_{\overline{H}}} \\ \textit{other } H^{(j)}(B_{mn}) &= 0. \end{aligned}$$

Chapter 9. Limit of Eta Invariant

9.1. Eta invariant.

The eta invariant of $\eth_\epsilon$ is given for each $\epsilon > 0$ by the formula (1.1):

$$\eta(\eth_\epsilon) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{\frac{1}{2}} \operatorname{Tr}\left(\eth_\epsilon e^{-t\eth_\epsilon^2}\right) \frac{dt}{t}. \tag{9.1}$$

At $\epsilon = 0$, the *b*-eta invariant is defined by

$$b\text{-}\eta(\eth_{M_\pm}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{\frac{1}{2}} b\text{-}\operatorname{Tr}\left(\eth_{M_\pm} e^{-t\eth_{M_\pm}^2}\right) \frac{dt}{t}, \tag{9.2}$$

where *b*-Tr is defined in section 2.3. We will show below that this is independent of the choice of boundary defining function in the definition of *b*-Tr.

As in the previous chapter we split the operator $\eth_\epsilon$ into two pieces, $\eth_\epsilon^\perp$, orthogonal to the finite number of eigenfunctions corresponding to the L^2 null space of $\eth_0$ and $0 \in \operatorname{spec} \operatorname{RN}(\eth)$, and the projection Π_ϵ onto these eigenfunctions. Then

$$\eta(\eth_\epsilon) = \eta(\eth_\epsilon^\perp) + \eta(\Pi_\epsilon) = \eta(\eth_\epsilon^\perp) + \eta_{\mathrm{fd}}(\epsilon), \tag{9.3}$$

where $\eta_{\mathrm{fd}}(\epsilon)$ is just the signature of Π_ϵ, that is, the dimension of the space of eigenfunctions with positive eigenvalue minus the dimension of the space of eigenfunctions with negative eigenvalue. Hence η_{fd} takes values in $\{0, \pm 1 \cdots \pm N\}$, where N is the rank of Π_ϵ. The $\eth^\perp$ part of (9.3) we calculate by applying the Pushforward theorem to the integral (9.1), replacing $\eth_\epsilon$ with $\eth_\epsilon^\perp$. As this involves the trace of $\eth_\epsilon e^{-t\eth_\epsilon^2}$ we first consider the diagonal submanifold of X^2_{LHs}.

9.2. The diagonal of the Logarithmic heat space.

The diagonal Δ_{LHs} of X^2_{LHs} is the lift of $\Delta_{\mathrm{Ls}} \times [0,\infty]_\tau \subset X^2_{\mathrm{Ls}}$ to X^2_{LHs}. It is transversal to all boundaries and has the form

$$\begin{aligned}\Delta_{\mathrm{LHs}} = \big[\Delta_{\mathrm{Ls}} \times [0,\infty]_\tau; (B_{33} \cap \Delta_{\mathrm{Ls}}) \times \{\tau = \infty\}; (B_{00} \cap \Delta_{\mathrm{Ls}}) \times \{\tau = \infty\};\\ (B_{11} \cap \Delta_{\mathrm{Ls}}) \times \{\tau = \infty\}\big].\end{aligned}$$

We define the degrees of boundary hypersurfaces of Δ_{LHs} to be those of the corresponding hypersurfaces of X^2_{LHs}. On a compact manifold without boundary, N, one has the canonical density bundle isomorphism

$$\Omega^{\frac{1}{2}}(N \times N)_{\restriction\Delta} \equiv \Omega(\Delta).$$

This means that, given a half-density on $N \times N$ representing the Schwartz kernel of a (suitable) operator, one can restrict to the diagonal and integrate the resulting density to obtain the trace. Here, our version of this isomorphism is, from (4.16),

$$\Omega_D^{\frac{1}{2}}(X^2_{\mathrm{LHs}})_{\restriction\Delta_{\mathrm{LHs}}} \otimes \left|\frac{d(\operatorname{ilg}\epsilon)}{(\operatorname{ilg}\epsilon)^2}\right|^{\frac{1}{2}} \equiv \Omega_D(\Delta_{\mathrm{LHs}}).$$

Hence, if ι is the inclusion $\Delta_{\mathrm{LHs}} \hookrightarrow X^2_{\mathrm{LHs}}$ and p is the map $X^2_{\mathrm{LHs}} \to [0, \mathrm{ilg}\,\epsilon_0]_{\mathrm{ilg}\,\epsilon}$, we can express the eta invariant for $\eth^{\perp}_{\epsilon}$ as

$$\eta(\eth^{\perp}_{\epsilon}) = \frac{1}{\sqrt{\pi}} p_*\Big(t^{\frac{1}{2}} \cdot \iota^*\big(\operatorname{tr} \eth^{\perp}_{\epsilon} e^{-t\eth^{\perp 2}_{\epsilon}}\big) \cdot \Big|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\Big|^{\frac{1}{2}}\Big) \cdot \Big|\frac{d(\mathrm{ilg}\,\epsilon)}{(\mathrm{ilg}\,\epsilon)^2}\Big|^{-1}. \tag{9.4}$$

9.3. Asymptotic expansion of η as $\epsilon \to 0$.

Consider the integral (9.4). To apply the Pushforward Theorem (Theorem 2.3), we must check the integrability condition $\mathcal{K}(G) > d(G)$ for all G such that $p(G) = [0, \mathrm{ilg}\,\epsilon_0]$, that is, for $G = B_{\mathrm{tf}}$, $B_{\mathrm{t}\,\infty}$. Proposition 8.2 asserts that the integrand is rapidly vanishing at $B_{\mathrm{t}\,\infty}$. At B_{tf} it is well known that the heat kernel has growth of order $t^{-n/2}$, which would appear to be a problem. However after taking the pointwise trace, Patodi in [24] showed that 'fantastic cancellations' occur in the integrand and the growth is of order $t^{1/2}$, making (9.4) integrable. This also follows rather simply using Getzler's rescaling (see [11]).

The structure of the integrand on the double heat space follows readily from Theorem 8.2.

PROPOSITION 9.1. *On X^2_{LHs}, the integrand I in (9.4) is in $\mathcal{C}^\infty(X^2_{\mathrm{LHs}}; \Omega^{\frac{1}{2}}_D)$ and the top terms at each diagonal face at* $\mathrm{ias}\,\epsilon = 0$ *are*

$$I^{(1)}(B^{\infty}_{mn}) = \frac{1}{\sqrt{\pi}} \mathfrak{T}(T^{\frac{1}{2}} \operatorname{tr} \mathrm{RN}(\eth_\epsilon) e^{-T\,\mathrm{RN}(\eth_\epsilon)^2})$$
$$I^{(0)}(B_{00}) = \frac{1}{\sqrt{\pi}} t^{\frac{1}{2}} \operatorname{tr} \eth_{\overline{M}} e^{-t\eth^2_{\overline{M}}}$$
$$I^{(0)}(B_{11}) = \frac{1}{\sqrt{\pi}} t^{\frac{1}{2}} \operatorname{tr} \eth_{\overline{H}} e^{-t\eth^2_{\overline{H}}}$$
$$I^{(0)}(B_{33}) = \frac{1}{\sqrt{\pi}} t^{\frac{1}{2}} \operatorname{tr} \eth_{\overline{H}} e^{-t\eth^2_{\overline{H}}}.$$

PROOF: The last three lines are immediate from Theorem 8.2. The first line follows for B^{∞}_{22} since there $\eth = \eth_H + (\mathrm{ias}\,\epsilon)\,\mathrm{RN}(\eth) + O((\mathrm{ias}\,\epsilon)^2)$ and $t^{1/2} = (\mathrm{ias}\,\epsilon)^{-1} T^{1/2}$. This determines the other top terms on B^{∞}_{mn} by compatibility. ∎

