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A TRANSITION CONTROL SYSTEM

by

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1955

Signature of Authors ______________________________

____
Dept. of Aeronautical Engineering, May 23, 1955

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Thesis Supervisor

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Chairman, Departmental Committee on Graduate Students

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A TRANSITION CONTROL SYSTEM

by

John C. Herther

Malcolm R. Malcomson

Submitted to the Department of Aeronautical Engineering on May 23, 1955, in partial fulfillment of the requirements for the degree Master of Science.

ABSTRACT

The purpose of this thesis is to conduct a preliminary study of a transition control system for placing a vehicle in an orbit about the earth. A brief resume of the overall problem, its history and its necessity is included.

Computers which determine the control quantities that are necessary for vehicle placement in a preferred orbit are to operate during coasting flight from cutoff to apogee. The purpose of these computers is to generate commands to a rocket motor control system. Proposal for such computers is included, and a vehicle orientational control system is presented and analyzed. For the sake of simplicity in this investigation, certain assumptions have been made which may not be quite true physically, however they are representative of the actual situation, and do not invalidate any conclusions which indicate qualitative results which may be expected of the control system. Since simplicity of the computers is desirable due to limitations in space and weight in a vehicle of this type, a perturbation technique is employed in the development of the mathematical models which are simulated.

The accuracy of the mathematical models was determined to be within tolerable limits based on the amount of orbit distortion allowable. Errors in the measuring system used during rocket thrusting have a negligible effect on the computed quantities.

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Using transistors and printed circuits for the electronic gear involved in the computers, it is felt that the total weight of the two computers proposed may be of the order of magnitude of a few pounds, and the space occupied by the computers to be about the size of Volume II of Instrument Engineering. The additional control possible through the use of such computers justifies their existence in the vehicle control system.

Thesis Supervisor: Walter Wrigley

Title: Associate Professor of Aeronautical Engineering

[redacted]

[redacted]

May 23, 1955

Professor Leicester F. Hamilton
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

Dear Professor Hamilton:

In accordance with the regulations of the faculty, we hereby submit a thesis entitled A Transition Control System in partial fulfillment of the requirements for the degree of Master of Science without specification.

John C. Herther

Malcolm R. Malcomson

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ACKNOWLEDGEMENT

The authors express their appreciation to the personnel of the Instrumentation Laboratory, Massachusetts Institute of Technology, who assisted in the preparation of this thesis. Particular thanks are due to Dr. Walter Wrigley, who as the thesis supervisor, inspired and guided the entire project, to Dr. Joseph De Lisle for his interest and encouragement, and to William Kehl and Myron Smith for their help in the theoretical aspects of the problem.

Additional thanks are due to Dr. C. S. Draper for his indirect inspiration and Mr. Forrest Houston for his constructive criticism in the publication.

Final laurels must be given to the wife and fiancee of the authors for their patience and fortitude during the hectic days of thesis preparation.

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OBJECT

The object of this thesis is to propose a system which will properly control a vehicle on the basis of initial conditions of a ballistic trajectory at burnout in order to place it in a desired orbit.

CHAPTER 1

INTRODUCTION

A. Background

1.1 Motivation of Investigation

This thesis is a preliminary study of a control system for placing a vehicle in an approximately circular orbit about the earth.

References for the thesis are listed in the bibliography. Much of the basic data for the problem was extracted from the results of an extensive study by the Rand Corporation on the feasability of placing a vehicle in an orbit about the earth. Progress Reports 1 and 2 of the Instrumentation Laboratory also furnished basic information for the formulation of the investigation. These progress reports concern vehicle control while in the orbit, and propose methods for maintaining the vehicle in a stable orientation after entry into the orbit.

1.2 Scope of Thesis

The vehicle will be the payload of a multi-step rocket used for propelling the vehicle to orbital altitude. The ascent to orbital altitude takes place in two distinct phases:

1) the thrust or powered phase,
2) the coasting phase on a ballistic trajectory.

(see Figure 1-1)

The portion of the ascent phase to the orbit being considered in this thesis is from the cutoff of the last rocket stage until the vehicle is in an approximately circular orbit at an altitude of about three hundred statute miles.

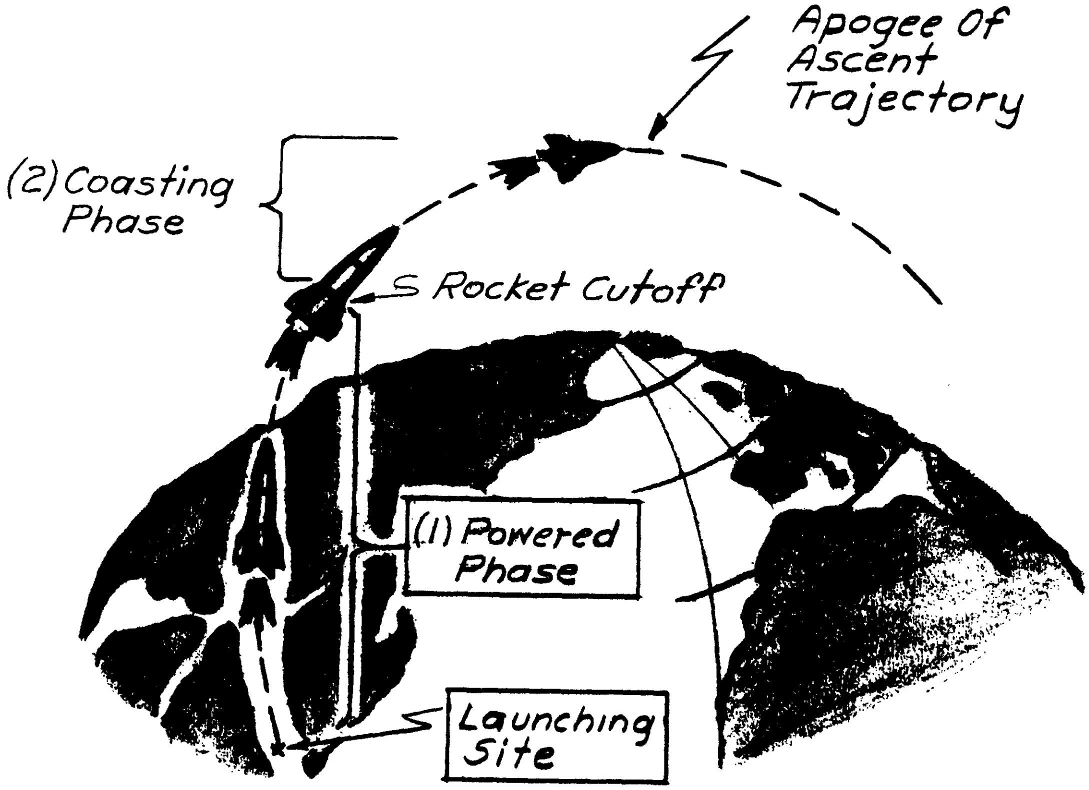

Fig 1-1 Ascent To Orbital Altitude.

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It is assumed that a guidance system, such as [redacted] will provide control for the rocket during the thrust period in order to give it the proper initial conditions to place it on an elliptical ballistic trajectory whose apogee is at the desired orbital altitude. The apogee of the ascent trajectory is defined as that point of the orbit which is at a maximum altitude above the earth, and is identical with orbital altitude of the vehicle when it becomes a circular satellite.

1.3 Vehicle Coasting Phase

During the coasting phase, the vehicle translates as though all of its mass were concentrated at its center of mass, and the equations of motion for a particle in an inverse square central force field apply. Since the work done in this thesis is based on the equations of motion of the vehicle while on the coasting ascent trajectory, a derivation due to P. A. Lapp is included for completeness in Chapter 2.

The vehicle center of mass traverses a ballistic trajectory which is an ellipse with one focus at the center of the earth. Although there are an infinite number of elliptical trajectories whose apogees are at an altitude of three hundred statute miles, a unique trajectory is completely determined by the magnitude and direction of the vehicle velocity at cutoff altitude.

1.4 Equilibrium Condition for a Circular Orbit

In order for the vehicle to become a circular satellite, the two forces acting on it must be equal in magnitude and opposite in direction; these being the gravitation attraction force of the earth, and the centrifugal force due to motion about the center of the earth. The gravitation attraction force is a function of the radius squared, and the centrifugal force is a function of both the velocity squared and the radius. Thus for a given altitude above the earth (or radius), the velocity must be a specific value for the orbit to be circular rather than elliptical. A control system whose function is that of placing a vehicle in a circular orbit must control the magnitude and direction of the vehicle at the apogee of its ascent trajectory.

1.5 Vernier Velocity

The velocity of a vehicle whose path of ascent is that of an ellipse will always have less velocity at apogee than that required for a circular satellite at that altitude. Consequently, vernier velocity is defined as that velocity which must be added to the velocity of the vehicle at apogee in order to place it in a circular orbit about the earth. To add the required vernier velocity at apogee, a rocket thrust must be imparted to the vehicle, which necessitates a rocket thrust control program to act at apogee on the basis of the amount of vernier velocity required.

1.6 Equipment in the Vehicle for Measurement During Thrusting

Measurement of position and velocity relative to the earth may be accomplished using a measuring system proposed by P. A. Lapp as part of the thrust phase guidance system. This measuring system is the data gathering portion for use during rocket burning stages and is based on the physical sensing of vehicle acceleration. Lapp's proposal for the measuring system includes a gyro monitored platform, stabilized with respect to inertial space, on which are mounted three integrating accelerometers. An integrating accelerometer is a device which receives as its input a specific force and produces an output signal proportional to the time integral of the input. Specific force is the total force acting on the accelerometer which is the vector sum of the inertial reaction force and the gravitation attraction force. The output of the accelerometer, being the integral of the specific force, is proportional to the total free fall velocity of the vehicle. To indicate velocity relative to an inertial frame, the effect of the gravitation acceleration must be removed from the output velocity indicated by the integrating accelerometer unit. Lapp proposes a gravitational computer to compute this gravitation acceleration, integrate it, and subtract it from the free fall velocity. To compute the gravitation acceleration, position with respect to the earth must be known.

By considering the center of the earth as an inertial point, and at launching giving the stabilized platform a preferred orientation, the velocity relative to the earth is the velocity with respect to an inertial frame as computed by the method described previously. This quantity is integrated to yield position relative to the earth, which in turn is utilized as input information for the gravitational computer in a closed loop fashion to continuously generate the vehicle velocity and position relative to the earth in the stabilized coordinate system maintained in the pre-launching orientation by the gyros.

B. Statement of the Problem

The purpose of the satellite vehicle requires that the orbit be one of nearly constant altitude. Although the shape of the earth is that of a geoid, a close approximation would be that of an ellipsoid and an elliptical orbit would be one of nearly constant altitude. For simplicity in presenting concepts, a spherical earth is assumed, thus an orbit of constant altitude is in the shape of a circle. Definite limitations have been set forth determining the amount of orbit distortion allowable for the actual vehicle. For this thesis, these limitations will impose the same type of accuracy requirements on the control system for attaining a circular orbit as those required in the actual control system to be used to place the vehicle in a preferred elliptical orbit.

It is assumed that the data gathering portion of the thrust phase guidance system can very accurately measure the actual kinematic properties of the vehicle with respect to the earth. Specifications for the vehicle velocity measurement are of the order of magnitude of 1 foot/second (1 part in 25000) and 1 milliradian in velocity direction measurement.

An ideal thrust control system would be one which would operate in such a fashion as to accept preset commands for vehicle cutoff velocity magnitude and direction at a definite altitude and achieve these exactly. However at this time it is felt that the thrust control system will not achieve the exact cutoff conditions desired even though the actual conditions may be measured quite accurately during flight.

If an ideal thrust control system were available, it would be possible to completely predict the coasting ascent trajectory prior to launching, and preset an amount of vernier velocity to be imparted at apogee of the coasting ascent and predict the exact time at which the

[redacted]

vernier thrust is to be imparted. Realizing that such an ideal thrust control system is not possible yet still desiring to place the vehicle in a circular orbit, it seems possible to make further use of the accuracy possible with existing measuring equipment. This thesis proposes computers for determining control parameters for placing a vehicle in a circular orbit based on measuring actual parameters existing at cutoff and computing during the coasting flight.

1.7 Vernier Velocity Computer

A computer will be proposed for determining the amount of vernier velocity required to place it in a circular orbit on the coasting ascent trajectory which the vehicle is actually traversing.

1.8 Time of Flight from Cutoff to Apogee Computer

An additional computer is necessary to determine when the vernier thrust used for providing the necessary vernier velocity will be imparted to the vehicle. The time of flight of the vehicle from the instant of rocket cutoff to apogee of the actual trajectory which the vehicle is traversing is the chosen quantity for signalling the vernier thrust control system to begin operation.

1.9 Orientation of Vernier Thrust for Apogee Burst

At this time, the vernier thrust rocket's position and type of mounting in the vehicle are not specified, and several suggested arrangements are listed as follows:

1) The vernier thrust rocket may be mounted so as to be fired out of the tail of the vehicle using the same rocket motor that was used during the powered portion of the ascent, provided that this stage has not been dropped. (Figure 1-2, Case 1)

2) The rocket may be mounted so as to be fired out of the side of the vehicle. (Figure 1-2, Case 2)

3) The rocket may be mounted on gimbals, and the direction of the thrust controlled relative to the vehicle.

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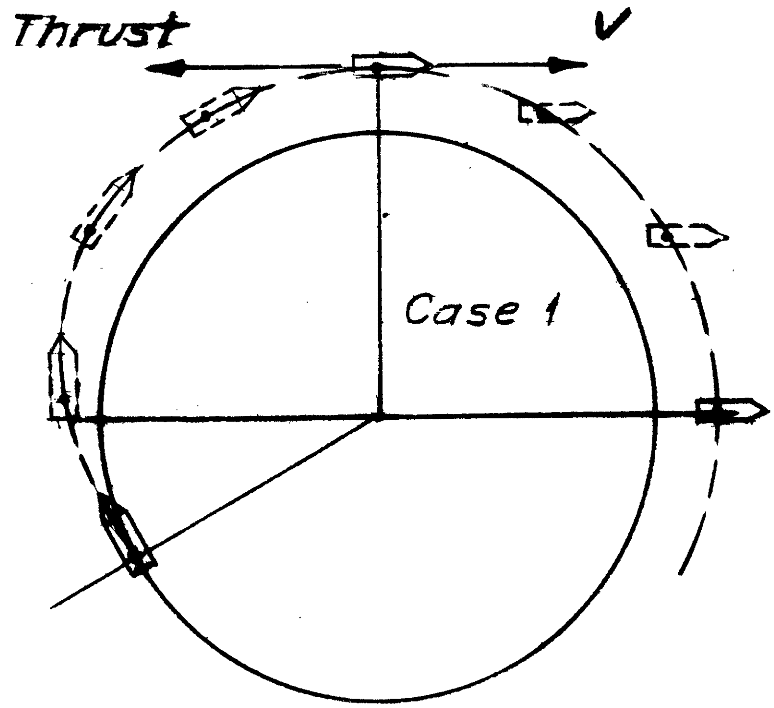
Methods for mounting Vernier thrust Rocket in Vehicle.



Rocket Directed Out Tail Of Vehicle

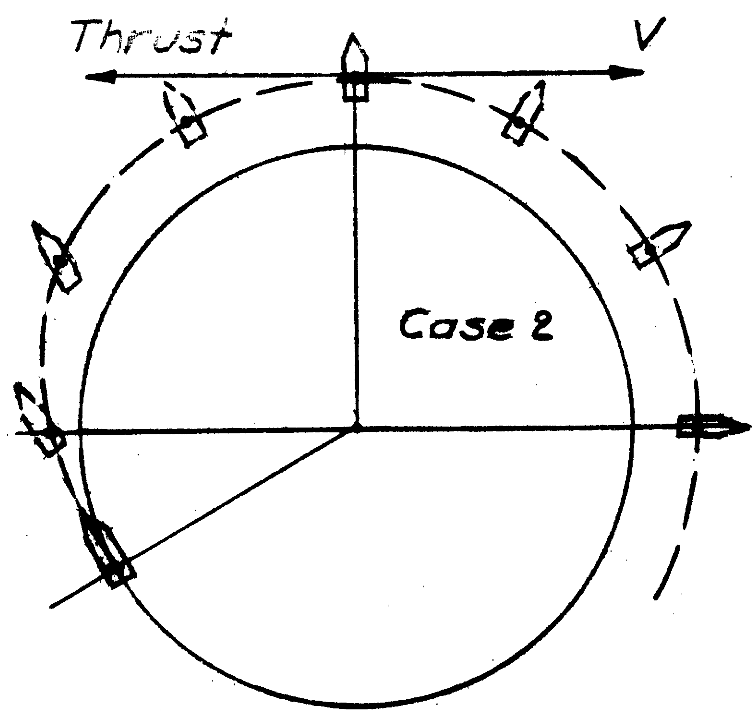


Fig.1-2 Rocket Directed Out Of Side Of Vehicle

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The mounting of the rocket motor on gimbals does not seem to have any particular advantage at this time because of the additional weight and complexity of equipment required.

An advantage for mounting as described in Case 1 is the avoiding of duplication of equipment, thus saving weight. However, if this method is used it can be seen from Figure 1-2 Case 1 that the vehicle will have to be pitched through a large angle in order to properly orient the vehicle for the vernier thrust to be imparted at apogee. Then in order to properly orient the vehicle in its nose up position for the circular orbit, the vehicle will again have to be rotated through another large angle. These two operations require powerful torquing devices which otherwise would be unnecessary.

From Figure 1-2 Case 2 it can be seen that the vehicle need only be turned through a small angle in order to fire the vernier thrust rocket out of the side of the vehicle at apogee, then it is in the proper orientation for the circular orbit. This method has distinct advantages over the other two, and is judged by the authors as the most promising. It should be pointed out at this time that the amount of vernier thrust required is very small compared with the thrust required during the powered ascent, and only a very small additional rocket motor is necessary. It also may be advantageous to have dropped the second stage at burnout so as to have the vehicle completely prepared for its circular orbit and place it in its nose up position as soon as possible.

1.10 Means for Application of Torque to Vehicle for Orientation

Assuming that the rocket motor for vernier thrust will be rigidly mounted in the vehicle, yet not <u>necessarily</u> in the manner indicated by Case 1 or 2, a means for applying torque to turn the vehicle about its center of mass must be provided. Suggested methods for providing this torque are as follows:

1) Small rocket motors that produce thrusts of equal magnitude might be mounted opposing each other at some distance from the vehicle center of mass to provide the torque for turning the vehicle, but it would be difficult to control these motors to the accuracy required in vehicle orientation.

[redacted]

[redacted]

2) Two spinning flywheels mounted on a platform could be used to provide torque. This is a method proposed by Convair for controlling the attitude of the WS-107A [redacted] ICBM for re-entry into the earth's atmosphere.

3) Flywheels which are being accelerated could be used for applying torque. This method is proposed in M. I. T. Progress Report 2 for damping out oscillations during the transient after entry into the circular orbit, the vehicle being stable inherently under the action of gravitational torque while in the circular orbit.

It would be advantageous to use the same torquing means for orientation control during the ascent phase as in the orbit to avoid equipment duplication, therefore accelerating flywheels will be investigated in this thesis.

The orientation control system, consisting of flywheels and associated equipment for their control, must be designed to receive orientation commands, and to minimize the effect of disturbing torques that the vehicle may be subjected to while in flight. The command signal is the desired orientation of the vehicle, as determined by a command orientation computer: and the disturbing torques are a combination of torques due to gravitational attraction (dumbbell effect), rotating equipment, etc.

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CHAPTER 2

ASCENT TRAJECTORIES

2.1 Introduction

After launching, the vehicle will be controlled during the thrusting stages by a guidance system similar to that used in an Intercontinental Ballistic Missile to attain a prescribed velocity, both in magnitude and direction, at a cutoff altitude above the earth. The velocity direction and magnitude will determine a preferred trajectory of ascent. The selection of this trajectory will represent an engineering compromise between the many problems associated with the vehicle's ascent.

2.2 Object of Chapter

The object of this chapter is to acquaint the reader with the trajectory equations that will specify the vehicles motion during its ascent from cutoff (occurring at 350,000') to the time at which additional thrust will be imparted to the vehicle. The additional thrust is necessary to place the vehicle in a circular orbit about the earth. Emphasis will be on placing before the reader the final trajectory equations of motion, with the intermediate steps leading to the solution found in Appendix A.

2.3 Assumptions Leading to the Derivation of the Coasting Trajectory Equations

1. Earth is a spherical non-rotating body.
2. The portion of the vehicle's trajectory under consideration takes place outside of the earths atmosphere, and therefore aerodynamic forces are neglected.

[redacted]

3. The only external force acting on the vehicle is that due to gravitational attraction.

2.4 Trajectory Equations of Motion

The parameters associated with the trajectory of the vehicle are illustrated in Figure 2-1.

