STATISTICAL APPROACHES TO THE TOMOGRAPHIC RECONSTRUCTION OF FINITELY PARAMETERIZED GEOMETRIC OBJECTS *

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Abstract

Tomographic reconstruction in 2 dimensions is concerned with the reconstruction of a positive, bounded function $f(x, y)$ and its compact domain of support $\mathcal{O}$ from noisy and possibly sparse samples of its Radon-transform projections, $g(t, \theta)$. If the pair $(f, \mathcal{O})$ is referred to as an object, a finitely parameterized object is one in which both $f(x, y)$ and $\mathcal{O}$ are determined uniquely by a finite number of parameters. For instance, a binary N-sided polygonal object in the plane is uniquely specified by exactly $2N$ parameters which may be the vertices, normals to the sides, etc. In this work we study the optimal reconstruction of finitely parameterized objects from noisy projections. In specific, we focus our study on the optimal reconstruction of binary polygonal objects from noisy projections. We show that when the projections are corrupted by Gaussian white noise, the optimal Maximum Likelihood (ML) solution to the reconstruction problem is the solution to a nonlinear optimization problem. This optimization problem is formulated over a parameter space which is a finite dimensional Euclidean space. We also demonstrate that in general, the moments of an object can be estimated directly from the projection data and that using these estimated moments, a good initial guess for the numerical solution to the nonlinear optimization problem may be constructed. Finally, we study the performance of the proposed algorithms from both statistical and computational viewpoints.

1

1 Introduction

2 The Reconstruction Problem

The Radon-Transform [9, 10] of a function $f(x,y)$ defined over a compact domain of the plane $\mathcal{O}$ is given by

$$g(t,\theta) = \int_{\mathcal{O}} f(x,y)\delta(t - \omega \cdot [x,y]^T)dxdy. \tag{1}$$

For every fixed t and θ, $g(t,\theta)$ is simply the line-integral of f over $\mathcal{O}$ in the direction $\omega = [\cos(\theta),\ \sin(\theta)]^T$, where $\delta(t - [cos(\theta),\ sin(\theta)] \cdot [x,y]^T)$ is a delta function on a line at angle $\theta + \pi/2$ and distance t from the origin. Let us now assume that the function $f(x,y)$ is known to within some set of unspecified parameters. For instance, $f(x,y)$ may be known or assumed to be of the form $f(x,y) = ax + by$ with a and b unknown. Further assume that the region $\mathcal{O}$ over which f is defined is specified by a finite number of parameters. For instance, the region may be known to be circular with the parameters denoting the center of the circle and its radius unknown. Under the above assumptions, the tomographic reconstruction problem at hand is exactly a parameter estimation problem. Other efforts in the parametric study of tomographic reconstruction problems have been carried out in [1, 2, 3] where the reconstruction problem has been studied in the framework of object detection and estimation in 2 and 3-D.. The estimation of the parameters that uniquely specify the function f and the region $\mathcal{O}$ is the concern of this paper. More specifically, we will focus our attention on binary, polygonal objects. We shall define these as follows.

Definition 1 *An object* $(f,\mathcal{O})$ *is termed binary and polygonal if*

- *The function* $f(x,y)$ *is an indicator function over the region* $\mathcal{O}$. *i.e.* $f(x,y) = 1$ *whenever* $(x,y) \in \mathcal{O}$, *and* $f(x,y) = 0$ *otherwise.*

- *The boundary of the compact region* $\mathcal{O}$ *is a simple polygon.*

To denote the direct dependence of the projection data on the parameters we wish to estimate, we let V denote a matrix of appropriate dimensions that contains all the relevant parameters and denote the Radon Transform of a binary polygonal object with these parameters as $g(t,\theta,V)$. In what follows, V contains the vertices of an N-sided polygonal region $\mathcal{O}$ and we write

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}, \tag{2}$$

where $v_i = [x_i, y_i]$ denote the Cartesian coordinates of the ith vertex of the polygonal region. The tomographic reconstruction problem now can be stated as the problem of optimal estimation of the parameter matrix V given noisy samples of the projections $g(t,\theta)$.

