

# FINDING MAXIMAL BENEFIT/MAXIMAL CARDINALITY ASSIGNMENTS

by

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## Abstract

Given an assignment problem where we want to match persons to objects, and a benefit for matching any given person-object pair, we consider the problem of finding a set of pairs that has maximal cardinality with the further property that it offers maximal total benefit over all sets of pairs of maximal cardinality. We show that this problem naturally decomposes into three feasible assignment problems, two asymmetric and one symmetric. We also discuss how this decomposition property can be used algorithmically.

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## 1. INTRODUCTION

We consider a bipartite matching problem where we have  $m$  persons and  $n$  objects that we want to match on a one-to-one basis. There is a given benefit  $a_{ij}$  for matching person  $i$  with object  $j$  and we want to assign persons to objects so as to maximize the total benefit. We are given a set  $\mathcal{A}$  of pairs  $(i, j)$  that can be matched. By an *assignment* we mean a set  $S$  of person-object pairs  $(i, j) \in \mathcal{A}$  such that each person  $i$  and each object  $j$  is involved in at most one pair from  $S$ . The number of pairs in  $S$  is referred to as the *cardinality* of  $S$ . If the cardinality of  $S$  is  $\min\{m, n\}$ , so that either every person is assigned or every object is assigned, we say that  $S$  is *feasible*; otherwise  $S$  is said to be *infeasible*. If a feasible assignment exists the problem is said to be feasible, and otherwise it is said to be infeasible. The *benefit* of an assignment  $S$  is the sum of the benefits of its pairs. In the classical assignment problem we want to find a feasible assignment that has maximal benefit over all feasible assignments.

Many assignment problems of interest in practice are not known a priori to be feasible. Thus an algorithm to solve the problem must also include a mechanism to detect infeasibility. Once the problem is found to be infeasible, one may then be interested in obtaining a maximal benefit assignment within the set of assignments with maximal cardinality. This problem, which we call *maximal benefit/maximal cardinality problem*, may be viewed as an extension of the classical assignment problem and is the focal point of this paper.

The main contribution of this paper is to provide a characterization of the set of maximal cardinality assignments, denoted  $\mathcal{C}$ , and the set of maximal benefit/maximal cardinality assignments, denoted  $\mathcal{C}^*$ . In particular, we show that if the original assignment problem is infeasible, then it can be decomposed into three feasible assignment problems. These problems are obtained by dividing (in a particular way to be discussed in the next section) the set of persons  $\{1, \dots, m\}$  in three disjoint sets  $I_1$ ,  $I_2$ , and  $I_3$ , and by dividing the set of objects  $\{1, \dots, n\}$  in three disjoint sets  $J_1$ ,  $J_2$ , and  $J_3$ . The  $k$ th component problem ( $k = 1, 2, 3$ ), is the assignment problem with set of persons  $I_k$ , set of objects  $J_k$ , and set of pairs that can be matched

$$\mathcal{A}_k = \{(i, j) \in \mathcal{A} \mid i \in I_k, j \in J_k\}.$$

(It is possible that  $I_3$  and  $J_3$  are empty, in which case the third component problem is trivial.) The decomposition can be easily effected once a maximum cardinality assignment is obtained.

Our characterizations are given in terms of the three component problems as follows: Let  $\mathcal{C}_k$  be the set of feasible assignments of the  $k$ th component problem. We show that

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3,$$

in the sense that the pairs of each maximal cardinality assignment  $S \in \mathcal{C}$  can be obtained as the union of the pairs of three feasible assignments  $S_k \in \mathcal{C}_k$ ,  $k = 1, 2, 3$ . Furthermore, we show that the set  $\mathcal{C}^*$  of maximal benefit/maximal cardinality assignments of the original problem is

$$\mathcal{C}^* = \mathcal{C}_1^* \times \mathcal{C}_2^* \times \mathcal{C}_3^*,$$

where  $\mathcal{C}_k^*$  is the set of optimal (maximal benefit/feasible) assignments of the  $k$ th component problem.

The characterization just stated can be used to solve the maximal benefit/maximal cardinality problem in a number of ways. One possibility is to first decompose it into the three component problems using a maximal cardinality matching algorithm and then to find optimal assignments of these problems using any of the known algorithms. This approach has worst-case complexity of optimal order, but is practically cumbersome and inefficient. For example, if the original problem turned out to be feasible, the maximum cardinality matching phase would be entirely wasted.