Of the six boundary faces at $\mathrm{ias}\,\epsilon = 0$, four, B_{00}, B_{11}, B^{∞}_{00} and B^{∞}_{11} have degree one and the other two, B_{33}, B^{∞}_{33} have degree two. Using the Pushforward theorem and the result above, we see that for small ϵ

$$\eta(\eth_\epsilon) \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}([0, \mathrm{ilg}\,\epsilon_0]_{\mathrm{ilg}\,\epsilon}),$$

where $\mathcal{E}$ is the index set

$$\mathcal{E} = \{(-1,0)\} \cup \{(n,0),(n,1) \mid n \in \mathbb{N}\},$$

that is, asymptotically

$$\eta(\eth_\epsilon) \sim \sum_{(n,k)\in\mathcal{E}} a_{n,k} (\mathrm{ias}\,\epsilon)^n (\log(\mathrm{ias}\,\epsilon))^k.$$

The $(-1,0)$ term is from B_{33} and the $(0,1)$ term is from $B_{11} \cap B_{33}^{\infty}$ and $B_{00} \cap B_{33}^{\infty}$. Let us compute the coefficients $a_{-1,0}$ and $a_{0,1}$. The term $a_{-1,0}$ is an integral with integrand $I^{(0)}(B_{33})$. Consider the pointwise trace

$$\operatorname{tr} \eth_{\overline{H}} e^{-t\eth_{\overline{H}}^2} = \operatorname{tr}\left(\eth_H + \gamma D_{(\log x'/x)}\right) e^{-t\eth_H^2} e^{-|(\log x'/x)|^2/4t} \restriction \Delta_{\mathrm{Ls}}.$$

Note that $e^{-t\eth_H^2}$ is diagonal with respect to the grading of the spinor bundle $S = S^+ \oplus S^-$, while $\eth_{\overline{H}}$ is the sum of a piece which is off diagonal with respect to the grading of S and a piece which kills $e^{-|(\log x'/x)|^2/4t}$ at $\Delta_{\mathrm{Ls}} = \{(\log x'/x) = 0\}$. Hence this term is identically zero, and therefore $a_{-1,0} = 0$. This also proves the remark made above that (9.2) is independent of the choice of boundary defining function. In fact, after taking the pointwise trace, the integral is absolutely convergent.

Next, $a_{0,1}$ is an integral over $B_{11} \cap B_{33}^{\infty}$ and $B_{00} \cap B_{33}^{\infty}$. The integrand is

$$I^{(1)}(B_{33}^{\infty}) = \mathfrak{T}\big(T^{\frac{1}{2}}\, \mathrm{RN}(\eth_\epsilon)^{\perp} e^{-T\,\mathrm{RN}(\eth_\epsilon)^{\perp 2}}\big).$$

To compute this, refer to the explicit formula (5.4) for $e^{-t\,\mathrm{RN}(\Delta)}$. The integral is at $T = 0$, so the only terms that contribute are

$$\frac{1}{\sqrt{4\pi t}}\left(e^{-|s-s'|^2/4t}\,\mathrm{Id} + e^{-|2-s-s'|^2/4t} S_+ + e^{-|-2-s-s'|^2/4t} S_-\right).$$

As noted in chapter 5, $\operatorname{tr}\gamma = \operatorname{tr}\gamma S_{\pm} = 0$. Hence $a_{0,1}$ is also zero.

It follows that the top term in the expansion of η is $a_{0,0}$, which is constant up to $\epsilon = 0$. Hence, $\eta(\eth_\epsilon^{\perp})$ has a limit at $\epsilon = 0$, which is

$$\begin{aligned} a_{0,0} = {} & \frac{1}{\sqrt{\pi}} \int_0^{\infty} t^{-\frac{1}{2}}\, dt \int_{\overline{M}} \operatorname{tr}\left(\eth_{\overline{M}} e^{-t\eth_{\overline{M}}^2}\right) \\ & + \frac{1}{\sqrt{\pi}} \int_0^{\infty} t^{-\frac{1}{2}}\, dt \int_{\overline{H}} \operatorname{tr}\left(\eth_{\overline{H}} e^{-t\eth_{\overline{H}}^2}\right) \\ & + \frac{1}{\sqrt{\pi}} \int_0^{\infty} T^{-\frac{1}{2}}\, dT \int_{-1}^{1} ds \operatorname{tr} \mathfrak{T}\big(\mathrm{RN}(\eth_\epsilon) e^{-T\,\mathrm{RN}(\eth_\epsilon)^2}\big). \end{aligned}$$

The first integral gives $\eta_b(\eth_{\overline{M}})$, the second gives zero, since $I^{(1)}(B_{33}^{\infty}) = 0$ and the third is the eta invariant of $\mathrm{RN}(\eth)$. Hence we have shown

$$\lim_{\epsilon \downarrow 0} \eta(\eth_\epsilon^{\perp}) = \eta_b(\eth_{M_+}) + \eta_b(\eth_{M_-}) + \eta(\mathrm{RN}(\eth))$$

and therefore

$$\eta(\eth_\epsilon) = \eta_b(\eth_{M_+}) + \eta_b(\eth_{M_-}) + \eta_{\mathrm{fd}}(\epsilon) + \eta(\mathrm{RN}(\eth)) + O((\mathrm{ias}\,\epsilon)\log(\mathrm{ias}\,\epsilon)).$$

In fact, it is not hard to see that all logarithmic terms $a_{n,1}$, $n > 0$ vanish. First, there are no contributions to $a_{n,1}$ from the faces at finite time becuase they have no terms in their Taylor series at this level. So a logarithmic term can only come from the corners $B^{\infty}_{00} \cap B^{\infty}_{33}$ or $B^{\infty}_{11} \cap B^{\infty}_{33}$. On B^{∞}_{00}, $\frac{\partial}{\partial t} + \Delta_\epsilon$ as the model operator $(\text{ias}\,\epsilon)\frac{\partial}{\partial T} - \Delta_{\overline{M}}$. Hence $H^{(1)}(B^{\infty}_{00})$ takes values in $\mathcal{C}^\infty([0,\infty]_{\sqrt{T}}; \text{null}\,\Delta_{\overline{M}} \otimes \text{null}\,\Delta_{\overline{M}})$ and inductively it follows that

$$H^{(n)}(B^{\infty}_{00}) \in \mathcal{C}^\infty\big([0,\infty]_{\sqrt{T}}; [\text{span}\;\Phi^{(0)} \dots \Phi^{(n-1)}]^2\big),$$

where the $\Phi^{(i)}$ are defined in section 6.1. Hence, $H^{(n)}$ has, at this corner, only a finite expansion $\sum_{j=o}^{n} c_k(\text{ilg}\,x)^{-j}$, with no terms $(\text{ilg}\,x)^j$, $j > 0$. Therefore, near B^{∞}_{33}, the heat kernel is $T^{-1/2}$ times a smooth function of T, s, y and ias ϵ:

$$e^{-t\Delta^{\perp}} \sim \sum_{k\geq 1} T^{-\frac{1}{2}}(\text{ias}\,\epsilon)^k b_k(s,y,T)|dy\frac{dT}{T}||\frac{d(\text{ilg}\,\epsilon)}{(\text{ilg}\,\epsilon)^2}|d(1-s).$$

We get just the density factor $d(1-s)$, smooth up to B^{∞}_{00} and B^{∞}_{11} because, in the notation of Theorem 2.3, $\text{ex}(B^{\infty}_{33}) = 2$, $\text{ex}(B^{\infty}_{00})$, $\text{ex}(B^{\infty}_{11}) = 1$. A similar result holds for B^{∞}_{11}. It is explicit from this presentation in coordinates that the integral is smooth in ias ϵ. This proves Theorem 1.3.

