The Role of Tachyons and Dilatons in Off-Shell String Field Theory

by

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Abstract

We use the techniques of covariant closed string field theory to investigate off-shell properties of the tachyon and the dilaton. In our study of the tachyon we investigate its effective potential in the tree approximation. For this purpose we derive general formulae for calculation of the tree-level off-shell amplitudes and discuss their relation to the effective potential. We derive explicit modular invariant formulae for tachyonic amplitudes and the coefficients in the tachyonic potential and prove that closed string polyhedra, among all possible choices of string vertices, yield a tachyon potential which is as small as possible order by order in the string coupling constant. We apply our general analysis to the case of four-tachyon interaction. We investigate both the elementary coupling and the coupling mediated by massive intermediate states. We show that the elementary coupling presents the major contribution to the four-tachyon interaction. We also show that the fourth order term in the tachyonic potential destroys a local minimum that exists in the cubic approximation. We complete the proof of off-shell dilaton theorem by proving it for the matter part of the full dilaton field. Our proof is based on the observation that a particular linear combination of the ghost and the matter parts of the dilaton becomes BRST trivial in an extended complex which incorporates the string center of mass as a legal operator. We argue that in an off-shell approach this is a natural choice which does not lead to any contradiction when string amplitudes are treated as distributions in the momentum space. We present a complete analysis of the BRST cohomology of the extended complex. We show that all the states capable, according to the dilaton theorem, of changing the string coupling constant belong to the same cohomology class in the extended complex. In addition we prove that for the $D = 2$ string, for which this cohomology class is trivial, the string coupling constant is not an observable parameter of the background. We show that this observation is true in general—in the backgrounds where the dilaton generates a trivial cohomology class the coupling constant is not an observable parameter.

Thesis Supervisor: Barton Zwiebach
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to my wife Mariya

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Chapter 1

Introduction

1.1 Motivations

String Theory originated in late sixties in an attempt to construct a theory of strong interactions [1–4]. It was unsuccessful because it contradicted experimental data. The most apparent inconsistency with the experiment was the proliferation of massless fields and the appearance of a scalar field with negative mass squared—the tachyon. Yet it was soon realized that massless states, which were phenomenologically undesirable in the theory of hadrons, can describe Yang-Mills gauge fields [5,6] and the gravitational field of General Relativity [7]. Based on these discoveries a new philosophy was proposed in 1974 by Scherk and Schwarz [8] who suggested that String Theory can be viewed as a unification of gauge theories with gravity. Quantum theory of gauge fields was a tremendous phenomenological and theoretical success which culminated with the formulation of the Standard Model. On the other hand, despite numerous attempts, a fully satisfactory quantum field theory of General Relativity had never been constructed. Thus String Theory became the only satisfactory quantum theory which naturally combines gauge fields with gravity. Since then String Theory took a stage as the most (the only?) promising candidate for the unified theory of Nature [9].

String Theory is particularly attractive as a unified theory because it appears to have no adjustable parameters. Consistency conditions leave only five possibilities for String Theory in 10 dimensions (and no consistent theory exists in $D > 10$). They are a theory of nonorientable (type I) open and closed superstrings with gauge group $SO(32)$ [10], two theories of orientable (type IIA and IIB) closed superstrings [11], and two heterotic string theories with $SO(32)$ and $E_8 \times E_8$ gauge groups [12]. In each of these theories there is only one dimensionless parameter, the string coupling constant, which can be related to a vacuum expectation value of a scalar field—the dilaton. When we start building models with lower space-time dimension by compactifying, *à la* Kaluza-Klein, the extra dimensions on some small compact manifolds, the resulting theories seem to have a lot of adjustable dimensionless parameters (*i.e.*, the ratios of compactification radii for toroidal compactifications). All these parameters, however, can be interpreted as vacuum condensates of different scalar fields present in the theory. This said, an open question remains: how does the theory decide which vacuum to choose or, in other words, why do we live in four dimensions with four fundamental forces?

In order to find the true vacuum in quantum field theory, one has to calculate the effective potential of the theory and find its minimum. The effective potential is the generating functional for zero-momentum Green functions and these are off-shell amplitudes, unless the corresponding fields are massless. A pragmatic point of view is that only massless fields are important for phenomenology, since already the first mass level is of order of Planck mass, 10^{19}GeV, and cannot significantly affect the low energy physics. Restricting the attention to massless fields enormously simplifies the problem of determining the effective (low energy) action and indeed, the low energy behavior of superstrings has been successfully identified with supergravity theories.

The study of these low energy effective theories has revealed many surprising connections between different string theories. Recent progress in this direction has given further evidence that String Theory is the right way to go in our quest for the theory of everything, and at the same time, it became likely that String Theory is not the end of the story. Higher dimensional extended objects will supposedly play an important role in the future developments.

Despite a lot of progress in superstring theory, very little we know about the global theory, the one which would incorporate all the fields of String Theory and provide mechanisms for their condensation to the known vacuum values. String Field Theory (SFT) is a possible candidate for such global theory. Ideally, SFT should be formulated as a quantum field theory without an explicit reference to any particular vacuum or background. Different backgrounds should appear as solutions to the classical equations of motion. Perturbation theory around them should coincide with a theory of strings propagating in this background.

Significant progress has been achieved in formulating such a theory for the case of bosonic strings. The complete quantum theory of covariant closed strings was constructed [13]. This theory was formulated around a background represented by any choice of $c = 26$ conformal field theory (CFT)[1]. A string field was associated with every state in this CFT, and the full quantum action was constructed as a series expansion in string field products. The theory obtained is not manifestly background independent, since a choice of a background has to be made before we can construct the action and even then the action is defined only locally. Nevertheless, there is strong evidence that this local action can be extended to a space of string field configurations not already known (CFT state space should appear as a tangent space at a stable point). The primary piece of this evidence is the proof of local background independence. Roughly speaking, local background independence means that two string field actions built around two nearby backgrounds (or CFTs) are the same up to a string field redefinition [14, 15].

A manifestly background independent formulation of string field theory with a globally defined string action arising non-perturbatively from some kind of geometrical principle is clearly desirable in order to tell how nature chooses a background

[1]The standard flat background would be represented by a CFT of 26 free bosons ∂X^{μ}.

for us to live. At the moment, no such principle has been proposed, and the locally background independent string field theory is probably the best tool we have to try to find it. In this work we will use this theory to obtain results which cannot be obtained without it.

Our first subject is the tachyonic potential. We previousely mentioned the tachyon which has negative mass squared. On the string field theory language, this means that its potential starts with a negative quadratic term ($V(\tau) = m^2\tau^2 + \cdots$). This indicates that bosonic string background is unstable under tachyon condensation. Closed String Field Theory predicts [[16] that the full potential includes infinitely many terms and thus a possibility exists that it has a minimum whose position would determine[2] the value of the tachyon condensate $\langle\tau\rangle$.

The issue of dilaton is another example where String Field Theory is already capable of giving new insights. As we mentioned above, the dilaton vacuum expectation value determines the coupling constant in string theory. This property established on the level of on-shell string amplitudes can be proven for the full string field action [17, 18].

Our study of the dilaton will lead to a better understanding of the role of the string center of mass operator. This is the conjugate operator to the string momentum and can be represented by a differentiation operator $\hat{x}_0^\mu = \partial/\partial p_\mu$ or infinitesimal shift in momentum space. A shift the momentum space can move a physical state away from its mass-shell; therefore, it cannot be properly realized in the space of physical string states. This calls for an off-shell treatment possible only in the context of string field theory.

Let us speculate on the role of these developments in the overall picture of string theory before we turn to a more detailed summary of our results. We can certainly justify our interest in bosonic string theory just by saying that it provides a toy model where many ideas can be tested before they are applied to more realistic models, but let us try to say more.

During the last year a distinction between more realistic and less realistic string theories has become vague at best. With evidence of string dualities becoming more compelling we begin to understand that two perturbatively different string theories can in fact be just different expansions of the same non-perturbative theory. It is likely that all the consistent superstring theories are related in this way.

The bosonic string does not seem to fit into this picture. The low energy effective action approach is flawed by the presence of the tachyon and since supersymmetry is absent, none of the semiclassical methods that work so well for the superstring can be applied. Nevertheless, at least some evidence has been presented that may reserve a place for the bosonic string in the future theory. This was an observation made a few years ago by Berkovits and Vafa [19] who suggested that bosonic strings may be viewed as a particular class of vacua for $N = 1$ superstrings. If this is the case then there should exist a (probably non-perturbative) mechanism for tachyon condensation in the bosonic string theory such that the theory with a condensate does have supersymmetry.

[2] In section 2.1.2 we will analyze this statement more critically

1.2 Summary of the results

Covariant String Field Theory [13,20] of closed bosonic strings is based on the quantization procedure of Batalin and Vilkovisky [21,22] (BV) and requires a choice of a particular background to be made. Once a choice of background (which, roughly speaking, is a conformal field theory) is made, the BV master action can be found as a non-polynomial functional $S(\Psi)$, where the string field Ψ plays the role of coordinates on the state space of the corresponding conformal field theory. Although the full expression for the action $S(\Psi)$ is known, it is given as a power series in Ψ, each term of which is an integral of some differential form over a specific subspace of the moduli space $\widehat{\mathcal{P}}_{g,n}$ of Riemann surfaces of genus g with n punctures and a choice of local coordinate (up to a phase) around each puncture. These subspaces, called string vertices, must satisfy certain geometrical recursion relations [23] dictated by the BV master equation on $S(\Psi)$. These recursion relations do not specify completely the vertices but there is one solution, given by closed string polyhedra [24], which in many aspects is the most interesting. This solution can be described in terms of minimal area metrics or, equivalently, in terms of quadratic differentials of special kind [25–28]. These quadratic differentials were first studied by Strebel [29] and we will call them Strebel quadratic differentials.

In chapter 2 we derive explicit formulae for evaluation of the classical (genus zero) closed string action and for calculation of arbitrary off-shell amplitudes. The formulae require a parameterization, in terms of some moduli space coordinates, of the family of local coordinates needed to insert the off-shell states on Riemann surfaces.

The focus is then turned to the evaluation of the tachyon potential as a power series in the tachyon field. The expansion coefficients in this series are shown to be geometrical invariants of Strebel quadratic differentials. We show that so defined coefficients, among all possible choices of string vertices, yield the tachyon potential which is as small as possible order by order in the string coupling constant. Our discussion in section 2.4 emphasizes the geometrical meaning of off-shell amplitudes.

In chapter 3.1.4 we apply general formulae for the tachyonic potential to calculate this potential up to the order of τ^4. The evaluation of the coefficient in front of τ^4 is non-trivial because it involves integration over a complicated region in the moduli space of four-punctured spheres and the integrand is defined implicitly in terms of a Strebel quadratic differential. The later problem is reduced to a single equation involving elliptic functions. Surprisingly, the same equation appears in calculation of the effective tachyonic potential at this order. Numerical values obtained show that the bare potential calculated up to the fourth order has no local minimum and that massive states provide only a tiny correction compared to the bare four-tachyon interaction.

The second part of the thesis is devoted to the dilaton. The zero momentum dilaton consists of two parts both annihilated by the BRST charge. We call them the ghost dilaton and the matter dilaton respectively. Only one linear combination (the dilaton) of the ghost dilaton and the matter dilaton remains on-shell for some non-zero values of the momentum. The other combination (the longitudinal graviton) has to be considered as a discrete state at zero momentum. The matter dilaton and the ghost dilaton are two very different states. For example, the matter dilaton is a primary field and the ghost one is not. We show that they, nevertheless, are closely related. We show that both ghost and matter dilatons affect the coupling constant while the longitudinal graviton does not.

We will explain that the reason why the longitudinal graviton does not change the coupling constant is that it can be gauged away with the aid of a gauge parameter which grows linearly in space-time. An ordinary BRST complex used to describe physical states of string does not include such field configurations. We show how the BRST complex can be extended to include field configurations which are polynomial in space-time coordinates and calculate its cohomology. We find that the cohomology of the extended complex has a number of features with good physical interpretation. The most noticeable features are the absence of longitudinal graviton in cohomology and the correspondence between ghost number one cohomology and Poincaré group, as expected from the string field theory. We suggest that for each uncompactified direction in space-time one has to extend the BRST complex by the polynomials in the correspondent coordinates. As a bonus we will see that this approach resolves the problem of doubling of physical states when at least one uncompactified direction is present.

The dilaton theorem has to be reinterpreted for the case of non-critical strings. For a non-critical string the ghost part of the dilaton becomes a BRST trivial state due to the presence of a Liouville field. A detailed analysis of the role of the dilaton for the $D = 2$ string is presented in section 4.7.

Chapter 2

Off-shell closed string amplitudes

2.1 Introduction to off-shell calculations

A manifestly background independent field theory of strings should define the conceptual framework for string theory and should allow the precise definition and explicit computation of nonperturbative effects. The present version of quantum closed string field theory [13, 20], developed explicitly only for the case of bosonic strings, while not manifestly background independent, was proven to be background independent for the case of nearby backgrounds [14,15,30]. The proof indeed uncovered structures that are expected to be relevant to the conceptual foundation of string theory. At the computational level one can ask if present day string field theory allows one to do new computations, in particular computations that are not defined in first quantization. While efficient computation may require the manifestly background independent formulation not yet available, it is of interest to attempt new computations with present day tools.

Off-shell amplitudes are not naturally defined without a field theory. Indeed, while the basic definition of an off-shell string amplitude *is* given in first quantization, off-shell string amplitudes are only interesting if they obey additional properties such as permutation symmetry and consistent factorization. These properties are automatically incorporated when the off-shell amplitudes arise from a covariant string field theory [31].

Off-shell string amplitudes are obtained by integrating over the relevant moduli space of Riemann surfaces differential forms that correspond to the correlators of vertex operators inserted at the punctures of the surfaces and antighost line integrals. The vertex operators correspond to non-primary fields of the conformal field theory. In contrast, in on-shell string amplitudes the vertex operators are always primary fields. In order to insert non-primary fields in a punctured Riemann surface we must choose an analytic local coordinate at every puncture. The moduli space of Riemann surfaces of genus g and N punctures is denoted as $\overline{\mathcal{M}}_{g,N}$, and the moduli space of such surfaces with choices of local coordinates at the punctures is denoted as $\widehat{\mathcal{P}}_{g,N}$.[1] An off-shell amplitude is just an integral over a subspace of $\widehat{\mathcal{P}}_{g,N}$. Typically, the relevant

[1]The local coordinate at each puncture is defined only up to a constant phase.

subspaces of $\widehat{\mathcal{P}}_{g,N}$ are sections over $\overline{\mathcal{M}}_{g,N}$. Such sections are obtained by making a choice of local coordinates at every puncture of each surface in $\overline{\mathcal{M}}_{g,N}$. In closed string field theory, the use of minimal area metrics allows one to construct these sections using the vertices of the theory and the propagator. Off-shell amplitudes arising in open string field theory have been studied by Bluhm and Samuel [32–35].

While interesting in their own right, off-shell amplitudes are not physical observables. More relevant is the evaluation of the string action for any choice of an off-shell string field. This computation would be necessary in evaluating string instanton effects. The string action, apart for the kinetic term, is the sum of string interactions each of which is defined by a *string vertex*, namely, a subspace $\mathcal{V}_{g,N}$ of $\widehat{\mathcal{P}}_{g,N}$. Typically $\mathcal{V}_{g,N}$ is a section over a compact subspace of $\overline{\mathcal{M}}_{g,N}$. Therefore, given an off-shell string field, the contribution to the string action arising from a particular interaction corresponds to a *partially integrated* off-shell amplitude. The classical potential of a field theory in flat Minkowski space is a simple example of the above considerations; it amounts to the evaluation of the action for field configurations that are spacetime constants. Ideally we would like to compute, for the case of bosonic strings formulated around the twenty-six dimensional Minkowski space, the complete classical potential for the string field. This may be eventually possible but we address here the computation of the classical potential for some string modes. In particular we focus in the case of the tachyon of the closed bosonic string.

For the case of open strings some interesting results have been obtained concerning the classical *effective* potential for the tachyon [36]. This potential takes into account the effect of all other fields at the classical level. In the context of closed string field theory only the cubic term in the tachyon potential is known [37]. The possible effects of this term have been considered in Refs. [38–40]. Our interest in the computation of the closed string tachyon potential was stimulated by G. Moore [41] who derived the following formula for the potential $V(\tau)$ for the tachyon field $\tau(x)$

$$V(\tau) = -\tau^2 - \sum_{n=3}^{\infty} \mathrm{v}_N \frac{\tau^N}{N!}, \qquad \text{where} \quad \mathrm{v}_N \sim \int_{\mathcal{V}_{0,N}} \Big(\prod_{I=1}^{N-3} d^2 h_I\Big) \prod_{I=1}^{N} \left|h'_I(0)\right|^{-2} . \quad (2.1.1)$$

This potential is the tachyon potential with all other fields set to zero. It is not an effective potential. It is fully nonpolynomial, and starts with a negative sign quadratic term (the symbols appearing in the expression for v_N will be defined in section 2.2). The calculation of the tachyon potential amounts to the calculation of the constant coefficients v_N for $N \geq 3$. For the cubic term, since $\mathcal{V}_{0,3}$ is a point, the integral is actually not there, and the evaluation of the coefficient of v_3 is relatively straightforward. The higher coefficients are difficult to compute since they involve integrals over the pieces of moduli spaces $\mathcal{V}_{0,N}$.

We will rewrite Eq. (2.1.1) in PSL$(2,\mathbb{C})$ invariant form in order to understand the geometrical significance of the coefficients v_N and to set up a convenient computation scheme. Moreover, we will obtain a generalization of Eq. (2.1.1) valid for any component field of the string field theory. The expression will be given in the operator formalism and will be PSL$(2,\mathbb{C})$ invariant.

2.1.1 Extremal property

We will show that the polyhedral vertices of closed string field theory are the solution to the problem of minimizing recursively the coefficients in the expansion defining the tachyon potential. That is, the choice of the Witten vertex, among all possible choices of cubic string vertices[2] minimizes v_3. Once the three string vertex is chosen, the region of moduli space to be covered by the four string vertex is fixed. The choice of the standard tetrahedron for the four string vertex, among all possible choices of four string vertex filling the required region, will minimize the value of v_4. This continues to be the case for the complete series defining the classical closed string field theory. This fact strikes to us as the string field theory doing its best to obtain a convergent series for the tachyon potential. It is also interesting that a simple demand, that of minimizing recursively the coefficients of the tachyon potential, leads uniquely to the polyhedral vertices of classical closed string field theory. It has been clear that the consistency of closed string field theory simply depends on having a choice of string vertices giving a cover of moduli space. The off-shell behavior, however, is completely dependent on the choice of vertices, and one intuitively feels there are choices that are better than other. We see here nice off-shell behavior arising from polyhedra.

2.1.2 A minimum in the potential?

In calculating the tachyon potential we must be very careful about sign factors. The relative signs of the expansion coefficients are essential to the behavior of the series. We find that all the even terms in the tachyon potential, including the quadratic term, come with a negative sign, and all the odd terms come with a positive sign. It then follows, by a simple sign change in the definition of the tachyon field, that all the terms in the potential have negative coefficients. This implies that there is no global minimum in the potential since the potential is not bounded from below. Moreover, there is no local minimum that can be identified without detailed knowledge of the complete series defining the tachyon potential. If the series defining the potential has no suitable radius of convergence further complications arise in attempting to extract physical conclusions. We were not able to settle the issue of convergence, but present some work that goes in this direction. In estimating the coefficient v_N we must perform an integral of the tachyon off-shell amplitude over $\mathcal{V}_{0,N}$. In this region the tachyon amplitude varies strongly. In the middle region the amplitude is lowest, and if this were the dominating region, we would get convergence. In some corners of $\mathcal{V}_{0,N}$ the amplitude is so big that, if those corners dominate, there would be no radius of convergence.

It is important to emphasize that only the tachyon effective potential (or the full string field potential) is a significant object. The tachyon potential is not by itself sufficient to make physical statements. A stable critical point of this potential may not even be a critical point of the complete string field potential. The effects of the infinite number of massive scalar fields must be taken into account. Our results, making

[2]Vertices are defined by coordinate curves surrounding the punctures and defining disks. The disks should not have finite intersection.

unlikely the existence of a stable critical point reinforce the sigma-model arguments that suggest that bosonic strings do not have time independent stable vacua [42], but are not conclusive. (See also Ref. [43] for a discussion of tachyonic ambiguities in the sigma model approach to the string effective action.) The calculation of the full string field potential, or the tachyon effective potential is clearly desirable. We discuss in appendix A string field redefinitions, and argue that it does not seem possible to bring the string action into a form (such as one having a purely quadratic tachyon potential) where one can easily rule out the existence of a local minimum.

2.1.3 General off-shell amplitudes

Since general off-shell computations do not have some of the simplifying circumstances that are present for the tachyon (such as being primary, even off-shell), we derive a general formula useful to compute arbitrary off-shell amplitudes. This formula, written in the operator formalism, gives the integrand for generic string amplitudes as a differential form in $\widehat{\mathcal{P}}_{0,N}$. The only delicate point here is the construction of the antighost insertions for Schiffer variations representing arbitrary families of local coordinates (local coordinates at the punctures as a function of the position of the punctures on the sphere). Particular cases of this formula have appeared in the literature. If the family of local coordinates happens to arise from a metric, the required antighost insertions were given in Ref. [44]. Antighost insertions necessary for zero-momentum dilaton insertions were calculated in Refs. [45].

2.1.4 Organization of the contents of this chapter

We now give a brief summary of the contents of the present chapter. In section 2.2 we explain what needs to be calculated to extract the tachyon potential, set up our conventions, and summarize all our results on the tachyon potential. In section 2.3 we prepare the grounds for the geometrical understanding of the off-shell amplitudes. We review the definition of the mapping radius of punctured disks and study its behavior under PSL$(2,\mathbb{C})$ transformations (the conformal maps of the Riemann sphere to itself). We show how to construct PSL$(2,\mathbb{C})$ invariants for punctured spheres equipped with coordinate disks, by using the mapping radii of the punctured disks and coordinate differences between punctures, both computed using an arbitrary uniformizer. We review the extremal properties of Jenkins-Strebel quadratic differentials [29], and show how our PSL$(2,\mathbb{C})$ invariants, in addition to having extremal properties, provide interesting (and seemingly new) functions on the moduli spaces $\overline{\mathcal{M}}_{0,N}$.[3] In section 2.4 we compute the off-shell amplitude for scattering of N tachyons at arbitrary momentum, and give the answer in terms of integrals of PSL$(2,\mathbb{C})$ invariants. This formula is the off-shell extension of the Koba-Nielsen formula. At zero momentum and partially integrated over moduli space, it gives us, for each N, the coefficient v_N of the tachyon potential. We show why these coefficients are minimized recursively by the string

[3] These functions are analogous to the function that assigns to an unpunctured Riemann surface the area of the minimal area metric on that surface.

vertices defined by Strebel differentials. In section 2.5 we do large$-N$ estimates for the coefficients v_N of the tachyon potential in an attempt to establish the existence of a radius of convergence for the series. The measure of integration is computed exactly for corners in $\mathcal{V}_{0,N}$ representing a planar configuration for the tachyon punctures on the sphere. We are also able to estimate the measure of integration for a uniform distribution of punctures on the sphere. In section 2.6 we give the operator formalism construction for general differential forms on $\widehat{\mathcal{P}}_{0,N}$ labelled by arbitrary off-shell states.

2.2 String action and the tachyon potential

In this section we will show what must be calculated in order to obtain the tachyon potential. This will help to put in perspective the work that will be done in the next few sections. We will also give some of the necessary conventions, and we will comment on the significance of the tachyon potential and the limitations of our results. All of our results concerning the tachyon potential will be summarized here.

The full string field action is a non-polynomial functional of the infinite number of fields, and from the component viewpoint, a non-polynomial function of an infinite number of spacetime fields. Here we consider only the part of it which contains the tachyon field $\tau(x)$. We will call this part the tachyonic action $S^{\text{tach}}(\tau)$. It is a nonpolynomial, non-local functional of the tachyonic field $\tau(x)$. In order to introduce the string field configuration associated to the tachyon field $\tau(x)$ we first a Fourier transform

$$\tau(p) = \int d^D x \ \tau(x) \, e^{-ipx} \,, \tag{2.2.1}$$

and use $\tau(p)$ to define the tachyon string field $|T\rangle$ as follows

$$|T\rangle = \int \frac{d^D p}{(2\pi)^D} \, \tau(p) \, c_1 \bar{c}_1 |\mathbf{1}, p\rangle. \tag{2.2.2}$$

In the conformal field theory representing the bosonic string, the tachyon vertex operator is given by $T_p = c\bar{c}e^{ipX}$ and is of conformal dimension $(L_0, \overline{L}_0) = (-1 + p^2/2, -1+p^2/2)$. The conformal field theory state associated to this field is $T_p(0)|\mathbf{1}\rangle = c_1\bar{c}_1|\mathbf{1}, p\rangle$. This state is BRST invariant when we satisfy the on-shell condition $L_0 = \overline{L}_0 = 0$, which requires $p^2 = 2 = -M^2$ (this is the problematic negative mass squared of the tachyon). The above representative T_p for the cohomology class of the physical tachyon is particularly nice, because this tachyon operator remains a primary field even off-shell ($p^2 \neq 2$).

The tachyonic action is then given by evaluating the string field action $S(|\Psi\rangle)$ for $|\Psi\rangle = |T\rangle$:

$$S^{\text{tach}}(\tau) = S\ (\,|\Psi\rangle = |T\rangle\,)\,, \tag{2.2.3}$$

where

$$S(\Psi) = \frac{1}{2}\langle\Psi|c_0^- Q|\Psi\rangle + \sum_{N=3}^{\infty} \frac{\kappa^{N-2}}{N!}\{\Psi^N\}_{\mathcal{V}_{0,N}}, \tag{2.2.4}$$

and κ is the closed string field coupling constant (see [46]). This action satisfies the classical master equation $\{S, S\} = 0$ when the string vertices $\mathcal{V}_0 = \sum_{N\geq 3} \mathcal{V}_{0,N}$ are chosen to satisfy the recursion relations $\partial\mathcal{V}_0 + \frac{1}{2}\{\mathcal{V}_0, \mathcal{V}_0\} = 0$ (see Ref. [30])

Let us verify that the above definitions lead to the correctly normalized tachyon kinetic term

$$S^{\text{tach}}_{\text{kin}} = \frac{1}{2}\langle T|c_0^- Q|T\rangle\,. \tag{2.2.5}$$

Recall that the BRST operator Q is of the form $Q = c_0 L_0 + \bar{c}_0 \bar{L}_0 + \cdots$, where the dots denote the terms which annihilate $|T\rangle$. Moreover, acting on the state $c_1\bar{c}_1|\mathbf{1}, p\rangle$ the operators L_0 and $\bar{L}_0$ both have eigenvalue $p^2/2 - 1$. We then find

$$S^{\text{tach}}_{\text{kin}} = \frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\int \frac{d^Dp'}{(2\pi)^D}\langle -p', \mathbf{1}|c_{-1}\bar{c}_{-1}c_0^- c_0^+ c_1\bar{c}_1|p, \mathbf{1}\rangle\; \tau(p')\,(p^2-2)\,\tau(p). \tag{2.2.6}$$

We follow the conventions of Ref. [46] where

$$\langle -p', \mathbf{1}|c_{-1}\bar{c}_{-1}c_0^- c_0^+ c_1\bar{c}_1|p, \mathbf{1}\rangle \equiv (2\pi)^D\,\langle -p', \mathbf{1}^c|p, \mathbf{1}\rangle = (2\pi)^D\delta^D(p'+p)\,, \tag{2.2.7}$$

and $c_0^\pm = \frac{1}{2}(c_0 \pm \bar{c}_0)$. Using this we finally find

$$S^{\text{tach}}_{\text{kin}} = -\frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\tau(-p)(p^2-2)\tau(p)\,, \tag{2.2.8}$$

which is indeed the correctly normalized kinetic term.[4] The N-th term in the expansion of the tachyonic action requires the evaluation of string multilinear functions

$$S^{\text{tach}}_{0,N}(\tau) = \frac{\kappa^{N-2}}{N!}\{T^N\}_{\mathcal{V}_{0,N}}\,, \tag{2.2.9}$$

and this will be one of the main endeavors in this paper. The answer will be of the form

$$\{T^N\}_{\mathcal{V}_{0,N}} = \int \prod_I \frac{dp_I}{(2\pi)^D}\,(2\pi)^D\delta\Big(\sum p_I\Big)\cdot V_N(p_1,\cdots p_N)\,\tau(p_1)\cdots\tau(p_N)\,, \tag{2.2.10}$$

where V, the function we will be calculating, is well defined up to terms that vanish upon use of momentum conservation. To extract from this the tachyon potential we evaluate the above term in the action for spacetime constant tachyons $\tau(x) = \tau_0$, which gives $\tau(p) = \tau_0(2\pi)^D\delta(p)$, and as a consequence

$$S^{\text{tach}}_{0,N}(\tau_0) = \frac{\kappa^{N-2}}{N!}V_N(\mathbf{0})\,\tau_0^N\cdot(2\pi)^D\delta(\mathbf{0})\,. \tag{2.2.11}$$

[4]We work in euclidean space with positive signature, and the action S should be inserted in the path integral as $\exp(S/\hbar)$, which is a convenient convention in string field theory. The euclidean action S is of the form $S = -\int d^Dx(K+V)$, where K and V stand for kinetic and potential terms respectively.

Since the infinite $(2\pi)^D\delta(\mathbf{0})$ factor just corresponds to the spacetime volume, the tachyon potential will read

$$V(\tau) = -\tau^2 - \sum_{N\geq 3} \frac{\kappa^{N-2}}{N!} \mathrm{v}_{_N}\, \tau^N\,, \tag{2.2.12}$$

where we have used the fact that the potential appears in the action with a minus sign. Here the expansion coefficients $\mathrm{v}_{_N}$ are given by

$$\mathrm{v}_{_N} \equiv V_{_N}(\mathbf{0})\,. \tag{2.2.13}$$

We will see that the coefficient v_3 is given by[5]

$$\mathrm{v}_3 = -\frac{3^9}{2^{11}} \approx -9.61\,. \tag{2.2.14}$$

Analytic work, together with numerical evaluation as we will show in the next chapter gives

$$\mathrm{v}_4 = 72.39 \pm 0.01\,. \tag{2.2.15}$$

Therefore, to this order the tachyon potential reads

$$V(\tau) = -\tau^2 + 1.602\kappa\tau^3 - 3.016\kappa^2\tau^4 + \cdots\,, \tag{2.2.16}$$

and gives no local minimum for the tachyon. The general form for $\mathrm{v}_{_N}$ will be shown to be given by

$$\mathrm{v}_{_N} = (-)^N \frac{2}{\pi^{N-3}} \int_{\mathcal{V}_{0,N}} \prod_{I=1}^{N-3} \frac{dx_{_I} dy_{_I}}{\rho_{_I}^2} \frac{1}{\rho_{_{N-2}}^2(0)\rho_{_{N-1}}^2(1)\rho_{_N}^2(\infty)}\,, \tag{2.2.17}$$

where the quantities $\rho_{_I}$, called mapping radii, will be discussed in the next section. Since the integrand is manifestly positive, $\mathrm{v}_{_N}$ will be positive for even N and negative for odd N. Note that by a sign redefinition of the tachyon field we can make all terms in the tachyon potential negative. Therefore the tachyon potential is unbounded from below and cannot have a global minimum. A local minimum may or may not exist. Even these statements should be qualified if the series defining the tachyon potential has no suitable radius of convergence. We will study the large-N behavior of the coefficients $\mathrm{v}_{_N}$ in section 2.5, but we will not be able to reach a definite conclusion as far as the radius of convergence goes.

Even if one could establish the existence of a local minimum for the tachyon potential, the question remains whether it represents a vacuum for the whole string field theory. One way to address this question would be to compute the effective potential for the tachyon. For a complete understanding of the string field potential we should actually examine all zero-momentum Lorentz scalar fields appearing in the theory. This would include physical scalars, unphysical scalars and trivial scalars. Since even the number of physical scalars at each mass level grows spectacularly fast [47], a more stringy way to discuss the string field potential is clearly desirable.

[5]The value quoted here agrees with that quoted in Ref. [38] after adjusting for a factor of two difference in the definition of the dimensionless coupling constant.

2.3 Geometrical preliminaries

In the present section we will begin by reviewing the definition of mapping radius of a punctured disk. While this object requires a choice of local coordinate at the puncture, it is possible to use it to construct conformal invariants of spheres with punctured disks *without* having to make choices of local coordinates at the punctures. We will discuss in detail those invariants. We review the extremal properties of the Strebel quadratic differentials and explain how to calculate mapping radii from them. The invariants relevant to the computation of tachyon amplitudes are shown to have extremal properties as well.

2.3.1 Reduced modulus

Given a punctured disk D, equipped with a chosen local coordinate z vanishing at the puncture, one can define a conformal invariant called the mapping radius ρ_D of the disk. It is calculated by mapping conformally the disk D to the unit disk $|w| \leq 1$, with the puncture going to $w = 0$. One then defines

$$\rho_D \equiv \left|\frac{dz}{dw}\right|_{w=0} . \tag{2.3.1}$$

Alternatively one may map D to a round disk $|\xi| \leq \rho_D$, with the puncture going to $\xi = 0$, so that $|dz/d\xi|_0 = 1$. The reduced modulus M_D of the disk D is defined to be

$$M_D \equiv \frac{1}{2\pi} \ln \rho_D . \tag{2.3.2}$$

Clearly, both the mapping radius and the reduced modulus depend on the chosen coordinate. If we change the local coordinate from z to z', also vanishing at the puncture, we see using Eq. (2.3.1) that the new mapping radius ${\rho'}_D$ is given by

$${\rho'}_D = \rho_D \left|\frac{dz'}{dz}\right|_{z=0}, \qquad \rightarrow \qquad \frac{\rho_D}{|dz|} = \text{invariant.} \tag{2.3.3}$$

Thus the mapping radius transforms like the inverse of a conformal metric g, for which the length element $g|dz|$ is invariant. For the reduced modulus we have

$$M'_D = M_D + \frac{1}{2\pi} \ln\left|\frac{dz'}{dz}\right|_{z=0} . \tag{2.3.4}$$

2.3.2 PSL(2, $\mathbb{C}$) invariants

It should be noted that the above transformation property 2.3.3 is not in contradiction with the conformal invariance of the mapping radius. Conformal invariance just states that if we map a disk, *and* carry along the chosen local coordinate at the puncture, the mapping radius does not change. This brings us to a point that will be quite important. Throughout this paper we will be dealing with punctured disks on the Riemann sphere. How will we choose local coordinates at the punctures? It will be done as follows: we will choose a global uniformizer z on the sphere and keep it fixed. If a punctured disk D has its puncture at $z = z_0$, then the local coordinate at the puncture will be taken to be $(z - z_0)$ (the case when $z_0 = \infty$ will be discussed later). Consider now an arbitrary PSL$(2, \mathbb{C})$ map taking the sphere into itself

$$z \to f(z) = \frac{az + b}{cz + d}, \qquad (2.3.5)$$

Under this map a disk D centered at $z = z_0$ will be taken to a disk $f(D)$ centered at $z = f(z_0)$. According to our conventions, the local coordinate for D is $z - z_0$ and the local coordinate for $f(D)$ is $z - f(z_0)$. This latter coordinate is not the image of the original local coordinate under the map. Therefore the mapping radius *will transform*, and we can use 2.3.3 to find

$$\rho_{f(D)} = \rho_D \left| \frac{df}{dz} \right|_{z_0}, \qquad \to \qquad \rho_D = |cz_0 + d\,|^2 \, \rho_{f(D)} \,. \qquad (2.3.6)$$

The off-shell amplitudes will involve the mapping radii of various disks. Moreover, they must be PSL$(2, \mathbb{C})$ invariant. How can that be achieved given that we do not have a PSL$(2, \mathbb{C})$ invariant definition of the mapping radius? Let us first examine the case when we have two punctured disks on a sphere. The data is simply a sphere with two marked points and two closed Jordan curves each surrounding one of the points. We will associate a PSL$(2, \mathbb{C})$ invariant to this sphere. The invariant is calculated using a uniformizer, but is independent of this choice. Choose any uniformizer z on the sphere, and denote the disks by $D_1(z_1)$ and $D_2(z_2)$, where z_1 and z_2 are the positions of the punctures. We now claim that

$$\chi_{12} \equiv \frac{|z_1 - z_2|^2}{\rho_{D_1(z_1)}\, \rho_{D_2(z_2)}}\,, \qquad (2.3.7)$$

is a PSL$(2, \mathbb{C})$ invariant (in other words, it is independent of the uniformizer, or, it is a conformal invariant of the sphere with two punctured disks). Indeed, under the PSL$(2, \mathbb{C})$ transformation given in Eq. (2.3.5) we have that

$$|z_1 - z_2| = |cz_1 + d\,| \cdot |cz_2 + d\,| \cdot |f(z_1) - f(z_2)|\,, \qquad (2.3.8)$$

and it follows immediately from this equation and Eq. (2.3.6) that

$$\frac{|z_1 - z_2|^2}{\rho_{D_1(z_1)}\, \rho_{D_2(z_2)}} = \frac{|f(z_1) - f(z_2)|^2}{\rho_{f(D_1(z_1))}\, \rho_{f(D_2(z_2))}}\,, \qquad (2.3.9)$$

which verifies the claim of invariance of the object χ_{12}. It seems plausible that any PSL$(2, \mathbb{C})$ invariant built from mapping radii of two disks must be a function of χ_{12}. It is not hard to construct in the same fashion an PSL$(2, \mathbb{C})$ invariant of three punctured disks. Indeed, we have

$$\chi_{123} \equiv \frac{|z_1 - z_2|\,|z_1 - z_3|\,|z_2 - z_3|}{\rho_{D_1(z_1)}\, \rho_{D_2(z_2)} \rho_{D_3(z_3)}}\,, \qquad (2.3.10)$$

which is easily verified to be a conformal invariant. This invariant can be written in terms of the invariant associated to two disks, one sees that

$$\chi_{123} = (\,\chi_{12}\,\chi_{13}\,\chi_{23}\,)^{1/2}\,. \tag{2.3.11}$$

This shows the invariant χ_{123} of three disks is not really new. It is also clear that we can now construct many invariants of three disks. We can form linear combinations of complicated functions build using the invariants associated to all possible choices of two disks from the three available ones. Nevertheless, the particular invariant χ_{123} given above will be of relevance to us later on. Let us finally consider briefly the case of four punctured disks, and concentrate on invariants having a product of all mapping radii in the denominator. Let

$$\chi_{1234} \equiv \frac{|z_1 - z_2|\,|z_2 - z_3|\,|z_3 - z_4||z_4 - z_1|}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}\rho_{D_4(z_4)}}\,, \tag{2.3.12}$$

and, as the reader will have noticed, the only requisite for invariance is that, as it happens above, every z_I appear twice in the numerator. This can be done in many different ways; for example, we could have written

$$\chi'_{1234} \equiv \frac{|z_1 - z_2|^2\,|z_3 - z_4|^2}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}\rho_{D_4(z_4)}}\,, \tag{2.3.13}$$

and the ratio of the two invariants is

$$\frac{\chi'_4}{\chi_4} = |\lambda|\,, \quad \text{with} \quad \lambda = \frac{(z_1 - z_2)\,(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \{z_1, z_2; z_3, z_4\}\,, \tag{2.3.14}$$

which being independent of the mapping radii, and, by construction a conformal invariant of a four-punctured sphere, necessarily has to equal the cross-ratio of the four points (or a function of the cross-ratio). The cross-ratio, as customary, will be denoted by λ. It is the point where z_1 lands when z_2, z_3 and z_4 are mapped to zero, one and infinity, respectively.

2.3.3 Letting one puncture go to infinity

It is sufficient to consider the behavior of the invariant χ_{12}, given by

$$\chi_{12} \equiv \frac{|z_1 - z_2|^2}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}}\,, \tag{2.3.15}$$

which we have seen is independent of the chosen uniformizer. We must examine what happens as we change the uniformizer in such a way that $z_2 \to \infty$. Given one uniformizer z there is another one $w = 1/z$ that is well defined at $z = \infty$, the only point where z fails to define a local coordinate. This is why there is no naive limit to χ_2 as $z_2 \to \infty$. Using Eq. (2.3.3) we express the mapping radius of the second disk in terms of the mapping radius as viewed using the uniformizer induced by w. We have

$\rho_{D_2(z_2)} = \left|\frac{dz}{dw}\right|_{w_2} \rho_{D_2(w_2)} = |z_2|^2 \rho_{D_2(w_2)}$, and substituting into the expression for χ_{12} we find

$$\chi_{12} = \frac{|1 - z_1/z_2|^2}{\rho_{D_1(z_1)}\, \rho_{D_2(w_2)}}\,, \tag{2.3.16}$$

and we can now take the limit as $z_2 \to \infty$ without difficulty. Writing, for convenience, $\rho_{D_2(w_2=0)} \equiv \rho_{D_2(\infty)}$, we get $\chi_{12} = \rho^{-1}_{D_1(z_1)}\, \rho^{-1}_{D_2(\infty)}$. The apparent dependence of χ_{12} on the choice of point z_1 is fictitious. Any change of uniformizer $z \to az + b$ which changes z_1 leaving the point at infinity fixed, will change the uniformizer at infinity, and the product of mapping radii will remain invariant. The point z_1 can therefore be chosen to be at the origin, and we write our final expression for χ_2

$$\chi_{12} = \frac{1}{\rho_{D_1(0)}\, \rho_{D_2(\infty)}}\,. \tag{2.3.17}$$

Following exactly the same steps with χ_3 and χ_4 we obtain

$$\chi_{123} \equiv \frac{|z_1 - z_2|}{\rho_{D_1(z_1)}\, \rho_{D_2(z_2)} \rho_{D_3(\infty)}}\,, \tag{2.3.18}$$

$$\chi_{1234} \equiv \frac{|z_1 - z_2|\, |z_2 - z_3|}{\rho_{D_1(z_1)}\, \rho_{D_2(z_2)} \rho_{D_3(z_3)} \rho_{D_4(\infty)}}\,. \tag{2.3.19}$$

One could certainly take $z_1 = 0$ and $z_2 = 1$ for χ_{123}, and, $z_2 = 0$ and $z_3 = 1$ for χ_{1234}. It should be remembered that whenever a disk is centered at infinity, the local coordinate used is the inverse of the chosen uniformizer on the rest of the sphere.

2.3.4 Mapping radii and quadratic differentials

In this subsection we will review how one uses the Strebel quadratic differential on a punctured sphere to define punctured disks. These disks, called coordinate disks, define the local coordinates used to insert the off-shell states. We will show how one can use the quadratic differential to calculate the explicit form of the local coordinates, and the mapping radii of the coordinate disks. We will review the extremal properties of the Strebel quadratic differentials and then discuss the extremal properties of the PSL$(2, \mathbb{C})$ invariants.

We will concentrate on the Strebel quadratic differentials relevant for the restricted polyhedra of closed string field theory. The reader unfamiliar with these objects may consult Refs. [24,29]. The Strebel quadratic differential for a sphere with N punctures in $\mathcal{V}_{0,N}$ induces a metric where the surface can be constructed by gluing N semiinfinite cylinders of circumference 2π across their open boundaries. The gluing pattern is described by a restricted polyhedron, which is a polyhedron having N faces, each of perimeter 2π and, in addition, having all nontrivial closed paths longer than or equal to 2π. Each semiinfinite cylinder defines a punctured disk with a local coordinate w. The boundary $|w| = 1$ corresponds to the edge of the cylinder, to be glued to the polyhedron, and the puncture corresponds to $w = 0$.

The Strebel quadratic differential on the sphere is usually expressed as $\varphi = \phi(z)(dz)^2$, where z is a uniformizer in the sphere. At the punctures it has second order poles; if there is a puncture at $z = z_I$ the quadratic differential near z_I reads

$$\varphi = \Big(-\frac{1}{(z-z_I)^2} + \frac{b_{-1}}{z-z_I} + b_0 + b_1(z-z_I) + \cdots\Big)(dz)^2\,. \tag{2.3.20}$$

Moreover, as mentioned above, the quadratic differential defines a disk D_I on the sphere, with the puncture at z_I. A local coordinate w_I on D_I, such that D_I becomes a round disk can be found as follows. We set

$$z = \rho_I\, w_I + c_1 w_I^2 + c_2 w_I^3 + \cdots\,, \tag{2.3.21}$$

where $\rho_I, c_1, c_2 \cdots$ are constants to be determined. We have written ρ_I for the coefficient of w_I on purpose. If we can make the D_I disk correspond to the disk $|w_I| \leq 1$, then ρ_I is by definition the mapping radius of the disk D_I, since it is the value of $|d(z-z_I)/dw_I|$ at $w_I = 0$ (recall Eq. (2.3.1)). We will actually use the notation

$$z = h_I(w_I), \quad \text{and} \quad \rho_I = |h_I'(0)|\,. \tag{2.3.22}$$

Note that as explained in the previous subsection we are using the local uniformizer on the sphere to define the mapping radius.

Back to our problem of defining the w_I coordinate, we demand that the quadratic differential, expressed in w_I coordinates, takes the form

$$\varphi = -\frac{1}{w_I^2}(dw_I)^2. \tag{2.3.23}$$

Since the above form is invariant under a change of scale, $w_I \rightarrow a w_I$, we cannot determine by this procedure the constant ρ_I. If ρ_I is fixed, the procedure will fix uniquely the higher coefficients c_1, c_2, $\cdots$. While for general off-shell states the knowledge of the coefficients c_I is necessary, for tachyons we only need the mapping radius. This radius can be determined by the following method. Given a quadratic differential one must find an arbitrary point P lying on the boundary of the punctured disk D_I defined by the quadratic differential. Possibly, the simplest way to do this is to identify the zeroes of the quadratic differential and then sketch the critical trajectories to identify the various punctured disks and ring domains. One can then pick P to be a zero lying on the nearest critical trajectory surrounding the puncture. We now require $w_I(P) = 1$, and this will fix both the scale and the phase of the local coordinate. This requirement is satisfied by taking

$$w_I(z) = \exp\Big(i\int_{z(P)}^{z} \sqrt{\phi(\xi)}\, d\xi\Big)\,, \tag{2.3.24}$$

where we take the positive branch for the square root. If the integral can be done explicitly then the mapping radius is easily calculated by taking a derivative $\rho_I = |\frac{dw_I}{dz}|_{z_I}^{-1}$. If the integral cannot be done explicitly one can calculate the mapping radius by a limiting procedure. One computes $\rho_I = \lim_{\epsilon\to 0} |\frac{dw_I}{dz}|^{-1}_{z_I+\epsilon}$. This leads, using Eq. (2.3.20) to the following result

$$\ln \rho_I = \lim_{\epsilon \to 0} \left(\mathrm{Im} \int_{z_I+\epsilon}^{z(P)} \sqrt{\phi(\xi)} d\xi \; + \ln \epsilon \right). \tag{2.3.25}$$

The integration path is some curve in the disk D_I, and using contour deformation one can verify that the imaginary part of the integrand does not depend on the choice of P as long as P is on the boundary of D_I. When using equation Eq. (2.3.25) one must choose a branch for the square root, and keep the integration path away from the branch cut. The sign is fixed by the condition that the limit exist. Equation Eq. (2.3.25) and the recursive procedure indicated above allow us, in principle, to calculate the function $h_I(w_I)$ if we know explicitly the quadratic differential.

2.3.5 Extremal properties

Imagine having an N punctured Riemann sphere and label the punctures as P_1, P_2, $\cdots P_N$. Fix completely arbitrary local analytic coordinates at these punctures. Now consider drawing closed Jordan curves surrounding the punctures and defining punctured disks D_I, in such a way that the disks do not overlap (even though they might touch each other). Given this data we can evaluate the functional

$$\mathcal{F} = M_{D_1} + M_{D_2} + \cdots + M_{D_N} \,, \tag{2.3.26}$$

which is simply the sum of the reduced moduli of the various disks. This functional, of course depends on the shape of the disks we have chosen, and is well defined since we have picked some specific local coordinates at the punctures. We may try now to vary the shape of the disks in order to maximize $\mathcal{F}$. Suppose there is a choice of disks that maximizes $\mathcal{F}$, then, it will maximize $\mathcal{F}$ whatever choice of local coordinates we make at the punctures. This follows because upon change of local coordinates the reduced modulus of a disk changes by a constant which is independent of the disk itself (see Eq. (2.3.4)). The interesting fact is that the Strebel differential defines the disks that maximize $\mathcal{F}$ [29]. Using the relation between reduced modulus and mapping radius we see that the functional

$$(\rho_{D_1} \cdots \rho_{D_N})^{-1} = \exp(-2\pi \mathcal{F}) \,, \tag{2.3.27}$$

consisting of the inverse of the product of all the mapping radii, is actually minimized by the choice of disks made by the Strebel quadratic differential. This property will be of use to us shortly.

It is worth pausing here to note that the above definition of the functional $\mathcal{F}$ allows us to compare choices of disks given a *fixed* Riemann sphere. Since we have chosen arbitrarily the local coordinates at the punctures there is no reasonable way to compare the maximal values of $\mathcal{F}$ for two *different* spheres. It is therefore hard to think of $\mathrm{Max}(\mathcal{F})$ as a function on $\overline{\mathcal{M}}_{0,N}$. This is reminiscent of the fact that while for higher genus surfaces *without* punctures we can think of the area of the minimal area metric as a function on moduli space, it is not clear how to do this for punctured surfaces. The difficulty again is due to the regularization needed to render the area finite, this requires a choice of local coordinates at the punctures, and there is no simple way to compare the choices for different punctured surfaces. We now wish to emphasize that our earlier discussion teaches us how to define functions on $\overline{\mathcal{M}}_{0,N}$. These functions are interesting because they are simple modifications of $\exp[-2\pi\,\mathrm{Max}(\mathcal{F})]$ that turn out to be functions on $\overline{\mathcal{M}}_{0,N}$.

Indeed, consider the invariant χ_{1234} that was defined as

$$\chi_{1234} \equiv \frac{|z_1 - z_2|\,|z_2 - z_3|\,|z_3 - z_4||z_4 - z_1|}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}\rho_{D_4(z_4)}}\,. \tag{2.3.28}$$

Recall that the mapping radii entering in the definition of χ_{1234}, as well as the coordinate differences, are computed using the global uniformizer, and the invariance of χ_{1234} just means independence of the result on the choice of uniformizer. No choice is required to evaluate χ_{1234}. We obtain a function f on $\overline{\mathcal{M}}_{0,4}$ by giving a number for each four punctured sphere R_4 as follows. We equip the sphere R_4 with the Strebel quadratic differential $\varphi_S(R_4)$ and we evaluate the invariant χ_{1234} using the disks $D^I[\varphi_S(R_4)]$ determined by the differential. In writing

$$f(R_4) \equiv \chi_{1234}\,(\,D^I\,[\varphi_S(R_4)]\,)\,. \tag{2.3.29}$$

We claim that $f(R_4)$ actually is the lowest value that the invariant χ_{1234} can take for any choice of nonoverlapping disks in R_4

$$\chi_{1234}\,(\,D^I\,[\varphi_S(R_4)]\,) \leq \chi_{1234}\,(\,D^I\,[R_4]\,)\,. \tag{2.3.30}$$

To see this, fix a uniformizer such that three of the punctures lie at three points (say, $z = -1, 0, 1$) and the fourth puncture will lie at some fixed point, which depends on the choice of four punctured sphere. This fixes completely the numerator of χ and fixes the local coordinates at the punctures, necessary to compute the mapping radii. Therefore

$$\chi_{1234} \propto \left(\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}\rho_{D_4(z_4)}\right)^{-1} = \exp(-2\pi\mathcal{F})\,, \tag{2.3.31}$$

where we recognize that, up to a fixed constant, the invariant is simply related to the value of $\mathcal{F}$ evaluated with the chosen coordinates at the punctures. As we now vary the disks around the punctures, $\mathcal{F}$ will be maximized by the quadratic differential. This verifies that χ is minimized by the disks chosen by the quadratic differential (Eq. (2.3.30).)

We expect the function $f(R_4)$ to have a minimum for the most symmetric surface in $\mathcal{M}_{0,4}$, namely, for the regular tetrahedron $[\lambda = (1 + i\sqrt{3})/2]$. We have not proven this, but the intuition is that for the most symmetric surface we can get the disks of largest mapping radii. There is, of course the issue of the numerator of χ with the coordinate differences, which also varies as we move in moduli space. Still, one can convince oneself that the function $f(R_4)$ grows without bound as R_4 approaches degeneration.

2.3.6 Estimating mapping radii

As we have mentioned earlier, in defining the mapping radius of a punctured disk on the sphere we use a local coordinate at the puncture which is obtained from a chosen uniformizer on the sphere. While this mapping radius depends on the uniformizer, we are typically interested in functions, such as the χ functions, which are constructed out of mapping radii and coordinate differences, and are independent of the chosen uniformizer.

Consider now the sphere as the complex plane z together with a point at infinity. The two following facts are useful tools to estimate the mapping radius of a punctured disk centered at z_0.

- If the disk D is actually a round disk $|z - z_0| \leq R$, then the mapping radius $\rho_{_D}$ is precisely given by the radius of the disk: $\rho_{_D} = R$. This is clear since $w = (z - z_0)/R$ is the exact conformal map of D to a unit disk.

- If the disk D is not round but it is contained between two round disks centered at z_0 with radii R_1 and R_2, with $R_1 < R_2$, then $R_1 < \rho_{_D} < R_2$. This property follows from the superadditivity of the reduced modulus (see Ref. [48], Eq. (2.2.25)).

Given an N punctured sphere, the Strebel quadratic differential will maximize the product of the N mapping radii. We can obtain easily a bound $\rho_1\rho_2\cdots\rho_{_N} \geq R_1R_2\cdots R_{_N}$, where the $R_{_I}$ are the radii of non-overlapping round disks centered at the punctures with the sphere represented as the complex plane together with the point at infinity.

2.4 Off-shell amplitudes for tachyons

In this section we compute off-shell amplitudes for tachyons at arbitrary momentum. We first discuss the case of three tachyons and then the case of $N \geq 4$ tachyons which requires integration over moduli space. We examine the results for the case of zero-momentum tachyons obtaining in this way the coefficients $\mathrm{v}_{_N}$ of the tachyon potential. We explain why the choice of polyhedra for the string vertices, minimizes recursively the coefficients of the nonpolynomial tachyon potential.

2.4.1 Three point couplings

We will now examine the cubic term in the string field potential. Assume we are now given a three punctured sphere, and we want to calculate the general off-shell amplitude for three tachyons. We then must compute the correlator

$$A_{p_1p_2p_3} = \Big\langle c\bar{c}e^{ip_1X}(w_1 = 0)\; c\bar{c}e^{ip_2X}(w_2 = 0)\; c\bar{c}e^{ip_3X}(w_3 = 0)\Big\rangle\,. \tag{2.4.1}$$

In order to do this, we have to transform these operators from the local coordinates $w_{_I}$ to some uniformizer z. Let $w_{_I} = 0$ correspond to $z = z_{_I}$. We then have from the transformation law of a primary field

$$c\bar{c}e^{ip_I X}(w_{_I}=0) = c\bar{c}e^{ip_I X}(z=z_{_I})\left|\frac{dz}{dw_{_I}}\right|^{p_I^2-2}_{w_I=0} = c\bar{c}e^{ip_I X}(z_{_I})\ \rho_1^{p_I^2-2}\,, \tag{2.4.2}$$

where in the last step we have recognized the appearance of the mapping radius for the disk D_I. The correlator then becomes

$$\begin{aligned} A_{p_1p_2p_3} &= \left\langle c\bar{c}e^{ipX}(z_1)\ c\bar{c}e^{ipX}(z_2)\ c\bar{c}e^{ipX}(z_3)\right\rangle \frac{1}{\rho_1^{2-p_1^2}\rho_2^{2-p_2^2}\rho_3^{2-p_3^2}}\,, \\ &= \frac{|z_1-z_2|^{2+2p_1p_2}|z_2-z_3|^{2+2p_2p_3}|z_1-z_3|^{2+2p_1p_3}}{\rho_1^{2-p_1^2}\rho_2^{2-p_2^2}\rho_3^{2-p_3^2}} \\ &\qquad\qquad \times[-2(2\pi)^D\delta^D(\textstyle\sum p_{_I})]\,, \end{aligned} \tag{2.4.3}$$

where we made use of Eq. (2.2.7), which introduces an extra factor of -2 (shown in brackets) due to our convention $c_0^{\pm}=(c_0\pm\bar{c}_0)/2$. In order to construct a manifestly PSL$(2,\mathbb{C})$ invariant expression we use momentum conservation in the denominator to write

$$\begin{aligned} A_{p_1p_2p_3} &= \frac{|z_1-z_2|^{2+2p_1p_2}}{(\rho_1\rho_2)^{1+p_1p_2}}\ \frac{|z_2-z_3|^{2+2p_2p_3}}{(\rho_2\rho_3)^{1+p_2p_3}}\ \frac{|z_1-z_3|^{2+2p_1p_3}}{(\rho_1\rho_3)^{1+p_1p_3}} \\ &\qquad\qquad \times[-2(2\pi)^D\delta^D(\textstyle\sum p_{_I})]\,, \\ &= [-2(2\pi)^D\delta^D(\textstyle\sum p_{_I})]\cdot\displaystyle\prod_{I<J}^{3}[\chi_{_{IJ}}]^{1+p_Ip_J}\,, \end{aligned} \tag{2.4.4}$$

which is the manifestly PSL$(2,\mathbb{C})$ invariant description of the off-shell amplitude.