A more efficient approach would be to suitably modify an algorithm for finding an optimal assignment of a feasible problem, so that when the original problem turns out to be feasible, the algorithm works like its unmodified version, and when the original problem turns out to be infeasible, the algorithm implicitly or explicitly decomposes the problem and solves the three component problems. In Section 3, we discuss an appropriate modified version of the auction algorithm with  $\epsilon$ -scaling that requires a small amount of additional calculation to detect infeasibility in the first scaling phase. If the problem turns out to be infeasible, the modified algorithm constructs the three component problems, and uses additional scaling phases to solve these problems independently of each other.

We finally mention a “brute force” algorithmic possibility to solve the maximal benefit/maximal cardinality problem without using the theory of this paper. In this approach we convert the problem into one that is guaranteed to be feasible by adding an artificial object  $j_i$  for each person  $i$  and a corresponding artificial arc  $(i, j_i)$ . Then, if the benefits  $a_{ij_i}$  are chosen sufficiently small, thereby providing an inducement for using as few artificial objects as possible in an assignment, an optimal solution of the resulting problem yields a maximal cardinality/maximum benefit assignment; for this, it is sufficient to choose

$$a_{ij_i} = -\min\{m, n\} \left( \max_{(i,j) \in \mathcal{A}} a_{ij} - \min_{(i,j) \in \mathcal{A}} a_{ij} \right)$$

for each artificial arc  $(i, j_i)$  – see [Ber92]. The advantage of this approach is that existing algorithms can be used to solve the modified problem. There is a price, however. In addition to an increase in the number of objects by  $m$ , the approach of using low benefit artificial person-object pairs does not work well with the auction algorithm because it enlarges substantially the benefit range of the problem, thereby requiring additional scaling phases and possibly necessitating the use of floating point arithmetic. The modified version of the auction algorithm discussed in Section 3 does not suffer from this difficulty.

## 2. DECOMPOSITION OF AN INFEASIBLE PROBLEM

For each person  $i$ , we denote by  $A(i)$  the set of objects that can be matched with  $i$

$$A(i) = \{j \mid (i, j) \in \mathcal{A}\},$$

and for each object  $j$ , we denote by  $B(j)$  the set of objects that can be matched with  $j$

$$B(j) = \{i \mid (i, j) \in \mathcal{A}\}.$$

To avoid degenerate (and uninteresting) situations, we will assume that  $A(i)$  is nonempty for all  $i$  and  $B(j)$  is nonempty for all  $j$ . For any set  $T$ , the cardinality of  $T$  will be denoted by  $\text{card}(T)$ .

Suppose we are given an assignment  $S$ . An *augmenting path* with respect to  $S$  is a sequence of the form

$$(i, j_1, i_1, \dots, j_m, i_m, j)$$

where  $i$  is an unassigned person,  $j$  is an unassigned object,  $j_1 \in A(i)$ ,  $j \in A(i_m)$ ,  $j_{k+1} \in A(i_k)$  for  $k = 1, \dots, m-1$ , and  $j_k$  is assigned to  $i_k$  under  $S$  for  $k = 1, \dots, m$ . It is well-known that  $S$  has maximal cardinality if and only if there is no augmenting path with respect to  $S$ .

For a person  $i$  and an object  $j$ , we define an *alternating path (with respect to  $S$ ) starting at  $i$  and ending at  $j$*  to be a sequence of the form

$$(i, j_1, i_1, \dots, j_m, i_m, j)$$

where  $j_1 \in A(i)$ ,  $j \in A(i_m)$ ,  $j_{k+1} \in A(i_k)$  for  $k = 1, \dots, m-1$ , and  $j_k$  is assigned to  $i_k$  under  $S$  for  $k = 1, \dots, m$ . Note that the terminal object  $j$  could be either unassigned or assigned, and that for any pair  $(i, j) \in \mathcal{A}$ , the path  $(i, j)$  is viewed as an alternating path starting at  $i$  and ending at  $j$ .

For an object  $j$  and a person  $i$ , we define an *alternating path (with respect to  $S$ ) starting at  $j$  and ending at  $i$*  to be a sequence of the form

$$(j, i_1, j_1, \dots, i_m, j_m, i)$$

where  $i_1 \in B(j)$ ,  $i \in B(j_m)$ ,  $i_{k+1} \in B(j_k)$  for  $k = 1, \dots, m-1$ , and  $j_k$  is assigned to  $i_k$  under  $S$  for  $k = 1, \dots, m$ . Note that the terminal person  $i$  could be either unassigned or assigned, and that for any pair  $(i, j) \in \mathcal{A}$ , the path  $(j, i)$  is viewed as an alternating path starting at  $j$  and ending at  $i$ .

