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Recovering the Moments of a Function From Its Radon-Transform Projections: Necessary and Sufficient Conditions *

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Abstract

The question we wish to address in this paper is the following: To what extent does a limited set of noise-free Radon-Transform projections of a function $f(x,y)$ determine this function? This question has been dealt with in the mathematical literature to some extent. Noteworthy are results due to Volcic [4, 9], Fishburn et al [10], Falconer [3], Gardner [5, 11], Kuba [8] and other references contained therein. These results almost exclusively deal with the case when the function $f(x,y)$ is an indicator function over its domain of definition $\mathcal{O}$. There has also been some effort in the physics and engineering literature to answer this question. Some noteworthy examples are [17, 18]. In this paper, we will prove that one may uniquely recover the first p geometric moments [13] of a bounded, positive function $f(x,y)$, with compact support, from a fixed number p of Radon-Transform [6] projections. We further show that one can not uniquely recover any higher order moments of $f(x,y)$ from such limited information. The importance of this result lies in the fact that it directly shows to what extent a limited number of projections of a function determine the function. This, in essence, is a precise notion of the geometric complexity that a limited set of projections can support. In applications of this result to tomographic reconstruction problems such as in Medical Imaging [1], our result shows, in a quantitative way, how well one can theoretically expect to reconstruct an object being imaged in the absence of noise. From a more abstract viewpoint, it sheds some light on the reconstructability of certain elementary binary objects from a limited number of tomographic projections.

1 Background and Notation

The Radon-Transform of a function $f(x,y)$ defined over a compact domain of the plane $\mathcal{O}$ is defined by

$$g(t,\theta) \equiv \int_{\mathcal{R}^2} f(x,y)\delta(t - \omega \cdot [x,y]^T)dxdy. \tag{1}$$

For every fixed t and θ, $g(t,\theta)$ is simply the line-integral of f over $\mathcal{O}$ in the direction $\omega = [\cos(\theta),\ \sin(\theta)]^T$, where $\delta(t - [cos(\theta),\ sin(\theta)] \cdot [x,y]^T)$ is a delta function on a line at angle $\theta + (\pi/2)$ from the x-axis, and distance t from the origin.

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Not all functions $g(t,\theta)$, however, are Radon Transforms of some $f(x,y)$. Several well-known mathematical properties of the Radon transform known as the consistency relations specify valid 2-D Radon transforms. The Radon transform of some function $f(x,y)$ is constrained to lie in a particular functional subspace of the space of all real-valued functions $g(t,\theta): \mathcal{R} \times \mathbf{S}^1 \rightarrow \mathcal{R}$, where $\mathbf{S}^1$ is the unit circle. This subspace is characterized by the fact that g must be an even function of t, and that certain coefficients of the Fourier expansion of g must be zero [6, 7].

Let $\omega = [\cos(\theta), \sin(\theta)]$ denote a unit direction vector, and $\mathbf{x} = [x, y]$ a vector in $\mathcal{R}^2$. Using the definition of the Radon Transform we can write

$$\int_{-\infty}^{\infty} g(t,\omega)t^k dt = \int_{-\infty}^{\infty} t^k \int_{\mathbf{x}\in\mathcal{R}^2} f(\mathbf{x})\delta(t-\omega\cdot\mathbf{x})d\mathbf{x}dt = \int_{\mathbf{x}\in\mathcal{R}^2} f(\mathbf{x})(\omega\cdot\mathbf{x})^k d\mathbf{x} \tag{2}$$

The above identity clearly holds in higher dimensions as well. For our purposes, however, we shall only deal with $\mathcal{R}^2$. Assuming that the support, O, of $f(x,y)$ is contained within the unit disk, we can write the identity (2) as

$$H^{(k)}(\theta) \equiv \int_{-1}^{1} g(t,\theta)t^k dt = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \cos^{k-j}(\theta)\sin^j(\theta)\mu_{k-j,j} \tag{3}$$

where the right hand side is obtained by expanding the term $(\omega\cdot\mathbf{x})^k = (\cos(\theta)x + \sin(\theta)y)^k$ according to the binomial theorem, and where $\mu_{k-j,j}$ are the geometric moments of $f(x,y)$ defined as follows.