The law of Conservation of Angular Momentum states that the time rate of change of the angular momentum of the system is zero, or writing in equation form:

$$d/dt\,(mr^2\,\dot{\theta}) = 0 \tag{2-1}$$

which may be integrated directly to yield:

$$mr^2\,\dot{\theta} = h \tag{2-2}$$

where

h = is the constant magnitude of the angular momentum with respect to the center of the earth

The mass attraction between the vehicle and the earth maybe represented by:

$$F = -\,Em/r^2 \qquad \text{where } E = GM_e \tag{2-3}$$

and

F = mass attraction
G = Universal gravitational constant
M_e = mass of the earth
m = mass of the vehicle

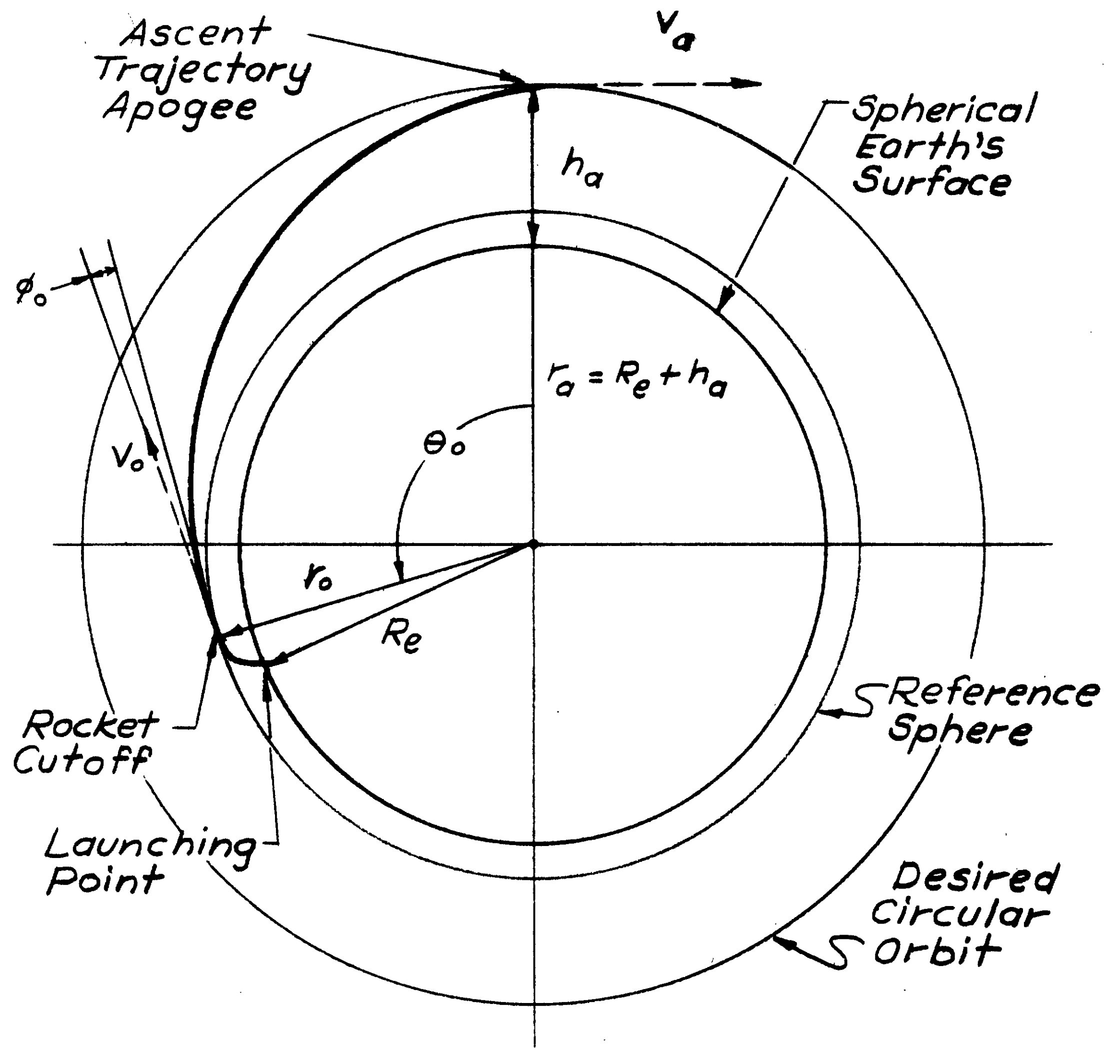
The zero potential surface is located at an infinite distance from the center of the earth; the potential energy of a mass m in an inverse square central force field involves only the radial distance and is written as follows:

$$V(r) = -\int_{-\infty}^{r} -\frac{Em}{r^2}\,dr = -\frac{Em}{r} \tag{2-4}$$

[redacted]

[redacted]

Fig. 2-1 Vehicle Coasting Ascent Phase
Elliptical Trajectory



[redacted]

Since the motion of the vehicle is in a conservative force field, the Conservation of Energy Theorem states that the total energy of the system is constant. Hence:

$$W = T + V \qquad (2\text{-}5)$$

where

W = total energy of the system

T = kinetic energy of the system

V = potential energy of the system

The kinetic energy of the system is:

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) \qquad (2\text{-}6)$$

Replacing T and V in Equation (2-5):

$$W = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) - \frac{mE}{r} \qquad (2\text{-}7)$$

Solving the above expression for $\dot{r}$:

$$\dot{r} = \sqrt{\frac{2}{m}\left(W + \frac{mE}{r} - \frac{h^2}{m r^2}\right)} \qquad (2\text{-}8)$$

By use of Equation (2-2) time can be eliminated from Equation (2-8). The result is:

$$dr = \frac{h\sqrt{\frac{2}{m}\left(W + \frac{mE}{r} - \frac{h^2}{mr^2}\right)}}{mr^2}\, d\theta \qquad (2\text{-}9)$$

Integrating the above expression:

$$r = \frac{E\left(1 - \frac{h}{mE}\left(\frac{W}{m} + \frac{m^2 E^2}{h^2}\right)\right.}{(h/m)^2} \qquad (2\text{-}10)$$

Equation (2-10) then may be simplified to:

$$r = \frac{r_o^2\, V_o^2\, \cos^2 \phi_o}{E(1 - e \cos \theta)} \qquad (2\text{-}11)$$

[redacted]

[redacted]

or rewriting the above in the more convenient form of a conic section in polar coordinates:

$$r = \frac{p}{(1 - e\cos\theta)} \tag{2-12}$$

where

p = semi-latus rectum of a conic section

e = eccentricity

θ = angle from the major axis of the ellipse to a line through a point on the ellipse to the focal point

For the ascent phase of the satellite vehicle, the value of $e < 1$.

An expression for $\dot{\theta}$ maybe obtained from Equation (2-2) to be as follows:

$$\dot{\theta} = \frac{h}{mr^2}$$

and the elimination of r and h from the above equation results in:

$$\dot{\theta} = \frac{E^2(1 - e\cos\theta)}{r_o^3 V_o^3 \cos^3\phi_o} \tag{2-13}$$

From Equation (2-10), an expression for θ may be written as follows:

$$\theta = \cos^{-1}\frac{(Er - r_o^2 V_o^2 \cos^2\phi_o)}{reE} \tag{2-14}$$

2.5 Vehicle Velocity

An expression for the velocity of the vehicle curing its coasting ascent is a parameter of interest. In later chapters of the thesis reference will be made to the vehicle's velocity at various points on the trajectory, and therefore an expression for velocity as a function of position is included at this time. The expression for velocity is obtained by the solution of Equation (2-7) where W (total energy) is replaced by its equivalent in terms of conditions at cutoff. The resultant expression for velocity is:

[redacted]

$$V = \sqrt{V_o^2 - \frac{2E}{r_o} + \frac{2E}{r}} \tag{2-15}$$

2.6 Time of Flight to Apogee

In addition to the trajectory equations that determine the position of the vehicle during its ascent, it also is of interest to know how long it takes the vehicle to travel from cutoff to apogee of the elliptical trajectory. The time of flight may be determined by integrating Equation (2-13) from cutoff to apogee, with the result being as follows:

$$t_a = \frac{r_o^2 V_o}{2E - r_o V_o^2} \left[\sin \phi_o - \frac{2E}{2 E r_o V_o^2 - r_o^2 V_o^4} \tan^{-1} \left(\frac{1+e}{1-e} \bullet \tan \frac{\theta_o}{2} \right) \right] \tag{2-16}$$

2.7 Summary of Chapter

Before reviewing the chapter, the reader once again is reminded that the intermediate steps leading to the development of the trajectory equations have been omitted for the purpose of relieving the reader from the minute details associated with their development. For the reader who is interested, the intermediate steps can be found in Appendix A. Because the vehicle's motion takes place in essentially that of a vacuum, the effect of aerodynamic forces have been neglected with only those associated with gravitation being considered. The trajectory of the vehicle during its ascent is that of an ellipse. The posi tion of the vehicle during its ascent is specified by the trajectory equations with respect to the center of the earth. The time that it takes the vehicle to travel from cutoff to apogee has been developed in terms of burn-out conditions. If burn-out conditions are measurable, it is then but a small task to instrument a time of flight computer to operate on these quantities to indicate when the vehicle is at apogee. A list of the more important equations will follow.

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LIST OF EQUATIONS

At cutoff:

$$r = r_o; \quad V = V_o; \quad \phi = \phi_o$$

$$\theta_o = \cos^{-1} \frac{(E r_o - r_o^2 V_o^2 \cos^2 \phi_o)}{r_o e E}$$

$$e = \frac{r_o V_o \cos \phi_o}{E} \sqrt{V_o^2 - \frac{2E}{r_o} + \frac{E^2}{r_o^2 V_o^2 \cos^2 \phi_o}}$$

Between cutoff and apogee:

$$r = \frac{r_o^2 V_o^2 \cos^2 \phi_o}{E(1 - e \cos \theta)}$$

$$\theta = \cos^{-1} \frac{(E r - r_o^2 V_o^2 \cos^2 \phi_o)}{r e E}$$

$$e = e \text{ at cutoff}$$

At apogee:

$$\theta = 0; \quad \phi = 0$$

$$r_a = \frac{r_o^2 V_o^2 \cos^2 \phi_o}{E(1 - e)}$$

$$V_a = \frac{E(1 - e)}{r_o V_o \cos \phi_o}$$

$$e = e \text{ at cutoff} = 1 - \frac{r_a V_a^2}{E}$$

[redacted]

CHAPTER 3

FOUNDATION FOR INSTRUMENTATION OF COMPUTERS WHICH ARE TO OPERATE DURING VEHICLE COASTING ASCENT PHASE

1) Vernier Velocity Computer
2) Time of Flight from Cutoff to Apogee Computer

3.1 Introduction

The need for a vernier velocity computer and a time of flight from cutoff to apogee computer has been established in Chapter 1. Both computers must operate during coasting flight on the basis of quantities measurable at rocket cutoff; these being the magnitude and direction of the vehicle velocity with respect to the earth, and the distance from the center of the earth.

From Appendices A and B, equations for time of flight to apogee (t_a) and vernier velocity (V_v) in terms of cutoff conditions ($r_o V_o \phi_o$) are as follows:

$$V_v = \frac{E}{r_o V_o \cos\phi_o}\left[\sqrt{1-e} - (1-e)\right] \qquad (3\text{-}1)$$

$$t_a = \frac{r_o^2 V_o}{ZE - r_o V_o^2}\left[\sin\phi_o + \frac{ZE}{\sqrt{ZE\, r_o V_o^2 - r_o^2 V_o^4}} \tan^{-1}\left(\sqrt{\frac{1+e}{1-e}} \bullet \tan\frac{\theta_o}{2}\right)\right] \qquad (3\text{-}2)$$

Using digital or analog simulation techniques it is possible to instrument a computer which will solve equations (3-1) and (3-2), however because of the many arithmetic operations necessary for the solution, a rather complex computer would be required. In order to place the vehicle on a coasting ascent trajectory whose apogee is at an altitude of three hundred miles, the ascent guidance system must control the rocket during burning stages quite accurately. There is some ideal trajectory which the guidance system must attempt to achieve. This trajectory may be minimum energy, minimum fuel to reach the preferred altitude, or an arbitrary trajectory selected on the basis of other engineering considerations. For our purposes, the choice of this ideal trajectory is immaterial, but the guidance system must operate so as to give the vehicle actual measurable cutoff conditions ($r_o V_o \phi_o$) which will in fact take the vehicle to orbital altitude. The need for a vernier velocity and time of flight to apogee computer arises from the fact that although the measuring system used in the ascent during rocket burning can readily determine the appropriate physical quantities for rocket control, the rocket control system may not be adequate to achieve a desired set of cutoff conditions. These computers are designed to accept the difference between a measured and a reference set of cutoff conditions and still provide means for placing the vehicle in a circular orbit. A general solution of the equations for V_v and t_a is not required because of the limited range allowed in cutoff conditions, hence instrumentation may be simplified. Knowing that the measured quantities differ only slightly from some reference set suggests application of a perturbation technique to be applied about this reference trajectory.

3.2 Series Expansion of Equations to be Instrumented

Let the expressions for vernier velocity and time of flight to apogee in terms of cutoff conditions be represented by:

$$y = f(r_o V_o \phi_o) \tag{3-3}$$

where:

$$y = V_v \text{ or } t_a$$

Since both of these functions are continuous and have derivatives, they may be expanded in a multi-variable Taylor's Series about a reference set of burnout conditions.

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$$y = y(\ ) + \frac{\partial y}{\partial r_o}(\ )(r_o - r_r) + \frac{\partial y}{\partial v_o}(\ )(V_o - V_r)$$

$$= \frac{\partial y}{\partial \phi_o}(\ )(\phi_o - \phi_r) + \frac{1}{2}\frac{\partial^2 y}{\partial r_o^2}(\ )(r_o - r_r)^2 + \frac{1}{2}\frac{\partial^2 y}{\partial v_o^2}(\ )(V_o - V_r)^2$$

$$+ \frac{1}{2}\frac{\partial^2 y}{\partial \phi_o^2}(\ )(\phi_o - \phi_r)^2 + \frac{1}{2}\frac{\partial^2 y}{\partial r_o \partial v_o}(\ )(r_o - r_r)(V_o - V_r)$$

$$+ \frac{1}{2}\frac{\partial^2 y}{\partial r_o \partial \phi_o}(\ )(r_o - r_r)(\phi_o - \phi_r) + \frac{1}{2}\frac{\partial^2 y}{\partial v_o \partial \phi_o}(\ )(V_o - V_r)(\phi_o - \phi_r) + \ldots \qquad (3\text{-}4)$$

where: ( ) represents quantity evaluated for $r_o V_o \phi_o$ reference cutoff conditions

r_r = reference r_o

V_r = reference V_o

ϕ_r = reference ϕ_o

A first attempt for determining vernier velocity and time of flight to apogee would be to utilize only the first term of the Taylor Series as an approximation for the function. Succeeding attempts to obtain more accuracy in the determination of the quantities would be by including more terms of the series to more closely approximate the function. Before an evaluation of the mathematical model which is to be simulated may be made, it is necessary to investigate the degree of accuracy that is required of the computers while in operation.

If the assumption is made that the expressions for the quantities are representative of the physical situation, then comparison can be made on the basis of computation of an "actual" quantity through Equations (3-1) or (3-2), and a computed quantity using whatever computer equation is to be instrumented.

It should be noted that the equation is a mathematical model of a physical situation which hopefully is indicative of a functional relationship between actual quantities; yet it will only be subject to the validity of the assumptions which have been made in the derivation of the equation.

3.3 Accuracy Requirements of the Computers

Since the vernier velocity computer is to determine the amount of velocity to be added at apogee, an error in the magnitude of the computed vernier velocity would result in the vehicle possessing the incorrect satellite velocity commensurate with a circular orbit at that altitude. Consequently the orbit would be distorted to that of an ellipse.

An error in the computed time of flight would result in the vernier velocity being added to the vehicle velocity at an incorrect orientation of the vernier rocket thrust. This acceleration produces an undesirable component of velocity normal to the earth resulting in a similar orbit distortion.

Because of design considerations concerning the purpose of the vehicle in the orbit, certain limitations have been imposed on the amount of orbit distortion allowable.

An investigation was conducted to determine the effect of errors in velocity magnitude and direction at the point of entry into the orbit. Plots were made showing variation in altitude as the vehicle makes one revolution about the earth as a function of errors in V_a and ϕ_a. (See Figures 3-1 and 3-2). For a given combination of errors in V_a and ϕ_a altitude may be read as a function of θ and the results superimposed to yield total orbit distortion. A sketch showing a typical combination is shown in Figure 3-3.

An examination of Figures 3-1 and 3-2 indicates that for distortion of a 300 mile circular orbit to be less than $\pm$ 17 miles, the velocity must be controlled to better than $\pm$ 25 feet/second and the direction controlled to better than 20 minutes of arc. At the present, a 15-20 mile error in altitude appears to be tolerable, yet even this latitude of orbit distortion imposes stringent requirements on the accuracy of the vernier velocity and time of flight to apogee computers.

The equations used in determining the effect of the errors described above are as follows:

$$r = \frac{p}{1 - e \cos \theta} \qquad (3\text{-}5)$$

[redacted]

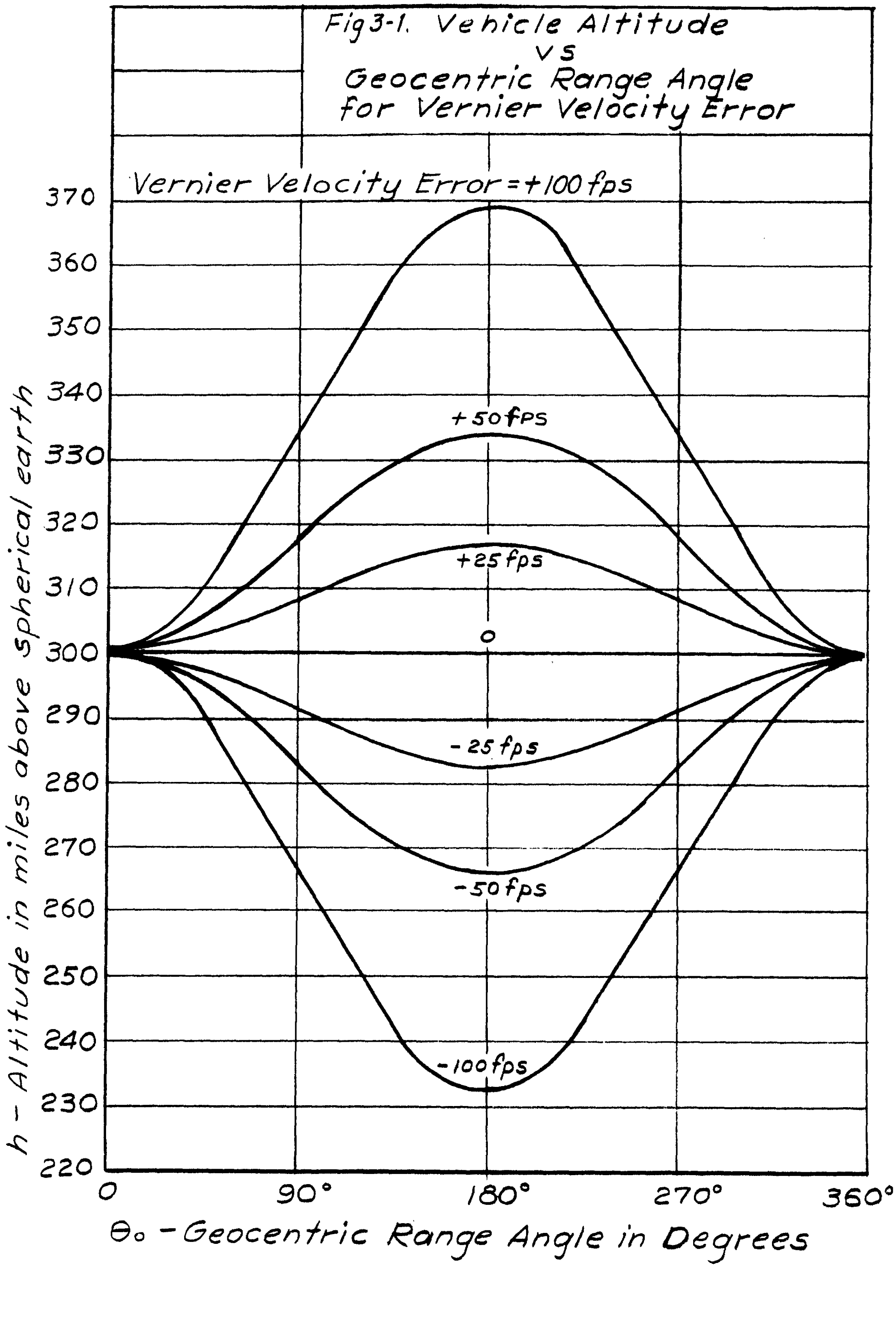


Fig 3-1. Vehicle Altitude vs Geocentric Range Angle for Vernier Velocity Error

[redacted]

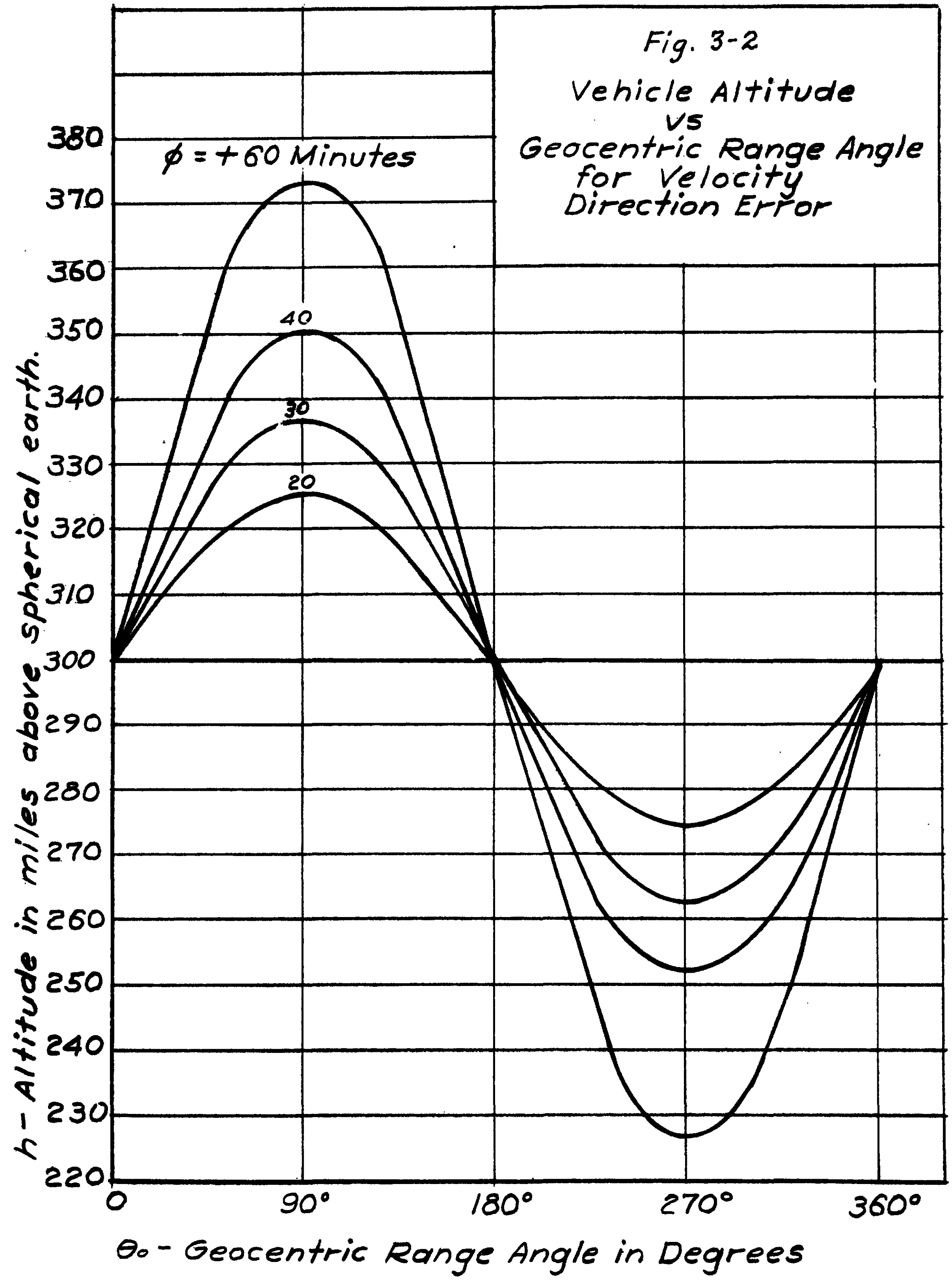
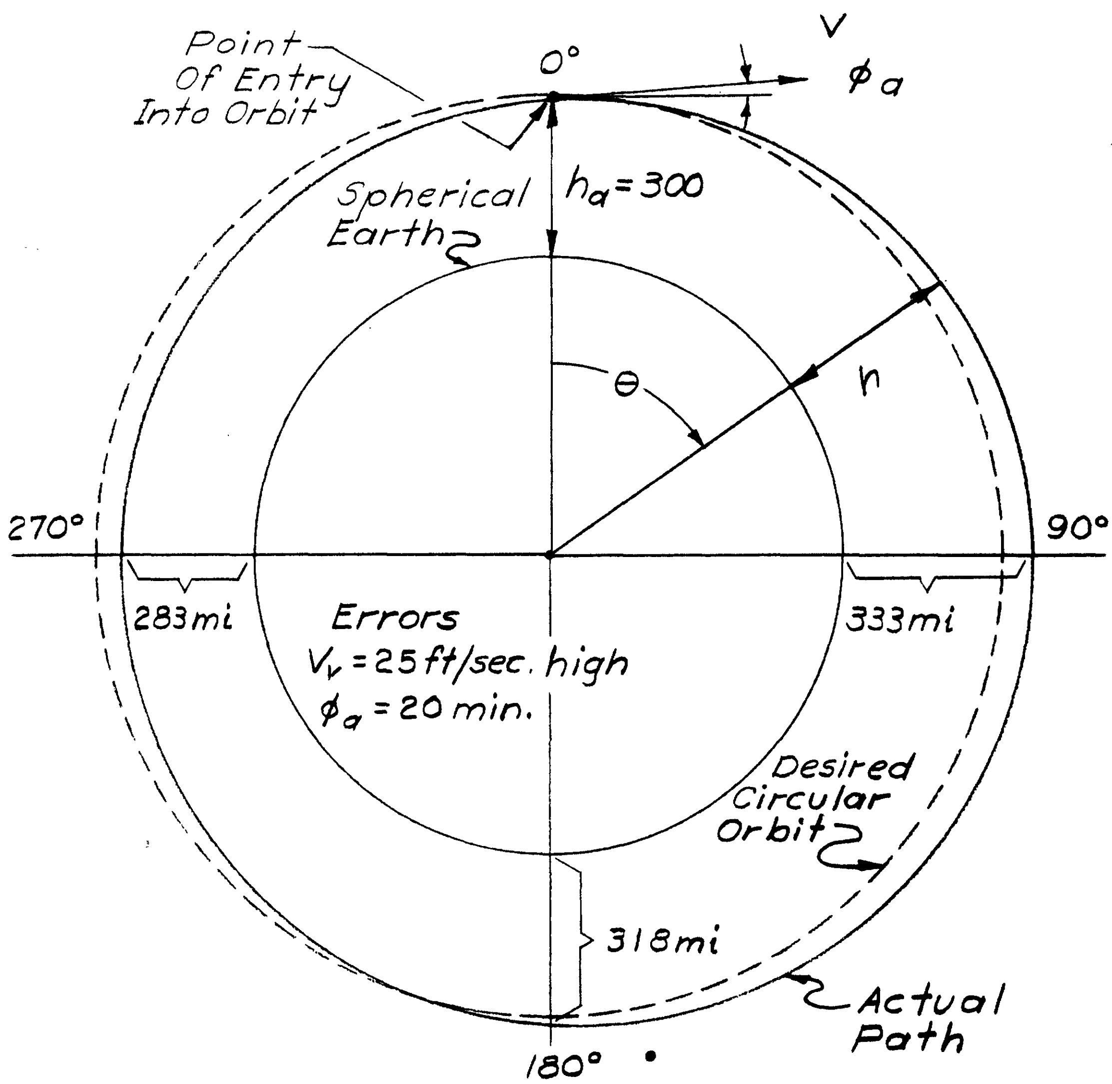


Fig. 3-2
Vehicle Altitude vs Geocentric Range Angle for Velocity Direction Error

[redacted]

[redacted]

Fig. 3-3 Perturbed Orbit Of Satellite Vehicle Due To Errors In V_v And ϕ_a'



h_a = Altitude of desired circular orbit.

h = Altitude of vehicle on perturbed trajectory in miles.