An explicit formula for $\eta(\text{RN}(\eth))$ in terms of the scattering matrix is given by Proposition 5.3.

Chapter 10. A Hodge Mayer-Vietoris cohomology sequence

10.1. Mayer-Vietoris sequence.

In the next two chapters we will study the Laplacian on a flat unitary bundle E over M. Since E is flat, the operator d on E-valued forms determined by the (flat) connection on E forms a twisted complex

$$0 \longrightarrow \Omega^0(M;E) \longrightarrow \Omega^1(M;E) \longrightarrow \Omega^2(M;E) \longrightarrow \dots$$

giving twisted cohomology groups $H^q(M;E)$. If M is the union of two open sets U, V then from the short exact sequence

$$0 \longrightarrow \Omega^*(M;E) \longrightarrow \Omega^*(U;E) \oplus \Omega^*(V;E) \longrightarrow \Omega^*(U \cap V;E) \longrightarrow 0$$

$$\alpha \longrightarrow (\alpha_{\restriction U}, \alpha_{\restriction V})$$

$$(\beta,\gamma) \longrightarrow \beta_{\restriction U\cap V} - \gamma_{\restriction U\cap V}$$

we get a long exact sequence in cohomology, the Mayer-Vietoris sequence:

$$\cdots \to H^{q-1}(U\cap V;E) \to H^q(M;E) \to H^q(U;E)\oplus H^q(V;E) \to H^q(U\cap V;E) \to \dots.$$

From now on the bundle E will be understood and will usually be dropped in notation.

If H splits M into two pieces $M_\pm$ then thickening each piece $M_\pm$ a little across H yields two open sets to which Mayer-Vietoris can be applied; the intersection $U \cap V$ then is $H \times (-\delta,\delta)$ whose (absolute) cohomology is naturally the same as that of H. If Z is a manifold with boundary, let us denote by $H^*_{\mathrm{abs}}(Z)$, $H^*_{\mathrm{rel}}(Z)$ the cohomology, and the cohomology relative to the boundary respectively. Then the Mayer-Vietoris sequence looks like

(10.1)
$$\longrightarrow H^{q-1}(H) \longrightarrow H^q(M) \xrightarrow{j^q} H^q_{\mathrm{abs}}(M_+) \oplus H^q_{\mathrm{abs}}(M_-) \xrightarrow{k^q} H^q(H) \longrightarrow$$

On a compact manifold M without boundary one has the Hodge Theorem. This states that the space of harmonic forms with respect to any metric is isomorphic to the de Rham cohomology and gives canonical choices (given the metric) for cohomology classes on M. We will want to have canonical choices of cohomology for all spaces in the above exact sequence for the computations in the next chapter. For this purpose we will develop a Hodge version of this exact sequence in the next two sections.

10.2. b-Hodge theory.

As a first step towards a Hodge version of the sequence (10.1), recall results from [17] on the Hodge theory of a manifold Z with boundary and exact b-metric.

This 'b-Hodge theory' works equally well with twisted cohomology groups, as the analysis presented there is valid for arbitrary bundles. In [17] relative and absolute Hodge cohomology groups $H^*_{b\text{-}\mathrm{abs,Ho}}(Z)$ and $H^*_{b\text{-}\mathrm{rel,Ho}}(Z)$ are defined, being smooth elements of the null space of Δ_Z. They have boundary-data maps BD into $H^*(\partial Z)$ such that $\mathrm{BD}(H^q_{b\text{-}\mathrm{abs,Ho}}(Z)) \oplus \mathrm{BD}(H^{q+1}_{b\text{-}\mathrm{rel,Ho}}(Z))$ is an orthogonal decomposition of $H^q(\partial Z)$. The null space of BD is the L^2 cohomology $H^q_{b\text{-}\mathrm{Ho}}(Z)$. The groups have an inner product given by the L^2 norm on L^2 cohomology and the inner product induced from $H^q(H)$ by BD. They can be assembled into a relative cohomology sequence

$$\longrightarrow H^{q-1}_{b\text{-}\mathrm{abs,Ho}}(Z) \xrightarrow{\mathrm{BD}} H^q(\partial Z) \xrightarrow{\mathrm{BD}^*} H^q_{b\text{-}\mathrm{rel,Ho}}(Z) \longrightarrow H^q_{b\text{-}\mathrm{abs,Ho}}(Z) \longrightarrow$$

The unlabelled map is the identity on the L^2 cohomology and is zero on its orthocomplement.

Let us denote the images $\mathrm{BD}(H^q_{b\text{-}\mathrm{abs,Ho}}(M_\pm))$ and $\mathrm{BD}^*(H^{q+1}_{b\text{-}\mathrm{rel,Ho}}(M_\pm)) \subset H^q(H)$ by $A^q_\pm$, $R^q_\pm$ respectively. Then we have $A^q_\pm = (R^q_\pm)^\perp = *R^{n-1-q}_\pm$, where $\dim M = n$.

10.3. Surgery Hodge theory.

Consider the implications of b-Hodge theory for the sequence (10.1) under surgery. We have

$$\begin{aligned} H^q(M) &\equiv \mathrm{im}\,(H^{q-1}(H) \to H^q(M)) \oplus \mathrm{im}\, j^q \\ &\equiv (\mathrm{im}\, k^{q-1})^\perp \oplus \ker k^q. \end{aligned}$$

By the above remarks,

$$\begin{aligned} (\mathrm{im}\, k^{q-1})^\perp &= (A^{q-1}_+ + A^{q-1}_-)^\perp = R^{q-1}_+ \cap R^{q-1}_- \\ \text{and } \ker k^q &= A^q_+ \cap A^q_- \oplus H^q_{b\text{-}\mathrm{Ho}}(M_+) \oplus H^q_{b\text{-}\mathrm{Ho}}(M_-). \end{aligned}$$

Hence,

(10.2)
$$\dim H^q(M) = \dim(R^{q-1}_+ \cap R^{q-1}_-) + \dim(A^q_+ \cap A^q_-) + \dim(H^q_{b\text{-}\mathrm{Ho}}(M_+) \oplus H^q_{b\text{-}\mathrm{Ho}}(M_-)).$$

Now recall the results on small eigenvalues proved in chapter 6. We showed that the eigenvalues $z(\epsilon)$ are continuous down to $\epsilon = 0$, with limiting behaviour $(\mathrm{ias}\,\epsilon)^2 z^2 + o((\mathrm{ias}\,\epsilon)^2)$, where $z = 0$ (corresponding to L^2 null space) or $z^2 = z_j^2 \in \operatorname{spec} \mathrm{RN}(\Delta_q)$. We have that the multiplicity of $0 \in \operatorname{spec} \mathrm{RN}(\Delta)_q$ is the dimension of the intersection $\Lambda^D_+ \cap \Lambda^D_-$. The intersection $\Lambda^D_+ \cap \Lambda^D_-$ is, for the Laplacian on q-forms, equal to $A^q_+ \cap A^q_- \oplus R^{q-1}_+ \cap R^{q-1}_-$, so the dimension of the space of eigenfunctions corresponding to $z = 0$ is

$$(10.3) \quad \dim H^q_{b\text{-}\mathrm{Ho}}(M_+) + \dim H^q_{b\text{-}\mathrm{Ho}}(M_-) + \dim(A^q_+ \cap A^q_-) + \dim(R^{q-1}_+ \cap R^{q-1}_-).$$

The cohomology $H^q(M)$ is, by ordinary Hodge theory, given for each $\epsilon > 0$ by the null space of Δ_q, so comparing (10.2) and (10.3) shows that *all* the eigenforms corresponding to $z = 0$ represent cohomology on M and therefore all eigenvalues corresponding to $z = 0$ remain identically zero for $\epsilon > 0$. In other words, the multiplicity of $0 \in \operatorname{spec} \Delta^q$ is constant as $\epsilon \downarrow 0$. This fact has some useful corollaries:

PROPOSITION 10.1. *The generalized inverse G of Δ_E defined for each $\epsilon > 0$ by*

$$G\Delta_E = \Delta_E G = \mathrm{Id} - \Pi_\epsilon$$

is in $\Psi^{-2,-1}(X^2_{\mathrm{Ls}}; \Omega^{\frac{1}{2}}_D X^2_{\mathrm{Ls}})$.