$$\mu_{p,q} = \int_{[x,y]\in\mathcal{O}} f(x,y)x^p y^q dxdy \tag{4}$$

The identity (3) was apparently first discovered by I.M. Gelfand and M.I. Graev in 1961 [19]. Note that the left hand side of (3) is simply the k^{th} order moment of the projection function $g(t,\theta)$ for a fixed angle θ, denoted by $H^{(k)}(\theta)$. Defining the vector of k^{th} order moments of f as

$$\mu^{(k)} = [\mu_{k,0},\ \mu_{k-1,1},\ \cdots,\ \mu_{0,k}]^T, \tag{5}$$

we have that

$$H^{(k)}(\theta) = D^{(k)}(\theta)\mu^{(k)} \tag{6}$$

$$D^{(k)}(\theta) = [\gamma_{k,0}\cos^k(\theta),\ \gamma_{k,1}\cos^{k-1}(\theta)\sin(\theta),\ \cdots,\ \gamma_{k,k-1}\cos(\theta)\sin^{k-1}(\theta),\ \gamma_{k,k}\sin^k(\theta)] \tag{7}$$

$$\gamma_{k,j} = \begin{pmatrix} k \\ j \end{pmatrix}, \tag{8}$$

so that $D^{(k)}(\theta)$ is a $1\times(k+1)$ matrix. Thus moments of order k of the projection *only* depend on moments of order k of $f(x,y)$. This observation will form the basis for the result that will be established in the next section.

2 The Result

Here we will show our main result, that given a fixed number p of noise free integral projections of a real valued function of 2 variables $f(x,y)$, we may uniquely recover the first p geometric moments of $f(x,y)$ but *cannot* uniquely recover any higher order moments.

The function of interest, $f(x,y)$, is completely and uniquely defined by the complete set of its (k-th order) geometric moments [12], $\mu^{(k)}$, for all $k \geq 0$. In particular, we define the vector of geometric moments of $f(x,y)$ up to order N by $\mathcal{M}_N = [\mu^{(0)^T}, \mu^{(1)^T}, \cdots, \mu^{(N)^T}]^T$. We assume that we are given (noise-free) integral projections of $f(x,y)$ at a fixed number of angles θ_i. Essentially we are given "cuts" of the Radon transform $g(t,\theta)$ of $f(x,y)$ at a finite number of θ. Note that each projection $g(t,\theta_i)$ itself is uniquely and completely defined by the complete set of *its* geometric moments $H^{(k)}(\theta_i)$ for all $k \geq 0$ In the previous section we showed that $H^{(k)}(\theta)$ and $\mu^{(k)}$ are related as follows:

$$H^{(k)}(\theta) = D^{(k)}(\theta)\mu^{(k)} \tag{9}$$

so that $D^{(k)}(\theta)$ is a $1 \times (k+1)$ matrix. In particular, for a given projection at angle θ, we have the following

$$\begin{aligned} \begin{bmatrix} H^{(0)}(\theta) \\ H^{(1)}(\theta) \\ \vdots \\ H^{(N)}(\theta) \end{bmatrix} &= \begin{bmatrix} D^{(0)}(\theta) & 0 & \cdots & 0 \\ 0 & D^{(1)}(\theta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D^{(N)}(\theta) \end{bmatrix} \begin{bmatrix} \mu^{(0)} \\ \mu^{(1)} \\ \vdots \\ \mu^{(N)} \end{bmatrix} & (10) \\ \mathcal{H}_N(\theta) &= \mathcal{D}_N(\theta)\mathcal{M}_N & (11) \end{aligned}$$

where the obvious association is made in the last equation.

Now consider the problem where we observe a number p of (noise free) projections $g(t, \theta_i)$, $i = 1, \ldots, p$, and wish to uniquely recover as much of $f(x, y)$ as possible. We look at this problem in the following way: we observe $H^{(k)}(\theta_i)$ $\forall k$, $i = 1, \ldots, p$ and wish to uniquely recover as many of the $\mu^{(j)}$ as possible. Note that we treat moments of a given order, j, *as a unit*, e.g. even though $\mu^{(1)}$ consists of *two* numbers, we consider it determined only if *both* are uniquely determined. Our main result is the following:

Theorem 1 **(p Moments From p Projections)** *Given p (line integral) projections of $f(x, y)$ at p different angles θ_i in $[0, \pi)$, one can uniquely determine the first p geometric moment vectors $\mu^{(j)}$, $0 \leq j < p$ of $f(x, y)$. Further this can be done using only the first p order geometric moments $H^{(k)}(\theta)$, $0 \leq k < p$ of the projections. Conversely, moments of $f(x, y)$ of higher order* cannot *be uniquely determined from p projections.*