θ = Geocentric range angle in degrees.

[redacted]

[redacted]

where:

$$p = r_a^2 V_a^2 \cos^2 \phi_2$$

$$e = \frac{r_o V_o}{E} \cos \phi_o \sqrt{r_o^2 - \frac{2E}{r_o} + \frac{E^2}{r_o^2 V_o^2 \cos^2 \phi_o}}$$

and $r_a V_a \phi_a$ are conditions at entry into circular orbit from the apogee of the ascent trajectory.

3.4 Family of Useful Trajectories for Coasting to Apogee

A unique trajectory is completely determined by cutoff conditions $r_o V_o \phi_o$, thus an infinite number of ascent trajectories exist for all combinations of the controllable conditions. A study of all elliptical trajectories beginning at an altitude above the earth at cutoff of 350,000 feet, and terminating at an altitude of 300 statute mile, has been made using Equations 3-1 and 3-2 as a basis for programming the Card Programmed Calculator (hereafter referred to as CPC) to calculate vernier velocity required and time of flight to apogee for reasonable values of V_o and ϕ_o. In particular, a maximum vernier velocity of 1500 feet per second was assumed, and cutoff velocity direction was assumed to range from 1^o to 5^o. The family of ascent trajectories considered may be seen from a sketch for a few typical ones shown in Figure 3-4. Figures 3-5, 3-6, 3-7 are plots showing variation of time of flight from rocket cutoff to apogee of the coasting ascent as a function of vernier velocity, cutoff velocity direction, and geocentric coasting angle respectively. In each plot, the data is plotted for constant values of the other parameter, hence an entire family of curves is presented.

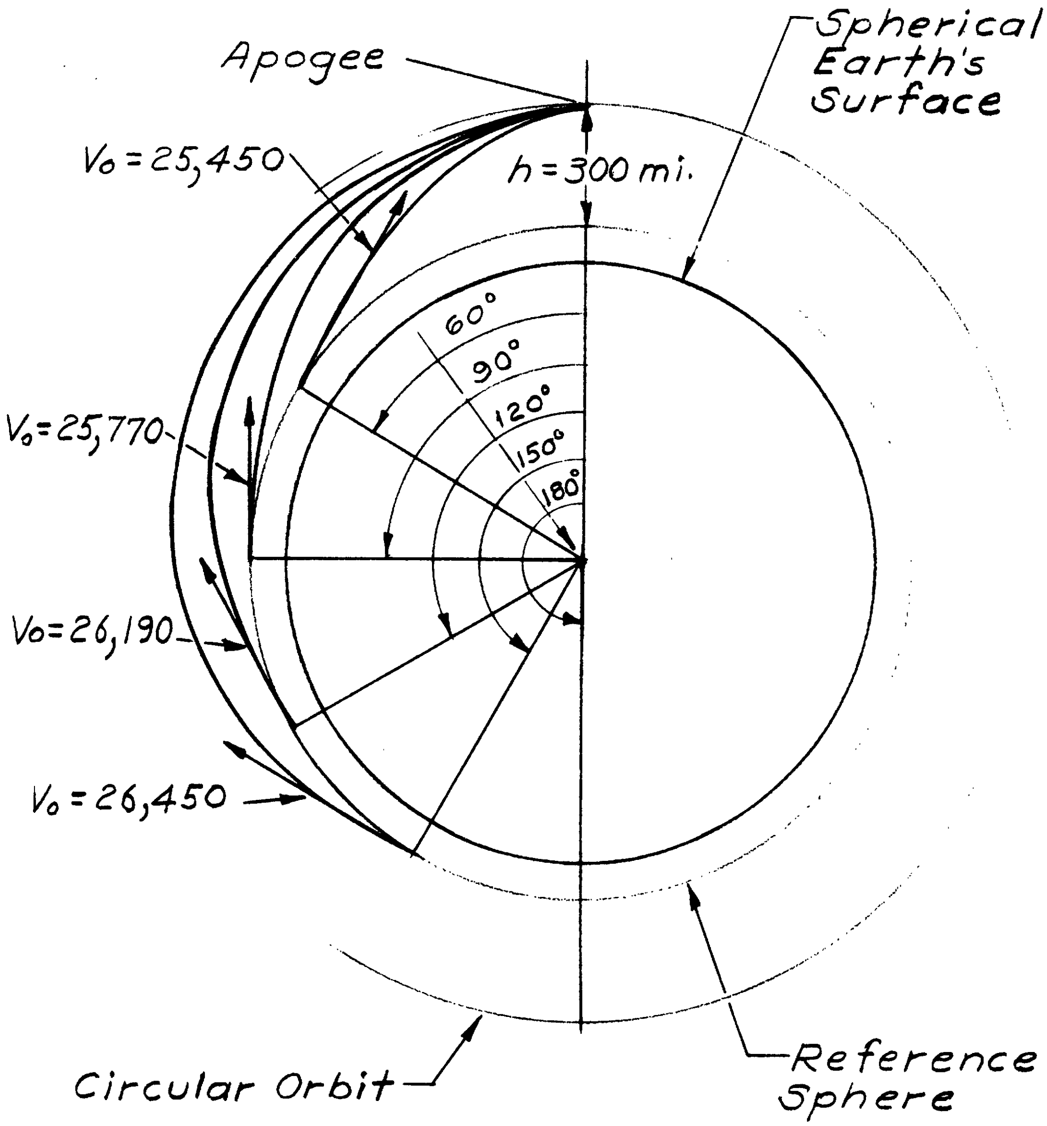
From Figure 3-5, it is interesting to note that time of flight is completely independent of cutoff velocity direction for a vernier velocity of approximately 680 feet/second. For vernier velocities less than this value, the time of flight increases for decreasing cutoff velocity direction angles, while the converse is true for vernier velocities greater than this value.

From Figure 3-6, it can be seen that time of flight is linearly related to cutoff velocity direction for a vernier velocity of about 700 feet/second. For vernier velocities other than this specific value,

[redacted]

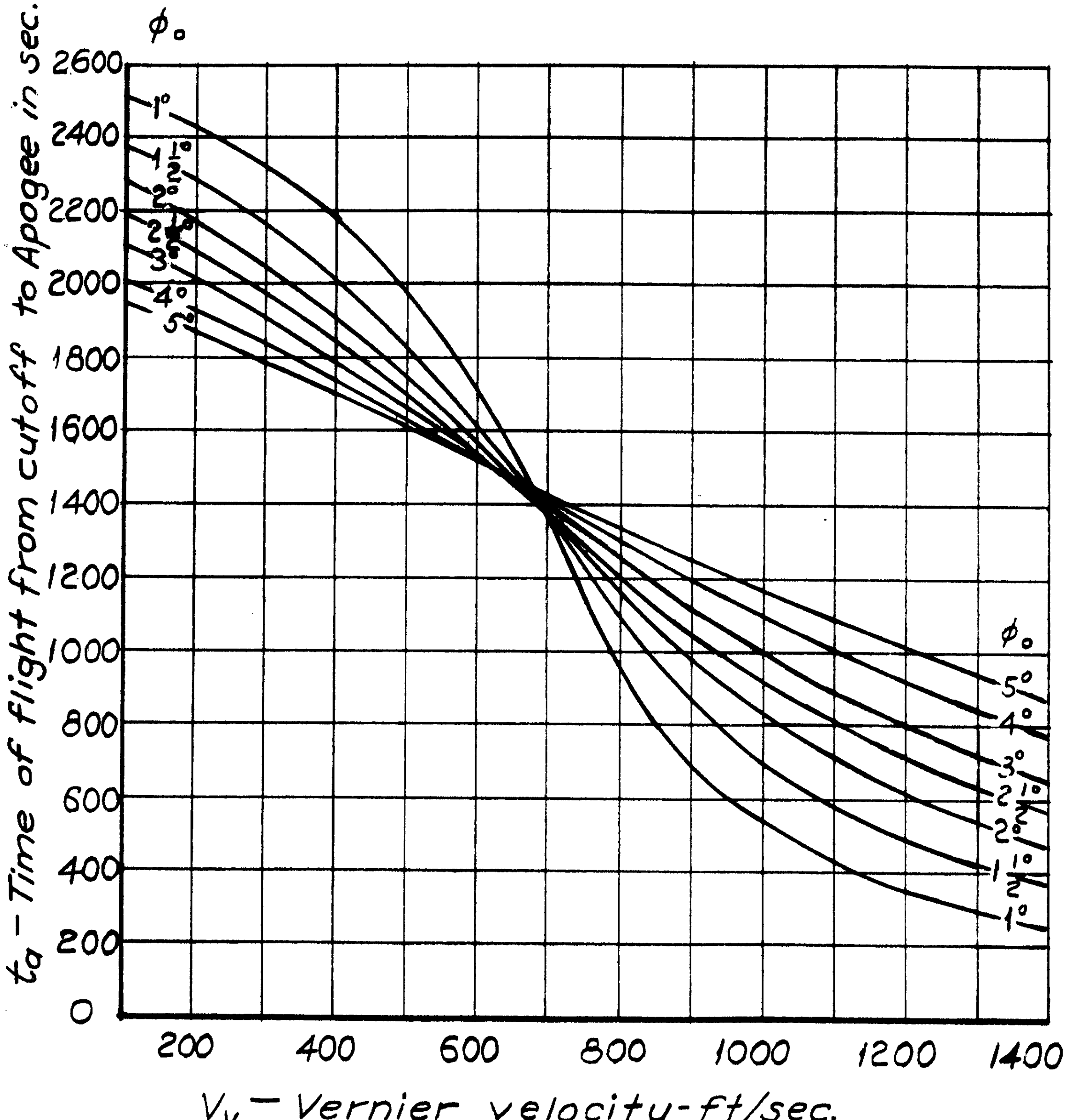
[redacted]

Fig. 3-4. Family Of Coasting Ascent Trajectories To 300 Mile Orbit.



[redacted]

Fig. 3-5 Vehicle time of flight from Cutoff to Apogee vs Vernier Velocity for Constant Cutoff velocity Direction



[redacted]

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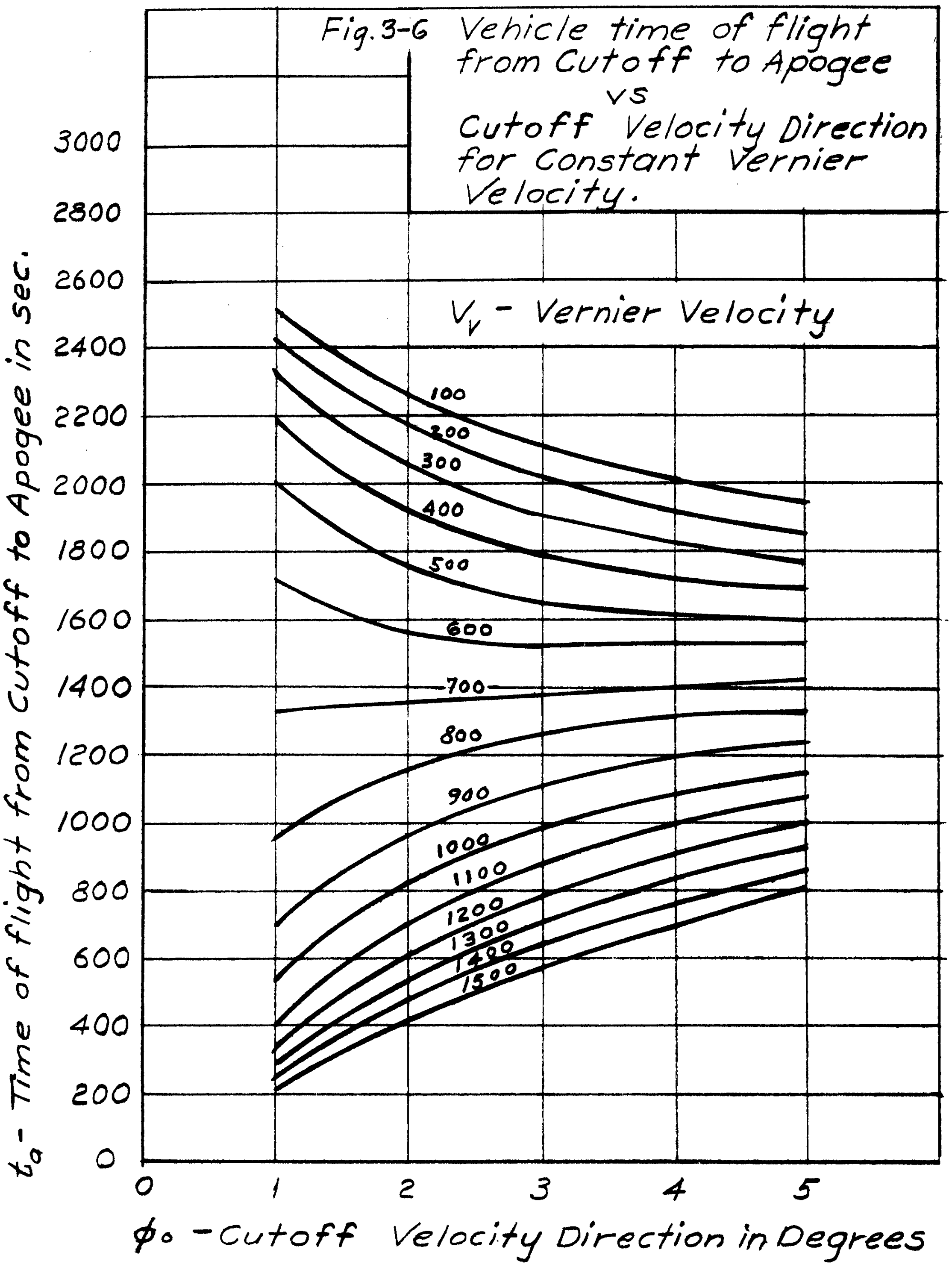
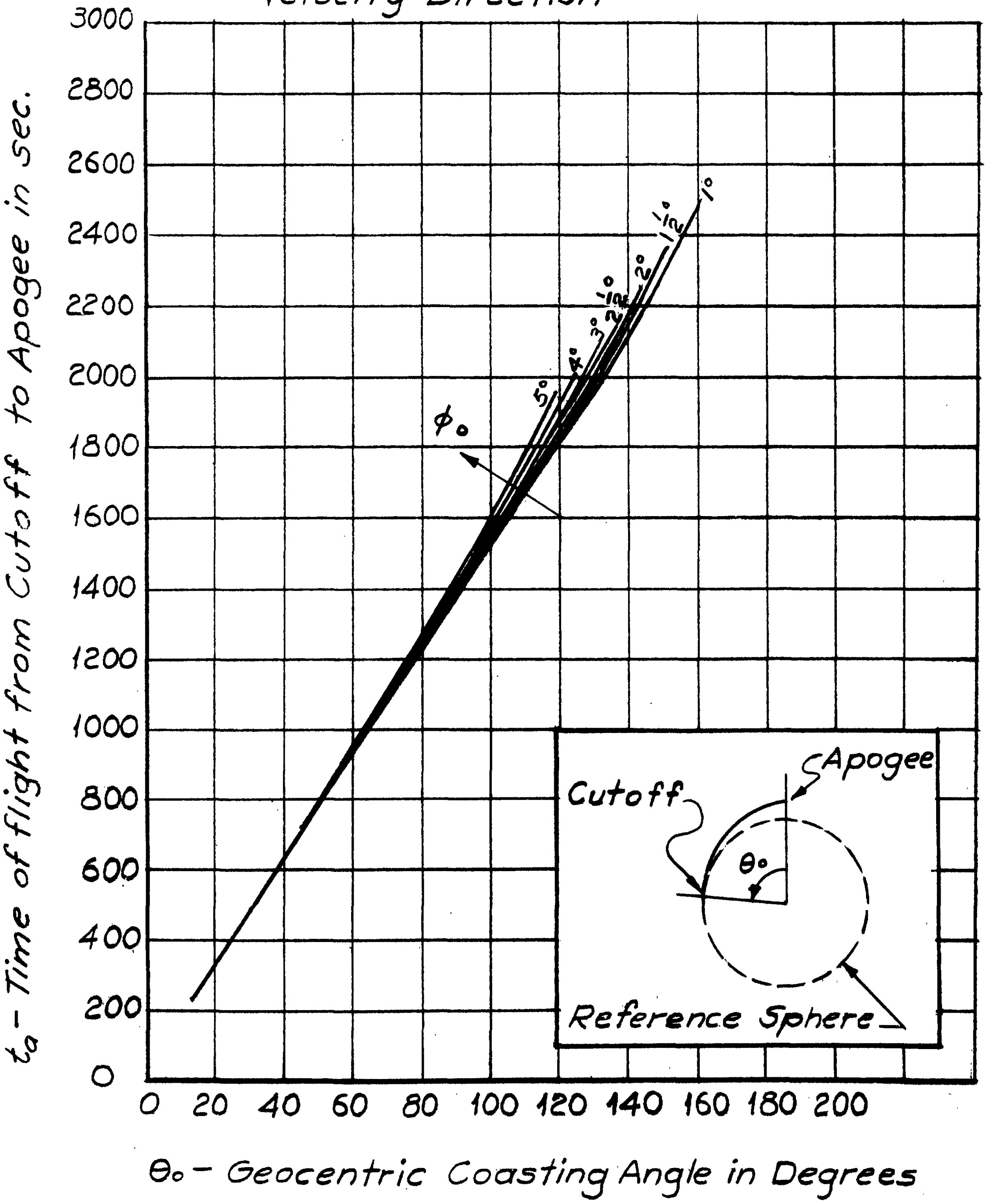


Fig. 3-6 Vehicle time of flight from Cutoff to Apogee vs Cutoff Velocity Direction for Constant Vernier Velocity.

[redacted]

[redacted]

Fig. 3-7 Vehicle time of flight from Cutoff to Apogee vs Geocentric Coasting Angle for Constant Cutoff Velocity Direction



[redacted]

[redacted]

a quite non-linear relation exists between time of flight and cutoff velocity direction.

From Figure 3-7, it can be seen that time of flight is practically directly proportional to geocentric coasting angle for a constant cutoff velocity direction, yet time of flight increases with cutoff velocity direction for a given geocentric coasting angle. It should be noted that the entire range of geocentric coasting angles from 0^{o} to 180^{o} represents the extreme trajectories from the standpoint of amount of vernier velocity required; the maximum amount of vernier velocity occurs on the trajectory with $\theta_o = 0^{o}$ for which case ϕ_o would have to be 90^{o}, and the minimum amount of vernier velocity occurring on the trajectory with $\theta_o = 180^{o}$ with ϕ_o something slightly greater than 0^{o}.

Figure 3-8 shows cutoff conditions V_o and ϕ_o vs θ_o plotted in the region of vernier velocity less than the critical value of 680 feet/second. In this plot, θ_o varies between 100^{o} and 130^{o}; these limitations being imposed on the basis that:

1) for $\theta_o \leq 100^{o}$ vernier velocities greater than 700 feet/second would be required. Fuel for thrust at apogee would be of excessive weight.

2) for $\theta_o \geq 130^{o}$ a large portion of the coasting ascent would take place at lower altitudes, increasing the effect of aerodynamic forces on the vehicle.

It can be seen from the plot that the smaller the geocentric coasting angle, the higher the cutoff velocity required for a given amount of vernier velocity.

3.5 Choice of a Reference Coasting Ascent Trajectory

In designing computers which will compute control commands for placing the vehicle in a circular orbit on the basis of measured cutoff conditions, the equations which relate the desired quantities to the measured quantities have been simplified using perturbation techniques since the measured quantities will not differ greatly from a reference set associated with a reference trajectory. At this time, it should be pointed out that no attempt is being made to track a reference trajectory or to in any way attempt to force the vehicle's path

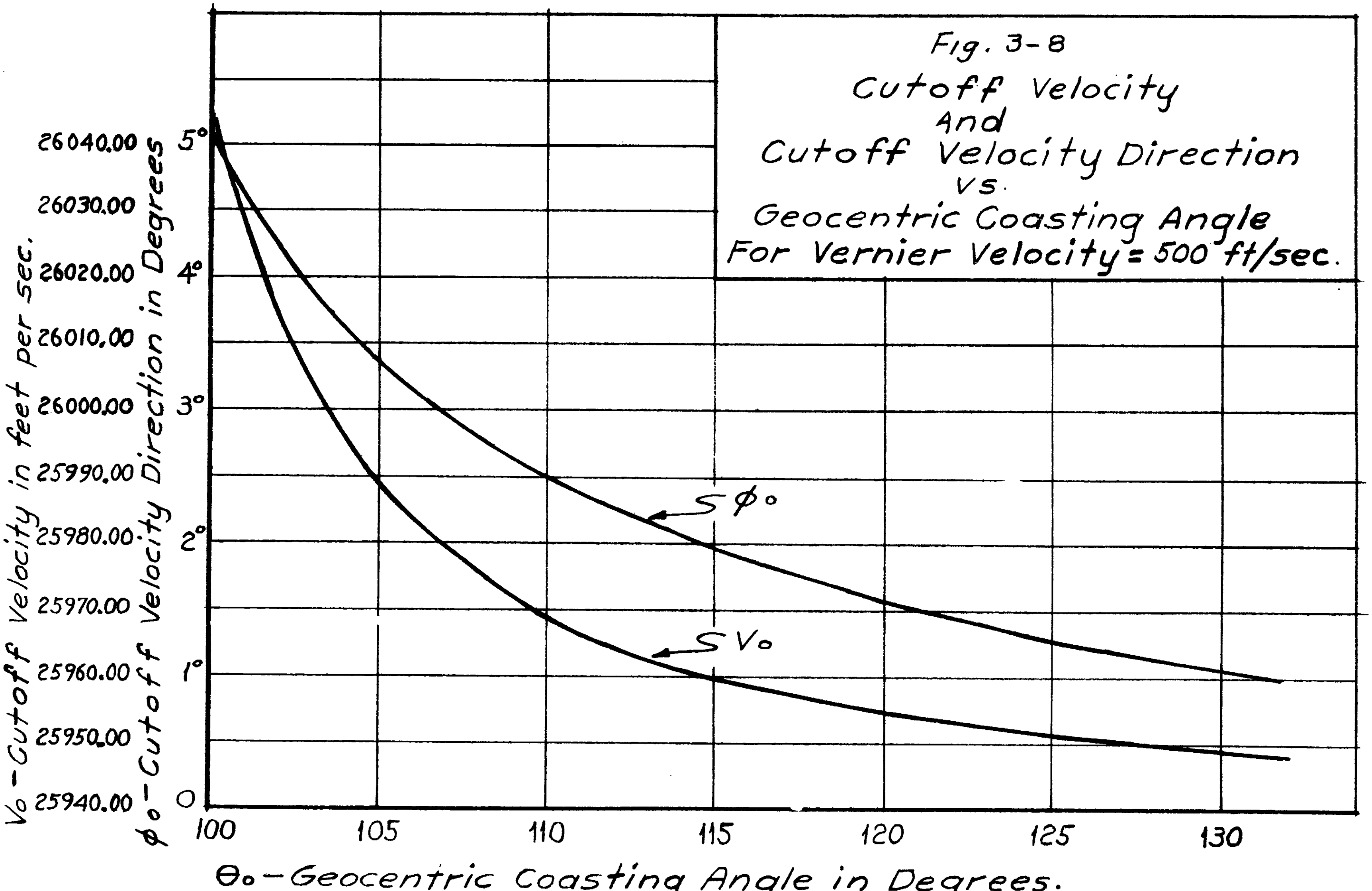


Fig. 3-8
Cutoff Velocity
And
Cutoff Velocity Direction
Vs.
Geocentric Coasting Angle
For Vernier Velocity = 500 ft/sec.

