

## Control of Vibrational Systems \*

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#### Abstract

Vibrational systems, characterized by their second-order or "spring-mass" nature, occur throughout engineering. Examples range from lightly damped structures, such as the proposed space station, to regional power system models. While the study of such vibrational systems has a long and rich history, concern has traditionally focused on the properties of *passive*, uncontrolled systems. The growing need to actively control large, complex systems of this type leads us to the study of controlled vibrational systems. We provide stability-assuring constraints for compensator design. These constraints directly use the initial system model, thus avoiding the need for order reduction techniques with the associated problems of "spill-over" from the full to the reduced system model.

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## 1 Introduction

Vibrational systems, characterized by their second-order or “spring-mass” nature, occur throughout engineering. Examples range from lightly damped structures, such as oil derricks and the proposed space station, to regional power system models. While the study of such vibrational systems has a long and rich history, traditionally concern has focused on properties of *passive* systems, where active control of the complete system dynamics has *not* been a primary objective [1, 2]. As a consequence, the development of special results, techniques, and tools for properly and *directly* exploiting the special features of vibrational systems has, for the most part, been neglected. Our paper presents constraints on vibrational system compensators that guarantee the stability of the compensated system, and that are relatively easy to use in a design setting. More detail may be found in [3].

In Section 2 we start by reviewing vibrational systems and their stability from the standpoint of this work. Section 3 brings in the control issue by considering the addition of a compensator to the vibrational system model given in Section 2. The compensator is incorporated into the model in a way that preserves the physical intuition inherent in the second-order form. Simple sufficiency conditions for the stability of the closed-loop system are then presented in the form of compensator design constraints in the spirit of [4]. A guide to notation is given as Appendix A. The proof of our main result is outlined in Appendix B.

## 2 Vibrational Systems

### System Definition

For this work, a vibrational system is taken to be any system that can be modeled by a set of coupled, second-order differential equations, as follows:

$$\begin{aligned} I\ddot{x} + D\dot{x} + Kx &= Pu \\ y &= Q\dot{x} + Rx \end{aligned} \tag{1}$$

where the superscript dots denote differentiation with respect to time, and  $x$ ,  $u$ , and  $y$  are  $n$ -,  $m$ -, and  $p$ -dimensional vectors of (generalized) positions, actuator inputs, and sensor outputs, respectively. The coefficient matrices  $D$  and  $K$  are square and real,  $I$  is the identity matrix, and  $P$ ,  $Q$ , and  $R$  are real matrices representing the external coupling of the system.

The model in (1) follows *naturally* from the widely used structural modeling schemes, such as the finite element method, and strong physical associations may be made with each term (e.g.  $D$  being a mass-normalized damping coefficient matrix and  $K$  a normalized spring or stiffness coefficient matrix [5]). We preserve the flavor of these interpretations by maintaining the natural second-order form in the work to follow.

### Latent Values, Latent Vectors and Stability

The stability of the uncontrolled system (1) is determined by the (Smith) zeros or latent values of the corresponding matrix polynomial [6, 7]:

$$V(s) = Is^2 + Ds + K \tag{2}$$

The zeros or latent values and associated latent vectors of (2) are respectively the set of solutions,  $s_i$  and  $v_i$ , to the equation:

$$V(s_i)v_i = 0, \quad v_i \neq 0$$

The system (1) is asymptotically stable if and only if the latent values of (2) are in the open left half-plane. We will use the term “stable” rather than “asymptotically stable” for convenience. Also for convenience, we will refer to the latent values of the matrix polynomial (2) themselves as being stable when the system (1) is stable. For later reference, we note that (Smith-McMillan) zeros or latent values  $s_i$  and associated latent vectors  $g_i \neq 0$  can also be defined for a *rational* matrix  $G(s)$  in the same way, namely via the condition  $G(s_i)g_i = 0$ .

A development entirely parallel to that given above may be accomplished using an equivalent first-order model of the system (1) in conjunction with standard state-space stability arguments. We adopt the given approach because it exhibits the special features of the vibrational systems under consideration.

