

The Structure of Paired Boson Superfluids

by

Mara S. Daniel

Submitted to the Department of Physics in partial fulfillment of the requirements for the degree of

Bachelor of Science


at the


MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 2004 [June 2004]

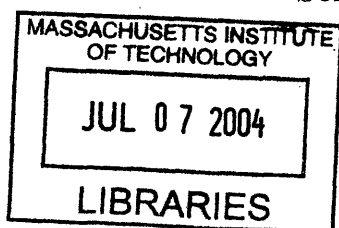
© 2004 Mara S. Daniel. All rights reserved.

The author hereby grants to MIT permission to reproduce and distribute publicly paper and electronic copies of this thesis document in whole or in part.

Author 
Department of Physics
May 7, 2004

Certified by 
Senthil Todadri
Assistant Professor
Thesis Supervisor

Accepted by
Professor David D. Pritchard
Senior Thesis Coordinator, Department of Physics



ARCHIVES

The Structure of Paired Boson Superfluids

by

Mara S. Daniel

Submitted to the Department of Physics
on May 7, 2004, in partial fulfillment of the
requirements for the degree of
Bachelor of Science

Abstract

This paper investigates classical and quantum mechanical models of superfluids and superfluid vortices using the Ginsburg Landau energy equation. Specifically, two types of superfluids are considered, ordinary superfluids where single bosons condense to form a strongly correlated system and superfluids where pairs of bosons condense to form a strongly correlated system while the single bosons remain uncondensed. First, the classical minimum energy configuration for an ordinary superfluid with and without a vortex was calculated. Additionally, the phase diagram for the exotic superfluid created by treating single bosons separately from pairs of bosons was determined as was the minimum energy state for each phase. Using these results, I then quantized the Ginsburg-Landau energy and investigated the possibility of excited states by creating small quantum mechanical oscillations about the classical minima. In the uniform superfluid, both the ordinary and exotic superfluids are able to support low energy excitations in the form of sound waves. In addition, the exotic superfluid has a gapped excitation that is a remanant of the uncondensed boson. Finally, the formalism for studying the modes of small oscillation about the classical minimum was developed for the superfluid vortex.

Thesis Supervisor: Senthil Todadri
Title: Assistant Professor

1 Introduction

Superconductors and superfluids are two examples of highly correlated systems that produce supercurrents, or currents that flow without resistance. In a superconductor, the supercurrent is comprised of electron pairs, known as Cooper pairs. These Cooper pairs are bosons which means that they can simultaneously occupy the same state.[3] In a superfluid, the molecules comprising the supercurrent are also bosons and the entire fluid flows without resistance. A supercurrent is produced in both superconductors and superfluids because each boson occupies the same quantum mechanical state. It costs a finite amount of energy to excite bosons to another momentum state, and therefore, the Cooper pairs or superfluid molecules remain in a coherent state and flow without resistance.

The macroscopic state of superfluids can be described by a complex number known as the order parameter, which is defined for every point in space. The order parameter, Ψ , represents a macroscopic variable similar to temperature or pressure. Ψ only has meaning when considered on length scales larger than the atomic scale and can vary over macroscopic distances. In the lowest energy state of a superfluid, the magnitude and phase of the order parameter are uniform throughout space. Spatial variation in the order parameter causes energy excitations in the superfluid. I will specifically investigate an excitation in the order parameter that forms a vortex in the superfluid. A vortex is a location where the phase winds by 2π forcing the current to flow in a circle.

Throughout this paper I will study a two dimensional superfluid. This superfluid will either be composed of bosons which have condensed to form a superfluid or pairs of bosons which have condensed to form a superfluid while the single bosons remain uncondensed. This second situation represents a unique state of a superfluid. A vortex excitation in this state will carry a magnetic flux that is half of the flux quantum, Φ_0 . $\Phi_0 = h/(2e)$ where e is the electron charge, and a vortex with two bosons condensed carries twice the charge of an ordinary vortex, so $e \rightarrow 2e$ and $\Phi_0 \rightarrow \Phi_0/2$. Experimental laboratories have not yet succeeded in condensing pairs of bosons without the single boson, but ultra cold atom groups may produce such a state in the future. The situation is relevant because the questions concerning such a fractional flux state are similar to questions raised in exotic quantum spin systems. These quantum spin systems are known as “spin-liquids” and spin liquids can support excitations with fractional spin. Additionally, it has been proposed that

high temperature superconductivity is produced from states that carry fractional spin. This physics is somewhat similar to superfluids with fractional flux. Since high temperature superconductivity is not yet well understood, further insight about fractional spin or flux could prove useful.

My specific task is to investigate the classical ground state energies in superfluid systems with and without vortex excitations. I will also calculate the phase diagram depicting when the single and double bosons condense when they are treated as separate particles. From these ground state classical energies, I will then quantize the systems about their classical minimums and observe the excited states that are produced when small quantum mechanical excitations are introduced into the system.

2 Background on Superfluids

Through out this paper I will be working with superfluids existing either in a superconductor or superfluid. My purpose is not to delve into the physics governing superconductors or superfluids, but merely to investigate the effects of vortices in superfluids. For this paper, it is sufficient to know that a superfluid is produced when bosons condense to form a highly correlated system. This superfluid is resistanceless and I will model it using the Ginsburg-Landau energy equation.

2.1 Order Parameters and Vortices

The quantity Ψ , called the order parameter, assigns each position within a superfluid a magnitude and phase. The superfluid is completely defined by specifying Ψ at every point and the lowest energy state of a superfluid occurs when Ψ is uniform at every point in the sample.

A vortex is a location within a superfluid where the phase winds by 2π causing the current flows in a circle, see Figure 1. Using symmetry arguments it is obvious that the magnitude of Ψ cannot depend on the phase, however the magnitude could vary with radius. The fist part of this paper will discuss how to calculate Ψ in a uniform superfluid and in the presence of a vortex.

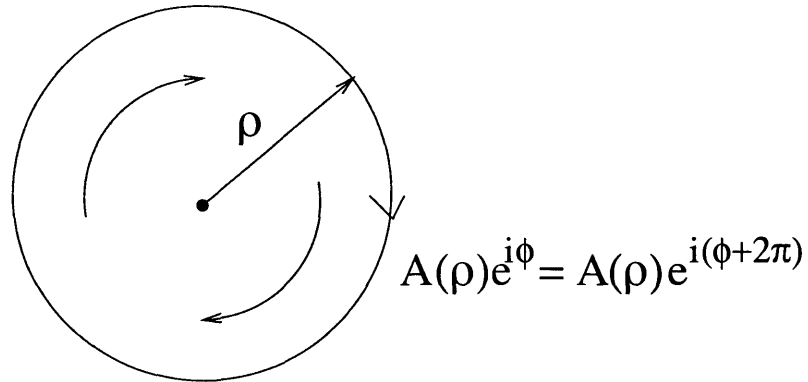


Figure 1: Schematic of a vortex. The current circles around a central point and the phase of Ψ winds by 2π with each revolution. ρ is the radius of the vortex.

2.2 Ginsburg-Landau Energy Equation

The Ginsburg Landau equation given by

$$E = \int d^2x \left[|\vec{\nabla}\Psi|^2 + r|\Psi|^2 + u|\Psi|^4 \right] \quad (1)$$

where E is the total energy of the system, Ψ is the order parameter which depends on position, and r and u are constants. The Ginsburg-Landau equation is simply an expansion of Ψ to fourth order plus a term, $|\vec{\nabla}\Psi|^2$, that requires any variation in Ψ to cost energy. I will assume that u is a positive number and that r can be either positive or negative. To calculate specific observable quantities with accurate precision, a more correct energy expression would be necessary; however, the Ginsburg Landau equation accurately models the physics of an ordinary superfluid system especially when the superfluid is near the phase transition making Ψ smaller than 1. By minimizing the Ginsburg Landau equation, I will be able to determine the classical ground state energy of the superfluid.