This confirms the intuitively plausible idea that one ought to have a cohomology element on M for every pair of elements on M_+ and M_- that match at H. We also get better regularity for the heat kernel $e^{-t\Delta_E}$ than one can expect for arbitrary generalized Laplacians (such as $\eth^2_\epsilon$):

COROLLARY 10.2. *The heat kernel $e^{-t\Delta_E}$ is smooth up to the boundary hypersurface $B_{\mathrm{t}\,\infty}$ (cf. Theorem 8.4).*

We call the range of the projector onto zero eigenvalues, Π_ϵ, the surgery Hodge cohomology of M, $H^*_{s\text{-}\mathrm{Ho}}(M)$. Using Proposition 10.1 we can determine the image of a cohomology class $[\alpha]$, where α is a closed E-valued form on M, in $H^*_{s\text{-}\mathrm{Ho}}(M)$. Lifting α to $\tilde\alpha$ on X_{Ls}, by the above proposition the image is given by $\Pi_\epsilon(\tilde\alpha)$. To analyse the behaviour of $\Pi_\epsilon\tilde\alpha$, let ϕ^0_i be an orthonormal basis for $H^*_{b\text{-}\mathrm{Ho}}(M_+) \oplus H^*_{b\text{-}\mathrm{Ho}}(M_-)$, let

$\psi^0_{j,+}$ be an orthonormal basis for $\{\psi \in H^*_{b\text{-}\mathrm{abs,Ho}}(M_+) \mid \mathrm{BD}(\psi) \in A^*_-\}$,

$\chi^0_{k,+}$ be an orthonormal basis for $\{\chi \in H^*_{b\text{-}\mathrm{rel,Ho}}(M_+) \mid \mathrm{BD}(\psi) \in R^*_-\}$.

and let $\psi^0_{j,-}$, $\chi^0_{k,-}$ be the corresponding elements of $H^*_{b\text{-}\mathrm{abs,Ho}}(M_-)$, $H^*_{b\text{-}\mathrm{rel,Ho}}(M_-)$. We can extend ϕ^0_i, $\psi^0_{j,\pm}$, $\chi^0_{k,\pm}$ to smooth functions ϕ_i, ψ_j, χ_k on X_{Ls} such that $(\phi_i, (\mathrm{ias}\,\epsilon)^{1/2}\psi_j, (\mathrm{ias}\,\epsilon)^{1/2}\chi_k)$ form an orthonormal basis of the zero eigenspace of Δ_E. This depends on the fact that they are flat to infinite order at B_1 and B_2, proved in Proposition 6.1. Near H, we can write $\alpha = \alpha_a + \alpha_r dx$ in product coordinates, where α_a, α_r are closed forms on H. At $\epsilon = 0$ near H, we should write $\tilde\alpha$ as a surgery form (see section 3.3): $\tilde\alpha = \alpha_a + \sqrt{x^2+\epsilon^2}\,\alpha_r(dx/\sqrt{x^2+\epsilon^2})$. We have

$$\Pi_\epsilon\tilde\alpha = \phi_i\langle\tilde\alpha,\phi_i\rangle + (\mathrm{ias}\,\epsilon)\psi_j\langle\tilde\alpha,\psi_j\rangle + (\mathrm{ias}\,\epsilon)\chi_k\langle\tilde\alpha,\chi_k\rangle.$$

Because the $dx/\sqrt{x^2+\epsilon^2}$ coefficient vanishes at B_1 and B_2, we have

$$\begin{aligned}\langle\tilde\alpha,\phi_i\rangle &= a_i(\mathrm{ias}\,\epsilon)\\ \langle\tilde\alpha,\psi_j\rangle &= (\mathrm{ias}\,\epsilon)^{-1} b_j(\mathrm{ias}\,\epsilon)\\ \langle\tilde\alpha,\chi_k\rangle &= c_k(\mathrm{ias}\,\epsilon),\end{aligned}$$

where a_i, b_j, c_k are smooth functions. Hence

$$\Pi_\epsilon\tilde\alpha = a_i\phi_i + b_j\psi_j + (\mathrm{ias}\,\epsilon)c_k\chi_k,$$

with (a_i, b_j, c_k) smooth functions of ϵ, linearly independent as $[\alpha]$ ranges over $H^*(M)$. We define an inner product on $H^*_{s\text{-}\mathrm{Ho}}(M)$ by setting

$$|\Pi_\epsilon\tilde\alpha|^2 = \sum_i |a_i(0)|^2 + 2\sum_j |b_j(0)|^2 + 2\sum_k |c_k(0)|^2. \tag{10.4}$$

This is independent of the choice of ϕ_i, ψ_j, χ_k as the norm only depends on the orthonormality of $\phi_i^0, \psi_{j,\pm}^0, \chi_{k,\pm}^0$. Hence we can split $H^*_{s\text{-}\mathrm{Ho}}(M)$ into three orthogonal subspaces:

$$
\begin{aligned}
H^*_{s\text{-}L^2,\mathrm{Ho}}(M) &= \{\Pi_\epsilon \tilde{\alpha} \mid b_j(0) = c_k(0) = 0\}; \\
H^*_{s\text{-}\mathrm{abs},\mathrm{Ho}}(M) &= \{\Pi_\epsilon \tilde{\alpha} \mid a_i(0) = c_k(0) = 0\}; \\
H^*_{s\text{-}\mathrm{rel},\mathrm{Ho}}(M) &= \{\Pi_\epsilon \tilde{\alpha} \mid a_i(0) = b_j(0) = 0\}
\end{aligned}
\tag{10.5}
$$

and an element of cohomology $[\alpha]$ is represented by the smooth harmonic form $\Pi_\epsilon \tilde{\alpha}$. Note that the L^2 norms of elements of these three subspaces have the leading behaviours $(\mathrm{ias}\,\epsilon)^k$ where $k = 0, -\frac{1}{2}, \frac{1}{2}$ respectively. In the Mayer-Vietoris sequence (10.1) $H^q_{s\text{-}\mathrm{rel},\mathrm{Ho}}(M)$ is the image of the connecting homomorphism $H^{q-1}(H) \to H^q(M)$ and the map j^q on $H^q_{s\text{-}\mathrm{Ho}}(M)$ is restriction to the boundary B_0. With the inner product (10.5), both these maps are isometries from the orthocomplement of the preceding map to their image.