2.1 Maximum Likelihood Formulation

We formulate our solution to the reconstruction problem for binary polygonal objects under the assumption that we are given a finite number of samples of the function $g(t,\theta)$ which are corrupted by white Gaussian noise. More specifically,

2

$$y_{i,j} = g(t_i, \theta_j, V^*) + w(t_i, \theta_j) \tag{3}$$

where V^* is the true polygonal region we wish to reconstruct, and where we are given $\{y_{i,j}| 1 \le i \le m, \ 1 \le j \le n\}$. The numbers $\{w(t_i, \theta_j)| 1 \le i \le m, \ 1 \le j \le n\}$ are assumed to be independent, identically distributed (i.i.d.) Gaussian random variables with variance σ^2.

The Maximum Likelihood (ML) [4] estimate, $\hat{V}_{ml}$, of the parameter matrix V is given by that value of V which makes the observed data most likely. More specifically,

$$\hat{V}_{ml} = argmax P(y|V), \tag{4}$$

where $P(y|V)$ denotes the conditional probability density of the observed data set y, given the parameter matrix V. It is well-known that given the assumption that the data is corrupted by i.i.d. Gaussian random noise, the solution to the above ML-estimation problem is precisely equivalent to the following Nonlinear Least Squares Error (NLSE) formulation

$$\hat{V}_{ml} = argmin \sum_{i,j} ||y_{i,j} - g(t_i, \theta_j, V)||^2. \tag{5}$$

The formulation (5) shows that the ML tomographic reconstruction problem for binary polygonal objects (and more generally all finitely parameterized objects) is equivalent to the nonlinear minimization problem given by (5). (It is the nature of the dependence of g on the parameter matrix V that makes this a *nonlinear* least squares problem.)

The minimization problem in (5) can be solved numerically in a variety of ways. Due to the fact that the parameters in V appear nonlinearly in the cost function given by (5), it is extremely difficult to compute explicit gradients for a numerical solution of the minimization problem. We have, therefore, used the Nelder-Mead algorithm [5] to numerically compute the ML estimate.

2.2 Basic Performance Results

In order to quantify some measure of performance of the above proposed reconstruction scheme, we first need to define an appropriate notion of signal-to-noise ratio (SNR). We define the **SNR per sample** as

$$SNR = 10 \log \frac{\sum_{i,j} g^2(t_i, \theta_j)/(m \times n)}{\sigma^2} \ (dB), \tag{6}$$

where $(m \times n)$ denotes the total number of samples given, and σ^2 is the variance of the i.i.d. noise in the projections. In this section we study the performance of our reconstruction scheme while assuming that the number of sides of the true object is known a-priori. This assumption will be relaxed in Section 4 where a more general formulation of the reconstruction problem is studied which allows for the estimation of the unknown number of sides as well. In order to ensure that the solution produced by the algorithm is the global minimum of the cost function, we have used the parameters of the true polygon as an initial guess for all the studies in this section. In Section 3 we demonstrate how to produce a reasonable initial guess for the minimization problem (5) directly from the data.