Equation 1 is relevant when investigating a superfluid with a single condensed boson; however, it will have to be modified to consider a pair of bosons separately

from a single boson. The equation I will use to model this scenario is given by

$$E = \int d^2x [|\vec{\nabla}\Psi_1|^2 + r_1|\Psi_1|^2 + u_1|\Psi_1|^4 + |\vec{\nabla}\Psi_2|^2 + r_2|\Psi_2|^2 + u_2|\Psi_2|^4 - g(\Psi_2^*\Psi_1^2 + \Psi_2\Psi_1^{*2}) + u|\Psi_1|^2|\Psi_2|^2] \quad (2)$$

where Ψ_1 is the order parameter for the single boson and Ψ_2 is the order parameter for the pair of bosons. The term multiplied by g says that there are two single bosons for every pair of bosons and the u term is a repulsion term. For simplicity, I will take $u = u_1 = u_2$ which makes the equation

$$E = \int d^2x [|\vec{\nabla}\Psi_1|^2 + r_1|\Psi_1|^2 + u|\Psi_1|^4 + |\vec{\nabla}\Psi_2|^2 + r_2|\Psi_2|^2 + u|\Psi_2|^4 - g(\Psi_2^*\Psi_1^2 + \Psi_2\Psi_1^{*2}) + u|\Psi_1|^2|\Psi_2|^2]. \quad (3)$$

3 Classical Treatment of the Ginsburg-Landau Equation

Throughout this section, I will treat the Ginsburg-Landau equation as a classical energy equation. This will allow me to calculate the value of Ψ which minimizes the energy for a uniform single boson and the functional form of Ψ which minimizes the energy of a single boson vortex. I will also determine the phase diagram for when single bosons and pairs of bosons condense as a function of r_1 and r_2 .

3.1 Uniform Superfluid with Single Condensed Boson

Returning to Equation 1,

$$E = \int d^2x [|\vec{\nabla}\Psi|^2 + r|\Psi|^2 + u|\Psi|^4],$$

it is obvious that if $r > 0$, the minimum energy configuration for the superfluid will occur with $\Psi = 0$. This state corresponds to a non-superfluid state. However, if $r < 0$, Equation 1 is minimized by a non-zero value of Ψ . In the case without a

vortex, or $\vec{\nabla}\Psi = 0$, the lowest energy state occurs when $(r|\Psi|^2 + u|\Psi|^4)$ is minimized,

$$\frac{d}{d\Psi} [r|\Psi|^2 + u|\Psi|^4] = 0.$$

Differentiating, solving for Ψ , and remembering that r is negative,

$$\Psi(\rho, \theta) = \sqrt{\frac{|r|}{2u}} e^{i\theta}. \quad (4)$$

For a uniform superfluid with a single condensed boson, Equation 4 gives the value of Ψ which minimizes the energy. The phase of Ψ is arbitrary, but in the ground state, θ will be the same for every location in the superfluid.

3.2 Single Condensed Boson with Vortex

What happens if $\vec{\nabla}\Psi \neq 0$? I still want to minimize the energy; however, now I must use the variational method to minimize the integrand. The variational method states that an integral is extremized when small variations about the extrema result in no change in the value of the integral. The process of varying the argument of the integral and setting the variation equal to zero results in a differential equation for Ψ . Varying Equation 1, the differential equation is given by

$$\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho} \frac{d\Psi}{d\rho} = \left(\frac{1}{\rho^2} + r + 2u|\Psi|^2 \right) \Psi. \quad (5)$$

This equation cannot be solved exactly for Ψ ; however, I can look at the behavior of Ψ in the limits of large and small ρ .

Far away from the vortex, as ρ becomes large, I expect the behavior of Ψ to approach the uniform, non-vortex result. This implies that $\frac{\partial \Psi}{\partial \rho} = 0$. Also, as $\rho \rightarrow \infty$, $1/(\rho^2) \rightarrow 0$. Evaluating Equation 5, in the limit of large ρ , $\Psi(\rho, \theta) = \sqrt{\frac{|r|}{2u}} e^{i\theta}$, as expected. From the differential equation, I can also determine how Ψ approaches the limit of $|\Psi| = \Psi_0 = \sqrt{\frac{|r|}{2u}}$. Plugging in $|\Psi| = \Psi_0 + g(\rho)$ to the differential equation and taking the limit as $\rho \rightarrow \infty$, $g(\rho)$ must be proportional to $e^{-c\rho}$ where $c > 0$. This result indicates that Ψ approaches the limit exponentially.

In the core of the vortex, as $\rho \rightarrow 0$, Ψ is required to be well behaved and not

blow up when $\rho = 0$. I will guess that for $\rho \ll 1$, Ψ is proportional to some power of ρ . Making this assumption, the terms term with r and the term with u are small compared to the other terms in Equation 5 and the differential equation reduces to

$$\rho^2 \left(\frac{\partial^2 \Psi}{\partial \rho^2} \right) + \rho \left(\frac{d\Psi}{d\rho} \right) = \Psi^2. \quad (6)$$

The solutions to this equation are either proportional to $1/\rho$ or ρ . Since I do not want Ψ to blow up at the origin, the physical solution for Ψ in the limit of small ρ is proportional to ρ .

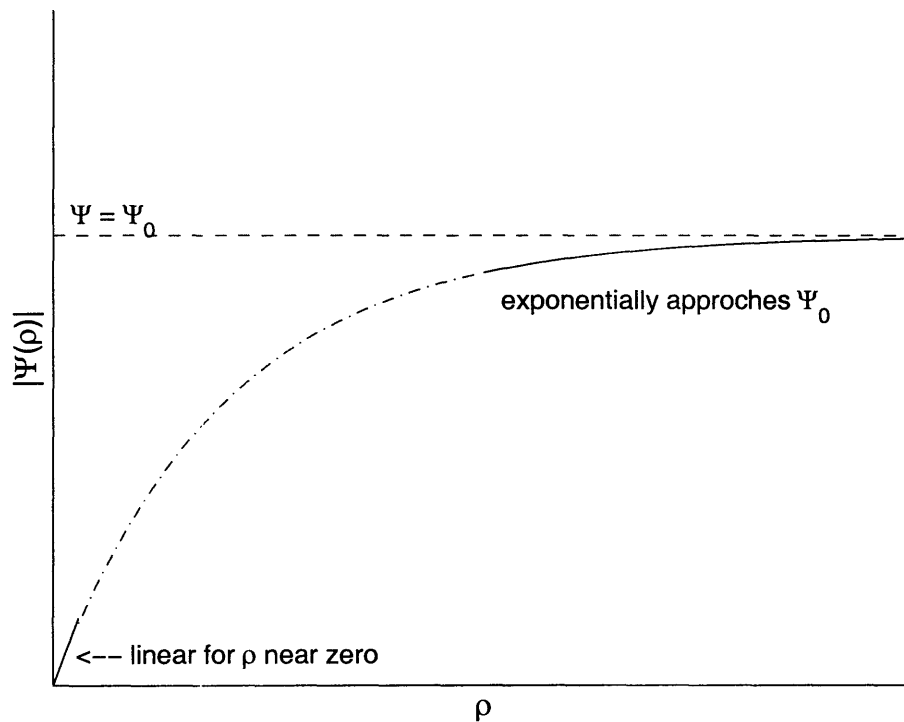


Figure 2: Approximate vortex profile for a single condensed boson.

The approximate vortex profile for a standard superfluid is shown in Figure 2. This result is important because the minimum energy configuration of a vortex will prove useful in later calculations.