Proof Consider (11) and suppose that we stack up our observations at the different angles to obtain:

$$\begin{aligned} \begin{bmatrix} \mathcal{H}_N(\theta_1) \\ \hline \mathcal{H}_N(\theta_2) \\ \hline \vdots \\ \hline \mathcal{H}_N(\theta_p) \end{bmatrix} &= \begin{bmatrix} \mathcal{D}_N(\theta_1) \\ \hline \mathcal{D}_N(\theta_2) \\ \hline \vdots \\ \hline \mathcal{D}_N(\theta_p) \end{bmatrix} \mathcal{M}_N & (12) \\ \mathbf{H}_N &= \mathbf{D}_N \mathcal{M}_N & (13) \end{aligned}$$

so that the matrix $\mathbf{D}_N$ relating $\mathcal{M}_N$ and the projections is $pN \times \frac{(N+1)(N+2)}{2}$. Clearly to be able to determine $\mathcal{M}_N$ uniquely we must have full column rank of $\mathbf{D}_N$. Now we may rearrange the rows of (12), grouping together the moments of the same order from all projections, without changing the column rank of $\mathbf{D}_N$.

This operation yields the following equivalent equation:

$$\left[\begin{array}{c} H^{(0)}(\theta_1) \\ H^{(0)}(\theta_2) \\ \vdots \\ H^{(0)}(\theta_p) \\ \hline H^{(1)}(\theta_1) \\ H^{(1)}(\theta_2) \\ \vdots \\ H^{(1)}(\theta_p) \\ \hline \vdots \\ \hline H^{(N)}(\theta_1) \\ H^{(N)}(\theta_2) \\ \vdots \\ H^{(N)}(\theta_p) \end{array}\right] = \left[\begin{array}{c|c|c|c} \begin{array}{c} D^{(0)}(\theta_1) \\ D^{(0)}(\theta_2) \\ \vdots \\ D^{(0)}(\theta_p) \end{array} & \bigcirc & \cdots & \bigcirc \\ \hline \bigcirc & \begin{array}{c} D^{(1)}(\theta_1) \\ D^{(1)}(\theta_2) \\ \vdots \\ D^{(1)}(\theta_p) \end{array} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & \bigcirc \\ \hline \bigcirc & \cdots & \bigcirc & \begin{array}{c} D^{(N)}(\theta_1) \\ D^{(N)}(\theta_2) \\ \vdots \\ D^{(N)}(\theta_p) \end{array} \end{array}\right] \left[\begin{array}{c} \mu^{(0)} \\ \hline \mu^{(1)} \\ \hline \vdots \\ \hline \mu^{(N)} \end{array}\right] \tag{14}$$

$$\left[\begin{array}{c} \tilde{H}_p^{(0)} \\ \hline \tilde{H}_p^{(1)} \\ \hline \vdots \\ \hline \tilde{H}_p^{(N)} \end{array}\right] = \left[\begin{array}{cccc} \tilde{D}_p^{(0)} & \bigcirc & \cdots & \bigcirc \\ \bigcirc & \tilde{D}_p^{(1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bigcirc \\ \bigcirc & \cdots & \bigcirc & \tilde{D}_p^{(N)} \end{array}\right] \left[\begin{array}{c} \mu^{(0)} \\ \hline \mu^{(1)} \\ \hline \vdots \\ \hline \mu^{(N)} \end{array}\right] \tag{15}$$

where $\tilde{H}_p^{(k)}$ is thus the collection of moments of order k in each projection and $\tilde{D}_p^{(k)}$ is the matrix relating moments of order k of the object to moments of order k in each of the p projections. Now since the overall matrix is block diagonal the problem of determining each $\mu^{(k)}$ decouples so that $\tilde{H}_p^{(k)} = \tilde{D}_p^{(k)}\mu^{(k)}$. In particular, we can determine $\mu^{(k)}$ uniquely if and only if the corresponding matrix block:

$$\tilde{D}_p^{(k)} = \begin{bmatrix} D^{(k)}(\theta_1) \\ D^{(k)}(\theta_2) \\ \vdots \\ D^{(k)}(\theta_p) \end{bmatrix} \tag{16}$$

has full column rank. Let us now examine the conditions when $\tilde{D}_p^{(k)}$ will have full column rank.