Chapter 11. Analytic Torsion and Reidemeister torsion

11.1. Analytic torsion.

In this chapter we compute the surgery limit of analytic torsion, obtaining Theorem 1.4, which is similar to the computation for the eta invariant in chapter 9. Comparing it to the surgery formula for R-torsion we obtain Theorem 1.5 and the corollaries then follow readily.

Let us return to the 'definition' of the zeta function:

$$\zeta_q(s) \text{ `}=\text{' } \frac{1}{\Gamma(s)}\int_0^\infty t^s \operatorname{Tr} e^{-t\Delta_q}\frac{dt}{t}. \tag{11.1}$$

This need not converge for any value of s. The usual remedy for this is to replace Δ_q by Δ_q', the operator projected off null modes, because Δ_q' decays exponentially at $t=\infty$. For surgery it is more natural to write the integral as

$$\frac{1}{\Gamma(s)}\int_0^C t^s \operatorname{Tr} e^{-t\Delta_q}\frac{dt}{t} + \frac{1}{\Gamma(s)}\int_C^\infty t^s \operatorname{Tr} e^{-t\Delta_q}\frac{dt}{t}.$$

It is well known that the heat kernel on the diagonal has an expansion at $t=0$

$$\operatorname{tr} e^{-t\Delta_q} \sim \sum_{j\geq 0} t^{-\frac{n}{2}+j} a_{-\frac{n}{2}+j}(M,\Delta_q,g_\epsilon)$$

where the a_k are local expressions involving the metric and its derivatives. Hence the first integral converges for $\operatorname{Re} s > n/2$ and continues meromorphically to the complex plane. The second converges for $\operatorname{Re} s<0$ and also continues to the complex plane. The sum of these two functions of s defined by analytic continuation is independent of C, so we take this sum to *define* (11.1). We do it this way because the same definition then works for the b-zeta function of a b-metric (replacing Trace by b-Trace). By Theorem 8.4, the heat kernel in this case also has an expansion at $t=\infty$ in powers of $t^{-1/2}$, so again the integral at $t=\infty$ meromorphically continues to all s.

Let us write down a more explicit formula for analytic torsion. The above regularization of $\int_0^\infty t^z dt/t$ is identically zero. So we may write, choosing $C=1$,

$$\zeta_q(s) = \frac{1}{\Gamma(s)}\int_0^1 t^s\frac{dt}{t}\int_M\Big\{\operatorname{tr} e^{-t\Delta_q} - \sum_{j=0}^{\frac{n-1}{2}} t^{-\frac{n}{2}+j}a_{-\frac{n}{2}+j}(M,\Delta_q,g_\epsilon)\Big\}$$
$$+\frac{1}{\Gamma(s)}\int_1^\infty t^s\frac{dt}{t}\int_M\Big\{\operatorname{tr} e^{-t\Delta_q} - \sum_{j=0}^{\frac{n-1}{2}} t^{-\frac{n}{2}+j}a_{-\frac{n}{2}+j}(M,\Delta_q,g_\epsilon)\Big\}.$$

The first integral converges absolutely near $s=0$; the second will not because of the constant term $\operatorname{proj}\operatorname{null}\Delta_q$ in the trace of the heat kernel. Pulling this term out and integrating, we obtain an expression for the zeta function in terms of integrals absolutely convergent near $s = 0$, which by differentiating in s at $s = 0$ gives us the formula:

$$\begin{aligned} T(M, g_\epsilon) = \frac{1}{2}\sum(-1)^q q\Bigg[\int_0^1 \frac{dt}{t}\int_M \Big\{\operatorname{tr} e^{-t\Delta_q} - \sum_{j=0}^{\frac{n-1}{2}} t^{-\frac{n}{2}+j} a_{-\frac{n}{2}+j}(M, g_\epsilon)\Big\} \\ + \int_1^\infty \frac{dt}{t}\int_M \Big\{\operatorname{tr} e^{-t\Delta'_q} - \sum_{j=0}^{\frac{n-1}{2}} t^{-\frac{n}{2}+j} a_{-\frac{n}{2}+j}(M, g_\epsilon)\Big\} + c\dim\operatorname{null}\Delta_q\Bigg], \end{aligned} \tag{11.2}$$

where $c = \partial_s\Gamma(s)_{\restriction s=1}$ is Euler's constant.

For the b-heat kernel we have a similar expansion at $t = 0$:

$$\operatorname{tr} e^{-t\Delta_{\overline{M},q}} \sim \sum_{j\geq 0} t^{-\frac{n}{2}+j} a_{-\frac{n}{2}+j}(M, \Delta_q, g_0).$$

At $t = \infty$, by Theorem 8.4,

$$b\text{-}\operatorname{Tr} e^{-t\Delta_{\overline{M},q}} \sim \dim L^2\operatorname{null}(\Delta_{\overline{M}_q}) + O(t^{-\frac{1}{2}}),$$

so following the same line of reasoning gives the expression for the b-analytic torsion: (11.3)

$$\begin{aligned} {}^bT(\overline{M}, g_0) = \frac{1}{2}\sum(-1)^q q\Bigg[\int_0^1 \frac{dt}{t} b\text{-}\operatorname{Tr}\Big\{e^{-t\Delta_q} - \sum_{j=0}^{\frac{n-1}{2}} t^{-\frac{n}{2}+j} a_{-\frac{n}{2}+j}(\overline{M}, \Delta_q, g_0)\Big\} \\ + \int_1^\infty \frac{dt}{t} b\text{-}\operatorname{Tr}\Big\{e^{-t\Delta'_q} - \sum_{j=0}^{\frac{n-1}{2}} t^{-\frac{n}{2}+j} a_{-\frac{n}{2}+j}(\overline{M}, \Delta_q, g_0)\Big\} + c\dim L^2\operatorname{null}\Delta_q\Bigg], \end{aligned}$$

Dai and Melrose [9] have shown, using algebraic properties of the Clifford module $\Lambda^* M$ that $\sum(-1)^q q\, a_k(\Delta_q, M, g)$ vanishes pointwise for all $k < -1/2$, so $a_{-1/2}$ is the only term one needs to consider. Here we need only the weaker result that $\sum(-1)^q q\, a_k(H \times \mathbb{R}, \Delta_q, g)$ vanishes for $k < -1/2$. This proves that

$$\sum(-1)^q q\int_M a_k(M, \Delta_q, g_\epsilon) \to \int_M a_k(\overline{M}, \Delta_q, g_0),\ k < -\frac{1}{2}.$$

Since also $a_k t^k$ is, for $k < -1/2$, smaller at $t = \infty$ than the terms we need to consider, this is enough for the computations in this thesis.

The analytic torsion $T(M, g)$ has a very simple dependence on the metric g. It can be described as follows. Let $\mu^q \in \bigwedge H^q(M)$ be fixed, and let V^q be the volume of μ^q with respect to the inner product on $H^q(M)$ induced by the inner product on orthonormal harmonic forms given by g. Then

$$T(M, g) = \Bigg(\prod_{q\text{ even}} V_q\Bigg)\Bigg(\prod_{q\text{ odd}} V_q\Bigg)^{-1} \overline{T}(M),$$

where $\overline{T}(M)$ is independent of the metric [27]. Thus, T can be thought of more invariantly as a metric on

$$\operatorname{detline} H^*(M) \equiv \bigwedge H^{\mathrm{even}}(M) \left(\bigwedge H^{\mathrm{odd}}(M)\right)^* .$$

Here $\bigwedge$ denotes the top exterior bundle of a vector space. In view of this, we will define analytic torsion with respect to a choice of volume elements $\mu^i \in \bigwedge H^i(M)$ by

$$T(M,\mu^i) = T(M,g) \prod_{q \text{ even}} [\mu^q \mid \omega^q] \Big(\prod_{q \text{ odd}} [\mu^q \mid \omega^q]\Big)^{-1}$$

where ω^i are volumes corresponding to an orthonormal basis of harmonic forms with respect to the metric g and $[\mu^q \mid \omega^q]$ denotes the determinant of the change of basis matrix T^q, if $\mu^q = T^q(\omega^q)$. The comments above ensure that $T(M,\mu^i)$ is well defined, independent of the metric. In analysing the limit of T under surgery, it is convenient to use volumes corresponding to a fixed set of cohomology classes (independent of ϵ). This means $T(M,\mu^i)$ is constant in ϵ; using the pushforward theorem, we can calculate the value in terms of ${}^bT(M_\pm, g_0)$ and the determinants of the reduced normal operators.