3.3 Uniform Superfluid with Two Bosons Condensed

I will now examine the possibility of treating a pair of bosons separately from a single boson. A pair of bosons still is a boson (not a fermion) so I can consider the possibility that this new particle will condense to form a superfluid separately from the single boson. This leads to three possible phases of the superfluid: a) neither the single boson nor the pair of bosons are condensed, b) both the single boson and the pair of bosons are condensed, c) the pair of bosons are condensed but the single boson is not. The fourth permutation, having the single boson condensed but the pair of bosons in a non-superfluid state is not possible. When the single boson condensed it forces the pair of bosons to condense as well. This is because of the term in Equation 3, $g(\Psi_2^*\Psi_1^2 + \Psi_2\Psi_1^{*2})$. If Ψ_1^2 is non-zero, Ψ_2 would also become non-zero.

Let me review what it means mathematically for bosons to either condense into a superfluid phase or to remain uncondensed in a non-superfluid phase. Looking at Equation 1, the basic difference depends on the sign of r . If $r > 0$, the equation is minimized by having $\Psi = 0$. This represents the non-superfluid phase of matter when the bosons are not condensed. If $r < 0$, the minimum energy occurs with a non-zero value of Ψ . If $\vec{\nabla}\Psi = 0$, signifying that no vortex exists, then the Ginsburg-Landau equation is minimized for $\Psi = \Psi_0 = \sqrt{\frac{|r|}{2u}}$. In the presence of a vortex, the ground state is given by the approximate shape I found in Figure 2. These mathematical results will continue to hold when pairs of bosons are considered separately from single bosons, if the magnitude of r is large. If $r_1 \gg 0$ and $r_2 \gg 0$, both Ψ_1 and Ψ_2 will remain uncondensed, and if $r_1 \ll 0$ and $r_2 \ll 0$, a superfluid state will form. Why could these relationships fail if r_1 or r_2 are close to zero? Let me first return to the equation I will use to model the individual bosons separately from the pairs of bosons and then I will address this question.

The energy equation used to model the two boson case is given by Equation 3

$$E = \int d^2x [|\vec{\nabla}\Psi_1|^2 + r_1|\Psi_1|^2 + u|\Psi_1|^4 + |\vec{\nabla}\Psi_2|^2 + r_2|\Psi_2|^2 + u|\Psi_2|^4 - g(\Psi_2^*\Psi_1^2 + \Psi_2\Psi_1^{*2}) + u|\Psi_1|^2|\Psi_2|^2].$$

My goal is to determine a phase diagram for the three possible phases I discussed above. The axes in the phase diagram will be r_1 and r_2 , both of which can take

any positive or negative value. I mentioned before that certain behavior is expected if the magnitudes of r_1 and r_2 get really big, and let me now review that behavior. First, what happens if r_1 and r_2 are both large and positive? Equation 3 is obviously minimized if $\Psi_1 = \Psi_2 = 0$ which makes neither the single boson or pair of bosons condensed. Similarly, if r_1 and r_2 are both large and negative, then the energy will be minimized when both Ψ_1 and Ψ_2 have non-zero values. Therefore, both the single boson and the double boson will be condensed. What if r_1 is negative but r_2 is positive? I have already determined that if the single boson is condensed, the pair of bosons must also be condensed, therefore I expect in this case both Ψ_1 and Ψ_2 to be condensed. The final case, r_1 large and positive and r_2 large and negative, is where I expect to see this strange case where the single boson is not condensed, but the pair of bosons has condensed to form a superfluid.

This qualitative thinking provides a framework for what to expect at large values for r_1 and r_2 , but what happens when r_1 and r_2 are small? I am looking for something that would force Ψ_1 to be different than expected given the sign of r_1 . r_1 is multiplied by $|\Psi_1|^2$. The term $u|\Psi_1|^2|\Psi_2|^2$ also has a $|\Psi_1|^2$ in it. If r_1 is small and negative, but u is positive, there might be an $r_{1,effective}$ which is actually positive and forces Ψ_1 to remain in a non-superfluid phase. In the following calculation, I will be looking for an $r_{1,effective}$ which will determine when Ψ_1 is condensed.

Since I am just looking for the phase diagram, it is sufficient to consider the uniform order parameter configuration; therefore, for simplicity I will set $\vec{\nabla}\Psi_1 = \vec{\nabla}\Psi_2 = 0$. Minimizing the energy first with respect to Ψ_1 and then with respect to Ψ_2 , I obtain the following equations:

$$\frac{\partial E}{\partial \Psi_1} = \Psi_1^*(r_1 + 2u|\Psi_1|^2 + u|\Psi_2|^2) - 2g\Psi_2^*\Psi_1 = 0 \quad (7)$$

$$\frac{\partial E}{\partial \Psi_2} = \Psi_2^*(r_2 + 2u|\Psi_2|^2 + u|\Psi_1|^2) - g(\Psi_1^*)^2 = 0 \quad (8)$$

Remembering that $\Psi_1 = |\Psi_1|e^{i\theta_1}$ and $\Psi_2 = |\Psi_2|e^{i\theta_2}$, these equations easily tell us how the phase of Ψ_1 is related to the phase of Ψ_2 . The phase associated with each of the terms must be equal, so equating only the phases in the above equations,

$$-\theta_1 = \theta_1 - \theta_2 \qquad \theta_2 = 2\theta_1.$$

Both of these equations imply that θ_2 must be twice that of θ_1 . This is an interesting result because it says that the phase of the superfluid due to the double boson winds twice as quickly as the phase due to the single boson. In other words, when θ_2 winds by 2π making a full circle around the vortex, θ_1 only winds by π .

Discarding the phase dependence in Equation 7 and only considering the magnitudes of Ψ_1 and Ψ_2 , it is obvious that one solution is $\Psi_1 = 0$. Substituting $\Psi_1 = 0$ into the Equation 8 and solving for Ψ_2 , $|\Psi_2| = \sqrt{\frac{-r_2}{2u}}$ or $|\Psi_2| = 0$. $|\Psi_2| = \sqrt{\frac{-r_2}{2u}}$ is only valid when $r_2 < 0$ because $|\Psi_2|$ must be real. Therefore, $|\Psi_2| = 0$ must be valid only when $r_2 > 0$. As expected from my qualitative arguments, if $\Psi_1 = 0$ ($r_1 > 0$), Ψ_2 can either be zero ($r_2 > 0$) meaning both the single and double boson are not condensed or non-zero ($r_2 < 0$) meaning Ψ_1 is not condensed, but Ψ_2 is condensed.

Using Equation 7 the second solution for $|\Psi_1|$ is

$$|\Psi_1|^2 = \frac{1}{2u} [2g|\Psi_2| - u|\Psi_2|^2 - r_1].$$

The quantity in brackets is the $|r_{1,effective}|$ I was looking for! For one condensed boson, $|\Psi|^2 = \frac{|r|}{2u}$. In order for Ψ_1 to be condensed, this $r_{1,effective}$ must be greater than zero. Solving this condition for r_1 and plugging in that $|\Psi_2|^2 = \frac{|r_2|}{2u}$ if $|\Psi_2|$ is condensed, I find that for both Ψ_1 and Ψ_2 to be condensed

$$r_1 \leq 2g\sqrt{\frac{|r_2|}{2u}} - \frac{|r_2|}{2} \quad (9)$$

The full phase diagram showing the states of Ψ_1 and Ψ_2 as a function of r_1 and r_2 is shown in Figure 3. Radzihovsky et.al. independently derived the same result very recently.[8]

3.4 Vortex State with Condensed Boson Pairs

I have derived the phase diagram for when Ψ_1 and Ψ_2 condense in a uniform superfluid, but does the presence of a vortex alter the state of either the single or double boson? The most intriguing physical questions arise from the superfluid phase where the double bosons condense and the single bosons remain uncondensed. The