To study the average performance of the algorithm, a Monte-Carlo study was done to establish average reconstruction errors at various SNR's. Two polygons were chosen for the performance
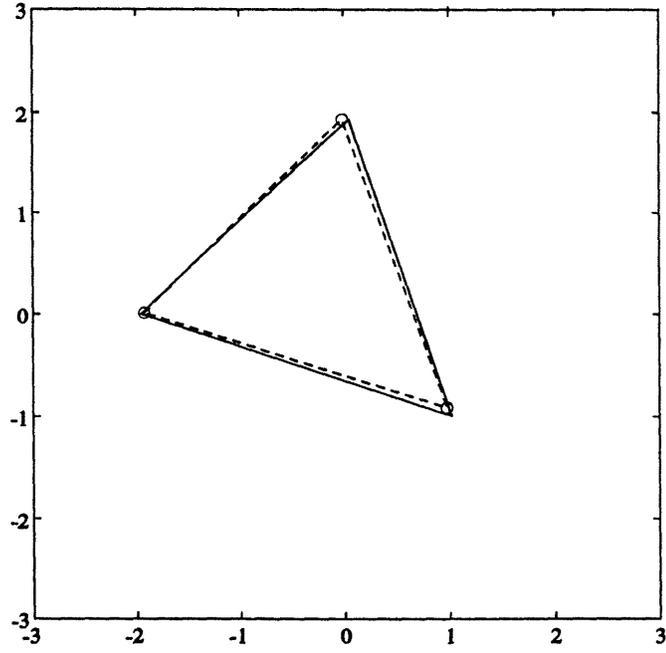
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Figure 1: Sample reconstruction of a triangle at 0 dB SNR: True(-), Reconstruction(- -)
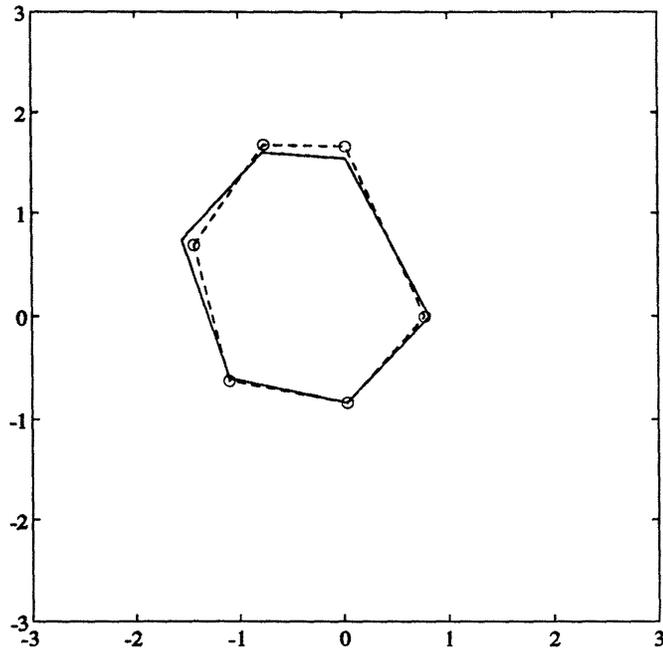

Figure 2: Sample reconstruction of a hexagon at 0 dB SNR: True(-), Reconstruction(- -)
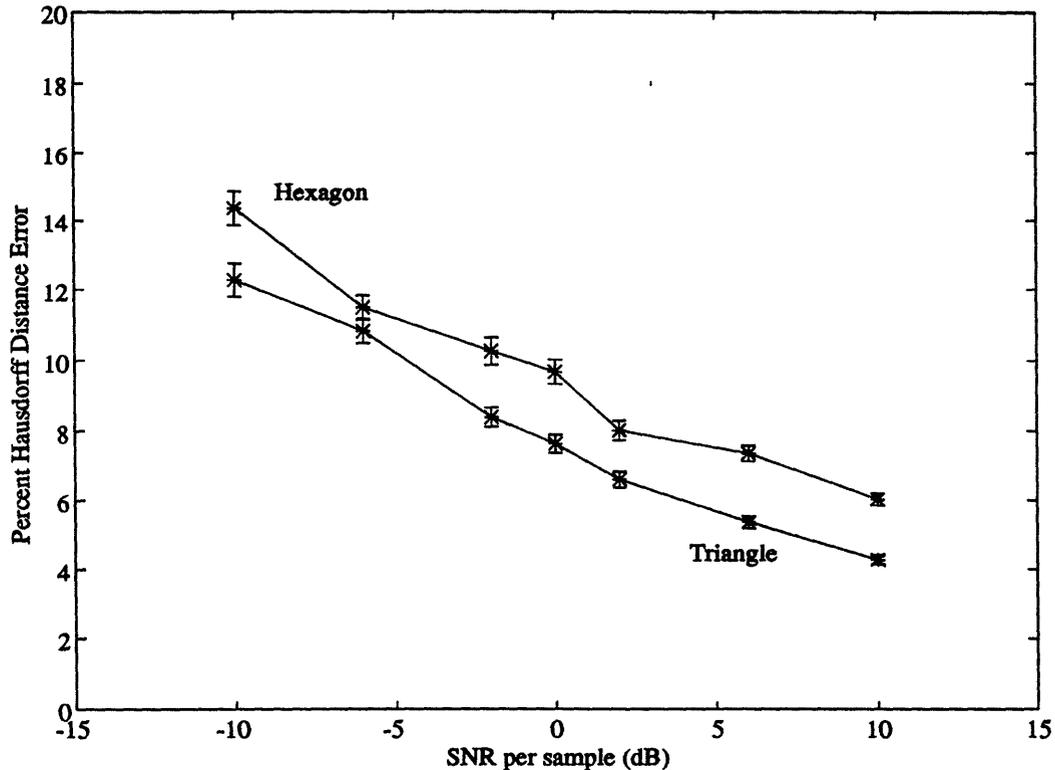
Figure 3: Mean performance curves for ML reconstructions of a triangle and a hexagon