Several classical results on the stability of the vibrational system (1) exist in the literature, with the famous Kelvin-Tait-Chetaev (KTC) Theorem being among the most powerful. Examples of such results and their extensions (for passive systems) may be found in [5, 8, 9, 10, 11] and serve as models of the types of results we would like to obtain for controlled systems.

### 3 Application of Dynamic Feedback

#### The Compensator

To the system (1) we now apply the following general control law:

$$\begin{aligned} \dot{z} &= Az + By \\ u &= Cz + Fy \end{aligned} \tag{3}$$

with transfer function:

$$H(s) = F + C(sI - A)^{-1}B \tag{4}$$

where  $A$  is an  $r \times r$  dynamic matrix,  $B$  an  $r \times p$  input matrix,  $C$  a  $m \times r$  output matrix,  $F$  an  $m \times p$  feedthrough matrix, and where the system (3) is assumed to be stable. The eigenvalues  $\lambda_i$  of  $A$  are thus in the open left half-plane. We further assume that  $A$  is diagonalizable, and  $T$  is taken to be a transformation of  $A$  to diagonal form, that is  $T^{-1}AT = \text{diag}[\lambda_i]$ .

Application of this compensator to (1) yields a closed-loop system that can be described in the Laplace transform (or frequency) domain by:

$$\left[ s^2I + sD(s) + \mathcal{K}(s) \right] X(s) = \xi(s) \tag{5}$$

where  $D(s) \equiv D - PH(s)Q$  and  $\mathcal{K}(s) \equiv K - PH(s)R$ , and  $\xi(s)$  is a vector determined by the initial conditions in (1) and (3). The closed-loop system is thus presented in a second-order form, but with new *frequency-dependent* “damping” and “stiffness” matrices (assuming the compensator is truly dynamic, and not just a constant gain  $F$ ). A similar formulation may be found in some of the power systems literature, see for instance the application of robustness theory to power systems in [12].

### Closed-Loop Stability

Certainly one task of interest is the determination of closed-loop system stability solely from the properties of  $\mathcal{D}(s)$  and  $\mathcal{K}(s)$ , the *compensated* damping and stiffness matrices. Such extensions will be presented in what follows, in the form of *sufficiency* conditions for the stability of the compensated system (5). Note first that the characteristic polynomial  $\phi(s)$  of the closed-loop system is simply:

$$\phi(s) = \det W(s) \cdot \det(sI - A) \quad (6)$$

where  $W(s)$  denotes the matrix in square brackets in (5), i.e.

$$W(s) = s^2I + s\mathcal{D}(s) + \mathcal{K}(s) \quad (7)$$

The closed-loop system is stable if and only if  $\phi(s)$  has no zeros in the closed right half-plane (RHP). Now since (3) is assumed to be stable,  $\det(sI - A)$  is nonzero in RHP. Hence  $\phi(s)$  has a zero at a point  $s_0$  in RHP if and only if  $\det W(s)$  becomes 0 at  $s = s_0$ , which is in turn equivalent to:

$$W(s_0)w_0 = 0, \quad w_0 \neq 0. \quad (8)$$

Equivalently, the closed-loop system is stable if and only if  $W(s)$  has no latent values in RHP. The following lemma, which involves checking the stability of a *family* of complex but *non-dynamic* second-order matrix polynomials, therefore constitutes a sufficient condition for stability of (5):

**Lemma 1** *The matrix second-order system (5) is stable if the following (frozen) complex matrix polynomial is stable for every  $\zeta$  in the closed right half-plane:*

$$\widetilde{W}(\zeta, s) = s^2I + s\mathcal{D}(\zeta) + \mathcal{K}(\zeta) \quad (9)$$