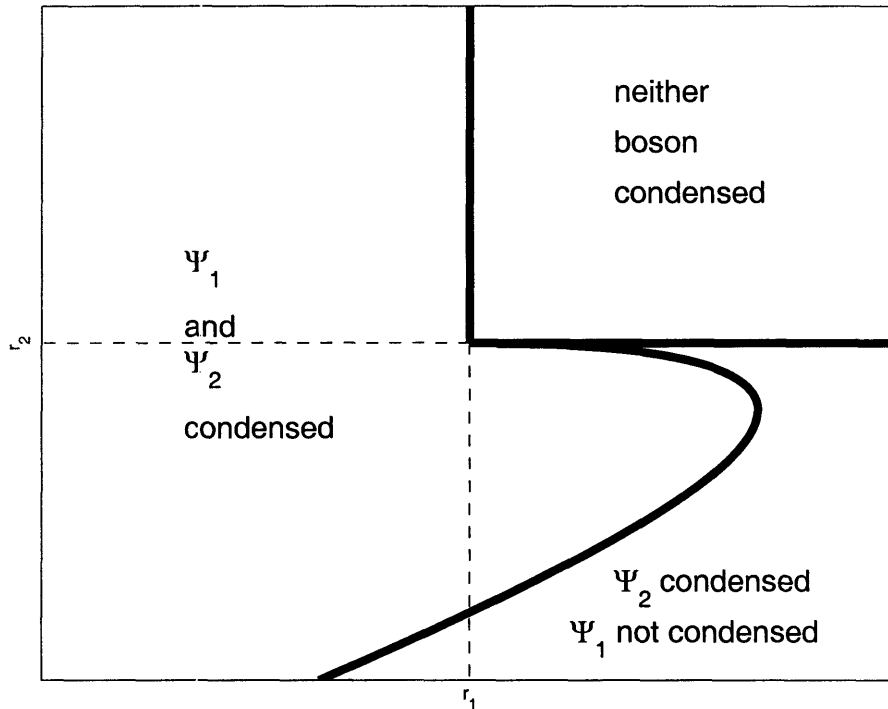


Figure 3: Phase diagram for when Ψ_1 and Ψ_2 are condensed or uncondensed. If r_1 and r_2 are greater than zero, than neither boson can condense. If r_1 is large and negative, than both Ψ_1 and Ψ_2 are condensed. If $r_1 \gg 0$ and $r_2 \ll 0$, then the double boson not condensed, but the single boson is not condensed. The phase boundary where the double boson is condensed and the single boson changes from a condensed to uncondensed state is given by $r_1 = 2g\sqrt{\frac{|r_2|}{2u}} - \frac{|r_2|}{2}$

quantity that determines whether Ψ_1 condenses is

$$r_{1,eff} = 2g|\Psi_2| - u|\Psi_2|^2 - r_1. \quad (10)$$

If $r_{1,eff} > 0$, Ψ_1 is uncondensed, and if $r_{1,eff} < 0$, Ψ_1 condenses. Since $r_{1,eff}$ depends on Ψ_2 , if Ψ_2 varies with ρ , $r_{1,eff}$ will also vary with ρ . In a paired boson vortex, Ψ_2 will have approximately the same shape as the order parameter for the single boson vortex shown in Figure 2. This implies that for $\Psi_2 < 1$, $r_{1,eff}$ will decrease as ρ decreases. If r_1 is less than zero, the value of $r_{1,eff}$ will eventually become negative in the vortex core. A plot of $r_{1,eff}$ as a function of ρ is shown in Figure 4.

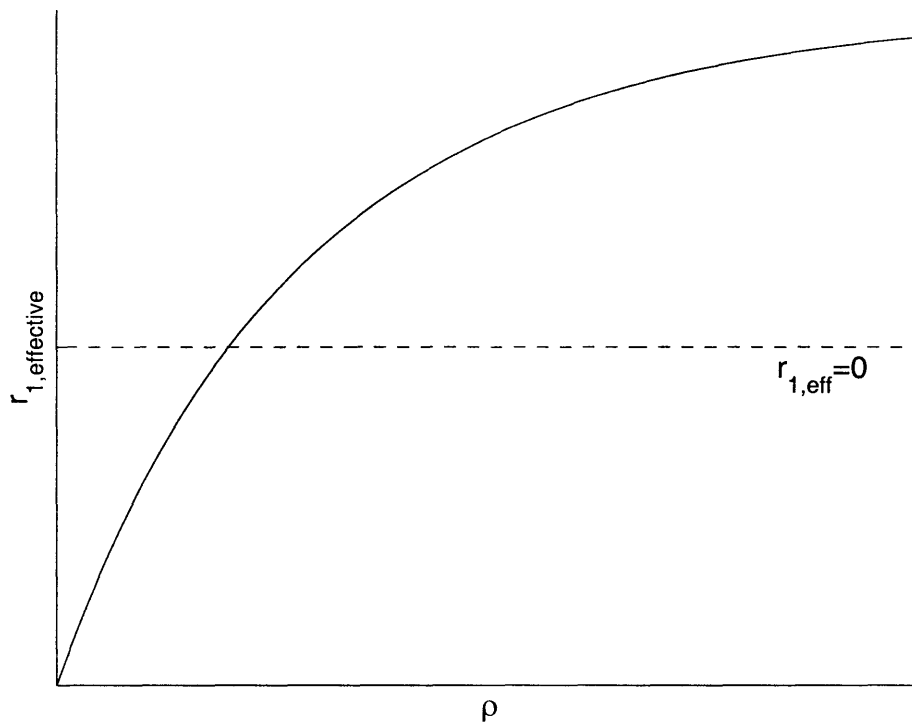


Figure 4: This diagram shows $r_{1, effective}$ as a function of ρ in the presence of a vortex. The value of r_1 is taken to be negative. In the core of the vortex, $r_{1, eff}$ drops below zero implying that single bosons may be able to exist in the core of a paired boson vortex even when they are forbidden in the bulk.

It is obvious from Figure 4 that $r_{1, eff}$ decreases as ρ decreases, but what effect does that have on Ψ_1 ? As long as $r_{1, eff}$ is greater than zero, Ψ_1 remains uncondensed. However, if $r_{1, eff}$ becomes negative, Ψ_1 can want to form a condensate. This suggests that in the presence of a vortex, Ψ_1 may condense or be bound to the vortex if $r_1 < 0$ and ρ becomes small enough. It is not apparent from these equations what state the single boson would occupy in the vortex core. It is possible that a single boson could be bound to the double boson vortex or that multiple single bosons could condense to form an excited state of a superfluid. Because the single bosons must be classically confined to the region where $r_{1, eff} < 0$, single bosons would exist in an excited superfluid state (not the ground state) in the core of a paired boson vortex. More can be learned about the possibility of having single bosons to the double boson vortex by

investigating small quantum mechanical oscillations in the paired boson vortex state.