[redacted]

to coincide with this reference, rather the computers operate on the basic philosophy that the vehicle measured conditions differ from the reference set, and computations are performed based on these differences to still place the vehicle in a circular orbit.

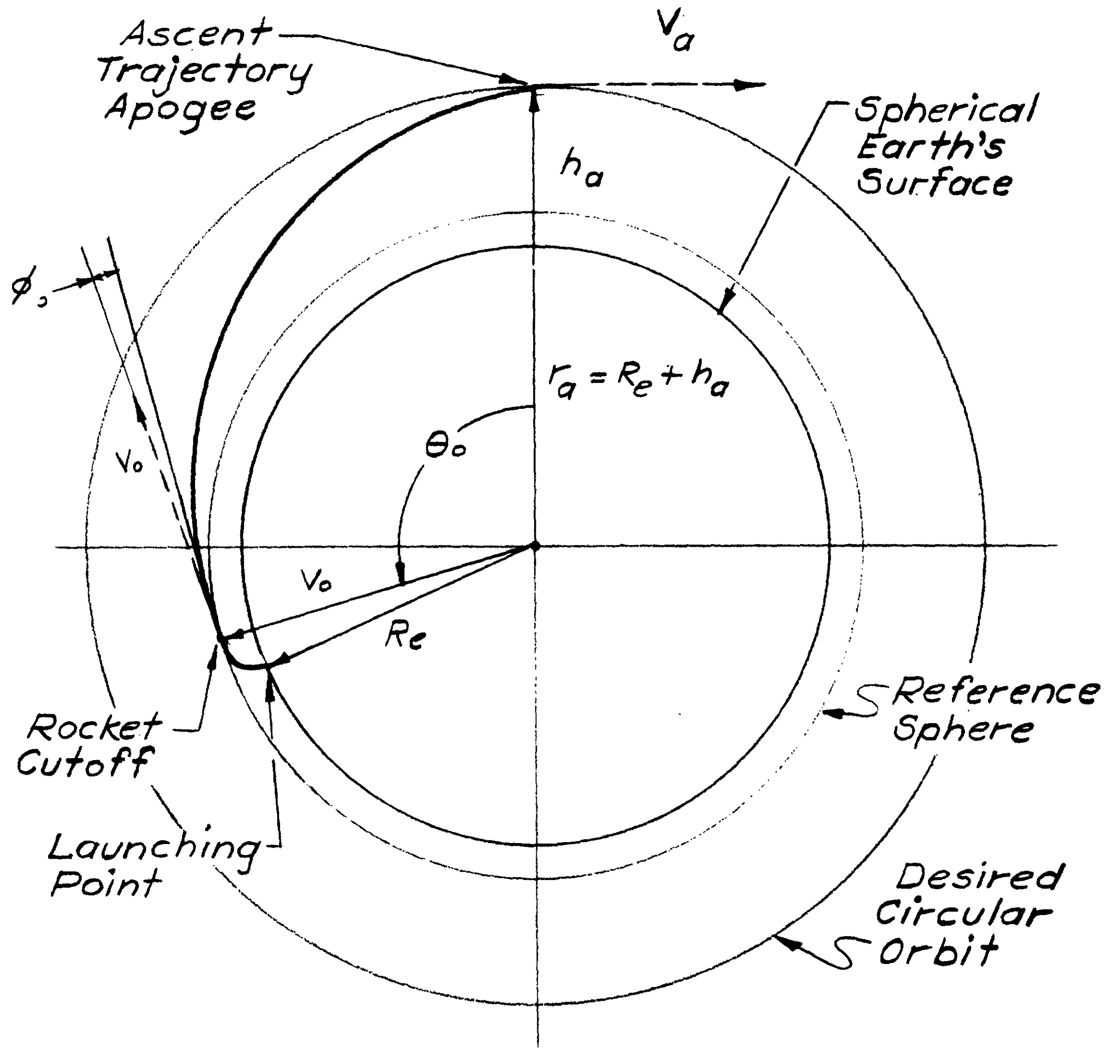
The reference trajectory which is actually to be used is immaterial, the important idea is the concept of considering perturbations about some reference set of cutoff conditions.

The choice of the reference trajectory will be made on the basis of an engineering compromise between many factors involved in the vehicle design. One such factor is the problem of carrying fuel for the vernier thrust stage at apogee, obviously this should tend toward a minimum. This implies that a reference trajectory be chosen on the basis of a minimum vernier velocity required. A second problem exists because of the assumption that the coasting portion of the ascent trajectory takes place in a vacuum and therefore aerodynamic forces could be neglected. Existing knowledge of atmospheric conditions at the altitudes of interest indicates this to be a reasonable assumption, nevertheless, it does not appear desirable to subject the vehicle to a long time of flight from cutoff to apogee during which any existing aerodynamic forces can act on the vehicle.

A more extensive study of the problem of an optimum ascent trajectory has been conducted by the Rand Corporation (Rand Memorandum 1207) and from the results of their studies, came a proposal of an optimum trajectory of ascent as one having a vernier velocity of 500 feet/second, a geocentric coasting angle of approximately 90°, and a time of flight from cutoff to apogee of about 25 minutes. The authors chose a reference trajectory based on a required vernier velocity of 500 feet/second, and a set of cutoff conditions which would provide such a required vernier velocity on the basis of the ascent trajectory study discussed previously. Figure 3-9 is a sketch showing the reference trajectory, and a table of parameters which determine it. The same data which was presented in Figures 3-5, 3-6, and 3-7 is shown on Figures 3-10, 3-11, and 3-12 for smaller increments of vernier velocity.

[redacted]

Fig. 3-9
Vehicle Coasting Ascent Phase With Reference Trajectory Parameters.



Reference Trajectory Parameter

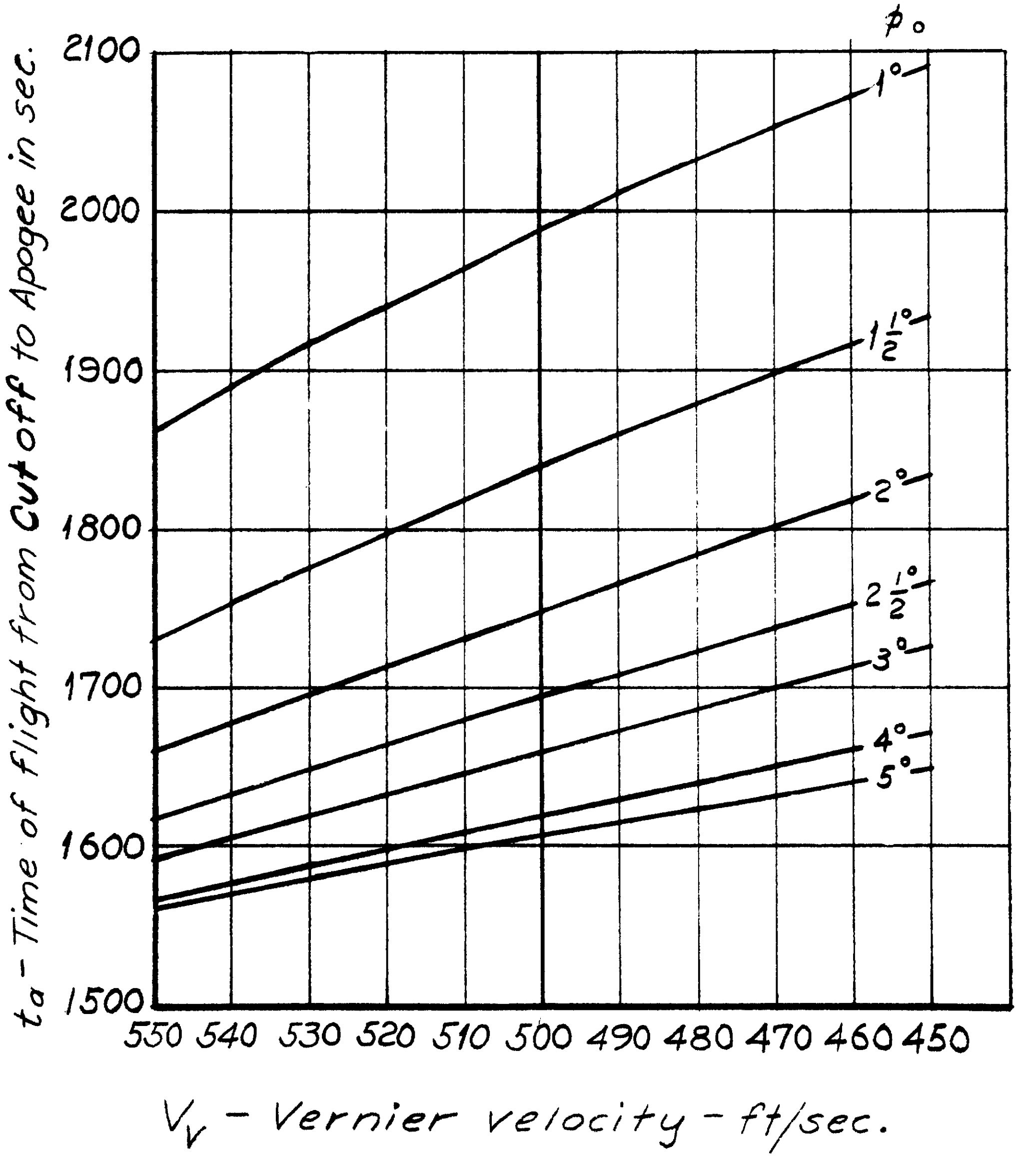
$R_e = 20.9264 \times 10^6$ ft
$h_a = 300$ miles
$= 1.22900 \times 10^6$ ft
$r_a = 22.51040 \times 10^6$ ft
$r_o = 21.28140 \times 10^6$ ft
$V_s = 25{,}028$ ft/sec
$E = GM_e = 1.41008 \times 10^{16}$ ft^3/sec^2

$\theta_o = 106°37'$
$\phi_o = 3°$
$V_o = 25{,}980$ ft/sec
$V_a = 24{,}528$ ft/sec
$V_r = 500$ ft/sec
$t_a = 1659.17$ sec
$= 27.653$ min

[redacted]

[redacted]

Fig. 3-10 Vehicle time of flight from Cutoff to Apogee vs Vernier Velocity for Constant Cutoff Velocity Direction



[redacted]

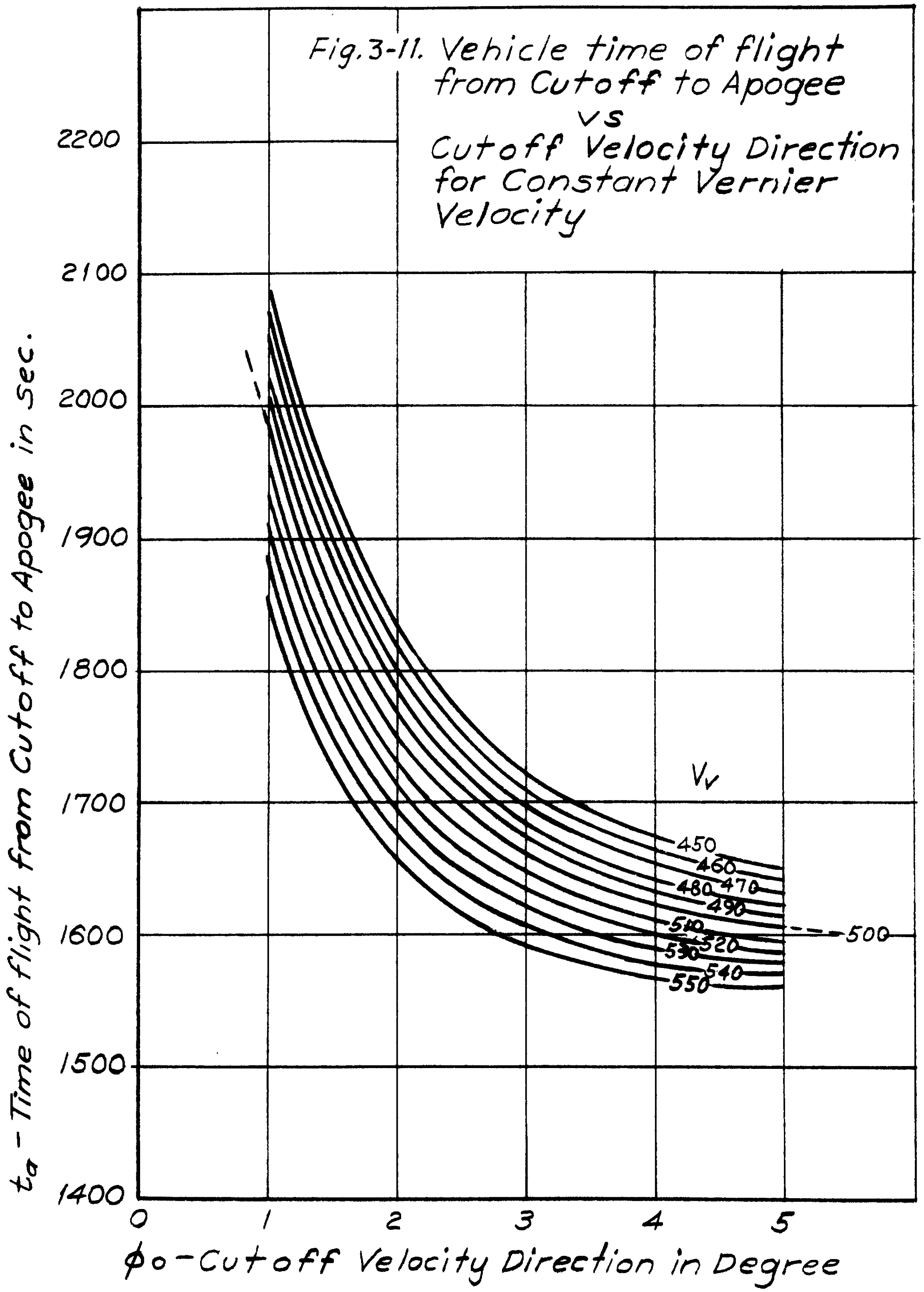


Fig. 3-11. Vehicle time of flight from Cutoff to Apogee vs Cutoff Velocity Direction for Constant Vernier Velocity

[redacted]

[redacted]

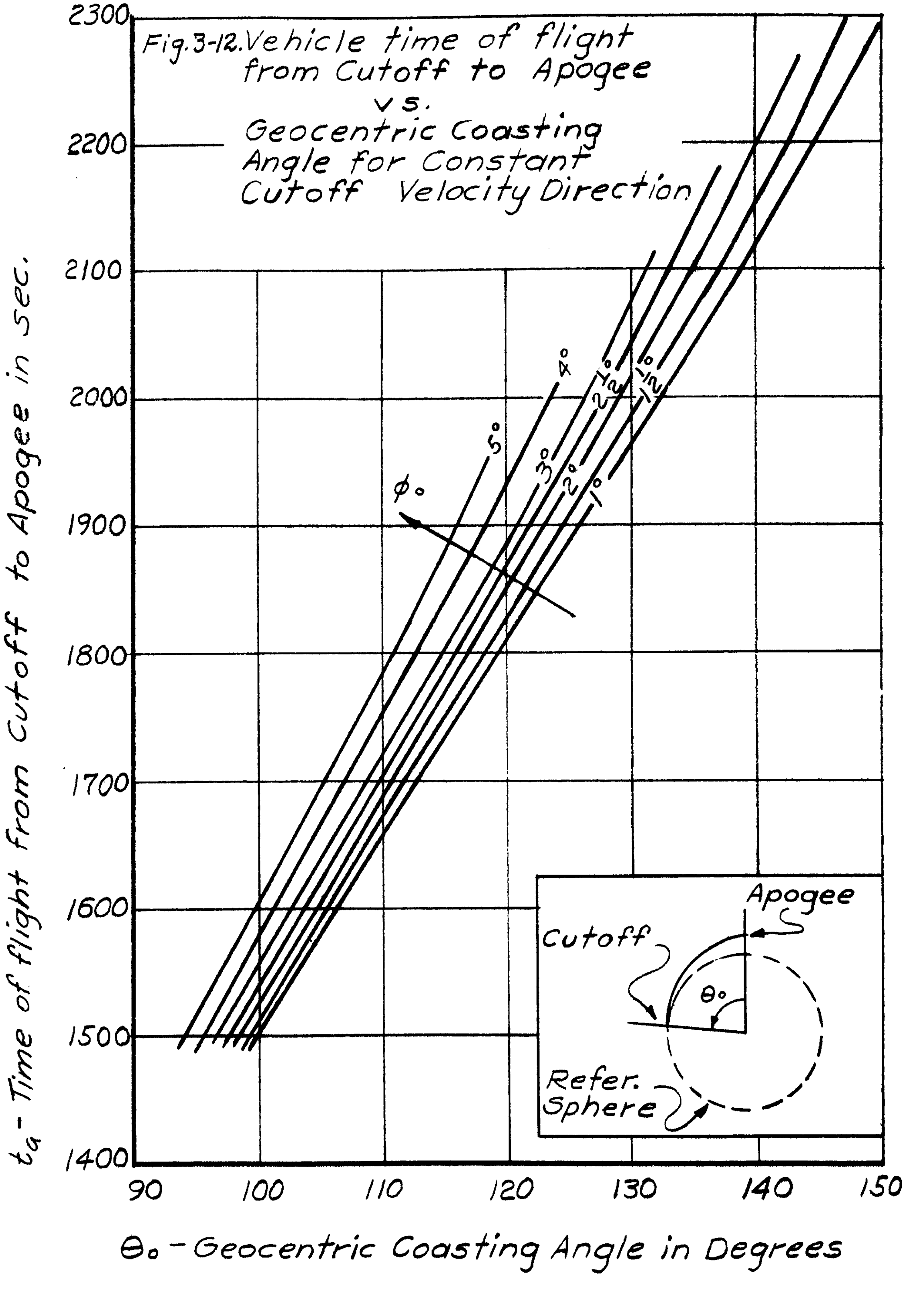


Fig. 3-12. Vehicle time of flight from Cutoff to Apogee vs. Geocentric Coasting Angle for Constant Cutoff Velocity Direction

[redacted]

An estimate of the weight of the fuel required for the vernier thrust is of the order of magnitude of 100 lbs., for the required 500 feet per second vernier velocity.

3.6 Utilization of the Reference Trajectory

The development of the computer equations is based on the concept of a Multi-variable Taylor's Series expansion of the vernier velocity and time of flight expressions about a reference set of cutoff conditions. The first term of the series is the vernier velocity and time of flight for the reference trajectory. The next group of terms in each expression is a change in the quantity from the reference value based on a linear difference between the measured and reference set of cutoff conditions, additional terms of the series result in changes of the quantity proportional to the square of the difference, etc. The reference set of quantities will be stored in the computers, and the coefficients of the other terms of the series used will act as coefficients on measurable differences at cutoff.

3.7 Summary

The purpose of this chapter was to introduce the concept of expanding the vernier velocity and time of flight from rocket cutoff to apogee expressions in a Taylor's series expansion about a reference set of cutoff conditions. The family of trajectories whose apogee are at an altitude of three hundred statute miles above the surface of the earth was investigated, and curves presented showing functional relationships between desired quantities and measurable cutoff quantities.

Based on a Rand recommendation of a reasonable vernier velocity of 500 feet/second, a set of cutoff conditions was arbitrarily chosen to determine a reference trajectory. Some of the problems leading to the choice of a reference trajectory to be used for an actual ascent were discussed.

The following two chapters will be devoted to the development of a vernier velocity computer and a time of flight to apogee computer based on the perturbation concept. An evaluation of these computers will be made, and block diagrams for their functional layout will be

[redacted]

included. Determination of the effect of errors in measurement of cutoff conditions on the computation will be discussed.

[redacted]

CHAPTER 4

VERNIER VELOCITY COMPUTER

4.1 Introduction

In Chapter 3, proposed instrumentation of a vernier velocity computer was to be accomplished by approximating the generalized expression for the required value of vernier velocity to place a vehicle in a 300 mile circular orbit about the earth by a multi-variable Taylor series expanded about some reference set of cut-off conditions. The coefficients of the Taylor series represent partial derivatives of the vernier velocity expression with respect to the cutoff conditions (r_o, V_o, ϕ_o). The magnitudes of the partials may be found by substituting the selected reference set of cutoff conditions (r_{oref}, V_{oref}, ϕ_{oref}) into the coefficient expressions and evaluating. An expression for the proposed vernier velocity will be found, and an evaluation of the accuracy for the expression simulated undertaken.

4.2 Generalized Vernier Velocity Expression

The vernier velocity, is defined as that velocity which must be added to the velocity of the vehicle at apogee in order to place it in a circular orbit about the earth at apogee altitude. This value will be computed in flight during the coasting ascent phase on the basis of measurable cutoff conditions. From the definition of vernier velocity:

$$V_v = V_s - V_a \tag{4-1}$$

where

V_v = vernier velocity
V_s = circular satellite velocity
V_a = vehicle velocity at apogee

An expression for the vernier velocity can be found in terms of cutoff conditions of the ballistic trajectory existing at burnout. A relation between the circular satellite velocity and the position vector from the center of earth may be found by equating the gravitation attraction force of the vehicle to its centrifugal force with respect to the center of the earth. The circular orbit imposes the condition that these two forces be equal, colinear and in opposite direction. The radius vector from the center of the earth is the sum of the altitude and the radius of the earth. From the equating of these forces, an expression for V_s can be found as follows:

$$V_s = \sqrt{E/r_a} \qquad (4\text{-}2)$$

where,

E_s = constant = GM_e on earth's surface

$r_a = R_e + h_a$

= radius of circular orbit from the center of the earth

From the Conservation of Angular Momentum, the following expressions relating missile apogee conditions to burnout conditions have been obtained. They are as follows:

$$V_a = \frac{E(1-e)}{r_o V_o \cos \phi_o} \qquad (4\text{-}3)$$

$$r_a = \frac{r_o^2 V_o^2 \cos^2 \phi_o}{E(1-e)}$$

where

V_a = vehicle velocity at apogee

V_o = vehicle velocity at cutoff

r_a = radius of circular orbit from the center of the earth

r_o = radial distance from the center of earth at burnout

ϕ_o = angle between velocity vector and local horizontal at burnout

[redacted]

e = eccentricity of elliptical trajectory
E = constant = 1.41008×10^{16} ft. 3/sec.^{2}

From Equations (4-2), (4-3) and (4-4) an expression for vernier velocity in terms of cutoff conditions is found to be:

$$V_v = \frac{E}{r_o V_o \cos\phi_o}\left[\sqrt{1 - e} - (1 - e)\right] \tag{4-5}$$

where

$$e = \frac{r_o V_o \cos\phi_o}{E}\sqrt{V_o^2 - \frac{2E}{r_o} + \frac{E^2}{r_o^2 V_o^2 \cos^2\phi_o}}$$

From Chapter 3,the amount of vernier velocity required can be found by expanding the necessary vernier velocity in a multi-variable Taylor series about some reference value. In addition if only the first term of the series is considered in the development of the vernier velocity computer, the computer equation is simply:

$$V_{v_{actual}} = V_{v_{ref}} \tag{4-6}$$

This implies that the value of vernier velocity precalculated for the reference trajectory would be near enough to the correct value, and no vernier velocity computer would be necessary. In the event that the vehicle does not possess the reference set of cutoff conditions at burn-out, no provisions would be available to compensate for the difference. As a result the vehicle would not attain the preferred orbit. If the differences at burn-out were of sufficient magnitude, the vehicle could realize an orbit off-set from the preferred orbit greater then 15 miles either side; equipment contained in the vehicle would not function properly.