We can now use the above result to extract the cubic coefficient of the tachyon potential. By definition, the operator formalism bra $\langle V_{123}^{(3)}|$ representing the three punctured sphere must satisfy

$$\langle V_{123}^{(3)}|\,(c_1\bar{c}_1|\mathbf{1},p_1\rangle)^{(1)}\,(c_1\bar{c}_1|\mathbf{1},p_2\rangle)^{(2)}\,(c_1\bar{c}_1|\mathbf{1},p_3\rangle)^{(3)} = A_{p_1p_2p_3}\,. \tag{2.4.5}$$

Moreover, the multilinear function representing the cubic interaction is given by

$$\begin{aligned} \{T\}_{\mathcal{V}_{0,3}} &\equiv \langle V_{123}^{(3)}|T\rangle^{(1)}|T\rangle^{(2)}|T\rangle^{(3)}\,, \\ &= \int\frac{dp_1}{(2\pi)^D}\frac{dp_2}{(2\pi)^D}\frac{dp_3}{(2\pi)^D}\,A_{p_1p_2p_3}\,\tau(p_1)\tau(p_2)\tau(p_3)\,, \\ &= \int\prod_{I=1}^{3}\frac{dp_{_I}}{(2\pi)^D}\,(2\pi)^D\delta^D(\textstyle\sum p_{_I}) \\ &\qquad \times(-2)\displaystyle\prod_{I<J}^{3}[\chi_{_{IJ}}]^{1+p_Ip_J}\cdot\tau(p_1)\tau(p_2)\tau(p_3)\,, \end{aligned} \tag{2.4.6}$$

where use was made of the definition of the string tachyon field in Eq. (2.2.2), of Eq. (2.4.5), and of Eq. (2.4.4). Comparison with Eq. (2.2.10), and use of Eq. (2.2.13) now gives

$$\mathrm{v}_3 = -2\cdot\prod_{I<J}^{3}[\chi_{_{IJ}}] = -2\cdot\frac{|z_1-z_2|^2|z_2-z_3|^2|z_1-z_3|^2}{\rho_1^2\rho_2^2\rho_3^2} = -2\cdot\chi_{123}^2\,, \tag{2.4.7}$$

in terms of the PSL(2, $\mathbb{C}$) invariant χ_{123}. It follows from the extremal properties discussed earlier that the minimum value possible for v_3 is achieved for the Strebel quadratic differential defining the Witten vertex. We will calculate the minimum possible value for v_3 in section 2.5.1.

2.4.2 The off-shell Koba-Nielsen formula

We now derive a formula for the off-shell scattering amplitude for N closed string tachyons at arbitrary momentum. The final result will be a manifestly PSL(2, $\mathbb{C}$) invariant expression. The computation is simplified because the tachyon vertex operator is primary even off-shell, and because its ghost structure is essentially trivial.

We work in the z-plane and fix the position of the three last insertions at z_{N-2}, z_{N-1} and z_N. The positions of the first $N-3$ punctures will be denoted as $z_1, z_2, \cdots z_{N-3}$. We must integrate over the positions of these $N-3$ punctures. Each will therefore give a factor

$$dx_I \wedge dy_I \; b(\frac{\partial}{\partial x})\, b(\frac{\partial}{\partial y}) = dx_I \wedge dy_I \;\; 2i\, \bar{b}_{-1} b_{-1} = -dz_I \wedge d\bar{z}_I \, \bar{b}_{-1} b_{-1} \,, \tag{2.4.8}$$

where $z_I = x_I + iy_I$. There is a subtlety here, each of the antighost oscillators refers to the z plane, while the ghost oscillators in each tachyon insertion $c_1^{w_I} \bar{c}_1^{w_I} |0,p\rangle^{w_I}$ refer to the local coordinate w_I, where $z = h_I(w_I)$. Transforming the antighost oscillators we obtain $b_{-1} = [h_I'(0)]^{-1} b_{-1}^{w_I} + \cdots$, where the dots indicate antighost oscillators $b_{n\geq 0}^{w_I}$ that annihilate the tachyon state. For the antiholomorphic oscillator we have $\bar{b}_{-1} = \left[\overline{h_I'(0)}\right]^{-1} \bar{b}_{-1}^{w_I} + \cdots$. Therefore each of the integrals will be represented by

$$2i\, dx_I \wedge dy_I \; \frac{1}{\rho_I^2} \, |0,p\rangle^{w_I} = 2i\, dx_I \wedge dy_I \; \frac{1}{\rho_I^{2-p_I^2}} \, |0,p\rangle \,, \tag{2.4.9}$$

where $\rho_I = |h_I'(0)|$ is the mapping radius of the I-th disk. The Koba-Nielsen amplitude will therefore be given by

$$\begin{aligned} A_{p_1 \cdots p_N} = \left(\frac{i}{2\pi}\right)^{N-3} (2i)^{N-3} \int \prod_{I=1}^{N-3} \frac{dx_I dy_I}{\rho_I^{2-p_I^2}} \frac{1}{\rho_{N-2}^{2-p_{N-2}^2}\, \rho_{N-1}^{2-p_{N-1}^2}\, \rho_{N-2}^{2-p_{N-2}^2}} \\ \times \left\langle e^{ip_1 X(z_1)} \cdots e^{ip_{N-3} X(z_{N-3})}\, c\bar{c} e^{ip_{N-2} X(z_{N-2})}\, c\bar{c} e^{ip_{N-1} X(z_{N-1})}\, c\bar{c} e^{ip_N X(z_N)} \right\rangle , \end{aligned} \tag{2.4.10}$$

where the correlator is a free-field correlator in the complex plane. We will not include in the amplitude the coupling constant factor κ^{N-3}. The extra factor $(i/2\pi)^{N-3}$ included in the formula above is well-known to be necessary for consistent factorization, and has been derived in closed string field theory.[6] We then have

$$\begin{aligned} A_{p_1\cdots p_N} = (-)^N \cdot \frac{2}{\pi^{N-3}} \int \prod_{I=1}^{N-3} \frac{dx_I dy_I}{\rho_I^{2-p_I^2}} \\ \times \frac{|(z_{N-2}-z_{N-1})(z_{N-2}-z_N)(z_{N-1}-z_N)|^2}{\rho_{N-2}^{2-p_{N-2}^2}\rho_{N-1}^{2-p_{N-1}^2}\rho_N^{2-p_N^2}} \\ \times \prod_{I<J}^{N} |z_I - z_J|^{2p_I p_J} \cdot \left[(2\pi)^D \delta\left(\sum p_I\right)\right] . \end{aligned} \tag{2.4.11}$$

Using momentum conservation, and the definition of the invariants χ_{IJ} and χ_{IJK}, we can write the above as

$$A_{p_1\cdots p_N} = (-)^N \frac{2}{\pi^{N-3}} \int \prod_{I=1}^{N-3} \frac{dx_I dy_I}{\rho_I^2}\, \chi^2_{N-2,N-1,N} \cdot \prod_{I<J}^{N} \chi_{IJ}^{p_I p_J} \cdot \left[(2\pi)^D \delta\left(\sum p_I\right)\right], \tag{2.4.12}$$

It follows immediately from the transformation law for the mapping radius that the measure $dz_I \wedge d\bar{z}_I / \rho_I^2$ is PSL(2, $\mathbb{C}$) invariant. Therefore the above result is a manifestly PSL(2, $\mathbb{C}$) invariant off-shell generalization of the Koba-Nielsen formula. For the case of four tachyons it reduces to an off-shell version of the Virasoro-Shapiro amplitude

$$A_{p_1\cdots p_4} = \frac{2}{\pi} \int \frac{dx_1 dy_1}{\rho_1^2}\, \chi^2_{234} \cdot \prod_{I<J}^{4} \chi_{IJ}^{p_I p_J} \cdot \left[(2\pi)^D \delta\left(\sum p_I\right)\right] . \tag{2.4.13}$$

If we choose to place the second, third, and fourth punctures at zero, one and infinity respectively, we end with

$$A_{p_1\cdots p_4} = \frac{2}{\pi} \int dx\, dy \frac{|z|^{2p_1p_2}|z-1|^{2p_1p_3}}{\rho_1^{2-p_1^2}\rho_2^{2-p_2^2}\rho_3^{2-p_3^2}\rho_4^{2-p_4^2}} \cdot \left[(2\pi)^D \delta\left(\sum p_I\right)\right] . \tag{2.4.14}$$

Another expression can be found where the variables of integration are cross-ratios. We define the cross ratio

$$\lambda_I \equiv \{z_I, z_{N-2}; z_{N-1}, z_N\} = \frac{(z_I - z_{N-2})(z_{N-1}-z_N)}{(z_I - z_N)(z_{N-1}-z_{N-2})}, \tag{2.4.15}$$

and it follows that

$$dz_I \wedge d\bar{z}_I = \frac{d\lambda_I \wedge d\bar{\lambda}_I}{|\lambda_I|^2} \frac{|z_I - z_N|^2 |z_I - z_{N-2}|^2}{|z_N - z_{N-2}|^2}, \tag{2.4.16}$$

leading to

$$\begin{aligned} A_{p_1\cdots p_N} = 2(-)^N \left(\frac{i}{2\pi}\right)^{N-3} \int \prod_{I=1}^{N-3} \left[\frac{d\lambda_I \wedge d\bar{\lambda}_I}{|\lambda_I|^2} \left(\frac{\chi_{I,N-2,N-1,N}}{\chi_{N-2,N-1,N}}\right)^2\right] \cdot \chi^2_{N-2,N-1,N} \\ \times \prod_{I<J}^{N} \chi_{IJ}^{p_I p_J} \cdot \left[(2\pi)^D \delta\left(\sum p_I\right)\right] . \end{aligned} \tag{2.4.17}$$

[6]The value used here appears in Ref. [15], where a sign mistake of Ref. [46] was corrected.

For the case of four tachyons the above result reduces to

$$A_{p_1\cdots p_4} = \frac{i}{\pi}\int \frac{d\lambda\wedge d\bar{\lambda}}{|\lambda|^2}\,\chi^2_{1234}\cdot\prod_{I<J}^{4}\chi_{IJ}^{p_I p_J}\cdot\Big[(2\pi)^D\delta\Big(\sum p_I\Big)\Big]\,. \tag{2.4.18}$$

In the above expressions the λ integrals extend over the whole sphere. This formula with a slight modification will be used in chapter 3 to calculate the fourth order tachion term in the effective tachion potential.

Having obtained manifestly PSL$(2,\mathbb{C})$ invariant expressions valid for arbitrary momenta, we now go back to our particular case of interest, which is the case when all the momenta are zero. It is simplest to go back to Eq. (2.4.10) to obtain

$$A_{1\cdots N} = (-)^N\frac{2}{\pi^{N-3}}\int\prod_{I=1}^{N-3}\frac{dx_I dy_I}{\rho_I^2}\;\frac{|z_{N-2}-z_{N-1}|^2|z_{N-2}-z_N|^2|z_{N-1}-z_N|^2}{\rho^2_{N-2}\rho^2_{N-1}\rho^2_N}\times[(2\pi)^D\delta(\mathbf{0})]\,, \tag{2.4.19}$$

and for the case $N=4$

$$A_{1\cdots 4} = \frac{2}{\pi}\int\frac{dx_1\,dy_1}{\rho_1^2}\;\frac{|z_2-z_3|^2|z_2-z_4|^2|z_3-z_4|^2}{\rho_2^2\rho_3^2\rho_4^2}[(2\pi)^D\delta(\mathbf{0})]\,. \tag{2.4.20}$$

These are the expressions we shall be trying to estimate. If we set the three special points appearing in the above expressions to zero, one and infinity, we find (see section 2.3.3)

$$A_{1\cdots N} = (-)^N\frac{2}{\pi^{N-3}}\int\prod_{I=1}^{N-3}\frac{dx_I dy_I}{\rho_I^2}\;\frac{1}{\rho^2_{N-2}(0)\rho^2_{N-1}(1)\rho^2_N(\infty)}\,[(2\pi)^D\delta(\mathbf{0})]\,, \tag{2.4.21}$$

and in particular case $N=4$

$$A_{1\cdots 4} = \frac{2}{\pi}\int\frac{dx_1\,dy_1}{\rho_1^2}\;\frac{1}{\rho_2^2(0)\rho_3^2(1)\rho_4^2(\infty)}\,[(2\pi)^D\delta(\mathbf{0})]\,. \tag{2.4.22}$$

Let us now use the above results to extract the quartic and higher order coefficient of the tachyon potential. By definition, the operator formalism bra representing the collection of N-punctured spheres must satisfy

$$\int_{\mathcal{V}_{0,N}}\langle\Omega_N^{(0)0,N}|\,(c_1\bar{c}_1|\mathbf{1},p_1\rangle)^{(1)}\cdots(c_1\bar{c}_1|\mathbf{1},p_N\rangle)^{(N)} = A_{p_1\cdots p_N}(\mathcal{V}_{0,N})\,, \tag{2.4.23}$$

where the $\mathcal{V}_{0,N}$ argument of $A_{p_1\cdots p_N}(\mathcal{V}_{0,N})$ indicates that the off-shell amplitude has only been partially integrated over the subspace $\mathcal{V}_{0,N}$. The corresponding multilinear function is given by

$$\begin{aligned}\{T\}_{\mathcal{V}_{0,N}} &\equiv \int_{\mathcal{V}_{0,N}}\langle\Omega_N^{(0)0,N}|T\rangle^{(1)}\cdots|T\rangle^{(N)}\\ &= \int\prod_{I=1}^{N}\frac{dp_I}{(2\pi)^D}\,A_{p_1\cdots p_N}(\mathcal{V}_{0,N})\,\tau(p_1)\cdots\tau(p_N)\,,\end{aligned} \tag{2.4.24}$$

where we made use of the definition of the string tachyon field in Eq. (2.2.2), and of Eq. (2.4.23). Reading the value of the amplitude at zero momentum, and by virtue of Eq. (2.2.10) and Eq. (2.2.13) we get

$$\begin{aligned} \mathrm{v}_N &= (-)^N \frac{2}{\pi^{N-3}} \int_{\mathcal{V}_{0,N}} \prod_{I=1}^{N-3} \frac{dx_I dy_I}{\rho_I^2} \, \frac{|z_{N-2}-z_{N-1}|^2 |z_{N-2}-z_N|^2 |z_{N-1}-z_N|^2}{\rho_{N-2}^2 \rho_{N-1}^2 \rho_N^2}, \\ &= (-)^N \frac{2}{\pi^{N-3}} \int_{\mathcal{V}_{0,N}} \prod_{I=1}^{N-3} \frac{dx_I dy_I}{\rho_I^2} \, \frac{1}{\rho_{N-2}^2(0)\rho_{N-1}^2(1)\rho_N^2(\infty)} \, . \end{aligned} \tag{2.4.25}$$

Note here the pattern of signs. All v_N for N even come with positive sign, and all v_N for N odd come with a negative sign. Including the overall minus sign in passing from the action to the potential (Eq. (2.2.12)), and the sign redefinition $\tau \to -\tau$, all coefficients of the tachyon potential become negative.

It is worthwhile to pause and reflect about the above pattern of signs. In particular, since v_4 turned out to be positive, the quartic term in the tachyon potential is negative, as the quadratic term is. While the calculations leading to the sign factors are quite subtle, we believe that the result should have been expected. In closed string field theory, the elementary four point interaction changes if we include stubs in the three string vertex. Both the original interaction and the one for the case of stubs must have the same sign, because they only differ by the region of integration over moduli space, and the integrand, as we have seen, has a definite sign. On the other hand, the interaction arising from the stubbed theory would equal the original interaction plus a collection of Feynman graphs with two three-string vertices and with one propagator whose proper time is only partially integrated. Such terms, for completely integrated propagators and massive intermediate fields would give a contribution leading to a potential unbounded below. For partially integrated propagators they also contribute such kind of terms, both if the field is truly massive, or if it is tachyonic. This indicates that one should have expected an unbounded below elementary interaction.

It is now simple to explain why the choice of restricted polyhedra (polyhedra with all nontrivial closed paths longer than or equal to 2π [49]) for closed string vertices minimizes recursively the expansion coefficients of the tachyon potential. We have seen that v_3 is minimized by the Witten vertex. At the four point level we then have a missing region $\mathcal{V}_{0,4}$. In the parametrization given by the final form in Eq. (2.4.25) the region of integration corresponding to $\mathcal{V}_{0,4}$ is fixed. At each point in this region, the integrand, up to a constant, is given by $1/\prod_i \rho_i^2 = \exp(-4\pi\mathcal{F})$, and as explained around Eq. (2.3.27), this quantity is minimized by the choice of coordinate disks determined by the Strebel differential. Since the integrand is positive definite throughout the region of integration, and, at every point is minimized by the use of the Strebel differential, it follows that the integral is minimized by the choice of the Strebel differential for the string vertex. That is precisely the choice that defines the restricted polyhedron corresponding to the standard four closed string vertex. It is clear that the above considerations hold for any $\mathcal{V}_{0,N}$. The minimum value for v_N is obtained by using polyhedra throughout the region of integration. Therefore, starting with the three string vertex, we are led recursively by the minimization procedure to the restricted polyhedra of closed string field theory.

2.5 Estimates for the tachyon potential

The present section is devoted to estimates of the tachyon potential. While more analytic work on the evaluation of the tachyon potential may be desirable, here we will get some intuitive feeling for the growth of the coefficients v_N for large N. The aim is to find if the tachyon potential has a radius of convergence. We will not be able to decide on this point, but we will obtain a series of results that go in this direction.

For every number of punctures N, there is a configuration of these punctures on the sphere for which we can evaluate exactly the measure of integration. This is the configuration where the punctures are "equally separated" in a planar arrangement. These configurations appear as a finite number of points in the boundary of $\mathcal{V}_{0,N}$, and in some sense are the most problematic. The large N behavior of the measure at those points is such that if the whole integrand were to be dominated by these points the tachyon potential would seem to have no radius of convergence. The shape of $\mathcal{V}_{0,N}$ around those points, however, is such that the contributions might be suppressed.

In each $\mathcal{V}_{0,N}$ there are configurations where the punctures are distributed most symmetrically. It is intuitively clear that at these configurations the measure is in some sense lowest. It is possible to estimate this measure for large N, and conclude that, if dominated by this contribution, the tachyon potential should have some radius of convergence.

The behavior of the measure for the tachyon potential is such that the measure grows as we approach degeneration, and if $\mathcal{V}_{0,4}$, for example, was to extend over all of $\overline{\mathcal{M}}_{0,4}$ the naive integral would be infinite. This infinity is not physical, because we do not expect infinite amplitude for the scattering of four zero-momentum tachyons. We explain how analytic continuation of the contribution from the Feynman graphs removes this apparent contradiction.

We begin by presenting several exact results pertaining two, three, and four-punctured spheres.

2.5.1 Evaluation of invariants

Consider first the invariant χ_{12} of a sphere with two punctured disks (Eq. (2.3.7)). The disks may touch but they are assumed not to overlap. Since the mapping radii can be as small as desired, the invariant χ_{12} is not bounded above. It is actually bounded below, by the value attained when we have a Strebel quadratic differential. We can take the sphere punctured at zero and infinity, and the quadratic differential to be $\varphi = -(dz)^2/z^2$. While this differential does not determine a critical trajectory, we can take it to be any closed horizontal trajectory, say $|z| = 1$. This divides the z-sphere into two disks, one punctured at $z = 0$ with unit disk $|z| \leq 1$ and the other punctured at $z = \infty$, or $w = 0$, with $w = 1/z$, and with unit disk $|w| \leq 1$. It follows that their mapping radii are both equal to one. We therefore have, using Eq. (2.3.17),

$$\widehat{\chi}_{12} \equiv \text{Min}\,(\chi_{12}) = \text{Min}\Big(\frac{1}{\rho_{D_1(0)}\,\rho_{D_2(\infty)}}\Big) = 1\,. \qquad (2.5.1)$$

We now consider three punctured spheres and try to evaluate the minimum value of the invariant χ_{123} defined in Eq. (2.3.10). This minimum is achieved for the three punctured sphere corresponding to the Witten vertex. In this vertex we can represent the sphere by the z plane, with the punctures at $e^{\pm i\pi/3}, -1$. The disk surrounding the puncture at $z = e^{i\pi/3}$ is the wedge domain $0 \leq \text{Arg}(z) \leq 2\pi/3$. This domain is mapped to a unit disk $|w| \leq 1$ by the transformations

$$t = z^{3/2}\,, \quad w = \frac{t-i}{t+1}\,. \qquad (2.5.2)$$

One readily finds that the mapping radius ρ of the disk is given by $\rho = |\frac{dz}{dw}|_{w=0} = 4/3$. Furthermore, the distance $|z_I - z_J|$ between any of the punctures is equal to $\sqrt{3}$. Therefore back in Eq. (2.3.10) we obtain

$$\widehat{\chi}_{123} \equiv \text{Min}\,(\chi_{123}) = \Big(\frac{3\sqrt{3}}{4}\Big)^3\,. \qquad (2.5.3)$$

This result implies that the minimum possible value for $|\text{v}_3|$ is realized with

$$\text{v}_3 = -3^9/2^{11} quad \text{ (see Eq. (2.4.7))}. \qquad (2.5.4)$$

Another computation that is of interest is that of the most symmetric four punctured sphere, a sphere where the punctures are at $z = 0, 1, \infty$, and at $z = \rho = e^{i\pi/3}$. The Strebel quadratic differential for this sphere can be found to be

$$\varphi = \frac{(2z+1)(2z-\rho)(2z-\bar{\rho})}{(z-1)^2(z-\rho^2)^2(z-\bar{\rho}^2)^2}\cdot\frac{(dz)^2}{z^2}\,. \qquad (2.5.5)$$

Here the poles are located at the points $z = 0,\ 1,\ \rho^2$, and $\bar{\rho}^2$. The zeroes are located at $z = -\frac{1}{2},\ \frac{1}{2}\rho,\ \frac{1}{2}\bar{\rho}$, and ∞. One can use this expression for a calculation of the mapping radii.

2.5.2 The measure at the planar configuration

In each $\mathcal{V}_{0,N}$, for $N \geq 4$ there is a set of symmetric planar configurations for the punctures. They correspond to the surfaces obtained by Feynman diagrams constructed using only the three string vertex, and with all the propagators collapsed with zero twist angle. We will consider the case of N punctures and give an exact evaluation for the measure. This will be done in the frame where three punctures are mapped to the standard points $z = 0, 1, \infty$, and the rest of the punctures will be mapped to the points $z_1, z_2, \cdots z_{N-3}$ lying on the real line in between $z = 0$ and $z = 1$. Consequently,

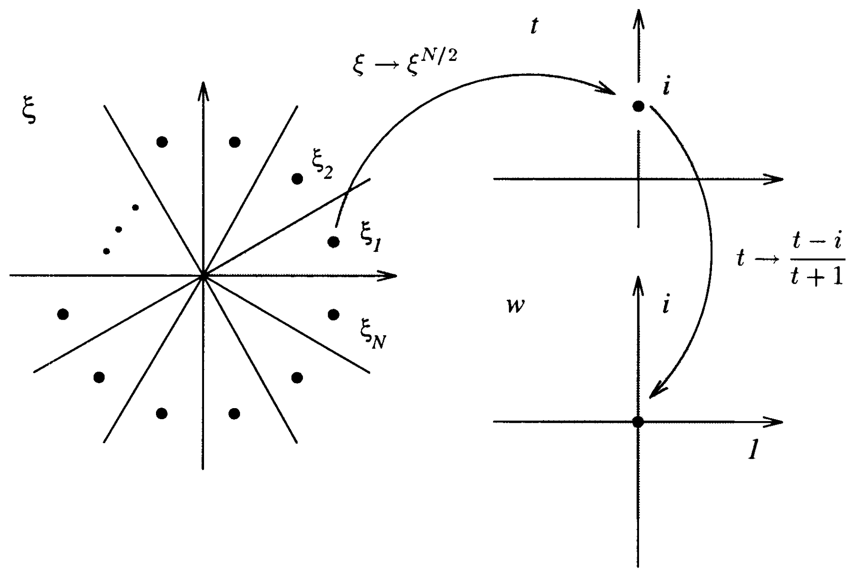


Figure 2-1: Planar configuration of punctures on the sphere. Shown are the maps from the ring domain associated with a specific puncture to the unit disk.

we will take $z_{N-2} = 1, z_{N-1} = \infty$ and $z_N = 0$. The measure we will calculate will be defined as

$$d\mu_N \equiv \prod_{I=1}^{N-3} \frac{dx_I dy_I}{\rho_I^2} \frac{|z_{N-2} - z_{N-1}|^2 |z_{N-2} - z_N|^2 |z_{N-1} - z_N|^2}{\rho_{N-2}^2 \rho_{N-1}^2 \rho_N^2}, \tag{2.5.6}$$

which, up to constants, is the measure that appears in Eq. (2.4.19). The result will be of the form $d\mu_N = f_N \prod_{k=1}^{N-3} dx_k dy_k$, where f_N is a number depending on the number of punctures.

We begin the computation by using a ξ plane where we place all the N punctures equally spaced on the unit circle $|\xi| = 1$. We thus let ξ_k be the position of the k-th puncture, with

$$\xi_k = \exp((2k-1)i\pi/N)\,, \quad k = 1, 2, \cdots N\,. \tag{2.5.7}$$

In this presentation the ring domain surrounding a puncture, say the first one, is the wedge domain $0 \le \mathrm{Arg}(\xi) \le 2\pi/N$ (see Fig. 2-1) The mapping radius can be computed exactly by mapping the wedge to the unit disk $|w| \le 1$, via $t = \xi^{N/2}$ and $w = \frac{t-i}{t+1}$. The result is $\rho = 4/N$, and picking the three special punctures to be ξ_{N-2}, ξ_{N-1} and ξ_N, we find

$$d\mu_N = 64 \cdot \sin^4\Big(\frac{\pi}{N}\Big) \cdot \sin^2\Big(\frac{4\pi}{N}\Big) \cdot \Big(\frac{N}{4}\Big)^{2N} \cdot \prod_{k=1}^{N-3} d^2\xi_k\,, \tag{2.5.8}$$

where $d^2\xi_k = d\mathrm{Re}\,\xi_k\, d\mathrm{Im}\,\xi_k$. This is the measure, but in the ξ plane. In order to transform it to the z-plane we need the PSL$(2,\mathbb{C})$ transformation that will satisfy

$z(\xi_N) = 0, z(\xi_{N-1}) = 1$, and $z(\xi_{N-2}) = \infty$. The desired transformation is

$$z = \frac{\xi - e^{-i\pi/N}}{\xi - e^{-3i\pi/N}} \cdot \beta, \quad \text{with} \quad |\beta| = \frac{\sin(\pi/N)}{\sin(2\pi/N)}, \tag{2.5.9}$$

and a small calculation gives

$$d^2\xi_k = \left|\frac{dz}{d\xi}\right|_{\xi_k}^{-2} dx_k dy_k = 4 \cdot \frac{\sin^2(2\pi/N)}{\sin^4(\pi/N)} \cdot \sin^4\Big(\frac{(k+1)\pi}{N}\Big) \cdot dx_k dy_k \,. \tag{2.5.10}$$

This expression, used in Eq. (2.5.8) gives us the desired expression for the measure

$$\begin{aligned} d\mu_N \;=\; & 4^{2N-3} \cdot \sin^4\Big(\frac{\pi}{N}\Big) \cdot \sin^2\Big(\frac{4\pi}{N}\Big) \cdot \Big(\frac{N}{4}\Big)^{2N} \cdot \Big(\frac{\cos(\pi/N)}{\sin(\pi/N)}\Big)^{2N-6} \\ & \times \Big[\prod_{k=1}^{N-3} \sin^4\Big(\frac{(k+1)\pi}{N}\Big)\Big] \cdot \prod_{k=1}^{N-3} dx_k\, dy_k \,. \end{aligned} \tag{2.5.11}$$

This is an exact result, valid for all $N \geq 4$. For the case of $N = 4$ it gives $d\mu_4 = 256\, dxdy$.[7] Let us now consider the leading behavior of this measure as $N \to \infty$. The only term that requires some calculation is the product $a_N \equiv \prod_{k=1}^{N-3} \sin^4((k+1)\pi/N)$. One readily finds that as $N \to \infty$

$$\ln a_N \;\sim\; 4 \cdot \frac{N}{\pi} \int_0^\pi d\theta \ln(\sin\theta) = -4N\ln(2) \quad \Rightarrow \quad a_N \sim 4^{-2N}, \tag{2.5.12}$$

and using this result, we find the large N behaviour of the planar measure

$$d\mu_N \;\sim\; 4^5 \cdot \pi^6 \cdot \Big[\frac{N^2}{4\pi}\Big]^{2N-6} \cdot \prod_{k=1}^{N-3} dx_k dy_k \,. \tag{2.5.13}$$

This was the result we were after. We see that this measure grows like N^{4N}. This growth is so fast that presents an obstruction to a simple proof of convergence for the series defining the tachyon potential. Indeed, a very naive estimation would not yield convergence. Let us see this next.

Let us assume that this planar uniform configuration is indeed the point in $\mathcal{V}_{0,N}$ for which the measure is the largest. This statement requires explanation, since the numerical coefficient appearing in front of a measure can be changed by PSL$(2,\mathbb{C})$ transformations. Thus given any other configuration in $\mathcal{V}_{0,N}$ with a puncture at $0, 1$ and ∞ we do a transformation $z \to az$ with $a = 1/z_{\max}$, where $z_{\max}$ is the position of the puncture farther away from the origin. In this way we obtain a configuration with all the punctures in the unit disk, the same two punctures at zero and infinity, and some puncture at one. At this point the measures can be compared and we expect

[7]For $N = 4$ the measure can also be calculated exactly for the configuration with cross ratio equal to $(1+i\sqrt{3})/2$. As we will become clear in chapter 3, at this configuration $d\mu_4 = \frac{2^{11}}{3^4\sqrt[3]{2}}\, dx\, dy \approx 20.07\, dx\, dy$. This corresponds to the measure at the "center" of $\mathcal{V}_{0,4}$ and is indeed much smaller than the measure $256\, dx\, dy$ at the corners of $\mathcal{V}_{0,4}$.

the planar one to be larger. It is now clear from the construction that the full $\mathcal{M}_{0,N}$ is overcounted if we fix two punctures, one at zero, the other at infinity, and among the rest we pick one at a time to be put at one, while the others are integrated all over the inside of the unit disk. If we estimate this integral using our value for the measure in the worst configuration we get that each coefficient $\mathrm{v}_{_N} < [(N-2)\pi^{N-3}]N^{4N-12}$, where the prefactor in brackets arises from the above described integrals (this prefactor does not really affect the issue of convergence). The growth N^{4N} rules out the possibility of convergence. This bound is quite naive, but raises the possibility that there may be no radius of convergence for the tachyon potential.

2.5.3 The most uniform distribution of punctures

The corners of $\mathcal{V}_{0,N}$ turned out to be problematic. Since we expect the measure for the coefficients of the tachyon potential to be lowest at the most symmetric surfaces, we now estimate the measure at this point in $\mathcal{V}_{0,N}$ for large N. The estimates we find are consistent with some radius of convergence for the tachyon potential if the integrals are dominated by these configurations.

It is possible to do a very simple estimate. To this end consider the z plane and place one puncture at infinity with $|z| \geq 1$ its unit disk. In this way its mapping radius is just one. All other punctures will be distributed uniformly inside the disk. Because of area constraint we can imagine that each puncture will then carry a little disk of radius r, with $N\pi r^2 \sim \pi$ fixing the radius to be $r \sim 1/\sqrt{N}$. The mapping radius of each of these disks will be r. Another of the disks will be fixed at 0, and another to $2r$. We can now estimate the measure, which is the integrand in Eq. (2.4.19), where in dealing with the three special punctures we make use of Eq. (2.3.18). We have then

$$d\mu_{\text{sym}} \sim \frac{(2r)^2}{r^2 \cdot r^2 \cdot 1} \prod_{I=1}^{N-3} \frac{dx_{_I} dy_{_I}}{r^2} \sim N^{N-2} \prod_{I=1}^{N-3} dx_{_I} dy_{_I} \,. \tag{2.5.14}$$

Since all the punctures, except for the one at infinity, are inside the unit disk, we can compare the measure given above with the measure in the planar configuration. In that case the measure coefficient went like N^{4N} and now it essentially goes like N^N, which is much smaller, as we expected.[8] We can also repeat the estimate we did for the integration over moduli space for the planar configuration, and again, we just get an extra multiplicative factor of N, which is irrelevant. Therefore, if we assume this configuration dominates we find $\mathrm{v}_{_N} \sim N^N$ and the tachyon potential $\sum \frac{\mathrm{v}_N}{N!}\tau^N$ would have some radius of convergence.

[8]Notice that if the punctures in the planar configuration had remained in the boundary of the unit disk then the measure would have only diverged like N^{2N}. The conformal map that brought them all to the real line between 0 and 1 introduced an extra factor of N^{2N}. This suggests that the divergence may actually not be as strong as it seems at first sight.

2.5.4 Analytic continuation and divergences

Here we want to discuss what happens if we do string field theory with stubs of length l. It is well-known that as the length of the stubs goes to infinity $l \to \infty$ then the region of moduli space corresponding to $\mathcal{V}_{0,4}$ approaches the full $\mathcal{M}_{0,4}$. In this case the coefficient of the quartic term in the potential will go into the full off-shell amplitude for scattering of four tachyons at zero momentum. We will examine the measure of integration for the tachyon potential as we approach the boundaries of moduli space and see that we would get a divergence corresponding to a tachyon of zero momentum propagating over long times. We believe this divergence is unphysical, and that the correct approach is to define the amplitude by analytic continuation from a region in the parameter space of the external momenta where the amplitude converges. When the full off-shell amplitude is built from the vertex contribution and the Feynman diagram contribution, analytic continuation is necessary for the Feynman part.

We therefore examine the off-shell formula for the evaluation of the four string vertex for general off-shell tachyons. What we need is the expression given in Eq. (2.4.14) integrated over $\mathcal{V}_{0,4}$

$$A_{p_1\cdots p_4} = \frac{2}{\pi}\int_{\mathcal{V}_{0,4}} dx\, dy\, \frac{|z|^{2p_1p_2}|z-1|^{2p_1p_3}}{\rho_z^{2-p_1^2}\rho_0^{2-p_2^2}\rho_1^{2-p_3^2}\rho_\infty^{2-p_4^2}}\,, \tag{2.5.15}$$

where we have added subscripts to the mapping radii in order to indicate the position of the punctures. Let us now examine what happens as we attempt to integrate with $z \to 0$, corresponding to a degeneration where punctures one and two collide. In this region ρ_1 and ρ_∞ behave as constants, and we have that

$$A_{p_1\cdots p_4} \sim \int_{|z|<c} dx\, dy\, \frac{|z|^{2p_1p_2}}{\rho_z^{2-p_1^2}\rho_0^{2-p_2^2}}\,, \tag{2.5.16}$$

As the puncture at z is getting close to the puncture at zero it is intuitively clear that the mapping radii $\rho_z \sim \rho_0 \sim |z|/2$ as these are the radii of the "largest" nonintersecting disks surrounding the punctures. Therefore

$$A_{p_1\cdots p_4} \sim \int_{|z|<c} \frac{dx\, dy}{|z|^{4-p_1^2-p_2^2-2p_1p_2}} = \int_{|z|<c} \frac{dx\, dy}{|z|^{4-(p_1+p_2)^2}}\,, \tag{2.5.17}$$

and we notice that the divergence is indeed controlled by the momentum in the intermediate channel. If all the momenta were set to zero before integration, we get a divergence of the form $\int dr/r^3$. But the way to proceed is to do the integral in a momentum space region where we have no divergence

$$A_{p_1\cdots p_4} \sim \int_{|z|<c} \frac{dr}{r^{3-(p_1+p_2)^2}} \sim \frac{1}{[2-(p_1+p_2)^2]}\,, \tag{2.5.18}$$

and the final result does not show a divergence for $p_1 = p_2 = 0$. Notice also that the denominator in the result is nothing else than $L_0 + \bar{L}_0$ for the intermediate tachyon, if that tachyon were on shell, we would get a divergence due to it.

2.6 A formula for general off-shell amplitudes on the sphere

In this section we will derive a general formula for off-shell amplitudes on the sphere. In string field theory those amplitudes are defined as integrals over subspaces $\mathcal{A}_N$ of the moduli space $\widehat{\mathcal{P}}_{0,N}$ of N-punctured spheres with local coordinates, up to phases, at the punctures. We will assume a real parameterization of $\mathcal{A}_N$ and derive an operator expression which expresses the amplitude as a multiple integral over these parameters. The new point here is the explicit description of the relevant antighost insertions necessary to obtain the integrand, and the discussion explaining why the result does not depend neither on the parameterization of the subspace nor on the choice of a global uniformizer on the sphere.

We also give a formula for the case when the space $\mathcal{A}_N$ is parametrized by complex coordinates. For this case we will emphasize the analogy between the **b**–insertions for moduli and the **b**–insertions necessary to have PSL$(2,\mathbb{C})$ invariance. Finally, we will show how the general formula works by re-deriving the off–shell Koba-Nielsen amplitude considered earlier.

2.6.1 An operator formula for N–string forms on the sphere

Recall that the state space $\mathcal{H}$ of closed string theory consists of the states in the conformal theory that are annihilated both by $L_0 - \bar{L}_0$ and by $b_0 - \bar{b}_0$. Following [46] we now assign an N–linear function on $\mathcal{H}$ to any subspace $\mathcal{A}$ of $\widehat{\mathcal{P}}_{0,N}$. The multilinear function is defined as an integral over $\mathcal{A}$ of a canonical differential form

$$\{\Psi_1, \Psi_2, \ldots, \Psi_N\}_{\mathcal{A}} = \int_{\mathcal{A}} \Omega^{(\dim \mathcal{A} - \dim \mathcal{M}_{0,N})0,N}_{\Psi_1\Psi_2\ldots\Psi_N} \,. \tag{2.6.1}$$

One constructs the forms by verifying that suitable forms in $\mathcal{P}_{0,N}$ do lead to well defined forms in $\widehat{\mathcal{P}}_{0,N}$. This is the origin of the restriction of the CFT state space to $\mathcal{H}$. The canonical $2(N-3)+k$–form $\Omega^{(k)0,N}$ on $\widehat{\mathcal{P}}_{0,N}$ is defined by its action on $2(N-3)+k$ tangent vectors $V_I \in T_{\Sigma_P}\widehat{\mathcal{P}}_{0,N}$ as

$$\Omega^{(k)0,N}_{\Psi_1,\Psi_2,\ldots\Psi_N}(V_1, \ldots, V_{2(N-3)+k}) = \left(\frac{i}{2\pi}\right)^{N-3} \langle \Sigma_P | \mathbf{b}(\mathbf{v}_1) \cdots \mathbf{b}(\mathbf{v}_{2(N-3)+k}) | \Psi^N \rangle \,. \tag{2.6.2}$$

Here the surface state $\langle \Sigma_P|$ is a bra living in $(\mathcal{H}^*)^{\otimes N}$ and represents the punctured Riemann surface Σ_P. The symbol $\mathbf{v}_I$ denotes a Schiffer variation representing the tangent V_I, and

$$\mathbf{b}(\mathbf{v}) = \sum_{I=1}^{N} \oint_{w_I=0} \frac{dw_I}{2\pi i} v^{(I)}(w_I) b^{(I)}(w_I) + \oint_{\bar{w}_I=0} \frac{d\bar{w}_I}{2\pi i} \bar{v}^{(I)}(\bar{w}_I) \bar{b}^{(I)}(\bar{w}_I) \,. \tag{2.6.3}$$

Recall that a Schiffer variation for an N-punctured surface (in our present case a sphere) is an N–tuple of vector fields $\mathbf{v} = \left(v^{(1)}, v^{(2)}, \cdots, v^{(N)}\right)$ where the vector $v^{(k)}$

is a vector field defined in the coordinate patch around the k–th puncture.[9] Let w_k be the local coordinate around the k–th puncture P_k ($w_k(P_k) = 0$). The variation defines a new N–punctured Riemann sphere with a new chosen local coordinates $w'_k = w_k + \epsilon v^{(k)}(w_k)$. The new k-th puncture is defined to be at the point P'_k such that $w'_k(P'_k) = 0$. It follows that the k–th puncture is shifted by $-\epsilon v^{(k)}(P_k)$. For any tangent $V \in T_P\widehat{\mathcal{P}}_{0,N}$ there is a corresponding Schiffer vector. Schiffer vectors are unique up to the addition of vectors that arise from the restriction of holomorphic vectors on the surface minus the punctures.

Note that the insertions $\mathbf{b}(\mathbf{v})$ are invariantly defined, they only depend on the Schiffer vector, and do not depend on the local coordinates. Indeed b is a primary field of conformal dimension 2 or a holomorphic 2–tensor. Being multiplied by a holomorphic vector field v it produces a holomorphic 1–form, whose integral $\oint_{w=0} \frac{dw}{2\pi i} v(w)b(w)$ is well-defined and independent of the contour of integration.

In order to evaluate Eq. (2.6.1) we choose some real coordinates $\lambda_1, \ldots, \lambda_{\dim\mathcal{A}}$ on $\mathcal{A}$. Let $\{V_{\lambda_k}\} = \partial/\partial\lambda_k$ be the corresponding tangent vectors, and let $\{d\lambda_k\}$ be the corresponding dual one-forms, *i.e.* $d\lambda_k(V_{\lambda_l}) = \delta_{k,l}$. Using $\{V_{\lambda_k}\}$ we can rewrite Eq. (2.6.1) as

$$\begin{aligned}\{\Psi_1, \cdots, \Psi_N\}_{\mathcal{A}} = \left(\frac{i}{2\pi}\right)^{N-3} \int d\lambda_1 \cdots d\lambda_{\dim\mathcal{A}} \\ \times\langle\Sigma_P|\mathbf{b}(\mathbf{v}_{\lambda_1})\cdots\mathbf{b}(\mathbf{v}_{\lambda_{\dim\mathcal{A}}})|\Psi_1\rangle\cdots|\Psi_N\rangle\,.\end{aligned} \tag{2.6.4}$$

In order to continue we must parameterize $\mathcal{A}$ as it sits in the moduli space $\widehat{\mathcal{P}}_{0,N}$. Let w_I be a local coordinate around the I–th puncture. Given a global uniformizer z on the Riemann sphere we can represent w_I by as an invertible analytic function $w_I(z)$ defined on some disk in the z–plane which maps the disk to a standard unit disk $|w_I| < 1$. The inverse map $h_I = w_I^{-1}$ is therefore an analytic function on a unit disk. Therefore, N functions $h_I(w_I)$ define a point in $\widehat{\mathcal{P}}_{0,N}$, namely the sphere with N punctures at $h_I(0)$ and local coordinates given by $w_I(z) = h_I^{-1}(z)$. The embedding of $\mathcal{A}$ in $\widehat{\mathcal{P}}_{0,N}$ is then represented by a set of N holomorphic functions parameterized by the real coordinates λ_k on $\mathcal{A}$: $\{h_1(\{\lambda_k\}; w_1), \cdots, h_N(\{\lambda_k\}; w_N)\}$. It is well known how to write the state $\langle\Sigma_P|$ in terms of h_I's (see [50, 51]).

$$\begin{aligned}\langle\Sigma_P| = 2\cdot \int \prod_{I=1}^{N} dp_I\, (2\pi)^D \delta^D\left(\sum p_I\right) \bigotimes_{I=1}^{N} \langle \mathbf{1}^c, p_I| \int d^2\zeta^1 d^2\zeta^2 d^2\zeta^3 \\ \times \exp\left(E(\alpha) + F(b,c) - \sum_{n=1}^{3}\sum_{m\geq -1}\left(\zeta^n M_m^{nJ} b_m^{(J)} - \bar{\zeta}^n \overline{M_m^{nJ}} \bar{b}_m^{(J)}\right)\right).\end{aligned} \tag{2.6.5}$$

where repeated uppercase indices $I, J\cdots$, are summed over the N values they take.

[9] In general the vector fields $v^{(k)}$ are defined on some annuli around the punctures and do not extend holomorphically to the whole coordinate disk, in order to represent the change of modulus of the underlying non-punctured surface (see [46]). For $g = 0$ the underlying surface is the Riemann sphere and has no moduli. Therefore, the Schiffer vectors can be chosen to be extend throughout the coordinate disk.

In here

$$E(\alpha) = -\frac{1}{2}\sum_{n,m\geq 0}\left(\alpha_n^{(I)}\mathcal{N}_{nm}^{IJ}\alpha_m^{(J)} + \bar{\alpha}_n^{(I)}\overline{\mathcal{N}_{nm}^{IJ}}\bar{\alpha}_m^{(J)}\right)\,,$$
$$F(b,c) = \sum_{\substack{n\geq 2\\ m\geq -1}}\left(c_n^{(I)}\tilde{\mathcal{N}}_{nm}^{IJ}b_m^{(J)} + \bar{c}_n^{(I)}\overline{\tilde{\mathcal{N}}_{nm}^{IJ}}\bar{b}_m^{(J)}\right)\,. \tag{2.6.6}$$

The vacua satisfy $\langle \mathbf{1}^c, p|\mathbf{1}, q\rangle = \delta(p-q)$, and the odd Grassmann variables ζ^n are integrated using

$$\int d^2\zeta^1 d^2\zeta^2 d^2\zeta^3\ \zeta^1\bar{\zeta}^1\ \zeta^2\bar{\zeta}^2\ \zeta^3\bar{\zeta}^3 \equiv 1\,. \tag{2.6.7}$$

Note that the effect of this integration is to give the product of six antighost insertions coming from the last term in the exponential in Eq. (2.6.5). The minus sign in front of this term is actually irrelevant (as the reader can check) but it was included for later convenience. A bar over a number means complex conjugate while a bar over an operator is used in order to distinguish the left–moving modes from right–moving ones. The *Neumann coefficients* $\mathcal{N}_{mn}^{IJ}$ and $\tilde{\mathcal{N}}_{mn}^{IJ}$ are given by the following formulae:

$$\begin{aligned}
\mathcal{N}_{00}^{IJ} &= \begin{cases}\log(h_I'(0)), & I=J\\ \log(h_I(0)-h_J(0)), & I\neq J\end{cases}\\
\mathcal{N}_{0n}^{IJ} &= \frac{1}{n}\oint_{w=0}\frac{dw}{2\pi i}w^{-n}h_J'(w)\frac{-1}{h_I(0)-h_J(w)}\,,\\
\mathcal{N}_{mn}^{IJ} &= \frac{1}{m}\oint_{z=0}\frac{dz}{2\pi i}z^{-m}h_I'(z)\frac{1}{n}\oint_{w=0}\frac{dw}{2\pi i}w^{-n}h_J'(w)\frac{1}{(h_I(z)-h_J(w))^2}\,,\\
\tilde{\mathcal{N}}_{mn}^{IJ} &= \frac{1}{m}\oint_{z=0}\frac{dz}{2\pi i}z^{-m+1}(h_I'(z))^2\frac{1}{n}\oint_{w=0}\frac{dw}{2\pi i}w^{-n-2}(h_J'(w))^{-1}\frac{1}{h_I(z)-h_J(w)}\,.
\end{aligned} \tag{2.6.8}$$

Moreover,

$$M_m^{nJ} = \oint_{w=0}\frac{dw}{2\pi i}w^{-m-2}(h_J'(w))^{-1}[h_J(w)]^{n-1},\quad n=1,2,3\,. \tag{2.6.9}$$

Antighost insertions

Now let us show how to take the $\mathbf{b}$-insertions into account. In order to calculate the $\mathbf{b}$-insertion associated to a tangent vector $V\in T_\Sigma\widehat{\mathcal{P}}_{0,N}$, we must find the Schiffer vector (field) that realizes the deformation of the surface Σ specified by V. Consider a line $c(t)$ in $\widehat{\mathcal{P}}_{0,N}$ parameterized by the real parameter t: $c:[0,1]\to\widehat{\mathcal{P}}_{0,N}$, such that $\Sigma = c(0)$, and the tangent vector to the curve is $V = c_*\left(\frac{d}{dt}\right)$. We will see now how one can use this setup to define in a natural way a vector field on the neighborhood of the punctures of the Riemann surface Σ. This vector field is the Schiffer vector.

We can represent the curve $c(t)$ by N functions $h_I(t;w)$, holomorphic in w, and parameterized by t. Choose a fixed value w_0 of the w disk. We now define a map $f^{w_0}: c(t)\to\Sigma$ from the curve $c(t)$ to a curve on the surface Σ. The function f^{w_0} takes $c(t)$ to $z_I(t) = h_I(t,w_0)$ for each value of t. We can now use the map f^{w_0} to produce a push-forward map of vectors $f_*^{w_0}: Tc\to T_{h_I(0,w_0)}\Sigma$. In this way we can

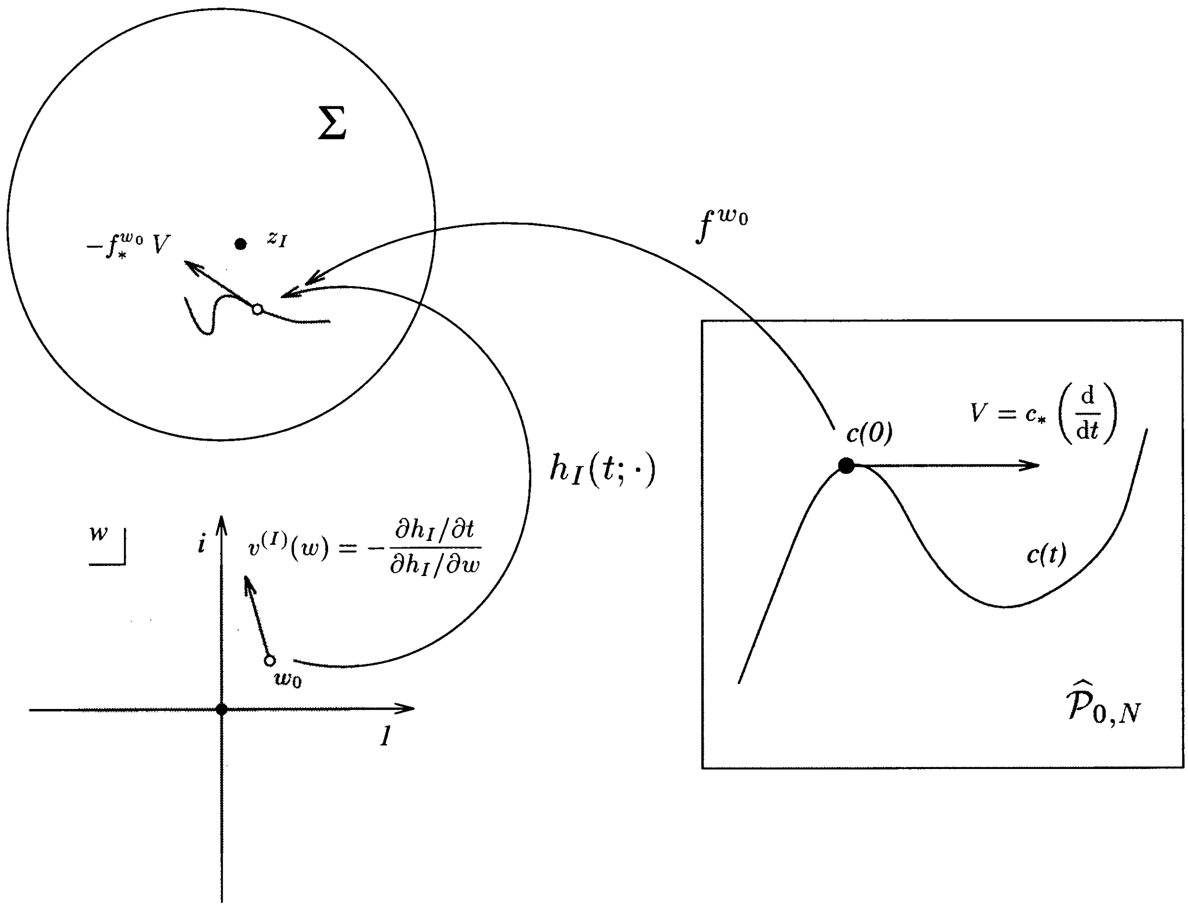


Figure 2-2: We show how to obtain a Schiffer vector field associated to a tangent vector in $\widehat{\mathcal{P}}_{0,N}$. Shown are the Riemann surface Σ and the local coordinate plane w.

produce the vector $f_*^{w_0}V \in T_{z_I(0,w_0)}\Sigma$. By varying the value of w_0 we obtain a vector *field* on the neighborhood of the I-th puncture. We claim that this vector field, with a minus sign, is the Schiffer vector. In components, and with an extra minus sign, the pullback gives

$$v^{(I)}(z) = -\frac{\partial h_I}{\partial t}(t; w_I(z)) \;, \tag{2.6.10}$$

It is useful to refer the Schiffer vector to the local coordinate w_I. We then find, by pushing the vector further

$$v^{(I)}(w_I) = -\left(\frac{\partial h_I}{\partial w_I}\right)^{-1} \cdot \frac{\partial h_I(t; w_I)}{\partial t}. \tag{2.6.11}$$

By definition, the Schiffer vector $\mathbf{v}(V)$ corresponding to the vector V is given by the collection of vector fields $\mathbf{v} = \big(v^{(1)}(w_1), \cdots, v^{(N)}(w_N)\big)$. If we define the vector $V_{\lambda_k} \in T\widehat{\mathcal{P}}_{0,N}$ to be the tangent associated to the coordinate curve parameterized by

λ_k, we then write the Schiffer variation for V_{λ_k} as:

$$v^{(I)}_{\lambda_k}(w_I) = -\frac{1}{h'_I(w_I)}\frac{\partial h_I(\lambda; w_I)}{\partial \lambda_k}, \tag{2.6.12}$$

where $h'_I(w_I) \equiv (\partial h_I/\partial w_I)$.

Before proceeding any further, let us confirm that the 'natural' vector we have obtained is indeed the Schiffer vector. This is easily done. Let p denote a point in the Riemann surface Σ and let $w_I(p)$ denote its local coordinate. By definition, the Schiffer vector defines a new coordinate $w'_I(p)$ as $w'_I(p) = w(p) + dt\, v^I(w(p))$, where t is again a parameter for the deformation. Since the z-coordinate of the point p does not change under the deformation, we must have that $h_I(t+dt, w'_I(p)) = h_I(t, w_I(p))$. Upon expansion of this last relation one immediately recovers Eq. (2.6.11).

One more comment is in order. What happened to the usual ambiguity in choosing Schiffer vectors? Schiffer vectors are ambiguous since there are nonvanishing N-tuples that do not induce any deformation. This happens when the N-tuples can be used to define a holomorphic vector on the surface minus the punctures. In our case the ambiguity is due to the fact that the functions $h_I(\lambda, w)$ can be composed with any PSL$(2,\mathbb{C})$ transformation S in the form $S\circ h_I(\lambda, w)$. We will come back to this point later.

Using Eq. (2.6.3) we can now write $\mathbf{b}(\mathbf{v}^\lambda)$ as

$$\mathbf{b}(\mathbf{v}_{\lambda_k}) = -\sum_{m\geq -1}(B^{kJ}_m b^{(J)}_m + \overline{B^{kJ}_m}\bar{b}^{(J)}_m), \tag{2.6.13}$$

where, as usual, the repeated index J is summed over the number of punctures, and

$$B^{kJ}_m = \oint_{w=0}\frac{dw}{2\pi i}\, w^{-m-2}\,\frac{1}{h'_J(w)}\frac{\partial h_J(\lambda, w)}{\partial \lambda_k}. \tag{2.6.14}$$

The range $m \geq -1$ has been obtained because the Schiffer vectors can be chosen to be holomorphic and not to have poles at the punctures (this will not be the case for higher genus surfaces).