4 Quantum Mechanical Treatment of the Ginsburg-Landau Equation

In this section I will continue to use the Ginsburg-Landau energy equation to model a superfluid; however I will now take a quantum mechanical rather than classical approach. The goal in this section is to determine if small quantum mechanical oscillations about the classical minimum of a superfluid produce excited states. If higher energy states are generated, I will try to calculate the energy of these excitations. In order to make the Ginsburg-Landau equation quantum mechanical, the order parameter must be redefined as a quantum operator. Consequently, commutation relations must be specified for Ψ . Small oscillations about a minimum energy can be modeled as a harmonic oscillator, therefore, the commutation relations for Ψ are

$$\begin{aligned} [\Psi(x), \Psi(y)] &= [\Psi^\dagger(x), \Psi^\dagger(y)] = 0 \\ [\Psi(x), \Psi^\dagger(y)] &= \delta(x - y). \end{aligned}$$

Taking Ψ to be an operator and rearranging Equation 1, I can write the energy equation, now a Hamiltonian, for a single boson superfluid as

$$\hat{H} = \int d^2x [\hat{\Psi}^\dagger (-\nabla^2 + r) \hat{\Psi} + u \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi}]. \quad (11)$$

The approach to calculating the energy of any possible excited states will be to begin with the ground state energy for Ψ I found in the previous section and perturb it slightly. This is identical to modeling each point in space as its own harmonic oscillator. The goal is to then to diagonalize the Hamiltonian such that it takes the form

$$H = \sum_n [C_n a_n^\dagger a_n] \quad (12)$$

where a_n^\dagger and a_n are the raising and lowering operators respectively for the harmonic oscillator, and C_n will give the energy of the n^{th} harmonic oscillator. This energy is the quantity that will give information about what type of excitation the superfluid

can support.

4.1 Uniform Superfluid with Single Boson Condensed

From an earlier calculation, I know that the the energy of a uniform superfluid with a single boson condensed is minimized when $\Psi = \sqrt{\frac{|r|}{2u}} e^{i\theta}$. θ is arbitrary, so I will choose it to be zero making

$$\Psi_{min} = \Psi_0 = \sqrt{\frac{|r|}{2u}}.$$

I want to plug in this minimum energy solution plus a small variation into Equation 11. Therefore, I will make the substitution that

$$\hat{\Psi}(x) = \Psi_0 + \hat{\phi}(x) \tag{13}$$

$$\hat{\Psi}(x)^\dagger = \Psi_0 + \hat{\phi}(x)^\dagger \tag{14}$$

where $\hat{\phi}$ is a small perturbation. Using the commutations relations for Ψ , and Equations 13 and 14, the commutation relations for ϕ are

$$\begin{aligned} [\phi(x), \phi(y)] &= [\phi^\dagger(x), \phi^\dagger(y)] = 0 \\ [\phi(x), \phi^\dagger(y)] &= \delta(x - y). \end{aligned}$$

Plugging Equation 13 and 14 into Equation 11 all of the terms to first order in ϕ cancel because I am expanding about a minimum. The constant terms just add a minimum ground state energy onto the Hamiltonian, but do not include any new physics. I will therefore neglect the constant terms. After expanding the Hamiltonian to second order in ϕ and simplifying,

$$H = \int d^2x [\phi^\dagger(-\nabla^2 + |r|)\phi + \frac{|r|}{2}(\phi\phi + \phi^\dagger\phi^\dagger)]. \tag{15}$$

My goal is to diagonalize the Hamiltonian to find the energy of any excited states. This proves much simpler if I first Fourier transform $\phi(x)$ to ϕ_k in momentum space.

Under this transformation the Hamiltonian take the form

$$H = \sum_{k>0} \left[\phi_k^\dagger \phi_k (-k^2 + |r|) + \frac{|r|}{2} (\phi_k \phi_{-k} + \phi_k^\dagger \phi_{-k}^\dagger) \right]. \quad (16)$$

This Hamiltonian is almost diagonal. I just need to find a method of decoupling the ϕ_k oscillators from the ϕ_{-k} oscillators in order to determine the energy spacing between eigenstates.

The method I will employ to diagonalize the Hamiltonian is known as the Bogolubov transformation.[6] First, I will define

$$\Gamma_k = \begin{pmatrix} \phi_k \\ \phi_{-k}^\dagger \end{pmatrix} \quad \text{and} \quad \Gamma_k^\dagger = \begin{pmatrix} \phi_k^\dagger & \phi_{-k} \end{pmatrix}$$

in order to simplify to

$$H = \sum_{k>0} \left[\Gamma_k^\dagger h_k \Gamma_k \right] \quad (17)$$

where

$$h_k = \begin{pmatrix} k^2 + |r| & |r| \\ |r| & k^2 + |r| \end{pmatrix}.$$

The goal is to find a linear transformation to put the Hamiltonian into the form

$$H = \sum_{k>0} \left[\Gamma_k^\dagger h_k \Gamma_k \right] = \sum_{k>0} \left[\gamma_k^\dagger U_k^\dagger h_k U_k \gamma_k \right] = \sum_{k>0} \left[\gamma_k^\dagger D_k \gamma_k \right] \quad (18)$$

where $\Gamma_k = U_k \gamma_k$, D_k is a diagonal matrix defined as $D_k = U_k^\dagger h_k U_k$ and

$$\gamma_k = \begin{pmatrix} a_k \\ a_k^\dagger \end{pmatrix}.$$

I will require that a^\dagger and a are raising and lowering operators of the harmonic oscillator, consequently,

$$\begin{aligned} [a_k, a_{k'}] &= [a_k^\dagger, a_{k'}^\dagger] = 0 \\ [a_k, a_{k'}^\dagger] &= \delta_{kk'}. \end{aligned}$$

In order to solve for D_k , I need to determine the relationship between U and U^\dagger . It is not necessarily accurate to assume that the transformation to diagonalize the Hamiltonian is unitary. Therefore, UU^\dagger may not equal the identity operator. To derive this relationship, I will start with the known commutation relations between ϕ_k and a_k . First, I will rename the elements of Γ_k to

$$\Gamma_k = \begin{pmatrix} \phi_k \\ \phi_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \Gamma_{1k} \\ \Gamma_{2k} \end{pmatrix}$$

In terms of Γ , the commutation relations now become

$$\begin{aligned} [\Gamma_{1k}, \Gamma_{2k}^\dagger] &= [\Gamma_{2k}, \Gamma_{1k}^\dagger] = 0 \\ [\Gamma_{1k}, \Gamma_{1k}^\dagger] &= 1 & [\Gamma_{2k}, \Gamma_{2k}^\dagger] &= -1. \end{aligned}$$

Changing the subscripts 1 and 2 to a and b where a and b are summed from 1 to 2, the commutation relation becomes

$$[\Gamma_{ak}, \Gamma_{bk}^\dagger] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_{ab}^z. \quad (19)$$

What I want is a relationship between U and U^\dagger , so I will substitute $\Gamma_k = U_k \gamma_k$ into Equation 19.

$$[\Gamma_{ak}, \Gamma_{bk}^\dagger] = [U_{aa'} \gamma_{a'}, \gamma_{b'}^\dagger U_{b'b}^\dagger] = U_{aa'} [\gamma_{a'}, \gamma_{b'}^\dagger] U_{b'b}^\dagger = \sigma_{ab}^z. \quad (20)$$

Because γ_k follows the same commutation relations as Γ_k , $[\gamma_a, \gamma_b^\dagger] = \sigma_{ab}^z$. Therefore, simplifying the above equation,

$$U \sigma^z U^\dagger = \sigma^z. \quad (21)$$

My goal is still to find D_k which is a diagonal matrix that gives the energy spacing between the eigenstates of each harmonic oscillator. I started with the equation

$$U_k^\dagger h_k U_k = D_k.$$

Multiplying each side of the equation on the left by $U\sigma^z$ results in the eigenvalue equation

$$\sigma^z h_k U_k = U_k \sigma^z D_k \quad (22)$$

where the rows of U_k are the eigenvectors of $\sigma^z h_k$ and the diagonal matrix elements of $\sigma^z D_k$ are the eigenvalues of $\sigma^z h_k$. In other words, all I have to do to find the energy spacing of the harmonic oscillators is find the eigenvalues of the matrix $\sigma^z h_k$. This is a relatively simple algebraic calculation, so I will spare you from the math. After taking into account σ^z acting on both h_k and D_k , both non-zero matrix elements of the 2×2 diagonal matrix D_k have the same value

$$k\sqrt{k^2 + 2|r|}. \quad (23)$$

Substituting this back into the Hamiltonian given by Equation 18,

$$H = \sum_{k>0} \left[k\sqrt{k^2 + 2|r|} (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) \right] \quad (24)$$