To avoid this possibility, a computer can be developed to make corrections for any differences existing in actual cutoff conditions from the reference set.

From Equation (3-5) of Chapter 3 the vernier velocity in Equation (4-5) may be written as:

$$V_v = V_{v_{ref}} + \frac{\partial V_{v_{ref}}}{\partial r_o} \Delta r_o + \frac{\partial V_{v_{ref}}}{\partial V_o} \Delta V_o + \frac{\partial V_{v_{ref}}}{\partial \phi_o} \Delta \phi_o \qquad (4\text{-}7)$$

$$+ \text{ higher order terms}$$

where

$$\Delta r_o = r_o - r_{o_{ref}}$$

$$\Delta V_o = V_o - V_{o_{ref}}$$

$$\Delta \phi_o = \phi_o - \phi_{o_{ref}}$$

Appendix B contains the complete expressions for the first order partial derivatives of the vernier velocity in terms of cutoff conditions. The higher derivatives were omitted because of their length and complexity but can be obtained by successive differentiations of the first order partials.

4.4 Evaluation of the Partial Derivatives

For the problem of placing the vehicle in a particular orbit about the earth, namely 300 miles, another method exists simpler than that contained in Appendix B for getting the partial derivatives of interest. In order to attain an orbit of this height, the apogee of the vehicle coasting ascent trajectory was forced to a value of 300 miles above the earth for various sets of cutoff conditions through Equation (A-39) of Appendix A. In this manner the orbital velocity and distance from the center of earth can be treated as constants.

This being the case, the partial derivatives located in Equation (4-1) can be obtained in the following way. From Equation (4-1) the vernier velocity is seen to be:

$$V_v = V_s - V_a$$

from which the quantities V_s and V_a may be eliminated by use of Equations (4-2) and (A-38). These expressions are:

$$V_s = \sqrt{E/r_a}$$

[redacted]

$$V_a = \frac{r_o V_o}{r_a} \cos \phi_o$$

Substitution of these expressions into V_v gives:

$$V_v = \sqrt{E/r_a} - \frac{r_o V_o}{r_a} \cos \phi_o \qquad (4\text{-}8)$$

where

E, r_a are constants

The partial derivatives of Equation (4-8) with respect to cutoff conditions r_o, V_o, ϕ_o through the second order terms are as follows:

$$\frac{\partial V_v}{\partial r_o} = - \frac{V_o}{r_a} \cos \phi_o \qquad (4\text{-}9)$$

$$\frac{\partial V_v}{\partial V_o} = - \frac{r_o}{r_a} \cos \phi_o \qquad (4\text{-}10)$$

$$\frac{\partial V_v}{\partial \phi_o} = \frac{r_o V_o}{r_a} \sin \phi_o \qquad (4\text{-}11)$$

$$\frac{\partial^2 V_v}{\partial r_o^2} = 0 \qquad (4\text{-}12)$$

$$\frac{\partial^2 V_v}{\partial V_o^2} = 0 \qquad (4\text{-}13)$$

$$\frac{\partial^2 V_v}{\partial \phi_o^2} = \frac{r_o V_o}{r_a} \cos \phi_o \qquad (4\text{-}14)$$

$$\frac{\partial^2 V_v}{\partial V_o \partial r_o} = - \frac{1}{r_a} \cos \phi_o \qquad (4\text{-}15)$$

$$\frac{\partial^2 V_v}{\partial \;{}_o \partial r_o} = \frac{V_o}{r_a} \sin \phi_o \qquad (4\text{-}16)$$

[redacted]

$$\frac{\partial^2 V_v}{\partial r_o \partial V_o} = -\frac{1}{r_a} \cos \phi_o \qquad (4\text{-}17)$$

$$\frac{\partial^2 V_v}{\partial \phi_o \partial V_o} = \frac{r_o}{r_a} \sin \phi_o \qquad (4\text{-}18)$$

$$\frac{\partial^2 V_v}{\partial r_o \partial \phi_o} = \frac{V_o}{r_a} \sin \phi_o \qquad (4\text{-}19)$$

$$\frac{\partial^2 V_v}{\partial V_o \partial \phi_o} = \frac{r_o}{r_a} \sin \phi_o \qquad (4\text{-}20)$$

The partial derivatives can then be evaluated for the reference set of cutoff conditions found in Figure 3-9. The results can be found tabulated in Table 4-1.

An examination of Table 4-1 indicates that as a first approach to the development of a vernier velocity computer only the first order terms of the Taylor series expansion need be considered provided sufficient accuracy is obtainable. From the standpoint of computer instrumentation, the fewer terms of the Taylor series that need be considered, the simpler the computer. If the evaluation of computer should fail to provide the necessary accuracy, additional terms of the series may be used to more closely approximate the function.

4.5 Vernier Velocity Computer Equation

Considering only the first order terms of the Taylor series, the vernier velocity computer equation is:

$$V_v = V_{v_{ref}} + A\Delta r_o + B\Delta V_o + C\Delta\phi_o \qquad (4\text{-}21)$$

where

$$A = \frac{\partial V_{v_{ref}}}{\partial r_o}$$

[redacted]

TABLE 4-1

Partial Derivative Values			
1st Order Term	2nd Order Term		
$\frac{\partial V_{v_{ref}}}{\partial r_o} = -1.15 \times 10^{-3}$	$\frac{\partial^2 V_{v_{ref}}}{\partial r_o^2} = 0$	$\frac{\partial^2 V_{v_{ref}}}{\partial V_o \partial r_o} = -4.43 \times 10^{-8}$	$\frac{\partial^2 V_{v_{ref}}}{\partial \phi_o \partial r_o} = -1.05 \times 10^{-6}$
$\frac{\partial V_{v_{ref}}}{\partial V_o} = -0.945$	$\frac{\partial^2 V_{v_{ref}}}{\partial V_o^2} = 0$	$\frac{\partial^2 V_{v_{ref}}}{\partial r_o \partial V_o} = -4.43 \times 10^{-8}$	$\frac{\partial^2 V_{v_{ref}}}{\partial \phi_o \partial V_o} = 8.64 \times 10^{-4}$
$\frac{\partial V_{v_{ref}}}{\partial \phi_o} = 22.5$	$\frac{\partial^2 V_{v_{ref}}}{\partial \phi_o^2} = 7.48$	$\frac{\partial^2 V_{v_{ref}}}{\partial r_o \partial \phi_o} = 1.05 \times 10^{-6}$	$\frac{\partial^2 V_{v_{ref}}}{\partial V_o \partial \phi_o} = 8.64 \times 10^{-4}$

V = ft/sec , r = ft , ϕ = degrees

[redacted]

$$B = \frac{\partial V_{v_{ref}}}{\partial V_o}$$

$$C = \frac{\partial V_{v_{ref}}}{\partial \phi_o}$$

From Table 4-1, the numerical values of the first order partial derivatives can be substituted into the computer equation which can then be rewritten as:

$$V_v = V_{v_{ref}} - 1.15 \times 10^{-3} \Delta r_o - 0.95 \Delta V_o + 22.5 \Delta \phi_o \qquad (4\text{-}22)$$

From this equation it can be seen that for the Δr_o term to become appreciable, $\Delta r_o \geqq$ 1000 ft. It is assumed that the ascent thrust guidance system will operate such as to have rocket cutoff occur within a thousand feet of the reference, hence this term can be neglected from the following analysis.

The remainder of the chapter will be devoted to evaluating the vernier velocity computer over a reasonable range of cutoff values from the preferred set, and to discuss errors associated with the computer equation.

4.6 Accuracy of Mathematical Model Instrumented

In the evaluation of the vernier velocity computer equation, a maximum deviation in vehicle cutoff velocity and velocity direction of $\pm$ 40 ft/sec and $\pm \frac{1}{2}$ degree will be considered.

The method of evaluation will be to introduce various values of ΔV_o and $\Delta \phi_o$ into the computer Equation (4-21) and then solve the equation for the amount of vernier velocity that the vehicle would receive upon reaching apogee from the computer assuming ideal operation. The computed value will then be compared against the known value of vernier velocity required to place the vehicle in an intended orbit about the earth. The known value was obtained by programming the generalized expression for vernier velocity on the Card Programmed Calculator (IBM digital computer) located at the M.I.T. Instrumentation Laboratory.

[redacted]

[redacted]

From Chapter 3 the accuracy requirements on the vernier velocity computer was established on the basis that the vehicle vernier velocity be controlled at apogee to better then 25 ft/sec, and the velocity direction controlled to better than 20 minutes of arc. To provide control to this order of magnitude, it is assumed that measuring equipment is available in the vehicle for measuring cutoff conditions to the nearest foot per second in vehicle velocity and the nearest milliradian in velocity direction.

The computed values for the range considered all fall within the limit of accuracy established in Chapter 3 i.e., for ΔV_o = 40 ft/sec and $\Delta \phi_o$ = 1/2 degree, the computer value was found to be within 1.5 ft/sec of the correct value. The results of the computer evaluation assuming ideal components and no error introduced due to measuring cutoff conditions can be found in Table 4-2.

4.7 Computer Equation Errors

The errors that arise out of the computer equation are the result of one's going from a physical situation to a mathematical model. This type of error is generally referred to as an error in formulation. In the proposed computer an error stems from the omission of terms greater then first order in the Taylor series from the computer equation.

Nevertheless, the computer equation generates an answer that is within 1.5 ft/sec of the correct value over the entire range of cutoff conditions considered. One point of interest extrapolated from the results of the evaluation is the consistency with which the computed value differs from the known value between 1 and 0.50 ft/sec. Assuming the value that was obtained from the Card Programmed Calculator is an "actual" value, the error associated with the computer equation is a result of neglecting the higher order terms of the series.

This being the case, a bias value could be introduced into the vernier velocity computer to reduce the computer equation error.

[redacted]

VERNIER VELOCITY ERRORS

TABLE 4-2

ϕ_o	V_o	$V_{v(act)}$	$V_{v(comp)}$	$(E)V_v$
2°40'	25946	525	524.6	-0.4
2°50'	25950	525	524.7	-0.3
3°	25954	525	524.6	-0.4
3°10'	25958	525	524.6	-0.4
3°20'	25962	525	524.5	-0.5
2°40'	25957	515	514.2	-0.8
2°50'	25960	515	515.2	+0.2
3°	25964	515	515.1	+0.1
3°10'	25968	515	515.1	+0.1
3°20'	25972	515	514.1	-0.9
2°40'	25967	505	504.8	-0.2
2°50'	25971	505	504.8	-0.2
3°	25975	505	504.7	-0.3
3°10'	25979	505	504.7	-0.3
3°20'	25983	505	504.7	-0.3
2°40'	25973	500	449.1	-0.9
2°50'	25976	500	500.1	+0.1
3°	25980	500	500.0	0.0
3°10'	25984	500	500.0	0.0
3°20'	25988	500	499.0	-1.0
2°40'	25978	495	494.6	-0.4
2°50'	25982	495	494.4	-0.6
3°	25985	495	495.3	+0.3
3°10'	25989	495	494.3	-0.7
3°20'	25994	495	494.3	-0.7
2°40'	25989	485	484.0	-1.0
2°50'	25992	485	485.0	0.0
3°	25996	485	484.9	-0.1
3°10'	26000	485	484.9	-0.1
3°20'	26009	485	484.8	-0.2
2°40'	25999	475	474.6	-0.4
2°50'	26003	475	474.5	-0.5
3°	26007	475	474.5	-0.5
3°10'	26011	475	474.5	-0.5
3°20'	26015	475	474.6	-0.4

Where: ϕ_{oref} = 3 degrees

$$V_{v(comp)} = V_{v(ref)} - 0.945\,\Delta r_o - 22.5\,\Delta\phi_o$$

$$(E)V_v = V_{v(comp)} - V_{v(act)}$$

[redacted]

[redacted]

4.8 Summary

In this chapter the development of the vernier velocity equation was undertaken in generalized form. From this the equation of the proposed computer considering only the first order terms of the Taylor series expansion was developed. An evaluation of the equation simulated was then undertaken for a range of cutoff conditions differing from those of the reference set. The accuracy of the mathematical model was found to be adequate from the standpoint of computer requirements determined in Chapter 3. The error in computed vernier velocity was found to be less than 1 ft/sec which would have a very small effect on orbit distortion. The error contained in the proposed vernier velocity computer equation due to considering only the first order terms of the multi-variable Taylor series was discussed and a method of reducing the error pointed out. However, prior to introducing a bias to correct for this error the computations involved in checking computer accuracy should be carried out to a higher degree of accuracy to reduce the effect of round off errors.

[redacted]

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CHAPTER 5

TIME OF FLIGHT FROM CUTOFF TO APOGEE COMPUTER

5.1 Introduction

In Chapter 3, proposed instrumentation of a time of flight from cutoff to apogee computer was to be accomplished by approximating the expression for time of flight during coasting by a multi-variable Taylor's series expanded about some reference set of cutoff conditions. In Appendix C, the expressions for the partial derivatives of the time of flight expression with respect to the cutoff conditions (r_o, v_o, ϕ_o) may be found. The magnitudes of the partial derivatives may be found by substituting the reference set of cutoff conditions. However another means of partial derivative evaluation is available which will indicate the degree of approximation of the function by taking only a limited number of terms. This method will be used. An expression for the computer will be found, and a block diagram of the the computers will be included along with an evaluation of the accuracy of the expressions simulated. Propagation of errors in measured cutoff conditions through the computers will be discussed and accuracy requirements of computer components will be determined.

5.2 Evaluation of Computer Coefficients

It is proposed to approximate the time of flight expression by a linear combination of terms involving small differences in measured cutoff conditions from some reference set. The expression to be simulated is of the form:

$$t_a = A \Delta r_o + B \Delta v_o + C \Delta \phi_o + t_{ref} \qquad (5-1)$$

where:

$$A = \frac{\partial t_a}{\partial r_o}$$

$$B = \frac{\partial t_a}{\partial v_o}$$

$$C = \frac{\partial t_a}{\partial \phi_o}$$

t_r = reference time of flight

The partial derivative of time of flight with respect to initial position was evaluated by substitution into the expression (C-6) from Appendix C and was found to be of the order of magnitude of 1 second change in time of flight for a thousand foot deviation in initial position from the reference position. It is assumed that the ascent thrust guidance system will operate such as to have rocket cutoff occur within a few thousand feet of the reference, hence this term is neglected in the time of flight computer to simplify instrumentation. In the event that larger changes in initial position are to be anticipated, an additional linear term may be later added for computer simulation. The other partial derivatives will be evaluated on the assumption that the initial cutoff position is the reference cutoff position.

From Chapter 3 the accuracy requirement on the time of flight computer was established on the basis that the vehicle velocity be controlled at apogee to better than 25 fps. at apogee, and the velocity direction controlled to better than 20 minutes of arc. To provide control to this order of magnitude, it is assumed that measuring equipment is available in the vehicle for measuring cutoff conditions to the nearest foot per second in vehicle velocity and the nearest milliradian in velocity direction.

Using the Card Programmed Calculator, the time of flight was calculated on the basis of Equation (2-16) for various values of vernier velocity ranging from 525 to 475 feet per second in 5 fps intervals, while varying velocity direction at cutoff from 2 $1/2^o$ to 3 $1/2^o$ in 10 minutes of arc intervals. A plot of time of flight from cutoff to apogee vs. cutoff velocity (Figure 5-1) shows that time of flight is directly proportional to cutoff velocity. The coefficient B in Equation 5-1 is

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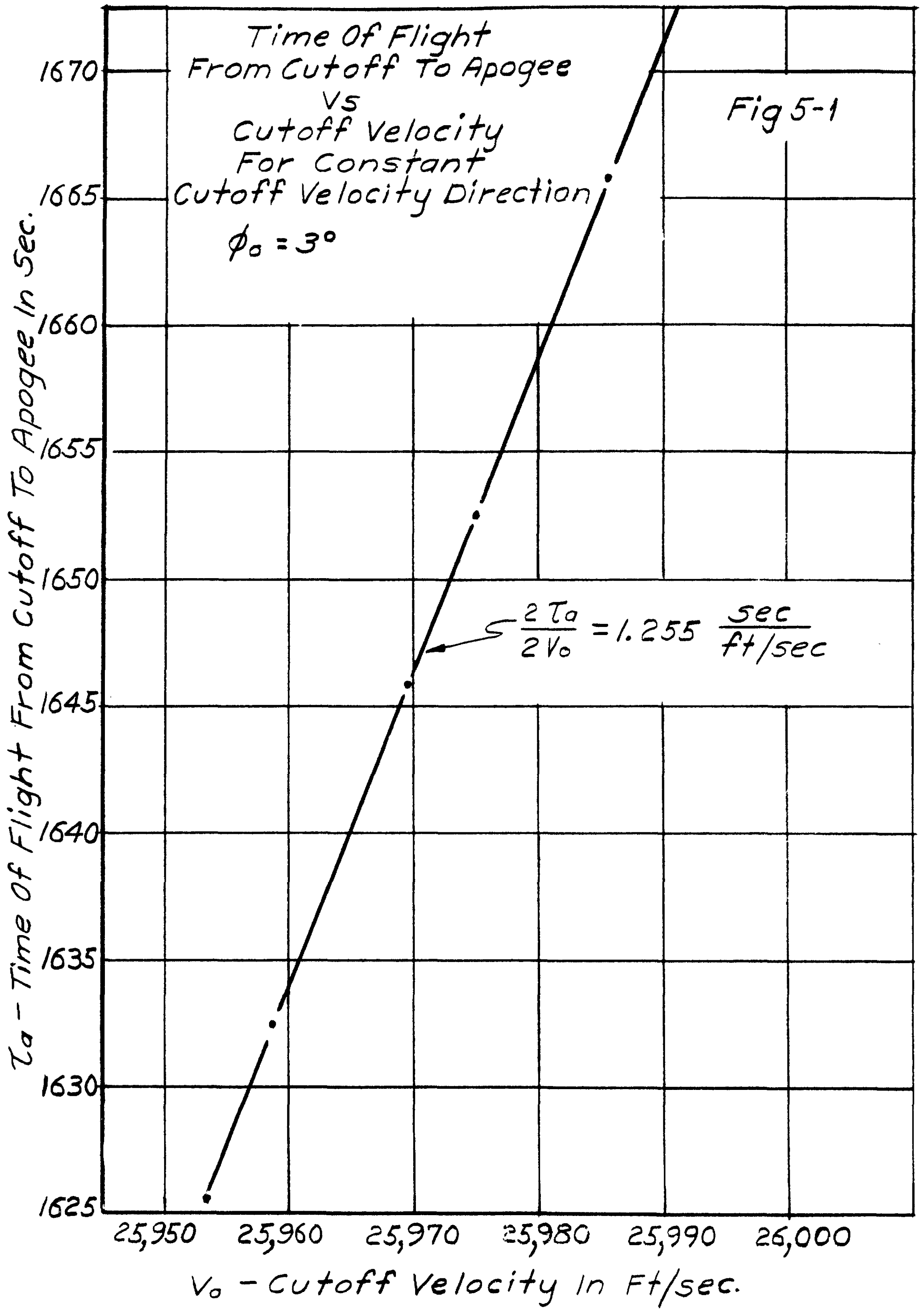


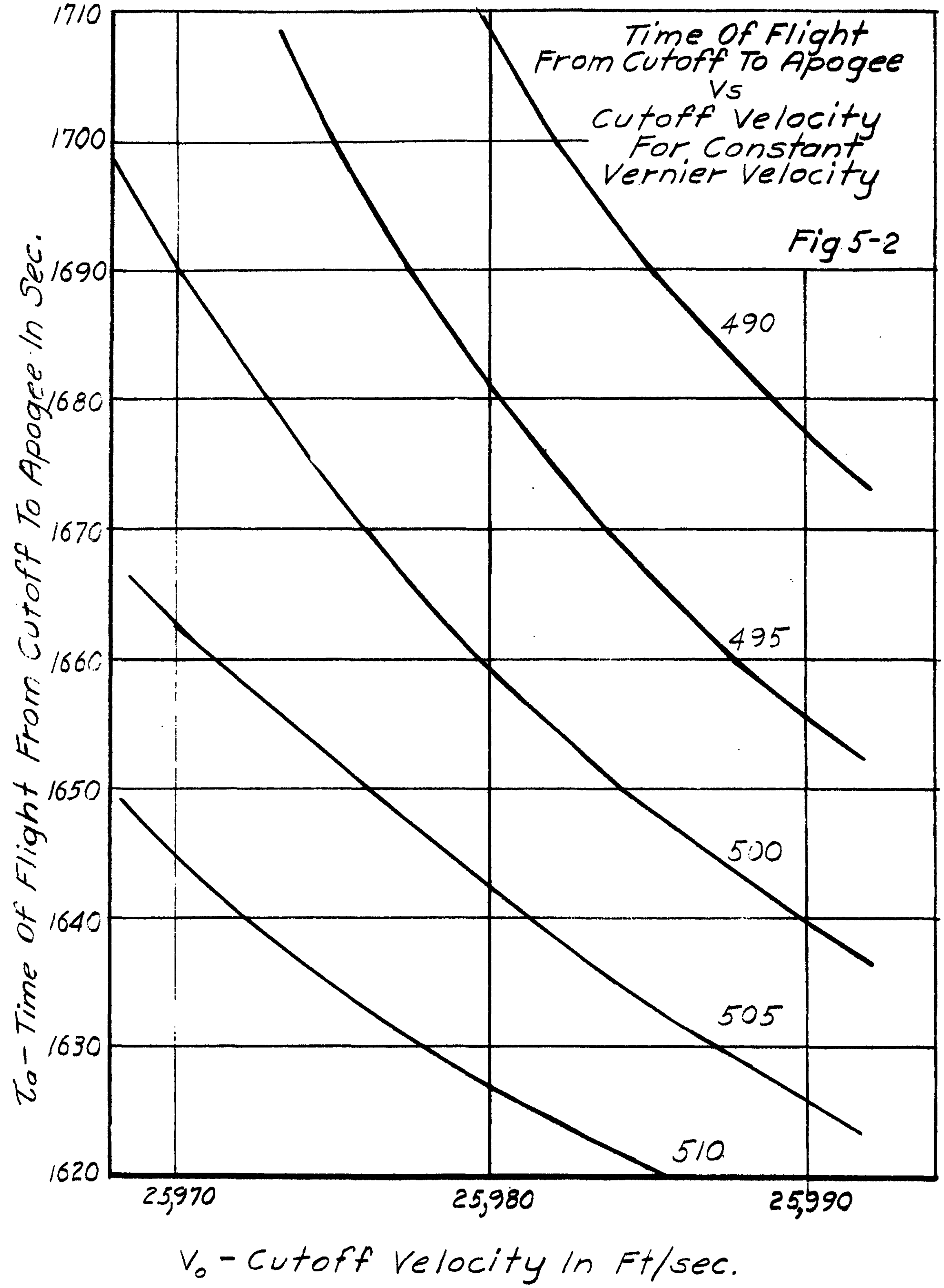
[redacted]

the slope of the t_a vs. v_o curve, and is found to be 1.255 sec/foot per second. A plot of time of flight from cutoff to apogee vs. cutoff velocity for constant vernier velocity (Figure 5-2) and a plot of time of flight from cutoff to apogee vs. cutoff velocity direction for constant vernier velocity (Figure 5-3) may be read simultaneously to yield a plot of time of flight from cutoff to apogee vs. cutoff velocity direction for constant cutoff velocity. This curve, also on Figure 5-3 shows the relation between time of flight from cutoff to apogee and the cutoff velocity direction which the coefficient C in Equation (5-1) assumes to be linear. From the curve, the slope in the vicinity of the reference cutoff velocity direction of 3^o is approximately 90 seconds/degree. The slope of the t_a vs. ϕ_o curve for constant v_o does not change radically in the range of interest, hence the curvature may be neglected, and only linear variation need be considered. In the event that a more accurate approximation to the curve is desired, a polynomial approximation to the curve may be made which would make the time increment due to perturbations of cutoff velocity direction from the reference proportional to the square of the difference in addition to the linear difference. It is felt that the additional accuracy in the approximation of the time of flight expression obtained by the polynomial representation does not warrant the additional complexity which would be required in the computer instrumented on this basis.