studies. The first was a triangle which is depicted in solid lines in Figure 1. The second was a hexagon which is depicted in solid lines in Figure 2. In each case, 1000 samples of the projections of these objects were collected in the form of 50 equally spaced projections in the interval $(0, \pi]$ (n=50), and 20 samples per projection (m=20). The field of view (extent of measurements in the variable t) was chosen as twice the maximum width of the object in each case. The samples were then corrupted by Gaussian white noise to yield data sets at several signal to noise ratios. At each signal to noise ratio, 100 reconstructions were done using independent sample paths of the corrupting white noise. The average reconstruction error was then computed and is displayed against the SNR in Figure 3. The error bars denote the 95% confidence intervals for the computed mean values.

The reconstruction error was measured in terms of the Hausdorff distance [6] between the ML estimate and the true polygon. The Hausdorff metric is a proper notion of "distance" between two nonempty compact sets in the plane and it is defined as follows. Let $d(x, R_1)$ denote the minimum distance between the point x in the plane and the compact set R_1. i.e.

$$d(x, R_1) = min\{\|x - p\| : p \in R_1\}. \tag{7}$$

Now given two non-empty compact sets in the plane, R_1 and R_2, the Hausdorff distance between them is defined as

$$\mathcal{H}(R_1, R_2) = max\{sup_{x \in R_1} d(x, R_1), sup_{x \in R_2} d(x, R_2)\} \tag{8}$$

In essence, the Hausdorff metric is a measure of the largest distance by which R_1 and R_2 differ. Our actual performance measure as displayed in Figure 3 is in terms of the *percent* Hausdorff distance

5

between the true object and the reconstruction. Sample reconstructions of the triangle and hexagon at $SNR = 0$ dB are displayed in Figures 1 and 2, respectively.

3 The Initial Guess

Given the highly nonlinear nature of the dependence of the cost function in 5 on the parameters in V, it appears evident that given a poor initial condition, a typical numerical optimization algorithm may converge to a local minimum of the cost function. We use the projection data directly to compute a reasonable initial guess that is sufficiently close to the true global minimum as to result in convergence to it. We do this by estimating the moments of the object from the projections directly and then using (some) of these moments to compute a reasonable initial guess. It is important to emphasize that a key reason why the use of moments is beneficial is that the amount of computation involved in arriving at an initial guess using the moments is far smaller than the amount of computation (number of iterations) required to converge to the global minimum given a poor initial guess. A careful development of exactly how the moments of an object (not necessarily binary and polygonal) are related to those of the projections may be found in [7, 11]. We describe a simple algorithm based on the ideas in [7, 11] to generate initial guesses. In short, we wish to construct a regular polygon with the same area and center of mass as that estimated from the data. The details of the algorithm for constructing the initial guess follow.

It is easy to prove [7] that

$$\int_{-\infty}^{\infty} g(t, \theta) t^k dt = \int_{\mathcal{R}^2} f(x, y)(x \cos(\theta) + y \sin(\theta))^k dx dy. \tag{9}$$

Specializing this result for $k = 0, 1$, and noting that $f(x, y)$ is an indicator function when the objects in question are binary, we arrive at the following conclusions.

1. The area of any binary object is equal to the area of any one of its projections.

2. The center of mass of any binary object denoted by (x_c, y_c) is given by

$$x_c \cos(\theta) + y_c \sin(\theta) = t_c(\theta), \tag{10}$$

where $t_c(\theta)$ denotes the center of mass of the projection $g(t, \theta)$ at the angle θ.

It is then clear that if we have projections at two or more distinct, known angles, we can estimate both the area and the center of mass of the binary polygonal object we wish to reconstruct. Assume that we know that we are looking for a binary polygonal object with N sides. An algorithm to compute the initial condition is as follows.