Let us now treat the $\mathbf{b}$-insertions in a way similar to that used for the zero modes in Eq. (2.6.5). Let ζ_I and η_I be anti-commuting variables, then

$$\int d\xi_1\ldots d\xi_n e^{\xi_1\eta_1+\cdots+\xi_n\eta_n} = \int d\xi_1 e^{\xi_1\eta_1}\cdots\int d\xi_n e^{\xi_n\eta_n} = \eta_1\cdots\eta_n. \tag{2.6.15}$$

This observation allows us to represent the product of $\mathbf{b}$-insertions in Eq. (2.6.2) as an integral of an exponent.

$$\mathbf{b}(\mathbf{v}_{\lambda_1})\cdots\mathbf{b}(\mathbf{v}_{\lambda_{\dim\mathcal{A}}}) = \int\prod_{k=1}^{\dim\mathcal{A}} d\xi^k \exp\Big(-\sum_{k=1}^{\dim\mathcal{A}}\sum_{m\geq -1}\xi^k\left(B^{kJ}_m b^{(J)}_m + \overline{B^{kJ}_m}\bar{b}^{(J)}_m\right)\Big), \tag{2.6.16}$$

where the ξ^n's are real Grassmann odd variables. The multilinear product Eq. (2.6.1) now assumes the form

$$
\begin{aligned}
\{\Psi^N\}_{\mathcal{A}} = 2\left(\frac{i}{2\pi}\right)^{N-3} \int \prod_{I=1}^{N} d^D p_{{}_I} (2\pi)^D \delta^D \left(\sum_{I=1}^{N} p_{{}_I}\right) \\
\times \bigotimes_{I=1}^{N} \langle \mathbf{1}^c, p_{{}_I} | \int \prod_{k=1}^{\dim \mathcal{A}} d\lambda_k \prod_{k=1}^{\dim \mathcal{A}} d\xi^k \\
\times \int d^2\zeta^1 d^2\zeta^2 d^2\zeta^3 \exp\Bigg(E(\alpha) + F(b,c) \\
- \sum_{k=1}^{3} \sum_{m\geq -1} \left(\zeta^k M_m^{kJ} b_m^{(J)} - \bar{\zeta}^k \overline{M_m^{kJ}} \bar{b}_m^{(J)} \right) \\
- \sum_{k=1}^{\dim \mathcal{A}} \sum_{m\geq -1} \xi^k \left(B_m^{kJ} b_m^{(J)} + \overline{B_m^{kJ}} \bar{b}_m^{(J)} \right) \Bigg) |\Psi^N\rangle,
\end{aligned}
\tag{2.6.17}
$$

The above formula together with equations (2.6.6),(2.6.8), (2.6.9) and (2.6.14)gives a closed expression for a multi-linear form associated with $\mathcal{A} \subset \widehat{\mathcal{P}}_{0,N}$.

The resemblance of the last two terms appearing in the exponential is not a coincidence. While the last term arose from the antighost insertions for moduli, the first term, appearing already in the description of the surface state $\langle\Sigma|$, can be thought as the antighost insertions due to the Schiffer vectors that represent PSL$(2,\mathbb{C})$ transformations. This is readily verified. Consider the sphere with uniformizer z. The six globally defined vector fields are given by $v_k(z) = z^k$, and $v_k'(z) = iz^k$ with $k = 0,1,2$. Referring them to the local coordinates one sees that $v_k^I(w) = [h_{{}_I}(w)]^k / h_{{}_I}'(w)$ and $v'^I_k(w) = i[h_{{}_I}(w)]^k / h_{{}_I}'(w)$. As a consequence

$$
\mathbf{b}(v_k) = \sum_{m\geq -1} \left(M_m^{kJ}\, b_m + \overline{M_m^{kJ}}\, \bar{b}_m \right) , \tag{2.6.18}
$$

$$
\mathbf{b}(v_k') = i \sum_{m\geq -1} \left(M_m^{kJ}\, b_m - \overline{M_m^{kJ}}\, \bar{b}_m \right) , \tag{2.6.19}
$$

where the M coefficients were defined in Eq. (2.6.9). It is clear that the product of the six insertions precisely reproduces the effect of the first sum in the exponential of Eq. (2.6.17).

In order to be used in Eq. (2.6.17) the subspace $\mathcal{A}$ is parametrized by some coordinates λ_k. The expression for the multilinear form is independent of the choice of coordinates; it is a well-defined form on $\mathcal{A}$.[10] Once the parametrization is chosen, the space $\mathcal{A}$ has to be described by the N functions $h_{{}_I}(\lambda, w)$. These functions, as we move on $\mathcal{A}$ are defined up to a *local* linear fractional transformation. At every point in moduli space we are free to change the uniformizer. Let us see why Eq. (2.6.17)

[10]This is easily verified explicitly. Under coordinate transformations, the product $\prod d\lambda_i$ transforms with a Jacobian, and the product of antighost insertions, as a consequence of Eq. (2.6.14) transforms with the inverse Jacobian.

has local PSL$(2,\mathbb{C})$ invariance. The bosonic Neumann coefficients $\mathcal{N}^{IJ}_{nm}$ are PSL$(2,\mathbb{C})$ invariant by themselves for $n, m \geq 1$. The sums $\sum_{IJ} \alpha_0^{(I)} \mathcal{N}^{IJ}_{00} \alpha_0^{(J)} = \sum_{IJ} \mathcal{N}^{IJ}_{00} p_I p_J$ and $\sum_J \mathcal{N}^{IJ}_{m0} \alpha_0^{(J)} = \sum_J \mathcal{N}^{IJ}_{m0} p_J$ can be shown to be invariant due to momentum conservation $\sum p_J = 0$. A detailed analysis of PSL$(2,\mathbb{C})$ properties of the ghost part of a surface state have been presented by LeClair *et al.* in [50] for open strings. Their arguments can be readily generalized to our case. The only truly new part that appears in Eq. (2.6.17) is the last term in the exponential.

Under global PSL$(2,\mathbb{C})$ transformations, namely transformations of the form $h \to (ah+b)/(ch+d)$, with a, b, c and d independent of λ, this term is invariant because so is every coefficient B^{kJ}_m. Since a general local PSL$(2,\mathbb{C})$ tranformation can be written locally as a global transformation plus an infinitesimal local one, we must now show invariance under infinitesimal local transformations. These are transformations of the form

$$\widetilde{h}_I = h_I + a(\lambda) + b(\lambda) h_I + c(\lambda) h_I^2 , \tag{2.6.20}$$

for a, b, and c small. A short calculation shows that

$$-\frac{1}{\widetilde{h}'_J} \frac{\partial \widetilde{h}_I}{\partial \lambda_k} = -\frac{1}{h'_J} \frac{\partial h_I}{\partial \lambda_k} - \frac{1}{h'_J} \Big[\frac{\partial a}{\partial \lambda_k} + \frac{\partial b}{\partial \lambda_k} h_I + \frac{\partial c}{\partial \lambda_k} h_I^2 \Big] . \tag{2.6.21}$$

On the left hand side we have the new Schiffer vector, and the first term the right hand side is the old Schiffer vector. We see that they differ by a linear superposition of the Schiffer vectors v^I_k and v'^I_k introduced earlier in our discussion of PSL$(2,\mathbb{C})$ transformations (immediately above Eq. (2.6.18).) It follows that the extra contributions they make to the antighost insertions vanish when included in the multilinear form because the multilinear form already includes the antighost insertions corresponding to the Schiffer vectors generating PSL$(2,\mathbb{C})$.

Complex coordinates

In some applications the subspace $\mathcal{A}$ has even dimension and can be equipped with complex coordinates. Let $\dim^c \mathcal{A}$ denote the complex dimension of $\mathcal{A}$ and let

$$\{\lambda_k\} = \{\lambda_1, \cdots, \lambda_{\dim^c \mathcal{A}}\}$$

be a set of complex coordinates. The subspace $\mathcal{A}$ can now be represented by the collection of functions $\{h_I(\{\lambda_k\}, \{\bar{\lambda}_k\}; w_I)\}$ with $I = 1, \cdots, N$. In order to derive a formula for this case we simply take the earlier result for two real insertions and pass to complex coordinates. We thus consider

$$\Omega^2 = d\lambda_1 \wedge d\lambda_2 \; d\xi^1 d\xi^2 \; \exp\Big[-\sum_{k=1}^{2} \xi^k \left(B^{kJ}_m b^{(J)}_m + \overline{B^{kJ}_m} \bar{b}^{(J)}_m \right) \Big] , \tag{2.6.22}$$

Using complex coordinates $\lambda'_1 = \lambda_1 + i\lambda_2$ and $\xi'^1 = \xi^1 + i\xi^2$, and letting $\int d^2\xi' \, \xi' \bar{\xi}' \equiv 1$, we can write the above as

$$\Omega^2 = d\lambda'_1 \wedge d\bar{\lambda}'_1 \; d^2\xi'_1 \; \exp\Big[-\xi'^1 \left(B^{1J}_m b^{(J)}_m + \overline{B^{\bar{1}J}_m} \bar{b}^{(J)}_m \right) + \bar{\xi}'^1 \left(B^{\bar{1}J}_m b^{(J)}_m + \overline{B^{1J}_m} \bar{b}^{(J)}_m \right) \Big] , \tag{2.6.23}$$

where we have defined

$$
\begin{aligned}
B_m^{kJ} &= \oint_{w=0} \frac{dw}{2\pi i} w^{-m-2} \frac{1}{h'_J(w)} \frac{\partial h_J(\lambda, \bar{\lambda}; w)}{\partial \lambda_k}, \\
B_m^{\bar{k}J} &= \oint_{w=0} \frac{dw}{2\pi i} w^{-m-2} \frac{1}{h'_J(w)} \frac{\partial h_J(\lambda, \bar{\lambda}; w)}{\partial \bar{\lambda}_k}.
\end{aligned} \tag{2.6.24}
$$

Note that the $B_m^{\bar{k}J}$ coefficients do not vanish because the embedding of $\mathcal{A}$ in $\widehat{\mathcal{P}}_{0,N}$ need not be holomorphic and, as a consequence, the derivatives $\partial h_I / \partial \bar{\lambda}_k$ need not vanish. Using this result we can now rewrite Eq. (2.6.17) for the case of complex coordinates for moduli

$$
\begin{aligned}
\{\Psi^N\}_{\mathcal{A}} = 2\left(\frac{i}{2\pi}\right)^{N-3} & \int \prod_{I=1}^{N} d^D p_I (2\pi)^D \delta^D \left(\sum p_I\right) \\
& \times \bigotimes_{I=1}^{N} \langle \mathbf{1}^c, p_I | \int_{\mathcal{A}} \prod_{k=1}^{\dim^c \mathcal{A}} d\lambda_k \wedge d\bar{\lambda}_k \prod_{k=1}^{\dim^c \mathcal{A}} d^2 \xi^k \\
\times \prod_{k=1}^{3} \mathrm{d}^2 \zeta^k \exp \Bigg(& E(\alpha) + F(b,c) - \sum_{k=1}^{3} \left(\zeta^k M_m^{kJ} b_m^{(J)} - \bar{\zeta}^k \overline{M_m^{kJ}} \bar{b}_m^{(J)} \right) \\
- \sum_{k=1}^{\dim^c \mathcal{A}} & \left[\xi^k \left(B_m^{kJ} b_m^{(J)} + \overline{B_m^{\bar{k}J}} \bar{b}_m^{(J)} \right) - \bar{\xi}^k \left(B_m^{\bar{k}J} b_m^{(J)} + \overline{B_m^{kJ}} \bar{b}_m^{(J)} \right) \right] \Bigg) |\Psi^N\rangle,
\end{aligned} \tag{2.6.25}
$$

where $m \geq -1$ for the implicit oscillator sum. It is useful to bring out the similarity between the antighost insertions for $\mathrm{PSL}(2, \mathbb{C})$ and those for moduli. In order to achieve this goal we introduce $\xi^{\dim^c \mathcal{A}+k} \equiv \zeta^k$ for $k = 1, 2, 3$.

$$
\begin{aligned}
\{\Psi^N\}_{\mathcal{A}} = 2\left(\frac{i}{2\pi}\right)^{N-3} & \int \prod_{I=1}^{N} d^D p_I (2\pi)^D \delta^D \left(\sum p_I\right) \\
& \times \bigotimes_{I=1}^{N} \langle \mathbf{1}^c, p_I | \int_{\mathcal{A}} \prod_{k=1}^{\dim^c \mathcal{A}} d\lambda_k \wedge d\bar{\lambda}_k \\
\times \prod_{k=1}^{\dim^c \mathcal{A}+3} d^2 \xi^k \cdot \exp \Bigg(& E(\alpha) + F(b,c) \\
& - \sum_{k=1}^{\dim^c \mathcal{A}+3} \left(\xi^k \mathcal{B}_m^{kJ} b_m^{(J)} - \bar{\xi}^k \overline{\mathcal{B}_m^{kJ}} \bar{b}_m^{(J)} \right) \\
- \sum_{k=1}^{\dim^c \mathcal{A}} & \left(\xi^k \overline{B_m^{\bar{k}J}} \bar{b}_m^{(J)} - \bar{\xi}^k B_m^{\bar{k}J} b_m^{(J)} \right) \Bigg) |\Psi^N\rangle,
\end{aligned} \tag{2.6.26}
$$

where the script style $\mathcal{B}$ matrix elements are defined as

$$
\mathcal{B}_m^{kJ} = \begin{cases} B_m^{kJ}, & \text{for } k \leq \dim^c \mathcal{A} \\ M_m^{(k-\dim^c \mathcal{A})J}, & \text{for } k - \dim^c \mathcal{A} = 1, 2, 3. \end{cases} \tag{2.6.27}
$$

This concludes our construction of off-shell amplitudes as forms on moduli spaces of punctured spheres.

2.6.2 Application to off-shell tachyons

Let us see how the formulae derived above work for the case of N tachyons with arbitrary momenta. This particular example allows us to confirm our earlier calculation of off-shell tachyon amplitudes. Specifically, we are going to evaluate the multilinear function $\{\tau_{p_1},\cdots,\tau_{p_N}\}$ where $|\tau_{p_i}\rangle = c_1\bar{c}_1|\mathbf{1},p_i\rangle$. In this case the state to be contracted with the bra representing the multilinear function is $|\tau^N\rangle = \otimes_{I=1}^N c_1^{(I)}\bar{c}_1^{(I)}|\mathbf{1},p_{_I}\rangle$. Upon contraction with this state we will only get contributions from $b_{-1}^{(I)}$, $\bar{b}_{-1}^{(I)}$, and the matter zero modes $\alpha_0^{(I)} = \bar{\alpha}_0^{(I)} = ip_{_I}$.

We will use as moduli the complex coordinates $z_1,\cdots,z_{N-3}$ representing the position of the first $(N-3)$ punctures. Therefore, $\lambda_k = z_k$, for $k=1,\ldots,N-3$, and we must use Eq. (2.6.26) to calculate the multilinear function. Our setting of the z-coordinates as moduli implies that the functions $h_{_J}$ take the form

$$h_{_J}(z,\bar{z},w) = z_{_J} + a(z,\bar{z})w + \cdots\,. \tag{2.6.28}$$

It then follows that

$$M_{-1}^{kJ} = \oint_{w=0}\frac{dw}{2\pi i}\frac{1}{w}(h'_{_J}(w))^{-1}[h_{_J}(w)]^{k-1} = \frac{z_{_J}^{k-1}}{h'_{_J}(0)}\,, \tag{2.6.29}$$

and furthermore

$$B_{-1}^{kJ} = \oint_{w=0}\frac{dw}{2\pi i}\frac{1}{w}\frac{1}{h'_{_J}(w)}\frac{\partial h_{_J}}{\partial\lambda_k} = \frac{\delta^{kJ}}{h'_{_J}(0)}\,, \tag{2.6.30}$$

while $B_m^{\bar{k}J} = 0$, since $\partial h_{_J}/\partial\bar{z}_k = 0$. With this information, back in Eq. (2.6.26) we find

$$\begin{aligned}\{\tau_{p_1},\cdots,\tau_{p_N}\}_{\mathcal{A}} = 2\left(\frac{i}{2\pi}\right)^{N-3}(2\pi)^D\delta^D(\mathbf{0})\int\prod_{k=1}^{N-3}dz_k\wedge d\bar{z}_k\prod_{k=1}^{N}d^2\xi^k \\ \times\langle\mathbf{1}^c|\exp\left(E(\alpha)-\sum_{k=1}^{N}\left(\xi^k\mathcal{B}_{-1}^{kJ}b_{-1}^{(J)}-\bar{\xi}^k\overline{\mathcal{B}_{-1}^{kJ}}\bar{b}_{-1}^{(J)}\right)\right)\prod_{I=1}^{N}c_1^{(I)}\bar{c}_1^{(I)}|\mathbf{1}\rangle\,.\end{aligned} \tag{2.6.31}$$

We can now calculate the bosonic contribution from $E(\alpha)$

$$\begin{aligned}\exp(E(\alpha)) &= \exp\Big(-\frac{1}{2}\sum_{I,J=1}^{N}\left(\alpha_0^{(I)}\mathcal{N}_{00}^{IJ}\alpha_0^{(J)}+\bar{\alpha}_0^{(I)}\overline{\mathcal{N}_{00}^{IJ}}\bar{\alpha}_0^{(J)}\right)\Big)\\ &= \exp\Big(-\frac{1}{2}\sum_{I,J=1}^{N}\left(\mathcal{N}_{00}^{IJ}+\overline{\mathcal{N}_{00}^{IJ}}\right)p_{_I}p_{_J}\Big)\\ &= \prod_{I<J}\left(\frac{|h_{_I}(0)-h_{_J}(0)|^2}{|h'_{_I}(0)|\cdot|h'_{_J}(0)|}\right)^{p_Ip_J} = \prod_{I<J}\chi_{IJ}^{p_Ip_J}\,,\end{aligned} \tag{2.6.32}$$

where we have used the expression for Neumann coefficients Eq. (2.6.8), momentum conservation, and Eq. (2.3.7). We thus obtain

$$\{\tau_{p_1},\cdots,\tau_{p_N}\}_{\mathcal{A}} = \frac{2}{\pi^{N-3}}\,(2\pi)^D\delta^D(\mathbf{0})\int\prod_{k=1}^{N-3} dx_k\,dy_k \prod_{I<J}\chi_{IJ}^{p_I p_J}$$
$$\times\prod_{k=1}^{N} d^2\xi^k\langle\mathbf{1}^c|\exp\Big(-\sum_{k=1}^{N}\big(\xi^k\mathcal{B}^{kJ}_{-1}b^{(J)}_{-1}-\bar\xi^k\overline{\mathcal{B}^{kJ}_{-1}}\bar b^{(J)}_{-1}\big)\Big)\prod_{I=1}^{N}c^{(I)}_1\bar c^{(I)}_1|\mathbf{1}\rangle\,. \tag{2.6.33}$$

Consider the second line of the above equation. The effect of the ghost state is to select the term in the exponential proportional to the product of all antighosts. Since

$$\langle\mathbf{1}^c|\prod_{I=1}^{N}b^{(I)}_{-1}\bar b^{(I)}_{-1}\prod_{I=1}^{N}c^{(I)}_1\bar c^{(I)}_1|\mathbf{1}\rangle = (-)^N\,, \tag{2.6.34}$$

we can write the second line of Eq. (2.6.31) using a second set of Grasmmann variables η^k

$$(-)^N\int\prod_{k=1}^{N}d^2\xi^k\,d^2\eta^k\exp\Big[\sum_{k,p=1}^{N}\Big(-\,\xi^k\,\mathcal{B}^{kp}_{-1}\,\eta^p+\bar\xi^k\,\overline{\mathcal{B}^{kp}_{-1}}\,\bar\eta^p\Big)\Big] = (-)^N\,\det(\mathcal{B})\,\det(\bar{\mathcal{B}})\,, \tag{2.6.35}$$

where in the last step we used a standard formula in Grassmann integration. We can now use equations (2.6.29), (2.6.30), and (2.6.27) to calculate $|\det\mathcal{B}|^2$. We find

$$\begin{aligned}|\det\mathcal{B}|^2 &= |(z_N-z_{N-2})(z_N-z_{N-1})(z_{N-2}-z_{N-1})|^2\prod_{I=1}^{N}\frac{1}{|h'_I(0)|^2}\,,\\ &= \chi^2_{N-2,N-1,N}\prod_{I=1}^{N-3}\frac{1}{\rho_I^2}\,,\end{aligned} \tag{2.6.36}$$

where we made use of the definition of the mapping radius and of Eq. (2.3.10). We can now assemble the final form of the tachyon multilinear function. Back in Eq. (2.6.33) we have

$$\{\tau_{p_1},\cdots,\tau_{p_N}\}_{\mathcal{A}} = (-)^N\frac{2}{\pi^{N-3}}\prod_{I=1}^{N-3}\frac{dx_I dy_I}{\rho_I^2}\,\chi^2_{N-2,N-1,N}\prod_{I<J}\chi_{IJ}^{q_I q_J}\cdot(2\pi)^D\delta(\mathbf{0})\,. \tag{2.6.37}$$

which agrees precisely with the off-shell Koba-Nielsen formula Eq. (2.4.13).

Chapter 3

Effective tachyonic potential up to fourth order

3.1 Introduction

The following expression for the classical tachyonic potential has been obtained by G. Moore [41] and was proven in Ref. [52]:

$$V(t) = -t^2 - \sum_{N=3}^{\infty} \mathrm{v}_N \frac{t^N}{N!}, \tag{3.1.1}$$

where

$$\mathrm{v}_N \equiv (-)^N \frac{2}{\pi^{N-3}} \int_{\mathcal{V}_{0,N}} (\prod_{I=1}^{N-3} d^2\xi_I) \prod_{I=1}^{N} |w_I'(\xi_I)|^2 . \tag{3.1.2}$$

The global uniformizer ξ is chosen such that the coordinates of the last three punctures are $\xi_{N-2} = 0$, $\xi_{N-1} = 1$ and $\xi_N = \infty$. $w_I(\xi)$ denotes the local coordinate around the I-th puncture and the derivative at infinity is to be taken with respect to $1/\xi$. The integration in (3.1.2) has to be performed over $\mathcal{V}_{0,N}$, the region of the moduli space which can not be covered by the string diagrams with a propagator. We will distinguish the *missing region* or the *string vertex* $\mathcal{V}_{0,N}$ from the *Feynman region* $\mathcal{F}_{0,N} = \mathcal{M}_{0,N} \backslash \mathcal{V}_{0,N}$.

The cubic term does not require integration and can be easily evaluated (see Eq. (2.5.4)).

$$\mathrm{v}_3 = -\frac{3^9}{2^{11}} \approx -9.61. \tag{3.1.3}$$

For $N > 3$ there are two major obstacles to evaluation of (3.1.2): firstly, we need a description of $\mathcal{V}_{0,N}$ and secondly we have to define the local coordinates w_I. Unfortunately, the string field theory defines the vertex and the local coordinates implicitly in terms of a quadratic differential of special type and its invariants. Finding the quadratic differential is a difficult problem on its own and even when an analytic expression for it is known to find the desired invariants is still not trivial. In this article we will deal mostly with the fourth order term:

$$\mathrm{v}_4 \equiv \frac{2}{\pi} \int_{\mathcal{V}_{0,4}} \mu, \tag{3.1.4}$$

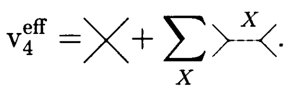
where μ is the measure of integration on the moduli space. It can be expressed in terms of local coordinates $w_i(\xi)$ as

$$\mu = \left| w_1'(\lambda)\, w_2'(0)\, w_3'(1)\, w_4'(\infty) \right|^2 d^2\lambda. \tag{3.1.5}$$

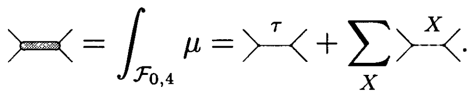
As before the global uniformizer ξ is fixed by placing three punctures at $\xi = 0$, 1 and ∞. The coordinate of the fourth puncture λ provides a coordinate on the moduli space $\mathcal{M}_{0,4}$. We will use a notation $d^2\lambda$ to denote the standard measure $d\,\mathrm{Re}\ \lambda\, d\,\mathrm{Im}\ \lambda$ on the complex plane of λ.

3.1.1 Effective potential

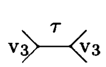
The bare tachyonic potential defined by (3.1.1) and (3.1.2) is not a physical quantity because the tachyon is coupled to the other fields in the string field theory. In order to calculate an effective potential (which is physical) one has to perform a summation over all the diagrams with intermediate non-tachyon states. Thus the effective four tachyon coupling constant $\mathrm{v}_4^{\mathrm{eff}}$ consists, in the tree approximation, of the elementary coupling $\mathrm{v}_4 =$ ⨯ and the sum over infinite number of diagrams with intermediate massive states X. We can write it schematically as


(3.1.6)

Instead of summing over all massive states we will calculate the full sum over all the states including the tachyon as an integral over the Feynman region $\mathcal{F}_{0,4} = \mathcal{M}_{0,4} \backslash \mathcal{V}_4$

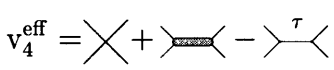


The first term with an intermediate tachyon can be easily evaluated in terms of the three-tachyon coupling constant v_3:



$$= 3 \cdot \frac{\mathrm{v}_3^2}{p^2 + m^2} = -\frac{3}{2}\mathrm{v}_3^2, \tag{3.1.7}$$

where $p = 0$ is the momentum of a propagating tachyon and $m_\tau^2 = -2$ is its mass squared. The factor of three comes from the sum over three channels each giving the same contribution. Combining the above equations we find



$$= \mathrm{v}_4 + \frac{2}{\pi} \int_{\mathcal{F}_{0,4}} \mu + \frac{3}{2}\mathrm{v}_3^2. \tag{3.1.8}$$

We will see that the integral in (3.1.8) is divergent and has to be found by analytic continuation.

For the case of $\mathcal{M}_{0,4}$ the invariants of the quadratic differential can be expressed in terms of elliptic integrals. Our discussion will involve an extensive use of elliptic functions and their q-expansions. These q-expansions prove to be a powerful tool in numerical calculations.

This chapter is organized as follows. First of all we will derive a general formula for the four-tachyon amplitude. We express the amplitudes in terms of invariants (χ_{ij}) of a four-punctured sphere with a choice of local coordinates. Then in section 3.3 we will review some basic properties of quadratic differentials and show how a quadratic differential defines local coordinates in general. In section 3.4 we will apply the general construction of section 3.3 to the case of $\mathcal{M}_{0,4}$. We introduce integral invariants a, b and c associated with a quadratic differential with four second order poles. In section 3.5 we show that the integrals over $\mathcal{V}_{0,4}$ and $\mathcal{F}_{0,4}$ can be easily evaluated if we know the integrand in terms of a and b. In sects. 3.6, 3.7 and 3.8 we express the measure of integration as a function of a and b. We reduce the problem to a single equation involving elliptic functions, which we solve approximately in two limits: one corresponding to a long propagator and an arbitrary twist angle and the other corresponding to both propagator and twist being small. For the intermediate region we solve the equation numerically. Finally we calculate the contribution of the Feynman diagrams (3.1.8) in section 3.9 and the elementary coupling (3.1.4) in section 3.10.

3.2 Four tachyon off-shell amplitude

In this section we will derive a formula for the scattering amplitude of four tachyons with arbitrary momenta. Although for the tachyonic potential we only need the amplitude at zero momentum, the integral which defines it is divergent and we are forced to treat it as an analytic continuation from the region in the momentum space where it converges. We will give the details on the origin of this divergence in section 3.9.

A general formula for the tree level off-shell amplitude is given by Eq. (2.4.17). For the case of four tachyons it gives the off-shell Koba-Nielsen formula

$$\Gamma_4(p_1,p_2,p_3,p_4) = \frac{2}{\pi}\int \frac{d^2\lambda}{|\lambda(1-\lambda)|^2}\,|\chi_{1234}|^2\cdot\prod_{i<j}|\chi_{ij}|^{p_ip_j}, \tag{3.2.1}$$

which expresses the four tachyon amplitude in terms of PSL$(2,\mathbb{C})$ invariants λ, χ_{ij} and χ_{1234}. The first invariant is just the cross ratio of the poles which we define as[1]

$$\lambda = \frac{z_1-z_2}{z_1-z_3}\cdot\frac{z_3-z_4}{z_2-z_4}. \tag{3.2.2}$$

The χ invariants can be expressed in terms of the mapping radii ρ_i as

$$\chi_{ij} = \frac{(z_i-z_j)^2}{\rho_i\rho_j}. \tag{3.2.3}$$

[1]Here we use a different cross ratio to that in the previouse chapter. In order to use the formulae of chapter 2 one has to change λ to $\lambda/(1-\lambda)$

Unlike those in Ref. [52] the χ invariants and mapping radii used here are complex numbers. We achieve the complexification by keeping the phases of the local coordinates. Thus, here ρ_i is given by

$$\rho_i = \frac{1}{w_i'(z_i)},$$

and not just the absolute value. The last invariant χ_{1234} can be expressed in terms of χ_{ij} as

$$\chi_{1234}^2 = \chi_{12}\chi_{23}\chi_{34}\chi_{41}, \tag{3.2.4}$$

By definition $\chi_{ij} = \chi_{ji}$ and thus for a four punctured sphere we have $\binom{4}{2} = 6$ different invariants. We will call a choice of local coordinates symmetric if the local coordinates do not change under the symmetries of the Riemann surface. Specifically, if S is an automorphism of a punctured Riemann surface Σ which maps the i-th puncture to the j-th puncture, we require that

$$w_j(S(\sigma)) = w_i(\sigma), \tag{3.2.5}$$

where $\sigma \in \Sigma$ belongs to the i-th coordinate patch. It is well known, that in most cases this condition can only be satisfied up to a phase (see Ref. [16]). Nevertheless, for a general four-punctured sphere the phases can be retained. Four-punctured spheres have a unique property: there exists a non-trivial symmetry group which acts on *any* four-punctured sphere. This group consists of the automorphisms which interchange two distinct pairs of punctures. One can easily check that these automorphisms exist for any $\Sigma \in \mathcal{M}_{0,4}$. One can visualize this symmetry by placing the punctures at the vertices of a rectangle—the symmetry group then becomes the group of the rectangle $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are a couple of four-punctured spheres which have a larger symmetry group: a tetrahedral symmetry in the case of $\lambda = \exp(\pi i/3)$, which is the most symmetric case or the symmetry group of the square for $\lambda = -1,\ 1/2$, or 2. It is not possible to realize the symmetry conditions for these larger groups if we wish to retain the phases, therefore we can require that (3.2.5) holds only for $S \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

For symmetric local coordinates the six χ-invariants are not independent. Using $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry one can prove that

$$\begin{aligned} \chi_{12} &= \chi_{34} \equiv \chi_s, \\ \chi_{14} &= \chi_{23} \equiv \chi_t, \\ \chi_{13} &= \chi_{24} \equiv \chi_u. \end{aligned} \tag{3.2.6}$$

Furthermore, due to the transformation properties of the mapping radii

$$\chi_s/\chi_u = -\lambda, \qquad \text{and} \qquad \chi_t/\chi_u = \lambda - 1, \tag{3.2.7}$$

and thus

$$\chi_u + \chi_s + \chi_t = 0. \tag{3.2.8}$$

Equations (3.2.6) and (3.2.8) show that for a symmetric choice of local coordinates there are only two independent χ-invariants.

Now we can rewrite the Koba-Nielsen formula in terms of χ_s, χ_t, χ_u and the Mandelstam variables

$$\Gamma_4(s,t,u) = \frac{2}{\pi}\int \frac{|\chi_s\chi_t|^2 d^2\lambda}{|\lambda(1-\lambda)|^2}\,|\chi_s|^{t+u-s}|\chi_t|^{u+s-t}|\chi_u|^{s+t-u}. \tag{3.2.9}$$

Note that the momentum dependent part of (3.2.9) is manifestly symmetric with respect to s, t and u. Let us show that the momentum independent part is symmetric as well. First of all we introduce a differential one-form

$$\gamma_4(s,t,u) = \frac{\chi_s\chi_t d\lambda}{\lambda(1-\lambda)}\chi_s^{\frac{t+u-s}{2}}\chi_t^{\frac{u+s-t}{2}}\chi_u^{\frac{s+t-u}{2}}. \tag{3.2.10}$$

Given a differential one-form $\omega = \omega(\lambda)d\lambda$ we can define the corresponding measure as $\mu = |\omega|^2 = |\omega(\lambda)|^2 d^2\lambda$. The measure of integration in (3.2.9) is just $|\gamma_4(s,\, t,\, u)|^2$ and we rewrite the Koba-Nielsen formula as

$$\Gamma_4(s,\, t,\, u) = \frac{2}{\pi}\int |\gamma_4(s,\, t,\, u)|^2. \tag{3.2.11}$$

Consider the momentum independent part of γ_4:

$$\gamma_4^{(0)} = \gamma_4(0,0,0) = \frac{\chi_s\chi_t d\lambda}{\lambda(1-\lambda)} = \chi_s d\chi_t - \chi_t d\chi_s, \tag{3.2.12}$$

where we have made use of (3.2.7). We can now use $\chi_u + \chi_s + \chi_t = 0$ and show that

$$\gamma_4^{(0)} = \chi_t d\chi_u - \chi_u d\chi_t = \chi_u d\chi_s - \chi_s d\chi_u, \tag{3.2.13}$$

and hence that $|\gamma_4^{(0)}|^2$ is totally symmetric.

The following expression for $\gamma_4^{(0)}$ although not explicitly symmetric is very simple and will be particularly useful latter. Using Eq. (3.2.7) we can rewrite Eq. (3.2.12) as

$$\gamma_4^{(0)} = \chi_u^2 d\lambda. \tag{3.2.14}$$

In the spirit of the string field theory we distinguish the contribution from the Feynman region $\mathcal{F}_{0,4} \subset \mathcal{M}_{0,4}$ (the surfaces which can be sewn out of two Witten's vertices and a propagator) and the missing region $\mathcal{V}_{0,4} = \mathcal{M}_{0,4}\backslash\mathcal{F}_{0,4}$. The later appears in the string field theory as the elementary four tachyon coupling

$$\mathrm{v}_4 = \frac{2}{\pi}\int_{\mathcal{V}_{0,4}} |\chi_u|^4 d^2\lambda. \tag{3.2.15}$$

3.3 How a quadratic differential defines local coordinates

As we mentioned above, the definition of off-shell string amplitudes requires use of local coordinates around the punctures of a Riemann surface. In this section we describe how the local coordinates can be specified by a quadratic differential of special type.

Given a local coordinate in some region of a Riemann surface, a quadratic differential can be written as $\phi = \varphi(z)(dz)^2$. $\varphi(z)$ is called the 'function element' of the quadratic differential. Although the value of the function element at a particular point does depend on the choice of the coordinate, its zeros and poles are coordinate-independent. The second order poles of quadratic differentials play a similar role to the simple poles of Abelian differentials. The residue $\mathrm{Res}_p\phi$ (the coefficient of the most singular term in the Laurent expansion of the function element) of a quadratic differential ϕ at a second order pole p is coordinate independent.

Given a Riemann surface $\Sigma \in \mathcal{M}_{G,N}$ of genus G with N punctures we define the space $\mathcal{D}_{G,N}(\Sigma)$ of quadratic differentials with second order poles at each puncture and the space $\mathcal{D}^R_{G,N}(\Sigma) \subset \mathcal{D}_{G,N}(\Sigma)$ restricted by the condition $\mathrm{Res}\,\phi = -1$ at every pole. The space $\mathcal{D}_{G,N}(\Sigma)$ is finite dimensional with $\dim\,\mathcal{D}_{G,N}(\Sigma) = 3\,G - 3 + 2\,N$. Furthermore,

$$\dim\,\mathcal{D}^R_{G,N}(\Sigma) = \dim\,\mathcal{D}_{G,N}(\Sigma) - N = 3\,G - 3 + N$$

is equal to the dimension of the moduli space $\mathcal{M}_{G,N}$. We consider the spaces of quadratic differentials with N second order poles $\mathcal{D}_{G,N}$ and $\mathcal{D}^R_{G,N}$ as fiber bundles over $\mathcal{M}_{G,N}$.

With a quadratic differential ϕ we associate a contact field $\phi > 0$. The integral lines of this field are called horizontal trajectories. We define a critical horizontal trajectory as one which starts at a zero of the quadratic differential and the critical graph as the set of horizontal trajectories which start and end at the zeros.

Let $\widehat{\mathcal{P}}_{G,N}$ be the moduli space of the genus G Riemann surfaces with N punctures and a choice of local coordinate up to a phase around each puncture. One can think of $\widehat{\mathcal{P}}_{G,N}$ as of a space of surfaces with N punctures and a closed curve (coordinate curve) drawn around each puncture. Due to the Riemann mapping theorem, there is a unique (up to phase) holomorphic map from the interior of a curve to the unit circle, which takes the puncture to 0. This map defines a local coordinate. Keeping this description in mind one can define an embedding $\Phi : \mathcal{D}^R_{G,N} \to \widehat{\mathcal{P}}_{G,N}$ using the critical graph of a quadratic differential to define a set of coordinate curves.

We can describe Φ more explicitly. Let $\phi \in \mathcal{D}^R_{G,N}$ be a quadratic differential. By definition of $\mathcal{D}^R_{G,N}$ it has N second order poles with residue -1. Let p be such a pole. Then, there exists a local coordinate w in the vicinity of p such that

$$\phi = -\frac{(dw)^2}{w^2}. \qquad (3.3.1)$$

Indeed, let z be some other coordinate and

$$\phi = \varphi(z)(dz)^2. \qquad (3.3.2)$$

We can find $w(z)$ solving the differential equation $i\,dw/w = \varphi^{1/2}(z)dz$. The solution is given by

$$w(z) = \exp\left(-i\int_{z_0}^{z}\varphi^{1/2}(z')dz'\right). \tag{3.3.3}$$

The point z_0 may be chosen arbitrarily and, so far, the local coordinate is defined by (3.3.1) only up to a multiplicative constant. Moreover (3.3.1) does not change when we substitute $1/w$ for w, which is equivalent to the change of sign of the square root in (3.3.3). The latter arbitrariness can be easily fixed by imposing the condition $w(p) = 0$. The inverse map $z = h(w)$ is a holomorphic function of the local coordinate, which can be analytically continued to a disk of some radius r. We can always rescale w so that $r = 1$. this fixes the scale of w. Now we have to show that the coordinate curves corresponding to this set of local coordinates form the critical graph of the quadratic differential. Indeed, the coordinate curve given by $|w| = 1$ is a horizontal trajectory of the quadratic differential which is equal to $(dw)^2/w^2$. Let us show that it has at least one zero on it. By definition $h(w)$ is holomorphic inside the unit disk and can not be analytically continued to a holomorphic function on a bigger disk. Yet $h'(w) = dz/dw = 1/(w(z)\varphi^{1/2}(z))$ and thus $h(w)$ is holomorphic at w unless $\varphi(h(w)) = 0$, or w is the coordinate of a zero of ϕ. We conclude then, that there is at least one zero on the curve $w(z) = 1$. Finally we can write a closed expression for the local coordinates associated with the quadratic differential ϕ:

$$w(z) = \exp\left(-i\int_{z_0}^{z}\sqrt{\phi}\right), \tag{3.3.4}$$

where the sign of the square root is fixed by $\mathrm{Res}_p\sqrt{\phi} = i$ and z_0 is a zero of ϕ. In general for each pole one has to select a zero to use in (3.3.4), but for the most interesting case when critical graph is a polyhedron choosing a different zero alters only the phase of $w(z)$.

So far a quadratic differential defines the local coordinates, but it is not itself defined by the underlying Riemann surface because the dimension of $\mathcal{D}_{G,N}$ is twice as big as the dimension of $\mathcal{M}_{G,N}$. In order to fix the quadratic differential we need an extra $3\,G - 3 + 2N$ complex or $6\,G - 6 + 4N$ real conditions. In the next section we will describe these conditions for the case $G = 0$, $N = 4$.

3.4 Quadratic differentials with four second order poles

In this section we focus on the case of a four-punctured sphere, $G = 0$ and $N = 4$. We define the integral invariants a, b and c of a quadratic differential which control the behavior of its critical horizontal trajectories. We find explicit formulae for these invariants in terms of Weierstrass elliptic functions.

Consider a meromorphic quadratic differential on a sphere which one has four second order poles. Given a uniformizing coordinate z on the sphere we can write the quadratic differential as

$$\phi = \frac{Q(z)}{\prod_{i=1}^{4} (z - z_i)^2} (dz)^2. \tag{3.4.1}$$

In order for ϕ to be holomorphic at $z = \infty$ the polynomial Q should be of degree less than or equal to 4:

$$Q(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0. \tag{3.4.2}$$

So far we have a five-dimensional complex linear space $\mathcal{D}_{0,4} = \mathbb{C}^5$ of quadratic differentials. When we restrict ourselves to quadratic differentials with the residues[2] equal -1 at every pole we define a one-dimensional complex affine subspace $\mathcal{D}^R_{0,4} \in \mathcal{D}_{0,4}$. Now we want to parameterize $\mathcal{D}^R_{0,4}$ in such way that coordinates do not depend on the choice of global uniformizer z. The following combinations of the coordinates of the poles and the zeros are invariant: the cross ratio of the poles,

$$\lambda^{\text{poles}} = \frac{z_1 - z_2}{z_1 - z_3} \cdot \frac{z_3 - z_4}{z_2 - z_4}, \tag{3.4.3}$$

which parameterize the underlying $\mathcal{M}_{0,4}$, and the cross ratio of the zeros,

$$\lambda^{\text{zeros}} = \frac{e_1 - e_2}{e_1 - e_3} \cdot \frac{e_3 - e_4}{e_2 - e_4}, \tag{3.4.4}$$

which fixes the quadratic differential. Such a parameterization is particularly useful because it separates the fibers of $\mathcal{D}^R_{0,4}$ in an obvious way.

Another parameterization can be obtained as follows. Let γ_{ij} be a set of smooth curves connecting e_i and e_j in such a way that they form a tetrahedron with the poles z_i on the faces. The integrals

$$I_{ij} = \int_{\gamma_{ij}} \sqrt{\phi}$$

are well defined and do not depend on the deformation of γ_{ij}. By contour deformation we can show that the integrals along the opposite edges of the tetrahedron are equal. Let

$$\begin{aligned} a &= I_{12} = I_{34}, \\ b &= I_{23} = I_{14}, \\ c &= I_{31} = I_{24}. \end{aligned} \tag{3.4.5}$$

Again, by contour deformation arguments, $a + b + c = 2\pi$ and thus we have only two independent complex parameters a and b which can be used as coordinates on $\mathcal{D}^R_{0,4}$.

[2]We call the coefficient of the $\frac{(dz)^2}{(z-z_0)^2}$ in the Laurent expansion of a quadratic differential near the point z_0 the residue of the quadratic differential. One can easily see that the residue does not depend on the choice of a local coordinate.

So far λ^{poles} and λ^{zeros} are analytic functions of a and b. Note that we propose here a point of view regarding the a, b, c-parameters differing from that of Ref. [24]. In that paper a, b and c were real by definition and provided a real parameterization of the moduli space $\mathcal{M}_{0,4}$, while here they are complex and parameterize $\mathcal{D}^R_{0,4}$. This will be useful to give a unified description of the Strebel and Feynman regions as we will show later on in section 3.5.

In general integrals in (3.4.5) are complete elliptic integrals of the third kind. In order to evaluate them we will need the following lemma.

Reduction Lemma. *Let ϕ be a quadratic differential on the sphere such that in a uniformizing coordinate z it is given by*

$$\phi = \frac{Q(z)}{\prod_{i=1}^{4}(z-z_i)^2}\,(dz)^2.$$

where $Q(z)$ is a polynomial of degree four. The square root of ϕ defines an Abelian differential on the Riemann surface Σ of $\sqrt{Q(z)}$. Since $Q(z)$ has degree four, Σ is a torus. Let the periods of the torus be $2\,\omega_1$ and $2\,\omega_2$. The Abelian differential $\sqrt{\phi}$ has periods ω_1 and ω_2 if all the poles of ϕ have equal residues.

Proof. The proof is based on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the four-punctured sphere. Let us show that a quadratic differential $\phi \in \mathcal{D}_{0,4}$ with equal residues is invariant under these symmetries. It is convenient to fix the uniformizing coordinate z on the sphere so that the zeros of the quadratic differential have coordinates ± 1 and $\pm k$. Using this coordinate we can write any quadratic differential $\phi \in \mathcal{D}_{0,4}$ with equal residues as

$$\phi = C\frac{(z^2-k^2)(z^2-1)}{(\zeta^2 z^2-k^2)^2(z^2-\zeta^2)^2}\,(dz)^2. \tag{3.4.6}$$

where ζ is a position of one of the poles and C is an arbitrary constant. The symmetry group is generated by two transformations which can be written as

$$S_1:\ z \rightarrow -z, \quad \text{and} \quad S_2:\ z \rightarrow k/z. \tag{3.4.7}$$

We can extend this symmetry to the Riemann surface of ϕ which is a torus given by

$$w^2 = (z^2-k^2)(z^2-1). \tag{3.4.8}$$

The generators S_k act on w by

$$S_1:\ w \rightarrow -w, \quad \text{and} \quad S_2:\ w \rightarrow -k\,\frac{w}{z^2}. \tag{3.4.9}$$

Clearly, (3.4.9) together with (3.4.7) define the symmetries of the torus given by (3.4.8). A holomorphic Abelian differential on the torus $du = dz/w$ is invariant under these transformations and therefore S_k are translations of the torus. By definition

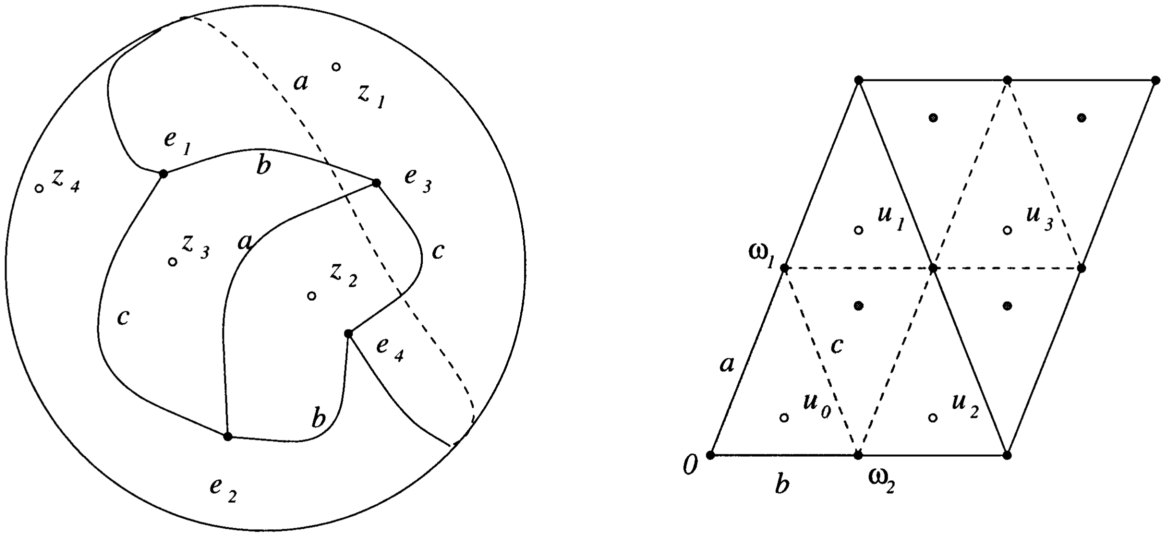


Figure 3-1: The sphere and the torus

$S_k^2 = 1$ and we conclude that S_k is a translation by half a period, $S_k(u) = u + \omega_k$. The square root of the quadratic differential can be written in terms of du as

$$\sqrt{\phi} = \sqrt{C}\frac{w^2 du}{(\zeta^2 z^2 - k^2)(z^2 - \zeta^2)}. \tag{3.4.10}$$

Expression (3.4.10) is invariant under S_k and therefore $\sqrt{\phi}$ has periods ω_1 and ω_2. QED.

Let u be a coordinate on the torus and $[2\omega_1, 2\omega_2]$ be its periods. For a quadratic differential $\phi \in \mathcal{D}^R$ the reduction lemma states that if $\sqrt{\phi} = f(u)du$ then $f(u)$ has periods $[\omega_1, \omega_2]$. The quadratic differential has four second order poles with residue -1, and four simple zeros. Thus, $\sqrt{\phi}$ has eight poles with residue $\pm i$ and four double zeros, or equivalently, $f(u)$ has two poles and a double zero in its fundamental parallelogram. In Fig. 3-1 we show the sphere and the torus with the positions of the poles and zeros marked. The shaded region on the torus is the fundamental parallelogram of $f(u)$.

Any meromorphic function with two periods (an elliptic function) can be written in terms of two basic elliptic functions — the Weierstrass $\wp$-function and its derivative $\wp'$ (see Ref. [56]). Let u_0 be the position of a pole which is inside the parallelogram $[\omega_1, \omega_2]$. An elliptic function having two poles with residue $\pm i$ and a double zero is uniquely defined by the positions of the zero an one of the poles. Let the zero be at $u = 0$ (we can always shift u by a constant in order to achieve this), and the pole

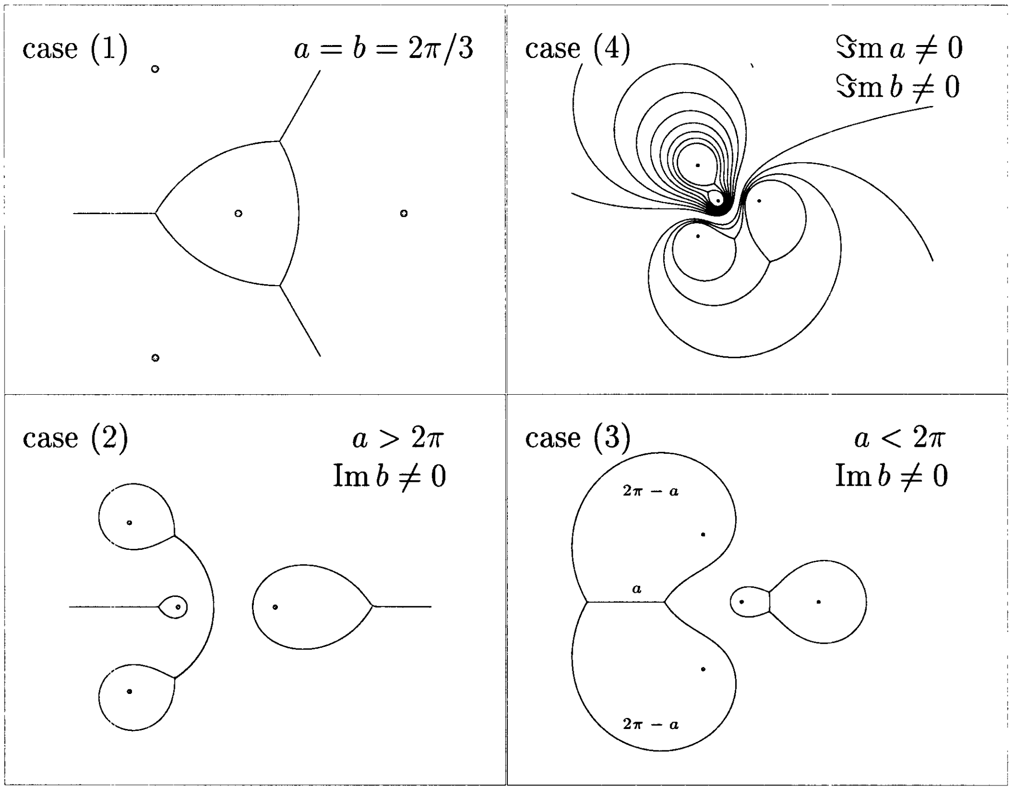


Figure 3-2: Four different kinds of the critical graph

with residue i be at $u = u_0$, then

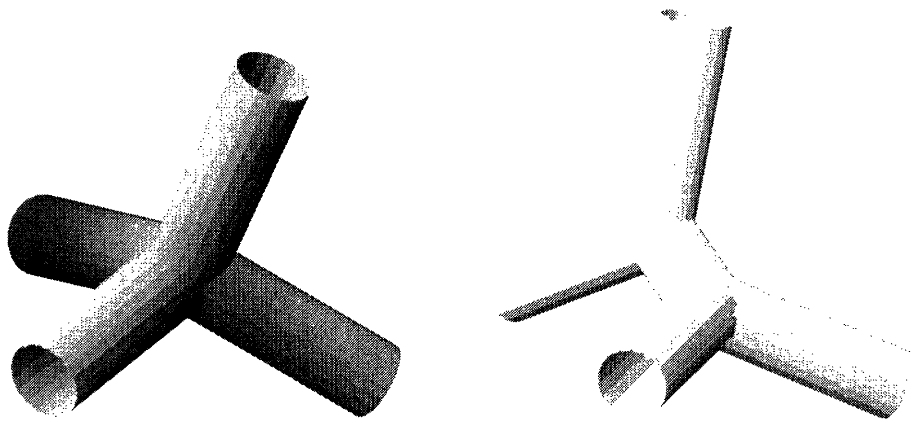
$$\begin{aligned} f(u) &= \frac{i\wp'(u_0)}{\wp(u) - \wp(u_0)} \\ &= i(\zeta(u+u_0) - \zeta(u-u_0) - 2\zeta(u_0)) \\ &= i\frac{d}{du}\left(\ln\frac{\sigma(u+u_0)}{\sigma(u-u_0)} - 2\zeta(u_0)u\right), \end{aligned} \tag{3.4.11}$$

where $\wp$, ζ and σ are the corresponding Weierstrass functions for the lattice $[\omega_1,\ \omega_2]$. Using (3.4.5) and (3.4.11) we can calculate a and b:

$$\begin{aligned} a &= -\int_0^{\omega_1} f(u)\,du = -2\,\pi - 2\,i\,\left(\zeta(u_0)\,\omega_1 - \eta_1\,u_0\right), \\ b &= \int_0^{\omega_2} f(u)\,du \ = -2\,\pi + 2\,i\,\left(\zeta(u_0)\,\omega_2 - \eta_2\,u_0\right). \end{aligned} \tag{3.4.12}$$

See Fig. 3-1 to justify the limits of integration. The values of a and b define the geometry of the critical horizontal trajectories. Using the last equation in (3.4.11) we can write the quadratic differential as $\phi = (dv)^2$, where

$$v(u) = i\,\ln\frac{\sigma(u_0+u)}{\sigma(u_0-u)} - 2\,i\,\zeta(u_0)\,u. \tag{3.4.13}$$



(a) Strebel (b) Feynman

Figure 3-3: Riemann surfaces corresponding to Strebel (a) and Feynman (b) quadratic differentials

On the v plane horizontal trajectories are horizontal lines. From (3.4.13) and (3.4.12) we can see that on the v-plane the zeros of ϕ are at $v(0) = 0$, $v(\omega_1) = a$, $v(\omega_2) = b$. Thus when a and b are real any three of the zeros are connected by one horizontal trajectory and the critical graph is a tetrahedron. If only a is real the critical horizontal trajectories form two separate connected graphs. When $a \leq 2\pi$ the zeros are connected in two pairs, each pair having three horizontal trajectories traversing from one zero to the other. When $a \geq 2\pi$ we have a different picture, with each pair of zeros having one trajectory passing between them and the others forming two tadpoles. Finally, when none of the a, b or c is real, two of the three critical trajectories leaving a zero collide on their way around a pole and come back forming a tadpole and the other becomes infinite. Figure 3-2 illustrates these four cases.

3.5 Four-string vertex and Feynman region

In this section we show how integral invariants can be used to find the four-string vertex. The use of complex values of the integral invariants will allow us to describe the quadratic differentials used to define local coordinates in the string vertex and Feynman regions similarly using particular constraints imposed on the possible values of the invariants.