This is finally the diagonal Hamiltonian, now what does it mean? What this Hamiltonian represents is an infinite number of harmonic oscillators where each oscillator is labeled by a different value for k . a_k^\dagger and a_k are creation and annihilation operators respectively for the k^{th} harmonic oscillator and $k\sqrt{k^2 + 2|r|}$ specifies the energy. What type of oscillations are produced in the superfluid? These oscillations are sound waves because as k goes to zero, the Hamiltonian goes to zero as a polynomial of k . Remember, however, that this Hamiltonian is an approximation and only valid for low level excitations about the classical minimum of the system. To obtain this result I only kept the second order terms in ϕ so that the superfluid could be modeled as a harmonic oscillator. This approximation is valid as long as we only consider low energy excitations. Understanding higher energy excitations, would require inventing a method to solve these equations while keeping higher order terms.

4.2 Uniform Superfluid with Two Bosons Condensed

After carefully deriving the diagonalized Hamiltonian for the single condensed boson case, extending the derivation to the uniform case with two condensed bosons is

relatively straight forward. My goal is to again find the low energy excitations in the superfluid. The difference now is there are two separate excitations to consider, the single boson and the double boson. As before, I will start by rearranging the energy equation and treating Ψ_1 and Ψ_2 as quantum operators. For this case, where I am considering a pair of bosons separately from a single boson, the Hamiltonian takes the form

$$\begin{aligned} \hat{H} = \int d^2x & [\hat{\Psi}_1^\dagger (-\nabla^2 + r_1) \hat{\Psi}_1 + u \hat{\Psi}_1^\dagger \hat{\Psi}_1^\dagger \hat{\Psi}_1 \hat{\Psi}_1 \\ & + \hat{\Psi}_2^\dagger (-\nabla^2 + r_2) \hat{\Psi}_2 + u \hat{\Psi}_2^\dagger \hat{\Psi}_2^\dagger \hat{\Psi}_2 \hat{\Psi}_2 \\ & - g(\hat{\Psi}_2^\dagger \hat{\Psi}_1^2 + \hat{\Psi}_2 \hat{\Psi}_1^{\dagger 2}) + u \hat{\Psi}_1^\dagger \hat{\Psi}_1 \hat{\Psi}_2^\dagger \hat{\Psi}_2]. \end{aligned} \quad (25)$$

I want to look at the specific situation where pairs of bosons condense, but the single bosons remain uncondensed. I found that the classical ground state solution for this configuration is given by $\Psi_1 = 0$ and $|\Psi_2| = \Psi_{20} = \sqrt{\frac{|r_2|}{2u}}$. In order to take small variations about this ground state solution, I will substitute the following expressions in to Equation 25

$$\begin{aligned} \hat{\Psi}_1 &= \hat{\phi}_1 & \hat{\Psi}_2 &= \hat{\Phi}_{20} + \hat{\phi}_2 \\ \hat{\Psi}_1^\dagger &= \hat{\phi}_1^\dagger & \hat{\Psi}_2^\dagger &= \hat{\Phi}_{20} + \hat{\phi}_2^\dagger. \end{aligned}$$

The commutation relations for Ψ_1 , Ψ_2 , ϕ_1 , and ϕ_2 carry over directly from the previous example and single boson operators always commute with double boson operators. I will substitute these expressions into Equation 25 and, as in the previous example, expand to second order in ϕ . Taking the Fourier transform of the Hamiltonian to separate the oscillators into their different modes,

$$\begin{aligned} H = \sum_{k>0} & [\phi_{1k}^\dagger \left(r_1 + \frac{|r_2|}{2} + k^2 \right) \phi_{1k} + \phi_{2k}^\dagger (|r_2| + k^2) \phi_{2k} + \\ & \frac{|r_2|}{2} (\phi_{2k} \phi_{-2k} + \phi_{2k}^\dagger \phi_{-2k}^\dagger) - g \Psi_{20} (\phi_{1k} \phi_{-1k} + \phi_{1k}^\dagger \phi_{-1k}^\dagger)]. \end{aligned} \quad (26)$$

I can simplify the Hamiltonian in the same way I did before,

$$H = \sum_{k>0} \left[\Gamma_k^\dagger h_k \Gamma_k \right].$$

This time, however, h_k is a 4×4 matrix and Γ_k represents a 4 component vector,

$$\Gamma_k = \begin{pmatrix} \phi_{1k} \\ \phi_{-1k}^\dagger \\ \phi_{2k} \\ \phi_{-2k}^\dagger \end{pmatrix},$$

where the subscript 1 indicates the single boson and the subscript 2 indicates the double boson. Although the matrices have gotten bigger, there are not any cross terms between ϕ_1 and ϕ_2 . This means that the problem breaks down into two 2×2 matrix problems. In Equation 4.2,

$$h_k = \begin{pmatrix} h_{1k} & 0 \\ 0 & h_{2k} \end{pmatrix}. \quad (27)$$

where

$$h_{1k} = \begin{pmatrix} k^2 + r_1 + |r_2|/2 & -2g\Psi_{20} \\ -2g\Psi_{20} & k^2 + r_1 + |r_2|/2 \end{pmatrix} \quad \text{and} \quad h_{2k} = \begin{pmatrix} k^2 + |r| & |r| \\ |r| & k^2 + |r| \end{pmatrix}.$$

Following the logic from the previous problem, the two eigenvalue equations I need to solve in order to find the energy spacing are

$$\sigma^z h_{1k} U_{1k} = U_{1k} \sigma^z D_{1k} \quad \text{and} \quad \sigma^z h_{2k} U_{2k} = U_{2k} \sigma^z D_{2k} \quad (28)$$

where U_{1k} and U_{2k} are 2×2 matrices analogous to the U matrix in the previous example for the single boson and double boson respectively. D_{1k} and D_{2k} give the energy spacing for the single and double boson respectively assuming it is possible to create a low level excitation in this superfluid configuration.

For the condensed double boson, the matrix and therefore the energy is the same

as the single boson case,

$$D_{2k} = k\sqrt{k^2 + 2|r|} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (29)$$

For the single uncondensed boson, solving the eigenvalue equation for Ψ_{1k} ,

$$D_{1k} = \sqrt{\left(k^2 + r_1 + \frac{|r_2|}{2}\right)^2 - 4g^2\Psi_{20}^2} \quad (30)$$

The eigenvalues of the Hamiltonian must be real numbers because the energy of a system is an observable quantity. Therefore, in order for there to be low level excitations in the single boson, the relation

$$k^2 + r_1 + \frac{|r_2|}{2} \geq 2g\Psi_{20} \quad (31)$$

must hold. Notice that setting $k = 0$ (the lowest energy oscillator) produces same relationship I derived in Equation 9 which gives the classical requirement for having Ψ_1 uncondensed but Ψ_2 condensed. The conditions for having low level excitations in the single uncondensed boson is the same condition as having a superfluid with only the double bosons condensed.

What do these results mean? In this approximation, there can always be low level excitations in the superfluid created by the condensed double boson. These excitations have the same energy as the single boson superfluid and also represent sound waves. In order to create excitations in the single boson, however, there must be a minimum energy supplied to the system. Without this minimum energy, it is not possible to create a single boson.