**Proof:** The lemma can be proven by contradiction. First note that  $\mathcal{D}(\zeta)$  and  $\mathcal{K}(\zeta)$  are finite for all  $\zeta$  in RHP because of the assumed stability of (3). Now, suppose the hypotheses of the lemma are true but the system (5) is unstable. In this case  $W(s)$  in (7) has at least one latent root  $s_0$  in RHP, and so  $\widetilde{W}(s_0, s)$  must possess a root in RHP also (namely at  $s = s_0$ ). But this contradicts the hypothesis that  $\widetilde{W}(\zeta, s)$  is stable for every  $\zeta$  in the right half-plane.  $\square$

The study of the stability of families of matrix polynomials is an area of current interest [13, 14, 15, 16, 17]. We use the approach in [17, 3] to further simplify our analyses by embedding the frozen matrix polynomials (9) in the study of complex *scalar* polynomials, for which simple and exact stability conditions are known. To this end we make the following conservative definition of stability [17]:

**Definition 1 (Strongly Stable Systems)** *A complex-coefficient matrix polynomial  $P(s) = Is^n + P_1s^{n-1} + P_2s^{n-2} + \dots + P_n$  is strongly stable if the derived scalar polynomial defined by  $p(s, v) = v^*P(s)v$  is stable for all unit-magnitude complex vectors  $v$ . A set of matrix polynomials is labeled *strongly stable* if all of its elements are strongly stable.*

(Here  $M^*$  is used to denote the complex conjugate transpose of the complex matrix or vector  $M$ .) The reason for the terminology is that the domain of strongly stable  $P(s)$  is contained *within* the stability domain comprising all stable  $P(s)$  – stability of  $p(s, v)$  for all  $v$  is only a sufficient condition for stability of  $P(s)$ . Thus, strongly stable matrix polynomials are stable, but not necessarily vice

versa. Our definition, however, may be used to obtain a condition that depends solely on the properties of the coefficient matrices.

In particular, suppose we let  $M_H \equiv (M + M^*)/2$  denote the Hermitian part of the complex matrix  $M$ ,  $M_S \equiv (M - M^*)/2j$  the skew-Hermitian part,  $\underline{\lambda}(\cdot)$  and  $\overline{\lambda}(\cdot)$  the minimum and maximum eigenvalue of the argument respectively, and  $\Re(\cdot)$  and  $\Im(\cdot)$  the real and imaginary part of the argument. The coefficients  $p_i(v)$  of  $p(s, v)$  are given by  $p_i(v) = v^* P_i v$ . We then have that  $\underline{\lambda}[P_{i_H}] \leq \Re[p_i(v)] \leq \overline{\lambda}[P_{i_H}]$ , and similarly for the other parts of the coefficients. Thus the coefficients of the scalar polynomial  $p(v, s)$  are constrained to lie in bounded intervals. Powerful results exist for the stability of such interval polynomials [18, 19], involving only certain extreme polynomials of the whole set.

The concept of strong stability is now used to simplify our situation, at the cost of a corresponding increase in conservativeness. Combining Definition 1 and Lemma 1 yields the following result:

**Result 1** *The matrix second-order system (5) is stable if the following frozen matrix polynomial is strongly stable for every  $\zeta$  in the closed right half-plane:*

$$\widetilde{W}(\zeta, s) = s^2 I + s \mathcal{D}(\zeta) + \mathcal{K}(\zeta) \quad (10)$$

As suggested above, this result is attractive when the special structure of the vibrational system and its compensator, specifically the complex coefficient matrices  $\mathcal{D}(\zeta)$  and  $\mathcal{K}(\zeta)$ , can be exploited to reduce the stability test to a finite one, perhaps at only a few extreme points in the complex plane.

The usefulness of the above conditions, despite their inherent conservativeness, may be demonstrated by applying them to the second-order matrix polynomial governing the *open-loop* system (2). The following results are obtained:

**Result 2** *The open-loop system (2) is strongly stable (and thus stable) if  $D$  is symmetric and positive definite and*

$$\underline{\lambda}^2[D] \underline{\lambda}[K_H] > \overline{\lambda}^2[K_S].$$

**Result 3** *The open-loop system (2) is strongly stable if  $K$  is symmetric and positive definite and*

$$\underline{\lambda}[D_H] > 0$$

The first result implies that the real damping, represented by the symmetric  $D$  and the symmetric part of the spring forces, represented by  $K_H$ , should be “large” compared to the circulatory forces represented by  $K_S$  [5]. The condition in the second result can be recognized as being *identical* to the sufficiency part of the KTC Theorem [10], a result that is itself necessary *and* sufficient for stability.