As was shown in Ref. [24] the elementary interaction can be found using so-called 'Strebel quadratic differentials'. A Strebel quadratic differential is a quadratic differential whose critical graph is a polyhedron, or, — as the analysis in section 3.4 shows — all the integral invariants are real. For the case of four-punctured spheres we define the *Strebel constraint* by

$$\text{Im } a = \text{Im } b = \text{Im } c = 0. \tag{3.5.1}$$

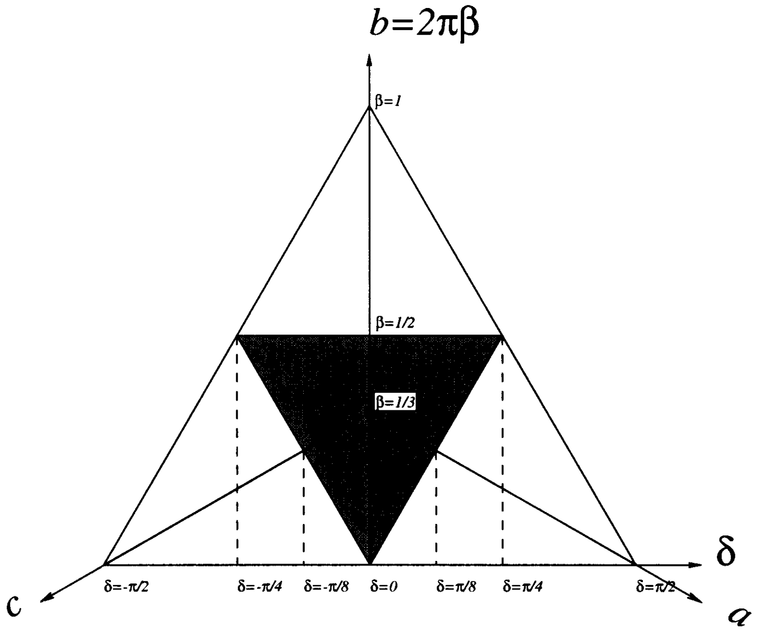


Figure 3-4: The four-string vertex $\mathcal{V}_{0,4}$ on the $a+b+c=2\pi$ plane

Given a quadratic differential $\phi = \varphi(z)(dz)^2$ one can naturally define a metric by $g = |\varphi(z)|dzd\bar{z}$. Since $\varphi(z)$ is meromorphic, this metric has zero curvature at every point where $\varphi(z) \neq 0$:

$$R = -\frac{4}{|\varphi(z)|^2}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\log|\varphi(z)| = 0. \tag{3.5.2}$$

Therefore if we cut the sphere along the critical graph it will break into pieces each isometric to a cylinder. For the Strebel quadratic differential the four-punctured sphere breaks into four semi-infinite cylinders each of circumference 2π (Fig. 3-3). In order to reconstruct the Riemann surface one has to glue these four cylinders along the edges of a tetrahedron with the sides equal a, b and c (see Ref. [16]).

Due to the Strebel theorem [29] one can use real positive values of the integral parameters ($a+b+c=2\pi$) in order to parameterize $\mathcal{M}_{0,4}$. It is well known that we actually need two copies of the abc triangle $a+b+c=2\pi$ to cover the whole whole $\mathcal{M}_{0,4}$. This parameterization is very useful because we can easily describe the four-string vertex $\mathcal{V}_{0,4}$ which is given by (see Ref. [24])

$$a > \pi, \quad b > \pi, \quad \text{and} \quad c > \pi. \tag{3.5.3}$$

In Fig. 3-4 we present the view at abc triangle along the line $a=b=c$. The shaded region corresponds to $\mathcal{V}_{0,4}$.

In order to calculate the contribution of Feynman diagrams we have to define the measure μ in the Feynman region of the moduli space. We will achieve this goal by

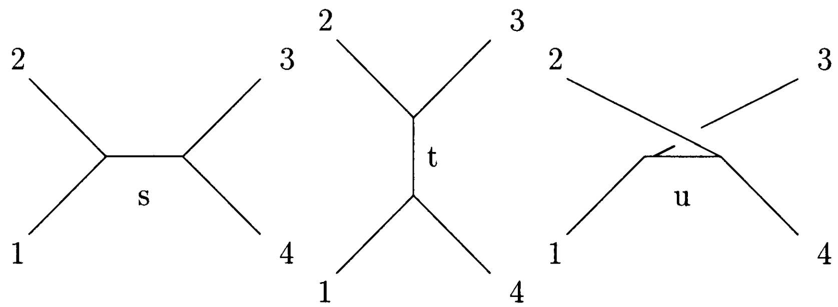


Figure 3-5: Three Feynman diagrams built with the three-vertex and propagator that enter in the computation of the four tachyon amplitude

finding the corresponding quadratic differential for each Riemann surface or a string diagram in the Feynman region.

A Feynman string diagram for the four-string scattering is a Riemann surface obtained by gluing together five cylinders with circumference 2π: four semi-infinite cylinders representing the scattering strings and one finite cylinder representing an intermediate string or a propagator. There are three topologically inequivalent ways to glue these cylinders together corresponding to the three channels s, t and u. For each channel we can vary the length of the propagator l and the twist angle θ. This construction defines three non-intersecting regions in the moduli space $\mathcal{F}_s$, $\mathcal{F}_t$ and $\mathcal{F}_u$ each naturally parameterized by $l > 0$ and $0 < \theta < 2\pi$. A Feynman string diagram can be easily constructed using a quadratic differential with complex integral invariants. Take a look at the case 4 in Fig. 3-2, which shows the critical graph of a quadratic differential which has one of the integral invariants (a) real and less then 2π. The correspondent Riemann surface consists of two pairs of semi-infinite cylinders glued to a finite cylinder with length $|\,\mathrm{Im}\ b|$ and circumference $4\pi - 2a$. If we define the twist θ as an angle between two zeros on the propagator we obtain $\theta = \mathrm{Re}\ b$. Thus we conclude that in order to define a a Feynman string diagram a quadratic differential should have one integral invariant equal to π and another equal to $\theta + il$. We define three *Feynman constraints* corresponding to the diagrams in Fig. 3-5 by

$$\begin{aligned} F_s: &\quad a = \pi, \quad c = \theta + i\,l; \\ F_t: &\quad c = \pi, \quad b = \theta + i\,l; \\ F_u: &\quad b = \pi, \quad a = \theta + i\,l. \end{aligned} \tag{3.5.4}$$

By definition the length of the propagator $l > 0$ and the twist θ is between zero and 2π. It is convenient to combine l and θ into one complex variable $\varepsilon = e^{i\theta - l}$ (for different channels ε is equal to either e^{ia} or to e^{ib} or to e^{ic}). Different values of ε correspond to different Riemann surfaces or different points in $\mathcal{M}_{0,4}$. Therefore each Feynman constraint defines a section over the correspondent region in the moduli space. We define three regions $\mathcal{F}_s$, $\mathcal{F}_t$ and $\mathcal{F}_u$ as the projections of the correspondent sections on $\mathcal{M}_{0,4}$. Each of these regions can be naturally parameterized by $|\varepsilon| < 1$. We can summarize this construction on the following diagram

$$\begin{array}{ccc} U = \{|\varepsilon| < 1\} & \xrightarrow{F_{s,t,u}} & \mathcal{D}^R_{0,4} \\ & & \downarrow \\ & & \mathcal{F}_{s,t,u} \subset \mathcal{M}_{0,4}. \end{array} \tag{3.5.5}$$

We also obtain an alternative description of the four-string vertex: $\mathcal{V}_{0,4} = \mathcal{M}_{0,4} \backslash (\mathcal{F}_s \cup \mathcal{F}_t \cup \mathcal{F}_u)$. One can easily see that this agrees with (3.5.3).

Both the Feynman and the Strebel constraints define two-dimensional subspaces in the four-dimensional $\mathcal{D}^R_{0,4}$, but these subspaces are quite different. The Strebel constraint defines a global section of $\mathcal{D}^R_{0,4}$ over $\mathcal{M}_{0,4}$. This is a result known as the Strebel theorem [29]. The section defined by the Strebel constraint is not holomorphic because the constraint is given in terms of real functions on $\mathcal{D}^R_{0,4}$ (3.5.1). The Feynman constraints are defined by fixing a value of one of the three holomorphic functions on $\mathcal{D}^R_{0,4}$: $a = \pi$, $b = \pi$ or $c = \pi$. It is well known that the Feynman constraints define holomorphic sections only over a part of $\mathcal{M}_{0,4}$, namely over the Feynman regions $\mathcal{F}_{s,t,u}$.

Using complex integral invariants allows us to treat the four-string vertex and the Feynman regions in a unified manner by imposing some extra conditions (3.5.1) and (3.5.3) or (3.5.4) on a, b and c and integrating over simple regions which they define.

At this point we face a dilemma: the measure of integration in the formulae defining the four-tachyon amplitude (3.2.12) is given in terms of χ-invariants. On the other hand, the regions of integration for in the definition of the elementary four-tachyon coupling and the formula defining the massive states correction are given in terms of a, b and c. Therefore, our next goal will be to relate the χ-invariants and a, b and c. We will proceed in two steps: in the section 3.6 we will solve the system (3.4.12) and find the torus modulus $\tau = \omega_1/\omega_2$ and and the position of the pole u_0 in terms of a and b. Then, in section 3.7, we will express the χ invariants in terms of τ and u_0.

3.6 The main equation

In this section we will explore the system (3.4.12). Let us fix the scale of the coordinate on the torus so that $\omega_1 = \tau$ and $\omega_2 = 1$, then the system (3.4.12) can be written as

$$\begin{cases} \alpha = 1 + \dfrac{i}{\pi}\Big(\zeta(u_0;\,\tau)\,\tau - \eta_1(\tau)\,u_0\Big) \\ \beta = 1 - \dfrac{i}{\pi}\Big(\zeta(u_0;\,\tau) - \eta_2(\tau)\,u_0\Big) \end{cases} \tag{3.6.1}$$

where

$$\alpha = \frac{a}{2\pi} \qquad \text{and} \qquad \beta = \frac{b}{2\pi}. \tag{3.6.2}$$

This is a system of two equations for two complex variables τ and u_0, and its solution should define $\tau(\alpha, \beta)$ and $u_0(\alpha, \beta)$. In the present form it is extremely hard to solve. Fortunately we can reduce this system to a single equation defining $\tau(\alpha, \beta)$. Using the Legendre relation $\eta_2(\tau)\tau - \eta_1(\tau) = 2\pi i$ we can deduce that the system (3.6.1) is equivalent to

$$\begin{cases} u_0 = \dfrac{1-\beta}{2}\tau + \dfrac{1-\alpha}{2}, \\ \zeta(u_0) = \dfrac{1-\beta}{2}\eta_1(\tau) + \dfrac{1-\alpha}{2}\eta_2(\tau). \end{cases} \tag{3.6.3}$$

Now we can eliminate u_0 and get

$$\zeta\left(\frac{1-\beta}{2}\tau + \frac{1-\alpha}{2}\,;\,\tau\right) = \frac{1-\beta}{2}\,\eta_1(\tau) + \frac{1-\alpha}{2}\,\eta_2(\tau). \tag{3.6.4}$$

This equation plays the major role in our approach to the four-string amplitude problem. If we knew its solution $\tau(\alpha, \beta)$ we would know the solution to the system (3.6.1) because $u_0(\alpha, \beta)$ is given by:

$$u_0(\alpha, \beta) = \frac{1-\beta}{2}\tau(\alpha, \beta) + \frac{1-\alpha}{2}. \tag{3.6.5}$$

We will refer to (3.6.4) as the *main equation*.

In this section we will discuss the symmetries of this equation and find two regions for α and β which correspond to large values of Im τ. When Im τ is large the ζ function can be expanded as a series with respect to a small parameter $q_\tau = \exp(2\pi i\tau)$. We will call this series the q-series. We will use a truncated q-series to find approximate solutions of the main equation. Then we return to the Strebel case of real α and β and investigate the map from the *abc* to the τ plane.

3.6.1 Symmetries

Recall that α and β represent three invariants a, b and c which satisfy $a+b+c = 2\pi$. A permutation of a, b and c is equivalent to a permutation of the zeros. The torus modulus τ is closely related to the cross ratio of the zeros, and permutation of the zeros results in a modular transformation on the τ plane. More specifically:

$$\begin{aligned} a \leftrightarrow b &\equiv \alpha \leftrightarrow \beta & &\equiv \tau \rightarrow -1/\tau, \\ b \leftrightarrow c &\equiv \beta \rightarrow 1-\alpha-\beta & &\equiv \tau \rightarrow (2-\tau)/(1-\tau), \\ c \leftrightarrow a &\equiv \alpha \rightarrow 1-\alpha-\beta & &\equiv \tau \rightarrow -(1-\tau)/(2-\tau). \end{aligned} \tag{3.6.6}$$

One can easily check that the transformations (3.6.6) do not violate (3.6.4) using modular properties of the ζ-function.

Using the addition theorem for the ζ function (see Ref. [57]) one can show that the change of α and β to $-\alpha$ and $-\beta$ does not change Eq. (3.6.4). This is quite obvious because the integral invariants are defined up to a common sign which comes from the ambiguity in taking a square root.

3.6.2 $\beta \to 0$ limit

Let us rewrite the second equation of the system (3.6.1) using a q expansion for the Weierstrass ζ-function (see Ref. [56, page 248])

$$\zeta(u) = \eta_2 u + \pi i \frac{q_u + 1}{q_u - 1} + 2\pi i \sum_{n=1}^{\infty} \left[\frac{q_\tau^n q_{u_0}}{1 - q_\tau^n / q_{u_0}} - \frac{q_\tau^n \, q_{u_0}}{1 - q_\tau^n \, q_{u_0}} \right], \tag{3.6.7}$$

where we use a notation $q_x = \exp(2\pi i x)$. Terms linear in u in the expression for β cancel and we get

$$\frac{\beta}{2} = \sum_{n=0}^{\infty} \left[\frac{q_\tau^n (q_\tau / q_{u_0})}{1 - q_\tau^n (q_\tau / q_{u_0})} - \frac{q_\tau^n \, q_{u_0}}{1 - q_\tau^n \, q_{u_0}} \right]. \tag{3.6.8}$$

The reason why we collected the terms q_τ / q_{u_0} will be clear in a moment. Exponentiating the first equation in (3.6.3) we can express q_{u_0} in terms of q_τ, α and β as

$$q_{u_0} = -q_\tau^{\frac{1}{2}} e^{-\pi i (\alpha + \beta \tau)}. \tag{3.6.9}$$

If we substitute the value of q_{u_0} from eqn. (3.6.9) in to eqn. (3.6.8) we will get an equation which is equivalent to the main equation. Analyzing equations (3.6.8) and (3.6.9) we conclude that in the limit $\beta \to 0$

$$q_\tau \sim \beta^2 \qquad \text{and} \qquad q_{u_0} \sim \beta.$$

Therefore, in this limit $q_{u_0} \sim q_\tau / q_{u_0}$ which is reflected in the way we wrote (3.6.8). Moreover, q_τ being small in this limit allows us to find an approximate solution to the main equation. The first two terms in β-expansion of q_τ give

$$q_\tau = -\frac{\beta^2}{16 \cos^2 \delta} - \frac{i \sin \delta}{16 \cos^3 \delta} \left(2 \ln \frac{4 \cos \delta}{\beta} - 1 \right) \beta^3 + \mathrm{O}(\beta^4), \tag{3.6.10}$$

where

$$\delta = \pi \left(\alpha + \frac{\beta - 1}{2} \right). \tag{3.6.11}$$

Taking the logarithm of (3.6.10) we find

$$\tau = \frac{1}{2} + \frac{i}{\pi} \ln \frac{4 \cos \delta}{\beta} + \frac{\tan \delta}{2\pi} \left(2 \ln \frac{4 \cos \delta}{\beta} - 1 \right) \beta + \mathrm{O}(\beta^2). \tag{3.6.12}$$

This solution is valid for complex values of α and β. Therefore it can be used both for the vertex and the Feynman regions. The limit $\beta \to 0$ corresponds to the corner of the vertex (see Fig. 3-4) for real β and to the limit of short propagator and small twist for Im $\beta > 0$.

3.6.3 Im $\alpha \to \infty$ limit

There is another region where Im $\tau(\alpha,\ \beta) \to \infty$. This is the case when $0 < \beta < 1$ is a fixed real number and $\alpha \to i\,\infty$. Indeed, from equation (3.6.8) we derive that in the limit $q_\tau \to 0$ and finite β

$$q_{u_0} = -\frac{\beta}{2-\beta}, \tag{3.6.13}$$

so that q_{u_0} is finite unless $\beta = 0$ or $\beta = 2$. According to equation (3.6.3)

$$\alpha = 1 - 2\,u_0 + (1-\beta)\,\tau. \tag{3.6.14}$$

As we have seen, u_0 is finite as Im $\tau \to \infty$ and thus, for real β and $\alpha \to \infty$

$$\operatorname{Im}\tau \sim \frac{\alpha}{1-\beta}. \tag{3.6.15}$$

Let us set $\beta = 1/2$, which corresponds to the Feynman (u-channel) constraint. This constraint makes τ an analytic function of α. The first equation of (3.6.3) can now be written as

$$u_0 = \frac{1}{4}\tau + \frac{1-\alpha}{2}, \quad \text{or} \quad q_{u_0} = -q_\tau^{1/4}\,q_\alpha^{-1/2}, \tag{3.6.16}$$

and collecting the terms of the same order, we can rewrite (3.6.8) as

$$\frac{1}{4} = -\frac{q_{u_0}}{1-q_{u_0}} + \sum_{n=1}^{\infty}\left[\frac{q_\tau^n/q_{u_0}}{1-q_\tau^n/q_{u_0}} - \frac{q_\tau^n\,q_{u_0}}{1-q_\tau^n\,q_{u_0}}\right]. \tag{3.6.17}$$

Using (3.6.16) we can iterate (3.6.17) and find q_τ as a power series in q_α.

$$q_\tau = \frac{1}{3^4}\,q_\alpha^2 + \frac{512}{3^{10}}\,q_\alpha^4 + \frac{94720}{3^{15}}\,q_\alpha^6 + \frac{167118848}{3^{22}}\,q_\alpha^8 + \mathrm{O}(q_\alpha^{10}), \tag{3.6.18}$$

or

$$\begin{aligned}\tau(\alpha,\ \tfrac{1}{2}) &= 2\,\alpha + \frac{2\,i\,\ln 3}{\pi} \\ &\quad -\frac{i}{\pi}\left(\frac{256}{3^6}q_\alpha^2 + \frac{76544}{3^{12}}q_\alpha^4 + \frac{99552256}{3^{19}}q_\alpha^6 + \mathrm{O}(q_\alpha^8)\right).\end{aligned} \tag{3.6.19}$$

The appearance of the powers of 3 in the coefficients is quite remarkable. Formula (3.6.19) provides a good approximation for τ at large values of Im α. It shows that in this limit τ is a linear function of α with a finite intercept $2\,\ln 3/\pi$. For small values of Im α Eq. (3.6.19) does not work, but we can still find an approximate formula. All we have to do is exchange α and β in (3.6.12). According to symmetry relations (3.6.6) this is equivalent to $\tau \to -1/\tau$, therefore for $\beta = 1/2$ and small α, we have

$$-\frac{1}{\tau} = \frac{i}{\pi}\ln\frac{4\,i}{\alpha} + \mathrm{O}(\alpha^2). \tag{3.6.20}$$

In Fig. 3-6 we show the result of numerical solution of the main equation together with the first order approximations for small and large Im α.

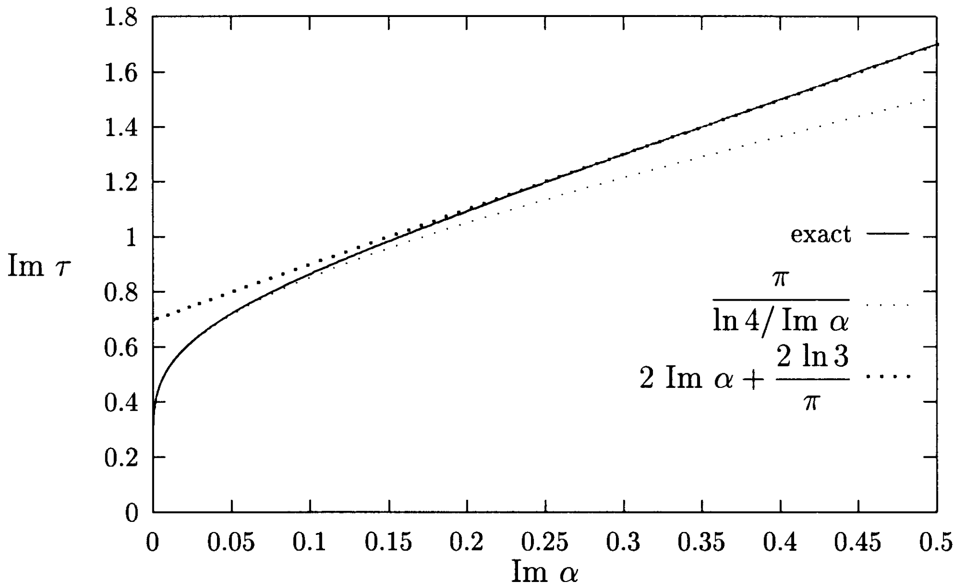


Figure 3-6: Solution of the main equation for $\beta = 1/2$ and imaginary α

3.6.4 The Strebel case

We now return to the case when α and β are real and represent a point on the equilateral triangle $a + b + c = 2\pi$ where a, b and c are real and positive. Strebel's theorem guarantees the existence of a solution to Eq. (3.6.4) for every point on the abc triangle. Indeed, τ is related to λ^{zeros} by a modular function of level 2, namely $\lambda(\tau)$ (see Ref. [29, page 254]). This function maps it's fundamental domain Γ, defined by

$$\Gamma = \{\tau :\ -1 < \text{Re}\ \tau \leq 1,\ |2\,\tau - 1| \geq 1 \text{ and } |2\,\tau + 1| > 1\},$$

bijectively to the whole complex plane. Therefore the existence of a Strebel differential is equivalent to the existence of a solution to the main equation in the fundamental domain of $\lambda(\tau)$.

Two Strebel differentials such that the zeros and poles of one are complex conjugate to those of the other have the same set of a, b and c invariants. Therefore in the fundamental domain Γ of $\lambda(\tau)$ we should have two solutions to the main equation. These two solutions correspond to conjugate values of $\lambda(\tau)$ and therefore are symmetric with respect to the imaginary axis on the τ plane. Finally, we conclude that for every point inside the abc triangle there exist a solution to Eq. (3.6.4) satisfying $0 \leq \tau \leq 1$ and $|2\,\tau - 1| \geq 1$. We will call this region $\Gamma/2$.

The main equation defines a map from the abc plane to $\Gamma/2$. Some information about this map can be obtained from the symmetry. According to Eq. (3.6.6) the line $a = c$ ($\delta = 0$) on the abc plane maps on to the line Im $\tau = 1/2$. Similarly, $a = b$ maps on to the circle $|\tau| = 1$ and $b = c$ on to $|\tau - 1| = 1$, and we conclude that the most symmetric point ($a = b = c = 2\pi/3$) is mapped to

$$\tau\left(\alpha = \frac{1}{3},\ \beta = \frac{1}{3}\right) = e^{\frac{\pi i}{3}}. \tag{3.6.21}$$

According to (3.6.12), the whole line $b = 0$ maps on to the single point $\tau = 1/2 + i\,\infty$, and therefore the other two sides of the abc triangle $a = 0$ and $c = 0$ are correspondingly mapped to 0 and 1 respectively. This seemingly leads to a contradiction at the corners. For example when $a = b = 0$ the solution must be $\tau = 0$ because $a = 0$; on the other hand it should be $\tau = 1/2 + i\,\infty$ because $b = 0$, but at the same time it should be somewhere on the unit circle $|\tau| = 1$ because $a = b$. In fact there is no contradiction because if we rewrite the main equation for this case we get

$$\zeta\left(\frac{\tau}{2} + \frac{1}{2}\,;\,\tau\right) = \frac{1}{2}\eta_1(\tau) + \frac{1}{2}\eta_2(\tau), \tag{3.6.22}$$

which is valid for *any* value of τ. The arbitrariness of τ does not contradict the Strebel theorem which guarantees the uniqueness of the quadratic differential because as we will show in the next section, the point $\alpha = \beta = 0$ corresponds to $\lambda^{\text{poles}} = 0$ which is excluded from $\mathcal{M}_{0,4}$. It is interesting to investigate how the solution to Eq. (3.6.4) behaves in the vicinity of a corner.

The corner $a = b = 0$ corresponds to $\delta = -\pi/2$ (see Fig. 3-4). It is problematic to use the expression (3.6.10) because the coefficients diverge as $\delta \to -\pi/2$.

Let α and β be small, but not both equal to zero. Recall, that $\zeta'(u) = -\wp(u)$ and $\wp(\tau/2 + 1/2) = e_3(\tau)$. When we keep only first order terms in α and β in Eq. (3.6.4), we find

$$-e_3(\tau)\left(\frac{\alpha}{2} + \frac{\beta}{2}\tau\right) = \frac{\alpha}{2}\eta_2(\tau) + \frac{\beta}{2}\eta_1(\tau), \tag{3.6.23}$$

then, using the Legendre relation to exclude $\eta_1(\tau)$, we get

$$(\eta_2(\tau) + e_3(\tau))\left(\frac{\alpha}{\beta} + \tau\right) = 2\pi i. \tag{3.6.24}$$

Inspecting Eq. (3.6.24), we conclude that the limiting value of τ depends on the ratio $r = \beta/\alpha$. Moreover for any value of τ there exists an r such that

$$\lim_{\alpha \to 0} \tau(\alpha,\ r\,\alpha) = \tau.$$

From Eq. (3.6.24) we can even find the ratio in terms of τ:

$$r = \left[\frac{2\pi i}{\eta_2(\tau) + e_3(\tau)} - \tau\right]^{-1}. \tag{3.6.25}$$

It is hard to tell what values of τ correspond to real r. For large Im τ we may use the q expansion

$$\eta_2(\tau) + e_3(\tau) = 8\pi^2 \sum_{n=0}^{\infty} \frac{q_\tau^{n+\frac{1}{2}}}{1 + q_\tau^{n+\frac{1}{2}}}, \tag{3.6.26}$$

and solve (3.6.24) approximately for $\tau(r)$

$$\tau(r) = \frac{1}{2} + \frac{i}{\pi}\ln\frac{4\pi}{r} + \left(\frac{i}{2} - \frac{1}{\pi}\ln\frac{4\pi}{r}\right)\frac{r}{\pi} + \mathrm{O}(r^2). \tag{3.6.27}$$

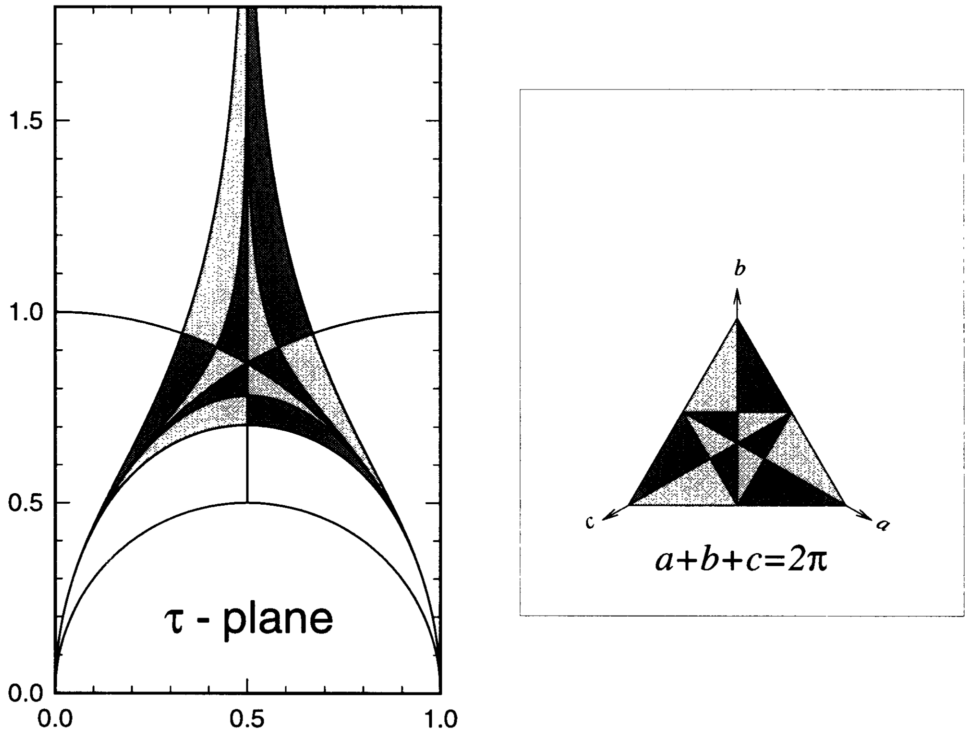


Figure 3-7: Solution of the main equation for real α and β. Corresponding regions on the $a+b+c=2\pi$ and τ-planes are shaded with the matching grey levels.

One can check that (3.6.10) yields the same result in the limit of small α, β and β/α. It is interesting that the map of the *abc* triangle to the τ plane does not cover $\Gamma/2$. It is mapped to a curved triangle. The sides of the original triangle ($a=0$, $b=0$ and $c=0$) become the corners ($\tau=0$, $\tau=\infty$ and $\tau=1$), while the corners blow up and become sides. Fig. 3-7 represents a map from the $a+b+c=2\pi$ plane to the τ-plane. The corresponding regions of the τ and *abc* planes are shaded with matching gray levels on the plot.

3.7 Infinite products

In this section we will perform the second step of the program announced at the end of section 3.5. We will derive explicit formulae for the χ invariants as functions of the torus modular parameter τ and the position of the pole u_0. We will find $\lambda^{\text{poles}} = -\chi_s/\chi_u$ and λ^{zeros}. The latter will be found as a special case $u_0=0$ of a formula defining λ^{poles}.

Recall that the χ invariants are defined in terms of the positions of the poles and the mapping radii as

$$\chi_{ij} = \frac{(z_i - z_j)^2}{\rho_i \rho_j}. \tag{3.7.1}$$

As before, the coordinate u on the torus is fixed by $\omega_2 = 1$. We can choose the coordinate z on the sphere so that $z = \wp(u/2)$. The positions of the poles on the sphere are given by

$$z_i = \wp\left(\frac{u_i}{2}\right), \quad u_i = u_0 + \omega_i, \qquad i = 1, \ldots, 4, \tag{3.7.2}$$

where $\omega_1 = \tau$, $\omega_2 = 1$, $\omega_3 = 1 + \tau$ and $\omega_4 = 0$. So far, the only nontrivial part of Eq. (3.7.1) are the mapping radii. Due to the translational symmetry all four mapping radii of the coordinate disks on the torus are equal and we denote their common value value by ρ. According to the general procedure described in section 3.3, in order to calculate ρ we have to find a local coordinate w around u_0 such that locally

$$\varphi = -\frac{(dw)^2}{w^2}, \qquad \text{and} \qquad w(0) = 1.$$

The last condition fixes the scale of w as well as its phase. From equation (3.4.11) we derive

$$w(u) = \frac{\sigma(u - u_0)}{\sigma(u + u_0)} e^{2\,\zeta(u_0)\,u}. \tag{3.7.3}$$

Note that $w(u)$ is just an exponent of the function $v(u)$ introduced in section 3.4, $w(u) = \exp(i\,v(u))$. The mapping radius is the inverse of the derivative of $w(u)$ at u_0.

$$\rho^{-1} = w'(u_0) = \frac{e^{2\,\zeta(u_0)\,u_0}}{\sigma(2\,u_0)}. \tag{3.7.4}$$

When we go from the torus to the sphere we make a change of coordinates from u to $z = \wp(u/2)$, therefore each mapping radius picks up a factor of $(d/du)\wp(u/2)$ and we find

$$\rho_i = \frac{1}{2}\wp'\left(\frac{u_i}{2}\right)\rho. \tag{3.7.5}$$

Now we can combine equations (3.7.2), (3.7.4) and (3.7.5) with Eq. (3.7.1) to obtain

$$\chi_{ij} = 4\,\frac{\left(\wp\left(\frac{u_i}{2}\right) - \wp\left(\frac{u_j}{2}\right)\right)^2}{\wp'\left(\frac{u_i}{2}\right)\wp'\left(\frac{u_j}{2}\right)}\,\rho^{-2}. \tag{3.7.6}$$

This expression can be rewritten in terms of Weierstrass σ-functions. In order to do that we need the formulae for the difference of two $\wp$ functions (see Ref. [56, page 243])

$$\wp(u) - \wp(v) = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)}, \tag{3.7.7}$$

and their derivatives

$$\wp'(u) = -\frac{\sigma(2\,u)}{\sigma^4(u)}. \tag{3.7.8}$$

The latter formula is just the derivative of (3.7.7) with respect to v at the point $v = u$. Now we see that the powers of $\sigma(u_i/2)$ cancel and we get

$$\chi_{ij} = 4\frac{\sigma^2\left(u_0 + \frac{\omega_i+\omega_j}{2}\right)\sigma^2\left(\frac{\omega_i-\omega_j}{2}\right)}{\sigma(u_0+\omega_i)\sigma(u_0+\omega_j)}\rho^{-2}. \tag{3.7.9}$$

The prefactor of ρ^{-2} in this expression is an elliptic function of u_0 with periods ω_1 and ω_2, which was not obvious from Eq. (3.7.6) because it was written in terms of elliptic functions of $u/2$. This extra periodicity enforces the symmetry relations (3.2.6). We can further simplify Eq. (3.7.9) by introducing a new function $\varphi(u)$ which is closely related to the Weierstrass $\sigma(u)$ (see Ref. [56, page 246]),

$$\varphi(u) = e^{-\frac{1}{2}\eta_2 u^2} q_u^{\frac{1}{2}} \sigma(u), \tag{3.7.10}$$

where $q_u = e^{2\pi i u}$ and $\eta_2 = \frac{1}{2}\zeta(\frac{1}{2})$ is a quasi-period of the ζ-function. This function has the following properties

$$\varphi(u+1) = \varphi(u), \qquad \text{and} \qquad \varphi(u+\tau) = -\frac{1}{q_u}\varphi(u). \tag{3.7.11}$$

We can use φ to replace σ in Eq. (3.7.9) and we find

$$\begin{aligned}
\chi_s = \chi_{12} = \chi_{34} &= -4\,\frac{q_{u_0}}{q_\tau^{\frac{1}{2}}}\,\frac{\varphi^2\left(u_0+\frac{\tau+1}{2}\right)\varphi^2\left(\frac{\tau+1}{2}\right)}{\varphi^2(u_0)}\rho^{-2},\\
\chi_t = \chi_{14} = \chi_{23} &= 4\,\frac{q_{u_0}}{q_\tau^{\frac{1}{2}}}\,\frac{\varphi^2\left(u_0+\frac{\tau}{2}\right)\varphi^2\left(\frac{\tau}{2}\right)}{\varphi^2(u_0)}\rho^{-2},\\
\chi_u = \chi_{13} = \chi_{24} &= 4\,\frac{\varphi^2\left(u_0+\frac{1}{2}\right)\varphi^2\left(\frac{1}{2}\right)}{\varphi^2(u_0)}\rho^{-2}.
\end{aligned} \tag{3.7.12}$$

The cross ratio of the poles does not depend on ρ and we can find it as

$$\lambda^{\text{poles}}(u_0) = -\frac{\chi_s}{\chi_u} = \frac{q_{u_0}}{q_\tau^{\frac{1}{2}}}\frac{\varphi^2\left(u_0+\frac{\tau+1}{2}\right)\varphi^2\left(\frac{\tau+1}{2}\right)}{\varphi^2\left(u_0+\frac{1}{2}\right)\varphi^2\left(\frac{1}{2}\right)}. \tag{3.7.13}$$

In the special case, $u_0 = 0$, this gives the cross ratio of the zeros

$$\lambda^{\text{zeros}} = \lambda^{\text{poles}}(0) = q_\tau^{-\frac{1}{2}}\frac{\varphi^4\left(\frac{\tau+1}{2}\right)}{\varphi^4\left(\frac{1}{2}\right)}. \tag{3.7.14}$$

The φ-function has a simple infinite product expansion in terms of q_u and q_τ (see Ref. [56, page 247]):

$$\varphi(u;\,\tau) = (2\pi i)^{-1}(q_u - 1)\prod_{n=1}^{\infty}\frac{(1-q_\tau^n q_u)(1-q_\tau^n/q_u)}{(1-q_\tau^n)^2}. \tag{3.7.15}$$

This product converges as a power series with ratio q_τ for small values of q_τ. Note that by symmetry we can always choose τ to lie in the fundamental region defined by $|\text{Re}\,\tau| \le 1/2$ and $|\tau| \ge 1$. The maximum value of $|q_\tau|$ in this region is obtained at $\tau = (\pm 1 + i\sqrt{3})/2$, therefore

$$|q_\tau| \le \exp(-\pi\sqrt{3}) \approx 0.00433.$$

Such a small value of $|q_\tau|$ makes the product (3.7.15) very useful for numerical calculations.

Equations (3.7.13) and (3.7.14) together with Eq. (3.7.15) provide infinite product expansions for λ^{poles} and λ^{zeros}. In order to find similar products for χ's we have to express the mapping radius ρ in terms of the function φ,

$$\rho^{-1} = \frac{e^{2u_0\zeta(u_0)}}{e^{2\eta_2 u_0^2} q_{u_0}^{-1} \varphi(2u_0)} = \frac{q_{u_0} e^{2u_0(\zeta(u_0) - \eta_2 u_0)}}{\varphi(2u_0)} = \frac{q_{u_0}^{\beta}}{\varphi(2u_0)}, \tag{3.7.16}$$

where we use the second equation of the system (3.6.1) for β.

For future reference we present here the products for all χ's.

$$\begin{aligned}
\chi_s &= -4\,\frac{q_{u_0}^{1+2\beta}}{q_\tau^{\frac{1}{2}}} \frac{1}{(1+q_{u_0})^2(1-q_{u_0})^4} \times \\
&\quad \times \prod_{n=1}^{\infty} \frac{(1+q_\tau^{n-\frac{1}{2}})^4}{(1-q_\tau^n q_{u_0}^2)^2(1-q_\tau^n/q_{u_0}^2)^2} \frac{(1+q_\tau^{n-\frac{1}{2}} q_{u_0})^2(1+q_\tau^{n-\frac{1}{2}}/q_{u_0})^2}{(1-q_\tau^n q_{u_0})^2(1-q_\tau^n/q_{u_0})^2}, \\
\chi_t &= 4\,\frac{q_{u_0}^{1+2\beta}}{q_\tau^{\frac{1}{2}}} \frac{1}{(1+q_{u_0})^2(1-q_{u_0})^4} \times \\
&\quad \times \prod_{n=1}^{\infty} \frac{(1-q_\tau^{n-\frac{1}{2}})^4}{(1-q_\tau^n q_{u_0}^2)^2(1-q_\tau^n/q_{u_0}^2)^2} \frac{(1-q_\tau^{n-\frac{1}{2}} q_{u_0})^2(1-q_\tau^{n-\frac{1}{2}}/q_{u_0})^2}{(1-q_\tau^n q_{u_0})^2(1-q_\tau^n/q_{u_0})^2}, \\
\chi_u &= 16\,\frac{q_{u_0}^{2\beta}}{(1-q_{u_0})^4} \times \\
&\quad \times \prod_{n=1}^{\infty} \frac{(1+q_\tau^n)^4}{(1-q_\tau^n q_{u_0}^2)^2(1-q_\tau^n/q_{u_0}^2)^2} \frac{(1+q_\tau^n q_{u_0})^2(1+q_\tau^n/q_{u_0})^2}{(1-q_\tau^n q_{u_0})^2(1-q_\tau^n/q_{u_0})^2}.
\end{aligned} \tag{3.7.17}$$

It is not so easy to show that the sum of these products is zero as required by Eq. (3.2.8).

Dividing the first equation of (3.7.17) by the third we find an infinite product for the cross ratio of the poles:

$$\lambda^{\text{poles}} = -\frac{\chi_s}{\chi_u} = \frac{1}{4} \frac{q_{u_0}}{(1+q_{u_0})^2} \prod_{n=1}^{\infty} \frac{(1+q_\tau^{n-\frac{1}{2}} q_{u_0})^2(1+q_\tau^{n-\frac{1}{2}}/q_{u_0})^2}{(1+q_\tau^n q_{u_0})^2(1+q_\tau^n/q_{u_0})^2}. \tag{3.7.18}$$

3.8 From a, b and c to χ_s, χ_t and χ_u

In this section we combine the results of the previous two and investigate how χ invariants depend on a, b and c.

3.8.1 Exact results

There are very few cases when the χ invariants can be found exactly. These are the cases when we know the solution of the main equation. Such a solution is available for example in the case of a degenerate quadratic differential i.e. when any of a, b or c is zero. According to Eq. (3.6.10) $b = 0$ corresponds to $q_\tau = 0$ and $q_{u_0}/q_\tau^{1/2} = -i\, e^{i\delta}$ and the χ invariants are found to be

$$\chi_s = 4\,\frac{1-\sin\delta}{\cos\delta}e^{i\delta}, \quad \chi_t = 4\,\frac{1+\sin\delta}{\cos\delta}e^{i\delta}, \quad \chi_u = -\chi_s - \chi_t = -\frac{8}{\cos\delta}e^{i\delta}.$$

Note that for real δ all the χ have the same phase, and therefore the cross ratio λ^{poles} is real:

$$\lambda^{\text{poles}} = \frac{1-\sin\delta}{2}. \tag{3.8.1}$$

The small parameter q_τ is also exactly zero in the limit Im $\alpha \to \infty$ (see section 3.6). In this limit, q_{u_0} is given by Eq. (3.6.13), and the χ invariants are

$$\chi_s = \infty, \quad \chi_t = \infty, \quad \chi_u = -\chi_s - \chi_t = 16\,\frac{q_{u_0}^{2\beta}}{(1-q_{u_0})^4}. \tag{3.8.2}$$

These results can also be obtained by an elementary approach. For example, in the case $b = 0$ and real a and c, we can choose the uniformizing coordinate z so that the poles of the quadratic differential are located at the vertices of a rectangle and the two degenerate zeros are at 0 and ∞. From symmetry, the horizontal trajectories are the symmetry lines and we can find the mapping radii by making a conformal transformation.

The other case of infinite Im α corresponds to the degeneracy of the poles. In this limit, two poles collide and we effectively have a three punctured sphere. For the case $\beta = \frac{1}{2}$, this sphere is the Witten vertex and the χ_u in the formula (3.8.2) gives correct value $|\chi| = 3^3/2^4$.

The only nontrivial point where an exact solution is still available is the most symmetric point $a = b = c$. In this case

$$\tau = e^{\frac{i\pi}{3}},$$

which corresponds to the so-called equianharmonic case in the theory of elliptic functions. In this case, all the necessary values of the Weierstrass functions can be evaluated explicitly in terms of elementary functions (see Ref. [57]) and we obtain

$$\chi_s = \frac{2^5\sqrt[3]{2}}{3^2}e^{-\frac{2\pi i}{3}}, \quad \chi_t = \frac{2^5\sqrt[3]{2}}{3^2}e^{\frac{2\pi i}{3}}, \quad \chi_u = \frac{2^5\sqrt[3]{2}}{3^2}. \tag{3.8.3}$$

The upper left picture on Fig. 3-2 shows the critical graph for this case. It is formed by three straight lines connecting the first three zeros with the last placed at infinity and three arcs connecting two finite zeros having the center at the third.

3.8.2 Approximate results

For other values of the integral invariants no exact solution for the main equation is available, but we can still solve perturbatively as we did in section 3.6 and find approximate formulae for the χ's. Consider the case of large Im α and $\beta = \frac{1}{2}$. In section 3.6 we found the solution of the main equation up to the 8-th order of q_α (see Eq. (3.6.18)).

$$q_\tau = \frac{1}{3^4} q_\alpha^2 + \frac{512}{3^{10}} q_\alpha^4 + \frac{94720}{3^{15}} q_\alpha^6 + \frac{167118848}{3^{22}} q_\alpha^8 + \mathrm{O}(q_\alpha^{10}). \tag{3.8.4}$$

We can find q_{u_0} as

$$q_{u_0} = -\frac{q_\tau^{\frac{1}{4}}}{q_\alpha^{\frac{1}{2}}} = -\frac{1}{3} - \frac{128}{3^7} q_\alpha^2 - \frac{15488}{3^{12}} q_\alpha^4 - \frac{7280128}{3^{18}} q_\alpha^6 + \mathrm{O}(q_\alpha^8). \tag{3.8.5}$$

Using these values to substitute in (3.7.12) we get the following approximate formulae for χ's

$$\begin{aligned}
\chi_s &= \frac{3^6}{2^8}\frac{1}{q_\alpha} + \frac{3^3}{2^5} + \frac{3^2}{2^6} q_\alpha + \frac{5}{2\cdot 3^3} q_\alpha^2 + \frac{1609}{2^7\cdot 3^6} q_\alpha^3 \\
&\quad + \frac{343}{2\cdot 3^8} q_\alpha^4 + \frac{16981}{2^5\cdot 3^{11}} q_\alpha^5 + \frac{163174}{3^{15}} q_\alpha^6 + \mathrm{O}(q_\alpha^7), \\
\chi_t &= -\frac{3^6}{2^8}\frac{1}{q_\alpha} + \frac{3^3}{2^5} - \frac{3^2}{2^6} q_\alpha + \frac{5}{2\cdot 3^3} q_\alpha^2 - \frac{1609}{2^7\cdot 3^6} q_\alpha^3 \\
&\quad + \frac{343}{2\cdot 3^8} q_\alpha^4 - \frac{16981}{2^5\cdot 3^{11}} q_\alpha^5 + \frac{163174}{3^{15}} q_\alpha^6 + \mathrm{O}(q_\alpha^7), \\
\chi_u &= -\frac{3^3}{2^4} - \frac{5}{3^3} q_\alpha^2 - \frac{343}{3^8} q_\alpha^4 - \frac{326348}{3^{15}} q_\alpha^6 + \mathrm{O}(q_\alpha^8),
\end{aligned} \tag{3.8.6}$$

and the cross ratio

$$\lambda = -\frac{\chi_s}{\chi_u} = \frac{3^3}{2^4} q_\alpha^{-1} + \frac{1}{2} - \frac{11\, q_\alpha}{2^2\cdot 3^3} - \frac{1621}{2^3\cdot 3^8} q_\alpha^3 - \frac{413941}{2\cdot 3^{15}} q_\alpha^5 + \mathrm{O}(q_\alpha^7). \tag{3.8.7}$$

As expected $\chi_s + \chi_t + \chi_u = 0$ up to this order.

For small β an approximate solution to the main equation is given by Eq. (3.6.12). We can use this approximate solution together with the infinite products (3.7.17) and find χ's, but if we leave all the terms the expressions become too complicated. We will need the full expression depending on a and b only for the case of real a and b. Most interesting is the dependence of λ on a and b, which describes a map from abc triangle to $\mathcal{M}_{0,4}$. Keeping only the first non-vanishing terms in both the real and imaginary part of λ^{poles}, we can write

$$\lambda^{\text{poles}} = \frac{1-\sin\delta}{2} - i\,\frac{\cos\delta}{2}\left(1 + \ln\frac{4\cos\delta}{\beta}\right)\beta + \mathrm{O}(\beta^2). \tag{3.8.8}$$

The image of the missing region $\mathcal{V}_{0,4}$ under this map is presented in Fig. 3-8. The approximate expression above is valid in the vicinity of $\lambda = 1/2$.

3.9 Summing the Feynman diagrams

In this section we compute the part of the tachyonic amplitude which comes from the Feynman diagrams. We show how to express this partial amplitude in terms of an integral over a part of the moduli space. We analyze analytic properties of this integral and show that it has no singularity at zero momentum. Our analysis allows us to calculate the Feynman part of the amplitude at zero momentum.

We define the partial or Feynman amplitude as an integral over the Feynman region of $\mathcal{M}_{0,4}$ (see Eq. (3.2.11)):

$$\Gamma_4^{\rm Feyn} = \frac{2}{\pi} \int_{\mathcal{F}_{0,4}} |\gamma_4(s,\, t,\, u)|^2 \,. \tag{3.9.1}$$

Instead of integrating over the Feynman region we can integrate over three unit disks $|\varepsilon| < 1$, one for each channel. Consider for example the contribution of the u channel:

$$\Gamma_4^{(u)}(s,\, t,\, u) = \frac{2}{\pi} \int_{\mathcal{F}_u} |\gamma_4(s,\, t,\, u)|^2. \tag{3.9.2}$$

We can find $\gamma_4(s,\, t,\, u)$ from equation (3.2.9) which we rewrite as

$$\gamma_4(s,\, t,\, u) = \frac{\lambda'(\varepsilon) d\varepsilon}{\lambda(\varepsilon)^{\frac{m^2}{2}-s}(1-\lambda(\varepsilon))^{\frac{m^2}{2}-t}\chi^{\frac{m^2}{2}+2}}. \tag{3.9.3}$$

In terms of ε the region of integration $\mathcal{F}_u$ is just the unit disk $|\varepsilon| < 1$. Recall that (see Eq. (3.8.7))

$$\lambda(\varepsilon) = \frac{3^3}{2^4}\,\varepsilon^{-1} + \frac{1}{2} - \frac{11\,\varepsilon}{2^2\cdot 3^3} - \frac{1621}{2^3\cdot 3^8}\,\varepsilon^3 - \frac{413941}{2\cdot 3^{15}}\,\varepsilon^5 + \mathrm{O}(\varepsilon^7), \tag{3.9.4}$$

is of order of ε^{-1} for small ε and $\chi_u = \mathrm{O}(1)$. Therefore we can represent $\gamma_4(s,t,u)$ in the u channel as

$$\gamma_4(s,t,u) = \varepsilon^{-2-\frac{u}{2}} \sum_{n=0}^{\infty} c_n(s,t)\varepsilon^n \, d\varepsilon. \tag{3.9.5}$$

We can now evaluate the integral in Eq. (3.9.2). If the coefficients c_n vanish sufficiently fast for $n \to \infty$, this integral converges for Re $u < -2$ and is given by

$$\Gamma_4^{(u)}(s,\, t,\, u) = \sum_{n=0}^{\infty} \frac{4\,|c_n(s,t)|^2}{2n-2-u}. \tag{3.9.6}$$

Note that π from the prefactor in Eq. (3.9.2) cancels with the area of the unit disk. Equation (3.9.6) shows that the amplitude has an analytic continuation to the whole region Re $u > -2$, except for even integer values of u, where it has first order poles. These poles correspond to the spectrum of the closed string.

In order to find the constants $c_n(s,t)$ it is sufficient to find the series expansion for $\gamma(s,t,0)$. For the tachyonic potential we need only $c_n(0,0)$, so let us restrict ourselves to this case,

$$\gamma_4^{(0)} = \gamma_4(0,\, 0,\, 0) = \chi_u^2 \, d\lambda. \tag{3.9.7}$$

First of all, recall that χ_u is an even function of ε and that $\lambda(-\varepsilon) = 1 - \lambda(\varepsilon)$. Indeed, when we make a twist by π it is equivalent to an exchange of the poles z_2 and z_3. This exchange does not affect $\chi_u = \chi_{14}$ but changes λ to $1 - \lambda$. We can now conclude that $\gamma_4^{(0)}$ is even with respect to ε and $c_n(0,0) = 0$ for odd n. In particular, this means that massless states ($n = 1$) are decoupled from the tachyon. The sum over massive states which appears in Eq. (3.1.6) is given by

$$\sum_X \succ\!\overset{X}{-}\!\prec = 3 \sum_{k=1}^{\infty} \frac{2\,|c_{2k}(0,0)|^2}{2\,k - 1}, \tag{3.9.8}$$

where we have introduced an extra factor of 3 which comes from summation over three channels. Each term in the series corresponds to a particular mass level and can be found by summing corresponding Feynman diagrams. For example, on the lowest mass level there is only one state — the tachyon, and we therefore conclude that

$$\frac{4\,|c_0|^2}{-2} = \succ\!\overset{T}{-}\!\prec = \frac{\mathrm{v}_3^2}{-2}, \qquad \text{or} \qquad c_0 = \frac{1}{2}\mathrm{v}_3^2 = \frac{3^9}{2^{12}}. \tag{3.9.9}$$

One can similarly evaluate the Feynman diagrams for some other massive levels and thus evaluate some more c_n. An alternative way to do this is to use the series for χ_u and λ from section 3.7 (see Eq. (3.8.6)) and evaluate $\gamma_4^{(0)}$ directly as

$$\gamma_4^{(0)} = \chi_u^2 d\lambda = \left(\frac{3^9}{2^{12}}\,\varepsilon^{-2} + \frac{1377}{2^{10}} + \frac{1399}{2^{11}}\,\varepsilon^2 + \frac{4504241}{2^9 \cdot 3^9}\,\varepsilon^4 + \mathrm{O}(\varepsilon^6) \right) d\varepsilon. \tag{3.9.10}$$

Although we can in principle find as many coefficients c_n as we want, it is very inefficient to evaluate v_4 summing the series because it converges very slowly. The reason for this poor convergence is that the series for $\gamma_4^{(0)}$ diverges at $\varepsilon = 1$. Indeed, $\varepsilon = 1$ corresponds to $\beta = 0$ and we can use approximate formulae for $\lambda(\varepsilon)$ and $\chi_u(\varepsilon)$ in the vicinity of this point to get:

$$\gamma_4^{(0)} = \left(8 \ln \left(\frac{8\pi}{1 - \varepsilon} \right) + \mathrm{O}(1 - \varepsilon) \right) d\varepsilon. \tag{3.9.11}$$

Looking at the first term of this expansion we conclude that

$$c_n \sim \frac{1}{n}, \qquad \text{for} \qquad n \to \infty. \tag{3.9.12}$$

Therefore the series in Eq. (3.9.8) converges as slowly as $\sum n^{-3}$.

Instead of summing the series we therefore decide to calculate the integral itself. First of all, we have to regularize $\gamma_4^{(0)}$ by subtracting the divergent term $(c_0/\varepsilon^2)d\varepsilon$. We can then evaluate convergent integral numerically:

$$\boxed{\sum_X \succ\!\overset{X}{-}\!\prec = \frac{6}{\pi} \int_{|\varepsilon|<1} \left| \gamma_4^{(0)} - \frac{c_0}{\varepsilon^2}\,d\varepsilon \right|^2 \approx 6.011.} \tag{3.9.13}$$

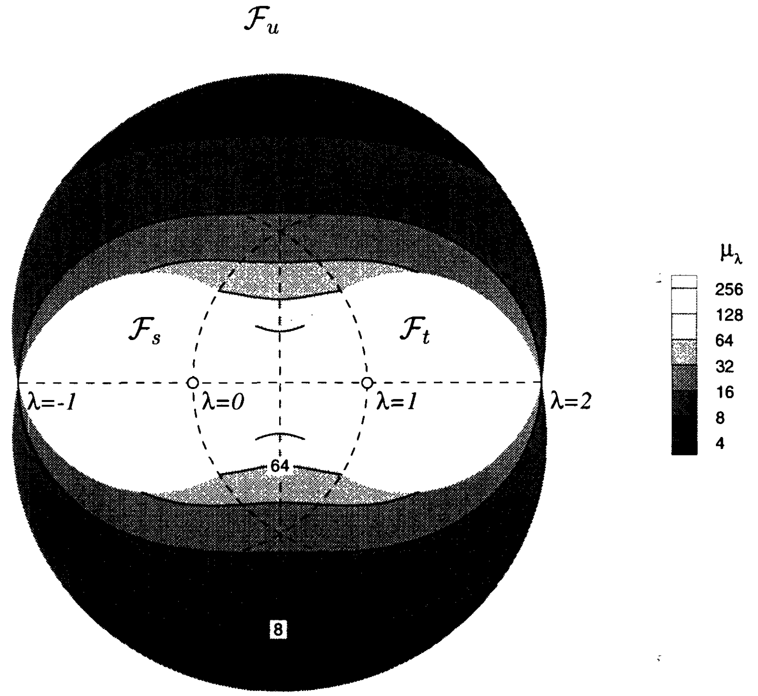


Figure 3-8: The moduli space $\mathcal{M}_{0,4}$ and the measure of integration μ_λ

3.9.1 Historical remarks

Calculations of the Feynman region contribution to the closed string amplitude are very similar to those in the case of the open string. Indeed in the open string we have to consider the same differential form γ_4 and integrate it along the real interval $[-1, 1]$ in order to get a contribution from one channel. The results of this section have been found in the realm of open string in the works of Kostelecký and Samuel [58, 59]. Using different methods to those applied here, they were able to find the quartic term in the effective potential. The series expansion analogous to Eq. (3.9.10) has been found in Ref. [60] up to order ε^2 and it was verified that the coefficients agree with what one gets from the Feynman diagrams with an intermediate massive state.

3.10 Bare four tachyon coupling constant and full effective potential

As we saw in the previous section the four punctured spheres which can be obtained from Feynman diagrams do not cover the moduli space $\mathcal{M}_{0,4}$. The contribution of the rest of $\mathcal{M}_{0,4}$ can be introduced in the string field theory as an elementary 4-string coupling In this section we evaluate this elementary coupling for the case of four tachyons.