6

Initial Guess Algorithm:

- Construct an N-sided regular polygon centered at the origin with vertices on the unit circle. The vertices of this polygon are simply given by:

$$\tilde{V}_0 = \begin{bmatrix} \cos(0) & \sin(0) \\ \cos(2\pi/N) & \sin(2\pi/N) \\ \vdots & \vdots \\ \cos(2\pi(N-1)/N) & \sin(2\pi(N-1)/N) \end{bmatrix} \qquad (11)$$

- Compute the area of $\tilde{V}_0$ and denote this by A_0. Let $\overline{V}_0$ denote a regular polygon centered at the origin which has the estimated area from the projections. This is given by a scaling of $\tilde{V}_0$ as follows

$$\overline{V}_0 = \sqrt{\frac{\hat{A}}{A_0}}\tilde{V}_0, \qquad (12)$$

where $\hat{A}$ is the estimated area from the projections.

- Translate $\overline{V}_0$ to the estimated center $(\hat{x}_c, \hat{y}_c)$ to obtain the initial guess V_0 as follows

$$V_0 = \overline{V}_0 + \begin{bmatrix} \hat{x}_c & \hat{y}_c \\ \hat{x}_c & \hat{y}_c \\ \vdots & \vdots \\ \hat{x}_c & \hat{y}_c \end{bmatrix} \qquad (13)$$

The above algorithm produces an initial guess which, by construction, has the same area and center of mass as that which is estimated from the data. The formulation begs the question of how one would incorporate higher order moments in this process to perhaps construct a better initial guess. In fact, the use of moments can in itself constitute a reconstruction algorithm if a sufficient number of moments are estimated and used. Hence, two distinct issues must be considered. The first is the tradeoff between the computational complexity of an algorithm using moments exclusively [7, 11] versus an algorithm that relies on numerical optimization to solve (5). The second issue is that estimated moments of higher order are more inaccurate [7, 11] (have larger variances). Hence, an estimated set of moments may turn out to be inconsistent. i.e. may not satisfy the conditions for a sequence of numbers to be the moments of a positive, bounded function. This inconsistency may undermine the effectiveness of an algorithm that relies exclusively on the estimated moments. With these two important issues in mind, the best approach we see is to use one technique to guide the other. In other words, use some of the moments to construct an initial guess that is near the global minimum of (5), and then iterate numerically on (5).

In the studies reported in this paper, we have only used the estimated areas and centers to produce initial guesses. Our experiments have shown that given a sufficiently small number of sides and a sufficiently large SNR, the initial guesses produced by the above algorithm are sufficiently close to the basin of the global minimum of (5) to converge to the true ML estimate.

4 The Number of Sides of Reconstruction

4.1 The Minimum Description Length Formulation

In this section we study the performance of our proposed solution having relaxed the assumption of prior knowledge of the number of sides. Without prior knowledge of the number of sides, we must modify our approach. We have done this by considering the *Minimum Description Length* (MDL) principal [8]. In this approach, the cost function to be minimized is formulated such that the global minimum of the cost corresponds to a model of least order that explains the data best. The MDL approach in essence extends the Maximum Likelihood principal by including a term that measures the model complexity. In the present context, the model complexity refers to the number of sides of the polygonal reconstruction. The ML approach minimizes the likelihood function given by

$$L_{ML}(y|V) = -\log(P(y|V)) \tag{14}$$

which amounts to minimizing the sum of the squared errors as described in (5). In contrast, the MDL minimizes the likelihood function given by

$$L_{MDL}(y|V) = -\log(P(y|V)) + \frac{N}{2}\log d, \tag{15}$$

where d is the number of samples of $g(t, \theta)$, ($d = mn$), and N refers to the number of sides of the reconstruction. Roughly speaking, the MDL cost is proportional to the number of bits required to model the observed data set with a model of order N, hence the term *Minimum Description Length*. More specifically, the MDL cost function yields the following optimization problem

$$\hat{V}_{MDL} = argmin\{\sigma^{-2}\sum_{i,j}\|y_{i,j} - g(t_i, \theta_j, V)\|^2 + N\log(d)\}, \tag{16}$$

where the optimization is now performed over both V and N. A reconstruction of the hexagon shown in Figure 2 was performed using the MDL formulation and the same data set as ($n = 50$, $m = 20$, $SNR = 0$) the ML reconstruction discussed in Section 2.2. The resulting reconstructed hexagon is identical to that shown in Figure 2, except that in the MDL formulation, the algorithm was not supplied with a number of sides. The algorithm successfully estimated the optimal number of sides to be 6. This result is shown in Figure 4 where a plot of minimum MDL cost vs. the number of sides is presented. As shown in the figure, the global minimum was reached for $N = 6$.