### Controller Constraints

We now apply Result 1 to the closed-loop system (5) to obtain results with the classical flavor of Results 2 and 3, that can be of direct use in the design of the controller (3). These conditions are

$$Q = 0, D = D^T > 0$$

Positive Feedback (PCT)* = $T^{-1}BR$	$\frac{\lambda^2[D] (\delta[K_H, (PFR)_H] - \kappa) > (\bar{\lambda}[K_s] + \bar{\lambda}[(PFR)_S] + \kappa/2)^2}{}$
Negative Feedback (PCT)* = $-T^{-1}BR$	$\frac{\lambda^2[D] (\delta[K_H, (PFR)_H]) > (\bar{\lambda}[K_s] + \bar{\lambda}[(PFR)_S] + \kappa/2)^2}{}$

Table 1: Position feedback

$$R = 0, K = K^T > 0$$

Positive Feedback (PCT)* = $T^{-1}BQ$	$\delta[D_H, (PFQ)_H] > \kappa$
Negative Feedback (PCT)* = $-T^{-1}BQ$	$\delta[D_H, (PFQ)_H] > 0$

Table 2: Velocity feedback

in the form of constraints on the controller parameters  $A$ ,  $B$ ,  $C$ , and  $F$  that guarantee the stability of the closed-loop system. We first define the following two parameters:

$$\kappa \equiv \frac{\bar{\lambda}[(PCT)(PCT)^*]}{-\max(\Re[\lambda_i])}$$

$$\delta[A, B] \equiv \lambda[A] - \bar{\lambda}[B]$$

where, as noted before,  $\lambda_i$  are the eigenvalues of the compensator. The first parameter  $\kappa$  is a positive scalar related to the overall coupling between the compensator and the controlled structure, while the second quantity  $\delta[\cdot, \cdot]$  provides a measure of the “distance” between the eigenstructures of two matrices. With these quantities and our previous notation we have the following result:

**Result 4 (Controller Constraints)** *The closed-loop system (1), (3) is strongly stable (hence stable) if the compensator is stable ( $\text{Re}[\lambda_i] < 0$ ) and the conditions of any of the cases in Tables 1, 2, or 3 are satisfied.*

The proof is a straightforward, but tedious, application of Result 1, together with the repeated use of Rayleigh quotient bounding techniques. A sketch of the proof is provided in Appendix B.

### Controller Design Guidelines

The constraints in Tables 1–3 have a number of interesting properties. The equality constraints  $(PCT)^* = \pm T^{-1}BQ$  or  $(PCT)^* = \pm T^{-1}BR$ , may be viewed as a type of generalized actuator and sensor co-location constraint. Further, the inequality conditions imply certain trends or directions in which to “push” the controller design parameters for greater stability. Consider, for example, the case of velocity feedback in Table 2. For positive dynamic feedback, given in the upper row, the