Substituting for $D^{(k)}(\theta_i)$ from (7) we find that $\tilde{D}_p^{(k)}$ is of the form:

$$\tilde{D}_p^{(k)} = \left[\begin{array}{c|c|c|c|c} \gamma_{k,0}\cos^k(\theta_1) & \gamma_{k,1}\cos^{k-1}(\theta_1)\sin(\theta_1) & \cdots & \gamma_{k,k-1}\cos(\theta_1)\sin^{k-1}(\theta_1) & \gamma_{k,k}\sin^k(\theta_1) \\ \gamma_{k,0}\cos^k(\theta_2) & \gamma_{k,1}\cos^{k-1}(\theta_2)\sin(\theta_2) & \cdots & \gamma_{k,k-1}\cos(\theta_2)\sin^{k-1}(\theta_2) & \gamma_{k,k}\sin^k(\theta_2) \\ & & \vdots & & \\ \gamma_{k,0}\cos^k(\theta_p) & \gamma_{k,1}\cos^{k-1}(\theta_p)\sin(\theta_p) & \cdots & \gamma_{k,k-1}\cos(\theta_p)\sin^{k-1}(\theta_p) & \gamma_{k,k}\sin^k(\theta_p) \end{array}\right] \tag{17}$$

and the matrix $\tilde{D}_p^{(k)}$ is $p \times (k+1)$. Now $\tilde{D}_p^{(k)}$ will have full column rank if and only if its columns are independent. Note that we must have $p > k$. The columns will be independent if and only if there is no set of α_i (not all zero) such that:

$$\alpha_0\cos^k(\theta_i) + \alpha_1\cos^{k-1}(\theta_i)\sin(\theta_i) + \cdots + \alpha_{k-1}\cos(\theta_i)\sin^{k-1}(\theta_i) + \alpha_k\sin^k(\theta_i) = 0 \qquad \forall\theta_i,\ 1 \le i \le p \tag{18}$$

In particular, this will be true (when $p > k$) if the homogeneous trigonometric polynomial of order k defined by (18) has at most k roots θ in $[0, \pi)$. This motivates the following lemma, which we prove in Appendix A.

Lemma 1 (Roots of Homogeneous Trigonometric Polynomial of order k) *A homogeneous trigonometric polynomial of order k of the following form:*

$$\alpha_0 \cos^k(\theta) + \alpha_1 \cos^{k-1}(\theta)\sin(\theta) + \cdots + \alpha_{k-1}\cos(\theta)\sin^{k-1}(\theta) + \alpha_k \sin^k(\theta) \tag{19}$$

vanishes for at most k distinct values of θ in $[0, \pi)$.

Given this lemma we see that $\tilde{D}_p^{(k)}$ will have full column rank if and only if $p > k$, i.e. the number of projections p greater than the order k of the moment vector $\mu^{(k)}$ that we are interested in! In particular, we can achieve this full column rank if $p = k+1$, so we use no more than the first p moments of the projection. Also note that this implies that if $\tilde{D}_p^{(n)}$ is of full column rank for some n then so is $\tilde{D}_p^{(k)}$ for all $k < n$.

Now the preceding arguments essentially show the first two statements of the theorem. Given p projections the matrices $\tilde{D}_p^{(k)}$ will have full column rank for $0 \leq k < p$. Thus we can uniquely find the moments $\mu^{(k)}$ from the corresponding $\tilde{H}_p^{(k)}$ and $\tilde{D}_p^{(k)}$. In particular, note that due to the block diagonal structure of (15), using higher order moments of the projections is of no help in determining a given moment $\mu^{(k)}$.

3 Conclusion

We have shown that to uniquely specify the first p moments of a function $f(x, y)$ one needs exactly p integral projections at distinct angles in the interval $[0, \pi)$, and that this number is both necessary and sufficient. The importance of this result lies in the fact that it directly shows to what extent a limited number of projections of a function determine the function. From a practical standpoint this is quite important. In applications of the above result to tomographic reconstruction problems such as in Medical Imaging [1], our result shows, in a quantitative way, how well one can theoretically expect to reconstruct an object being imaged in the absence of noise. From a more abstract viewpoint, it sheds some light on the reconstructability of certain elementary binary objects from a limited number of tomographic projections. For instance, it is known [2, 12] that any binary ellipse is uniquely determined from 3 projections at distinct angles. It is also known that the moment sets $\{\mu^{(0)}, \mu^{(1)}, \mu^{(2)}\}$ uniquely determine a binary ellipse in the plane [15]. Our result shows essentially that 3 projections suffice to uniquely determine the set of numbers $\{\mu^{(0)}, \mu^{(1)}, \mu^{(2)}\}$ hence showing that 3 projections suffice to uniquely determine any binary ellipse, thereby providing an alternate proof.