5.3 Accuracy of Mathematical Model Instrumented

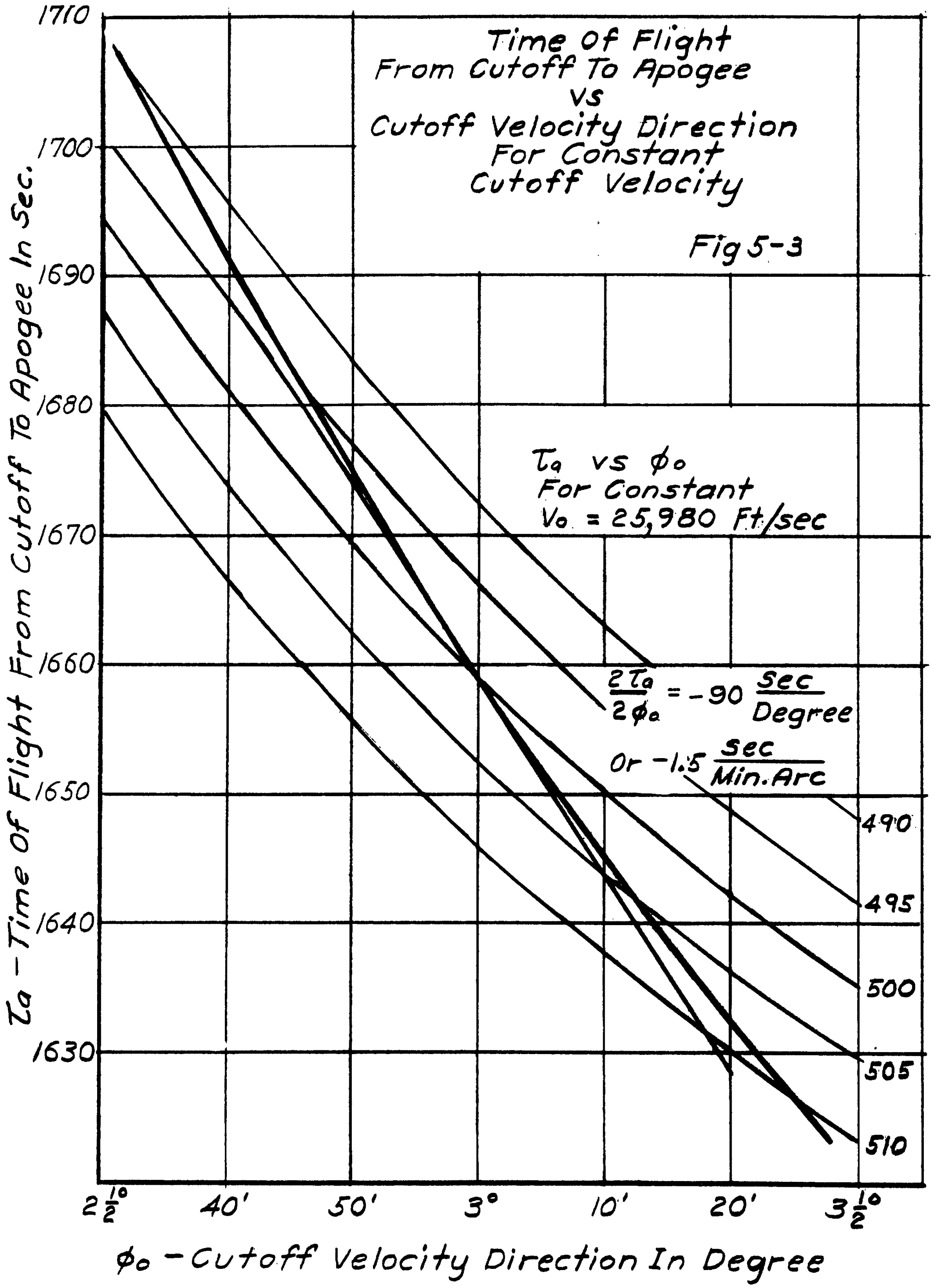
The time of flight from cutoff to apogee using the actual expression has been calculated using the CPC, for various values of v_o and ϕ_o. Using these same perturbations on initial conditions at cutoff, a time of flight was evaluated using the approximate expression which is to be simulated by the computer. The difference between the computer time of flight and the actual time of flight is an error due to the fact that the computer expression is only an approximation. The error in computed time of flight is plotted vs. difference in cutoff velocity for $\pm$ 20 minutes of arc difference in cutoff velocity direction. (See Table 5-1) Inspection of the table indicates that the maximum error in computer time of flight is -6.0 seconds for maximum perturbations of cutoff conditions.

[redacted]



Time Of Flight
From Cutoff To Apogee
Vs
Cutoff Velocity
For Constant
Vernier Velocity
Fig 5-2
490
495
500
505
510
1710
1700
1690
1680
1670
1660
1650
1640
1630
1620
25,970
25,980
25,990
τ_a - Time Of Flight From Cutoff To Apogee In Sec.
V_o - Cutoff Velocity In Ft/sec.

[redacted]



[redacted]

[redacted]

TABLE 5-1

ϕ_o	V_o	$t_{a(act)}$	$t_{a(comp)}$	$(E)t_a$
2°40'	25946	1645	1646	+1
2°50'	25950	1634	1636	+2
3°	25954	1626	1626	0
3°10'	25958	1618	1616	-2
3°20'	25962	1612	1606	-6
2°40'	25957	1659	1660	+1
2°50'	25960	1649	1649	0
3°	25964	1639	1639	0
3°10'	25968	1631	1629	-2
3°20'	25972	1624	1620	-4
2°40'	25967	1674	1673	-1
2°50'	25971	1662	1663	+1
3°	25975	1653	1653	0
3°10'	25979	1644	1643	-1
3°20'	25983	1636	1633	-3
2°40'	25973	1681	1680	-1
2°50'	25976	1669	1669	0
3°	25980	1659	1659	0
3°10'	25984	1650	1649	-1
3°20'	25988	1642	1639	-3
2°40	25978	1688	1687	-1
2°50'	25982	1676	1677	+1
3°	25985	1666	1665	-1
3°10'	25989	1657	1655	-2
3°20'	25994	1649	1647	-2
2°40'	25989	1703	1700	-3
2°50'	25992	1690	1689	-1
3°	25996	1679	1679	0
3°10'	26000	1669	1699	0
3°20'	26009	1661	1659	-2
2°40'	25999	1717	1713	-4
2°50'	26003	1704	1702	-2
3°	26007	1692	1693	+1
3°10'	26011	1682	1683	+1
3°20'	26015	1673	1673	0

Where: $\phi_{oref} = 3^\circ$ $V_{oref} = 25980$fps $t_{aref} = 1659$ sec

$$t_{a(comp)} = t_{aref} - 90\,\Delta\phi_o + 1.255\,\Delta V_o$$

$$(E)t_a = t_{a(comp)} - t_{a(act)}$$

5.4 Block Diagram of Proposed Vernier Velocity and Time of Flight to Apogee Computers

Figure 5-4 is a layout for the computers simulated on the basis of the Taylor's series approximations for the vernier velocity and time of flight to apogee functions.

The measuring system is part of the thrust phase guidance system which will continually indicate velocity magnitude and direction during rocket burning. The computers will accept data at the instant of rocket cutoff, and compute the required amount of vernier velocity required at apogee of the ascent trajectory, and compute the time at which this vernier thrust must be imparted.

Electrical signals proportional to velocity magnitude and direction will be fed into a pair of summing amplifiers, which will have output voltages proportional to the difference between measured and reference conditions. Signals proportional to these perturbations will be attenuated by a constant amount by potentiometers, and summed with a reference quantity to yield a signal proportional to the desired quantity.

The vernier velocity signal will serve as a command for a thrust control system which will control the rocket motors that will be fired at a time which is indicated by the time of flight computer.

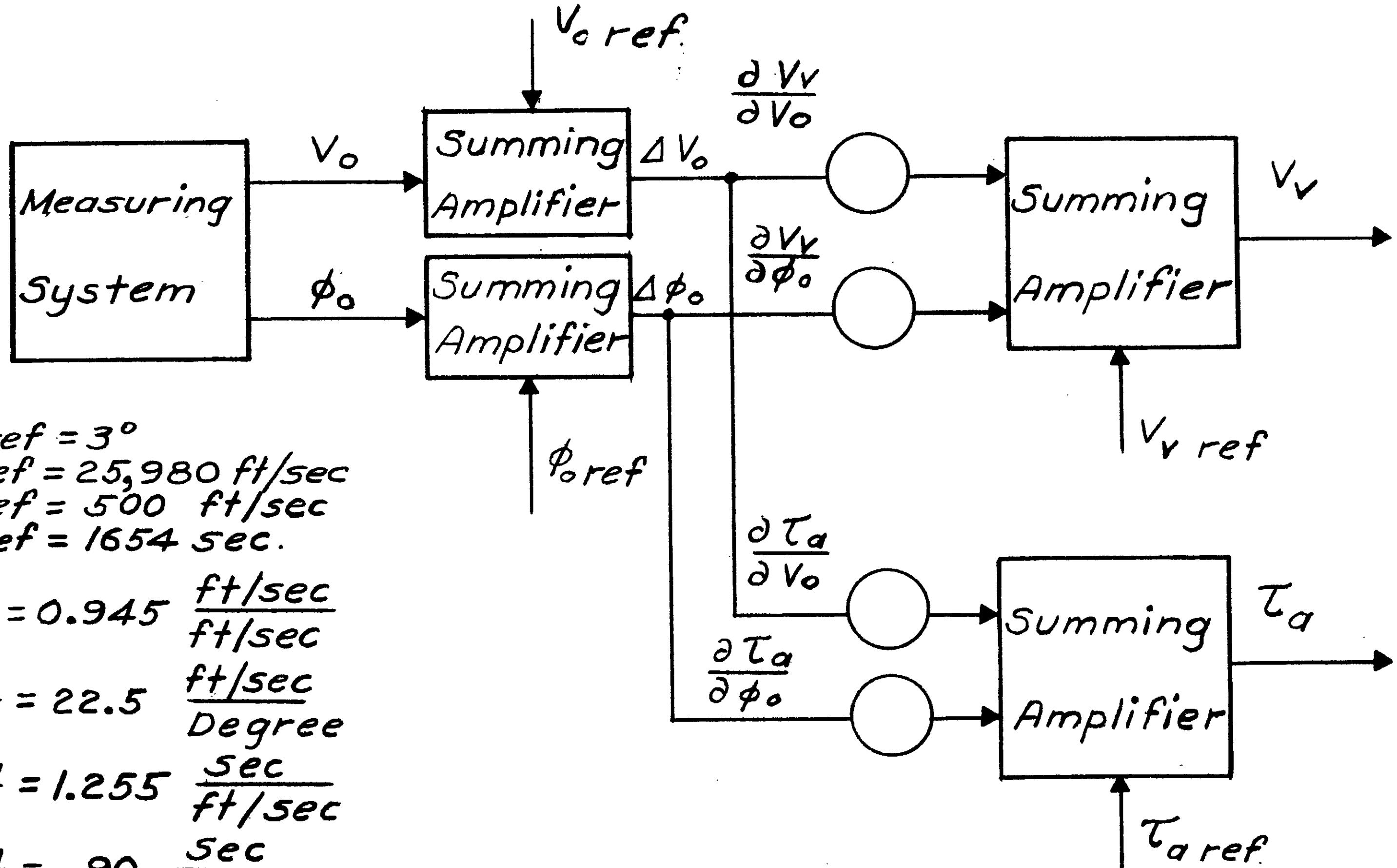
5.5 Propagation of Errors in Measured Cutoff Conditions

Errors in measured cutoff conditions will be propagated through the computers. Errors in measured cutoff velocity of 1 foot/second and .1 degree in measured cutoff velocity direction were chosen as representative. Table 5-2 shows the resulting errors in vernier velocity and time of flight computed.

TABLE 5-2

Measurement Error	Vernier Velocity Error	Time of Flight Error
$(E)V_o$ = 1 fps.	1 fps	1.25 sec
$(E)\phi_o$ = .1 deg.	2 fps	9 sec

Fig 5-4 Proposed Vernier Velocity & Time Of Flight To Apogee Computers.



$\phi_o ref = 3°$
$V_o ref = 25{,}980$ ft/sec
$V_v ref = 500$ ft/sec
$\tau_a ref = 1654$ sec.

$$\frac{\partial V_v}{\partial V_o} = 0.945 \ \frac{ft/sec}{ft/sec}$$

$$\frac{\partial V_v}{\partial \phi_o} = 22.5 \ \frac{ft/sec}{Degree}$$

$$\frac{\partial \tau_a}{\partial \phi_o} = 1.255 \ \frac{sec}{ft/sec}$$

$$\frac{\partial \tau_a}{\partial \phi_a} = -90 \ \frac{sec}{Degree}$$

[redacted]

5.6 Summary

A vernier velocity computer, and a time of flight from cutoff to apogee computer have been instrumented utilizing relatively simple mathematical expressions relating these quantities to measurable cut-off conditions V_o and ϕ_o. Very definite limitations have been imposed on the amount that the measured conditions may differ from the reference set in order to achieve a certain percentage accuracy in the computer outputs. Greater computer accuracy may be achieved by addition of more terms to the equations simulated. It is felt that the critical shortage of space and weight in a vehicle for orbiting the earth will necessitate a compromise between accuracy required of computers such as these and additional equipment required to achieve that accuracy. The maximum error in computed time of flight was found to be 6 seconds.

[redacted]

[redacted]

CHAPTER 6

THRUST ORIENTATION CONTROL SYSTEM

6.1 Introduction

During the coasting ascent phase of the vehicle's flight to the orbit, it is necessary to orient the vernier thrust rocket so that it will be aligned with the vehicle velocity vector at apogee. In Chapter 1, several means for mounting of the vernier rocket in the vehicle were suggested, and a decision was made that the rocket was to be rigidly mounted in the vehicle to be fired out of the tail (Case 1) or out of the side (Case 2). Orientation of the thrust for a rigid mounting of the vernier rocket requires that the vehicle be reoriented during coasting flight.

It is assumed that the control system will maintain the vehicle yaw and roll angles nulled to zero, and only turning about the pitch axis is necessary for proper vehicle alignment. A derivation of the equations of motion for the vehicle about its center of mass under the action of an interfering torque and a flywheel orientation torque will be presented. An analysis of a control system for vehicle orientation will be performed. It is assumed that the required angular orientation is known either by direct measurement or computation for Case 1 or 2.

6.2 Definition of Axis Systems

Considering the center of the earth (E) to be an inertial point. Let the:

Z axis be directed along the position vector $\overline{r}$

X axis be perpendicular to Z and lying in the plane of the trajectory

[redacted]

Y axis be orthogonal to both to form a right handed axis system. (See Figure 6-2)

The vehicle axes x, y, z are principal axes with their origin located at the center of mass of the vehicle. It is presumed that the vernier thrust will be imparted out of the tail of the vehicle or out of the side of the vehicle, so in order to make the following analysis apply for both cases, define the axes as follows.

Let the:

x axis be directed along the negative vernier thrust axis of the vehicle

y and z axes be principal axes aligned to form a right handed axis system.

AXIS SYSTEMS

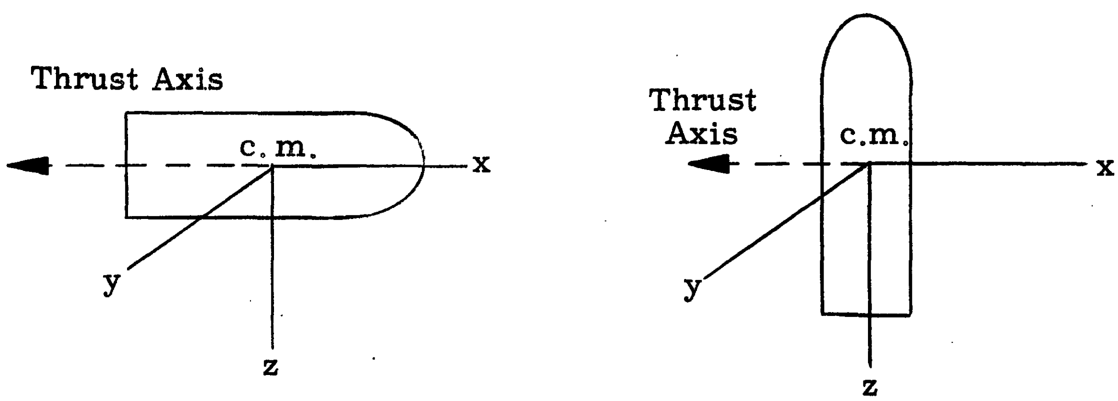


FIGURE 6-1

6.3 Equations of Motion about Vehicle Center of Mass

The total momentum of the system consists of the momentum of the vehicle and the flywheels:

$$\overline{H}_t = \overline{H}_v + \overline{H}_w \tag{6-1}$$

The time rate of change of the total angular momentum with respect to inertial space is equal to the applied torque.

[redacted]

[redacted]

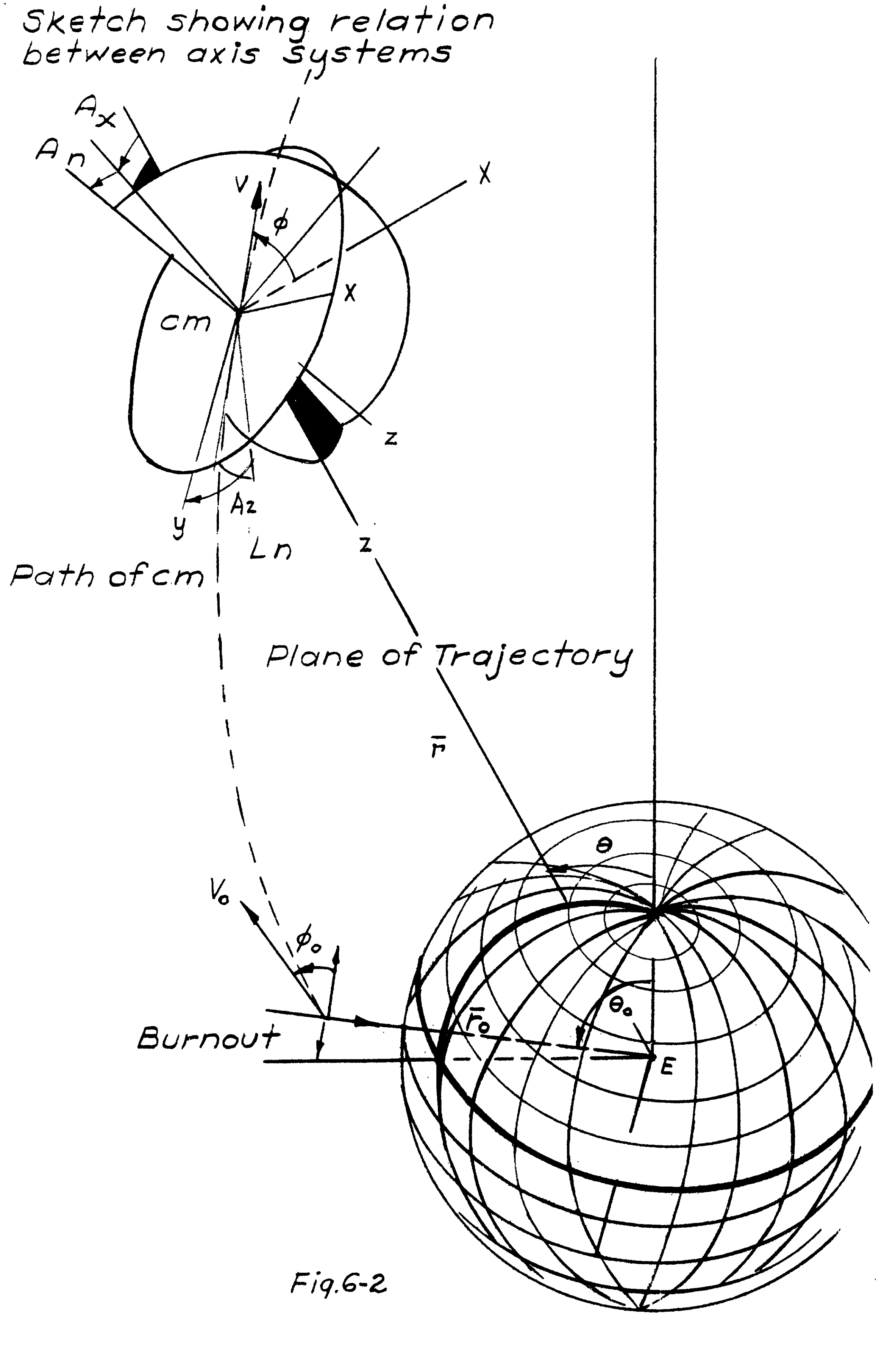


Fig.6-2

[redacted]

[redacted]

TABLE 6-1

Euler Angle Transformation Between Axis Systems

Define an Eulerian angle transformation between these two sets of axes as follows:

Rotate through angle A_x about the X axis.

Rotate through angle A_n about the new Y axis. (1n)

Rotate through angle A_z about final z axis. (z)

The transformation between axes may be expressed in matrix notation as follows:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = Q \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = Q^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \qquad Q^{-1} = \text{transpose of } Q$$

$$Q = \begin{bmatrix} \cos A_n \cos A_z & -\cos A_n \sin A_z & \sin A_n \\ \sin A_x \sin A_n \cos A_z + \cos A_x \sin A_z & -\sin A_x \sin A_n \sin A_z - \cos A_x \cos A_z & -\sin A_x \cos A_n \\ -\cos A_x \sin A_n \cos A_z + \sin A_x \sin A_z & \cos A_x \sin A_n \sin A_z + \sin A_x \cos A_z & \cos A_x \cos A_n \end{bmatrix}$$

[redacted]

[redacted]

$$\left(\frac{d\overline{H}_t}{dt}\right)_I = \left(\frac{d\overline{H}_v}{dt}\right)_I + \left(\frac{d\overline{H}_w}{dt}\right)_I = \overline{M}_{app} \qquad (6\text{-}2)$$

Let: I_x and I_y and I_z be principal moments of inertia of the vehicle about the vehicle axes x, y, z.

The total angular velocity of the vehicle with respect to an inertial frame is

$$\overline{W}_{I-v} = \overline{W}_{I-c.m.} + \overline{W}_{c.m.-v} \qquad (6\text{-}3)$$

Where:

$\overline{W}_{I-c.m.} = -\dot{\theta}(t)\,\overline{i}_Y$ angular velocity of vehicle center of mass with respect to the center of the earth.

$W_{c.m.-v} = \dot{A}_X \overline{i}_X + \dot{A}_n \overline{i}_n + \dot{A}_z \overline{i}_Z$ angular velocity of the vehicle about its center of mass in terms of Euler angles.

Expressing the total angular velocity of the vehicle with respect to inertial space in terms of components along the vehicle axes x, y, z:

$$\overline{W}_{I-v} = \overline{W}_{i-vx}\overline{i}_x + \overline{W}_{I-vy}\overline{i}_y + W_{I-vz}\overline{i}_z \qquad (6\text{-}4)$$

where

$$W_{I-vx} = -q_{21}\dot{\theta}(t) + q_{11}\dot{A}_x + \dot{A}_n \sin A_z$$

$$W_{I-vy} = -q_{22}\dot{\theta}(t) + q_{12}\dot{A}_x + \dot{A}_n \cos A_z$$

$$W_{I-vz} = -q_{23}\dot{\theta}(t) + q_{13}\dot{A}_x + \dot{A}_z$$

The angular momentum of the vehicle may be written:

$$\overline{H}_v = I_x W_{I-vx}\overline{i}_x + I_y W_{I-vy}\overline{i}_y + I_z W_{I-vz}\overline{i}_z \qquad (6\text{-}5)$$

Assume that the three flywheels are mounted in the vehicle so that their spin axes coincide with the vehicle axes, x, y, z. Let I, J, and K be the moments of inertia of the wheels, and w_x, w_y and w_z be the spin angular velocities of the wheels aligned with the x, y and z axes respectively.