11.2. Surgery formula for analytic torsion.

Let us compute the limit of T under analytic surgery, using the formula (11.2). $T(M,g_\epsilon)$ is given by the pushforward

$$\begin{aligned} \log T(M,g_\epsilon) = \sum_{q=0}^{n} \frac{(-1)^q q}{2} \Big\{ & p_{1*}\iota^*\big(\operatorname{tr} e^{-t\Delta_q} - \sum_{j=0}^{\frac{n-1}{2}} t^{-\frac{n}{2}+j} a_{-\frac{n}{2}+j}(M,g_\epsilon)\big) \Big|\frac{d(\operatorname{ilg}\epsilon)}{(\operatorname{ilg}\epsilon)^2}\Big|^{\frac{1}{2}} \\ & + p_{2*}\iota^*\big(\operatorname{tr} e^{-t\Delta'_q} - \sum_{j=0}^{\frac{n-1}{2}} t^{-\frac{n}{2}+j} a_{-\frac{n}{2}+j}(M,g_\epsilon)\big) \cdot \Big|\frac{d(\operatorname{ilg}\epsilon)}{(\operatorname{ilg}\epsilon)^2}\Big|^{-1} + c \dim \operatorname{null} \Delta_q \Big\} \end{aligned} \tag{11.4}$$

where ι is the inclusion of the diagonal of the logarithmic heat space and p_1, p_2 are the maps from the short time heat space, respectively long time heat space defined in chapter 8 with $C = 1$. As with the eta invariant, we get an asymptotic expansion

$$T(M,g_\epsilon) \sim \sum_{(n,k)\in\mathcal{E}} a_{n,k} (\operatorname{ias}\epsilon)^n (\log(\operatorname{ias}\epsilon))^k .$$

where $\mathcal{E}$ is the index set

$$\mathcal{E} = \{(-1,0)\} \cup \{(n,0),(n,1) \mid n \in \mathbb{N}\}.$$

The torsions with respect to fixed cohomology classes is

$$\log T(M,\mu^i) = \log T(M,g_\epsilon) + \sum (-1)^q \log[\mu^q \mid \omega^q]. \tag{11.5}$$

Let us use cohomology classes μ^q corresponding to an orthonormal basis of surgery Hodge cohomology as defined by (10.4). Then the sum of log determinants in (11.5) is, from the discussion in the last chapter, of the form $C \cdot \log(\text{ias}\,\epsilon) + O(\text{ias}\,\epsilon)$, where explicitly

$$C = \frac{1}{2}\sum(-1)^q \dim(A_+^q \cap A_-^q) - \dim(R_+^{q-1} \cap R_-^{q-1}).$$

We know *a priori* that $T(M, \mu^i)$ is independent of ϵ; since this correction contains no constant term in ϵ, it follows that $T(M, \mu^i) = a_{0,0}$. Hence we need to calculate the single term $a_{0,0}$.

The $a_{0,0}$ term comes from B_{11}, B_{00} and B_{33}^{∞}. From B_{00} we get (after including the null space term $c\dim L^2\,\text{null}(M_\pm)$) precisely the b-analytic torsion of $M_\pm$. From B_{11} we get nothing, because the heat kernel $e^{-t\Delta_{H\times\mathbb{R}}}$ is translation invariant in the $\mathbb{R}$-direction and so its b-trace vanishes identically. From B_{33}^{∞} we get something which appears difficult to calculate, as it appears below two other terms in the asymptotic expansion. To compute this term, first consider the one dimensional case. The eigenvalues of Δ_ϵ on $[-1,1]_x$ with surgery metric and boundary conditions as in chapter 5 scale as $(\text{ias}\,\epsilon)^2$. Hence the zeta function for Δ_ϵ satisfies $\zeta_\epsilon = (\text{ias}\,\epsilon)^{-2s}\zeta$, with ζ the zeta function of the reduced normal Laplacian on $[-1,1]_s$. The value of $\zeta(0)$ is $\dim \text{null}\,\Delta$. Hence,

$$\log\det\Delta_\epsilon = -\frac{\partial}{\partial s}(\text{ias}\,\epsilon)^{-2s}\zeta(s) = 2\log(\text{ias}\,\epsilon)\dim\text{null}\,\text{RN}(\Delta) + \log\det\text{RN}(\Delta).$$

Calculating $\log\det\Delta_\epsilon$ as a pushforward (11.2), the constant term in $\text{ias}\,\epsilon$ is the sum of a contribution from B_{33}^{∞} and the term $c\dim\text{null}\,\text{RN}(\Delta)$. There are no constant terms coming from B_{00} or B_{11} as the Laplacian Δ_0 is translation invariant on these faces and therefore the b-trace vanishes identically. Since in the general case $H^{(1)}(B_{33}^{\infty})$ is the transfer of $e^{-T\,\text{RN}(\Delta)}$ by Theorem 8.4, we have, in general, that the contribution from B_{33}^{∞} plus $c\dim\text{null}\,\text{RN}(\Delta)$ term is equal to $\log\det\text{RN}(\Delta)$. Since $c\dim H^q(M) = c\dim L^2\,\text{null}(\Delta_{\overline{M},q}) + c\dim\text{null}\,\text{RN}(\Delta_q)$, we obtain Theorem 1.4:

$$a_{0,0} = \log {}^bT(M_+, g_0) + \log {}^bT(M_-, g_0) + \frac{1}{2}\sum(-1)^q q\,\log\det\text{RN}(\Delta_q).$$

Next we substitute in the expressions for $\Lambda_\pm^D(\text{RN}(\Delta_q))$, $\Lambda_\pm^N(\text{RN}(\Delta_q))$ above and use Proposition 5.2. Let us write $A_\pm^{q,r}$, $R_\pm^{q,r}$ for the reduced spaces analogous to $\Lambda_\pm^{D,r}$, $\Lambda_\pm^{N,r}$, and write

$$S_\pm^{q,r} \equiv \text{proj}\,A_\pm^{q,r} - \text{proj}\,R_\pm^{q,r},$$

so that $S^r_\pm(\Delta_q) = S^{q,r}_\pm \oplus -S^{q-1,r}_\pm$. Then we have

$$\begin{aligned}
&\frac{1}{2}\sum(-1)^{q+1}q\,\log\det \mathrm{RN}(\Delta_q) = \\
&\frac{1}{2}\sum(-1)^{q+1}q\Big\{2(\log 2)\big[\dim(A^q_+\cap A^q_-\oplus R^{q-1}_+\cap R^{q-1}_-) \\
&\qquad +\dim(R^q_+\cap R^q_-\oplus A^{q-1}_+\cap A^{q-1}_-)\big] \\
&+(\log 2)\big[\dim(A^q_+\cap R^q_-\oplus R^{q-1}_+\cap A^{q-1}_-)+\dim(R^q_+\cap A^q_-\oplus A^{q-1}_+\cap R^{q-1}_-)\big] \\
&\qquad +\log\det(\mathrm{Id}-S^{q,r}_+S^{q,r}_-)+\log\det(\mathrm{Id}-S^{q-1,r}_+S^{q-1,r}_-)\Big\}. \\
&=\sum(-1)^q(\log 2)\big[\dim(A^q_+\cap A^q_-)+\dim(R^q_+\cap R^q_-)+\frac{1}{2}\dim(A^q_+\cap R^q_-) \\
&\qquad +\frac{1}{2}\dim(R^q_+\cap A^q_-)\big]+\frac{1}{2}\log\det(\mathrm{Id}-S^{q,r}_+S^{q,r}_-).
\end{aligned}$$