4.2 Polygonal Reconstruction of Non-Polygonal Objects

So far we have been operating under the assumption that the objects to be reconstructed are all polygonal. In this section we wish to study the performance of our proposed algorithms for a case when the underlying, true object is actually an ellipse. We will use the MDL formulation presented in the previous section and will study the optimal reconstructions at different signal to noise ratios.

Let the true be defined as a binary ellipse the boundary of which is given by

$$(x - \frac{1}{2})^2 + \frac{(y + \frac{1}{2})^2}{9/4} = 1. \tag{17}$$

The above relation defines an ellipse centered at the point $(1/2, -1/2)$ with major and minor axes aligned with the coordinate axes with lengths 1 and 3/2, respectively. This ellipse is illustrated in
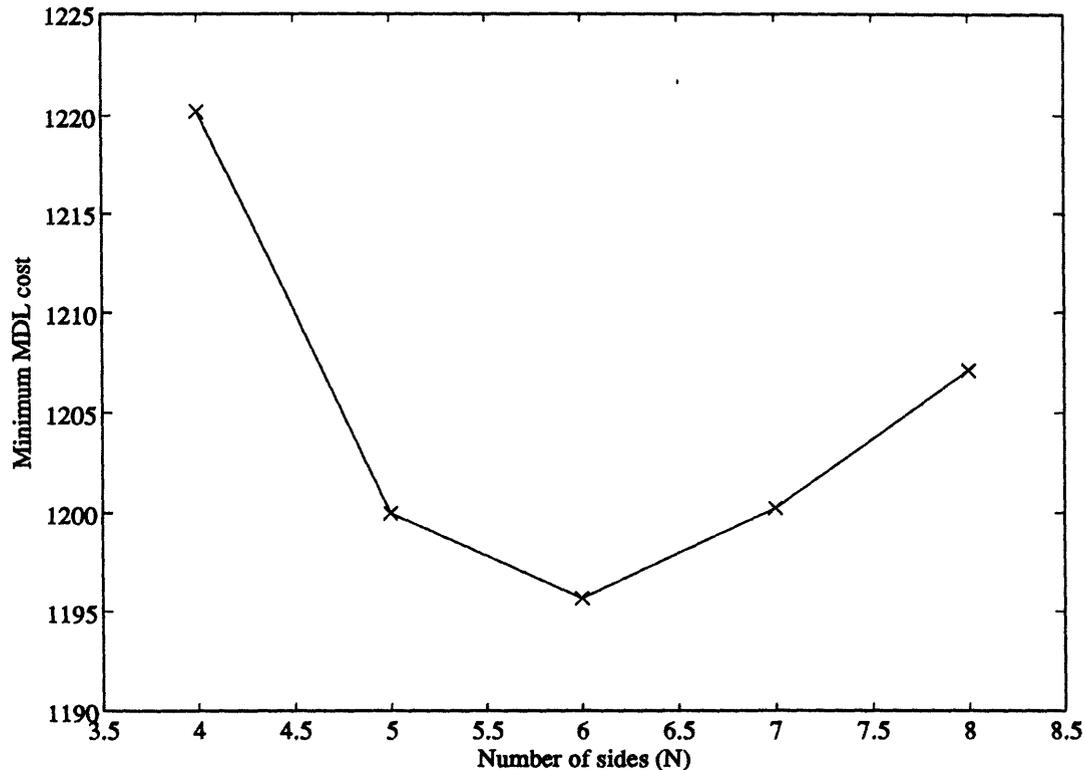
8

Figure 4: Minimum MDL cost vs number of sides for the hexagon in Figure 2

Figure 5. 1000 noisy samples of the Radon-Transform of this ellipse were generated (m=20, n=50) at SNR's of 0 and 5 dB respectively for 50 different sample paths of the corrupting noise. For each set of data, reconstructions were performed using the ML algorithm with 3, 4, 5, 6, 7, and 8 sides. The MDL cost was then computed for each of these reconstructions. The mean minimum MDL cost for each case was then computed and is presented in the top part of Figure 5. The curve corresponding to the SNR= 5 dB case displays its minimum at $N = 5$. This indicates that the optimal Minimum Description Length reconstruction uses 5 sides. The five-sided reconstruction of the ellipse is displayed on the lower left plot of Figure 5. The curve corresponding to the SNR= 0 dB case displays its minimum at $N = 6$. This indicates that the optimal Minimum Description Length reconstruction uses 6 sides. Note however, that the MDL cost curve has become rather flat for this low SNR. This hints to the fact that with increasing noise intensity, it becomes more difficult to discern how many sides might yield the best reconstruction. Furthermore, note that the optimal number of sides increased from 5 to 6 when the SNR decreased from 5 to 0. For the sake of completion, Figure 6 contains (from top to bottom) the noiseless sinogram [1] of the underlying ellipse, the noisy sinogram at SNR= 0 and the sinogram of the optimal 6-sided reconstruction displayed in figure 5. We can see that the reconstructed sinogram is, to the naked eye, virtually indistinguishable from the noiseless sinogram. This is further evidence that the proposed algorithms work well.