$$D = D^T, K = K^T, R = \alpha Q$$

Positive Feedback $(PCT)^* = T^{-1}BQ$ $\delta[D, (PFQ)_H] > \kappa$	$\alpha > 0$ $(\delta[K, (PFR)_H] - \alpha\kappa)(\delta[D, (PFQ)_H] - \kappa)^2$ $> \alpha^2 (\bar{\lambda}[(PFQ)_S] + \kappa/2)^2$
	$\alpha < 0$ $(\lambda[K] - \alpha\lambda[(PFQ)_H])(\delta[D, (PFQ)_H] - \kappa)^2$ $> (\bar{\lambda}[(PFQ)_S] + \kappa/2)^2 (\alpha^2 + \alpha\delta[(PFQ)_H, D])$
Negative Feedback $(PCT)^* = -T^{-1}BQ$ $\delta[D, (PFQ)_H] > 0$	$\alpha > 0$ $\delta[K, (PFR)_H] \delta^2[D, (PFQ)_H]$ $> \alpha^2 (\bar{\lambda}[(PFQ)_S] + \kappa/2)^2$
	$\alpha < 0$ $(\lambda[K] - \alpha\lambda[(PFQ)_H] + \alpha\kappa) \delta^2[D, (PFQ)_H]$ $> (\bar{\lambda}[(PFQ)_S] + \kappa/2)^2 (\alpha^2 + \alpha\delta[(PFQ)_H, D] - \alpha\kappa)$

Table 3: Position and velocity feedback

conditions are more readily satisfied if the maximum eigenvalue of  $(PCT)(PCT)^*$  is made smaller and the eigenvalues of the compensator,  $\lambda_i$ , are moved farther to the left in the complex plane. These two actions serve to decrease the coupling parameter  $\kappa$ .

Conversely, the conditions may be viewed as placing a restriction on the magnitude of  $\kappa$  and hence  $(PCT)(PCT)^*$ , thus serving as a kind of gain limit on the dynamic part of the compensator. This interpretation appeals to intuition, since we are dealing with a positive feedback through this path and, at high enough gain, positive feedback tends to create instability. Note that no such limit exists for the case of negative velocity feedback, given in the bottom right of Table 2. Again, this is in keeping with intuition, since velocity feedback places zeros at the origin. Similar observations may be made for the other cases and serve as guidelines to the designer and as ties to the classical single-input/single-output case.

## 4 Example

To illustrate the use of the conditions of Result 4, a simple scalar example investigating the dynamic feedback of velocity in an undamped second-order scalar system is given next. Consider the system shown in Figure 1, defined by the equations:

$$\begin{aligned} \ddot{x} + kx &= u \\ y &= \dot{x} \end{aligned} \tag{11}$$

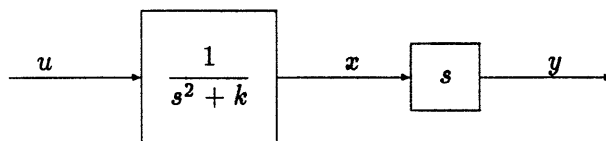


Figure 1: Scalar second-order system.

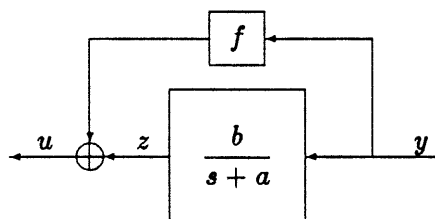


Figure 2: Controller for system.

To this system add the dynamic controller shown in Figure 2 and described by the equations:

$$\begin{aligned} \dot{z} &= -az + by \\ u &= z + fy \end{aligned} \quad (12)$$

Combining the system (11) and the controller (12) and expressing the result in second-order form, as described earlier, we obtain the closed-loop system:

$$s^2 + \left[ -f + \frac{-b}{s+a} \right] s + k = 0 \quad (13)$$

Applying Result 4 yields the following two sets of *sufficient* conditions for the stability of (13):

$$\text{either } \left\{ \begin{array}{l} k > 0 \\ a > 0 \\ b = 1 \\ f < -1/a \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} k > 0 \\ a > 0 \\ b = -1 \\ f < 0 \end{array} \right\}$$

Only the first condition in both sets involves the uncompensated system. The remaining three conditions concern the compensator parameters, over which we have full control. Clearing fractions and applying the Routh-Hurwitz criterion to (13), the *necessary and sufficient* stability conditions can be verified to be:

$$\begin{aligned} a &> f \\ ka &> 0 \\ k &> af + b \\ -(kf)/(a - f) &> af + b \end{aligned}$$

It can be easily checked that the exact conditions are satisfied when the sufficient conditions are satisfied.