Let us rewrite the last term. We have

$$\mathrm{Id}-S^{q,r}_+S^{q,r}_- = 2(\mathrm{proj}\,A^{q,r}_+\,\mathrm{proj}\,R^{q,r}_- + \mathrm{proj}\,R^{q,r}_+\,\mathrm{proj}\,A^{q,r}_-).$$

As $A^{q,r}_+$, $R^{q,r}_-$ and $R^{q,r}_+$, $A^{q,r}_-$ form two pairs of orthocomplements and all have the same dimension, it follows

$$\frac{1}{2}\log\det(\mathrm{Id}-S^{q,r}_+S^{q,r}_-) = \frac{\log 2}{2}(\dim A^{q,r}_+ + \dim R^{q,r}_-) + \log\det(A^{q,r}_+ \to R^{q,r}_-),$$

where the last operator is orthogonal projection. Our formula for the analytic torsion is therefore

$$\begin{aligned}
&\log T(M,\mu^i) = \log {}^bT(M_+,g_0) + \log {}^bT(M_-,g_0) \\
&+\sum(-1)^q\Big\{(\log 2)\big[\dim(A^q_+\cap A^q_-)+\dim(R^q_+\cap R^q_-)+\frac{1}{2}\dim(A^q_+\cap R^q_-)+ \\
&\frac{1}{2}\dim(R^q_+\cap A^q_-)\big]+\log\det(A^{q,r}_+\to R^{q,r}_-)+\frac{\log 2}{2}(\dim A^{q,r}_+ + \dim R^{q,r}_-)\Big\}.
\end{aligned}$$

Since $\dim(A^q_+\cap A^q_-)+\dim(R^q_+\cap R^q_-)+\dim(A^q_+\cap R^q_-)+\dim(R^q_+\cap A^q_-)+\dim A^{q,r}_+ + \dim R^{q,r}_- = \dim H^q(H)$, we get as our final formula

$$\begin{aligned}
&\log T(M,\mu^i) = \log {}^bT(M_+,g_0) + \log {}^bT(M_-,g_0) + \\
&\sum(-1)^q\Big\{\frac{\log 2}{2}\big[\dim(A^q_+\cap A^q_-)+\dim(R^q_+\cap R^q_-)\big]+\log\det(A^{q,r}_+\to R^{q,r}_-)\Big\} \qquad (11.6)\\
&\qquad +\frac{\log 2}{2}\chi_E(H).
\end{aligned}$$

11.3. Reidemeister torsion.

In this section we will review R-torsion and then compute the surgery formula for R-torsion in the same situation for which we computed the surgery formula for analytic torsion above. Let

$$0 \longrightarrow V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \xrightarrow{d^2} \dots$$

be a complex of finite dimensional vector spaces with inner product. Suppose preferred bases $\mu^i \in \bigwedge H^i$ in cohomology are given. The torsion is an alternating product of determinants defined as follows. Let $B^i \subset V^i$ be the image of d^{i-1}, let b^i be a basis of B_i and $\tilde{b}_i$ be an independent set in V^{i-1} mapping to b^i. Let c^i denote an orthonormal basis of V^i. Using the notation [|] as in section 11.1, the torsion is given by

$$\tau(V,d,\mu^i) = \prod_{q \text{ even}} [b^q, \tilde{b}^{q-1}, \mu^q \mid c^q]\Big(\prod_{q \text{ odd}} [b^q, \tilde{b}^{q-1}, \mu^q \mid c^q]\Big)^{-1}. \tag{11.7}$$

This definition is independent of the choice of $\tilde{b}^i$ and c^i. It is also independent of the choice of b^i as a change of basis will introduce into the product identical Jacobian factors in the numerator and denominator. Naturally, it does depend on the choice of volumes in cohomology. More invariantly, one can define the torsion as a metric on the determinant line of the cohomology of V.

Given a simplicial decomposition of a manifold M, we have a cochain complex

$$0 \longrightarrow C^0(M) \xrightarrow{d^0} C^1(M) \xrightarrow{d^1} C^2(M) \xrightarrow{d^2} \dots$$

whose elements are $\mathbb{R}$-valued (linear functionals on) formal sums of cells. Taking the inner product on C^q that makes each cell orthonormal, and choosing volumes in cohomology $H^*(M)$, we have a torsion defined, called the R-torsion of M. The important property (see [21]) of this quantity is that the torsion is invariant under subdivision of the cells comprising M, and is therefore a topological invariant of (M, μ^i). It is not, however, a homotopy invariant. One can also take a flat bundle E over M and look at cochains with values in E. (This is equivalent to passing to the trivial bundle on the cover of M to which E corresponds, and working with cochains with are invariant under the group action whose quotient gives $E \to M$.) From here on, we shall assume that a flat bundle E over M is given; it will usually be dropped from notation.

Notice that, with volume elements in cohomology given by the volumes of orthonormal harmonic forms with respect to some metric on M, the R-torsion has the same dependence on the metric as does the analytic torsion. This was one of the reasons that lead Ray and Singer to conjecture in [26] that the analytic and R-torsions of a closed manifold were equal.

To calculate the behaviour of R-torsion under surgery, we will use the following well known formula. Let

$$0 \longrightarrow K_1 \xrightarrow{k_1} K_2 \xrightarrow{k_2} K_3 \longrightarrow 0$$

be an exact sequence of complexes with inner product, such that the induced volumes are compatible. This means that, if a_1^q is an orthonormal basis for K_1^q, a_2^q an orthonormal basis in K_2^q and a_3^q an independent set in K_2^q mapping to an orthonormal basis of K_3^q, then

$$[k_1^q(a_1^q), a_3^q \mid a_2^q] = 1. \tag{11.8}$$

Let elements of cohomology μ_i^q be given in $H^q(K_i)$. With these chosen volume elements, the long exact sequence in cohomology

$$\longrightarrow H^{q-1}(K_3) \longrightarrow H^q(K_1) \xrightarrow{k_1^q} H^q(K_2) \xrightarrow{k_2^q} H^q(K_3) \longrightarrow$$

is an acyclic complex $\mathcal{H}$ with volumes. The formula we need is

$$\tau(K_2) = \tau(K_1)\tau(K_3)\tau(\mathcal{H}). \tag{11.9}$$

Suppose that we have a simplicial decomposition of M such that H, and therefore M_+ and M_-, are subcomplexes. Let us apply (11.9) to the exact sequence in simplicial cohomology:

$$0 \longrightarrow C^*_{\mathrm{rel}}(M_\pm) \xrightarrow{i} C^*(M) \xrightarrow{p} C^*_{\mathrm{abs}}(M_\mp) \longrightarrow 0$$