Suppose that the original assignment problem is infeasible and that  $S$  is a maximal cardinality assignment. We define the subset of persons  $I_1$  and the subset of objects  $J_1$  as follows:

$$I_1 = \{i \mid \text{there is an alternating path starting at an unassigned object } j \text{ and ending at } i\}, \quad (1)$$

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$$J_1 = \{j \mid j \text{ is unassigned or is assigned to some } i \in I_1\}. \quad (2)$$

Similarly, we define the subsets

$$J_2 = \{j \mid \text{there is an alternating path starting at an unassigned person } i \text{ and ending at } j\}, \quad (3)$$

$$I_2 = \{i \mid i \text{ is unassigned or is assigned to some } j \in J_2\}. \quad (4)$$

Finally, we define

$$I_3 = \{i \mid i \in \{1, \dots, m\}, i \notin I_1, i \notin I_2\}, \quad (5)$$

$$J_3 = \{j \mid j \in \{1, \dots, n\}, j \notin J_1, j \notin J_2\}. \quad (6)$$

The following proposition establishes some basic facts:

**Proposition 1:** Assume that the original assignment problem is infeasible and that  $S$  is a maximal cardinality assignment. Then the following hold for the sets  $I_1, J_1, I_2, J_2, I_3,$  and  $J_3$  corresponding to  $S$  as per Eqs. (1)-(6):

- (a)  $I_1, J_1, I_2,$  and  $J_2$  are all nonempty.
- (b) All persons in  $I_1$  are assigned to objects in  $J_1$  and all objects in  $J_2$  are assigned to persons in  $I_2$ .
- (c) There is no pair  $(i, j) \in \mathcal{A}$  such that  $i \notin I_1$  and  $j \in J_1$ , or  $i \in I_2$  and  $j \notin J_2$ .
- (d) If  $I_3$  and  $J_3$  are nonempty, then all persons in  $I_3$  are assigned to objects from  $J_3$  and all objects in  $J_3$  are assigned to persons in  $I_3$ .
- (e) The sets  $I_1, I_2,$  and  $I_3$  are disjoint, and the sets  $J_1, J_2,$  and  $J_3$  are disjoint.

**Proof:** (a)  $J_1$  is nonempty because it contains all unassigned objects, and there is at least one unassigned object since  $S$  is infeasible. Also  $I_1$  is nonempty since otherwise we would have that  $B(j)$  is empty for all unassigned objects  $j$ , which has been excluded by assumption. Similarly,  $I_2$  and  $J_2$  are shown to be nonempty.

(b) By construction, for every person  $i \in I_1$ , there is an unassigned object  $j \in J_1$  and an alternating path starting at  $j$  and ending at  $i$ . If  $i$  were unassigned, this path would be an augmenting path with respect to  $S$ , which is impossible since  $S$  has maximal cardinality. Thus  $i$  is assigned and by the definition of  $J_1$ , it must be assigned to some object in  $J_1$ . The proof that all objects in  $J_2$  are assigned to persons in  $I_2$  is similar.

(c) We argue by contradiction. If  $(i, j) \in \mathcal{A}$ ,  $i \notin I_1$ , and  $j \in J_1$ , then  $j$  cannot be unassigned since then the pair  $(i, j)$  would constitute an alternating path starting at  $j$  and ending at  $i$ , which contradicts the fact that  $i \notin I_1$ . If  $j$  is assigned, say to person  $\bar{i}$ , then by part (b),  $\bar{i} \in I_1$ , and there

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must exist an alternating path starting at some unassigned object  $\bar{j}$  and ending at  $\bar{i}$ . This path, when extended by the pairs  $(\bar{i}, j)$  and  $(i, j)$  forms an alternating path starting at  $\bar{j}$  and ending at  $i$ , contradicting the fact that  $i \notin I_1$ . The proof that there is no pair  $(i, j) \in \mathcal{A}$  with  $i \in I_2$ , and  $j \notin J_2$  is similar.

(d) By construction, all unassigned objects belong to  $J_1$ , and all unassigned persons belong to  $I_2$ . Therefore, since  $J_3 \cup J_1 = \emptyset$ , all objects in  $J_3$  are assigned, and since  $I_3 \cup I_2 = \emptyset$ , all persons in  $I_3$  are assigned. We also have by part (b) that all persons in  $I_1$  are assigned to objects in  $J_1$ , and by construction, all persons in  $I_2$  that are assigned are assigned to persons in  $J_2$ . Thus the only possibility is that the objects in  $J_3$  are assigned to persons in  $I_3$ . A similar argument shows that all persons in  $I_3$  are assigned to objects in  $J_3$ .