## 5 Conclusions

We have shown that stabilizing compensator constraints may be obtained for controlled vibrational systems by working directly with their second-order form. These constraints are in terms of the original system and compensator parameters and require no reduced-order model, thereby avoiding the possibility of “spill-over” from the original (and usually large) model to the reduced model. The constraints are sufficient conditions for stability, and as such guarantee stability of the compensated system. The cost for such guarantees is conservativeness of the associated constraints. In spite of this conservativeness, our stability conditions appear useful and have the benefit of retaining the physical intuition associated with the original second-order system model.

### A Notation

$\mathcal{D}(s)$	Compensated damping matrix, $D - PH(s)Q$ .
$\mathcal{K}(s)$	Compensated stiffness matrix, $K - PH(s)R$ .
$M^*$	Conjugate transpose of $M$ .
$\bar{\lambda}(M)$	Maximum eigenvalue of $M$ .
$\underline{\lambda}(M)$	Minimum eigenvalue of $M$ .
$M_H$	Hermitian part of $M$ , $(M + M^*)/2$ .
$M_S$	Skew-Hermitian part of $M$ , $(M - M^*)/2j$ .
$\Re(a)$	Real part of $a$ .
$\Im(a)$	Imaginary part of $a$ .
$\kappa$	The coupling parameter, $-\bar{\lambda}[(PCT)(PCT)^*] / \max(\Re[\lambda_i])$
$\delta(A, B)$	The difference in the eigenstructures of $A$ and $B$ , $\underline{\lambda}[A] - \bar{\lambda}[B]$

Table 4: Notation

### B Proof of Controller Constraints

This appendix presents a proof of Result 4, the controller constraints of Section 3. Because of the tedium involved, only one case is proven. The other cases are shown in a similar manner. The most complicated case of Result 4 concerns position and velocity feedback. Suppose that we consider the case of positive feedback given by the first row of Table 3. For this case we have that  $D = D^T$ ,  $K = K^T$ ,  $R = \alpha Q$  with  $\alpha > 0$  a scalar and  $(PCT)^* = T^{-1}BQ$ . The system under study is then

$$W(s) = Is^2 + \mathcal{D}(s)s + \mathcal{K}(s) \quad (14)$$

where  $\mathcal{D}(s) = D - PH(s)Q$  and  $\mathcal{K}(s) = K - \alpha PH(s)Q$ . The compensator transfer function  $H(s)$  is given by:

$$H(s) = F + C(sI - A)^{-1}B = F + CT \text{diag} [(s - \lambda_i)^{-1}] T^{-1}B$$

where  $T^{-1}AT = \text{diag}(\lambda_i)$ . Combination of the system and compensator  $H(s)$  then yields

$$\begin{aligned} \mathcal{D}(s) &= D - PFQ - PCT \text{diag} [(s - \lambda_i)^{-1}] T^{-1}BQ \\ \mathcal{K}(s) &= K - \alpha PFQ - \alpha PCT \text{diag} [(s - \lambda_i)^{-1}] T^{-1}BQ \end{aligned} \quad (15)$$

Let the scalar polynomial  $p(s, \zeta, v)$  be defined by

$$p(s, \zeta, v) \equiv s^2 + d(\zeta, v)s + k(\zeta, v) \quad (16)$$

where  $d(\zeta, v) \equiv v^* \mathcal{D}(\zeta)v$  and  $k(\zeta, v) \equiv v^* \mathcal{K}(\zeta)v$ . Then recall that the system (14) is stable if  $p(s, \zeta, v)$  is stable for all  $\zeta$  in the *right* half-plane and all unit complex vectors  $v$ . For a fixed  $\zeta, v$  the complex scalar polynomial (16) will be stable if and only if:

$$\begin{aligned} \Re[d(\zeta, v)] &> 0 \\ \Re[d(\zeta, v)] \Im[d(\zeta, v)] \Im[k(\zeta, v)] + \Re^2[d(\zeta, v)] \Re[k(\zeta, v)] - \Im^2[k(\zeta, v)] &> 0 \end{aligned} \quad (17)$$