A somewhat more subtle instance of the usefulness of our result occurs in the reconstruction of binary polygonal objects in the plane. A fascinating theorem due to Davis [16, 14] states that a triangle in the plane in uniquely determined by its moments of up to order 3. i.e.$\{\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \mu^{(3)}\}$. Furthermore, Davis has, in essence, provided an explicit algorithm for reconstructing the triangle from this set of numbers. Our result would imply that exactly 4 projections are sufficient to determine this set of moments. Hence, together with the work of Davis, our result provides a closed form solution to the problem of reconstructing a triangular region in the plane from only 4 tomographic projections in the absence of noise.

A Proof of Lemma 1

Proof: Let $p(\theta)$ denote the homogeneous polynomial in question. i.e.

$$p(\theta) = \alpha_0 \cos^k(\theta) + \alpha_1 \cos^{k-1}(\theta)\sin(\theta) + \cdots + \alpha_{k-1}\cos(\theta)\sin^{k-1}(\theta) + \alpha_k \sin^k(\theta). \quad (20)$$

- <u>**CASE I:**</u> Assume that $p(\pi/2) \neq 0$. Then we can write $p(\theta)$ as

$$p(\theta) = \cos^k(\theta) q(\theta). \quad (21)$$

where $q(\theta)$ has no roots at $\theta = \pi/2$ and

$$q(\theta) = \alpha_0 + \alpha_1 \tan(\theta) + \cdots + \alpha_{k-1}\tan^{k-1}(\theta) + \alpha_k \tan^k(\theta). \quad (22)$$

Letting $u = \tan(\theta)$ we observe that the right hand side of (22) is simply a polynomial of order k in u. By the Fundamental Theorem of Algebra [20], this polynomial has at most k real roots. This is to say that there exist at most k values $u_i \in \mathcal{R}$ such that $q(\tan^{-1}(u_i)) = 0$. Given this, we have that the roots of $q(\theta)$ are

$$\theta_i = \tan^{-1}(u_i). \quad (23)$$

We know that the function $\tan^{-1}$ is one-to-one over the interval $[0, \pi)$. Since $q(\pi/2) \neq 0$ by assumption, it follows that there exist at most k angles $\theta_i \in [0, \pi)$ for which $q(\theta_i) = 0$.

- <u>**CASE II:**</u> Let $r_0(\theta) = p(\theta)$, and define the functions $r_i(\theta)$ for $1 \leq i \leq k$ as follows.

$$r_i(\theta) = \begin{cases} \frac{r_{i-1}(\theta)}{cos(\theta)} & \text{if } r_{i-1}(\pi/2) = 0 \\ 1 & \text{if } r_{i-1}(\pi/2) \neq 0 \end{cases} \quad (24)$$

If $p(\pi/2) = 0$, then we have

$$p(\pi/2) = \alpha_k \sin^k(\pi/2) = \alpha_k = 0. \quad (25)$$

Therefore we have that

$$p(\theta) = \cos(\theta)(\alpha_0 \cos^{k-1}(\theta) + \cdots + \alpha_{k-1}\sin^{k-1}(\theta)) = \cos(\theta) r_1(\theta) \quad (26)$$

If $r_1(\theta)$ does not vanish at $\pi/2$, Case I shows that it has at most $k-1$ roots in $[0, \pi)$, which together with $\cos(\theta) = 0$ give at most k roots for $p(\theta)$ in $[0, \pi)$.

From the definition of $r_i(\theta)$, it is clear that

$$r_i(\theta) = \begin{cases} \alpha_0 \cos^{k-i}(\theta) + \cdots + \alpha_{k-i}\sin^{k-i}(\theta) & \text{if } r_{i-1}(\pi/2) = 0 \\ 1 & \text{if } r_{i-1}(\pi/2) \neq 0 \end{cases} \quad (27)$$

Now suppose that $r_i(\pi/2) = 0$ for $i = 0, 1, \cdots, n-1$ and $r_n(\pi/2) \neq 0$, where $1 \leq n \leq k$. Again, from the definition of $r_i(\theta)$ it follows that

$$p(\theta) = \cos^n(\theta) r_n(\theta). \quad (28)$$

From Case I, $r_n(\theta)$ has at most $k-n$ roots in $[0, \pi)$, which along with $\cos^n(\theta) = 0$ give at most k roots for $p(\theta)$ in $[0, \pi)$. □

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