The angular momentum of the wheels may be written:

$$\bar{H}_W = I\, w_x \bar{i}_x + J\, w_y \bar{i}_y + K\, w_z \bar{i}_z$$

Noting:

$$\left(\frac{d\bar{\bar{H}}_x}{dt}\right)_I = \left(\frac{d\bar{\bar{H}}_x}{dt}\right)_x + \bar{W}_{I-x} \times \bar{H}_x$$

where

$$x = v \text{ or } w$$

Substituting the derivative expressions into (6-2) the component relations are as follows:

$$I_x \dot{W}_{I-vx} + W_{I-vy} W_{I-vz}(I_z - I_y) + I\dot{w}_x + W_{I-vy} K w_z - W_{I-vz} J w_y = M_x$$

(6-7)

$$I_y \dot{W}_{I-vy} + W_{I-vx} W_{I-vz}(I_x - I_z) + J\dot{w}_y + W_{I-vz} I w_x - W_{I-vx} K w_z = M_y$$

$$I_z \dot{W}_{I-vz} + W_{I-vx} W_{I-vy}(I_y - I_x) + K\dot{w}_z + W_{I-vx} J w_y - W_{I-vy} I w_x = M_z$$

The applied torque consists of a combination of interfering torques which the vehicle may be externally subjected to and is expressed in component form in the above equations. The above relations completely describe the motion of the vehicle about its center of mass under the action of the interfering torque, and the restoring torque of the flywheels which must resist the interfering torque, and turn the vehicle through the commanded orientation angle in pitch while nulling roll and yaw.

6.4 Simplification of Vehicle Equations of Motion

The expression for vehicle motion about its center of mass (Equations 6-7) describes the orientation of the vehicle about its center of mass in terms of components along the three vehicle axes (x, y, z).

From Progress Report 2 of the Instrumentation Laboratory a simplification of these equations results in elimination of cross coupling terms if use is made of a decoupling computer which is to be

[redacted]

used for modifying the torques provided by the flywheels in proportion to the cross coupling terms of the three equations. For a more detailed analysis of this computer, see Reference 2.

Since reorientation is to take place about the pitch axis and the other two axes have similar properties, only the y component of Equation (6-7) will be analyzed.

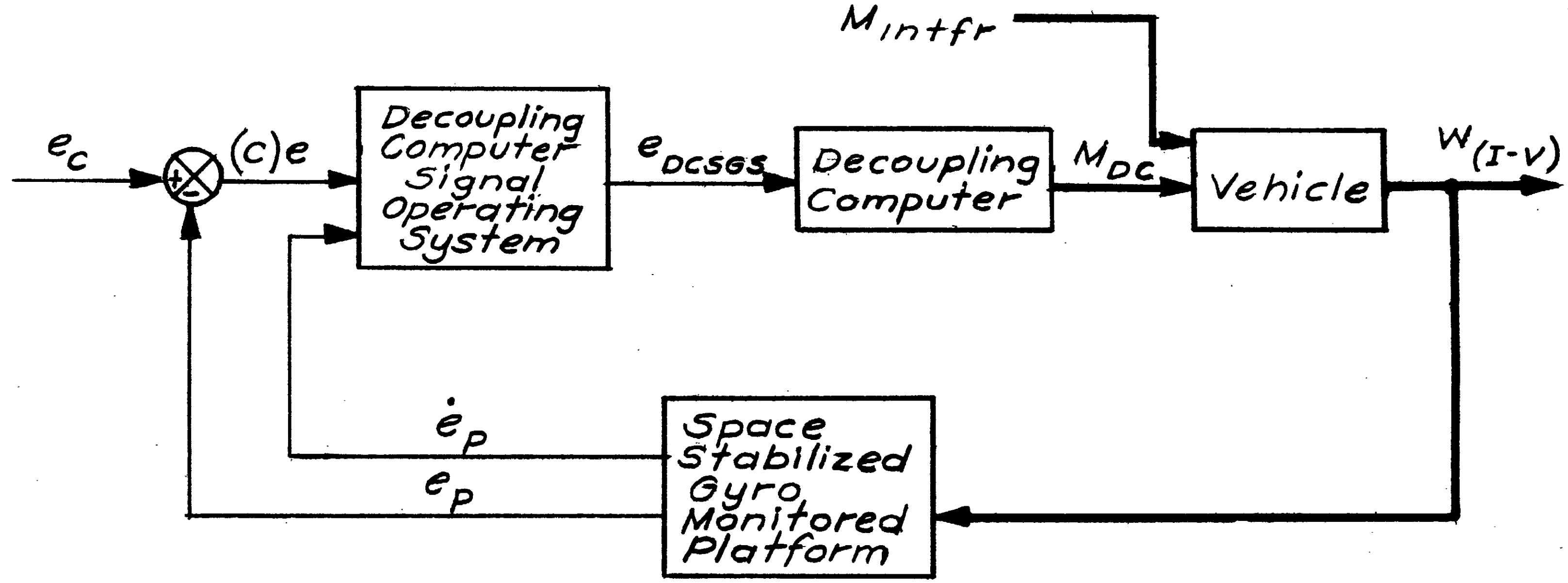
Assuming that the decoupling computer is performing its task adequately, the entire flywheel, motor, power amplifier, etc. may be considered as one functional component which delivers a torque about one axis for a voltage command for a torque about that axis. This may be represented functionally as a component which receives a voltage command in, and produces a torque output which is the product of a sensitivity of the device times the imput voltage. Since the decoupling computer is in the system, the torque about one axis produces an angular acceleration about one axis only. Thus the vehicle equation of motion in pitch reduces to:

$$I_y \dot{W}_{(I-v)y} - S_{dc(e;m)} e_{in} = M_{y(intfr)} \qquad (6\text{-}8)$$

6.5 Pitch Control System (Yaw and Roll Similar)

Figure 6-3 is a block diagram of a proposed pitch axis vehicle orientation control system. A space stabilized platform is to be the basic data gathering device for the control of the vehicle in flight. This platform is gyro monitored, and means are provided for measuring an angular difference between a space stabilized reference direction and a direction fixed to the body of the vehicle. It is also assumed that the rate of change of this angular orientation is also available as measurable data either by a signal directly from the gyros or a suitable tachometer arrangement.

The block representing the vehicle in Figure 6-3 is to be accelerated by the flywheel torque which is the output of the decoupling computer (dc). The vehicle is also accelerated by an interfering moment (M_{intfr}) which may arise from gravity unbalance, rotating machinery within the vehicle, cosmic radiation effects, etc. The output of this block representing the vehicle is the vehicle angular acceleration with respect to inertial space ($\ddot{\theta}_{I-v}$).



e = Signal Voltages
e_c = Input Signal Propotional To Preferred Vehicle Orientation.
$W_{(I-V)}$ = Vehicle Orientation With Respect To An Inertially Stabilized Reference Direction.

$M_{D.C.}$ = Output Torque Of Decoupling Computer
M_{intfr} = Interfering Moment
V = Vehicle
D.C. = Decoupling Computer
DCSGS = Decoupling Computer Signal Generating System
S.P = Space Stabilized Gyro Monitored Platform

Fig 6-3 Block Diagram Of A Vehicle Orientation Control System For A Single Axis.

The space stabilized gyro monitored platform (sp) measures the vehicle orientation with respect to a reference line in the vehicle, or orientation about the center of mass of the vehicle, and the time rate of change of this orientation ($\dot{\theta}_{(c.m-v)y}$).

A command signal which is proportional to the desired vehicle angular orientation at apogee is compared with the output orientation signal from the stable platform to yield a correction signal to be supplied to the decoupling computer. It would be possible to design a system to operate only on the basis of an angular orientation error as a command for the system, but the resulting system would not have any damping term present in its relating function. By adding a signal proportional to the rate of change of this error, a system is evolved which will perform the required angular orientation task yet still have damping present to insure that a solution be reached in an allowed time interval.

Due to the long interval of time allowed in reorienting the vehicle about its center of mass from its orientation at cutoff to the desired orientation at apogee, the time lags in the components of the system may be neglected. On this basis, the relating functions (RF) for the various components are as follows:

$$(RF)_{sp(\theta;e)} = S_{sp(\theta;e)}$$

$$(RF)_{sp(\dot{\theta};\dot{e})} = S_{sp(\theta;e)} \qquad (6\text{-}9)$$

$$(RF)_{dcsgs(e;e)} = S_{dcsgs_1(e;e)} - S_{dcsgs_2(\dot{e};e)} \qquad (6\text{-}10)$$

$$(RF)_{dc(e;M)} = S_{dc(e;M)} \qquad (6\text{-}11)$$

$$(C)e = e_c - e_p \qquad (6\text{-}12)$$

$$e_{dcsgs} = S_{dcsgs_1}(C)e - S_{dcsgs_2}\dot{e}_p \qquad (6\text{-}13)$$

$$M_{dc} = S_{dc(e;m)}\, e_{dcsgs} \qquad (6\text{-}14)$$

[redacted]

Solution of Equation (6-14) for M_{dc} and substitution into Equation (6-8) results in:

$$I_y \dot{W}_{(c.m.-v)y} - S_{dc(e;m)} \Big[S_{dcsgs((c)e;e)}\, e_c$$

$$- S_{dcsgs_1((C)e;e)} S_{sp(\theta;e)}\, \theta_{(c.m.-v)y}$$

$$- S_{dcsgs_2(e;e)} S_{sp(\;;e)}\, \dot{\theta}_{(c.m.-v)y} \Big] = \pm M_{intfr} \qquad (6\text{-}15)$$

Let

$$S_{dcsgs_2(e;e)} = \alpha S_{dcsgs_1((C)e;e)} \qquad (6\text{-}16)$$

Equation (6-16) can be reqritten as:

$$I_y \dot{W}_{(c.m.-v)y} - S_{dc} S_{dcsgs_2} \Big[e_c - S_{sp} (1+\alpha p)\theta \Big] = M_{intfr} \qquad (6\text{-}17)$$

It is convenient to define three new quantities, let:

$S_{pc(OL)}$ = open loop sensitivity of the pitch control system (stiffness)

$$= S_{dc(e;m)} S_{dcsgs_1((C)e;e)} S_{sp(\theta;e)} \qquad (6\text{-}18)$$

$S_{pc(e;\theta)}$ = command voltage (e_c) pitch angle (θ) sensitivity of the pitch control system

$$= \frac{1}{S_{sp(\theta;e)}} \qquad (6\text{-}19)$$

$S_{pc((C)e;m)}$ = error voltage - flywheels torque sensitivity of the pitch control system

$$= S_{dc(e;m)} S_{dcsgs_1((C)e;e)} \qquad (6\text{-}20)$$

Substituting these quantities into Equation (6-17) and collecting like terms, the overall system performance function is:

$$I_y \dot{W}_{(c.m.-v)\,y} + \alpha S_{pc(OL)}\, p\, \theta_{(c.m.-v)y}$$

$$+ S_{pc(OL)}\, \theta_{(c.m.-v)y} = S_{pc((C)e;m)}\, e_c \pm M_{intfr}$$

Noting that $\dot{W}_{(c.m.-v)y} = p^2 \theta_{(c.m.-v)y}$ (6-21)

[redacted]

[redacted]

The vehicle orientation control system about the pitch axis is of the conventional 2nd order form, and the non-dimensional curves from "Instrument Engineering" by Draper, McKay and Lees completely describe system response to various inputs.

Written non-dimensionally, the equation takes the form:

$$(p^2/w_n^2 + 2\,\zeta/w_n p + 1)\,\theta_{(c.m.-v)y} =$$

$$S_{pc(e;\theta)}\,e_c \pm M_{intfr}/S_{pc(OL)} \qquad (6\text{-}22)$$

where:

$$w_n = \sqrt{S_{pc(OL)}/I_y}$$

$$\zeta = \frac{1}{2}\alpha\sqrt{S_{pc(OL)}/I_y}$$

6.6 System Response to a Command Input

The pitch control system is to operate on the basis of a command from a device which senses the required reorientation angle either by a direct measurement, or a computation.

Depending on whether the vernier thrust rocket is mounted as shown in Figure 1-2 in the manner indicated by Case 1 or 2, the vehicle must be pitched through an angle of about 110° or about 20°. The command to the pitch control system is then a step input of angular reorientation of magnitude indicated by the pitch command system.

The total time of flight from cutoff to apogee is of the order of 1500-1700 seconds. Since the vehicle must be properly oriented by the time it reaches apogee of its ascent trajectory, the system is allowed about this amount of time to completely respond to its orientation command signal. It is therefore necessary to choose suitable system parameters (ζ and w_n) such that the system response time is less than 1500 seconds. The response time (RT) is defined as the time required for a step function response of the system to be within 95% of its final value

6.7 Effect of Interfering Moments on the Pitch Control System

The design of the pitch control system is dictated not only by the system response to a command signal but also by its performance in the presence of interfering moments. The source of these interfering moments has previously been cited, and at this time it seems wise to point out the type of time variations that may be expected. The gravitation torque due a dumbbell effect is essentially of constant magnitude. The torque due to rotating machinery, although very small, is referred to in Rand investigations as the primary source of external applied torque to the vehicle tending to accelerate it about its center of mass. The other miscellaneous torques, small in magnitude compared with the other two listed, may be of a random nature.

A torque of constant magnitude requires a high open loop gain to maintain the vehicle's orientation within a tolerable error band. (The effects of these orientation errors on orbit distortion has been discussed in Chapter 3.) The magnitude of the open loop gain which is determined from this consideration determines the natural frequency of the pitch control system, and the response time criteria remains for determination of system damping.

6.8 Orientation Control System Parameters

In order to estimate the range of system parameters required to control the vehicle's orientation during its coasting flight commensurate with the magnitude of the tolerable error determined in Chapter 3, the following representative values have been chosen:

$$I_y = 1060 \text{ slug-ft}^2$$

$$M_{intf} = 1 \text{ ounce-inch} = 0.00521 \text{ ft-lbs.}$$

$$(E)\phi_a = 20 \text{ minutes of arc} = 0.00582 \text{ radians}$$

From these values the open-loop sensitivity ($S_{pc(OL)}$) must be as follows:

$$\frac{M_{intf}}{S_{pc(OL)}} \leq (E)\phi_a$$

[redacted]

or

$$S_{pc(OL)} \geqq 0.895 \text{ ft-lb/rad}$$

The value of the natural frequency of the system must then be:

$$W_n = \sqrt{\frac{S_{pc(OL)}}{Iy}} \geqq 0.0291 \text{ rad/sec}$$

In Chapter 1 two methods are considered in this thesis by which the vehicle may be oriented during its coasting flight from cutoff to apogee. The methods are referred to in Chapter 1, Figure 1-2, as Case 1 and Case 2. The time interval during which the controlling of the vehicle must be accomplished prior to adding the vernier thrust is about 1500 seconds. To satisfy this condition the response time (RT) of the orientational system must be within the 1500 seconds available.

Various values of RT have been considered over this range and the system's parameters determined from 2nd order curves found in Instrument Engineering. The results of this investigation can be found in Table 6-2.

TABLE 6-2

RANGE OF SYSTEM PARAMETERS

RT	W_n	T_n	ζ
300 sec	0.0291 rad/sec	215 sec	0.3
600 sec	0.0291 rad/sec	215 sec	0.17
900 sec	0.0291 rad/sec	215 sec	0.11
1200 sec	0.0291 rad/sec	215 sec	0.08
1500 sec	0.0291 rad/sec	215 sec	0.07

From the above table it can be seen that for the system shown in Figure 6-3 to have a satisfactory response to interfering moments, and also possess a reasonable value of damping (ζ = 0.2 or greater), a short response time is necessary. Also, if it is desired to reduce the effect of the interfering moment, the system stiffness (as defined in Equation 6-18) would have to be increased. This results in a larger natural frequency (W_n) for the system. If in addition, it is desired to keep the response time of the system as large as possible to minimize

[redacted]

the size of flywheel motors required the system would be lightly damped. However if the system's damping is not of sufficient magnitude dynamic errors may result due to the inability of the pitch control system to settle on the desired steady state value within the allowable response time.

In the system shown a compromise must be made for system response time, stiffness and the amount of damping considered necessary. One important factor that should be considered in the choice, is the size of the motors required to accelerate the flywheels to obtain sufficient torque to control the vehicle.

If through the use of modifying networks the system damping can be increased with no loss in system stiffness a long system response time (750 seconds or greater) may permit a reduction in the motor sizes. However if a short response time for the system is chosen an analysis should be carried out considering the component dynamics associated with the decoupling computer, decoupling computer signal generating system and any lags that might exist in the gyro monitored space stabilized platform.

6.9 Summary

A simplified pitch control system was presented and analyzed on the basis of the task required; that being the reorienting of the vehicle for application of the vernier thrust, and stabilizing of the vehicle against any interfering torques tending to rotate it away from the desired orientation.

Many assumptions and simplifications have been made in this analysis, but it is intended that this system be a preliminary model to investigate the basic difficulties in such a reorientation problem.

The dominating factor in the design of such a system is the overall gain (stiffness) of the system between orientation angle and flywheel torque. The magnitude of this gain will determine the degree that the interfering moment will affect the vehicle orientation upon reaching apogee. Any existing error will result in orbit distortion. It should be pointed out that the application of torque by the flywheels

[redacted]

required that they be accelerated. Hence, a constant torque requirement imposed on the flywheels results in very large magnitudes of flywheel spin angular velocities. A saturation effect is present due to the fact that an unlimited wheel spin is not physically possible.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

A proposal was made for instrumenting a vernier velocity computer, and a time of flight to apogee computer to provide command signals for the vernier thrust rocket control system in order to place the vehicle in a preferred orbit. These computers accept measured vehicle velocity magnitude and direction at cutoff, and generate outputs on the basis of a perturbation technique from some reference set of cutoff conditions. The expressions for vernier velocity and time of flight in terms of cutoff conditions were derived under the assumptions that:

1) The earth is a non-rotating sphere.
2) Aerodynamic forces may be neglected during the coasting flight.

Errors arising due to measurement of cutoff conditions will be propagated through the computers, but the distortion of the orbit due to these errors is within allowable design limits.

Additional accuracy in the mathematical models simulated on the perturbation technique may be achieved by taking into account more terms of the Multi-variable Taylor's Series expansion of the functions. From the table showing the values of the partial derivatives of vernier velocity with respect to cutoff conditions (Table 4-1), it can be seen that the only additional terms which would have any appreciable effect on the vernier velocity would be the second partial with respect to cutoff velocity direction, yet, even this quantity is an order of magnitude (factor of 10) less than the first partial with respect to cutoff

[redacted]

velocity direction. From the plot of time of flight from cutoff to apogee vs. cutoff velocity direction for constant cutoff velocity (Figure 5-3) it can be seen that a slight amount of curvature exists in the function. Additional accuracy on the mathematical model for the time of flight expression may be realized by a polynomial expansion of the curve.

The accuracy of the series expansion of the expressions is deemed adequate, on the basis that the simplicity of the computers justifies a small tolerable error.

The overall computer accuracy which is finally made up as a circuit will of course depend on the linearity of the components used. Gain stabilized amplifiers will necessarily have to be used in the construction, and it is assumed that power supply fluctuations will not effect the overall accuracy of the computers since other control system components will place more stringent requirements on stability of excitation voltages than these computers.

Although the circuits may be constructed utilizing conventional miniaturized vacuum tubes, it is felt that a transistor circuit will result in an additional saving in space and weight both of the computers themselves, and the attendant power supply required with its inherent dissipation of heat problems.

7.2 Recommendations

Prior to construction of the computers, it is recommended that a study be made of the computers as simulated taking into account the ellipticity of the earth, effect of aerodynamic forces, and the effect of the earth's rotation. This may be accomplished by assuming a range in cutoff conditions about the reference and determining a quantity by the actual equations relating vernier velocity and time of flight to the three cutoff conditions, then determining the quantities that the computer would indicate. Variation in cutoff altitude should also be included in the event that it is felt that the thrust phase guidance system will not control the cutoff altitude to tolerable limits.

Additional analysis of the orientation control system from the standpoint of improving dynamic performance of the system, and elimination of the steady state error due to the effect of a constant

[redacted]

interfering torque acting on the vehicle should be conducted. An error analysis of the orientation of the control system similar to that undertaken by Covington should be performed. The possibility of using sample data techniques on a system which torques by short duration pulses should be investigated.

An investigation should be conducted concerning the dissipation of the angular momentum stored in the flywheels during the torquing of the vehicle in the coasting flight.

[redacted]

APPENDIX A

DERIVATION OF THE TRAJECTORY EQUATIONS OF MOTION*

For definition of symbols see page 105

Because vehicle flight occurs above 350,000 feet, it can be assumed that the vehicle motion takes place in a vacuum. Therefore, the Conservation of Energy and Angular Momentum Theorems completely describe the vehicle's motion.

$$\text{Potential Energy} = -\frac{mE}{r} \qquad \text{(A-1)}$$

$$\text{Kinetic Energy} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad \text{(A-2)}$$

The total energy is:

$$W = T + V$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{mE}{r} \qquad \text{(A-3)}$$

The vehicle energy at burn-out is:

$$\text{vehicle energy}_o = \frac{1}{2}mV_o^{\,2} - \frac{mE}{r_o} \qquad \text{(A-4)}$$

Equating total energy of vehicle to energy at burn-out:

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{mE}{r} = \frac{1}{2}mV_o^{\,2} - \frac{mE}{r_o} \qquad \text{(A-5)}$$

* The derivations included in this appendix have been extracted from P. A. Lapp's Sc.D. THESIS T-63.

[redacted]

[redacted]

or, canceling m:

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{E}{r} = \frac{1}{2}V_o^2 - \frac{E}{r_o} \qquad \text{(A-6)}$$

At this point it is necessary to obtain another relation between r and $\dot{\theta}$. This is obtained from Conservation of Angular Momentum. For motion in a central force field, Kepler's Second Law states that angular momentum is conserved. The equation is:

$$mr^2\dot{\theta} = h = \text{constant} \qquad \text{(A-7)}$$

The angular momentum of the vehicle above the reference sphere is equal to the vehicle's angular momentum at burn-out, or:

$$mr^2\dot{\theta} = h\Big|_{\substack{r = r_o \\ \theta = \theta_o}} = mr_oV_o\cos\phi_o \qquad \text{(A-8)}$$

Equations (A-6) and (A-8) represent the conservation of energy and angular momentum.