where:

$$\begin{aligned} \Re[d(\zeta, v)] &= v^* \mathcal{D}_H(\zeta) v & \Im[d(\zeta, v)] &= v^* \mathcal{D}_S(\zeta) v \\ \Re[k(\zeta, v)] &= v^* \mathcal{K}_H(\zeta) v & \Im[k(\zeta, v)] &= v^* \mathcal{K}_S(\zeta) v \end{aligned} \quad (18)$$

Since  $\mathcal{K}_S(\zeta) = \alpha \mathcal{D}_S(\zeta)$ , it follows that  $\Im[k(\zeta, v)] = \alpha \Im[d(\zeta, v)]$ , so that the inequalities in (17) that we must satisfy are equivalent to:

$$\begin{aligned} \Re[d(\zeta, v)] &> 0 \\ \alpha \Re[d(\zeta, v)] \Im^2[d(\zeta, v)] + \Re^2[d(\zeta, v)] \Re[k(\zeta, v)] - \Im^2[k(\zeta, v)] &> 0 \end{aligned} \quad (19)$$

Now it is straightforward to show that

$$\begin{aligned} \mathcal{D}_H(\zeta) &= D - (PFQ)_H - (PCT) \operatorname{diag} \left[ \frac{\Re(\zeta - \lambda_i)}{|\zeta - \lambda_i|^2} \right] (PCT)^* \\ \mathcal{D}_S(\zeta) &= - (PFQ)_S + (PCT) \operatorname{diag} \left[ \frac{\Im(\zeta - \lambda_i)}{|\zeta - \lambda_i|^2} \right] (PCT)^* \\ \mathcal{K}_H(\zeta) &= K - \alpha(PFQ)_H - \alpha(PCT) \operatorname{diag} \left[ \frac{\Re(\zeta - \lambda_i)}{|\zeta - \lambda_i|^2} \right] (PCT)^* \\ \mathcal{K}_S(\zeta) &= \alpha \mathcal{D}_S(\zeta) \end{aligned}$$

Note that only the last terms in the above expressions depend on  $\zeta$ . It can be shown that:

$$\begin{aligned} 0 &< v^* (PCT) \operatorname{diag} \left[ \frac{\Re(\zeta - \lambda_i)}{|\zeta - \lambda_i|^2} \right] (PCT)^* v < \kappa \\ -\kappa/2 &< v^* (PCT) \operatorname{diag} \left[ \frac{\Im(\zeta - \lambda_i)}{|\zeta - \lambda_i|^2} \right] (PCT)^* v < \kappa/2 \end{aligned} \quad (20)$$

where we have used the notion defined in Section 3. Using the inequalities (20) together with Rayleigh quotient bounds [20] we may show the following bounds on  $\Re[d(\zeta, v)]$ ,  $\Im[d(\zeta, v)]$ , and  $\Re[k(\zeta, v)]$ :

$$\begin{aligned} \Re[d(\zeta, v)] &> \underline{\lambda}(D) - \bar{\lambda}(PFQ)_H - \kappa \\ |\Im[d(\zeta, v)]| &< \bar{\lambda}(PFQ)_S + \kappa/2 \\ \Re[k(\zeta, v)] &> \underline{\lambda}(K) - \alpha \bar{\lambda}(PFQ)_H - \alpha \kappa \end{aligned} \quad (21)$$

where we have used the fact that  $-\underline{\lambda}(A_S) = \bar{\lambda}(A_S)$  for any real  $A$ .