The four-string vertex $\mathcal{V}_{0,4} = \mathcal{M}_{0,4} \backslash \mathcal{F}_{0,4}$ can be easily described in terms of integral invariants a, b and c introduced in section 3.4. The whole moduli space can be

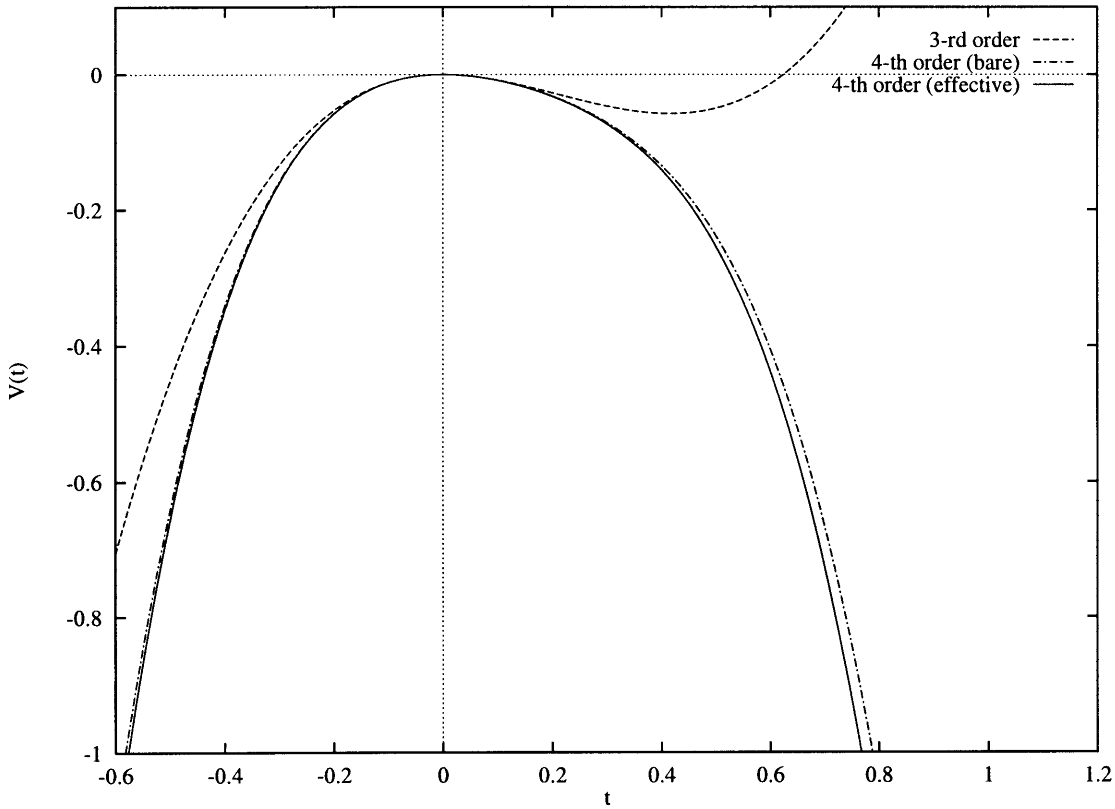


Figure 3-9: Tachyonic potential

parameterized by real values of these invariants varying from 0 to 2π restricted by the condition $a+b+c=2\pi$. In fact, each triple defines two points λ and $\bar{\lambda}$ in $\mathcal{M}_{0,4}$, so we need two copies of the *abc* triangle to cover it. The four-string vertex can now be described as a region in the *abc* triangle defined by

$$a > \pi, \quad b > \pi, \quad \text{and} \quad c > \pi. \tag{3.10.1}$$

The four tachyon coupling is given by the same integral as the amplitude, taken not over the whole moduli space, however, but rather restricted to only $\mathcal{V}_{0,4}$.

$$\mathrm{v}_4 = \frac{2}{\pi}\int_{\mathcal{V}_{0,4}} |\chi_u|^4 d^2\lambda, \tag{3.10.2}$$

where $d^2\lambda = d\,\mathrm{Im}\ \lambda\, d\,\mathrm{Re}\ \lambda$.

3.10.1 Numerical results

For the numerical calculations we use the complex secant method with the starting point given by (3.6.10) in order to solve the main equation. Then we calculate χ_u and λ^{poles} using the first few terms of the infinite products (3.7.17) and (3.7.18). Results are presented in Fig. 3-8 which shows the region of integration $\mathcal{V}_{0,4}$ and the contour plot of the measure $\mu_\lambda = |\chi_u|^4$. We perform calculations only for $\delta \geq 0$, $\beta \leq 1/3 - (2/3\pi)\,\delta$ which is 1/6 of the whole *abc* triangle (see Fig. 3-4). The values of $\lambda^{\rm poles}$ and μ_λ in the rest of the triangle are found from symmetry. As we can see μ_λ has its maximum value of $2^8 = 256$ at $\lambda = 1/2$ and drops exponentially as we go away from this point. Note, that the value of the measure μ_λ at the point where the unit circle intersects the boundary of $\mathcal{V}_{0,4}$ is equal to 64 exactly (at least up to machine precision 10^{-10}). We could not find any explanation to this curious fact.

We have performed numerical integration triangulating 1/12-th of $\mathcal{V}_{0,4}$ correspondent to $\delta \geq 0$ and $\beta \leq 1/3 - (2/3\pi)\,\delta$. Here we present the result of the calculation which involved about 500,000 triangles.

$$\boxed{\mathrm{v}_4 = \frac{2}{\pi}\int_{\mathcal{V}_{0,4}} |\chi_u|^4 d^2\lambda \approx 72.39} \tag{3.10.3}$$

Combining equations (3.10.3) and (3.9.13) we can finally write the tachyonic potential up to the fourth order:

$$V^{\rm eff}(t) = -t^2 + 1.60181\,t^3 - 3.267\,t^4 + \mathrm{O}(t^5). \tag{3.10.4}$$

We present the plot of the effective tachyonic potential computed up to the fourth term in Fig. 3-9. One can see that the fourth order term is big enough to destroy the local minimum suggested by the third order approximation (dashed line in the plot). The plot also shows the bare tachyonic potential computed up to the fourth order. One can see that the effective four-tachyon interaction gives only a small correction.

Chapter 4

The role of the dilaton

4.1 Introduction and summary

The soft-dilaton theorem is an old result in critical string theory. It is stated as a property of string amplitudes for on-shell vertex operators. A physical string amplitude involving a zero-momentum dilaton is written in terms of derivatives, with respect to the dimensionless coupling and the slope parameter, of the string amplitude with the dilaton suppressed [61, 62]. It is natural to ask what does this result tell us about the role of the dilaton *field* in the string action. There has been much work on the role of the dilaton in effective field theory limits of strings. Our interest here is on the role of the dilaton in the complete string action. This line of work began with Yoneya [63] who investigated the dilaton theorem using light cone string field theory. Subsequent studies [64–66] considered the dilaton in the context of covariantized light-cone string field theories.

A dilaton state has a component built by acting on the vacuum by operators from the ghost sector. This component is called the ghost-dilaton $|D_g\rangle$, and its relevance for the on-shell dilaton theorem was studied in Ref. [67, 68]. This work was extended recently to prove the off-shell "ghost-dilaton theorem" in covariant quantum closed string field theory [17, 18]. This result states that an infinitesimal shift along the zero-momentum ghost-dilaton changes the quantum string action, or more precisely, the path integral string measure, in a way equivalent to a shift in the dimensionless string coupling. This work showed concretely that conformal field theories and string backgrounds are not in one to one correspondence: while the ghost-dilaton deforms the string background, it does not deform the conformal field theory underlying the string background. The string background has at least one parameter which is absent in the conformal field theory.

A study of the ghost-dilaton alone is not enough to understand how the value of the string coupling can be changed in string theory. In critical string theory the zero-momentum "matter-dilaton" also shifts the coupling constant. Moreover, the properties of the ghost-dilaton depend on the matter sector of the conformal theory. In critical string theory it is a nontrivial BRST-physical state, but in two-dimensional string theory, for example, it becomes BRST-trivial. If the ghost-dilaton is trivial one may think that the string coupling cannot be changed. This is not correct, a shift of the ghost dilaton will always shift the string coupling. What happens is that the string coupling becomes unobservable. Thus the ghost-dilaton plays a fundamental role: if it is trivial the background has no string coupling constant parameter. This is one of the main results of this paper.

Another point we develop in detail is the analysis of conformal field theory deformations and string background deformations induced by the dimension $(1,1)$ primary field $\partial X\cdot\bar{\partial}X$. In critical string theory this is the matter-dilaton. We first consider the analog of this state in a conformal theory containing a single scalar field X living on the real line. We ask if $\partial X\bar{\partial}X$ deforms the conformal theory. The deformation involves integrating the two-form $\partial X\bar{\partial}X dz\wedge d\bar{z}$ over the surface, and this two form can be written as the exterior derivative $[d(X\bar{\partial}Xd\bar{z})]$. Stokes theorem cannot be used directly because $(X\bar{\partial}X)$ is not primary, and therefore $(X\bar{\partial}Xd\bar{z})$ is not a true one-form. Apart from a piece that can be absorbed by a redefinition of the basis of the conformal theory, we show that the deformation amounts to a scaling of correlators with a factor proportional to the Euler number of the underlying surface. Since the conformal theory has non-vanishing central charge, correlators depend on the scale factor of the metric and the deformation simply corresponds to a variation of this scale factor. Strictly speaking, the conformal field theory has been deformed: the correlators of the two theories on any *fixed* surface cannot be made to agree with a redefinition of the basis of states.

If the matter conformal theory includes twenty six scalars and is coupled to the ghost system, the operator $\partial X\cdot\bar{\partial}X$ does not deform the conformal theory. This result has been verified earlier to various degrees of completeness in interesting works by Mende [69] and Mahapatra *et.al.* [40]. Indeed, following [17] the deformation can be absorbed by a change of basis generated by the ghost-number operator. For integrated correlators this cannot be done, implying that the matter dilaton alters the string coupling. We emphasize, however, that the matter dilaton does not change the value of the dimensionful slope parameter α'.

In addition to the dilaton we also consider the graviton trace $\mathcal{G}$. This state, physical only at zero momentum, is the linear combination of the ghost-dilaton and the matter-dilaton that does not change the string coupling. The graviton trace can be written as Q acting on the state $|\xi\rangle$ created by $(cX\cdot\partial X-\bar{c}X\cdot\bar{\partial}X)$. This state is usually considered illegal since it uses the operator X which is not a scaling field of the conformal theory. We argue in this paper that $\mathcal{G}$ is legally BRST trivial. There is an immediate issue with this interpretation. The failure of $\mathcal{G}$ to decouple [44], easily verified in $\langle\mathcal{G},\text{phys},\text{phys}\rangle\neq 0$, seems to be in contradiction with the claim that $\mathcal{G}$ is BRST trivial: by contour deformation the BRST operator that occurs in $\mathcal{G}$ can be made to act on the physical states giving zero. We show that this is not a correct argument, the point being that correlators involving X are distributions. Careful use of delta functions confirms that $\mathcal{G}$ does not decouple.

Refining the discussion of Mahapatra *et.al.* [40] and, in agreement with them, we claim that the graviton trace $\mathcal{G}$ does not change any physical property of the string background. On the face of it this seems puzzling given the failure of decoupling for $\mathcal{G}$. There is no contradiction, however. Strictly speaking, a state leaves the physics of the background invariant if it appears as the inhomogeneous term of a nonlinear string field transformation that can be verified to leave the string action invariant. A state capable of having such behavior need not decouple. For $\mathcal{G}$ this is possible if correlators of operators including X's can be defined, and, once defined, obey the standard properties of sewing and action of the BRST operator. We discuss these matters and argue that the requisite nonlinear field transformation is simply a gauge transformation with gauge parameter $|\xi\rangle$.

Once we accept X in the gauge parameters we must accept it in the physical states as well. Having a larger set of gauge parameters, we lose some physical states; having a larger state space, we may also gain some new physical states. We formalize this setup via a new refined cohomology problem: BRST cohomology in the extended (semirelative) complex where X and powers of it are accepted as legal operators. The cohomology of this complex appears to capture accurately the idea of a string background: we lose the states that do not change the string background, as the graviton trace, and we gain no states describing new physics. In this cohomology the matter-dilaton is the same as the ghost-dilaton. Similarly, in two-dimensional string theory we show that the states in the semirelative cohomology that are trivial in the extended complex do not appear to change the string background. A detailed computation of BRST cohomology at various ghost numbers in the extended complex of critical string theory will be presented in the next chapter.

This chapter is organized as follows. In section two we set up notation and conventions. We discuss zero momentum physical states, and the uses of the X field operator. In section three we define the extended BRST complex, explain the failure of $\mathcal{G}$ to decouple, and argue that the usual BRST action on correlators holds in the extended complex. We summarize the properties of the new BRST cohomology which will be calculated in chapter 5. In section four we consider CFT deformations induced by the matter dilaton, and derive formulae for the integration of insertions of matter dilatons over spaces of surfaces. We use these results in section five to give a complete proof of the dilaton theorem in closed string field theory. In section six we prove that whenever the ghost-dilaton is BRST trivial the string coupling constant is not observable. Finally, in section seven we illustrate some of our work in the context of two-dimensional string theory.

4.2 Zero momentum states and uses of the $X^\mu(z,\bar{z})$ operator

In this section we begin by enumerating the zero-momentum physical states in critical string theory. This enables us to set our conventions and definitions for the dilaton, the ghost-dilaton, the matter dilaton, and the graviton trace. We then elaborate on the definition of the X^μ field operator and how it can be used to write all zero momentum states, with the exception of the ghost-dilaton, as BRST trivial states. The ghost-dilaton can be also be written in BRST trivial form, but, as is well known, in this case the gauge parameter is not annihilated by b_0^-. Finally, we discuss charges that can be constructed using currents that involve the field operator X^μ.

4.2.1 Zero momentum physical states in critical string theory

In this section we will list the ghost number two physical states of critical bosonic closed strings at zero momentum. These states are defined as cohomology classes of the semirelative BRST complex. We will find that only some of those states can be obtained as a zero momentum limit of physical states that exist for non-zero momentum (states corresponding to massless particles). This is a familiar phenomenon noted, for example, in Ref. [70] for the case of the fully relative closed string BRST cohomology. The present section will also serve the purpose of setting up definitions and conventions.

Let $|\Psi\rangle$ be the string field state and Q the BRST operator. Physical states are defined by

$$Q|\Psi\rangle = 0\,, \tag{4.2.1}$$

up to gauge transformations

$$\delta\,|\Psi\rangle = Q|\Lambda\rangle. \tag{4.2.2}$$

Here both $|\Psi\rangle$ and $|\Lambda\rangle$ must be annihilated by $b_0^- = b_0 - \bar{b}_0$. To look for states that can be physical at zero momentum the relevant part of the string field is

$$\begin{aligned} |\Psi\rangle = &\; E_{\mu\nu}\, c_1\alpha_{-1}^\mu \bar{c}_1\bar{\alpha}_{-1}^\nu|p\rangle \\ & -\overline{A}_\mu c_0^+ c_1\alpha_{-1}^\mu|p\rangle + A_\mu c_0^+\bar{c}_1\bar{\alpha}_{-1}^\mu|p\rangle \\ & +F\, c_1 c_{-1}|p\rangle - \overline{F}\,\bar{c}_1\bar{c}_{-1}|p\rangle + \cdots \end{aligned} \tag{4.2.3}$$

where $c_0^+ = (1/2)(c_0 + \bar{c}_0)$, and that of the gauge parameter

$$|\Lambda\rangle = \varepsilon_\mu c_1\alpha_{-1}^\mu|p\rangle - \bar{\varepsilon}_\mu\bar{c}_1\bar{\alpha}_{-1}^\mu|p\rangle + \varepsilon c_0^+|p\rangle + \cdots\,. \tag{4.2.4}$$

Let us work at zero momentum $p_\mu = 0$. Equation (4.2.1) gives $A_\mu = \overline{A}_\mu = 0$ and Eq. (4.2.2) gives us the gauge transformations $\delta F = -\delta\overline{F} = -\phi$. It follows that at zero momentum the d^2 degrees of freedom of $E_{\mu\nu}$ are unconstrained, the combination $F + \overline{F}$ is gauge invariant and unconstrained, and $F - \overline{F}$ can be gauged away. This gives a total of $d^2 + 1$ nontrivial BRST physical states in the semirelative complex. For $p_\mu \neq 0$, it is well known that there are $(d-2)^2$ nontrivial BRST states for each value of momentum satisfying $p^2 = 0$. These considerations indicate that we have $(4d-3)$ states that are only physical at zero momentum. These states are called discrete states.

The (d^2+1) zero-momentum physical states correspond to the following CFT fields

$$\begin{aligned} D_g &\equiv \tfrac{1}{2}\,(c\partial^2 c - \bar{c}\bar{\partial}^2\bar{c})\,, \\ D^{\mu\nu} &\equiv c\bar{c}\partial X^\mu\bar{\partial}X^\nu. \end{aligned} \tag{4.2.5}$$

The state associated to D_g is called the ghost-dilaton and we will refer to the state associated to the trace $\eta_{\mu\nu}D^{\mu\nu}$ as the matter dilaton. In addition we identify two relevant linear combinations of the ghost and the matter dilaton. The first combination is the zero-momentum dilaton

$$D \equiv \eta_{\mu\nu}D^{\mu\nu} - D_g \,, \tag{4.2.6}$$

which is the zero-momentum limit of the scalar massless state called the dilaton. It is recognized as such because the corresponding spacetime field transforms as a scalar under gauge transformations representing diffeomorphisms. The second state is the "graviton trace"

$$\mathcal{G} \equiv \eta_{\mu\nu}D^{\mu\nu} - \frac{d}{2}D_g \,. \tag{4.2.7}$$

This state corresponds to the trace of the graviton field in the convention where the gravity action is of the form $\int \sqrt{g}Rdx$ without a factor involving the dilaton (see, for example, Ref. [71]).

It is of interest to consider the gauge transformation generated by $|\Lambda\rangle$ for the case when we set $\varepsilon = 0$. We find

$$\begin{aligned} \delta E_{\mu\nu} &= p_\mu \overline{\varepsilon}_\nu + p_\mu \varepsilon_\mu, \\ \delta F &= -p^\mu \epsilon_\mu, \\ \delta \overline{F} &= -p^\mu \overline{\varepsilon}_\mu. \end{aligned} \tag{4.2.8}$$

Transforming to coordinate space we obtain the linearized gauge transformations

$$\begin{aligned} \delta E_{\mu\nu}(x) &= \partial_\mu \varepsilon_\nu(x) + \partial_\nu \overline{\varepsilon}_\mu(x) \,, \\ \delta F(x) &= -\partial^\mu \varepsilon_\mu \,, \\ \delta \overline{F}(x) &= -\partial^\mu \overline{\varepsilon}_\mu \,. \end{aligned} \tag{4.2.9}$$

Now consider the gauge parameters

$$\varepsilon_\nu(x) = C_{\mu\nu}x^\mu, \quad \overline{\varepsilon}_\mu(x) = C_{\mu\nu}x^\nu, \tag{4.2.10}$$

where $C_{\mu\nu}$ is a matrix of constants. Equation 4.2.9 implies that the following constant field configurations are pure gauge:

$$E_{\mu\nu}(x) = 2C_{\mu\nu} \,, \quad F(x) = \overline{F}(x) = C^\mu_\mu \,. \tag{4.2.11}$$

In string field theory the coefficient $E_{\mu\nu}(p)$ in Eq. (4.2.3) should be interpreted as a Fourier component of the space-time field $E_{\mu\nu}(x)$. The above spacetime constant field configurations must correspond to zero momentum states. The corresponding string field in 4.2.3 should be expected to be BRST trivial. This requires that

$$D^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} D_g \equiv \mathcal{G}^{\mu\nu}, \tag{4.2.12}$$

is BRST trivial

$$\mathcal{G}^{\mu\nu} = -\{\, Q\,, \xi^{\mu\nu}\,\}. \tag{4.2.13}$$

In the ordinary closed string BRST complex there is no state $\xi^{\mu\nu}$ satisfying 4.2.13. It is necessary to extend the BRST complex to include states corresponding to field configurations growing linearly in space-time. An example is the state $\lim_{z\to 0} X^\mu(z,\overline{z})|0\rangle$, which requires the consideration of $X^\mu(z,\overline{z})$ as a field operator. We analyze this next.

4.2.2 Definition of the X^μ operator

In the ordinary CFT of 26 free bosons only derivatives of $X^\mu(z,\overline{z})$ appear as conformal fields. These derivatives have the mode expansions

$$i\,\partial X^\mu(z,\overline{z}) = \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}}, \qquad i\,\overline{\partial} X^\mu(z,\overline{z}) = \sum_{n=-\infty}^{\infty} \frac{\overline{\alpha}_n^\mu}{\overline{z}^{n+1}}, \tag{4.2.14}$$

where the α_n^μ's are operators with commutation relations

$$[\alpha_m^\mu\,, \alpha_n^\nu] = m\,\eta^{\mu\nu}\delta_{n,m}\,, \qquad [\overline{\alpha}_m^\mu\,, \overline{\alpha}_n^\nu] = m\,\eta^{\mu\nu}\delta_{n,m}\,. \tag{4.2.15}$$

Formally integrating Eq. (4.2.14) we find

$$X^\mu(z,\overline{z}) = X_0^\mu - 2\,i\,\alpha_0^\mu \log|z| + \sum_{n\neq 0} \frac{i\,\alpha_n^\mu}{n z^n} + \sum_{n\neq 0} \frac{i\,\overline{\alpha}^\mu{}_n}{n\overline{z}^n}\,. \tag{4.2.16}$$

The zero mode operator X_0^μ, which appears in Eq. (4.2.16) as a constant of integration is not specified by Eq. (4.2.14) and has to be defined independently. It is standard to interpret it as the position operator for the center of mass, and taken to commute with all α's except the momentum operator α_0^μ

$$[X_0^\mu, \alpha_0^\nu] = i\,\eta^{\mu\nu}. \tag{4.2.17}$$

This interpretation is usually justified by canonical analysis of the two-dimensional quantum field theory. Eqs. (4.2.16) and (4.2.17) provide an abstract definition of $X^\mu(z,\overline{z})$ as an element of a Lie algebra.

4.2.3 A gauge parameter involving $X^\mu(z,\bar{z})$

Now we will try to use the X^μ field to find a gauge parameter which will generate $\mathcal{G}^{\mu\nu}$. Equations (4.2.4) and (4.2.10) suggest that the proper candidate is

$$\xi^{\mu\nu} = \frac{1}{2}\left(c\,{:}X^\nu\partial X^\mu{:} - \overline{c}\,{:}X^\mu\overline{\partial}X^\nu{:}\right). \tag{4.2.18}$$

We have to explain what the products like $:X^\mu \partial X^\nu:$ or $:X^\nu \overline{\partial} X^\mu:$ mean. Normal ordering amounts to placing annihilation operators to the right of creation operators. In our case it is not clear how to order the product of X_0^μ and α_0. We adopt the following definition

$$:X^\nu \partial X^\mu:(z,\overline{z}) \equiv \oint_z \frac{R(\,\partial X^\mu(w) X^\nu(z,\overline{z})\,)}{w-z}\frac{dw}{2\pi i}\,, \tag{4.2.19}$$

where R denotes the necessary radial ordering. Note that the integral is contour independent because ∂X^μ is a holomorphic field. As usual, to evaluate the integral we replace the contour around z by two constant radius contours, one with $|w| > |z|$, and the other with $|w| < |z|$. Because of radial ordering one must use different expansions for $1/(w-z)$ in the two contours. A small calculation gives

$$:X^\mu \partial X^\nu:(z,\overline{z}) = X^\mu(z,\overline{z}) \sum_{n\geq 0} \frac{-i\,\alpha_n^\nu}{z^{n+1}} + \sum_{n<0} \frac{-i\,\alpha_n^\nu}{z^{n+1}} X^\mu(z,\overline{z})\,, \tag{4.2.20}$$

which shows that the momentum zero mode α_0^μ appears to the right of the coordinate zero mode X_0^μ.

We now calculate the action of the BRST operator on $\xi^{\mu\nu}$:

$$\begin{aligned}\{Q,\xi^{\mu\nu}(z,\overline{z})\} &= \oint \Big(T_m(z)c(z) + :c(z)\partial c(z) b(z):\Big)\, \xi^{\mu\nu}(z,\overline{z})\,\frac{dz}{2\pi i} \\ &\quad + \oint \Big(\overline{T}_m(\overline{z})\overline{c}(\overline{z}) + :\overline{c}(\overline{z})\overline{\partial}\overline{c}(\overline{z}) b(\overline{z}):\Big)\, \xi^{\mu\nu}(z,\overline{z})\,\frac{d\overline{z}}{2\pi i}.\end{aligned} \tag{4.2.21}$$

Using Wick's theorem[1] to expand the operator product under the integral, we obtain

$$\{Q\,,\,\xi^{\mu\nu}\} = \frac{1}{4}\,\eta^{\mu\nu}(c\partial^2 c - \overline{c}\overline{\partial}^2\overline{c}) - c\overline{c}\partial X^\mu \overline{\partial} X^\nu\,, \tag{4.2.22}$$

and, as expected,

$$\mathcal{G}^{\mu\nu} = -\{Q, \xi^{\mu\nu}\}\,. \tag{4.2.23}$$

The ghost dilaton, the first state in Eq. (4.2.5), is a nontrivial state in the semirelative cohomology, but in the absolute complex it can be represented as [67, 68]

$$D_g = -\,\{Q\,,\chi_g\}\,, \tag{4.2.24}$$

where χ_g is given by

$$\chi_g = -\frac{1}{2}(\partial c - \overline{\partial}\overline{c})\,. \tag{4.2.25}$$

The state χ_g does not belong to $\widehat{\mathcal{H}}$ since it is not annihilated by b_0^-. Using χ_g we can represent the matter states of Eq. (4.2.5) as $\{Q, \cdot\}$. Indeed, let

$$\chi^{\mu\nu} \equiv \xi^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\chi_g\,. \tag{4.2.26}$$

[1] As explained in Ref. [72] Wick's theorem for composite operators is valid when normal ordering is defined as in Eq. (4.2.19).

$D_g = \frac{1}{2}(c\partial^2 c - \bar{c}\bar{\partial}^2 c)$	$D_g = -\{Q, \chi_g\}$	$\chi_g = -\frac{1}{2}(\partial c - \bar{\partial}\bar{c})$
$D^{\mu\nu} = c\bar{c}\partial X^\mu \bar{\partial} X^\nu$	$D^{\mu\nu} = -\{Q, \chi^{\mu\nu}\}$	$\chi^{\mu\nu} = \xi^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\chi_g$
$\mathcal{G}^{\mu\nu} = D^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}D_g$	$\mathcal{G}^{\mu\nu} = -\{Q, \xi^{\mu\nu}\}$	$\xi^{\mu\nu} = \frac{1}{2}(cX^\nu\partial X^\mu - \bar{c}X^\mu\bar{\partial}X^\nu)$
$D = D^\mu_\mu - D_g$ $= \mathcal{G}^\mu_\mu + \left(\frac{d-2}{2}\right) D_g$	$D_g = -\{Q, \chi_D\}$	$\chi_D = \chi^\mu_\mu - \chi_g$ $= \xi^\mu_\mu + \left(\frac{d-2}{2}\right)\chi_g$

Table 4.1: Summary of ghost and matter dilaton combinations

Unlike χ_g and $\xi^{\mu\nu}$, the state $\chi_m^{\mu\nu}$ can be verified to be a $(1,1)$ primary. Now combining Eq. (4.2.24) with Eq. (4.2.22) we obtain

$$D^{\mu\nu} = -\{Q, \chi^{\mu\nu}\}, \tag{4.2.27}$$

and conclude that $\chi^{\mu\nu}$ is the gauge parameter generating the matter states given by Eq. (4.2.5). We therefore conclude that all semirelative cohomology states at zero momentum can be represented as $\{Q, \cdot\}$ when we allow gauge parameters using the X operator and/or violating the $b_0^- = 0$ condition. This information is summarized in Table 4.1.

4.2.4 Properties of $:X^\nu \partial X^\mu: dz$

It is of interest to consider some additional properties of $:X^\nu\partial X^\mu: dz$ and $:X^\mu\bar{\partial}X^\nu: d\bar{z}$. Both of them have an exterior derivative proportional to the two-form

$$\partial X^\mu \bar{\partial} X^\nu dz \wedge d\bar{z} = -d\left(:X^\nu\partial X^\mu: dz\right) = d\left(:X^\mu\bar{\partial}X^\nu: d\bar{z}\right). \tag{4.2.28}$$

Nevertheless $:X^\nu\partial X^\mu: dz$ is not a one form because the field $:X^\nu\partial X^\mu:$ is not primary

$$T(z) :X^\nu\partial X^\mu: (w,\bar{w}) = \frac{-\eta^{\mu\nu}}{(z-w)^3} + \frac{:X^\nu\partial X^\mu: (w,\bar{w})}{(z-w)^2} + \frac{\partial(:X^\nu\partial X^\mu:)(w,\bar{w})}{z-w} + \cdots \,. \tag{4.2.29}$$

As a consequence we have an anomalous transformation law under analytic maps

$$:X^\nu\partial X^\mu: (z',\bar{z}')dz' = \left(:X^\nu\partial X^\mu: (z,\bar{z}) - \frac{1}{2}\eta^{\mu\nu}\frac{d^2z/dz'^2}{(dz/dz')^2}\right) dz. \tag{4.2.30}$$

See [50] for details. Similar results hold for $:X^\mu\bar{\partial}X^\nu: d\bar{z}$.

This implies that $:X^\nu\partial X^\mu: dz$ and/or $:X^\mu\bar{\partial}X^\nu: d\bar{z}$ cannot be integrated unambiguously over a contour unless we fix a coordinate in the vicinity of it. Nevertheless, we can show that an integral over a contractible path does not change if we make a coordinate transformation which is holomorphic inside the path. Indeed, according to Eq. (4.2.30), when we make a holomorphic change of coordinate the anomalous piece which appears in the transformation law is a holomorphic (or antiholomorphic) one-form whose integral over a contractible path is zero. Thus the integrals $\oint_\gamma :X^\nu\partial X^\mu: dz$ and $\oint_\gamma :X^\mu\overline{\partial}X^\nu: d\overline{z}$ are well defined if γ is contractible but they still depend on the choice of the contour γ itself. For later use we define

$$\mathcal{D}_X^{\mu\nu} = -\oint_{|z|=1} :X^\nu\partial X^\mu:(z,\overline{z})\frac{dz}{2\pi i} = \oint_{|z|=1} :X^\mu\overline{\partial}X^\nu:(z,\overline{z})\frac{d\overline{z}}{2\pi i}\ , \tag{4.2.31}$$

where the equality follows by use of 4.2.28. Using Eq. (4.2.20) we rewrite the above as

$$\mathcal{D}_X^{\mu\nu} = -\oint_{|z|=1}\frac{dz}{2\pi i}X^\nu(z,\overline{z})\sum_{n\geq 0}\frac{-i\,\alpha_n^\mu}{z^{n+1}} - \oint_{|z|=1}\frac{dz}{2\pi i}\sum_{n<0}\frac{-i\,\alpha_n^\mu}{z^{n+1}}X^\nu(z,\overline{z})\ . \tag{4.2.32}$$

The integration is done using Eq. (4.2.16). Since we integrate over the unit circle the logarithm in Eq. (4.2.16) vanishes, and we obtain

$$\mathcal{D}_X^{\mu\nu} = iX_0^\nu\alpha_0^\mu - \sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\,\overline{\alpha}_n^\nu\,. \tag{4.2.33}$$

This operator appeared earlier in the dilaton theorem analysis of refs. [63, 64].

4.3 Extended BRST complex

In this section we define an extended BRST complex where the coordinate zero mode X_0 acts as a linear operator. We will show how the BRST operator Q^{ext} acts on this complex and explain why BRST exact states in this complex may not decouple from physical correlations. We will explain, through an example why the usual BRST action on correlators holds in the extended complex. This, together with the fact that sewing also holds in the extended complex, implies that string field theory is well defined in the extended complex. The explicit computation of the BRST cohomology in this complex will be given in chapter 5. The present section concludes with a review of the results of this computation.

4.3.1 Definition of the extended complex

We define the extended space of states at any given momentum p as a tensor product of the original state space $\widehat{\mathcal{H}}_p$ with the space of polynomials of D variables x^μ:

$$\widehat{\mathcal{H}}_p^{\text{ext}} = \mathbb{C}\,[x^\mu]\otimes\widehat{\mathcal{H}}_p\,. \tag{4.3.1}$$

A state in this complex is written in the form $P\otimes v$ where $P\in\mathbb{C}\,[x^\mu]$ is a polynomial in x^μ, and $v\in\widehat{\mathcal{H}}_p$ is a vector from the original state space. The operator X_0^μ acts by multiplying the polynomial by x^μ. All the mode operators, except for α_0^μ, act as they acted on $\widehat{\mathcal{H}}_p$. For α_0^μ it is natural to define

$$\alpha_0^\mu\, P\otimes v = p^\mu P\otimes v - i\eta^{\mu\nu}\frac{\partial P}{\partial x^\nu}\otimes v\,, \tag{4.3.2}$$

preserving in this way the commutation relation $[X_0^\mu, \alpha_0^\nu] = i\eta^{\mu\nu}$. With the matter Virasoro generators written as

$$L_n = \frac{1}{2}\sum_m \eta_{\mu\nu}\; :\alpha_m^\mu\, \alpha_{n-m}^\nu:\,, \qquad \overline{L}_n = \frac{1}{2}\sum_m \eta_{\mu\nu}\; :\overline{\alpha}_m^\mu\, \overline{\alpha}_{n-m}^\nu:\,, \tag{4.3.3}$$

the BRST charge reads

$$Q^{\text{ext}} = \sum_n c_n L_{-n} - \frac{1}{2}\sum_{m,n}(m-n)\, :c_{-m}c_{-n}b_{m+n}: + \text{a.h.} \tag{4.3.4}$$

Although equations (4.3.3) and (4.2.2) have the standard form, when acting on the extended complex we must use Eq. (4.3.2). For the BRST operator this implies that

$$Q^{\text{ext}}\, P\otimes v = P\otimes Qv - i\frac{\partial P}{\partial x^\mu}\otimes \sum_{n=-\infty}^{\infty}(c_n\alpha_{-n}^\mu + \overline{c}_n\overline{\alpha}_{-n}^\mu)v - \eta^{\mu\nu}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\otimes c_0^+ v. \tag{4.3.5}$$

4.3.2 The failure of BRST decoupling

The analysis of section two shows that the states $\mathcal{G}^{\mu\nu}$ are BRST trivial in the extended complex. Indeed, as written in the table at the end of section 2.3, $\mathcal{G}^{\mu\nu} = \{Q\,,\, \xi^{\mu\nu}\}$, where $\xi^{\mu\nu}$ contains an explicit X operator. Correlators involving an explicit X operator should be evaluated using $X^\mu(z) = -i\,\frac{\partial}{\partial p_\mu}\exp(ipX(z))\Big|_{p=0}$, which operationally means evaluating the correlator with $X^\mu(z)$ replaced by $\exp(ipX(z))$ and evaluating the derivative $-i\frac{\partial}{\partial p_\mu}$ of the resulting correlator at $p=0$.

In general, the correlation functions of a BRST trivial state with physical states are known to vanish. Indeed, let $|\phi\rangle = -Q|\chi\rangle$ be BRST trivial, and $|\psi_k\rangle$ for $k = 1,\ldots,n$, be BRST physical ($Q|\psi_k\rangle = 0$). It follows by contour deformation that the BRST operator can be taken to act on the physical states giving

$$\langle \phi\,\psi_1\cdots\psi_n\rangle = \sum_{k=1}^{n}\langle \chi\,\psi_1\cdots Q\psi_k\cdots\psi_n\rangle = 0. \tag{4.3.6}$$

On the other hand, consider the three point function of the BRST trivial zero-momentum state $\mathcal{G}^{\mu\nu}$ with two tachyons $\tau_p(z,\overline{z}) = c\overline{c}\exp(ipX)(z,\overline{z})$. One readily verifies that for $\mu\neq\nu$

$$\langle \mathcal{G}^{\mu\nu}(z_1,\overline{z}_1)\,\tau_p(z_2,\overline{z}_2)\,\tau_q(z_3,\overline{z}_3)\rangle = 2\,p^\mu p^\nu\,|z_2-z_3|^{4-2p^2}(2\pi)^d\delta^d(p+q)\,, \tag{4.3.7}$$

and observes that this is not zero for on-shell tachyons ($p^2 = q^2 = 2$), in apparent contradiction with Eq. (4.3.6). Since the computation leading to Eq. (4.3.7) is beyond doubt we must find why the usual argument for decoupling fails.

The problem with Eq. (4.3.6) is that when $\phi = \mathcal{G}^{\mu\nu}$, the field X^μ is present in the correlators under the sum, and we cannot make sense of their on-shell values. This is because correlators are not functions but rather distributions. In the absence of X fields, however, we associate an ordinary function to a correlator because every correlator can be represented as an ordinary function of momenta times the standard momentum conserving δ-distribution

$$\langle \psi_1 \psi_2 \cdots \psi_n \rangle = F\left(p_1, p_2, \ldots, p_n\right) (2\pi)^d \delta^d \left(p_1 + p_2 + \cdots + p_n\right). \tag{4.3.8}$$

When we say that the correlator $\langle \psi_1 \psi_1 \cdots \psi_n \rangle$ vanishes for particular values of momenta what we really mean is that the function F vanishes. On the other hand, if the fields ψ_i contain X without derivatives, correlators have a more general structure

$$\begin{aligned} \langle \psi_1 \psi_2 \cdots \psi_n \rangle = \; & F\left(p_1, \ldots, p_n\right) \; (2\pi)^d \, \delta^d \left(p_1 + \cdots + p_n\right) \\ & + F^\mu(p_1, \ldots, p_n) \; (2\pi)^d \, \delta^d_{,\mu}(p_1 + \cdots + p_n) + \cdots . \end{aligned} \tag{4.3.9}$$

where $\delta^d_{,\mu}(p) \equiv \frac{d}{dp^\mu} \delta^d(p)$, and the dots indicate possible terms with higher derivatives of delta functions. A function F^μ that vanishes on-shell can contribute to the correlator if its derivative does not vanish on shell. This implies that the right hand side of Eq. (4.3.6) need not vanish when the correlator contains an X. We will illustrate this with an example. Rather than using the field $\mathcal{G}^{\mu\nu}$, the point can be made by considering open string theory where the analogous field is the zero-momentum photon $c\partial X^\mu = \{Q, X^\mu\}$. We therefore examine the correlator of this state with two tachyons

$$\langle c\partial X^\mu(x) \, ce^{ip_1 X(x_1)} \, ce^{ip_2 X(x_2)} \rangle = -ip_1^\mu \, |x_2 - x_1|^{2-p_1^2} \, (2\pi)^d \, \delta^d(p_1 + p_2), \tag{4.3.10}$$

where x, x_1 and x_2 denote the insertion points on the real axis. On the other hand, using the BRST property we are led to write

$$\begin{aligned} \langle c\partial X^\mu(x) \, ce^{ip_1 X(x_1)} \, ce^{ip_2 X(x_2)} \rangle = & -\langle X^\mu(x) \, \{Q, ce^{ip_1 X(x_1)}\} \, ce^{ip_2 X(x_2)} \rangle \\ & + \langle X^\mu(x) \, ce^{ip_1 X(x_1)} \, \{Q, ce^{ip_2 X(x_2)}\} \rangle . \end{aligned} \tag{4.3.11}$$

Using $\{Q, ce^{ipX(z)}\} = (\frac{p^2}{2} - 1) c\partial c e^{ipX(z)}$, we obtain

$$(I) = \langle X^\mu(x) \, \{Q, ce^{ip_1 X(x_1)}\} \, ce^{ip_2 X(x_2)} \rangle = \left(\frac{p_1^2}{2} - 1\right) |x_1 - x_2|^2 \langle X^\mu(x) \, e^{ip_1 X(x_1)} \, e^{ip_2 X(x_2)} \rangle , \tag{4.3.12}$$

and the remaining matter correlator is evaluated using the prescription given at the beginning of the section

$$(I) = -i \left(\frac{p_1^2}{2} - 1\right) |x_1 - x_2|^{2 + p_1 p_2} \left[p_1^\mu \log \frac{|x - x_1|}{|x - x_2|} \, (2\pi)^d \, \delta^d(p_1 + p_2) + (2\pi)^d \, \delta^d_{,\mu}(p_1 + p_2) \right]. \tag{4.3.13}$$

In this expression, the term including the ordinary delta function is unchanged under the simultaneous exchanges $x_1 \leftrightarrow x_2$ and $p_1 \leftrightarrow p_2$, and thus back in Eq. (4.3.11) we obtain

$$\begin{aligned}\langle c\partial X^\mu(x)\, ce^{ip_1X(x_1)}\, ce^{ip_2X(x_2)}\rangle &= \tfrac{i}{2}\,(p_1^2-p_2^2)|x_1-x_2|^{2+p_1p_2}\,(2\pi)^d\,\delta^d_{,\mu}(p_1+p_2)\,,\\ &= \tfrac{i}{2}\,p\cdot q\,|x_1-x_2|^{2+(p^2-q^2)/4}\,(2\pi)^d\,\delta^d_{,\mu}(p)\,,\end{aligned} \tag{4.3.14}$$

where $p^\mu = p_1^\mu + p_2^\mu$ and $q^\mu = p_1^\mu - p_2^\mu$. Since $xf(x)\delta'(x) = -f(0)\delta(x)$ we find

$$\langle c\partial X^\mu(x)\, ce^{ip_1X(x_1)}\, ce^{ip_2X(x_2)}\rangle = -i\,p_1^\mu|x_1-x_2|^{2-p_1^2}(2\pi)^d\,\delta^d(p_1+p_2)\,, \tag{4.3.15}$$

in agreement with Eq. (4.3.10), and confirms the failure of decoupling. This example also illustrates that with proper treatment of distributions the BRST property of correlators holds in the extended complex. There is no problem with the contour deformation arguments that are used to prove the BRST property. This will also be the case when we deal with integrated correlators. The sewing property of correlators is also preserved in the extended complex. This follows from the definition of correlators when a field X is present and the fact that the sewing ket is not changed. Since the consistency of string field theory depends only on the proper BRST action on correlators and sewing, the above arguments indicate that there is no difficulty in defining string field theory on the extended complex.

Let us comment on the significance of the BRST property for the issue of decoupling. The fact that the BRST property holds in the extended complex means that moduli space forms $\Omega_{|\Psi\rangle\cdots|\Psi\rangle}$ representing correlators of states $|\Psi\rangle$ in the extended complex satisfy $\Omega_{(\sum Q)|\Psi\rangle\cdots|\Psi\rangle} \sim d\Omega_{|\Psi\rangle\cdots|\Psi\rangle}$. Consider a simple case when we have two states, $|\mathcal{G}\rangle = -Q|\xi\rangle$, and a physical state $|\Psi_p\rangle$. Then $d\Omega_{|\xi\rangle|\Psi_p\rangle} \sim \Omega_{|\mathcal{G}\rangle|\Psi_p\rangle} + \Omega_{|\xi\rangle Q|\Psi_p\rangle}$. As we mentioned previously in this section, whenever we consider correlators involving ξ we must deal with distributions since they generally have the form indicated in Eq. (4.3.9) and the fact that $|\Psi_p\rangle$ is physical does not mean that $\Omega_{|\xi\rangle Q|\Psi_p\rangle}$ vanishes, since $Q|\Psi_p\rangle = 0$ only for on-shell values of momentum. We therefore conclude that a form $\Omega_{|\mathcal{G}\rangle|\Psi_p\rangle}$ representing the correlator of $\mathcal{G}$ with a physical state can be written as $\Omega_{|\mathcal{G}\rangle|\Psi_p\rangle} \sim d\Omega_{|\xi\rangle|\Psi_p\rangle} - \Omega_{|\xi\rangle Q|\Psi_p\rangle}$, and therefore it is not an exact form. Its integral over moduli space is non-zero, and $\mathcal{G}$ need not decouple. This failure to decouple is not an issue for the consistency of the string field theory action. Here only the BRST property, which ensures gauge invariance, is necessary. This property holds off-shell in the extended complex.

4.3.3 Cohomology of $Q^{\rm ext}$

While the extended BRST complex is larger than the original one, a priori, this has no immediate implication for the cohomologies. When we extend a complex we increase both the number of BRST closed states and the number of BRST trivial states. As we saw earlier, some zero momentum states that were physical in the original complex are trivial in the extended one. On the other hand there might be new solutions to $Q|\Psi\rangle = 0$ in the new complex. Explicite calculation of the cohomology of the extended complex $\widehat{\mathcal{H}}^{\rm ext}$ is presented in chapter 5. Indeed, we lose some states, in particular states that do not change the physics of the background, and states of peculiar ghost numbers. We gain some states, but the new states can be understood in terms of the old complex.

When we have a physical state $|v,p\rangle$ which remains physical under continuous variations of the momentum p, we can easily construct physical states in the extended complex by taking linear combinations of $|v,p_i\rangle$ where all $p_i \approx p$ are on-shell. Adjusting the coefficients we can get as many derivatives with respect to p as we want which we can interpret as factors of X_0^μ. Since mass-shells are not flat, in general we get nontrivial combinations of states with different numbers of X_0^μ. At non-zero momentum and ghost number two ($G=2$), all new physical states can be obtained from standard states by the above limiting procedure [73] .

Let us recall the structure of the semirelative cohomology. At non-zero momentum p the cohomology at $G=2$ and at $G=3$ can be represented by the states $c_1\bar{c}_1|v,p\rangle$, and $(c_0+\bar{c}_0)c_1\bar{c}_1|v,p\rangle$ respectively, where $|v,p\rangle$ is a dimension $(1,1)$ primary matter state. All $G=3$ states are trivial in the extended complex. Indeed, using Eq. (4.3.5) we can write

$$1\otimes(c_0+\bar{c}_0)c_1\bar{c}_1|v,p\rangle = Q^{\rm ext}\,\frac{p\cdot x}{p^2}\otimes c_1\bar{c}_1|v,p\rangle - \sum\frac{p_\mu}{p^2}\otimes c_1\bar{c}_1(c_{-n}\alpha_n^\mu+\bar{c}_{-n}\bar{\alpha}_n^\mu)|v,p\rangle\,. \tag{4.3.16}$$

The last sum must be Q trivial because, being annihilated by Q, b_0 and $\bar{b}_0$, it would represent a non-trivial relative cohomology class at $G=3$. Such a class does not exist.

Calculation of the cohomology of the extended complex at zero momentum is more delicate [73]. In the standard semirelative case the physical states go as follows. At $G=2$ we have the (d^2+1) states of section three, and at $G=3$ the (d^2+1) states obtained by multiplying the $G=2$ states by $(c_0+\bar{c}_0)$. There are $2d$ states at $G=1$ and at $G=4$, and one state at $G=0$ ($SL(2,\mathbb{C})$ vacuum) and at $G=5$. In the extended complex there are no zero-momentum physical states at $G>2$. There is an infinite tower of $G=2$ states which can be described as different limits of linear combination of massless states as all momenta are taken to zero. There is only one $G=0$ state, the $SL(2,\mathbb{C})$ vacuum, and there are $d(d+1)/2$ physical states at $G=1$ which contain no more than one X_0^μ. These are precisely the states that generate Poincare symmetry. While in the standard complex one gets the states that generate translations, the states generating Lorentz transformations are missing. They appear properly in the extended complex.

4.4 CFT Deformations and the matter dilaton

In this section we give a detailed analysis of the operator $\partial X\bar{\partial}X$ and its effect on conformal theories that include a free field X living on the open line. There are two cases of interest. In the first one, the conformal theory includes the ghost system and has total central charge of zero. We derive identities that show that $\partial X \bar{\partial} X$ induces a trivial deformation of the CFT, the ghosts playing a crucial role here.

We then consider the second case, when this operator appears in the context of the $c = 1$ matter conformal field theory of the free field (no extra ghosts). We use the definition of a $c \neq 0$ conformal theory in the operator formalism to show that the deformation *is not* strictly trivial. The detailed analysis shows that the deformation in question can be mostly eliminated by a change of basis in the conformal theory, but the scale of the world sheet metric is changed by the deformation.

The above results are certainly not controversial for the case of zero central charge. The triviality of the operator in question has been argued earlier at various levels of detail. In ref. [40], for example, the usual argument that such perturbation can be redefined away from the conformal field theory lagrangian is reviewed, along with a discussion from the viewpoint of gauge transformations in string field theory. In ref. [69] the deformation of the two-point function of stress-tensors is investigated explicitly. For the case of non-zero central charge our result appears to be new.

We then turn to integrals of string forms over moduli spaces of Riemann surfaces and consider their deformation by the insertion of the matter operator in question. The resulting integral expressions will be needed in section six in order to establish the complete dilaton theorems.

4.4.1 Operator $\partial X \bar{\partial} X$ in $c = 0$ CFT

Here we answer the following question: does the $(1,1)$ primary field $:\partial X^\mu \bar{\partial} X^\nu:$ define a non-trivial deformation of a conformal field theory? Being a $(1,1)$ primary we can write the corresponding deformation by integrating the field over the surface minus unit disks [74, 75]

$$\delta^{\mu\nu} \langle \Sigma_{g,n}| \equiv \frac{1}{2\pi i} \int_{\Sigma_{g,n} - \cup D_k} \langle \Sigma_{g,n+1}(z, \bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle \, dz \wedge d\bar{z} \ , \tag{4.4.1}$$

This deformation is trivial if there is an operator $\mathcal{O}^{\mu\nu}$ such that

$$\delta^{\mu\nu} \langle \Sigma_{g,n}| = -\langle \Sigma_{g,n} | \sum_{k=1}^{n} \mathcal{O}^{\mu\nu\,(k)}, \tag{4.4.2}$$

since this means that the deformation can be absorbed by a change of basis in the CFT, a change induced by the operator $\mathcal{O}^{\mu\nu}$. We will show that $\mathcal{O}^{\mu\nu}$ is given by

$$\mathcal{O}^{\mu\nu} = \mathcal{D}_X^{\mu\nu} + \frac{1}{6} \eta^{\mu\nu} G \,, \tag{4.4.3}$$

where $\mathcal{D}_X^{\mu\nu}$ was defined in Eq. (4.2.31) and G is the total ghost number operator. This shows that the the operator $:\partial X^\mu \bar{\partial} X^\nu:$ induces a trivial conformal field theory deformation.

We now give a simple proof of the above assertion. Note that the operator-valued two form that is being integrated can be written as

$$:\partial X^\mu \bar{\partial} X^\nu: dz \wedge d\bar{z} = d \left[\frac{1}{2} :X^\mu \bar{\partial} X^\nu: d\bar{z} - \frac{1}{2} :X^\nu \partial X^\mu: dz + (A^{\mu\nu}(z) dz - \overline{A}^{\mu\nu}(\bar{z}) d\bar{z}) \right], \tag{4.4.4}$$

where $A^{\mu\nu}(z)$ and $\overline{A}^{\mu\nu}(\bar{z})$ are arbitrary holomorphic or antiholomorphic operators, and are therefore are annihilated by the exterior derivative. They are important, however. The left hand side of the above equation is a well defined two-form, but the expression within brackets in the right hand side is not a well-defined one-form unless the A operators are suitably chosen. This is the case because the operators $:X^\mu\bar{\partial}X^\nu:$ and $:X^\nu\partial X^\mu:$ are not primary. We can obtain primary operators by choosing non-primary A and $\overline{A}$ operators. We take

$$:\partial X^\mu\bar{\partial}X^\nu:dz\wedge d\bar{z} = d\left[\frac{1}{2}\ :X^\mu\bar{\partial}X^\nu:d\bar{z} - \frac{1}{2}:X^\nu\partial X^\mu:dz + \frac{\eta^{\mu\nu}}{6}(G(z)dz - \overline{G}(\bar{z})d\bar{z})\right], \tag{4.4.5}$$

where $G(z) =:c(z)b(z):$ and $\overline{G}(\bar{z}) =:\bar{c}(\bar{z})\bar{b}(\bar{z}):$ are the holomorphic and antiholomorphic ghost currents. Indeed, since $T(z)G(w) \sim -3/(z-w)^3 + \cdots$, the anomaly in the transformation law of $:X^\nu\partial X^\mu:$ is canceled (see Eq. (4.2.30)). The expression inside the parenthesis is now a well-defined one form. This allows us to use Stokes's theorem to convert the integral over the surface minus the disks to an integral over the disks. Therefore, the original expression (4.4.1), written in correlator language as

$$\frac{1}{2\pi i}\int_{\Sigma_{g,n}-\cup D_k}\Big\langle\cdots:\partial X^\mu\bar{\partial}X^\nu:dz\wedge d\bar{z}\ \cdots\Big\rangle, \tag{4.4.6}$$

becomes

$$\frac{1}{2\pi i}\sum_{k=1}^{n}\Big\langle\cdots\oint_{\partial D_k}\frac{1}{2}(\ :X^\nu\partial X^\mu:dz-\ :X^\mu\bar{\partial}X^\nu:d\bar{z})^{(k)} - \frac{\eta^{\mu\nu}}{6}(G(z)dz-\overline{G}(\bar{z})d\bar{z})^{(k)}\ \cdots\Big\rangle \tag{4.4.7}$$

where the contour integrals are over the boundaries of the disks oriented as such. We now recognize that the contour integrals simply represent a single operator acting on each puncture, one at a time. The operator is just

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{1}{2}(\ :X^\nu\partial X^\mu:dz-\ :X^\mu\bar{\partial}X^\nu:d\bar{z})-\frac{\eta^{\mu\nu}}{6}(G(z)dz-\overline{G}(\bar{z})d\bar{z}) = -\Big(\mathcal{D}_X^{\mu\nu}+\eta^{\mu\nu}\frac{G}{6}\Big). \tag{4.4.8}$$

This concludes our proof of the triviality of the deformation.

4.4.2 Alternative analysis for $g = 0$

It is instructive to verify the main identity of the previous subsection for the case of the Riemann sphere the one for which an explicite operator expression for $\langle\Sigma_{g,n}|$ is given by Eq. (2.6.5). An explicite analysis is interesting for several reasons. First of all it will convince a sceptic that the prior methods do work despite a questionable use of X_0^μ operator. It will also further emphasise the importance of the momentum conserving delta function (and in part will be based on its properties.

According to Eq. (4.4.2)

$$\delta\langle\Sigma_N| = \langle\Sigma_N|\left(\frac{1}{6}G + \sum_{s=1}^{N}\mathcal{D}_X^{(s)}\right) = \langle\Sigma_N|\left(1+\sum_{s=1}^{N}\mathcal{D}_X^{(s)}\right), \tag{4.4.9}$$

where we suppress Lorentz indices and denote by $\mathcal{D}_X^{(s)}$ the generalized dilatation operator acting on the s-th state space.

$$\mathcal{D}_X^{(s)} = \sum_s \left(iX_0^s \alpha_0^s - \sum_{n\neq 0} \frac{1}{n} \alpha_n^s \bar{\alpha}_n^s \right) \tag{4.4.10}$$

Now Eq. (4.4.2) reduces to the following identity which we will verify by an explicite calculation.

$$\frac{2}{\pi} \int d^2z \langle \Sigma_{N+1}(z)|D\rangle^{(N+1)} = \langle \Sigma_N| = \left(1 + \sum_{s=1}^{N} \mathcal{D}_X^{(s)} \right), \tag{4.4.11}$$

where the dilaton state being inserted at the $N+1$ puncture is given by

$$|D\rangle^{(N+1)} = \frac{\alpha_{-1}^{N+1} \bar{\alpha}_{-1}^{N+1}}{|h'_{N+1}(0)|^2} \, |\mathbf{1}, 0\rangle_{N+1}. \tag{4.4.12}$$

We will show that (4.4.11) holds by direct evaluation of the both sides.