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[1]The sinogram is simply an intensity plot of the values of $g(t, \theta)$, where the horizontal axis is θ and the vertical axis is t
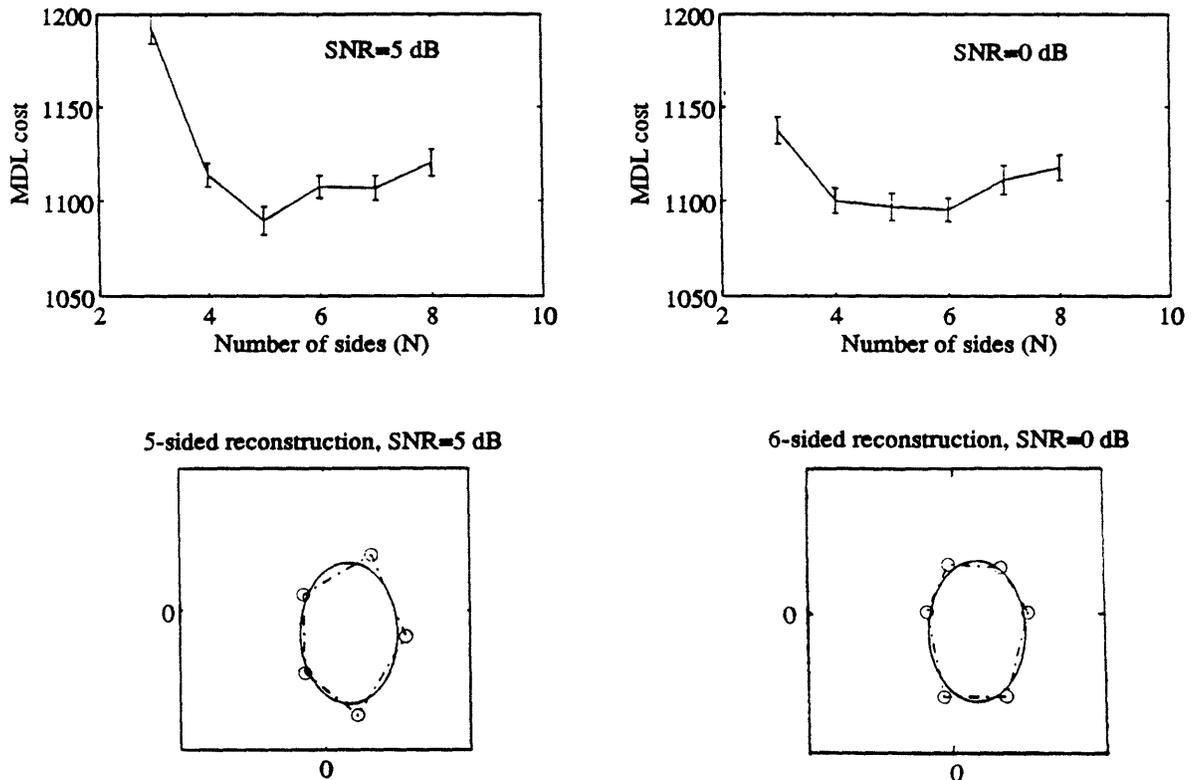
Figure 5: Minimum MDL costs and Sample Reconstructions for the Ellipse in Section 4.2.