4.3 Single Boson with Vortex

I will now return to the condensed single boson superfluid and impose a vortex solution. My goal is to construct low energy excitations which could be either scattering or bound states. The Hamiltonian remains the same as for the uniform single boson; however, imposing a vortex excitation in the superfluid means that the value of $\Psi(\rho, \theta)$ will depend on ρ and θ as shown in Figure 2. To solve the differential

equations I would have to assume a functional form for $\Psi(\rho, \theta)$, but for now, I will keep $\Psi(\rho, \theta)$ as general as possible,

$$\Psi(\rho, \theta) = \psi(\rho)e^{i\theta} = \psi e^{i\theta}. \quad (32)$$

In order to generate small oscillations about this ground state, I will approximate $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ in the following manner

$$\hat{\Psi}(x) = \psi e^{i\theta} + \hat{\phi}(x) \quad (33)$$

$$\hat{\Psi}(x)^\dagger = \psi e^{-i\theta} + \hat{\phi}(x)^\dagger. \quad (34)$$

Substituting the above relations into the Hamiltonian and keeping terms only to second order in ϕ , the Hamiltonian simplifies to

$$H = \int d^2x \left[\phi^\dagger (-\nabla^2 + \alpha) \phi + u\psi^2 (\phi^2 e^{-2i\theta} + \phi^{\dagger 2} e^{2i\theta}) \right] \quad (35)$$

where $\alpha = r + 4u\psi^2$.

As in the uniform case, my goal is to diagonalize the Hamiltonian. However, taking the Fourier transform of the above equation does not separate most of the eigenmodes as was accomplished previously. Instead of Fourier transforming the Hamiltonian and then diagonalizing the 2 remaining oscillators in k space, I will generalize those methods to diagonalize the infinite number of oscillators currently represented in position space.

The first step is to write the Hamiltonian in matrix form. As in the uniform case,

$$\Gamma(x) = \begin{pmatrix} \phi(x) \\ \phi^\dagger(x) \end{pmatrix} \quad \Gamma^\dagger(x) = \begin{pmatrix} \phi^\dagger(x) & \phi(x) \end{pmatrix}$$

and

$$\gamma_n = \begin{pmatrix} a_n \\ a_n^\dagger \end{pmatrix} \quad \gamma_n^\dagger = \begin{pmatrix} a_n^\dagger & a_{-n} \end{pmatrix}$$

where I have changed the subscript label from k to n to clarify that I am not transforming to momentum space. γ_n is defined as a linear transformation of $\Gamma(x)$ where

the Hamiltonian is diagonal in the γ_n basis. Therefore, ϕ is required to be a linear transformation of a and a^\dagger ,

$$\phi(x) = \sum_n [f_n(x)a_n + g_n(x)a_n^\dagger] \quad \phi(x)^\dagger = \sum_n [f_n^*(x)a_n^\dagger + g_n^*(x)a_n]. \quad (36)$$

Defining

$$\chi_n(x) = \begin{pmatrix} f_n^*(x) \\ g_n^*(x) \end{pmatrix} = \begin{pmatrix} f_n^* \\ g_n^* \end{pmatrix} \quad \chi_n^\dagger(x) = \begin{pmatrix} f_n(x) & g_n(x) \end{pmatrix} = \begin{pmatrix} f_n & g_n \end{pmatrix}$$

allows ϕ to be written as

$$\phi(x) = \sum_n \chi_n^\dagger(x)\gamma_n \quad \phi(x)^\dagger = \sum_n \gamma_n^\dagger\chi_n(x).$$

I will also need a matrix U to transform γ_n into $\Gamma(x)$. U is given by

$$U_n(x) = \begin{pmatrix} f_n & g_n \\ g_n^* & f_n^* \end{pmatrix} \quad U_n(x)^\dagger = \begin{pmatrix} f_n^* & g_n \\ g_n^* & f_n \end{pmatrix} \quad (37)$$

and

$$\Gamma(x) = \sum_n U_n(x)\gamma_n \quad \Gamma(x)^\dagger = \sum_n \gamma_n^\dagger U_n^\dagger(x). \quad (38)$$

I have defined numerous quantities, and before diagonalizing the Hamiltonian, I first need to specify the commutation relations or normalization requirements for each of these variables. The definitions for $\Gamma(x)$ and γ_n carry over from the uniform case. The elements of $\Gamma(x)$ and γ_n (ϕ and a_n respectively) obey the same commutation relations as the raising and lowering operators for the harmonic oscillator,

$$[\phi(x), \phi(y)^\dagger] = \delta(x - y) \quad \text{and} \quad [a_n, a_m^\dagger] = \delta_{m,n}.$$

What does this mean for the normalization of $f(x)$ and $g(x)$? Substituting Equa-

tion 36 into the commutation relation for ϕ , the commutator reduces to

$$\sum_n [f_n(x)f_n^*(y) - g_n(x)g_n^*(y)] = \delta(x - y)$$

or, in different notation

$$\sum_n [\chi_n^\dagger(x)\sigma^z\chi_n(y)] = \delta(x - y).$$

Extending this further to determine how U and U^\dagger relate,

$$\sum_n [U_n(x)\sigma^zU_n^\dagger(y)] = \delta(x - y)\sigma^z. \quad (39)$$

By manipulating the above equations, it can also be determined how U is normalized when integrated over x rather than summed over n . The result is

$$\int d^2x U_m^\dagger(x)\sigma^zU_n(x) = \delta_{m,n}\sigma^z. \quad (40)$$

Before this large quantity of math began, the Hamiltonian simplified to

$$H = \int d^2x \left[\phi^\dagger(-\nabla^2 + \alpha)\phi + u\psi^2(\phi^2e^{-2i\theta} + \phi^{\dagger 2}e^{2i\theta}) \right]$$

where $\alpha = r + 4u\psi^2$. Using the definitions above, the Hamiltonian further simplifies to

$$H = \int d^2x \Gamma^\dagger(x)h\Gamma(x)$$

where

$$h = \begin{pmatrix} \frac{1}{2}(-\nabla^2 + \alpha) & u\psi^2e^{2i\theta} \\ u\psi^2e^{-2i\theta} & \frac{1}{2}(-\nabla^2 + \alpha) \end{pmatrix}.$$

Substituting Equation 38 into the Hamiltonian for $\Gamma(x)$,

$$H = \int d^2x \Gamma^\dagger(x)h\Gamma(x) = \sum_{n,m} \int d^2x \gamma_n^\dagger U_n^\dagger(x)hU_m(x)\gamma_m.$$

My goal is to diagonalize the Hamiltonian so that it takes the form

$$H = \sum_n \gamma_n^\dagger \begin{pmatrix} d_{n,1} & 0 \\ 0 & d_{n,2} \end{pmatrix} \gamma_n$$

where $d_{n,1}$ and $d_{n,2}$ are the excited state energies. Notice the normalization condition given by Equation 40 would be helpful in simplifying the Hamiltonian if

$$hU_m(x) = \sigma^z U_m(x) D_m$$

where D_m is a diagonal matrix. Substituting this equation in the Hamiltonian,

$$\begin{aligned} H &= \sum_{n,m} \int d^2x \gamma_n^\dagger U_n^\dagger(x) hU_m(x) \gamma_m \\ &= \sum_{n,m} \int d^2x \gamma_n^\dagger U_n^\dagger(x) \sigma^z U_m(x) D_m \gamma_m \\ &= \sum_{n,m} \gamma_n^\dagger \delta_{m,n} \sigma^z D_m \gamma_m \\ &= \sum_n \gamma_n^\dagger \sigma^z D_n \gamma_n. \end{aligned}$$

This is exactly the simplification of the Hamiltonian I was looking for. Therefore, the eigenvalue equation I need to solve in order to diagonalize the Hamiltonian is given by

$$hU_m(x) = \sigma^z U_m(x) D_m. \quad (41)$$

The remainder of this section will be spent setting up the differential equations and discussing possible physical results. Unlike the uniform examples, the matrix h involves derivatives with respect to ρ and θ . Therefore, Equation 41 results in 2 sets of coupled differential equations for $f_m(x)$ and $g_m(x)$. $f_m(x)$ and $g_m(x)$ give the “quantity” of a_m and a_m^\dagger that produce an excitation $\phi(x)$ and can be thought of as the wave function of an excited state with energy $d_{m,1}$ or $d_{m,2}$ in a superfluid vortex. Solving these two sets of couple differential equations proves to be somewhat subtle. In order to look for solutions, it is first necessary to choose a functional form for $\psi(\rho)$. Additionally, I expect that oscillations can occur either in the ρ or θ directions.