Let us start with the right hand side. The bosonic part of the surface state corresponding to an N-punctured sphere Σ_N is given by

$$\langle \Sigma_N| = \langle 1 \ldots N| \exp\left(\frac{1}{2} \sum_{n,m\geq 0} \left(\alpha_n^r \mathcal{N}_{nm}^{rs} \alpha_m^s + \bar{\alpha}_n^r \overline{\mathcal{N}_{nm}^{rs}} \bar{\alpha}_m^s \right) \right), \tag{4.4.13}$$

where

$$\langle 1 \ldots N| = \int \prod_{r=1}^{N} dp_s \, (2\pi)^D \delta^D \left(\sum p_s \right) \bigotimes_{s=1}^{N} \langle \mathbf{1}^c, p_s| \tag{4.4.14}$$

is the string vertex in the point-like string limit. We will use the following properties of the $\langle \Sigma_N|$ state. Due to the commutation relations

$$[\alpha_m^r, \alpha_n^s] = [\bar{\alpha}_m^r, \bar{\alpha}_n^s] = m \, \delta^{rs} \delta_{m+n,0} \tag{4.4.15}$$

we have

$$\langle \Sigma_N| \alpha_{-m}^r = \langle \Sigma_N| \sum_{s=1}^{N} \sum_{n=0}^{\infty} m \, \mathcal{N}_{mn}^{rs} \, \alpha_n^s,$$

and

$$\langle \Sigma_N| \bar{\alpha}_{-m}^r = \langle \Sigma_N| \sum_{s=1}^{N} \sum_{n=0}^{\infty} m \, \overline{\mathcal{N}_{mn}^{rs}} \, \bar{\alpha}_n^s.$$

Using these relations we can evaluate the sum over negative modes in (4.4.11) as

$$\langle \Sigma_N| \sum_{s=1}^{N} \sum_{n>0} \alpha_{-n}^s \bar{\alpha}_{-n}^s = \langle \Sigma_N| \sum_{s,r,t=1}^{N} \sum_{l,m,n=0}^{\infty} n^2 \, \mathcal{N}_{mn}^{rs} \, \overline{\mathcal{N}_{ln}^{ts}} \, \alpha_m^r \, \bar{\alpha}_l^t. \tag{4.4.16}$$

Now consider the first term. It acts by differentiation with respect to momenta. The momenta appear in the surface state at two places: the delta function in the definition of $\langle 1 \ldots N|$ and in the exponent by alias α_0. Using properties of delta function one can show that

$$\langle 1 \ldots N| \sum_{s=1}^{N} \alpha_0^s = 0 \qquad \text{and} \qquad \langle 1 \ldots N| iX_0^s - iX_0^r = 0. \tag{4.4.17}$$

together with the commutation relation

$$[\alpha_0^r, iX_0^s] = \delta^{rs}, \tag{4.4.18}$$

relations (4.4.17) give

$$\langle 1 \ldots N| \sum_s iX_0^s \alpha_0^s = -\langle 1 \ldots N|. \tag{4.4.19}$$

Now we add a commutator with the exponent and get

$$\langle \Sigma_N| \sum_s \{p_s, iX_0^s\} = \langle \Sigma_N| - 1 + \sum_{s,t=1}^{N} \sum_{n=0}^{\infty} p_s \left(\mathcal{N}_{0n}^{st}\, \alpha_n^t + \overline{\mathcal{N}_{0n}^{st}}\, \bar{\alpha}_n^t \right). \tag{4.4.20}$$

Now we combine Eqs. (4.4.20) and (4.4.16) and see that Eq. (4.4.11) is equivalent to the following identity

$$\begin{aligned} &\frac{2}{\pi} \int d^2z \langle \Sigma_{N+1}(z)|D\rangle_{N+1} = \\ &\langle \Sigma_N| \sum_{r,t=1}^{N} \Bigg(\Big(\mathcal{N}_{00}^{rt} + \overline{\mathcal{N}_{00}^{rt}} + \sum_{s=1}^{N} \sum_{n=1}^{\infty} \mathcal{N}_{0n}^{rs}\, n \overline{\mathcal{N}_{n0}^{st}} \Big)\, p_r\, p_t \\ &\qquad + \sum_{m=1}^{\infty} \Big(\mathcal{N}_{0m}^{rt} + \sum_{s=1}^{N} \sum_{n=1}^{\infty} \overline{\mathcal{N}_{0n}^{rs}}\, n \mathcal{N}_{nm}^{st} \Big)\, p_r\, \alpha_m^t \\ &\qquad + \sum_{m=1}^{\infty} \Big(\overline{\mathcal{N}_{0m}^{rt}} + \sum_{s=1}^{N} \sum_{n=1}^{\infty} \mathcal{N}_{0n}^{rs}\, n \overline{\mathcal{N}_{nm}^{st}} \Big)\, p_r\, \bar{\alpha}_m^t \\ &\qquad + \sum_{l,m=1}^{\infty} \Big(- \frac{\delta_{lm} \delta^{rt}}{m} + \sum_{s=1}^{N} \sum_{n=1}^{\infty} \mathcal{N}_{ln}^{rs}\, n \overline{\mathcal{N}_{nm}^{st}} \Big)\, \alpha_l^r\, \bar{\alpha}_m^t \Bigg) \end{aligned} \tag{4.4.21}$$

Note that for the contact interaction the integral in the l.h.s. vanishes and we arrive to three conditions on the Neumann coefficients. These are exactly the same conditions as those set forth in [75, 76].

Let us evaluate the integral. Commuting the α_{-1} operators which appear in the definition of the dilaton (4.4.12) through the exponent we obtain

$$\begin{aligned} &\frac{2}{\pi} \int d^2z \langle \Sigma_{N+1}(z)|D\rangle_{N+1} = \\ &\langle \Sigma_N| \frac{2}{\pi} \int \frac{d^2z}{|h'_{N+1}(0)|^2} \sum_{r,s,t=1}^{N} \sum_{l,m,n=0}^{\infty} \mathcal{N}_{1m}^{N+1 r}(z)\, \overline{\mathcal{N}_{1n}^{N+1 s}}(\bar{z})\, \alpha_l^r \bar{\alpha}_m^t, \qquad . \end{aligned} \tag{4.4.22}$$

We expect that the terms containing the zero modes are different from those that don't. Therefore, we rewrite Eq. (4.4.22) as

$$\frac{2}{\pi}\int d^2z\langle\Sigma_{N+1}(z)|D\rangle_{N+1} = \\ \langle\Sigma_N|H(\mathbf{p}) + \sum_{t=1}^{N}\sum_{m=1}^{\infty} G^t_m(\mathbf{p})\alpha^t_m + \sum_{t=1}^{N}\sum_{m=1}^{\infty}\overline{G^t_m}(\mathbf{p})\bar{\alpha}^t_m + \sum_{r,t=1}^{N}\sum_{l,m=1}^{\infty} F^{rt}_{lm}\,\alpha^r_l\,\bar{\alpha}^t_m, \tag{4.4.23}$$

where

$$\begin{aligned} H(\mathbf{p}) &= \frac{2}{\pi}\int\frac{d^2z}{|h'_{N+1}(0)|^2}\sum_{r,t=1}^{N}\mathcal{N}^{N+1r}_{10}(z)\,\overline{\mathcal{N}^{N+1s}_{10}}(\bar{z})\,p_r\,p_t \\ G^t_m(\mathbf{p}) &= \frac{2}{\pi}\int\frac{d^2z}{|h'_{N+1}(0)|^2}\sum_{r=1}^{N}\mathcal{N}^{N+1r}_{1m}(z)\,\overline{\mathcal{N}^{N+1s}_{10}}(\bar{z})\,p_r \\ F^{rt}_{lm} &= \frac{2}{\pi}\int\frac{d^2z}{|h'_{N+1}(0)|^2}\mathcal{N}^{N+1r}_{1l}(z)\,\overline{\mathcal{N}^{N+1t}_{1m}}(\bar{z}). \end{aligned} \tag{4.4.24}$$

Let us calculate $H(\mathbf{p})$. By definition of the Neumann coefficients

$$\mathcal{N}^{N+1r}_{10}(z) = \frac{h'_{N+1}(0)}{z-h_r(0)}, \tag{4.4.25}$$

and thus

$$H(\mathbf{p}) = \frac{2}{\pi}\int\sum_{r,t=1}^{N}\frac{p_r\,p_t\,d^2z}{(z-h_r(0))\,(\bar{z}-\bar{h}_t(0))}. \tag{4.4.26}$$

Note, that it is important to perform a summation in Eq. (4.4.26) before taking the integral because otherwise the integrals would be logarithmically divergent. The integral of the sum is finite due to momentum conservation law $\sum p_r = 0$. The integral in Eq. (4.4.26) is an integral of an exact form and can be reduced to a sum of contour integrals. The region of integration is the whole complex plane without coordinate patches. One can check that due to momentum conservation law integral over the contour at infinity vanishes and we obtain

$$\begin{aligned} H(\mathbf{p}) = \sum_{s=1}^{N}\oint_{|w_s|=1}\sum_{r,t=1}^{N}\frac{p_r\,p_t}{4i}\log|h_s(w_s)-h_r(0)|^2 \\ \times\left(\frac{\bar{h}'_s(\bar{w}_s)d\bar{w}_s}{\bar{h}_s(\bar{w}_s)-\bar{h}_t(0)} - \frac{h'_s(w_s)dw_s}{h_s(w_s)-h_t(0)}\right), \end{aligned} \tag{4.4.27}$$

where we use the local coordinate w_s to perform the integration over the boundary of the s-th coordinate patch. In the expression (4.4.27) we recognize the generating function on the Neumann coefficients (see (2.6.8)).

$$\log|h_s(w_s)-h_r(0)|^2 = \delta^{sr}\log|w_s|^2 + \sum_{n=0}^{\infty}\mathcal{N}^{sr}_{n0}{w_s}^n + \overline{\mathcal{N}^{sr}_{n0}}{\bar{w}_s}^n, \tag{4.4.28}$$

and its derivatives

$$\frac{h_s'(w_s)}{h_s(w_s)-h_t(0)} = \frac{\delta^{st}}{w_s} + \sum_{n=1}^{\infty} n\,\mathcal{N}_{n0}^{sr} {w_s}^{n-1} \tag{4.4.29}$$

and

$$\frac{\bar{h}_s'(\overline{w}_s)}{\bar{h}_s(\overline{w}_s)-\bar{h}_t(0)} = \frac{\delta^{st}}{\overline{w}_s} + \sum_{n=1}^{\infty} n\,\overline{\mathcal{N}_{n0}^{sr}} {\overline{w}_s}^{n-1}. \tag{4.4.30}$$

Using equations (4.4.28), (4.4.29) and (4.4.30) we can evaluate the contour integrals and find

$$H(\mathbf{p}) = \sum_{r,t=1}^{N} \left(\mathcal{N}_{00}^{rt} + \overline{\mathcal{N}_{00}^{rt}} + \sum_{s=1}^{N}\sum_{n=1}^{\infty} \mathcal{N}_{0n}^{rs}\, n\, \overline{\mathcal{N}_{n0}^{st}} \right) p_r\, p_t \tag{4.4.31}$$

In order to calculate the coefficients $G_m^t(\mathbf{p})$ we introduce the generating function

$$G_{\mathbf{p}}(\mathbf{w}) = \sum_{t=1}^{N}\sum_{m=1}^{\infty} m\, G_m^t(\mathbf{p}) {w_t}^{m-1} = \frac{2}{\pi}\int \sum_{r,t=1}^{N} \frac{h_t'(w_t)}{(z-h_t(w_t))^2}\frac{p_r}{\bar{z}-\bar{h}_r(0)}\, d^2z \tag{4.4.32}$$

Now we can evaluate $G_{\mathbf{p}}(\mathbf{w})$ reducing the double integral (4.4.32) to a sum of contour integrals in the same manner as we evaluated $H(\mathbf{p})$.

$$\begin{aligned} G_{\mathbf{p}}(\mathbf{w}) = \sum_{s=1}^{N} \oint_{|w_s|=1} & \sum_{t,r=1}^{N} \frac{p_r}{4i} \frac{h_t'(w_t)}{h_s(w_s)-h_r(0)} \\ & \times \left(\frac{\bar{h}_s'(\bar{w}_s) d\overline{w}_s}{\bar{h}_s(\bar{w}_s)-\bar{h}_t(\overline{w}_t)} - \frac{h_s'(w_s) dw_s}{h_s(w_s)-h_t(w_t)} \right). \end{aligned} \tag{4.4.33}$$

The series Expansions from equations (4.4.29) and (4.4.30) together with

$$\frac{h_t'(w_t)}{h_s(w_s)-h_t(w_t)} = \delta^{st}\sum_{n=0}^{\infty} \frac{{w_t}^n}{{w_s}^{n+1}} + \sum_{m=0}^{\infty} m\,\mathcal{N}_{m0}^{st}\, {w_s}^{m-1}, \tag{4.4.34}$$

enable us to evaluate the integrals in Eq. (4.4.33), and we find

$$G_{\mathbf{p}}(\mathbf{w}) = \sum_{r,t=1}\sum_{m=1}^{\infty} \left(\mathcal{N}_{0m}^{rt} + \sum_{s=1}^{N}\sum_{n=1}^{\infty} \overline{\mathcal{N}_{0n}^{rs}}\, n\, \mathcal{N}_{nm}^{st} \right) p_r\, {w_t}^m \tag{4.4.35}$$

We calculate the remaining coefficients F_{lm}^{rt} by introducing the generating function

$$\begin{aligned} F(\mathbf{w},\overline{\mathbf{w}}) &= \sum_{r,s=1}^{N}\sum_{m,n=1}^{\infty} l\, m\, F_{lm}^{rt} {w_r}^{l-1} {\overline{w}_s}^{m-1} \\ &= \sum_{r,t} \frac{2}{\pi}\int \frac{h_r'(w_r)}{(z-h_r(w_r))^2}\frac{\bar{h}_t'(w_t)}{(\bar{z}-\bar{h}_t(\overline{w}_t))^2} d^2z, \end{aligned} \tag{4.4.36}$$

which we evaluate in the same fashion as $G_{\mathbf{p}}(\mathbf{w})$ and obtain

$$F(\mathbf{w},\overline{\mathbf{w}}) = \sum_{r,t=1}^{N}\sum_{l,m=1}^{\infty} \left(-\frac{\delta_{lm}\delta^{rt}}{m} + \sum_{s=1}^{N}\sum_{n=1}^{\infty} \mathcal{N}_{ln}^{rs}\, n\, \overline{\mathcal{N}_{nm}^{st}} \right) {w_r}^l\, {\overline{w}_t}^m \tag{4.4.37}$$

One can see that the formulae (4.4.31), (4.4.35) and (4.4.37) together with Eq. (4.4.22) give the same answer as the right hand side of Eq. (4.4.21).

4.4.3 Operator $\partial X \bar{\partial} X$ in $c \neq 0$ CFT

If we have any matter conformal theory coupled to the ghost conformal theory, the ghost number operator G acting on surface states will give

$$\langle \Sigma_{g,n} | \sum_{k=1}^{n} G^{(k)} = 6(1-g) \langle \Sigma_{g,n} |. \tag{4.4.38}$$

Using this result we recognize that the result of the previous subsection showing the triviality of the deformation can be written as

$$\begin{aligned} \delta \langle \Sigma_{g,n} | &= \frac{1}{2\pi i} \int_{\Sigma_{g,n} - \cup D_k} \langle \Sigma_{g,n+1}(z,\bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle \, dz \wedge d\bar{z} \\ &= -\langle \Sigma_{g,n} | \sum_{k=1}^{n} \mathcal{D}_X^{\mu\nu(k)} - (1-g) \, \eta^{\mu\nu} \langle \Sigma_{g,n} | \,, \end{aligned} \tag{4.4.39}$$

In this form, of course, the triviality of the deformation is not manifest since the second term in the right hand side is not written as a sum of linear operators acting on the surface state.

We now claim that equation (4.4.39) applies for the case when the matter conformal theory does not include the ghost conformal theory. By construction, Eq. (4.4.39) applies when the total conformal theory is the $c = 1$ matter theory coupled to the ghosts. The surface states in this total conformal theory are the tensor product of the surface states in the two separate conformal theories $\langle \Sigma_{g,n} | = \langle \Sigma_{g,n}^{c=1} | \otimes \langle \Sigma_{g,n}^{gh} |$. Since the operators in the right hand side of Eq. (4.4.39) are ghost-independent it is clear that we can factor out the ghost part $\langle \Sigma_{g,n}^{gh} |$ of the surface state. Since the insertion in the left hand side carries no ghost dependence the additional puncture is deleted in the ghost sector $\langle \Sigma_{g,n+1}(z,\bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle = \langle \Sigma_{g,n+1}^{c=1}(z,\bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle \otimes \langle \Sigma_{g,n}^{gh} |$. It follows that we can factor the ghost sector out of equation 4.4.39 totally, and we find

$$\begin{aligned} \delta \langle \Sigma_{g,n}^{c=1} | &= \frac{1}{2\pi i} \int_{\Sigma_{g,n} - \cup D_k} \langle \Sigma_{g,n+1}^{c=1}(z,\bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle \, dz \wedge d\bar{z} \\ &= -\langle \Sigma_{g,n}^{c=1} | \sum_{k=1}^{n} \mathcal{D}_X^{\mu\nu(k)} - (1-g) \, \eta^{\mu\nu} \langle \Sigma_{g,n}^{c=1} | \,. \end{aligned} \tag{4.4.40}$$

We can now address the issue of triviality. As mentioned earlier, the first term in the right hand side is just a similarity transformation. The second term is not. To give an interpretation to that term we recall the scaling properties of $c \neq 0$ CFT. Under a scale change of the metric on the surface the correlators change as

$$\Big\langle \cdots \Big\rangle_{\mathrm{g}e^\sigma} = \exp\left[\frac{c}{48\pi} S_L(\sigma; \mathrm{g}) \right] \Big\langle \cdots \Big\rangle_{\mathrm{g}} \tag{4.4.41}$$

where

$$S_L(\sigma; \mathrm{g}) = \int d^2\xi \sqrt{\mathrm{g}} \left(\frac{1}{2} \mathrm{g}^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma + R(\mathrm{g}) \sigma \right) \tag{4.4.42}$$

For constant σ we get $S_L(\sigma;\mathrm{g}) = \sigma \int d^2\xi \sqrt{\mathrm{g}}\, R(\mathrm{g}) = 4\pi\sigma\,(1-g)$, where g is the genus of the surface. This shows that for an infinitesimal scaling parameter σ the correlators scale as

$$\Big\langle \cdots \Big\rangle_{\mathrm{g}e^{\sigma}} = \Big(1 + \frac{c}{12}\sigma(1-g)\Big)\Big\langle \cdots \Big\rangle_{\mathrm{g}}\,. \tag{4.4.43}$$

This shows that the last term in Eq. (4.4.40) corresponds to a constant scale deformation of the metric on the two-dimensional surface. The deformation induced by $\partial X \bar{\partial} X$ is not completely trivial.

4.4.4 Generalization to spaces of surfaces

We must now extend the discussion of the $c = 0$ case to include spaces of surfaces. Since we will use the ghost sector in a nontrivial fashion we use the ghost number two primary states $D^{\mu\nu}$ defined in Eq. (4.2.5). Rather than integrating the matter dilaton over a single surface Σ, an operation that we can denote as $f_{D^{\mu\nu}}(\mathcal{K}\Sigma)$, we want to consider the object $f_{D^{\mu\nu}}(\mathcal{K}\mathcal{A})$, where $\mathcal{A}$ is a space of surfaces. We claim that the following result holds

$$f_{D^{\mu\nu}}(\mathcal{K}\mathcal{A}) + f_{\chi^{\mu\nu}}(\mathcal{L}\mathcal{A}) = \frac{1}{2}\,\eta^{\mu\nu}\,(2g-2+n) f(\mathcal{A})\,, \tag{4.4.44}$$

where $\chi^{\mu\nu}$ is the state defined in Eq. (4.2.26) and whose main property is that upon action by the BRST operator it gives us the matter dilaton state. The purpose of the present subsection is to prove equation (4.4.44).

We begin the proof of the above relation by evaluating $f_{D^{\mu\nu}}(\mathcal{K}\mathcal{A})$. Since the matter dilaton state can be written as $|D^{\mu\nu}\rangle = -Q|\xi^{\mu\nu}\rangle + \frac{1}{2}\eta^{\mu\nu}|D_g\rangle$ we find

$$f_{D^{\mu\nu}}(\mathcal{K}\mathcal{A}) = -\frac{1}{n!}\int_{\mathcal{K}\mathcal{A}} \langle \Omega^{[\mathcal{A}+2]g,n+1}|Q|\xi^{\mu\nu}\rangle + \frac{1}{2}\,\eta^{\mu\nu} f_{D_g}(\mathcal{K}\mathcal{A})\,. \tag{4.4.45}$$

Using the relation $f_{D_g}(\mathcal{K}\mathcal{A}) + f_{\chi_g}(\mathcal{L}\mathcal{A}) = (2g-2-n)f(\mathcal{A})$ [17], we rewrite the above as

$$f_{D^{\mu\nu}}(\mathcal{K}\mathcal{A}) + \frac{1}{2}\,\eta^{\mu\nu} f_{\chi_g}(\mathcal{L}\mathcal{A}) = -\frac{1}{n!}\int_{\mathcal{K}\mathcal{A}} \langle \Omega^{[\mathcal{A}+2]g,n+1}|Q|\xi^{\mu\nu}\rangle + \frac{1}{2}\,\eta^{\mu\nu}(2g-2+n)f(\mathcal{A})\,. \tag{4.4.46}$$

By virtue of Eq. (4.2.26), we see that Eq. (4.4.46) implies the desired result (4.4.44) if

$$f_{\xi^{\mu\nu}}(\mathcal{L}\mathcal{A}) = \frac{1}{n!}\int_{\mathcal{K}\mathcal{A}} \langle \Omega^{[\mathcal{A}+2]g,n+1}|Q|\xi^{\mu\nu}\rangle\,. \tag{4.4.47}$$

We must therefore establish this equation. Our first step is to replace the BRST operator, which is only acting on the additional puncture, by a sum of BRST operators acting on all punctures. The right hand side then becomes

$$\frac{1}{n!}\int_{\mathcal{K}\mathcal{A}} \langle \Omega^{[\mathcal{A}+2]g,n+1}|\sum_{k=1}^{n+1} Q^{(k)}|\xi^{\mu\nu}\rangle - \frac{1}{n!}\int_{\mathcal{K}\mathcal{A}} \langle \Omega^{[\mathcal{A}+2]g,n+1}|\xi^{\mu\nu}\rangle \sum_{k=1}^{n} Q^{(k)}\,. \tag{4.4.48}$$

The second integral can be seen to vanish identically. It involves the motion of the insertion over fixed surfaces, and thus includes two antighost insertions that have the property of annihilating the vacuum state. Since the $\xi^{\mu\nu}$ state only has a single ghost operator acting on the vacuum, the two antighost insertions will annihilate it. The first integral is rewritten by using the BRST property of forms and Stokes's theorem

$$\frac{1}{n!}\int_{\mathcal{KA}} d\,\langle\Omega^{[\mathcal{A}+1]g,n+1}|\xi^{\mu\nu}\rangle = \frac{1}{n!}\int_{\partial(\mathcal{KA})}\langle\Omega^{[\mathcal{A}+1]g,n+1}|\xi^{\mu\nu}\rangle\,. \tag{4.4.49}$$

We now recall that $\partial(\mathcal{KA}) = \mathcal{K}(\partial\mathcal{A}) + \mathcal{LA}$. The integral over $\mathcal{K}(\partial\mathcal{A})$ vanishes for exactly the same reason as quoted in the above paragraph; two antighosts insertions for position that annihilate the state. Thus the above term is simply

$$\frac{1}{n!}\int_{\mathcal{LA}}\langle\Omega^{[\mathcal{A}+1]g,n+1}|\xi^{\mu\nu}\rangle = f_{\xi^{\mu\nu}}(\mathcal{LA})\,. \tag{4.4.50}$$

This establishes the correctness of 4.4.47, and as a consequence concludes our proof of Eq. (4.4.44).

4.5 Complete dilaton theorem

In this section we write a general hamiltonian that induces string field diffeomorphisms relevant for the dilaton theorem. Such hamiltonian will take a form similar to that of Ref. [17] and will allow us to treat in a uniform way the dilaton, the ghost-dilaton, the matter-dilaton and the graviton-trace states. We explain what kind of deformations these various states produce, and emphasize that none of them leads to a change of the slope parameter α'. We then establish the complete dilaton theorem for critical closed string field theory. The main point is that the complete dilaton state can be written as

$$|D\rangle = -Q\,|\,\xi^{\mu}_{\mu}\rangle + \frac{(d-2)}{2}|D_g\rangle\,, \tag{4.5.1}$$

and therefore in the cohomology of the extended complex the complete dilaton is just proportional to the ghost-dilaton

$$|D\rangle \approx \frac{(d-2)}{2}|D_g\rangle\,. \tag{4.5.2}$$

In the extended complex only the ghost-dilaton changes the string coupling, and the above equation indicates that the complete dilaton changes the string coupling with a proportionality factor $(d-2)$.

4.5.1 A general hamiltonian

What we have in mind here is writing a hamiltonian $\mathbf{U}$ that generates a diffeomorphism F of the string field via a canonical transformation

$$F : |\Psi\rangle \;\rightarrow\; |\Psi\rangle + dt\,\{\,|\Psi\rangle, \mathbf{U}\,\}\,. \tag{4.5.3}$$

Associated to such canonical transformation we can imagine that a parameter λ of the string measure is shifted. This is expressed as

$$F^* \left\{ d\mu(\lambda) \exp\left(\frac{2}{\hbar} S(\lambda)\right) \right\} = d\mu(\lambda + d\lambda) \exp\left(\frac{2}{\hbar} S(\lambda + d\lambda)\right) , \qquad (4.5.4)$$

namely, the diffeomorphism pulls back the relevant *measure* of the theory with parameter λ to the measure of the theory with parameter $\lambda + d\lambda$. In order to express the requirement 4.5.4 explicitly we use the following two relations [15]

$$\begin{aligned} F^*(d\mu(\lambda)) &= \tfrac{\rho(\lambda)}{\rho(\lambda+d\lambda)}\, d\mu(\lambda + d\lambda)\ (1 + 2\, dt\, \Delta\mathbf{U}) , \\ F^*\{S(\lambda)\} &= S(\lambda) + dt\, \{S(\lambda), \mathbf{U}\} , \end{aligned} \qquad (4.5.5)$$

where $d\mu(\lambda) = \rho(\lambda) \prod d\psi$. Equation (4.5.4) then reduces to

$$\left(\frac{d\lambda}{dt}\right) \cdot \frac{d}{d\lambda} \left(S + \frac{1}{2} \hbar \ln \rho \right) = \{S, \mathbf{U}\} + \hbar \Delta \mathbf{U} \equiv \hbar \Delta_S \mathbf{U} . \qquad (4.5.6)$$

If the right hand side of the above equation is zero for some specific hamiltonian $\mathbf{U}$, we must conclude that the diffeomorphism does not change anything in the string field measure. The diffeomorphism is then a symmetry transformation.

The hamiltonian we will introduce depends on a pair of states $\mathcal{O}$ and χ related by a BRST operator:

$$\mathcal{O} + \{Q, \chi\} = 0 \quad \rightarrow \quad |\mathcal{O}\rangle = -Q|\chi\rangle. \qquad (4.5.7)$$

We demand that $|\mathcal{O}\rangle \in \widehat{\mathcal{H}}$, namely $b_0^-|\mathcal{O}\rangle = L_0^-|\mathcal{O}\rangle = 0$, but $|\chi\rangle$ need not be annihilated by b_0^-, and may involve the coordinate operator. Neither state needs to be primary. We define

$$\mathbf{U}_{\mathcal{O},\chi} = \mathbf{U}_{\mathcal{O}}^{[0,2]} - f_{\mathcal{O}}(\mathcal{B}_>) + f_\chi(\mathcal{V}_{0,3} + \{\mathcal{B}_{0,3}, \mathcal{V}\}). \qquad (4.5.8)$$

Note that this hamiltonian may involve a state χ which is *not* annihilated by b_0^- since this state only appears inserted on three punctured spheres and there is no problem defining the phase of a local coordinate on such collection of surfaces (see [17]). Using this hamiltonian we will be able to treat ghost and matter dilatons in a similar fashion. We now follow [17] to compute the right hand side of Eq. (4.5.6). Since the computation is rather similar, we will be brief. The first term of the right hand side gives

$$\begin{aligned} \{S, \mathbf{U}_{\mathcal{O},\chi}\} = & \frac{1}{\kappa} \{S, \mathbf{U}_{\mathcal{O}}^{[0,2]}\} + \{S, f_\chi(\underline{\mathcal{V}}_{0,3})\} \\ & - \frac{1}{\kappa} \{S, f_{\mathcal{O}}(\mathcal{B}_>)\} + \{\, S,\ f_\chi(\{\mathcal{B}_{0,3}, \mathcal{V}\})\ \} . \end{aligned} \qquad (4.5.9)$$

Making use of the identities given in Eqs. (2.47) and (2.48) of Ref. [17] we can rewrite Eq. (4.5.9) as

$$\{S, \mathbf{U}_{\mathcal{O},\chi}\} = \frac{1}{\kappa} f_{\mathcal{O}}(\underline{\mathcal{V}}) - f_{\mathcal{O}}(\underline{\mathcal{V}}_{0,3}) - f_\chi(\{\mathcal{V}, \underline{\mathcal{V}}_{0,3}\}) + \frac{1}{\kappa} f_{\mathcal{O}}\Big(\partial\mathcal{B}_> + \{\mathcal{V}, \mathcal{B}_>\}\Big) \\ - f_{\mathcal{O}}(\{\mathcal{B}_{0,3}, \mathcal{V}\}) - f_\chi\Big(\partial\{\mathcal{B}_{0,3}, \mathcal{V}\} + \{\mathcal{V}, \{\mathcal{B}_{0,3}\,, \mathcal{V}\}\}\Big)\,, \tag{4.5.10}$$

where each row in Eq. (4.5.10) is equal to the corresponding row in Eq. (4.5.9). Using the Jacobi identity in the fourth row, the geometrical recursion relations, and the definition $\partial\mathcal{B}_{0,3} = \mathcal{V}'_{0,3} - \underline{\mathcal{V}}_{0,3}$, we obtain

$$\{S, \mathbf{U}_{\mathcal{O},\chi}\} = \frac{1}{\kappa} f_{\mathcal{O}}\Big(\partial\mathcal{B}_> + \{\mathcal{V}, \mathcal{B}\} + \underline{\mathcal{V}}_>\Big) + f_\chi\Big(\{\mathcal{B}_{0,3}, \hbar\Delta\mathcal{V}\} - \{\mathcal{V}'_{0,3}, \mathcal{V}\}\Big)\,. \tag{4.5.11}$$

A similar calculation gives

$$\hbar\Delta\mathbf{U}_{\mathcal{O},\chi} = -\,\hbar f_\chi\,(\Delta\underline{\mathcal{V}}_{0,3}) + \frac{1}{\kappa} f_{\mathcal{O}}(\hbar\Delta\mathcal{B}_>) - \; f_\chi\,(\{\mathcal{B}_{0,3}\,, \hbar\Delta\mathcal{V}\})\,. \tag{4.5.12}$$

Eqs. (4.5.12) and (4.5.11) must be added to give the right hand side of Eq. (4.5.6). Doing this, and using the recursion relations

$$\partial\mathcal{B}_> \; = \kappa\overline{\mathcal{K}}\mathcal{V} - \hbar\Delta\mathcal{B}_> - \{\mathcal{V}, \mathcal{B}\} - \underline{\mathcal{V}}_> + \hbar\kappa\,\underline{\mathcal{V}}_{1,1}\,, \tag{4.5.13}$$

for the $\mathcal{B}$ spaces we finally find

$$\boxed{\Delta_S\,\mathbf{U}_{\mathcal{O},\chi} = \; f_{\mathcal{O}}(\overline{\mathcal{K}}\mathcal{V}) + f_\chi(\overline{\mathcal{L}}\mathcal{V}) + \hbar\,[\,f_{\mathcal{O}}(\underline{\mathcal{V}}_{1,1}) - \; f_\chi(\Delta\underline{\mathcal{V}}_{0,3})\,].} \tag{4.5.14}$$

Note that by definition, the term $\overline{\mathcal{L}}\mathcal{V}$ does not include vertices with zero punctures. Writing out the above equation more explicitly we have

$$\Delta_S\,\mathbf{U}_{\mathcal{O},\chi} = \textstyle\sum_{g,n\geq 1} f_{\mathcal{O}}(\overline{\mathcal{K}}\mathcal{V}_{g,n}) + f_\chi(\overline{\mathcal{L}}\mathcal{V}_{g,n}) \\ + \textstyle\sum_{g\geq 2} f_{\mathcal{O}}(\overline{\mathcal{K}}\mathcal{V}_{g,0}) \\ + \hbar\,[\,f_{\mathcal{O}}(\underline{\mathcal{V}}_{1,1}) - \; f_\chi(\Delta\underline{\mathcal{V}}_{0,3})\,]. \tag{4.5.15}$$

4.5.2 Application to the various dilaton-like states

In this section we will use the general hamiltonian Eq. (4.5.8) and its basic property Eq. (4.5.14) to calculate the effect of shifts of the ghost dilaton D_g, the matter dilaton D^μ_μ, the true dilaton D, and the graviton trace $\mathcal{G}_t$. We begin with the case of the ghost-dilaton, fully analyzed in Refs. [17, 18], as a way to use the present general formalism.

Ghost-dilaton

Since $D_g + \{Q, \chi_g\} = 0$, we consider the hamiltonian $\mathbf{U}_{D_g,\chi_g}$ (a simpler form of the ghost-dilaton hamiltonian will be given in the next section). The identities

$$f_{D_g}(\mathcal{K}\mathcal{V}_{g,n}) + f_{\chi_g}(\mathcal{L}\mathcal{V}_{g,n}) = (2g-2+n) f(\mathcal{V}_{g,n}), \tag{4.5.16}$$

established in Ref. [17], and the identities

$$\begin{aligned} f_{D_g}(\mathcal{K}\mathcal{V}_{g,0}) &= (2g-2) f(\mathcal{V}_{g,0})\,, \quad g \geq 2\,, \\ f_{\mathcal{D}_g}(\underline{\mathcal{V}}_{1,1}) &= f_{\chi_g}(\Delta \underline{\mathcal{V}}_{0,3}) = 0\,, \end{aligned} \tag{4.5.17}$$

established in [18] imply that Eq. (4.5.15) yields

$$\Delta_S \, \mathbf{U}_{\mathcal{D}_g,\chi_g} = \sum_{g,n} \hbar^g \kappa^{2g-2+n} \, (2g-2+n) f(\mathcal{V}_{g,n})\,. \tag{4.5.18}$$

Since the string action is given by $S = S_{0,2} + f(\mathcal{V}) + \hbar S_{1,0}$ and the kinetic term $S_{0,2}$, the elementary vacuum term $S_{1,0}$, and the measure $\ln \rho$ are all coupling constant independent we can write Eq. (4.5.18) as

$$\Delta_S \, \mathbf{U}_{\mathcal{D}_g,\chi_g} = \kappa \frac{d}{d\kappa} \left(S + \frac{1}{2} \hbar \ln \rho \right)\,. \tag{4.5.19}$$

This equation shows that the ghost dilaton changes the coupling constant κ. Comparing with Eq. (4.5.6) we see that $\frac{d\kappa}{dt} = \kappa$, or $\kappa = \kappa_0 e^t$. Here t plays the role of the vacuum expectation value of the ghost-dilaton.

Matter dilaton

Since $D^{\mu\nu} + \{Q, \chi^{\mu\nu}\} = 0$, we are led to consider the hamiltonian $\mathbf{U}_{\mathcal{D}^{\mu\nu},\chi^{\mu\nu}}$. This time Eq. (4.4.44) is relevant in the form

$$f_{D^{\mu\nu}}(\mathcal{K}\mathcal{V}_{g,n}) + f_{\chi^{\mu\nu}}(\mathcal{L}\mathcal{V}_{g,n}) = \frac{1}{2} \eta^{\mu\nu} \, (2g-2+n) f(\mathcal{V}_{g,n})\,. \tag{4.5.20}$$

Moreover we claim that

$$f_{D^{\mu\nu}}(\mathcal{K}\mathcal{V}_{g,0}) = \frac{1}{2} \eta^{\mu\nu} \, (2g-2) f(\mathcal{V}_{g,0}) \quad g \geq 2. \tag{4.5.21}$$

This follows from $|D^{\mu\nu}\rangle = -Q|\xi^{\mu\nu}\rangle + \frac{1}{2}\eta^{\mu\nu}|D_g\rangle$, the first equation in Eq. (4.5.17), and the vanishing of $f_{Q\xi^{\mu\nu}}(\mathcal{K}\mathcal{V}_{g,0})$. On the other hand

$$f_{\mathcal{G}^{\mu\nu}}(\underline{\mathcal{V}}_{1,1}) - f_{\xi^{\mu\nu}}(\Delta \underline{\mathcal{V}}_{0,3}) = 0\,, \tag{4.5.22}$$

by virtue of Eq. (4.2.12) and Stokes theorem which is valid here because b_0^- annihilates the state $\xi^{\mu\nu}$. Since $D^{\mu\nu} = \mathcal{G}^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu} D_g$, it now follows that

$$\begin{aligned} f_{D^{\mu\nu}}(\mathcal{V}_{1,1}) &= f_{\mathcal{G}^{\mu\nu}}(\mathcal{V}_{1,1}) + \tfrac{1}{2}\eta^{\mu\nu} f_{D_g}(\mathcal{V}_{1,1}) \\ &= f_{\xi^{\mu\nu}}(\Delta\underline{\mathcal{V}}_{0,3}) \qquad \text{(Using Eqs. (4.5.17) and (4.5.22))} \\ &= f_{\chi^{\mu\nu}}(\Delta\underline{\mathcal{V}}_{0,3}) - \tfrac{1}{2}\eta^{\mu\nu} f_{\chi_g}(\Delta\underline{\mathcal{V}}_{0,3}) \qquad \text{(Using Eq. (4.2.26))} \\ &= f_{\chi^{\mu\nu}}(\Delta\underline{\mathcal{V}}_{0,3}) \qquad \text{(Using Eq. (4.5.17))} \end{aligned} \tag{4.5.23}$$

and therefore

$$f_{D^{\mu\nu}}(\mathcal{V}_{1,1}) - f_{\chi^{\mu\nu}}(\Delta\underline{\mathcal{V}}_{0,3}) = 0\,. \tag{4.5.24}$$

The computation of $\Delta_S \mathbf{U}_{D^{\mu\nu},\chi^{\mu\nu}}$ is now straightforward. The terms in the right hand side of Eq. (4.5.15) have been evaluated in Eqs. (4.5.20), (4.5.21), and (4.5.24). We then find

$$\Delta_S\, \mathbf{U}_{D^{\mu\nu},\chi^{\mu\nu}} = \frac{1}{2}\eta^{\mu\nu} \cdot \kappa \frac{d}{d\kappa}\left(S + \frac{1}{2}\hbar \ln \rho\right)\,. \tag{4.5.25}$$

Note that the off-diagonal states ($\mu \neq \nu$) have no effect whatsoever, they ought to be interpreted as generating gauge transformations. Each one of the d diagonal states change the coupling constant. In particular, for the trace state we have

$$\Delta_S\, \mathbf{U}_{D^{\mu}_{\mu},\chi^{\mu}_{\mu}} = \frac{d}{2} \cdot \kappa \frac{d}{d\kappa}\left(S + \frac{1}{2}\hbar \ln \rho\right)\,. \tag{4.5.26}$$

The only effect of a shift of the matter dilaton D^{μ}_{μ} is a shift of the string coupling, with a strength proportional to the number of noncompact dimensions.

Complete dilaton

This state is written as $D = D^{\mu}_{\mu} - D_g = -\{Q\,,\,\chi^{\mu}_{\mu} - \chi_g\}$. Therefore

$$\Delta_S \mathbf{U}_{D,\chi_d} = \Delta_S \mathbf{U}_{D^{\mu}_{\mu},\chi^{\mu}_{\mu}} - \Delta_S \mathbf{U}_{D_g,\chi_g} = \left(\frac{d-2}{2}\right) \cdot \kappa \frac{d}{d\kappa}\left(S + \frac{1}{2}\hbar \ln \rho\right)\,, \tag{4.5.27}$$

where use was made of Eqs. (4.5.26) and (4.5.19). This equation shows that the only effect of a dilaton shift is changing the string coupling with a strength proportional to the number of noncompact dimensions minus two. This is the complete dilaton theorem in critical bosonic closed string field theory.

It is sometimes said that the effect of a dilaton is to change the dimensionless string coupling *and* the slope parameter α'. We believe such statements are at best misleading. We see in the above discussion that $\Delta_S\mathbf{U}$ amounts to just changing the dimensionless string coupling. The slope parameter is the only dimensionful parameter in the string theory, and, as such, it is not really a parameter but a choice of units. There is no invariant meaning to a change in α'. If the theory had another dimensionful parameter, say a compactification radius R, then there is a new dimensionless ratio $R/\sqrt{\alpha'}$ that can be changed in the theory. One can view such change, if so desired, as a change of the dimensionful radius of compactification, or equivalently, as a change in α'. Still, it should be remembered that only changes in dimensionless couplings have invariant meaning.

Graviton trace

The final case of interest is that of the states $\mathcal{G}^{\mu\nu}$ written as

$$\mathcal{G}^{\mu\nu} = D^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} D_g = -\{Q\,,\,\chi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\chi_g\}\,. \tag{4.5.28}$$

Therefore

$$\Delta_S \mathbf{U}_{\mathcal{G}^{\mu\nu},\xi^{\mu\nu}} = \Delta_S \mathbf{U}_{D^{\mu\nu},\chi^{\mu\nu}} - \frac{1}{2}\eta^{\mu\nu}\Delta_S \mathbf{U}_{D_g,\chi_g} = 0\,, \tag{4.5.29}$$

where use was made of Eqs. (4.5.25) and (4.5.19). This shows that none of the $\mathcal{G}^{\mu\nu}$ states deforms the string background. In particular, the graviton trace $\mathcal{G}$ does not deform the string background. Note that the hamiltonian $\mathbf{U}_{\mathcal{G}^{\mu\nu},\xi^{\mu\nu}}$ defines a string field transformation whose inhomogeneous term is indeed $\mathcal{G}^{\mu\nu}$, and leaves the string measure invariant. Our discussion of the extended complex indicates that a completely equivalent field transformation would be a gauge transformation generatred by $\xi^{\mu\nu}$.

4.6 The relevance of the ghost-dilaton

In this section we wish to consider the case when the ghost-dilaton becomes a trivial state in the standard semirelative BRST cohomology. This is not a hypothetical situation, it happens in $D = 2$ string theory, as will be reviewed in sect.8. If the ghost dilaton is trivial it may seem that it cannot change the coupling constant, leaving the possibility that other states change it. This is not the way things work out. Inspection of Ref. [17] reveals that the ghost-dilaton shifts the string coupling whether or not it is BRST trivial. It then seems clear that the string coupling should not be an observable. We give a proof that this is the case. Explicitly this means the following: while the string coupling is a parameter appearing in the string action its value can be changed by a string field redefinition having no inhomogeneous term.

If the ghost dilaton is trivial, it can be written as $|D_g\rangle = -Q|\eta\rangle$ with $|\eta\rangle$ a legal state in the standard semirelative complex. If the two-form associated to the motions of the state $|\eta\rangle$ vanishes, as is the case for $D = 2$ strings, the field redefinition changing the string coupling is simply a homogeneous field redefinition. This will be the case whenever $|\eta\rangle$ involves a single ghost field acting on the vacuum. If this is not the case, the field redefinition is nonlinear; while it lacks an inhomogeneous term at the classical level, there may be $\hbar$-dependent inhomogeneous terms. We do not know of an example where $|\eta\rangle$ is this complicated. If this happens the string coupling might not be completely unphysical at the quantum level.

4.6.1 Simplifying the ghost-dilaton hamiltonian

In this subsection we examine again the ghost-dilaton hamiltonian and show that it can be simplified considerably. The simplified hamiltonian will be of utility to show that a trivial ghost-dilaton implies an unphysical coupling constant.

The ghost-dilaton hamiltonian reads

$$\mathbf{U}_{D_g} = \mathbf{U}_{D_g}^{[0,2]} - f_{D_g}(\mathcal{B}_>) + f_{\chi_g}(\mathcal{V}_{0,3} + \{\mathcal{B}_{0,3}, \mathcal{V}\}). \tag{4.6.1}$$

We now show that the last term in this hamiltonian, involving χ_g can be replaced by a simpler term involving the ghost-dilaton. Consider the evaluation of $\Delta_S f_{\chi_g}(\mathcal{B}_{0,3})$

$$\begin{aligned} \Delta_S f_{\chi_g}(\mathcal{B}_{0,3}) &= \Delta f_{\chi_g}(\mathcal{B}_{0,3}) + \{S\,,\, f_{\chi_g}(\mathcal{B}_{0,3})\}\,, \\ &= -f_{\chi_g}(\Delta\mathcal{B}_{0,3}) - f_{Q\chi_g}(\mathcal{B}_{0,3}) - f_{\chi_g}(\partial\mathcal{B}_{0,3} + \{\mathcal{V}\,,\, \mathcal{B}_{0,3}\}\,)\,, \\ &= -f_{\chi_g}(\Delta\mathcal{B}_{0,3}) + f_{D_g}(\mathcal{B}_{0,3}) + f_{\chi_g}(\mathcal{V}_{0,3} + \{\mathcal{B}_{0,3}\,,\, \mathcal{V}\}\,)\,, \end{aligned} \tag{4.6.2}$$

where we made use of the relation $f_{\chi_g}(\mathcal{V}'_{0,3}) = 0$, which follows from sect.6.2 of Ref. [17]. Rearranging the terms in the equation we write

$$f_{\chi_g}(\mathcal{V}_{0,3} + \{\mathcal{B}_{0,3}\,,\, \mathcal{V}\}\,) \;=\; -f_{D_g}(\mathcal{B}_{0,3}) \;+\; f_{\chi_g}(\Delta\mathcal{B}_{0,3}) \;+\; \Delta_S f_{\chi_g}(\mathcal{B}_{0,3})\,. \tag{4.6.3}$$

Constant terms are irrelevant for hamiltonians since they do not generate transformations. We can therefore drop the second term in the above right hand side. Moreover, the third term in the right hand side can also be dropped since it is annihilated by Δ_S. It follows that we can replace $f_{\chi_g}(\mathcal{V}_{0,3}+\{\mathcal{B}_{0,3}\,,\, \mathcal{V}\}\,)$ by $(-f_{D_g}(\mathcal{B}_{0,3}))$ in the dilaton hamiltonian. We thus find that

$$\mathbf{U}_{D_g} = \mathbf{U}_{D_g}^{[0,2]} - f_{D_g}(\mathcal{B})\,, \tag{4.6.4}$$

is a hamiltonian equivalent to the original one, and by a slight abuse of notation we denote it with the same symbol. This hamiltonian is completely analogous to the background independence hamiltonians found in Ref. [15]. It is straightforward to verify that this hamiltonian has the desired properties. One computes

$$\begin{aligned} \Delta_S \mathbf{U}_{D_g} &= \{S, \mathbf{U}_{D_g}^{[0,2]}\} - \Delta f_{D_g}(\mathcal{B}) - \{S, f_{D_g}(\mathcal{B})\} \\ &= f_{D_g}\Big(\partial\mathcal{B} + \{\mathcal{V}\,,\, \mathcal{B}\} + \Delta\mathcal{B} \,+\, \underline{\mathcal{V}}\Big) \\ &= f_{D_g}\Big(\overline{\mathcal{K}}\mathcal{V} + \mathcal{V}'_{0,3} + \Delta\mathcal{B}_{0,3} \,+\, \underline{\mathcal{V}}_{1,1}\Big) \\ &= f_{D_g}(\overline{\mathcal{K}}\mathcal{V})\,, \end{aligned} \tag{4.6.5}$$

where use was made of the recursion relations (4.5.13) together with $\partial\mathcal{B}_{0,3} = \mathcal{V}'_{0,3} - \mathcal{V}_{0,3}$. In the last step we used $f_{D_g}(\mathcal{V}'_{0,3}) = 0$, which follows from $f_{\chi_g}(\mathcal{V}'_{0,3}) = 0$ and the BRST property, and, of the result of [18] that the two-form associated to the ghost-dilaton vanishes identically on the moduli space of once punctured tori. The fact that the ghost-dilaton hamiltonian can be written in the standard background independence form was not anticipated in [17] because both the identity $f_{D_g}(\mathcal{V}'_{0,3}) = 0$, and the understanding of the behavior of dilatons at genus one were missing.

4.6.2 Triviality of the ghost-dilaton and the string coupling

The ghost-dilaton theorem in string field theory, as established in Refs. [17,18] holds true whether or not the ghost-dilaton is BRST trivial or not. The ghost-dilaton hamiltonian will always have the effect of changing the string coupling. This ghost-hamiltonian produces a string field redefinition that includes an inhomogeneous term, a shift along the ghost-dilaton state. If the ghost-dilaton is trivial in the standard semirelative BRST cohomology, then it can be written as $|D_g\rangle = -Q|\eta\rangle$, where $|\eta\rangle$ is a standard vector in the closed string field theory state space. It then follows that there is another hamiltonian, the hamiltonian corresponding to a gauge transformation, that also has the property of inducing a shift along the direction of the ghost-dilaton. This gauge hamiltonian $\mathbf{U}_G$ reads

$$\mathbf{U}_G = \Delta_S \mathbf{U}_\eta^{[0,2]} = \mathbf{U}_{D_g}^{[0,2]} + f_\eta(\mathcal{V}) \,, \tag{4.6.6}$$

where the gauge invariance property follows from $\Delta_S \mathbf{U}_G = 0$. Since the gauge hamiltonian induces no change in the string action, it follows that the hamiltonian

$$\mathbf{U}_F \equiv \mathbf{U}_{D_g} - \mathbf{U}_G = -f_{D_g}(\mathcal{B}) - f_\eta(\mathcal{V}) \,, \tag{4.6.7}$$

still shifts the string coupling constant. The term $\mathbf{U}_{D_g}^{[0,2]}$ inducing the shift along the ghost dilaton is absent in $\mathbf{U}_F$ and therefore the hamiltonian $\mathbf{U}_F$ is a hamiltonian that induces a field redefinition without physical import; it does not change the vacuum expectation value of the string field. The fact that the string coupling parameter in the action can be changed by a field redefinition without an inhomogeneous term implies that the string coupling is unphysical. As we will see now, strictly, and in all generality, this is only the case for genus zero, or the $\hbar$-independent part of the string field redefinition generated by $\mathbf{U}_F$. In order to appreciate this point we now obtain a simple expression for the hamiltonian $\mathbf{U}_F$.

Our calculation begins by simplifying the expression for $f_{D_g}(\mathcal{B})$ in the ghost-dilaton hamiltonian by taking into account that $|D_g\rangle = -Q|\eta\rangle$. We consider

$$\{S \,,\, f_\eta(\mathcal{B})\} = -f_{Q\eta}(\mathcal{B}) \;-\; f_\eta\left(\partial\mathcal{B} + \{\mathcal{V} \,,\, \mathcal{B}\}\right). \tag{4.6.8}$$

Using $\Delta f_\eta(\mathcal{B}) = -f_\eta(\Delta\mathcal{B})$, we rewrite the above equation as

$$-f_{D_g}(\mathcal{B}) = -f_\eta\left(\partial\mathcal{B} + \{\mathcal{V} \,,\, \mathcal{B}\} + \Delta\mathcal{B}\right) - \Delta_S\, f_\eta(\mathcal{B}) \,. \tag{4.6.9}$$

Using the recursion relations (4.5.13) we find

$$-f_{D_g}(\mathcal{B}) = -f_\eta\left(\mathcal{V}'_{0,3} + \overline{\mathcal{K}}\mathcal{V} - \underline{\mathcal{V}}\right) - f_\eta\left(\Delta\mathcal{B}_{0,3} + \mathcal{V}_{1,1}\right) - \Delta_S\, f_\eta(\mathcal{B}) \,. \tag{4.6.10}$$

Since the left hand side is to be used in a hamiltonian we can drop the second term, being a constant, and the last term, being annihilated by Δ_S. It follows that back in Eq. (4.6.7) the hamiltonian $\mathbf{U}_F$ can be taken to be

$$\mathbf{U}_F = -f_\eta\,(\mathcal{V}'_{0,3} + \overline{\mathcal{K}}\mathcal{V})\,. \tag{4.6.11}$$

This is the simplest form of the hamiltonian. We see that at genus zero the hamiltonian is quadratic, and thus generates a field redefinition without an inhomogeneous term. Thus, at genus zero it is completely clear that the string coupling is unphysical. At higher genus there are, in principle, non-vanishing inhomogeneous terms arising from the surfaces $\overline{\mathcal{K}}\mathcal{V}_{g,1}$. This might mean that the string coupling is not fully unphysical at the quantum level. More likely, it may be that whenever the ghost-dilaton is trivial the two-form corresponding to the motion of the state $|\eta\rangle$ vanishes. This happens, for example, when each term in $|\eta\rangle$ only has one ghost field acting on the vacuum, as is the case in $D=2$ string theory. If the two-form vanishes the contribution from $\overline{\mathcal{K}}\mathcal{V}$ vanishes as well, and the hamiltonian $\mathbf{U}_F = -f_\eta\,(\mathcal{V}'_{0,3})$ simply generates a homogeneous field redefinition

$$\delta\,|\Psi\rangle_1 = -\{f_\eta\,(\mathcal{V}'_{0,3})\,,\,|\Psi\rangle_1\} = \Big[\,\langle V'_{1'23}\,|\eta\rangle_3|\mathcal{S}_{11'}\rangle\,\Big]\,|\Psi\rangle_2\,. \tag{4.6.12}$$

The linear operator acting on the string field has the interpretation of a contour integral of the conformal field operator $\eta(z,\bar z)$ using local coordinates induced by the special puncture in the string vertex $\mathcal{V}'_{0,3}$.

It is a straightforward calculation using $\overline{\mathcal{L}} = -\{\mathcal{V}'_{0,3}\,,\mathcal{V}\}$ to show that acting on the first part of the hamiltonian (4.6.11) the operator Δ_S gives

$$\begin{aligned}\Delta_S\Big(-f_\eta\,(\mathcal{V}'_{0,3})\Big) &= f_{D_g}(\mathcal{V}'_{0,3}) + f_\eta(\Delta\mathcal{V}'_{0,3}) - f_\eta(\overline{\mathcal{L}}\mathcal{V})\,,\\ &= -f_\eta\,(\overline{\mathcal{L}}\mathcal{V})\,.\end{aligned} \tag{4.6.13}$$

In the last step we have used the vanishing of $f_\eta(\Delta\mathcal{V}'_{0,3})$ which is readily established

$$\begin{aligned}f_\eta\,(\Delta\mathcal{V}'_{0,3}) &= f_\eta\,(\Delta\mathcal{V}_{0,3} - \partial\Delta\mathcal{B}_{0,3})\\ &= f_\eta\,(-\partial\,[\mathcal{V}_{1,1} - \Delta\mathcal{B}_{0,3}\,])\\ &= f_{D_g}\,(\mathcal{V}_{1,1} - \Delta\mathcal{B}_{0,3}) = 0\,.\end{aligned} \tag{4.6.14}$$

Using once more the vanishing of the ghost-dilaton two-form on the moduli space of punctured tori. It follows from Eqs. (4.6.13) and (4.6.11) that in general

$$\Delta_S\,\mathbf{U}_F = -f_\eta\,(\overline{\mathcal{L}}\mathcal{V}) - \Delta_S f_\eta\,(\overline{\mathcal{K}}\mathcal{V})\,. \tag{4.6.15}$$

Whenever the two-form associated with moving the state $|\eta\rangle$ vanishes we see that the effect of $\mathbf{U}_F$ reduces to inserting the state η (via $\overline{\mathcal{L}}$) on all the coordinate disks of the string vertices. One can readily verify that for an arbitrary $|\eta\rangle$ the relation $\Delta_S\,f_\eta\,(\overline{\mathcal{K}}\mathcal{V}) = f_{Q\eta}\,(\overline{\mathcal{K}}\mathcal{V}) - f_\eta(\overline{\mathcal{L}}\mathcal{V})$ holds. This confirms that 4.6.15 is equivalent to $\Delta_S\,\mathbf{U}_F = f_{D_g}(\overline{\mathcal{K}}\mathcal{V})$ as befits a hamiltonian that must change the string coupling.

4.7 Application to $D = 2$ string theory

In this section we consider the case of $D = 2$ string theory as an illustration of the ideas developed in the previous sections of this paper. We analyze zero-momentum physical states that are candidates for deformations of the string background. The semirelative BRST-physical states are seen to be trivial in the extended complex and should not deform the string background. We discuss why they do not appear to deform the conformal theory. Our analysis here is a refinement of that of Mahapatra, Mukherji and Sengupta [40]. We argue that states in the absolute cohomology that are not annihilated by b_0^- do not lead to CFT deformations, thus evading a possible conflict with background independence. Finally, noting that in this background the ghost-dilaton is BRST trivial, we explain how to apply our earlier considerations showing that the string coupling is unobservable.

4.7.1 Zero-momentum states and CFT deformations

Two dimensional string theory is based on a matter CFT including a Liouville field $\varphi(z, \bar{z})$ and a field $X(z, \bar{z})$. The holomorphic matter energy-momentum tensor reads $T_m = -\frac{1}{2}\,\partial X \partial X - \frac{1}{2}\partial\varphi\partial\varphi + \sqrt{2}\,\partial^2\varphi$. Due to the background charge, the field φ does not transform as a scalar. Under general analytic coordinate changes $z'(z)$

$$\varphi(z', \overline{z'}) = \varphi(z, \bar{z}) - 2\sqrt{2}\ln\left|\frac{dz'}{dz}\right|. \tag{4.7.1}$$

In this string theory an important operator $a(z)$ [77, 78] is obtained by taking the commutator of the BRST operator with the field φ

$$a(z) \equiv \frac{1}{\sqrt{2}}\{Q\,,\,\varphi(z, \bar{z})\} = \partial c + \frac{1}{\sqrt{2}}\,c\,\partial\varphi\,. \tag{4.7.2}$$

The operator $a(z)$ is trivial in semirelative cohomology of the extended complex, but it is nontrivial in the standard semirelative cohomology.