Here, relative cochains on $M_\pm$ are those which vanish at the boundary; absolute cochains are unrestricted at the boundary. With the usual inner products on these spaces (all delta functions on cells orthonormal) this short exact sequence is compatible on induced volumes, so (11.9) applies. Also, the relative and absolute torsions of a manifold with boundary are the same, so this equation is, in logarithmic form,

$$\log \tau(M) + \log \tau(M_+) + \log \tau(M_-) + \log \tau(\mathcal{H}),$$

where it is understood that $\log \tau(\mathcal{H})$ is measured with respect to the same choices of volumes that were used in to compute the other torsions. To use this result, we must calculate $\tau(\mathcal{H})$, where $\mathcal{H}$ is the long exact sequence

$$\longrightarrow H^{q-1}_{\mathrm{abs}}(M_\mp) \xrightarrow{c^{q-1}} H^q_{\mathrm{rel}}(M_\pm) \xrightarrow{i^q} H^q(M) \xrightarrow{p^q} H^q_{\mathrm{abs}}(M_\mp) \xrightarrow{c^q}$$

Since we want to compare $\tau(M)$ to the analytic torsion $T(M, \mu^i)$, we should take volumes in cohomology given by an orthonormal basis μ^i of surgery Hodge forms defined in equation (10.4), and volumes $\nu^i_\pm$ for the relative or absolute cohomology of $M_\pm$ given by b-Hodge theory. Now, let us split these spaces into subspaces corresponding to the images of the above maps and their orthogonal complements:

$$\begin{aligned}
H^q_{\mathrm{rel}}(M\pm) &\equiv \left[(R^{q-1}_\pm \ominus R^{q-1}_\mp)\right] \oplus \left[H^q_{b\text{-Ho}}(M_\pm) \oplus (R^{q-1}_+ \cap R^{q-1}_-)\right] \\
H^q(M) &\equiv \left[H^q_{b\text{-Ho}}(M_\pm) \oplus \sqrt{2}(R^{q-1}_+ \cap R^{q-1}_-)\right] \oplus \\
&\qquad \left[H^q_{b\text{-Ho}}(M_\mp) \oplus \sqrt{2}(A^q_+ \cap A^q_-)\right] \\
H^q_{\mathrm{abs}}(M\mp) &\equiv \left[H^q_{b\text{-Ho}}(M_\mp) \oplus (A^q_+ \cap A^q_-)\right] \oplus \left[(A^q_\mp \ominus A^q_\pm)\right].
\end{aligned}$$

The numerical factors are so the decomposition is isometric. The torsion then is given by

$$\begin{aligned}\log\tau(\mathcal{H}) = \sum(-1)^q \log\det i^q &\restriction H^q_{b\text{-Ho}}(M_\pm)\oplus R^{q-1}_+\cap R^{q-1}_- \\ -\log\det p^q &\restriction H^q_{b\text{-Ho}}(M_\mp)\oplus\sqrt{2}A^q_+\cap A^q_- \oplus\sqrt{2}R^{q-1}_+\cap R^{q-1}_- \\ +\log\det c^q &\restriction (A^q_\mp\ominus A^q_\pm).\end{aligned}$$

Let us recall what these maps are explicitly. The map i is the map which takes a relative form on $M_\pm$ and views it as a form on M. We then need to map it into Hodge cohomology, which is projection by Π_ϵ. On the space $H^q_{b\text{-Ho}}(M_\pm)$ this map is the identity and on $R^{q-1}_+\cap R^{q-1}_-$ this map is multiplication by $1/2$, which means it has log determinant $\log(1/\sqrt{2})(\dim R^{q-1}_+\cap R^{q-1}_-)$ on this factor. The map p is restriction to $M_\pm$; this is the identity on $H^q_{b\text{-Ho}}(M_\mp)$ and on $A^q_+\cap A^q_-$, which means it has log determinant $\log(1/\sqrt{2})(\dim A^q_+\cap A^q_-)$ on this factor. To find the image of the connecting homomorphism c one takes $\beta\in A^q_\mp\ominus A^q_\pm$, extends into M, applies d and regards the result as a form in $H^q_{\text{rel}}(M\pm)$; it is projection from $A^q_\mp\ominus A^q_\pm$ to $R^q_\pm\ominus R^q_\mp$. Thus,

$$\begin{aligned}\log\tau(\mathcal{H}) &= \sum(-1)^q\Big\{(-\frac{\log 2}{2})\dim(R^{q-1}_+\cap R^{q-1}_-)+\frac{\log 2}{2}\dim(A^q_+\cap A^q_-) \\ &\qquad +\log\det\left(A^q_\mp\ominus A^q_\pm\to R^q_\pm\ominus R^q_\mp\right)\Big\} \\ &= \sum(-1)^q\Big\{\frac{\log 2}{2}\big(\dim(A^q_+\cap A^q_-)+\dim(R^q_+\cap R^q_-)\big)+\log\det\left(A^{q,r}_+\to R^{q,r}_-\right)\Big\}\end{aligned}$$

The last map is orthogonal projection, and we used the notation of the previous section in the last line. Comparing with (11.6), we obtain Theorem 1.5.

11.4. Cheeger-Müller Theorem.

Finally we discuss how to obtain Corollaries 1.6 and 1.7 from Theorem 1.5. Cheeger's proof of the equality of T and τ runs along these lines: the equality for spheres, and therefore products of spheres, is already known, due to explicit calculations by Ray in [25]. If M is an arbitrary manifold, then $W = M\times[0,1]\setminus D^n$, where D^n is a small disc removed from the interior of W, is a cobordism between $2M$ and S^n. Analytic and R-torsion are both unchanged by a change of orientation and are multiplicative under disjoint union, so it suffices to prove the result for $2M$. Take a Morse function f on W that attains a minimum on $2M$, a maximum on the sphere, and has nonvanishing differential on the boundary. Then the level sets of f define a family of manifolds that are diffeomorphic expect when a critical point of f is crossed, when a cell is attached. Thus, from this one gets a finite collection of manifolds $M_0 = S^n, M_1,\dots M_N = 2M$, and M_{i+1} is obtained from M_i by removing a $S^k\times D^{n-k}$ and glueing in a $D^{k+1}\times S^{n-k-1}$ along the (common) boundary. If we can show that $T(M_{i+1}) = \tau(M_{i+1})$ assuming that $T(M_i) = \tau(M_1)$ then we are done. Note that all these surgeries involve a separating hypersurface, although sometimes it is disconnected.

Now we apply Theorem 1.5. First, with $M = S^k \times S^{n-k}$, and $M_\pm = S^k \times D^{n-k}$, we have

$$\log \frac{{}^bT(S^k \times D^{n-k})}{\tau(S^k \times D^{n-k})} = -\frac{1}{4}\chi(S^k \times S^{n-k}) = 0,$$

since n is odd. Applying it to M_i, $M_{i,+} = M_i \setminus S^k \times D^{n-k}$, $M_{i,-} = S^k \times D^{n-k}$ we get, assuming the result for M_i,

$$\log \frac{{}^bT(M_{i,+})}{\tau(M_{i,+})} = 0.$$

Applying it to M_{i+1}, $M_{i+1,+} = M_{i+1} \setminus D^{k+1} \times S^{n-k-1}$, $M_{i+1,-} = D^{k+1} \times S^{n-k-1}$ we get

$$\log \frac{T(M_{i+1})}{\tau(M_{i+1})} = \log \frac{{}^bT(M_{i+1,+})}{\tau(M_{i+1,+})}.$$

Since $M_{i,+} = M_{i+1,+}$, the Cheeger-Müller Theorem follows.

Corollary 1.7 is obtained by doubling N across its boundary and applying Theorem 1.5 and the Cheeger-Müller Theorem.

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