Small oscillations about the radial direction cost energy, however, the phase factor is arbitrary, and therefore any change in θ does not effect the energy. It should be possible to produce an excited state that changes the phase of $\Psi(\rho, \theta)$ and costs zero energy.

Returning to Equation 41, one pair of coupled differential equations is given by

$$-\frac{1}{2}\nabla^2 f_m(\rho, \theta) + 2u\psi^2 f_m(\rho, \theta) + u\psi^2 e^{2i\theta} g_m^*(\rho, \theta) = f_m(\rho, \theta) \left[d_{m,1} - \frac{r}{2} \right] \quad (42)$$

$$-\frac{1}{2}\nabla^2 g_m^*(\rho, \theta) + 2u\psi^2 g_m^*(\rho, \theta) + u\psi^2 e^{-2i\theta} f_m(\rho, \theta) = -g_m^*(\rho, \theta) \left[d_{m,1} + \frac{r}{2} \right]. \quad (43)$$

The second set of coupled differential equations is almost identical to the first, so nothing more can be learned by expanding them as well. I will assume that f_m and g_m^* are separable into functions of ρ and θ so that

$$f_m(\rho, \theta) = \beta(\rho)\xi(\theta) \quad \text{and} \quad g_m^*(\rho, \theta) = \alpha(\rho)\eta(\theta).$$

I will also assume that

$$\xi(\theta) = \tilde{\xi}_0 e^{\pm i\ell\theta} \quad \text{and} \quad \eta(\theta) = \tilde{\eta}_0 e^{\pm in\theta}.$$

because this is the angular dependence far away from a vortex. There is no reason for the phase dependence to change suddenly as the distance from a vortex changes. The θ dependence of each term in Equations 42 and 43 must be identical. Equating the phase dependence of each term, I find that

$$\ell = n + 2$$

or that the angular momentum of $f_m(\rho, \theta)$ is larger than the angular momentum for $g_m^*(\rho, \theta)$ by two units.

I can now simplify the differential equations for f_m and g_m^* to include only their

radial dependence. The resulting differential equations take the form

$$\frac{1}{2} \left(\frac{d^2 \beta}{d\rho^2} + \frac{1}{\rho} \frac{d\beta}{d\rho} \right) - \frac{1}{2\rho^2} \beta(\rho) - u\psi^2 \alpha(\rho) = \beta(\rho) \left(\frac{r}{2} + 2u\psi^2 - d_{m,1} \right) \quad (44)$$

$$\frac{1}{2} \left(\frac{d^2 \alpha}{d\rho^2} + \frac{1}{\rho} \frac{d\alpha}{d\rho} \right) - \frac{1}{2\rho^2} \alpha(\rho) - u\psi^2 \beta(\rho) = \alpha(\rho) \left(\frac{r}{2} + 2u\psi^2 + d_{m,1} \right). \quad (45)$$

In order to determine the types of states that are present in and around a vortex, it is necessary to solve these differential equations. I expect that there will be a zero energy solution which corresponds to the phase of $\Psi(\rho, \theta)$ changing value. I also expect to find the existence of low energy scattering solutions that become sound waves as they move away from a vortex and into the bulk of the superfluid. These excited states have not yet been worked out, but I am continuing work on this problem to derive the excited states associated with small oscillations in a superfluid vortex.

5 Conclusions

In this paper, I have succeeded in learning a great deal about the structure of superfluids. First I calculated the minimum energy of an ordinary superfluid state to be when $\Psi(\rho, \theta) = \sqrt{\frac{|r|}{2u}} e^{i\theta}$. For a vortex excitation, the minimum energy configuration is given by the vortex profile in Figure 2. I then considered a superfluid where pairs of bosons are treated separately from single bosons and derived the phase diagram for when each boson is condensed or uncondensed as a function of r_1 and r_2 . I found that for $r_1 \geq 2g\sqrt{\frac{|r_2|}{2u}} - \frac{|r_2|}{2}$ the superfluid phase is one where the double bosons are condensed, but the single bosons remain uncondensed. I then investigated excited states produced by small quantum mechanical oscillations about the classical minima. In the uniform case, low energy excitations produce sound waves in the superfluid. For condensed single boson and condensed double bosons, sound waves can always exist in the superfluid. To produce excitations in the uncondensed single boson while the double boson is condensed, there must be a minimum energy added to the system. Without this minimum energy, it is not possible for a single boson to exist in the bulk of the double boson superfluid.

Superfluid excitations in the presence of a vortex are less well understood. It is expected that there will be zero energy excited states that change the phase of

the order parameter and low energy excited states that correspond to the sound waves scattering off the vortex. I am continuing work in this area in order to arrive at a more definitive answer. In the exotic superfluid, the double boson will behave in the same manner as the single condensed boson. The more intriguing question is what happens to the single uncondensed boson. In the bulk of the superfluid, there is an energy gap that prevents the presence of single bosons below a certain energy. However, in the presence of a vortex it may be possible to have a single boson condensate or single bosons bound to the double boson vortex.

To extend this work even further, there are many directions to follow. The superfluid configuration with condensed double boson and uncondensed single bosons represents the more interesting problem. One set of topics involve interactions with this vortex with other particles. Because the single boson is not condensed, it costs energy to bring an additional single boson into the system. Is it possible to bind this single boson to the vortex? What are the statistics that result from this bound state? In the case of a superconductor, the boson carries a charge of $2e$ and the vortex carries a magnetic flux. What are the implications of having charge bound to magnetic flux? Answers to all of these questions will help decide the physics governing superfluids and possibly similar fractional charge, flux, or spin systems.

Acknowledgments

I would like to thank my thesis advisor Professor Senthil Todadri for his invaluable input and patience in helping me work through these problems.

References

- [1] Anderson, P. W., “Coherent matter field phenomena in superfluids”, *Basic Notions of Condensed Matter Physics*, Addison-Wesley, Massachusetts [1997]
- [2] Bardeen, John, R. Kümmel, A. Jacobs, L. Tewordt, “Structure of Vortex Lines in Pure Superconductors”, *Physical Review*, Vol 187, pg 556 [1969]
- [3] Feynman Lectures on Physics Volume III, Chapter 21 “A Seminar on Superconductivity” [1965]
- [4] De Gennes, P.G., *Superconductivity of Metals and Alloys*, Trans P.A. Pincus, Chapters 5-6, W.A. Benjamin, INC, New York [1966]
- [5] Griffith, D.J., *Introduction to Quantum Mechanics*, Prentice Hall, New Jersey, [1995]
- [6] Kittel, C., *Introduction to Solid State Physics*, Chapter 12, Wiley, New York. 5th Ed. [1976]
- [7] Pathria, R.K., *Statistical Mechanics*, Chapter 11, Pergamon Pres, New York [1972]
- [8] Radzihovsky, Leo, Jae Park, Peter Weichman, “Superfluid Transitions in Bosonic Atom-Molecule Mixtures near Feshbach Resonance”, *Cond-mat preprint*, [Dec, 2003]
- [9] Conversations with Professor Senthil Todadri.