Let us consider the absolute cohomology BRST physical states at ghost number two that can be formed without using exponentials of the free fields. The space of such states is spanned by a total of six states [78], the first three of which are states in the semirelative cohomology

$$\begin{aligned}
\mathcal{S}_1 &= \tfrac{1}{\sqrt{2}}\,c\bar{c}\,\partial X\bar{\partial}\varphi + c\partial X(\partial c + \bar{\partial}\bar{c})\,,\\
\mathcal{S}_2 &= \tfrac{1}{\sqrt{2}}\,c\bar{c}\,\bar{\partial}X\partial\varphi + \bar{c}\bar{\partial}X(\partial c + \bar{\partial}\bar{c})\,,\\
\mathcal{S}_3 &= c\bar{c}\partial X\bar{\partial}X\,,
\end{aligned} \tag{4.7.3}$$

and the other three are states that do not obey the semirelative condition

$$
\begin{aligned}
\mathcal{A}_1 &= \tfrac{1}{\sqrt{2}}\, c\bar{c}\,\partial X \bar{\partial}\varphi + c\partial X(\bar{\partial}\bar{c} - \partial c)\,, \\
\mathcal{A}_2 &= \tfrac{1}{\sqrt{2}}\, c\bar{c}\,\bar{\partial} X \partial\varphi + \bar{c}\bar{\partial} X(\partial c - \bar{\partial}\bar{c})\,, \\
\mathcal{A}_3 &= a \cdot \bar{a}\,.
\end{aligned}
\tag{4.7.4}
$$

The states in (4.7.3) are the only states in the semirelative cohomology at ghost number two under the condition of zero momenta [78].

Let us begin our analysis with the first two semirelative states. We first observe that they are trivial in in the extended complex

$$
\mathcal{S}_1(z,\bar{z}) = \Big\{Q\,,\, s_1\Big\}\,, \quad s_1 = -\frac{1}{\sqrt{2}}\, c\,\varphi\partial X\,, \tag{4.7.5}
$$

with an exactly analogous statement holding for $\mathcal{S}_2$. This indicates that such states do not deform the string background. Indeed, since the BRST property holds for the class of states containing φ, we may simply use the state s_1 as the gauge parameter in a string field gauge transformation.

As further confirmation that the string background is not changed, let us now see that if we try to use the state $\mathcal{S}_1$ to deform the underlying CFT the only possibility seems to be zero deformation. In order to use $\mathcal{S}_1$ to deform the CFT we must find the corresponding coordinate invariant two-form. We introduce a metric ρ on the Riemann surface and a brief computation using [17] gives

$$
\mathcal{S}_1^{[2]} = -\frac{1}{\sqrt{2}}\, dz\wedge d\bar{z}\,\Big[\partial X\,\bar{\partial}\,(\varphi - 2\sqrt{2}\ln\rho)\Big]\,. \tag{4.7.6}
$$

We see that only the first term in $\mathcal{S}_1$ has contributed to the result. Moreover, φ has been replaced by the coordinate invariant combination $(\varphi - 2\sqrt{2}\ln\rho)$. While the two-form is well-defined (it is coordinate invariant), it is not Weyl invariant. The difficulty of obtaining a coordinate invariant and Weyl invariant two-form was pointed out in [78]. At that time it was not clear how to obtain the two-form associated to *arbitrary* states. We now see that there is no Weyl-invariant two-form and thus no well-defined way to integrate the two-form on Riemann surfaces.

The analysis cannot stop here. A similar situation happens for the ghost-dilaton: its two-form is not Weyl independent. This dependence drops out of surface integrals when we add the integral of a suitable one-form over the boundary of the region of integration. That one-form is the one associated to the state $|\chi_g\rangle$ which acted by the BRST operator gives the ghost-dilaton. We therefore consider the state s_1 defined in (4.7.5) and construct the corresponding one-form. We find

$$
s_1^{[1]} = \frac{1}{\sqrt{2}}\, dz\,(\varphi - 2\sqrt{2}\ln\rho)\partial X\,. \tag{4.7.7}
$$

This one-form is coordinate invariant but it is not Weyl invariant. One readily verifies that $\mathcal{S}_1^{[2]} = d\, s_1^{[1]}$, as expected from 4.7.5. In the case of the ghost-dilaton the gauge parameter is not annihilated by b_0^-, and the one-form, sensitive to the phase of the local coordinates, is difficult to define globally. This time the state is annihilated by b_0^- and the one-form is phase independent. If we attempt to use the one-form to cancel out the Weyl dependence of the integral of the two-form, the relation $\mathcal{S}_1^{[2]} = d\, s_1^{[1]}$ holding globally, and Stokes theorem will imply that we get a total result of zero. This concludes our plausibility argument for the absence of a nontrivial CFT deformation induced by the semirelative states $\mathcal{S}_1$ and $\mathcal{S}_2$. The analysis of $\mathcal{S}_3$ will be done shortly.

Consider now the states that are not annihilated by b_0^-. These states, being outside the closed string state space, do not correspond to linearized solutions of the string equations of motion, and therefore do not represent deformations of string backgrounds. One can ask if such states can deform the CFT. If this were the case we would have a problem with background independence; we would have two nearby conformal theories giving rise to two string theories that are not related by a shift of the string field. As we will explain now we believe it is unlikely that states which are not annihilated by b_0^- define CFT deformations.

Consider the first state listed in (4.7.4). The associated two-form is found to be given by

$$\mathcal{A}_1^{[2]} = -\frac{1}{\sqrt{2}}\, dz \wedge d\bar{z} \left[\partial X\, \bar{\partial}\, (\varphi - 2\sqrt{2}\, i\theta) \right], \tag{4.7.8}$$

where θ is the phase of the quantity $a(z_0, \bar{z}_0)$ appearing in the definition of the local coordinate: $z - z_0 = a(z_0, \bar{z}_0) w + \mathcal{O}(w^2)$. The deformation induced by this state of the CFT partition function on a fixed surface would be given by integrating the above two-form over the complete surface. Due to the non-zero Euler number, the phase of the local coordinate cannot in general be defined globally throughout the surface and the integral is not well defined. It seems very unlikely that one can define a nontrivial CFT deformation using the states in the absolute cohomology that are not annihilated by b_0^-.

4.7.2 The coupling constant in $D = 2$ strings

The ghost-dilaton, always trivial in absolute cohomology, becomes trivial in semirelative cohomology for the background defining $D = 2$ string theory. Indeed, one readily verifies that

$$c\partial^2 c - \bar{c}\bar{\partial}^2 \bar{c} = -\frac{1}{\sqrt{2}} \left\{ Q\, , \, c\partial\varphi - \bar{c}\bar{\partial}\varphi \right\}. \tag{4.7.9}$$

Note that $c\partial\varphi - \bar{c}\bar{\partial}\varphi$ is fully legal; it is a state in the standard semirelative complex. Not only is the ghost-dilaton absent in $D = 2$ string theory, but now the last semirelative state $\mathcal{S}_3 = c\bar{c}\partial X \bar{\partial} X$, is recognized to be trivial in the extended complex. This is because $\mathcal{S}_3$ is equivalent to the ghost-dilaton in the extended complex.

Let us see explicitly why the string coupling is unobservable in this background. A change of coupling constant in string field theory amounts to scaling the string forms as

$$\langle \Omega^{[d]g,n} | \to \langle \Omega^{[d]g,n} | \Big(1 - \epsilon\, [2 - 2g - n] \Big), \tag{4.7.10}$$

where d is the degree of the form. On the other hand in $D = 2$ strings

$$\langle \Sigma_{g,n} \,|\, \sum_{i=1}^{n} \oint \frac{dz}{2\pi i} J^{(i)}(z) = -2\sqrt{2}\,(2-2g)\,\langle \Sigma_{g,n} \,|\, , \quad J^{(i)}(z) = \partial \varphi^{(i)}(z)\,. \qquad (4.7.11)$$

Since this current has no ghost dependence, an identical relation holds for string forms. Since one can always add to the above right hand side a contribution proportional to n by adding a constant to the charge associated to J, we see that the deformation (4.7.10) can be implemented by a similarity transformation induced by J. This means that a homogeneous string field redefinition changes the coupling constant making it unobservable. Indeed, the background we are considering, called the linear dilaton vacuum, has a coordinate dependent string coupling. The coupling is not observable since a shift of coupling is equivalent to a translation along the φ coordinate.

Chapter 5

Cohomology of the extended complex

5.1 Motivations for using extended complex

As it we argued in the previous chapter, the gauge parameters that include the zero mode of the X^μ operator have to be considered in order to prove the complete dilaton theorems. If we allow X^μ to appear in gauge parameters, it is natural to allow it to appear in the physical states as well. This calls for an extended version of the BRST complex where the zero mode of X^μ or, in other words, the string center of mass operator x_0^μ is well defined.

We will define the extended complex simply as a tensor product of the BRST complex with the space of polynomials of $D = 26$ variables. Our main objective is to calculate the semi-relative cohomology of this complex.

When we add new vectors to a complex, two phenomena may occur in cohomology. First, some vectors that used to represent nontrivial cohomology classes may become trivial, and second, some new cohomology states may appear. Our original motivation to use the extended complex was that the graviton trace $\mathcal{G}$, BRST-physical state, was trivial in the extended complex. As we will see, the extended complex provides many more examples of this kind. We will show that only one out of D^2+1 ghost number two zero-momentum BRST-physical states remains non-trivial in the extended complex. In the BRST complex, the spectrum of non-zero momentum physical states is doubled due to the presence of the ghost zero modes. We will see that there is no such doubling in the extended complex: only ghost number two states survive and all the ghost number three states become trivial[1]. Returning to the second phenomenon, the appearance of new physical states, we will be able to show that all such states can be obtained from the old ones by differentiation with respect to the continuous momentum parameter along the corresponding mass shells. In this sense no new

[1]The observation that ghost number three states are physical only in the case of finite space time volume was made in the book by Green, Schwarz, and Witten [79]. In the finite space time volume the momentum is discrete and $x_0^\mu = -i\partial/\partial p_\mu$ cannot be defined. A suggestions that a proper incorporation of the string venter of mass variables should remove the doubling has been made by M. Henneaux [80]

physical state will appear.

Zero momentum states will require special attention and the calculation of the cohomology of the zero momentum extended complex is technically the most difficult part of this work. We will find that the ghost number one discrete states at zero momentum correctly describe the global symmetries (Poincaré group) of the background.

The analysis presented below shows that the semi-relative cohomology of the extended complex correctly describes the physics of the closed bosonic string around the flat $D = 26$ background. The arguments that lead to this conclusion are the following:

1. Ghost number two non-discrete physical states are the same as in the BRST complex up to infinitesimal Lorentz transformations. In this sense the extended complex is as good as BRST (section 5.5.1).

2. There is no doubling of the physical states,—ghost number three BRST-physical states are trivial in the extended complex (section 5.4.2).

3. There is only one zero-momentum physical state at ghost number two, which can be represented by the ghost dilaton (section 5.4.3 and 4.2).

4. Ghost number one discrete states are in one to one correspondence with the generators of the Poincaré group (section 5.4.3).

This chapter is organized as follows. In section 5.2 we start by describing the extended complex and the nilpotent operator $\widehat{Q}$. We define a cohomology problem for the extended complex and explain what we are going to learn about its structure. In section 5.3 we formulate a simplified version of the problem in which we replace the closed string BRST complex by its chiral part. Section 5.3 is devoted to a detailed analysis of the cohomology of this complex. In section 5.4 we investigate the (semi-relative) cohomology of the full extended complex using the same methods as in section 5.3. At the end of section 5.3 and section 5.4 we will formulate two theorems which summarize our results on the structure of the cohomology of the chiral and the full extended complexes. We present a detailed analysis of the Lorentz group action on the cohomology of the extended complex in section 5.5. For the sake of completeness we add two appendices: B, where we review some basic algebraic facts which we use to calculate the cohomology and C, where we prove the results which are necessary to calculate the cohomology at zero momentum.

5.2 Definition of extended complex

When we describe a string propagating in flat uncompactified background, it seems natural to let the string center of mass operator x_0 act on the state space of the theory. This operator must satisfy the Heisenberg commutation relation

$$[\alpha_n^\mu, x_0^\nu] = -i\, \delta_{n,0}\, \eta^{\mu\nu}, \tag{5.2.1}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. In order to incorporate an operator with such properties we define an extended Fock space as a tensor product of an ordinary Fock space with the space of polynomials of D variables:

$$\widehat{\mathcal{F}}_p(\alpha,\overline{\alpha}) = \mathbb{C}[x^0,\ldots,x^{D-1}] \otimes \mathcal{F}_p(\alpha,\overline{\alpha}). \tag{5.2.2}$$

All operators except α_0^μ act only on the second factor which is an ordinary Fock space, x_0^μ operators act by multiplying the polynomials by the corresponding x^μ and, finally, the action of α_0 is defined by

$$\alpha_0^\mu = 1 \otimes p^\mu - i\eta^{\mu\nu}\frac{\partial}{\partial x^\nu} \otimes 1. \tag{5.2.3}$$

Note that in the extended Fock module the action of α_0^μ does not reduce to the multiplication by p^μ.

The extended Fock module is also a module over Virasoro algebra or, strictly speaking, over a tensor product of two Virasoro algebras corresponding to the left and right moving modes. The generators are given by the usual formulae

$$\begin{aligned} L_n &= \frac{1}{2}\sum_m \eta_{\mu\nu} :\alpha_m^\mu\, \alpha_{n-m}^\nu:, \\ \overline{L}_n &= \tfrac{1}{2}\textstyle\sum_m \eta_{\mu\nu} :\overline{\alpha}_m^\mu\, \overline{\alpha}_{n-m}^\nu: . \end{aligned} \tag{5.2.4}$$

The central charge of the extended Fock module is 26, the same as that of $\mathcal{F}_p(\alpha,\overline{\alpha})$, and we can use it to construct a complex with a nilpotent operator $\widehat{Q}$ (see refs. [79,81]). Following the standard procedure we define the extended complex as a tensor product of the extended Fock module with the ghost module $\mathcal{F}(b,c,\overline{b},\overline{c})$

$$\widehat{V}_p = \mathcal{F}(b,c,\overline{b},\overline{c}) \otimes \widehat{\mathcal{F}}_p(\alpha,\overline{\alpha}) \tag{5.2.5}$$

and introduce the nilpotent operator $\widehat{Q}$ as

$$\widehat{Q} = \sum_n c_n \widehat{L}_{-n} - \frac{1}{2}\sum_{m,n}(m-n) :c_{-m}c_{-n}b_{m+n}: + \text{a.h.}, \tag{5.2.6}$$

where we put a hat over Virasoro generators in order to emphasize that they are acting on the extended Fock space.

We can alternatively describe the extended complex as a tensor product of the BRST complex V_p with the space of polynomials $\mathbb{C}[x^0 \cdots x^{D-1}]$:

$$\widehat{V}_p = \mathbb{C}[x^0 \cdots x^{D-1}] \otimes V_p, \tag{5.2.7}$$

and express the nilpotent operator $\widehat{Q}$ in terms of the BRST operator Q as follows

$$\widehat{Q} = 1 \otimes Q - i\frac{\partial}{\partial x^\mu} \otimes \sum_n (c_n\alpha_{-n}^\mu + \overline{c}_n\overline{\alpha}_{-n}^\mu) - \Box \otimes c_0^+, \tag{5.2.8}$$

where $c_0^+ = (c_0 + \overline{c}_0)/2$, and $\Box = \eta^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\mu}$.

So far we have constructed an extended complex $\widehat{V}_p = \mathbb{C}[x^0 \cdots x^{D-1}] \otimes V_p$ equipped with a nilpotent operator $\widehat{Q}$ given by Eq. (5.2.8). One can easily check that Eq. (5.2.8) does define a nilpotent operator using $Q^2 = 0$ and commutation relations between α_n^μ operators.

5.2.1 Cohomology of extended complex

Our major goal is to calculate the cohomology of the operator $\widehat{Q}$ acting on the extended complex $\widehat{V}_p$ for different values of the momentum p. More precisely, we will be looking for the so-called semi-relative cohomology, which is the space of vectors annihilated by $\widehat{Q}$ and $b_0^- = b_0 - \bar{b}_0$ modulo the image of $\widehat{Q}$ acting on the vectors annihilated by b_0^- (see refs. [13, 68, 82, 83]).

In mathematical literature it is called the semi-infinite cohomology of the algebra $Vir \times Vir$ relative to its sub-algebra $\mathcal{L}_0^-$ generated by the central charge and $L_0^- = L_0 - \overline{L}_0$ with the values in the extended Fock module $\widehat{\mathcal{F}}_p$, and is denoted by

$$\widehat{\mathcal{H}}(Vir \times Vir, \mathcal{L}_0^-, \widehat{\mathcal{F}}_p)$$

(see refs. [84, 85]). We denote this cohomology by $H_S(\widehat{Q}, \widehat{V}_p)$.

Before we start the calculations, let us describe what kind of information about the cohomology we want to obtain. Ordinarily, we are looking for the dimensions of the cohomology spaces at each ghost number. In the presence of the x_0 operator these spaces are likely to be infinite dimensional (since multiplication by x does not change the ghost number). In order to extract reasonable information about the cohomology we have to use an additional grading by the degree of the polynomials in x. The operator $\widehat{Q}$ mixes vectors of different degrees and we can not a priori expect the cohomology states to be represented by homogeneous polynomials.[2] Instead of the grading on $\widehat{V}$ we have to use a decreasing filtration

$$\cdots \supset F_{-k}\widehat{V} \supset F_{-k+1}\widehat{V} \supset \cdots \supset F_0\widehat{V} = V, \tag{5.2.1}$$

where $F_{-k}\widehat{V}$ is the subspace of $\widehat{V}$ consisting of the vectors with x degree less or equal to k. The operator $\widehat{Q}$ respects this filtration in a sense that it maps each subspace $F_r\widehat{V}$ to itself:

$$\widehat{Q}: \; F_r\widehat{V} \to F_r\widehat{V}. \tag{5.2.2}$$

This allows us to define a filtration on the cohomology of $\widehat{Q}$. By definition $F_{-k}H(\widehat{Q}, \widehat{V})$ consists of the cohomology classes which can be represented by vectors with the x-degree less or equal to k. Although as we mentioned above there is no x-grading on the cohomology space, we can define a graded space which is closely related to it. We define

$$\mathrm{Gr}_r H(\widehat{Q}, \widehat{V}) = F_r H(\widehat{Q}, \widehat{V}) / F_{r+1} H(\widehat{Q}, \widehat{V}). \tag{5.2.3}$$

By definition $\mathrm{Gr}_r H(\widehat{Q}, \widehat{V})$ (for $r \leq 0$) consists of the cohomology classes which can be represented by a vector with x-degree $-r$ but not $-r+1$. These spaces carry a lot of information about the cohomology structure. For example if we know the dimensions of Gr_r for $r = 0, -1, \ldots, -k$ we can find the dimension of F_{-k} as their sum. On the other hand the knowledge of representatives of Gr_r states is not enough to find

[2] For the $p = 0$ extended complex and only for this case we will be able to show that the cohomology can be represented by homogeneous polynomials, but this will come out as a non-trivial result.

the representatives of cohomology classes. This is so because Gr_r spaces contain information only about the leading in x terms of the cocycles of $\widehat{Q}$.

To find the graded space $\mathrm{Gr}H = \bigoplus \mathrm{Gr}_r H$ we will use the machinery known as the method of spectral sequences. The idea is to build a sequence of complexes E_n with the differentials d_n such that $E_{n+1} = H(d_n, E_n)$ which converges to the graded space $\mathrm{Gr}H$. In our case we will be able to show that all differentials d_n for $n > 2$ vanish and thus $E_3 = E_4 = \cdots = E_\infty$. Therefore, we will never have to calculate higher then the third terms in the spectral sequence. We will give some more details on the application of the method of spectral sequences to our case in B.

5.3 Chiral extended complex

Before attempting a calculation of the cohomology of the full extended complex let us consider its chiral version. This is a warm up problem which, nevertheless, captures the major features.

We replace the Fock space $\mathcal{F}_p(\alpha, \overline{\alpha})$ by its chiral version, $\mathcal{F}_p(\alpha)$ which is generated from the vacuum by the left moving modes α_n only.

Repeating the arguments of the previous section we conclude that the chiral version of $\widehat{Q}$ is given by

$$\widehat{Q} = 1 \otimes Q - i\frac{\partial}{\partial x^\mu} \otimes \sum_n c_n \alpha^\mu_{-n} - \frac{1}{2}\eta^{\mu\nu}\Box \otimes c_0. \tag{5.3.1}$$

We will calculate the cohomology of chiral extended complex $\widehat{V}_p$ for three different cases: case $p^2 \neq 0$, which describes the massive spectrum; case $p^2 = 0$—the massless one; and case $p \equiv 0$, which besides the particular states from the massless spectrum describes a number of discrete states.

5.3.1 Massive states

Let us start the calculation of the cohomology of the chiral extended complex $\widehat{V}_p$ by considering the case of $p^2 \neq 0$. For this case the cohomology of the BRST complex is non-zero only for ghost number one and ghost number two. The cohomology contains the same number of ghost number one and two states which can be written in terms of dimension one primary matter states. Let $|v, p\rangle \in V_p$ be a dimension one primary state with no ghost excitations; then the following states,

$$c_0 c_1 |v, p\rangle \qquad \text{and} \qquad c_1 |v, p\rangle, \tag{5.3.1}$$

represent nontrivial cohomology classes and, moreover, each cohomology class has a representative of this kind (see ref. [86]).

We will calculate the cohomology of the extended complex in two steps. First, we extend the BRST complex by adding polynomials of one variable $\tilde{x} = (p \cdot x)$. The resulting space,

$$\widetilde{V}_p = \mathbb{C}[\tilde{x}] \otimes V_p, \tag{5.3.2}$$

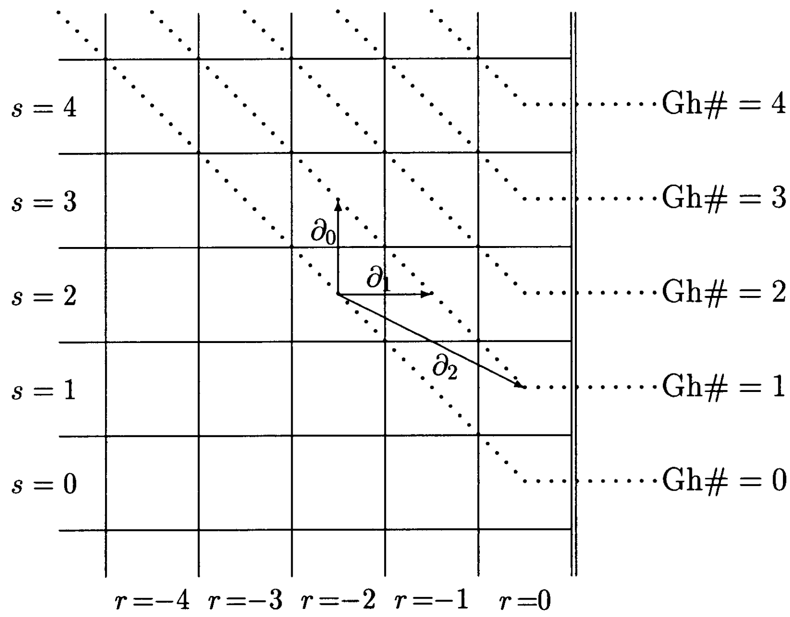


Figure 5-1: Anatomy of a double complex

is a subcomplex of $\widehat{V}_p$ and we define its cohomology as $H(\widetilde{Q},\widetilde{V}_p)$, where $\widetilde{Q}$ is the restriction of $\widehat{Q}$ on $\widetilde{V}_p$. Calculation of $\mathrm{Gr}H(\widetilde{Q},\widetilde{V}_p)$ is the objective of the first step. Second, we obtain the full extended space as a tensor product of $\widetilde{V}_p$ with the polynomials of the transverse variables

$$\widehat{V}_p = \mathbb{C}[\tilde{x}^1 \ldots, \tilde{x}^{D-1}] \otimes \widetilde{V}_p, \tag{5.3.3}$$

where

$$\tilde{x}^i = x^i - \frac{p^i(p\cdot x)}{p^2}. \tag{5.3.4}$$

Using $\mathrm{Gr}H(\widetilde{Q},\widetilde{V}_p)$ found in the first step, we will calculate $\mathrm{Gr}H(\widehat{Q},\widehat{V}_p)$.

Let us calculate $\mathrm{Gr}H(\widetilde{Q},\widetilde{V}_p)$. Beside the ghost number, complex $\widetilde{V}_p$ has an additional grading—the x-degree. According to these two gradings we can write $\widetilde{V}_p$ as a double sum

$$\widetilde{V}_p = \bigoplus_{r,s} \widetilde{E}_0^{r,s}(p), \tag{5.3.5}$$

where $\widetilde{E}_0^{r,s} = \tilde{x}^{-r} \otimes V_p^{(r+s)}$ is the space of ghost number $r+s$ states with $-r$ factors of $\tilde{x}$. Note that in our notations $r \leq 0$.

It will be convenient to represent a double graded complex like $\widetilde{V}_p$ graphically by a lattice (see Fig. 5-1) where each cell represents a space $\widetilde{E}_0^{r,s}$, columns represent the spaces with definite x-degrees and the diagonals represent the spaces with definite ghost numbers.

The action of $\widetilde{Q}$ on $\widetilde{V}_p$ can be easily derived from the general formula (5.3.1). Any vector from $\widetilde{E}_0^{-k,s}$ can be represented as $\tilde{x}^k \otimes |v,p\rangle$, where $|v,p\rangle \in V_p^{(s-k)}$ is a vector

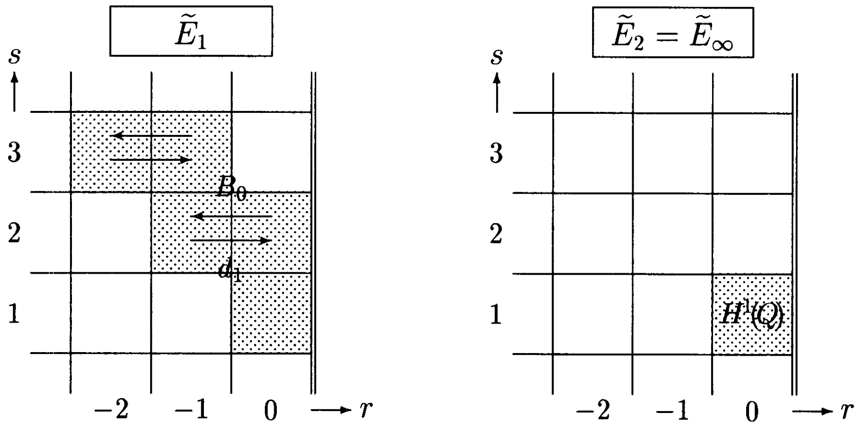


Figure 5-2: Spectral sequence for $\widetilde{V}_p$

from the BRST complex V_p with ghost number $s-k$. Applying $\widetilde{Q}$ to this state we obtain

$$\begin{aligned}\widetilde{Q}\,\tilde{x}^k \otimes |v,p\rangle &= \tilde{x}^k \otimes Q|v,p\rangle \\ &- ik\,\tilde{x}^{k-1} \otimes \sum_n c_n\,(p\cdot\alpha_{-n})\,|v,p\rangle \\ &- \frac{k\,(k-1)}{2}\,p^2\,\tilde{x}^{k-2} \otimes c_0\,|v,p\rangle\end{aligned} \tag{5.3.6}$$

According to Eq. (5.3.6) we decompose $\widetilde{Q}$ in the sum of operators with a definite x-degree

$$\widetilde{Q} = \partial_0 + \partial_1 + \partial_2, \tag{5.3.7}$$

where each ∂_n reduces the x-degree, or increases r, by n (see Fig. 5-1).

Now we start building the spectral sequence of the complex $(\widetilde{V}_p, \widetilde{Q})$. For a short review of the method see B. The first step, the calculation of $\widetilde{E}_1^{r,s} = \mathrm{Gr}_r H^{r+s}(\partial_0, \widetilde{V})$, reduces to the calculation of the cohomology of the BRST complex. Indeed, according to Eq. (5.3.6), $\partial_0 = 1 \otimes Q$, and therefore

$$\widetilde{E}_1^{r,s} = \tilde{x}^{-r} \otimes H^{(r+s)}(Q, V_p). \tag{5.3.8}$$

As we mentioned above, the BRST complex has nontrivial cohomology only at ghost numbers one and two. Thus the space $\widetilde{E}_1 = \bigoplus \widetilde{E}_1^{r,s}$ looks as shown in Fig. 5-2 (left), where shaded cells correspond to non-zero spaces.

The differential d_1 is induced on $\widetilde{E}_1$ by ∂_1, and acts from $\widetilde{E}_1^{r,s}$ to $\widetilde{E}_1^{r+1,s}$ as shown in Fig. 5-2 (left). Since there are no states below the ghost number one and above ghost number two, the cohomology of d_1 at ghost number one is given by its kernel:

$$\widetilde{E}_2^{r,1-r} = \ker d_1, \tag{5.3.9}$$

and at ghost number two by the quotient of $\widetilde{E}_2^{r,2-r}$ by the image of d_1:

$$\widetilde{E}_1^{r,2-r} = \widetilde{E}_2^{r,2-r}/\Im\mathrm{m} d_1. \tag{5.3.10}$$

We are going to show that d_1 establishes an isomorphism of the corresponding spaces and, therefore, the only non-empty component of $\widetilde{E}_2$ is $\widetilde{E}_2^{0,1} \simeq H^1(Q, V_p)$ as shown in Fig. 5-2 (right).

Consider an operator $B_0 = \tilde{x} \otimes b_0$. This operator is well defined on $\widetilde{E}_1$ *i.e.*, it maps cohomology classes to cohomology classes. On the other hand, its anticommutator with d_1 is given by

$$\{d_1, B_0\} = p^2\, \hat{k}, \tag{5.3.11}$$

where $\hat{k}$ the x-degree operator. The last equation shows that if $p^2 \neq 0$, nontrivial cohomology of d_1 may exist only in $\hat{k} = 0$ subspace of $\widetilde{E}_2$. Moreover, if we apply $\{d_1, B_0\} = d_1\, B_0 + B_0\, d_1$ to ghost number one states only the second term will survive because there are no ghost number zero states in $\widetilde{E}_1$. Thus we conclude that up to a diagonal matrix B_0 is an inverse operator to d_1. Since $\widetilde{E}_1^{r,1-r}$ and $\widetilde{E}_1^{r+1,1-r}$ have the same dimension and d_1 is invertible it is an isomorphism between $\widetilde{E}_2^{r,1-r}$ and $\widetilde{E}_2^{r+1,1-r}$ for any $r \leq 0$.

As shown in Fig. 5-2 (right), $\widetilde{E}_2$ contains only one non-empty component. This means that second differential d_2 and all higher are necessarily zero and the spectral sequence collapses at $\widetilde{E}_2 = \widetilde{E}_\infty$. Therefore, we conclude that

$$\mathrm{Gr}H(\widetilde{Q}, \widetilde{V}_p) = H^1(Q, V_p). \tag{5.3.12}$$

The second step in our program is trivial because the spectral sequence $\{\widehat{E}_n\}$ of the full complex

$$\widehat{V}_p = \mathbb{C}[\tilde{x}^1, \ldots, \tilde{x}^{D-1}] \otimes \widetilde{V}_p, \tag{5.3.13}$$

stabilizes at $\widehat{E}_1$ and

$$\widehat{E}_1 = \widehat{E}_\infty = \mathbb{C}[\tilde{x}_1, \ldots, \tilde{x}_{D-1}] \otimes \mathrm{Gr}H(\widetilde{Q}, \widetilde{V}_p). \tag{5.3.14}$$

This happens simply because, according to Eq. (5.3.12), $\mathrm{Gr}H(\widetilde{Q}, \widetilde{V}_p)$, and thus $\widehat{E}_1$, contains only ghost number one states and therefore d_1 and all higher differentials must vanish. Combining Eqs. (5.3.14) and (5.3.12) we obtain

$$\mathrm{Gr}H(\widehat{Q}, \widehat{V}_p) = \mathbb{C}[\tilde{x}^1, \ldots, \tilde{x}^{D-1}] \otimes H^1(Q, V_p). \tag{5.3.15}$$

This completes our analysis of the cohomology of the chiral extended complex for $p^2 \neq 0$.

5.3.2 Massless states

The analysis presented above can not be applied to the light-cone, $p^2 = 0$. We could, in principle, repeat all the arguments using $\xi \cdot x$ instead of $\tilde{x}$, where ξ is some vector for which $\xi \cdot p \neq 0$, to build $\widetilde{V}$ and this would work everywhere except at the origin of the momentum space, $p = 0$. Yet it is instructive to make a covariant calculation in this case. Since there is no covariant way to choose a vector ξ we can not apply our two step program. Instead we will start from scratch and build a spectral sequence for the whole module

$$\widehat{V}_p = \mathbb{C}[x^0, \ldots, x^{D-1}] \otimes V_p, \tag{5.3.1}$$

graded by the total x-degree.

According to Eq. (5.3.1), we can decompose $\widehat{Q}$ into a sum of operators of definite x-degree

$$\widehat{Q} = \partial_0 + \partial_1 + \partial_2, \tag{5.3.2}$$

where

$$\begin{aligned} \partial_0 &= 1 \otimes Q, \\ \partial_1 &= -i\frac{\partial}{\partial x^\mu} \otimes \sum_n c_n \alpha^\mu_{-n}, \\ \partial_2 &= -\frac{1}{2}\Box \otimes c_0. \end{aligned} \tag{5.3.3}$$

The first step is to find cohomology of ∂_0, which is just the tensor product of the BRST cohomology $H(Q, V_p)$ with the space of polynomials

$$E_1 = H(\partial_0, \widehat{V}_p) = \mathbb{C}[x^0 \cdots x^{D-1}] \otimes H(Q, V_p). \tag{5.3.4}$$

Multiplying the representatives of $H(Q, V_p)$ by arbitrary polynomials in x we obtain the following representatives of E_1 cohomology classes

$$\begin{aligned} &P_\mu(x) \otimes c_1 \alpha^\mu_{-1}|p\rangle, \\ &Q_\mu(x) \otimes c_0 c_1 \alpha^\mu_{-1}|p\rangle, \end{aligned} \tag{5.3.5}$$

where $P_\mu(x)$ and $Q_\mu(x)$ are polynomials in x that satisfy the transversality condition, $p^\mu Q_\mu(x) = p^\mu P_\mu(x) = 0$, and are not proportional to p_μ. These transversality conditions come from the same conditions on BRST cohomology classes at $p^2 = 0$. The first differential acts non-trivially from ghost number one to ghost number two states according to the following formula

$$d_1 : \quad P_\mu \to Q_\mu = -ip^\nu \frac{\partial}{\partial x^\nu} P_\mu. \tag{5.3.6}$$

It is easy to check that the map (5.3.6) is surjective and therefore $E_2^{r,s} = 0$ for $r+s = 2$. As expected the cohomology of the massless complex has a similar structure to that of the massive one. There are no cohomology states with ghost number two and there is an infinite tower of ghost number one states with different x-degree.

5.3.3 Cohomology of the zero momentum chiral complex

The zero momentum complex is exceptional. Already in the BRST cohomology we encounter additional "discrete" states at exotic ghost numbers (see ref. [87]). The cohomology is one dimensional at ghost numbers zero and three and D-dimensional at ghost numbers one and two. Explicit representatives for these classes can be written as given in Table 5.1.

Ghost #	Representatives	Dimension
3	$c_1 c_0 c_{-1}\|0\rangle$	1
2	$c_0 \alpha^{\mu}_{-1}\|0\rangle$	D
1	$\alpha^{\mu}_{-1}\|0\rangle$	D
0	$\|0\rangle$	1

Table 5.1: Chiral BRST cohomology at $p = 0$

Let us denote the direct sum of spaces $E_n^{r,s}$ with the same ghost number $m = r + s$ by $E_n^{(m)}$:

$$E_n^{(m)} \equiv \bigoplus_{r \leq 0} E_n^{r,m-r} \tag{5.3.1}$$

As usual, the cohomology of ∂_0 can be written in terms of the BRST cohomology as

$$E_1^{(m)} = H^m(\partial_0, \widehat{V}_0) = \mathbb{C}[x^0, \ldots, x^{D-1}] \otimes H^m(Q, V_0).$$

According to Table 5.1, $H^0(Q, V_0) \simeq H^3(Q, V_0) \simeq \mathbb{C}$. Therefore, $E_1^{(0)}$ and $E_1^{(3)}$ are isomorphic to the space of polynomials $\mathbb{C}[x^\mu]$. Similarly, $E_1^{(1)}$ and $E_1^{(2)}$ are isomorphic to the space of polynomial vector fields $\mathbb{C}^D[x^\mu]$. Evaluating the action of ∂_1 on the representatives (see Table 5.1), we get the following sequence

$$0 \to \mathbb{C}[x^\mu] \xrightarrow{\nabla} \mathbb{C}^D[x^\mu] \xrightarrow{0} \mathbb{C}^D[x^\mu] \xrightarrow{\nabla} \mathbb{C}[x^\mu] \to 0, \tag{5.3.2}$$

where the first nabla-operator is the gradient, which maps scalars to vectors and the second is the divergence, which does the opposite (and we have dropped an insignificant factor $-i$). It is convenient to interpret $\mathbb{C}[x^\mu] = \mathcal{O}^0$ and $\mathbb{C}^D[x^\mu] = \mathcal{O}^1$ as a space of polynomial zero and one-forms on the Minkowski space. The first nabla-operator in Eq. (5.3.2) will be interpreted as an exterior derivative d and the second as its Hodge conjugate δ. With new notation we rewrite the sequence in Eq. (5.3.2) as

$$0 \to \mathcal{O}^0 \xrightarrow{d} \mathcal{O}^1 \xrightarrow{0} \mathcal{O}^1 \xrightarrow{\delta} \mathcal{O}^0 \to 0. \tag{5.3.3}$$

This is illustrated by Fig. 5-3 (left). Note that individual cells in Fig. 5-3 correspond to the subspaces of homogeneous polynomials rather than whole $\mathcal{O}^1$ or $\mathcal{O}^0$. One can

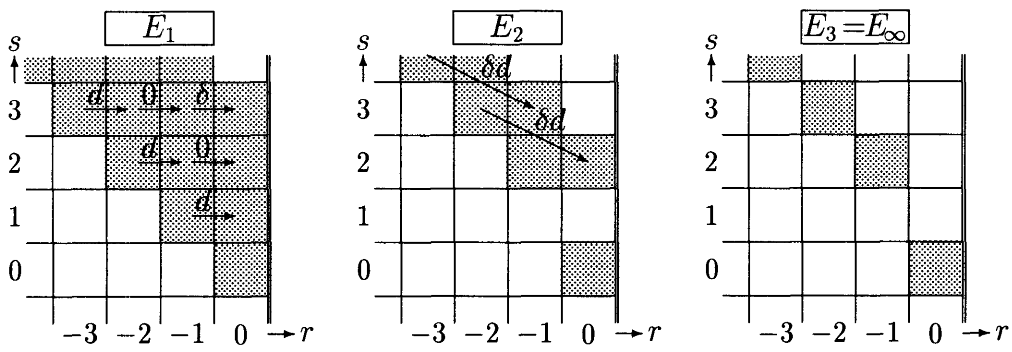


Figure 5-3: Spectral sequence for $\widehat{V}_0$

easily calculate cohomology of this complex and obtain

$$\begin{aligned} E_2^{(0)} &= \mathbb{C}, \\ E_2^{(1)} &= \mathcal{O}^1/d\mathcal{O}^0, \\ E_2^{(2)} &= \ker(\delta, \mathcal{O}^1), \\ E_2^{(3)} &= 0. \end{aligned} \tag{5.3.4}$$

Thus the second term in the spectral sequence has the structure presented by Fig. 5-3 (middle).

Now we have to calculate d_2. This differential acts as shown in Fig. 5-3 (middle) and can be found from the following formula (see B):

$$d_2 = \partial_2 - \partial_1 \partial_0^{-1} \partial_1. \tag{5.3.5}$$

Let $P = P_\mu dx^\mu \in \mathcal{O}^1$ be a polynomial one form. Corresponding state representing a $E_2^{(1)}$ cohomology class is given by

$$P_\mu \otimes c_1 \alpha_{-1}^\mu |0\rangle. \tag{5.3.6}$$

Applying ∂_1 to (5.3.6) we obtain

$$\partial_1 \; P_\mu \otimes \alpha_{-1}^\mu c_1 |0\rangle = -i \frac{\partial P_\mu}{\partial x_\mu} \otimes c_{-1} c_1 |0\rangle = \partial_0 \; i \frac{1}{2} \frac{\partial P_\mu}{\partial x_\mu} \otimes c_0 |0\rangle, \tag{5.3.7}$$

where we use the metric $\eta^{\mu\nu}$ to raise and lower indices. From Eq. (5.3.7) we derive that

$$\partial_1 \partial_0^{-1} \partial_1 \; P_\mu \otimes \alpha_{-1}^\mu c_1 |0\rangle = \partial_1 \; \frac{i}{2} \frac{\partial P_\mu}{\partial x_\mu} \otimes c_0 |0\rangle = \frac{1}{2} \frac{\partial^2 P_\mu}{\partial x_\mu \partial x^\nu} \otimes \alpha_{-1}^\nu c_1 c_0 |0\rangle. \tag{5.3.8}$$

With the definition of ∂_2 (see Eq. (5.3.3)) we immediately get

$$\partial_2 \; P_\mu \otimes \alpha_{-1}^\mu c_1 |0\rangle = \frac{1}{2} \Box P_\mu c_1 c_0 |0\rangle. \tag{5.3.9}$$

By adding the last two equations, and dropping an insignificant factor 1/2, we see that the second differential acts on $E_2^{(1)} = \mathbb{C}^D[x^\mu]/\nabla\mathbb{C}[x^\mu]$ as the following matrix differential operator:

$$(d_2)_\mu^\nu = \Box\delta_\mu^\nu - \partial_\mu\partial^\nu, \tag{5.3.10}$$

or, using the differential form interpretation $E_2^{(1)} = \mathcal{O}^1/d\mathcal{O}^0$,

$$d_2 = \Box - d\delta = \delta d. \tag{5.3.11}$$

In order to calculate $E_3 = H(d_2, E_2)$ we have to calculate the cohomology of the following complex:

$$0 \to \mathbb{C} \xrightarrow{0} \mathcal{O}^1/d\mathcal{O}^0 \xrightarrow{\delta d} \ker(\delta, \mathcal{O}^1) \to 0, \tag{5.3.12}$$

or equivalently

$$0 \to \mathcal{O}^0 \xrightarrow{d} \mathcal{O}^1 \xrightarrow{\delta d} \mathcal{O}^1 \xrightarrow{\delta} \mathcal{O}^0 \to 0. \tag{5.3.13}$$

The later sequence is known to be exact in the last two terms[3] and thus E_3 is given by

$$\begin{aligned} E_3^{(0)} &= \mathbb{C}, \\ E_3^{(1)} &= \ker(\delta d, \mathcal{O}^1)/d\mathcal{O}^0, \\ E_3^{(2)} &= E_3^{(3)} = 0. \end{aligned} \tag{5.3.14}$$

Looking at Fig. 5-3 (right) one can easily deduce that all differentials d_k for $k \geq 3$ vanish. This allows us to conclude that

$$\mathrm{Gr}_r H^{r+s}(\widehat{Q}, \widehat{V}_0) = E_\infty^{r,s} = E_3^{r,s}, \tag{5.3.15}$$

and GrH has the structure shown in Fig. 5-3 (right). Since cohomology is non-trivial only at one ghost number for every x-degree, we can easily find the dimensions of different Gr$_r H$ spaces. We present the summary of our results on the cohomology of the chiral extended complex in the following theorem.

Theorem 1 *The cohomology of the chiral extended complex $\widehat{V}_p$ admits a natural filtration (by the* minus *x-degree) $\mathcal{F}_r H(\widehat{V}_p, \widehat{Q})$. The cohomology can be described using the associated graded spaces*

$$\mathrm{Gr}_r H(\widehat{V}_p, \widehat{Q}) = \mathcal{F}_r H(\widehat{V}_p, \widehat{Q})/\mathcal{F}_{r+1} H(\widehat{V}_p, \widehat{Q})$$

as follows

1. *Non-zero momentum ($p \not\equiv 0$): cohomology is trivial unless $p^2 = n - 1$, where n is a non-negative integer. In the latter case*

$$\dim \mathrm{Gr}_r H^1(\widehat{V}_p, \widehat{Q}) = \binom{24-r}{24} d_n,$$
$$\dim H^l(\widehat{V}_p, \widehat{Q}) = 0 \qquad \text{for } l \neq 1,$$

[3] We would like to thank Jeffrey Goldstone for pointing this out to us.

where d_n denotes the number of the BRST states at mass level n. These numbers are generated by

$$\prod_{k=1}^{\infty}(1-z^k)^{-24}=\sum_{n=1}^{\infty}d_n z^n.$$

see ref. [79].

2. *Zero momentum ($p\equiv 0$): the cohomology appears at ghost number zero (the vacuum)*

$$\dim H^0(\widehat{V}_p,\widehat{Q})=1,$$

and ghost number one (physical states, recall that r is minus *x-degree).*

$$\dim \mathrm{Gr}_r H^1(\widehat{V}_p,\widehat{Q})=\begin{cases} 0 & \text{if } r=0;\\ \dfrac{D(D-1)}{2} & \text{for } r=-1;\\ \dfrac{D(D-2)(D+2)}{3} & \text{for } r=-2;\\ \chi_r & \text{for } r\leq -3.\end{cases}$$

where

$$\chi_r=D\binom{D-1-r}{D-1}+\binom{D-4-r}{D-1}-\binom{D-r}{D-1}-D\binom{D-3-r}{D-1}.$$

The cohomology is trivial for all the other ghost numbers

$$\dim H^l(\widehat{V}_p,\widehat{Q})=0 \qquad \text{for } l\neq 0,1.$$

5.4 Cohomology of full extended complex

In this section we will calculate the semi-relative cohomology of the full extended complex. We will use the same technique as for the chiral complex and obtain similar results.

5.4.1 Review of semi-relative BRST cohomology

By definition, the semi-relative cohomology $\widehat{\mathcal{H}}_S$ consists of Q invariant states annihilated by $b_0-\bar{b}_0$, modulo $Q|\Lambda\rangle$ where $|\Lambda\rangle$ is annihilated by $b_0-\bar{b}_0$. For future convenience we set $b_0^{\pm}=b_0\pm\bar{b}_0$ and $c_0^{\pm}=(1/2)(c_0\pm\bar{c}_0)$.

Semi-relative cohomology $\widehat{\mathcal{H}}_S$ of the BRST complex can be easily expressed in terms of relative cohomology $\widehat{\mathcal{H}}_R$. The latter consists of the Q invariant states annihilated both by b_0 and $\bar{b}_0$ modulo $Q|\Lambda\rangle$ where $|\Lambda\rangle$ is also annihilated by b_0 and $\bar{b}_0$ separately. We can express $\widehat{\mathcal{H}}_S$ in terms of $\widehat{\mathcal{H}}_R$ using the following exact sequence (see ref. [88]):

$$\cdots\to\mathcal{H}_R^n\xrightarrow{i}\mathcal{H}_S^n\xrightarrow{b_0^+}\mathcal{H}_R^{n-1}\xrightarrow{\{Q,c_0^+\}}\mathcal{H}_R^{n+1}\xrightarrow{i}\mathcal{H}_S^{n+1}\to\cdots. \tag{5.4.1}$$

The map i is induced in the cohomology by the natural embedding of the relative complex into the absolute complex. The relative cohomology can be found as a tensor product of the relative cohomologies of the left and right chiral sectors.

This is particularly easy for the case $p \neq 0$ because in this case the chiral relative cohomology is non-zero only at ghost number one (see [86]) and, therefore, $\widehat{\mathcal{H}}_R$ consists of ghost number two states only. The exact sequence (5.4.1) reduces in this case to

$$0 \to \widehat{\mathcal{H}}^3_S \xrightarrow{b_0^+} \widehat{\mathcal{H}}^2_R \to 0 \qquad \text{and} \qquad 0 \to \widehat{\mathcal{H}}^2_R \xrightarrow{i} \widehat{\mathcal{H}}^2_S \to 0, \tag{5.4.2}$$

which means that $\widehat{\mathcal{H}}^3_S$ is isomorphic to $\widehat{\mathcal{H}}^2_S$ which in turn is isomorphic to $\widehat{\mathcal{H}}^2_R$ and there are no semi-relative cohomology states at ghost numbers other than two or three. We can even write explicit representatives in terms of dimension $(1,1)$ primary matter states. Let $|v,p\rangle$ denote such a state. Representatives of semi-relative cohomology classes can be written as

$$c_1\bar{c}_1|v,p\rangle \qquad \text{and} \qquad c_0^+ c_1\bar{c}_1|v,p\rangle. \tag{5.4.3}$$

5.4.2 Semi-relative cohomology of $p \not\equiv 0$ extended complex

calculation of the extended cohomology for the semi-relative complex in the case of $p \not\equiv 0$ is line by line parallel to that of the chiral one. When we add polynomials of $\tilde{x} = (p \cdot x)$ the resulting cohomology is given by ghost number two BRST states only, and the semi-relative cohomology of the full extended complex is obtained by adding polynomials of transverse components of x:

$$\mathrm{Gr}\, \widehat{\mathcal{H}}_S(\widehat{Q}, \widehat{V}_p) = \mathbb{C}[\tilde{x}^1, \cdots, \tilde{x}^{D-1}] \otimes \widehat{\mathcal{H}}^2_S. \tag{5.4.4}$$

The case of the massless ($p^2 = 0$) complex may require some special consideration because there is no straightforward way to choose the transverse variables $\tilde{x}^i$ but the answer is the same.

In our opinion, an important result is that the extended cohomology does not contain ghost number three states. Let us explain in more details what happens to the ghost number three semi-relative states when we extend the complex by x_0. According to the results on extended cohomology for every state $c_0^+ c_1\bar{c}_1|v,p\rangle$ we should be able to find a vector $|w,p\rangle$ in the extended complex such that $c_0^+ c_1\bar{c}_1|v,p\rangle = \widehat{Q}|w,p\rangle$. We have found the leading part of $|w,p\rangle$ when we calculated the spectral sequence. It is given by[4] $\frac{(p\cdot x)}{p^2} c_1\bar{c}_1|v,p\rangle$. Applying $\widehat{Q}$ to this state we get

$$\widehat{Q}\frac{(p\cdot x)}{p^2} c_1\bar{c}_1|v,p\rangle = c_0^+ c_1\bar{c}_1|v,p\rangle + \sum_n p_\mu c_1\bar{c}_1(c_{-n}\alpha_n^\mu + \bar{c}_{-n}\bar{\alpha}_n^\mu)|v,p\rangle. \tag{5.4.5}$$

Thus in order to prove that $c_0^+ c_1\bar{c}_1|v,p\rangle$ is trivial we have to prove that the sum in the right hand side represents a $\widehat{Q}$ exact state. The latter is a direct consequence of the absence of relative cohomology of ghost number three. Indeed, this sum is annihilated by b_0, $\bar{b}_0$, and Q and if it were nontrivial it would have represented a non-zero cohomology class of ghost number three in the relative complex.

[4]This is not applicable to the massless case $p^2 = 0$ which has to be analyzed separately.

Ghost #	Representatives	Dimension
0	$\|0\rangle$	1
1	$c_1\alpha^\mu_{-1}\|0\rangle, \quad \bar{c}_1\bar{\alpha}^\mu_{-1}\|0\rangle$	$2D$
2	$c_1\alpha^\mu_{-1}\bar{c}_1\bar{\alpha}^\nu_{-1}\|0\rangle, \quad c_1c_{-1}\|0\rangle, \quad \bar{c}_1\bar{c}_{-1})\|0\rangle$	D^2+2
3	$c_1\alpha^\mu_{-1}\bar{c}_1\bar{c}_{-1}\|0\rangle, \quad c_1c_{-1}\bar{c}_1\bar{\alpha}^\nu_{-1}\|0\rangle$	$2D$
4	$c_1c_{-1}\bar{c}_1\bar{c}_{-1}\|0\rangle$	1

Table 5.2: Cohomology of the relative BRST complex at $p = 0$

5.4.3 Semi-relative cohomology of $p = 0$ extended complex

The calculation of the cohomology of the zero-momentum extended complex is based on the same ideas that we used for the chiral case but is technically more difficult. The major complication is due to a much larger BRST cohomology.

We can find the BRST cohomology of the zero momentum semi-relative complex using the long exact sequence we mentioned above (see Eq. (5.4.1)) which in this case breaks in to the following exact sequences:

$$\begin{aligned} 0 \to \widehat{\mathcal{H}}^0_R &\xrightarrow{i} \widehat{\mathcal{H}}^0_S \xrightarrow{b_0^+} 0, \\ 0 \to \widehat{\mathcal{H}}^1_R &\xrightarrow{i} \widehat{\mathcal{H}}^2_S \xrightarrow{b_0^+} \widehat{\mathcal{H}}^0_R \xrightarrow{\{Q,c_0^+\}} \widehat{\mathcal{H}}^2_R, \\ 0 \to \widehat{\mathcal{H}}^1_R &\xrightarrow{i} \widehat{\mathcal{H}}^3_S \xrightarrow{b_0^+} \widehat{\mathcal{H}}^0_R \xrightarrow{\{Q,c_0^+\}} \widehat{\mathcal{H}}^2_R, \\ \widehat{\mathcal{H}}^2_R \xrightarrow{\{Q,c_0^+\}} \widehat{\mathcal{H}}^4_R &\xrightarrow{i} \widehat{\mathcal{H}}^4_S \xrightarrow{b_0^+} \widehat{\mathcal{H}}^3_R \to 0, \\ 0 &\to \widehat{\mathcal{H}}^5_S \xrightarrow{b_0^+} \widehat{\mathcal{H}}^4_R \to 0. \end{aligned} \tag{5.4.1}$$

The cohomology of the relative complex can be found as a tensor product of the left and right relative cohomologies. The later can be found, for example, in ref. [86] and consists of ghost number zero, one, and two states. Therefore, the relative complex has non-trivial cohomologies at ghost numbers from zero through four and their representatives and dimensions are listed in Table 5.2.

Using the exact sequences (5.4.1) we can easily find the dimensions and representatives of the semi-relative cohomology as shown in Table 5.3.