5   Conclusion

In this paper, we have studied statistical techniques for the reconstruction of finitely parameterized geometric objects. More specifically, we have studied the reconstruction of binary polygonal objects in some detail. The reconstruction of such objects was posed as a parameter estimation problem for which the Maximum-Likelihood technique was proposed as a solution. It was shown that given a reasonably good initial guess for the numerical optimization, and prior knowledge of the number of parameters describing the geometric object in question, the proposed ML algorithm performs quite well under the Hausdorff measure. An algorithm was presented for computing a reasonable initial guess using moments of the object which can be estimated directly from the projection [7, 11]. It was indicated that moments can be used directly as a means for reconstruction [7, 11]. It was also shown that if the number of parameters describing the geometric object are not known, a Minimum Description Length criterion can be employed that simply generalized the ML framework to penalize the use of a large number of parameters for the reconstruction. The MDL approach was also shown to work successfully in estimating the number of sides and the underlying object itself for low signal to noise ratios. It was further shown that if the underlying object is not polygonal, but still binary, the proposed MDL algorithm is still capable of producing polygonal reconstructions with a reasonable number of sides and small error in the presence of noise with low signal to noise ratios.
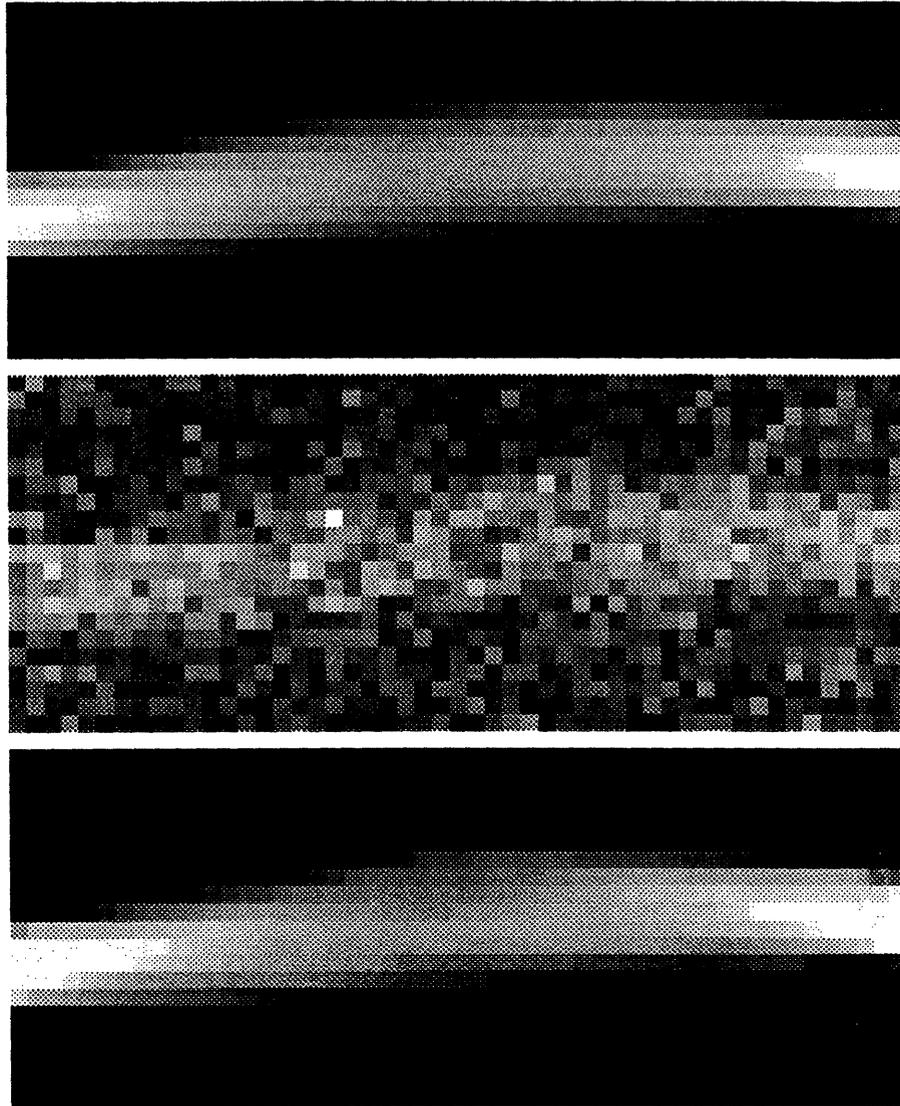
Figure 6: From Top to Bottom: Sinograms of I) Noiseless ellipse, II) Noisy data III) Reconstruction with 6-sided polygon

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