Using the bounds in (21), it is straightforward to show that the inequalities in (19), and hence those in (17), are satisfied for all  $\zeta$  and  $v$  if

$$\underline{\lambda}(D) - \bar{\lambda}(PFQ)_H - \kappa > 0$$

and

$$\left[\underline{\lambda}(D) - \bar{\lambda}(PFQ)_H - \kappa\right]^2 \left[\underline{\lambda}(K) - \alpha \bar{\lambda}(PFQ)_H - \alpha\kappa\right] > \alpha^2 \left[\bar{\lambda}(PFQ)_S + \kappa/2\right]^2$$

Again, using the notation defined in Section 3 and the fact that  $R = \alpha Q$ , these inequalities may be rewritten as follows:

$$\begin{aligned} \delta[D, (PFQ)_H] - \kappa &> 0 \\ (\delta[D, (PFQ)_H] - \kappa)^2 (\delta[K, (PFR)_H] - \alpha\kappa) &> \alpha^2 \left(\bar{\lambda}[(PFQ)_S] + \kappa/2\right)^2 \end{aligned} \quad (22)$$

The inequalities (22) are precisely the conditions given in Result 4 for positive feedback of position and velocity when  $\alpha > 0$ .

## References

- [1] R. Clough and J. Penzien, *Dynamics of Structures*. New York: McGraw-Hill, 1975.
- [2] L. Meirovitch, *Elements of Vibration Analysis*. New York: McGraw-Hill, 1975.
- [3] W. C. Karl, "Geometry of vibrational system stability domains with application to control," Master's thesis, Massachusetts Institute of Technology, Jan. 1984.
- [4] T. Yam, T. L. Johnson, and J. H. Lang, "Flexible system model reduction and control system design based upon actuator and sensor influence functions," *IEEE Transactions on Automatic Control*, vol. AC-32, pp. 573-582, 1987.
- [5] K. Huseyin, *Vibrations and Stability of Multiple Parameter Systems*. London: Noordhorr, 1978.
- [6] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*. New York: Academic, 1982.
- [7] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.
- [8] L. Kelvin and P. Tait, *Principles of Mechanics and Dynamics*. New York: Dover, 1962.
- [9] H. Beth, "On the stabilization of instable equilibrium by means of gyroscopic forces," *Philosophical Magazine*, Jan-Jun 1925.
- [10] E. E. Zajac, "The Kelvin-Tait-Chetaev theorem and extensions," *Journal of the Astronautical Sciences*, vol. 11, no. 2, pp. 46-49, 1964.
- [11] H. K. Wimmer, "Inertia theorems for matrices, controllability, and linear vibrations," *Linear Algebra and its Applications*, no. 8, 1978.
- [12] S. M. Chan and M. Athans, "Application of robustness theory to power system models," *IEEE Transactions on Automatic Control*, vol. AC-29, no. 1, 1984.
- [13] L. S. Shieh, M. M. Mehio, and M. D. Hani, "Stability of the second-order matrix polynomial," *IEEE Transactions on Automatic Control*, vol. AC-32, p. 231, Mar. 1987.

- [14] B. R. Barmish, M. Fu, and S. Saleh, "Stability of a polytope of matrices: Counterexamples," *IEEE Transactions on Automatic Control*, vol. 33, pp. 569–572, June 1988.
- [15] D. Petkovski, "Stability analysis of interval matrices: Improved bounds," *International Journal of Control*, vol. 48, no. 6, pp. 2265–2273, 1988.
- [16] Y. K. Foo and Y. C. Soh, "Stability analysis of a family of matrices," *IEEE Transactions on Automatic Control*, vol. 35, pp. 1257–1259, Nov. 1990.
- [17] W. C. Karl and G. C. Verghese, "A sufficient condition for the stability of interval matrix polynomials," *IEEE Transactions on Automatic Control*, Apr. 1993.
- [18] V. L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," *Differential'nye Uravneniya*, vol. 14, no. 11, pp. 2086–2088, 1978.
- [19] R. J. Minnichelli, J. J. Anagnost, and C. A. Desoer, "An elementary proof of Kharitonov's stability theorem with extensions," *IEEE Transactions on Automatic Control*, vol. AC-34, pp. 995–998, Sept. 1988.
- [20] G. Strang, *Linear Algebra and its Applications*. New York: Academic, 1980.