If time is eliminated from equations (A-6) and (A-8), and the subsequent expression solved for $\left(\frac{dr}{d\theta}\right)^2$, the following relation results:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{r_o^2V_o^2\cos^2\phi_o}\left[V_o^2 - 2E\left(\frac{1}{r_o} - \frac{1}{r}\right) - \frac{r_o^2V_o^2\cos^2\phi_o}{r^2}\right]$$

(A-9)

If the above expression is then integrated, an explicit relation for $d\theta$ may be obtained as follows:

$$d\theta = \frac{-d\,\dfrac{r_oV_o\cos\phi_o}{r}}{\sqrt{V_o^2 - 2E\left(\frac{1}{r_o} - \frac{1}{r}\right) - \dfrac{r_o^2V_o^2\cos^2\phi_o}{r}}} \qquad \text{(A-10)}$$

[redacted]

[redacted]

To solve for θ, let:

$$x = \frac{r_o V_o \cos \phi_o}{r} \tag{A-11}$$

Then:

$$\frac{2E}{r} = \left(\frac{r_o V_o \cos \phi_o}{r}\right)\left(\frac{2E}{r_o V_o \cos \phi_o}\right) = x \cdot \left(\frac{2E}{r_o V_o \cos \phi_o}\right) \tag{A-12}$$

Let:

$$a = -1, \quad b = \frac{2E}{r_o V_o \cos \phi_o}, \quad c = V_o^{\,2} - \frac{2E}{r_o} \tag{A-13}$$

Then equation (A-10) may be written in the following form:

$$d\theta = \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{A-14}$$

From Integral No. 163 of Mathematical Tables and Formulas by Burington, the above equation is seen to be:

$$\theta = -\frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{-2ax - b}{\sqrt{b^2 - 4ac}}\right) + \delta \tag{A-15}$$

where δ is the constant of integration. By substituting for a, b, c, and x from equations (A-11) and (A-13), r may be shown to be:

$$r = \frac{\dfrac{r_o^{\,2} V_o^{\,2} \cos^2 \phi_o}{E}}{1 - \dfrac{r_o V_o \cos \phi_o}{E} \sin(\theta - \delta)\sqrt{V_o^{\,2} - \dfrac{2E}{r_o} + \dfrac{E^2}{r_o^{\,2} V_o^{\,2} \cos^2 \phi_o}}} \tag{A-16}$$

The constant of integration, δ, may be evaluated by the definition of θ. For this problem, θ is measured relative to the major axis of an ellipse that constitutes a section of a conic and hence is zero at apogee. Therefore:

$$\frac{dx}{d\theta} = 0 \tag{A-17}$$

[redacted]

[redacted]

Differentiating equation (A-15):

$$\cos\,(\theta - \delta) = \frac{2a}{\sqrt{b^2 - 4ac}} \cdot \frac{dx}{d\theta} = 0 \qquad \text{(A-18)}$$

When $\frac{dx}{d\theta} = 0$, $\theta = 0$; therefore, $\cos\,(-\delta) = 0$, and:

$$\delta = \pm\, n\,\frac{\pi}{2}, \text{ where } n = \text{odd integer} \qquad \text{(A-19)}$$

By taking $\frac{d^2x}{d\theta^2}$, n is found to be unity and the sign negative for x to be a minimum at the apogee. Therefore:

$$\sin\,(\theta - \delta) = \sin\,(\theta - \frac{\pi}{2}) = \cos\,\theta$$

Equations (A-15) and (A-16) may then be written as:

$$\theta = \cos^{-1} \frac{rE - r_o^2 V_o^2 \cos^2\phi_o}{r\left(r_o V_o \cos\,\phi_o \sqrt{V_o^2 - \frac{2E}{r_o} + \frac{E^2}{r_o^2 V_o^2 \cos^2\phi_o}}\right)} \qquad \text{(A-20)}$$

$$r = \frac{\frac{r_o^2 V_o^2 \cos^2\phi_o}{E}}{1 - \left(\frac{r_o V_o \cos\,\phi_o}{E} \sqrt{V_o^2 - \frac{2E}{r_o} + \frac{E^2}{r_o^2 V_o^2 \cos^2\phi_o}}\right) \cos\,\theta} \qquad \text{(A-21)}$$

The above expression is the equation of a conic in polar coordinates (r, θ), whose eccentricity and semi-latus rectum are:

$$e = \frac{r_o V_o \cos\,\phi_o}{E} \sqrt{V_o^2 - \frac{2E}{r_o} + \frac{E^2}{r_o^2 V_o^2 \cos^2\phi_o}}$$

$$\text{(A-22)}$$

$$p = \frac{r_o^2 V_o^2 \cos^2\phi_o}{E}$$

[redacted]

[redacted]

and if substituted into equation (A-20), the more conventional representation of a conic is found:

$$r = \frac{p}{1 - e \cos \theta} \qquad \text{(A-23)}$$

The value of the eccentricity determines which type of conic the trajectory is. For:

$e = 0$ circle

$e < 1$ ellipse

$e = 1$ parabola

$e > 1$ hyperbola

For the case where the conic is an ellipse, the trajectory either returns to the earth or moves as a satellite about the earth. If a parabola or hyperbola, the vehicle has sufficient energy to escape the earth's gravitational field and continues off into space.

The time required by the vehicle to travel from cut-off to apogee may be obtained from equation (A-8):

$$dt = \frac{r^2}{r_o V_o \cos \phi_o} d\theta \qquad \text{(A-24)}$$

Replacing r in the above expression by equation (A-23) and substituting for p from (A-22), dt may be written as:

$$\int_o^{t_a} dt = \frac{r_o^3 V_o^3 \cos^3 \phi_o}{E^2} \int_o^{\theta_o} \frac{d\theta}{(1 - e \cos \theta)^2} \qquad \text{(A-25)}$$

Let:

$$I = \int_o^{\theta_o} \frac{d\theta}{(1 - e \cos \theta)^2} \qquad \text{(A-26)}$$

[redacted]

and:

$$z = \tan\frac{\theta}{2}, \quad d\theta = \frac{2dz}{1+z^2}, \quad \cos\theta = \frac{1-z^2}{1+z^2},$$

$$a = 1 + e, \quad c = 1 - e \qquad \text{(A-27)}$$

Then equation (A-26) may be put in the following form:

$$I = \int_0^{\tan\frac{\theta}{2}} \frac{dz}{(c + az^2)^2} + \int_0^{\tan\frac{\theta}{2}} \frac{z^2 dz}{(c+az^2)^2} \qquad \text{(A-28)}$$

Performing the above integrations results in the following expression:

$$I = \left[\left(\frac{1}{c} - \frac{1}{a}\right)\frac{z}{c + az^2} + \left(\frac{1}{c} + \frac{1}{a}\right)\frac{1}{\sqrt{ac}} \tan^{-1} z\sqrt{\frac{1+e}{1-e}}\,\right]_0^{\tan\frac{\theta_o}{z}}$$

(A-29)

By taking the limits and applying the substitutions of equation (A-27):

$$I = \frac{e \sin\phi_o}{(1-e^2)(1 - e\cos\theta_o)} + \frac{2}{(1-e^2)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{\theta_o}{2}\right)$$

(A-30)

The time of flight to apogee is then obtained by substituting equation (A-30) into equation (A-25), and if reduced to cut-off conditions is as follows:

$$t_a = \frac{r_o^2 V_o}{2E - r_o V_o^2}\left[\sin\phi_o + \frac{2E}{2Er_o V_o^2 - r_o^2 V_o^4} \tan^{-1} \left(\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{\theta_o}{z}\right)\right] \qquad \text{(A-31)}$$

From the Conservation of Angular Momentum, expressions relating missile apogee conditions to those existing at cut-off may be derived as follows:

[redacted]

[redacted]

$$r_aV_a = r_oV_o\cos\phi_o \qquad \text{(A-32)}$$

By definition, the flight path angle ϕ is zero at the apogee, and the energy equation is:

$$\frac{V_o^2}{2} - \frac{E}{r_o} = \frac{V_a^2}{2} - \frac{E}{r_a} \qquad \text{(A-33)}$$

Combining equations (A-32) and (A-33) to eliminate r_a:

$$V_a^2 - \frac{2E}{r_oV_o\cos\phi_o}V_a + \frac{2E}{r_o} - V_o^2 = 0 \qquad \text{(A-34)}$$

Equation (A-34) is a quadratic in V_a. The solution is:

$$V_a = \frac{E}{r_oV_o\cos\phi_o}\left(1 \pm \sqrt{1 + \frac{r_o^2V_o^4\cos^2\phi_o}{E} - 2\frac{r_oV_o^2\cos^2\phi_o}{E}}\right) \qquad \text{(A-35)}$$

Substituting e from equation (A-22), the above expression may be rewritten as:

$$V_a = \frac{E}{r_oV_o\cos\phi_o}(1 \pm e) \qquad \text{(A-36)}$$

and from equation (A-32):

$$r_a = \frac{r_o^2V_o^2\cos^2\phi_o}{E(1 \pm e)} \qquad \text{(A-37)}$$

Since for elliptical trajectories e must be positive, the negative sign should be used. Consequently:

$$V_a = \frac{E}{r_oV_o\cos\phi_o}(1 - e) \qquad \text{(A-38)}$$

$$r_a = \frac{r_o^2V_o^2\cos^2\phi_o}{E(1 - e)} \qquad \text{(A-39)}$$

[redacted]

Furthermore, if equations (A-38) and (A-39) are applied to equation (A-23), the trajectory equation may be written as:

$$r = \frac{\dfrac{r_a^{\,2} V_a^{\,2}}{E}}{1 - \left(1 - \dfrac{r_a V_a^{\,2}}{E}\right)\cos\theta} \qquad \text{(A-40)}$$

[redacted]

Definition of Symbols

m = mass of vehicle.

$E = 1.41008 \times 10^{16}\ ft^3/sec^2$.

r = radial distance from center of earth to vehicle center of gravity.

Θ = angle between major axis of ellipse and position radius.

W = total energy of vehicle at cut-off.

T = kinetic energy of vehicle.

V = potential energy of vehicle.

r_o, V_o, ϕ_o = cut-off conditions.

h = angular momentum of vehicle at cut-off.

t_a = time of flight from cut-off to apogee.

r_a, V_a = apogee conditions.

[redacted]

APPENDIX B

DERIVATION OF THE VERNIER VELOCITY COMPUTER EQUATION

For definition of symbols, see page 111.

From the definition of venier velocity in Chapter 4, the following equation results:

$$V_v = V_s - V_a \tag{B-1}$$

For a circular satellite vehicle, the gravitation attraction force must equal the centrifugal force. Hence, summing forces:

$$F = F_{gravitational} + F_{centrifugal} = 0 \tag{B-2}$$

where:

$$\begin{aligned} F_{gravitatational} &= -\frac{mE}{r^2} \\ F_{centrifugal} &= \frac{mV^2}{r} \end{aligned} \tag{B-3}$$

or equating forces and dividing through by m:

$$V = \sqrt{\frac{E}{r}} \tag{B-4}$$

For a circular satellite vehicle at a distance r_a from the center of the earth, equation (B-4) states that the satellite velocity must be:

$$V_s = \sqrt{\frac{E}{r_a}} \tag{B-5}$$

From equations (A-36) and (A-37) of Appendix A, expressions relating apogee to cut-off conditions are as follows:

$$V_a = \frac{E(1 - e)}{r_o V_o \cos \phi_o}$$

$$r_a = \frac{r_o^2 V_o^2 \cos^2 \phi_o}{E(1 - e)} \qquad \text{(B-6)}$$

Substitution of r_a into equation (B-5) gives an expression for V_s in terms of cut-off conditions:

$$V_s = \frac{E}{r_o V_o \cos \phi_o} \sqrt{1 - e} \qquad \text{(B-7)}$$

Replacing V_a and V_s by equations (B-6) and (B-7) respectively:

$$V_v = \frac{E}{r_o V_o \cos \phi_o} \left[\sqrt{1 - e} - (1 - e)\right] \qquad \text{(B-8)}$$

Considering only the first order term of a Taylor Series expansion, the Venier Velocity Computer Equation can be written as:

$$V_v = V_{v\ ref} + \Delta V_v \qquad \text{(B-9)}$$

where:

$V_{v\ ref}$ = Precalculated value of venier velocity based on a preferred set of cut-off conditions.

ΔV_v = Additional computed venier velocity due to actual cut-off conditions differing from the reference.

ΔV_v is defined as follows:

$$\Delta V_v = \frac{\partial (V_{v\ ref})}{\partial r_o} \Delta r_o + \frac{\partial (V_{v\ ref})}{\partial V_o} \Delta V_o + \frac{\partial (V_{v\ ref})}{\partial \phi_o} \Delta \phi_o \qquad \text{(B-10)}$$

where Δr_o, ΔV_o, $\Delta \phi_o$ represent the perturbations of the actual trajectory about the reference.

[redacted]

The partial derivatives are obtained by taking $\frac{\partial}{\partial r_o}, \frac{\partial}{\partial V_o}, \frac{\partial}{\partial \phi_o}$ of equation (B-7). They are as follows:

$$\frac{\partial (V_{v\ ref})}{\partial r_o} = - \frac{E}{r_o^2 V_o \cos \phi_o} \left[\sqrt{1 - e} - (1 - e)\right] + \frac{E}{r_o V_o \cos \phi_o} \left(\frac{\partial e}{\partial r_o}\right) \left[1 - \frac{1}{2\sqrt{1 - e}}\right] \qquad \text{(B-11)}$$

$$\frac{\partial (V_{v\ ref})}{\partial V_o} = - \frac{E}{r_o V_o^2 \cos \phi_o} \left[\sqrt{1 - e} - (1 - e)\right] + \frac{E}{r_o V_o \cos \phi_o} \left(\frac{\partial e}{\partial V_o}\right) \left[1 - \frac{1}{2\sqrt{1 - e}}\right] \qquad \text{(B-12)}$$

$$\frac{\partial (V_{v\ ref})}{\phi_o} = \frac{E \sin \phi_o}{r_o V_o \cos^2 \phi_o} \left[\sqrt{1 - e} - (1 - e)\right] + \frac{E}{r_o V_o \cos \phi_o} \left(\frac{\partial e}{\partial \phi_o}\right) \left[1 - \frac{1}{2\sqrt{1 - e}}\right] \qquad \text{(B-13)}$$

To get ΔV_v in terms of cut-off conditions, it is necessary to replace $\frac{\partial e}{\partial r_o}, \frac{\partial e}{\partial V_o}, \frac{\partial e}{\partial \phi_o}$ in terms of burn-out conditions. From equation (A-22) we know that:

$$e = \frac{r_o V_o \cos \phi_o}{E} \sqrt{V_o^2 - \frac{2E}{r_o} + \frac{E^2}{r_o^2 V_o^2 \cos^2 \phi_o}}$$

Taking the partial derivatives with respect to cut-off conditions gives the following expressions:

$$\frac{\partial e}{\partial r_o} = \frac{V_o \cos \phi_o}{E} \left[\frac{V_o^2 - E/r_o}{\left(V_o^2 - \frac{2E}{r_o} + \frac{E^2}{r_o^2 V_o^2 \cos^2 \phi_o}\right)^{1/2}} \right] \qquad \text{(B-14)}$$

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$$\frac{\partial e}{\partial V_o} = \frac{2r_o \cos \phi_o}{E} \left[\frac{V_o^{\,2} - E/r_o}{\left(V_o^{\,2} - \frac{2E}{r_o} + \frac{E^2}{r_o^{\,2} V_o^{\,2} \cos^2 \phi_o} \right)^{1/2}} \right] \tag{B-15}$$

$$\frac{\partial e}{\partial \phi_o} = \frac{r_o V_o \sin \phi_o}{E} \left[\frac{\frac{2E}{r_o} - V_o^{\,2} - \frac{2E^2}{r_o^{\,2} V_o^{\,2} \cos^2 \phi_o}}{\left(V_o^{\,2} - \frac{2E}{r_o} + \frac{E^2}{r_o^{\,2} V_o^{\,2} \cos^2 \phi_o} \right)^{1/2}} \right] \tag{B-16}$$

The values of the partial derivatives can be evaluated for the reference trajectory and placed in the venier velocity computer as coefficients to operate on measured perturbations.

Then the correct venier velocity is computed from the computer equation by the summation of the reference value and the correction due to the deviation of the actual trajectory from the reference.

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Definition of Symbols

V_v = venier velocity.

$V_{v\ ref}$ = venier velocity required for reference trajectory.

ΔV_v = additional venier velocity due to actual trajectory of ascent differing from the reference.

V_s = required satellite velocity.

V_a = vehicle velocity at apogee.

r_a = radius of circular orbit.

$E = 1.41008 \times 10^{16}\ ft^3/sec^2$.

e = eccentricity of elliptical trajectory.

r_o, V_o, ϕ_o = cut-off conditions.

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APPENDIX C

DERIVATION OF TIME OF FLIGHT COMPUTER EQUATION

For definition of symbols used, see page 118

In Chapter 3, the expression relating the actual time that the vehicle takes to reach apogee from cut-off can be expressed by a Taylor Series expansion about a reference trajectory. In addition, if only the first two terms of the series are considered, the computer equation may be written as:

$$t_a = t_{a\ ref} + \Delta t_a \tag{C-1}$$

where:

$t_{a\ ref}$ = precalculated reference time of flight obtained from evaluating generalized time of flight expression.

Δt_a = additional time of flight due to actual cut-off conditions differing from the reference.

From equation (A-31), the time of flight expression is:

$$t_a = \frac{r_o^2 V_o}{2E - r_o V_o^2}\left[\sin \phi_o + \frac{2E}{\sqrt{2Er_o V_o^2 - r_o^2 V_o^4}} \tan^{-1} \left(\sqrt{\frac{1+e}{1-e}} \cdot \tan \frac{\theta_o}{2}\right)\right] \tag{C-2}$$

[redacted]

and if evaluated for the reference trajectory, represents the value of $t_{a\ ref}$ found in equation (C-1).

The expression of Δt_a represents the first order terms of a multi-variable Taylor Series and may be written as:*

$$\Delta t_a = \frac{\partial t_{a\ ref}}{\partial r_o} \Delta r_o + \frac{\partial t_{a\ ref}}{\partial V_o} \Delta V_o + \frac{\partial t_{a\ ref}}{\partial \phi_o} \Delta \phi_o \qquad \text{(C-3)}$$

where Δr_o, ΔV_o, $\Delta \phi_o$ represent small perturbations about the reference condition.

The partial derivatives in equation (C-3) can be determined by differentiating equation (C-2) with respect to one variable, considering the other two variables as constants. The method of the differentiation is: Let equation (C-2) be represented by the following form:

$$t_a = A(B + CD) \qquad \text{(C-4)}$$

where:

$$A = \frac{r_o^2 V_o}{2E - r_o V_o^2}$$

$$B = \sin \phi_o$$

$$C = \frac{1}{\sqrt{2E r_o V_o^2 - r_o^2 V_o^4}} \qquad \text{(C-5)}$$

$$D = 2E \tan^{-1}\left(\sqrt{\frac{1+e}{1-e}} \cdot \tan \frac{\theta_o}{2}\right)$$

Then:

$$\frac{\partial t_a}{\partial r_o} = \frac{\partial A}{\partial r_o}(B + CD) + A\left(\frac{\partial B}{\partial r_o} + \frac{C \partial D}{\partial r_o} + \frac{D \partial C}{\partial r_o}\right) \qquad \text{(C-6)}$$

* I. S. Sokolnikoff: <u>Advanced Calculus</u>, McGraw-Hill, 1939, p. 320

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where:

$$\frac{\partial A}{\partial r_o} = \frac{4Er_oV_o - r_o^2V_o^3}{(2E - r_oV_o^2)^2}$$

$$\frac{\partial B}{\partial r_o} = 0$$

$$\frac{\partial C}{\partial r_o} = \frac{-\left[\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{\theta_o}{2}\right)\right]\left(EV_o^2 - r_oV_o^4\right)}{(2Er_oV_o^2 - r_o^2V_o^4)^{3/2}} \tag{C-7}$$

$$\frac{\partial D}{\partial r_o} = 2E\left\{\frac{1}{1 + \left(\frac{1+e}{1-e}\right)\tan^2\frac{\theta_o}{2}}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\left(\sec^2\frac{\theta_o}{2}\right)\left(\frac{1}{2}\right)\frac{\partial\theta_o}{\partial r_o}\right.\right.$$

$$\left.\left. + \tan\frac{\theta_o}{2} \cdot \left(\frac{1+e}{1-e}\right)^{1/2}\left(\frac{1}{1-e}\right)^2\frac{\partial e}{\partial r_o}\right]\right\}$$

$$\frac{\partial t_a}{\partial V_o} = \frac{\partial A}{\partial V_o}(B + CD) + A\left(\frac{\partial B}{\partial V_o} + \frac{C\partial D}{\partial V_o} + D\frac{\partial C}{\partial V_o}\right) \tag{C-8}$$

where:

$$\frac{\partial A}{\partial V_o} = \frac{2Er_o^2 + r_o^3V_o^2}{\left(2E - r_oV_o^2\right)^2}$$

$$\frac{\partial B}{\partial V_o} = 0$$

$$\frac{\partial C}{\partial V_o} = \frac{-\left[\tan^{-1}\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{\theta_o}{2}\right]\left(2Er_oV_o - 2r_o^2V_o^3\right)}{\left(2Er_oV_o^2 - r_o^2V_o^4\right)^{3/2}} \tag{C-9}$$

$$\frac{\partial D}{\partial V_o} = 2E\left\{\frac{1}{1 + \left(\frac{1+e}{1-e}\right)\tan^2\frac{\theta_o}{2}}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\left(\sec^2\frac{\theta_o}{2}\right)\left(\frac{1}{2}\right)\frac{\partial\theta_o}{\partial V_o}\right.\right.$$

$$\left.\left. + \tan\frac{\theta_o}{2} \cdot \left(\frac{1+e}{1-e}\right)^{1/2}\left(\frac{1}{1-e}\right)^2\frac{\partial e}{\partial V_o}\right]\right\}$$

$$\frac{\partial t_a}{\partial \phi_o} = \frac{\partial A}{\partial \phi_o} (B + CD) + A\left(\frac{\partial B}{\partial \phi_o} + C\frac{\partial D}{\partial \phi_o} + D\frac{\partial C}{\partial \phi_o}\right) \qquad \text{(C-10)}$$

where:

$$\frac{\partial A}{\phi_o} = 0$$

$$\frac{\partial B}{\partial \phi_o} = \cos \phi_o$$

$$\frac{\partial B}{\partial \phi_o} = 0$$

$$\frac{\partial D}{\partial \phi_o} = 2E\left\{\frac{1}{1 + \left(\frac{1+e}{1-e}\right)\tan^2\frac{\Theta_o}{2}}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\left(\sec^2\frac{\Theta_o}{2}\right)\left(\frac{1}{2}\right)\frac{\partial \Theta_o}{\partial \phi_o}\right.\right. \qquad \text{(C-11)}$$

$$\left.\left. + \tan\frac{\Theta_o}{2}\cdot\left(\frac{1+e}{1-e}\right)^{1/2}\left(\frac{1}{1-e}\right)^2\frac{\partial e}{\partial \phi_o}\right]\right\}$$

Expressions for $\frac{\partial e}{\partial r_o}$, $\frac{\partial e}{\partial V_o}$, $\frac{\partial e}{\partial \phi_o}$, can be obtained from equations (B-14), (B-15), (B-16), in terms of cut-off conditions for the elimination of the partials stated above, or their numerical values can be substituted directly. In addition, expressions for $\frac{\partial \Theta_o}{\partial r_o}$, $\frac{\partial \Theta_o}{\partial V_o}$, $\frac{\partial \Theta_o}{\partial \phi_o}$, have been obtained from P. A. Lapp's Sc.D. THESIS T-63; and upon evaluating these partials with respect to the reference trajectory, their values may be substituted in equations (C-6), (C-8), (C-10). The expressions are:

$$\frac{\partial \Theta_o}{\partial r_o} = \frac{EV_o^2 \cos \phi_o \sin \phi_o}{E^2 - (2Er_oV_o^2 - r_o V_o^4)\cos^2\phi_o}$$

$$\frac{\partial \Theta_o}{\partial V_o} = \frac{2r_oV_oE \cos \phi_o \sin \phi_o}{E^2 - (2Er_oV_o^2 - r_o^2V_o^4)\cos^2\phi_o}$$

[redacted]

$$\frac{\partial \Theta_o}{\partial \phi_o} = \frac{r_o V_o^{\,2} E \left(\cos 2\phi_o - \frac{r_o V_o^{\,2}}{E} \cos^2 \phi_o \right)}{E^2 - \left(2E r_o V_o^{\,2} - r_o^{\,2} V_o^{\,4} \right) \cos^2 \phi_o}$$

[redacted]

[redacted]

Definition of Symbols

t_a = time of flight of vehicle from cut-off to apogee.

$t_{a\ ref}$ = time of flight of vehicle from cut-off to apogee evaluated for reference trajectory.

Δt_a = additional time of flight due to an actual trajectory of ascent differing from the reference.

e = eccentricity of elliptical trajectory.

θ_o = angle between major axis of ellipse and position radius of vehicle at cut-off.

r_o, V_o, ϕ_o = 1.41008×10^{16} ft^3/sec^2.

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APPENDIX D

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