The extended module $\widehat{V}_0$ is given by

$$\widehat{V}_0 = \mathbb{C}[x^0, \ldots, x^{D-1}] \otimes V_0, \tag{5.4.2}$$

Ghost #	Representatives	Dimension
0	$\vert 0\rangle$	1
1	$c_1\alpha^\mu_{-1}\vert 0\rangle, \quad \bar{c}_1\bar{\alpha}^\mu_{-1}\vert 0\rangle$	$2D$
2	$c_1\alpha^\mu_{-1}\bar{c}_1\bar{\alpha}^\nu_{-1}\vert 0\rangle, \quad (c_1c_{-1} - \bar{c}_1\bar{c}_{-1})\vert 0\rangle$	D^2+1
3	$c_0^+c_1\alpha^\mu_{-1}\bar{c}_1\bar{\alpha}^\nu_{-1}\vert 0\rangle, \quad c_0^+(c_1c_{-1} - \bar{c}_1\bar{c}_{-1})\vert 0\rangle$	D^2+1
4	$c_0^+c_1\alpha^\mu_{-1}\bar{c}_1\bar{c}_{-1}\vert 0\rangle, \quad c_0^+c_1c_{-1}\bar{c}_1\bar{\alpha}^\mu_{-1}\vert 0\rangle$	$2D$
5	$c_0^+c_1c_{-1}\bar{c}_1\bar{c}_{-1}\vert 0\rangle$	1

Table 5.3: BRST cohomology of the semi-relative complex at $p = 0$

and the operator $\widehat{Q}$ has the following x-degree decomposition

$$\widehat{Q} = \partial_0 + \partial_1 + \partial_2, \tag{5.4.3}$$

where

$$\begin{aligned} \partial_0 &= 1 \otimes Q, \\ \partial_1 &= -i\frac{\partial}{\partial x^\mu} \otimes \sum_n (c_n\alpha^\mu_{-n} + \bar{c}_n\bar{\alpha}^\mu_{-n}), \\ \partial_2 &= -\frac{1}{2}\Box \otimes (c_0 + \bar{c}_0) = -\Box \otimes c^+. \end{aligned} \tag{5.4.4}$$

As it was the case for the chiral complex, the cohomology of ∂_0 coincides with the cohomology of the semirelative BRST module, $\widehat{\mathcal{H}}_S$ tensored with $\mathbb{C}[x^0, \ldots, x^{D-1}]$ and the first term in the spectral sequence is given by

$$E_1 = H_S(\partial_0, \widehat{V}_0) = \mathbb{C}[x^0, \ldots, x^{D-1}] \otimes \widehat{\mathcal{H}}_S \tag{5.4.5}$$

Using information from Table 5.3, we can parameterize the space

$$E_1 = \mathbb{C}[x^0 \cdots x^{D-1}] \otimes \widehat{\mathcal{H}}_S$$

as follows

$$\begin{aligned} (R^{[0]}) &= R^{[0]} \otimes |0\rangle, \\ (P^{[1]}_\mu, \overline{P}^{[1]}_\mu) &= P^{[1]}_\mu \otimes c_1\alpha^\mu_{-1}|0\rangle + \overline{P}^{[1]}_\mu \otimes \bar{c}_1\bar{\alpha}^\mu_{-1}|0\rangle, \\ (Q^{[2]}_{\mu\nu}, R^{[2]}) &= Q^{[2]}_{\mu\nu} \otimes c_1\alpha^\mu_{-1}\bar{c}_1\bar{\alpha}^\nu_{-1}|0\rangle + R^{[2]} \otimes (c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|0\rangle, \\ (Q^{[3]}_{\mu\nu}, R^{[3]}) &= Q^{[3]}_{\mu\nu} \otimes c_0^+c_1\alpha^\mu_{-1}\bar{c}_1\bar{\alpha}^\nu_{-1}|0\rangle + R^{[3]} \otimes c_0^+(c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|0\rangle, \\ (P^{[4]}_\mu, \overline{P}^{[4]}_\mu) &= P^{[4]}_\mu \otimes c_0^+c_1\alpha^\mu_{-1}\bar{c}_1\bar{c}_{-1}|0\rangle + \overline{P}^{[4]}_\mu \otimes c_0^+c_1c_{-1}\bar{c}_1\bar{\alpha}^\mu_{-1}|0\rangle, \\ (R^{[5]}) &= R^{[0]} \otimes c_0^+c_1c_{-1}\bar{c}_1\bar{c}_{-1}|0\rangle, \end{aligned} \tag{5.4.6}$$

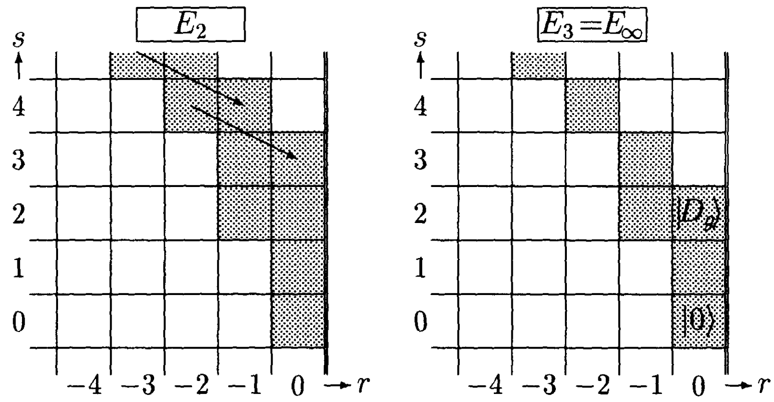


Figure 5-4: Spectral sequence for the zero momentum closed string states.

where $Q^{\mu\nu}$, P_μ and $\overline{P}_\mu$ and R are correspondingly tensor-, vector-, and scalar-valued polynomials of D variables x^μ. Applying ∂_1 to the representatives given by (5.4.6) and dropping Q-trivial states we can find that the first differential acts according to the following diagram

$$
\begin{array}{lll}
0: & (R^{[0]}) \searrow^{d_1} & (0) \\
1: & (P^{[1]}_\mu,\ \overline{P}^{[1]}_\mu) \searrow^{d_1} & (\partial_\mu R^{[0]},\ \partial_\mu R^{[0]}) \\
2: & (Q^{[2]}_{\mu\nu},\ R^{[2]}) \searrow^{d_1} & (\partial_\mu \overline{P}^{[1]}_\nu - \partial_\nu P^{[1]}_\mu,\ \partial^\mu \overline{P}^{[1]}_\mu - \partial^\mu P^{[1]}_\mu) \\
3: & (Q^{[3]}_{\mu\nu},\ R^{[3]}) \searrow^{d_1} & (0,\ 0) \\
4: & (P^{[4]}_\mu,\ \overline{P}^{[4]}_\mu) \searrow^{d_1} & (\partial^\nu Q^{[3]}_{\mu\nu} + \partial_\mu R^{[3]},\ \partial^\nu Q^{[3]}_{\nu\mu} + \partial_\mu R^{[3]}) \\
5: & (R^{[5]}) & (\partial_\mu P^{[4]} - \partial_\mu \overline{P}^{[4]})
\end{array}
\tag{5.4.7}
$$

Consider the cohomology of d_1. From the explicit formulae (5.4.7) it is clear that $E_2 = H(d_1, E_1)$ contains no ghost number five states and the only state that survives at ghost number zero is given by a constant $R^{[0]}$ and is the vacuum. It is less trivial to show that $E_2^{(4)} = 0$, and that $E_2^{(1)}$ contains only x-degree one and two states. We will prove these two results in C (lemmas 3 and 1). This shows that the space E_2 looks as shown in Fig. 5-4 (left). From the structure of E_2 we conclude that d_2 may only act from $E_2^{(2)}$ to $E_2^{(3)}$. In order to find

$$
d_2\,(Q^{[2]}_{\mu\nu},\ R^{[2]}) = (\partial_2 - \partial_1 \partial_0^{-1} \partial_1)\,(Q^{[2]}_{\mu\nu},\ R^{[2]}),
$$

we have to keep the ∂_0 trivial terms when we apply ∂_1 to $(Q^{[2]}_{\mu\nu},\ R^{[2]})$ which were dropped in Eq. (5.4.7). Using (5.4.6) and (5.4.4) we infer

$$\begin{aligned}\partial_1\,(Q^{[2]}_{\mu\nu},\ R^{[2]}) = \partial_0\Big(&(\partial^\nu Q^{[2]}_{\mu\nu} + \partial_\mu R^{[2]}) \otimes c_0^+ \bar{c}_1 \bar{\alpha}^\mu_{-1}|0\rangle \\ &-(\partial^\mu Q^{[2]}_{\mu\nu} + \partial_\nu R^{[2]}) \otimes c_0^+ c_1 \alpha^\nu_{-1}|0\rangle\Big)\,.\end{aligned} \tag{5.4.8}$$

Now we can apply ∂_1 to the argument of ∂_0 above and add the image of ∂_2 to obtain

$$d_2(Q^{[2]}_{\mu\nu},\ R^{[2]}) = (\delta Q^{[3]}_{\mu\nu},\ \delta R^{[3]}), \tag{5.4.9}$$

where

$$\begin{aligned}\delta Q^{[3]}_{\mu\nu} &= \Box Q^{[2]}_{\mu\nu} - \partial^\lambda \partial_\mu Q^{[2]}_{\lambda\nu} - \partial^\lambda \partial_\nu Q^{[2]}_{\mu\lambda} + 2\,\partial_\mu \partial_\nu R^{[2]}, \\ \delta R^{[3]} &= -\partial^\lambda \partial^\rho Q^{[2]}_{\lambda\rho} + 2\,\Box R^{[2]}.\end{aligned} \tag{5.4.10}$$

As we will prove in C, this map is surjective and therefore the cohomology at ghost number three is trivial. The structure of E_3 is presented in Fig. 5-4 (right). It is obvious that the spectral sequence collapses at E_3 and $E_\infty = E_3$. Therefore, we obtain the result that is similar to that in the chiral case (see Eq. (5.3.15)):

$$\mathrm{Gr}_r H^{r+s}(\widehat{Q}, \widehat{V}_0) = E^{r,s}_\infty = E^{r,s}_3. \tag{5.4.11}$$

Note that d_2 acts non-trivially only between ghost number two and ghost number three states, at the point where d_1 vanishes. This observation allows us to combine the calculation of E_2 and E_3 into one cohomology problem for the following complex

$$0 \to V^{(0)} \xrightarrow{d_1} V^{(1)} \xrightarrow{d_1} V^{(2)} \xrightarrow{d_2} V^{(3)} \xrightarrow{d_1} V^{(4)} \xrightarrow{d_1} V^{(5)} \to 0, \tag{5.4.12}$$

where $V^{(k)} = \bigoplus_r E_1^{r,k-r}$ are the ghost number k subspaces of E_1. This is very similar to what we did for the chiral complex (Eq. (5.3.12) and Eq. (5.3.13)), but, unfortunately, we lack a simple geometrical interpretation for the complex above.

Since d_1 and d_2 have x-degree -1 and -2 correspondingly, we can decompose the complex above into the sum of the following subcomplexes

$$0 \to V^{(0)}_{n+2} \xrightarrow{d_1} V^{(1)}_{n+1} \xrightarrow{d_1} V^{(2)}_{n} \xrightarrow{d_2} V^{(3)}_{n-2} \xrightarrow{d_1} V^{(4)}_{n-3} \xrightarrow{d_1} V^{(5)}_{n-4} \to 0, \tag{5.4.13}$$

where the subscripts refer to x-degree. Using this decomposition and the results of C one can easily calculate the dimensions of $\mathrm{Gr}_r H(\widehat{Q}, \widehat{V}_0) = E^{r,s}_3$ spaces.

Let us summarize our results on the semi-relative cohomology of the extended complex.

Theorem 2 *The semi-relative cohomology of the extended complex $\widehat{V}_p$, admits a natural filtration (by the* minus *x-degree) $\mathcal{F}_r H_S(\widehat{V}_p, \widehat{Q})$. The cohomology can be described using the associated graded spaces*

$$\mathrm{Gr}_r H_S(\widehat{V}_p, \widehat{Q}) = \mathcal{F}_r H_S(\widehat{V}_p, \widehat{Q}) / \mathcal{F}_{r+1} H_S(\widehat{V}_p, \widehat{Q})$$

as follows

1. *Non-zero momentum ($p \not\equiv 0$): cohomology is trivial unless $p^2 = 2n - 2$, where n is a non-negative integer. In the latter case*

$$\dim \mathrm{Gr}_r H^2_S(\widehat{V}_p, \widehat{Q}) = \binom{24-r}{24} d_n^2,$$
$$\dim H^l_S(\widehat{V}_p, \widehat{Q}) = 0 \qquad \text{for } l \neq 2$$

where d_n are generated by the following partition function

$$\prod_{k=1}^{\infty}(1 - z^k)^{-24} = \sum_{n=1}^{\infty} d_n z^n.$$

see ref. [79].

2. *Zero momentum case ($p \equiv 0$): the cohomology appears at ghost number zero (the vacuum)*

$$\dim H^0_S(\widehat{V}_p, \widehat{Q}) = 1,$$

ghost number one (global symmetries of the background, the Poincaré algebra),

$$\dim \mathrm{Gr}_r H^1_S(\widehat{V}_p, \widehat{Q}) = \begin{cases} D & \text{if } r = 0; \\ \dfrac{D(D-1)}{2} & \text{if } r = -1; \\ 0 & \text{otherwise;} \end{cases}$$

and ghost number two (physical states),

$$\begin{aligned}
\dim \mathrm{Gr}_0 H^2_S(\widehat{V}_p, \widehat{Q}) &= 1, \\
\dim \mathrm{Gr}_{-1} H^2_S(\widehat{V}_p, \widehat{Q}) &= \frac{D\,(D^2 - 3\,D + 8)}{6}, \\
\dim \mathrm{Gr}_{-2} H^2_S(\widehat{V}_p, \widehat{Q}) &= \frac{D(D-2)(D+2)}{3}, \\
\dim \mathrm{Gr}_{-3} H^2_S(\widehat{V}_p, \widehat{Q}) &= \frac{(D+2)\,(5\,D^3 - 16\,D^2 + 15\,D - 12)}{24}, \\
\dim \mathrm{Gr}_r H^2_S(\widehat{V}_p, \widehat{Q}) &= \chi_r \text{ for } r \leq -4.
\end{aligned}$$

where

$$\begin{aligned}
\chi_r = (D^2+1)&\left[\binom{D-1-r}{D-1} - \binom{D-3-r}{D-1}\right] \\
&+2D\left[\binom{D-r-4}{D-1} - \binom{D-r}{D-1}\right] \\
&+\binom{D-r+1}{D-1} - \binom{D-r-5}{D-1}.
\end{aligned}$$

There are no non-trivial cohomology states at any other ghost number,

$$\dim H^l_S(\widehat{V}_p, \widehat{Q}) = 0 \qquad \text{for } l \neq 0, 1, 2.$$

There is no evident structure among the ghost number two cohomology states. In the next section we are going to make some order in this zoo of zero momentum physical states using their transformation properties under the Lorentz group.

5.5 Lorentz group and extended complex

Since the cohomology of the extended complex is designed to describe the string theory near the flat background, the Lorentz group should act on it naturally, mapping the cohomology states to the cohomology states. In this section we investigate the action of the generators $J^{\mu\nu}$ of the infinitesimal Lorentz transformations.

5.5.1 Lorentz group acting on non-zero momentum extended complex

The generators of the infinitesimal Lorentz transformations, or the angular momentum operators of the closed string can be written as [79]

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + \overline{E}^{\mu\nu}, \tag{5.5.1}$$

where

$$\begin{aligned} l^{\mu\nu} &= x_0^\mu \alpha_0^\nu - x_0^\nu \alpha_0^\mu, \\ E^{\mu\nu} &= -i\sum_{n=1}^{\infty} \frac{1}{n}\,(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu), \\ \overline{E}^{\mu\nu} &= -i\sum_{n=1}^{\infty} \frac{1}{n}\,(\overline{\alpha}_{-n}^\mu \overline{\alpha}_n^\nu - \overline{\alpha}_{-n}^\nu \overline{\alpha}_n^\mu). \end{aligned} \tag{5.5.2}$$

Applying the zero mode part $l^{\mu\nu}$ to $\widehat{V}_p$, we obtain

$$l^{\mu\nu} = x^\mu \otimes p^\nu - x^\nu \otimes p^\mu + x^\mu \frac{\partial}{\partial x_\nu} \otimes 1 - x^\nu \frac{\partial}{\partial x_\mu} \otimes 1. \tag{5.5.3}$$

One can easily check that the $J^{\mu\nu}$ operators commute with $\widehat{Q}$ and therefore are well defined on the cohomology. Note that for $p \not\equiv 0$ these operators mix states of different x-degree.

Consider the massive case, $p^2 \not\equiv 0$. Recall that any physical state in the extended complex can be represented by

$$P(\tilde{x}^1, \cdots, \tilde{x}^{D-1}) \otimes c_1\bar{c}_1|v,p\rangle + \cdots, \tag{5.5.4}$$

where $|v,p\rangle$ is a dimension $(1,1)$ primary matter state with the momentum p and dots stand for the lower x-degree terms. Suppose the transverse components are chosen as

$$\tilde{x}^i = x^i - \frac{p^i(p\cdot x)}{p^2}. \tag{5.5.5}$$

We can obtain a state with the same leading term by applying the following operator to $c_1\bar{c}_1|v,p\rangle$

$$P\left(\frac{p_\mu J^{1\mu}}{p^2},\cdots,\frac{p_\mu J^{D-1\mu}}{p^2}\right)\, c_1\bar{c}_1|v,p\rangle = P(\tilde{x}^1,\cdots,\tilde{x}^{D-1})\otimes c_1\bar{c}_1|v,p\rangle+\cdots\,. \tag{5.5.6}$$

Together with the results on the cohomology of the extended complex obtained in the previous section, Eq. (5.4.4), this shows that the semi-relative cohomology of a $p^2\neq 0$ extended complex is spanned by the states which are generated from the standard physical states by the infinitesimal Lorentz boosts. Although the analysis above fails for the massless states $p^2=0$ $(p\not\equiv 0)$, the result is still the same. One can check it using the explicit formulae for the cohomology states at $p^2=0$.

We conclude that the ghost number two cohomology of the extended complex at $p\not\equiv 0$ has the same physical contents as that of the BRST complex.

5.5.2 Lorentz group and the zero momentum states

The Lorentz generators, $J^{\mu\nu}$, act on the zero-momentum states as linear operators of zero x-degree:

$$J^{\mu\nu}=x^\mu\frac{\partial}{\partial x_\nu}\otimes 1 - x^\nu\frac{\partial}{\partial x_\mu}\otimes 1 + 1\otimes(E^{\mu\nu}+\overline{E}^{\mu\nu}). \tag{5.5.1}$$

Consider the complexes

$$0\longrightarrow V^{(0)}_{n+2}\xrightarrow{d_1}V^{(1)}_{n+1}\xrightarrow{d_1}V^{(2)}_{n}\xrightarrow{d_2}V^{(3)}_{n-2}\xrightarrow{d_1}V^{(4)}_{n-3}\xrightarrow{d_1}V^{(5)}_{n-4}\longrightarrow 0. \tag{5.5.2}$$

which calculate $\mathrm{Gr}H_S(\widehat{V}_0,\widehat{Q})$. By definition $V^{(0)}_n$ and $V^{(5)}_n$ are the spaces of homogeneous polynomials of degree n. These spaces are reducible under the Lorentz group because the subspaces of the polynomials of the form $(x^\mu x_\mu)^k h_{n-2k}$ are invariant under $SO(1,D-1)$. Furthermore, if h_{n-2k} are harmonic, $\Box h_{n-2k}=0$, these subspaces form irreducible representations of $SO(1,D-1)$. We will denote these irreducible representations by $\mathbf{H}_n$. These representations can be alternatively described by Young tableaux as

$$\mathbf{H}_n=\overbrace{\Box\Box\cdots\Box}^{n}\,. \tag{5.5.3}$$

Now we can write the decomposition of $V^{(0)}_n$ or $V^{(5)}_n$ into irreducible representations as

$$V^{(0)}_n=V^{(5)}_n=\mathbf{H}_n+\mathbf{H}_{n-2}+\mathbf{H}_{n-4}+\cdots\,. \tag{5.5.4}$$

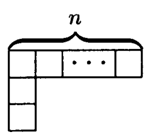
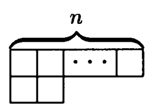
At the ghost numbers one and four spaces we find another kind of irreducible representations:

$$\mathbf{V}_n=\begin{array}{l}\Box\overbrace{\Box\cdots\Box\Box}^{n}\\ \Box\end{array}, \tag{5.5.5}$$

$V^{(0)}_{n+2}$	$V^{(1)}_{n+1}$	$V^{(2)}_{n}$	$V^{(3)}_{n-2}$	$V^{(4)}_{n-3}$	$V^{(5)}_{n-4}$
$\mathbf{H}_{n+2}$	$2\,\mathbf{H}_{n+2}$	$\mathbf{H}_{n+2}$			
$\mathbf{H}_{n}$	$4\,\mathbf{H}_{n}+2\,\mathbf{V}_{n}$	$5\,\mathbf{H}_{n}+\mathbf{S}_{n}$ $+\mathbf{A}_{n}+2\,\mathbf{V}_{n}$	$\mathbf{H}_{n}$		
$\mathbf{H}_{n-2}$	$4\,\mathbf{H}_{n-2}+2\,\mathbf{V}_{n-2}$	$6\,\mathbf{H}_{n-2}+\mathbf{S}_{n-2}$ $+\mathbf{A}_{n-2}+4\,\mathbf{V}_{n-2}$	$5\,\mathbf{H}_{n-2}+\mathbf{S}_{n-2}$ $+\mathbf{A}_{n-2}+2\,\mathbf{V}_{n-2}$	$2\,\mathbf{H}_{n-2}$	
$\mathbf{H}_{n-4}$	$4\,\mathbf{H}_{n-4}+2\,\mathbf{V}_{n-4}$	$6\,\mathbf{H}_{n-4}+\mathbf{S}_{n-4}$ $+\mathbf{A}_{n-4}+4\,\mathbf{V}_{n-4}$	$6\,\mathbf{H}_{n-4}+\mathbf{S}_{n-4}$ $+\mathbf{A}_{n-4}+4\,\mathbf{V}_{n-4}$	$2\,\mathbf{H}_{n-4}+2\,\mathbf{V}_{n-4}$	$\mathbf{H}_{n-4}$
$\mathbf{H}_{n-6}$	$4\,\mathbf{H}_{n-6}+2\,\mathbf{V}_{n-6}$	$6\,\mathbf{H}_{n-6}+\mathbf{S}_{n-6}$ $+\mathbf{A}_{n-6}+4\,\mathbf{V}_{n-6}$	$6\,\mathbf{H}_{n-6}+\mathbf{S}_{n-6}$ $+\mathbf{A}_{n-6}+4\,\mathbf{V}_{n-6}$	$2\,\mathbf{H}_{n-6}+2\,\mathbf{V}_{n-6}$	$\mathbf{H}_{n-6}$
...	...	...	...	...	...

Table 5.4: Decomposition of the complex (5.5.2) into irreducible representations of $SO(1, D-1)$ for $n \geq 2$

and, finally in the decompositions of $V^{(2)}_n$ and $V^{(3)}_n$ we will encounter

$\mathbf{A}_n =$  and $\mathbf{S}_n =$  . (5.5.6)

Suppose $n \geq 2$. Table 5.4 shows the decomposition of the whole complex (5.5.2) into irreducible representations. From the series of lemmas presented in C, we know that the complex (5.5.2) has cohomology only in $V^{(2)}_n$. Using Table 5.4 we conclude that

$$\mathrm{Gr}_{-n} H^2_S(\widehat{V}_0, \widehat{Q}) = \mathbf{H}_n + \mathbf{A}_n + \mathbf{S}_n, \tag{5.5.7}$$

which we obtain by "subtracting" the odd columns from column two and "adding" the even columns. A more detailed analysis shows that if we choose the representatives of $\mathrm{Gr}_{-n} H^2_S(\widehat{V}_0, \widehat{Q})$ so that they belong to $\mathbf{H}_n$, $\mathbf{A}_n$, or $\mathbf{S}_n$, they also will represent cohomology classes of $\widehat{Q}$, no lower x-degree corrections required.

Two exceptional cases, $n = 0$ and $n = 1$, have to be treated separately. The decompositions of the complex (5.5.2) into irreducible representations for these the first case is presented in Table 5.5. Using the results of C we can infer from Table 5.5

$V_2^{(0)}$	$V_1^{(1)}$	$V_0^{(2)}$
$\mathbf{H}_2$	$2\,\mathbf{H}_2$	$\mathbf{H}_2$
$\mathbf{H}_0$	$2\,\mathbf{H}_0 + 2\,\mathbf{V}_0$	$\mathbf{H}_0 + \mathbf{V}_0$

$V_3^{(0)}$	$V_2^{(1)}$	$V_1^{(2)}$
$\mathbf{H}_3$	$2\,\mathbf{H}_3$	$\mathbf{H}_3$
$\mathbf{H}_1$	$4\,\mathbf{H}_1 + 2\,\mathbf{V}_1$	$4\,\mathbf{H}_1 + \mathbf{A}_1 + 2\,\mathbf{V}_1$

Table 5.5: Decomposition of the complex (5.5.2) into irreducible representations for $n = 0$ (left) and $n = 1$ (right)

(left) that

$$\mathrm{Gr}_{-1}H^1_S(\widehat{Q},\widehat{V}_0) = \mathbf{V}_0 \quad \text{and} \quad \mathrm{Gr}_0 H^2_S(\widehat{Q},\widehat{V}_0) = \mathbf{H}_0, \tag{5.5.8}$$

and from Table 5.5 (right) that

$$\mathrm{Gr}_{-1}H^2_S(\widehat{Q},\widehat{V}_0) = \mathbf{H}_1 + \mathbf{A}_1; \tag{5.5.9}$$

and again, if we pick the representatives of $\mathrm{Gr}H_S$ from the irreducible representations, they will be annihilated by $\widehat{Q}$ and therefore represent the cohomology classes without lower x-degree corrections. For this case this can be easily checked by explicit calculation (see C).

It is tempting to interpret the irreducible representations $\mathbf{H}_n$, $\mathbf{S}_n$, and $\mathbf{A}_n$ as the dilaton, graviton, and antisymmetric tensor. If we do so, it is not quite clear why we have infinitely many irreducible representations for each field, and not just one. We speculate that these representations are related by infinitesimal shifts (the translational part of the Poincaré algebra), which acts on the spaces of polynomials by differentiation with respect to x^μ.

Appendix A

The tachyon potential and string field redefinitions.

Here we wish to discuss whether it is possible to make a field redefinition of the string field tachyon such that the string action is brought to a form where one could rule out the existence of a local minimum. Even at the level of open string field theory this seems hard to achieve. The tachyon potential is of the form $V \sim -\tau^2 + g\tau^3$. The cubic term produces a local minimum with a nonzero vacuum expectation value for τ. We are not allowed to just redefine τ to absorb the cubic term in the quadratic one; this is a non-invertible field redefinition. Using the massive fields is no help since the transformations must preserve the kinetic terms, and therefore should be of the form $\tau \rightarrow \tau + f(\phi_i, \tau)$ and $\phi_i \rightarrow \phi_i + g(\phi_i, \tau)$, with f and g functions that must start quadratic in the fields. Such transformations cannot eliminate the cubic term in the tachyon potential.

Let us examine the question of field redefinitions in a more stringy way. Assume it is possible to write the string action as

$$S = \int dx \, [\mathcal{L}(\nabla\tau, \phi_i) + \tau^2(1 + f(\phi_i))] \,, \tag{A.0.1}$$

namely, that one can separate out a term just depending on derivatives of the tachyon field, and all other fields, and a quadratic term for the tachyon potential. The term $f(\phi_i)$ was included to represent couplings to fields like dilatons or background metric. If the above were true we would expect no perturbatively stable minimum for the tachyon (the factor $(1 + f(\phi_i))$ is expected to be nonvanishing). We will now argue that the string action *cannot* be put in the form described in Eqn.Eq. (A.0.1) by means of a string field redefinition. It is therefore not possible to rule out a local minimum by such simple means.[1]

If Eqn.Eq. (A.0.1) holds, a constant infinitesimal shift of the tachyon field $\tau \rightarrow \tau + \epsilon$ would have the effect of shifting the action as

$$S \rightarrow S + 2\epsilon \int dx \, \tau(1 + f(\phi_i)) + \mathcal{O}(\epsilon^2) \,. \tag{A.0.2}$$

[1]It is not clear to us whether this result is in contradiction with that of Ref. [42], where presumably the relevant action is the effective action obtained after integrating out classically the massive fields.

We should be able to prove such "low-energy tachyon theorem" with string field theory. For this we must find the change in the string action as we shift the string field as follows

$$|\Psi\rangle_1 \to |\Psi\rangle_1 + \epsilon|T_0\rangle_1 + \epsilon\langle h^{(2)}_{23}|\Psi\rangle_2|\mathcal{S}_{13}\rangle + \cdots\,, \tag{A.0.3}$$

where $|T_0\rangle = c_1\bar{c}_1|\mathbf{1}\rangle$, the dots indicate quadratic and higher terms in the string field, $\langle h^{(2)}_{23}|$ is a symmetric bra, and $|\mathcal{S}_{13}\rangle$ is the sewing ket [15]. Indeed the transformation of the string field cannot be expected to be a simple shift along the zero-momentum tachyon, since the string field tachyon should differ from the tachyon appearing in Eq. (A.0.1). If we now vary the string action Eq. (2.2.4) we find

$$S \to S + \epsilon\langle\Psi|c_0^- Q|T_0\rangle + \frac{1}{2}\epsilon\left[\langle V^{(3)}_{123}|T_0\rangle_3 + \langle h^{(2)}_{12}|(Q_1+Q_2)\right]|\Psi\rangle_1|\Psi\rangle_2 + \cdots\,. \tag{A.0.4}$$

We must now see that by a suitable choice of $\langle h^{(2)}_{12}|$ the variation of the action takes the form required by Eq. (A.0.2). Indeed, the term $\epsilon\tau$ arises from the $\epsilon\langle\Psi|c_0^- Q|T_0\rangle$ term in Eq. (A.0.4) since $c_0^- Q|T_0\rangle$ can only couple to the tachyon field in Ψ. Assume the function $f(\phi_i)$ in Eq. (A.0.2) is zero, in that case there is no extra variation in the action, and we must require that

$$\langle V^{(3)}_{123}|T_0\rangle_3 + \langle h^{(2)}_{12}|(Q_1+Q_2) = 0\,. \tag{A.0.5}$$

This equation cannot have solutions; acting once more with (Q_1+Q_2) we find that Eq. (A.0.5) requires that $\langle V^{(3)}_{123}|Q_3|T_0\rangle_3 = 0$ which cannot hold (recall $Q|T_0\rangle \neq 0$). Even if $f(\phi_i)$ is not zero, we do not expect solutions to exist. In this case we still must have that Eq. (A.0.5) should be zero contracted with any arbitrary two states, except when one of them is a zero momentum tachyon. Again using states of the form $(Q_1+Q_2)|a\rangle_1|b\rangle_1$ where neither $|a\rangle$ nor $|b\rangle$ is a zero momentum tachyon, we see that again the equation cannot be satisfied. This shows that there is no simple "low-energy tachyon theorem" that rules out a local minimum.

Appendix B

Spectral sequence

In this section we review some basic facts about a particular type of the spectral sequence which we use in our analysis of the extended complex. This is not intended to be a complete introduction to the method. Our only goal is to introduce the spaces E_0, E_1 and E_2 equipped with differentials d_0, d_1 and d_2 acting on them. We will prove that these differentials have zero square and provide some motivations to why their cohomologies are related to $\mathrm{Gr}H$. For a more detailed analysis of the first three terms of a spectral sequence, the reader is referred to the book by Dubrovin, Fomenko and Novikov [89]. A general introduction to the spectral sequences from the physicist's point of view and further references can be found in refs. [90, 91].

Let (C, d) be a complex with additional grading $C = \bigoplus C_r$ such that the differential d can be written as

$$d = \partial_0 + \partial_1 + \partial_2, \tag{B.0.1}$$

where ∂_n maps C_r to C_{r+n}. Since d mixes vectors from different gradings we can not define grading on cohomology $H(d, C)$, but we can still define a decreasing filtration. By filtration of the element $x \in H(d, \mathrm{Gr}C)$ we will mean the smallest (negative) integer s such that x is representable by a cocycle

$$\overline{x} = x_r + x_{r+1} + \cdots, \tag{B.0.2}$$

where $x_r \in C_r$. We will denote the space of such vectors by $F_r H(d, C)$. Using the filtration we can define a graded space associated to the cohomology $H(d, C)$

$$\mathrm{Gr}H = \bigoplus_s \mathrm{Gr}_s H, \quad \text{where} \quad \mathrm{Gr}_s H = F_s H / F_{s+1} H. \tag{B.0.3}$$

The investigation of the spaces $\mathrm{Gr}_s H^n$ is carried out using the method of "successive approximations" based on what is called the "spectral sequence". The idea is to construct a sequence of complexes (E_n, d_n) such that $E_{n+1} = H(d_n, E_n)$ which converges to $\mathrm{Gr}H$

$$\mathrm{Gr}_s H^n = E_\infty^{s,n-s}. \tag{B.0.4}$$

Differentials d_n are acting on the spaces $E_n^{r,s}$ as follows

$$d_n : \quad E_n^{r,s} \longrightarrow E_n^{r+n,s-n+1} \tag{B.0.5}$$

For a complete description of the spectral sequence and the proof of the theorem which states that the spectral sequence converges to GrH we refer to [89,92]. Let us describe the first few terms of the spectral sequence. Suppose $\overline{x}$ given in Eq. (B.0.2) represent a cohomology class x in $F_sH(d,C)$. Applying $d = \partial_0 + \partial_1 + \partial_2$ to $\overline{x}$ we obtain

$$d\overline{x} = \partial_0\overline{x}_r + (\partial_1\overline{x}_r + \partial_0\overline{x}_{r+1}) + (\partial_2\overline{x}_r + \partial_1\overline{x}_{r+1} + \partial_0\overline{x}_{r+2}) \\ +(\partial_2\overline{x}_{r+1} + \partial_1\overline{x}_{r+2} + \partial_0\overline{x}_{r+3}) + \cdots. \tag{B.0.6}$$

where we enclosed in braces the terms from the same C_r space. It follows that

$$\partial_0\overline{x}_r = 0, \quad \partial_1\overline{x}_r = -\partial_0\overline{x}_{r+1}, \quad \partial_2\overline{x}_r = -\partial_1\overline{x}_{r+1} - \partial_0\overline{x}_{r+2}, \ldots, \tag{B.0.7}$$

form which we conclude that $\overline{x}_r$ is a ∂_0 cocycle and a ∂_1 cocycle modulo image of ∂_0. This suggests that the first approximation in the spectral sequence should be $E_1 = H(\partial_0, C)$ and $d_0 = \partial_0$, the second approximation is $E_2 = H(d_1, E_1)$, where d_1 is induced on $E_1 = H(\partial_0, C)$ by ∂_1. Using the second equation of (B.0.7) we can formally find $\overline{x}_{r+1}$ in terms of $\overline{x}_r$ as $\overline{x}_{r+1} = -\partial_0^{-1}\partial_1\overline{x}_r$ and rewrite the last equation of (B.0.7) as

$$(\partial_2 - \partial_1\partial_0^{-1}\partial_1)\overline{x}_r = -\partial_0\overline{x}_{r+2}. \tag{B.0.8}$$

The last formulae suggests that d_2 is induced on $E_2 = H(d_1, E_1)$ by $\partial_2 - \partial_1\partial_0^{-1}\partial_1$ and the third approximation is $E_3 = H(d_2, E_2)$.

Let us show that these differentials are well defined and square to zero. For d_0 the first is obvious since it acts on the same space as ∂_0 and $\partial_0^2 = 0$ follows from $d^2 = 0$ which is equivalent to

$$\partial_0^2 = \{\partial_0, \partial_1\} = \partial_1^2 + \{\partial_0, \partial_2\} = \partial_2^2 = 0. \tag{B.0.9}$$

In order to show that d_1 is well defined we have to show that ∂_1 maps ∂_0-closed vectors to ∂_0-closed vectors and ∂_0-trivial to ∂_0-trivial. This easily follows from the anticommutation relation $\{\partial_0, \partial_1\} = 0$. Let us prove that $d_1^2 = 0$. Suppose $x \in E_1$ can be represented by a cocycle $\overline{x} \in C$, $\partial_0\overline{x} = 0$. Then applying $\partial_1^2 = -\{\partial_0, \partial_2\}$ to $\overline{x}$ we obtain a trivial cocycle $\partial_1^2\overline{x} = -\partial_0\partial_2\overline{x}$. In cohomology this implies that $d_1^2 x = 0$.

Before we consider the differential d_2, let us describe the space E_2 on which it acts in greater details. By definition $E_2 = H(d_1, E_1)$, but E_1 in turn is the cohomology of the original complex with respect to ∂_0. Therefore, in order to find a d_1 cocycle we should start with a ∂_0 cocycle $\overline{x}$ and require that its image under ∂_1 is ∂_0 exact. Two cocycles $\overline{x}$ and $\overline{x}'$ represent the same d_1 cohomology class if $\overline{x}' - \overline{x} \in \Im\mathrm{m}\partial_0 + \partial_1 \ker \partial_0$. Let $\overline{x}$ be a d_1 cocycle, which means that

$$\partial_0\overline{x} = 0 \qquad \text{and} \qquad \partial_1\overline{x} = \partial_0\overline{y}. \tag{B.0.10}$$

We define $d_2\overline{x}$ as

$$d_2\overline{x} = \partial_2\overline{x} - \partial_1\overline{y} = (\partial_2 - \partial_1\partial_0^{-1}\partial_1)\overline{x}. \tag{B.0.11}$$

Let us show that the result is again a d_1 cocycle. Indeed, using the properties of ∂_n listed in Eq. (B.0.9) we obtain

$$\partial_0 d_2 \overline{x} = \partial_0 \partial_2 \overline{x} + \partial_0 \partial_1 \overline{y} = \partial_0 \partial_2 \overline{x} - \partial_1 \partial_0 \overline{y} = \{\partial_0, \partial_2\} \overline{x} - \partial_1^2 \overline{x} = 0$$

and

$$\partial_1 d_2 \overline{x} = \partial_1 \partial_2 \overline{x} - \partial_1^2 \overline{y} = \partial_0 \partial_2 \overline{y}.$$

Similarly we can prove that if $\overline{x}$ and $\overline{x}'$ belong to the same d_1 cohomology class then their d_2 images belong to the same class as well. This will finally establish the correctness of the definition of d_2 as an operator on E_2. In conclusion let us show that $d_2^2 = 0$. With $\overline{x}$ as above we have

$$\begin{aligned} d_2^2 \overline{x} =& \partial_2 (\partial_2 \overline{x} - \partial_1 \overline{y}) - \partial_1 \partial_0^{-1} \partial_1 (\partial_2 \overline{x} - \partial_1 \overline{y}) \\ =& \partial_1 \partial_2 \overline{y} + \partial_1 \partial_0^{-1} \partial_2 \partial_1 \overline{x} - \partial_1 \partial_0^{-1} \partial_0 \partial_2 \overline{y} - \partial_1 \partial_0^{-1} \partial_2 \partial_0 \overline{y} \\ =& \partial_1 \partial_2 \overline{y} - \partial_1 \partial_2 \overline{y} = 0 \end{aligned}$$

which completes our analysis of d_2.

Appendix C

Three lemmas

In this appendix we will calculate the cohomology of the following complex

$$0 \to V^{(0)} \xrightarrow{d^{(0)}} V^{(1)} \xrightarrow{d^{(1)}} V^{(2)} \xrightarrow{d^{(2)}} V^{(3)} \xrightarrow{d^{(3)}} V^{(4)} \xrightarrow{d^{(4)}} V^{(5)} \to 0, \qquad \text{(C.0.1)}$$

where $V^{(0)} \simeq V^{(0)} \simeq \mathbb{C}[x^0 \cdots x^{D-1}]$, $V^{(0)} \simeq V^{(0)} \simeq \mathbb{C}^{2D}[x^0 \cdots x^{D-1}]$ and $V^{(0)} \simeq V^{(0)} \simeq \mathbb{C}^{D^2+1}[x^0 \cdots x^{D-1}]$. Following the notations of section 5.4.3 we represent the elements of $V^{(0)}$ and $V^{(5)}$ as $(R^{[0]})$ and $(R^{[5]})$, elements of $V^{(2)}$ and $V^{(4)}$ as $(Q^{[2]}_{\mu\nu}, R^{[2]})$ and $(Q^{[3]}_{\mu\nu}, R^{[3]})$, and elements of $V^{(2)}$ and $V^{(4)}$ by $(P_\mu, \overline{P}_\mu)$. Differentials $d^{(n)}$ act as follows

$$\begin{array}{lcl}
(R^{[0]}) & \overset{d^{(0)}}{\searrow} & (0) \\
(P^{[1]}_\mu,\ \overline{P}^{[1]}_\mu) & \overset{d^{(1)}}{\searrow} & (\partial_\mu R^{[0]},\ \partial_\mu R^{[0]}) \\
(Q^{[2]}_{\mu\nu},\ R^{[2]}) & \overset{d^{(2)}}{\searrow} & (\partial_\mu \overline{P}^{[1]}_\nu - \partial_\nu P^{[1]}_\mu,\ \partial^\mu \overline{P}^{[1]}_\mu - \partial^\mu P^{[1]}_\mu) \\
(Q^{[3]}_{\mu\nu},\ R^{[3]}) & \overset{d^{(3)}}{\searrow} & (\delta Q^{[3]}_{\mu\nu},\ \delta R^{[3]}) \\
(P^{[4]}_\mu,\ \overline{P}^{[4]}_\mu) & \overset{d^{(4)}}{\searrow} & (\partial^\nu Q^{[3]}_{\mu\nu} + \partial_\mu R^{[3]},\ \partial^\nu Q^{[3]}_{\nu\mu} + \partial_\mu R^{[3]}) \\
(R^{[5]}) & & (\partial_\mu P^{[4]} - \partial_\mu \overline{P}^{[4]})
\end{array} \qquad \text{(C.0.2)}$$

where

$$\delta Q^{[3]}_{\mu\nu} = \Box Q^{[2]}_{\mu\nu} - \partial^\lambda \partial_\mu Q^{[2]}_{\lambda\nu} - \partial^\lambda \partial_\nu Q^{[2]}_{\mu\lambda} + 2\partial_\mu \partial_\nu R^{[2]} \qquad \text{(C.0.3)}$$

$$\delta R^{[3]} = -\partial^\lambda \partial^\rho Q^{[2]}_{\lambda\rho} + 2\Box R^{[2]} \qquad \text{(C.0.4)}$$

It is obvious that $d^{(4)}$ is surjective and the kernel of $d^{(0)}$ contains only constant polynomials. Thus we conclude that $H^0 = \mathbb{C}$ and $H^5 = 0$. The other cohomology spaces are described by the following lemmas

Lemma 1 *H^1 is finite dimensional and* $\dim H^1 = \frac{D(D+1)}{2}$. *$H^1$ cohomology classes can be represented by polynomials of degree no bigger than one.*

Proof. According to (C.0.2) H^1 is a quotient of the space S of solutions to the system of first order differential equations

$$\begin{cases} \partial_\mu P_\nu = \partial_\nu \overline{P}_\mu \\ \partial^\mu P_\mu = \partial^\mu \overline{P}_\mu \end{cases} \tag{C.0.5}$$

by the space T of trivial solutions $P_\mu = \overline{P}_\mu = \partial_\mu R$. Note that both S and T naturally decompose into direct sum of the spaces of homogeneous polynomials and so does the quotient

$$S = \bigoplus S^n, \quad T = \bigoplus T^n, \quad H^1 = S/T = \bigoplus S^n/T^n \tag{C.0.6}$$

We want to prove that $S^n = T^n$ for $n > 1$. Let P_μ and $\overline{P}_\mu$ be homogeneous polynomials of degree $n > 1$ that satisfy Eq. (C.0.5). First, it is obvious that $P_\mu = 0$ for every μ requires $\overline{P}_\mu = 0$. Indeed, if $P_\mu = 0$ for every μ then according to the first equation in Eq. (C.0.5) $\partial_\nu \overline{P}_\mu = 0$ for every μ and ν and since by assumption $\deg \overline{P}_\mu > 1$ this means $\overline{P}_\mu = 0$. Second, using the first equation of (C.0.5) twice we obtain

$$\partial_\alpha \partial_\nu P_\mu = \partial_\alpha \partial_\mu \overline{P}_\nu = \partial_\mu \partial_\alpha \overline{P}_\nu = \partial_\alpha \partial_\mu P_\mu. \tag{C.0.7}$$

Therefore for any α, μ and ν

$$\partial_\mu (\partial_\alpha P_\mu - \partial_\mu P_\alpha) = 0. \tag{C.0.8}$$

And since $\deg(\partial_\alpha P_\mu - \partial_\mu P_\alpha) = n - 1 > 0$ we conclude that $\partial_\alpha P_\mu - \partial_\mu P_\alpha = 0$ and there exist R such that $P_\mu = \partial_\mu R$. Subtracting a trivial solution $(\partial_\mu R, \partial_\mu R)$ from $(P_\mu, \overline{P}_\mu)$ we get another solution $(P'_\mu = 0, \overline{P}'_\mu = \overline{P}_\mu - \partial_\mu R)$. According to our first observation $P'_\mu = 0$ requires $\overline{P}'_\mu = 0$ and thus $\overline{P}_\mu = P_\mu = \partial_\mu R$.

It is easy to see that there are exactly $\frac{D(D-1)}{2}$ non-trivial solutions of degree one and D non-trivial constant solutions which can be written as

$$P_\mu = -\overline{P}_\mu = \xi_{[\mu\nu]} x^\nu, \quad \text{and} \quad P_\mu = -\overline{P}_\mu = const \tag{C.0.9}$$

where $\xi_{[\mu\nu]}$ is an antisymmetric tensor.

Lemma 2 $H^3 = 0$

Proof. This is the most difficult lemma in this work. We have to show that any solution of the system

$$\begin{aligned} \partial^\nu Q^{[3]}_{\mu\nu} + \partial_\mu R^{[3]} &= 0 \\ \partial^\nu Q^{[3]}_{\nu\mu} + \partial_\mu R^{[3]} &= 0 \end{aligned} \tag{C.0.10}$$

can be represented in the form

$$\begin{aligned} \delta Q^{[3]}_{\mu\nu} &= \Box Q^{[2]}_{\mu\nu} - \partial^\lambda \partial_\mu Q^{[2]}_{\lambda\nu} - \partial^\lambda \partial_\nu Q^{[2]}_{\mu\lambda} + 2\partial_\mu \partial_\nu R^{[2]} \\ \delta R^{[3]} &= -\partial^\lambda \partial^\rho Q^{[2]}_{\lambda\rho} + 2\Box R^{[2]} \end{aligned} \tag{C.0.11}$$

We will start from an arbitrary solution of the system (C.0.10) and will be modifying it step by step by adding the trivial solutions of the form (C.0.11) in order to get zero.

We can use the same arguments as in lemma 1 to consider only homogeneous polynomials of some degree m. Suppose $m \geq 1$. We will describe an iterative procedure which will allow us to modify $(Q^{[3]}_{\mu\nu},\ R^{[3]})$ so that $Q^{[3]}_{\mu\nu}$ will depend only on one variable, say x^0 and its only nonzero components be $Q^{[3]}_{\mu 0}$ and $Q^{[3]}_{0\nu}$.

If this is the case, the cocycle condition (Eq. (C.0.10)) tells us that

$$\partial_\mu R = \partial_0 Q_{\mu,0} = \partial_0 Q_{0,\mu}, \tag{C.0.12}$$

moreover, $Q_{\mu,0} = C_\mu x_0^m$ and $Q_{0,\mu} = \overline{C}_\mu x_0^m$. Furthermore, using Eq. (C.0.12) we conclude that $C_\mu = \overline{C}_\mu$. Integrating Eq. (C.0.12) we obtain

$$R^{[3]} = \sum_{i=1}^{D-1} mC_i x_i x_0^{m-1} + C_0 x_0^{26}. \tag{C.0.13}$$

One can check that such solution $(Q^{[3]}_{\mu\nu},\ R^{[3]})$ can be written in the form (C.0.11) with $Q^{[2]}_{\mu,\nu} = 0$ and

$$R^{[2]} = \frac{1}{2}\sum_{i=1}^{D-1}\left(\frac{C_1}{m+1}x_i x_0^{m+1} + \frac{C_0}{(m+1)(m+2)}x_0^{m+2}\right).$$

Now let us describe the procedure which reduces any solution to the abovementioned form. Our first objective is to get rid of x_i dependence for $i = 1..D-1$. Let us pick i. The following four step algorithm will make $(Q_{\mu\nu}, R)$ independent of x_i. We will see that when we apply the procedure to $(Q_{\mu\nu}, R)$ which does not depend on some other x_k it will not introduce x_k dependence in the output. This observation will allow us to apply the algorithm $D-1$ times and make $(Q_{\mu\nu}, R)$ depend only on x_0.

Step 1 Let us introduce some notations. For a polynomial P we will denote the minimal degree of x_i among all the monomials in P by $n_i(P)$. For a zero polynomial we formally set $n_i(0) = +\infty$. Given a matrix of polynomials $Q_{\mu,\nu}$, let

$$N_i(Q_{\mu\nu}) = \min_{\substack{\mu\neq i\\ \nu\neq i}} n_i(Q_{\mu\nu}) \tag{C.0.14}$$

Since $Q^{[3]}_{\mu\nu}$ are homogeneous polynomials of degree m we can write them as

$$Q^{[3]}_{\mu\nu} = \sum_{m_0+\cdots+m_{D-1}=m} C_{m_0\cdots m_{D-1},\mu\nu} x_0^{m_0}\cdots x_{D-1}^{m_{D-1}} \tag{C.0.15}$$

Let us show that it is possible to add a trivial solution to $(Q_{\mu\nu}, R)$ and increase $N_i(Q_{\mu\nu})$ by one. Indeed, let

$$Q^{[2]}_{\mu\nu} = \begin{cases} \displaystyle\sum_{m_0+\cdots+m_{D-1}=m} -\frac{C_{m_0\cdots m_{D-1},\mu\nu}\, x_0^{m_0}\cdots x_i^{m_i+2}\cdots x_{D-1}^{m_{D-1}}}{(m_i+1)(m_i+2)} & \text{for } \mu,\nu\neq i,\\ 0 & \text{otherwise,}\end{cases}$$

and $R^{[2]}=0$.

It is easy to see that

$$N_i(Q^{[3]}_{\mu\nu}+\delta Q^{[3]}_{\mu\nu})\geq N_i(Q^{[3]}_{\mu\nu})+1$$

where $\delta Q^{[3]}_{\mu\nu}$ comes from the trivial solution generated by $(Q^{[2]}_{\mu\nu},R^{[2]})$ according to (C.0.11). Repeating this procedure, we will increase $N_i(Q^{[3]}_{\mu\nu})$ at least by one every time. Since $Q^{[3]}_{\mu\nu}$ are homogeneous polynomials of degree m, N_i is either less then $m+1$ or equal $+\infty$. Therefore after a finite number of steps we will make $N_i=+\infty$ which means that all $Q^{[3]}_{\mu\nu}$ are zero for $\mu\neq i$ and $\nu\neq i$.

Step 2 Since $(Q^{[3]}_{\mu\nu},R^{[3]})$ is a solution to the system (C.0.10) we can write

$$\partial_i Q^{[3]}_{\mu i}=\partial_\mu R^{[3]}=\partial_i Q^{[3]}_{i\mu},\quad \mu\neq i$$

$$\partial^\nu Q^{[3]}_{\nu i}=\partial_i R^{[3]}=\partial^\nu Q^{[3]}_{i\nu}.$$

Suppose

$$R^{[3]}=\sum_{m_0+\cdots+m_{D-1}=m} D_{m_0\cdots m_{D-1}}x_0^{m_0}\cdots x_{D-1}^{m_{D-1}},$$

then we choose $R^{[2]}=0$,

$$Q^{[2]}_{ii}=\sum_{m_0+\cdots+m_{D-1}=m}\frac{D_{m_0\cdots m_{D-1}}}{(m_i+1)(m_i+2)}x_0^{m_0}\cdots x_i^{m_i+2}\cdots x_{D-1}^{m_{D-1}},$$

and $Q^{[2]}_{\mu\nu}=0$ for all the other μ and ν. It is easy to see that $\widetilde{Q}^{[3]}_{\mu\nu}=Q^{[3]}_{\mu\nu}+\delta Q^{[3]}_{\mu\nu}$ have the following properties:

- all $\widetilde{Q}^{[3]}_{\mu\nu}$ except $\widetilde{Q}_{ii}$ do not depend on x_i
- $\widetilde{Q}^{[3]}_{\mu\nu}=0$ for all $\mu\neq i$ and $\nu\neq i$.

Step 3 Suppose $(Q^{[3]}_{\mu\nu},R^{[3]})$ is of the form we obtained at the end of step 2. Since $\partial_i Q^{[3]}_{\mu i}=0$ for $\mu\neq i$, then $\partial_\mu R^{[3]}=0$ for $\mu\neq i$ and therefore, $R^{[3]}$ depends only on x_i. Thus there exists $R^{[2]}$ such that $R^{[3]}=-2\partial_i^2 R^{[2]}$ and $R^{[2]}$ depends only on x_i. adding a trivial solution generated by $(0,R^{[2]})$ we can make $R^{[3]}=0$.

Step 4 Using $R^{[3]}=0$ we can rewrite the system (C.0.10) as follows.

$$\partial_i Q^{[3]}_{ii}=-\sum_{\nu\neq i}\partial^\nu Q^{[3]}_{\nu i}.$$

Therefore, $Q^{[3]}_{ii}=Q^{[3]}_{ii,0}+x_i Q^{[3]}_{ii,1}$, where $Q^{[3]}_{ii,0}$ and $Q^{[3]}_{ii,1}$ do not depend on x_i. For every $Q^{[3]}_{ii,1}$ we can find a polynomial P which depends on the same set of variables and $\Box P=-Q^{[3]}_{ii,1}$. Choose $Q^{[2]}_{ii}=x_i\,P$, $R^{[2]}=0$ and $Q^{[2]}_{\mu\nu}=0$ for $(\mu,\nu)\neq(i,i)$. Adding the corresponding trivial solution we achieve that $Q^{[3]}_{\mu\nu}$ and $R^{[3]}$ do not depend on x_i.

Repeating this program $D-1$ times for each value of $i=1,\ldots,D-1$, we make $(Q^{[3]}_{\mu\nu}, R^{[3]})$ depend only on x_0. Now we can repeat the first step once again with $i=0$ and make $Q^{[3]}_{kj}=0$ for $k,j=1,\ldots D-1$. We have already proven that such solution is trivial.

Recall that in the very beginning of our analysis we have made an assumption that the polynomials have non-zero degree ($m\geq 1$). Therefore we have to consider this last case separately. If polynomials $Q^{[3]}_{\mu\nu}$ and $R^{[3]}$ are constant, they trivially satisfy the system (C.0.10). To show that any such constant solution can be represented in the form (C.0.11), it is sufficient to take $Q^{[2]}_{\mu\nu}=q_{\mu\nu,0}(x_0)^2/2+q_{\mu\nu,1}(x_1)^2/2$, which generates $(Q^{[3]}_{\mu\nu}, R^{[3]})$ if $q_{\mu\nu,0}$ and $q_{\mu\nu,1}$ are chosen so that

$$\begin{aligned}
q_{00,1}-q_{00,0}&=Q^{[3]}_{00},\\
q_{11,0}-q_{11,0}&=Q^{[3]}_{11},\\
q_{00,1}+q_{00,0}+q_{11,1}+q_{11,0}&=R^{[3]},\\
q_{0\nu,1}&=Q^{[3]}_{0\nu}\text{ for }\nu\neq 0,\\
q_{\mu0,1}&=Q^{[3]}_{\mu0}\text{ for }\mu\neq 0,\\
q_{1\nu,0}&=Q^{[3]}_{1\nu}\text{ for }\nu\neq 1,\\
q_{\mu1,0}&=Q^{[3]}_{\mu1}\text{ for }\mu\neq 1,\\
q_{\mu\nu,0}+q_{\mu\nu,1}&=Q^{[3]}_{\mu\nu}\text{ for }\mu,\nu>1.
\end{aligned}$$

This completes the proof of Lemma 2.

Lemma 3 $H^4=0$.

Proof. It is almost obvious that the image of $d^{(3)}$ covers the whole kernel of $d^{(4)}$ in $V^{(4)}$ because the space $V^{(3)}$ is much bigger than $V^{(4)}$ at every degree. Indeed we will show that it is sufficient to consider a subspace of $V^{(3)}$ spanned by the zeroth row and the zeroth column of the matrix $Q_{\mu\nu}$. Loosely speaking the row will cover P_μ and the column will cover $\overline{P}_\mu$.

Suppose $(P_\mu,\overline{P}_\mu)\in\ker d^{(4)}$ or equivalently

$$d^{(4)}(P_\mu,\overline{P}_\mu)=(\partial^\mu P_\mu-\partial^\mu\overline{P}_\mu)=0 \tag{C.0.16}$$

we want to show that subtracting vectors of the form $d^{(3)}(Q_{\mu\nu},0)$ from $(P_\mu,\overline{P}_\mu)$ we can get reduce it to zero. First of all we can easily get rid of the spacial components P_i and $\overline{P}_i$ for $i=1\cdots D-1$ using $Q_{\mu\nu}$ which has the only non-zero components given by

$$Q_{0i}=\int P_i dx^0\quad\text{and}\quad Q_{i0}=\int\overline{P}_i dx^0+C_i(x^1\cdots x^{D-1}) \tag{C.0.17}$$

This will reduce P_μ and $\overline{P}_\mu$ to the form $P_\mu=a\delta_{0,\mu}$ and $\overline{P}_\mu=\overline{a}\delta_{0,\mu}$. Furthermore, according to Eq. (C.0.16) polynomials a and $\overline{a}$ have the same derivative with respect to x^0. Thus varying say $C_1(x^1\cdots x^{D-1})$ in Eq. (C.0.17) we can achieve that $a=\overline{a}$. Finally if we have a vector given by $P_\mu=\overline{P}_\mu=a\delta_{0,\mu}$ we can use Q_{00} to reduce it to zero.

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