(e) We first note that  $I_3$  is disjoint from  $I_1$  and  $I_2$  by construction. To show that  $I_1$  and  $I_2$  are disjoint, we argue by contradiction. Let  $i \in I_1 \cup I_2$ . Then by part (b),  $i$  must be assigned to some object  $j$  that belongs to  $J_1 \cup J_2$ . Since  $i \in I_1$ , there must exist an alternating path starting at an unassigned object  $\bar{j}$  and ending at  $i$ . Similarly, there must exist an alternating path starting at some unassigned person  $\bar{i}$  and ending at  $j$ . By joining these paths, we obtain an augmenting path with respect to  $S$  that starts at  $\bar{j}$  and ends at  $\bar{i}$ , which is a contradiction, since  $S$  has maximal cardinality. The proof that the sets  $J_1$ ,  $J_2$ , and  $J_3$  are disjoint is analogous. **Q.E.D.**

For  $k = 1, 2, 3$ , let us define

$$\mathcal{A}_k = \{(i, j) \in \mathcal{A} \mid i \in I_k, j \in J_k\}, \quad (7)$$

and we consider the assignment problem having  $I_k$ ,  $J_k$ , and  $\mathcal{A}_k$  as the sets of persons, objects, and possible pairs, respectively; we call this the  $k$ th *component problem*. Let

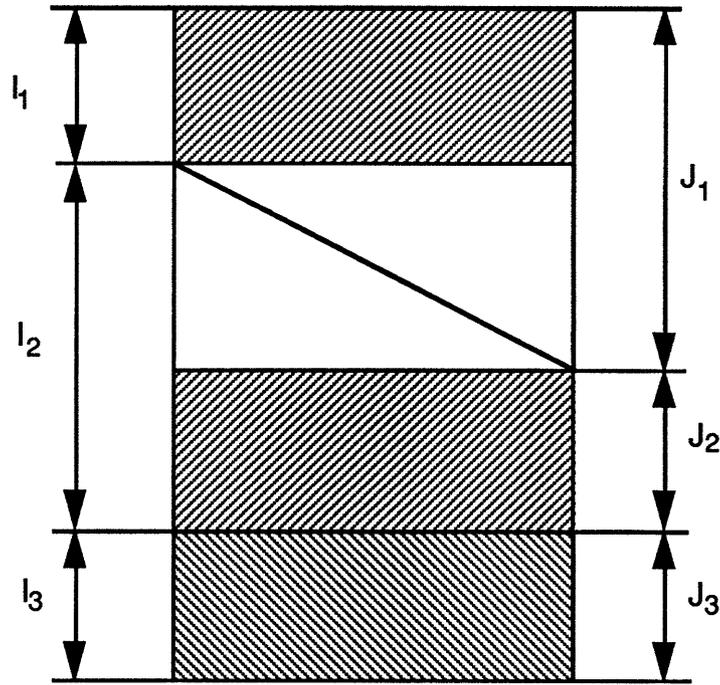
$$S_k = \{(i, j) \in S \mid (i, j) \in \mathcal{A}_k\}, \quad k = 1, 2, 3. \quad (8)$$

Note that *the 1st component problem has more objects than persons*, since all persons in  $I_1$  are assigned to objects in  $J_1$ , all unassigned objects belong to  $J_1$  by construction, and there exists at least one unassigned object since  $S$  is infeasible. Similarly, *the 2nd component problem has more persons than objects*. If  $I_3$  and  $J_3$  are nonempty, they must have the same cardinality, so in this case *the 3rd component problem has equal number of persons and objects*. However, it is also possible that  $I_3$  and  $J_3$  are empty. If this is the case, we say that the 3rd component problem is *trivial*; otherwise we say that this problem is *nontrivial*. Figure 1 illustrates the three component problems.

The following proposition is the main result of this section.

**Proposition 2:** Suppose that the original assignment problem is infeasible and that  $S$  is a maximal cardinality assignment. Consider the three component problems corresponding to  $S$ . Then:

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**Figure 1:** Schematic illustration of the three component problems. The shaded areas correspond to assigned pairs.

- (a) For  $k = 1, 2$ , the  $k$ th component problem is feasible and in fact  $S_k$  as given by Eq. (8) is a feasible assignment. The same is true for  $k = 3$  if the 3rd component problem is nontrivial.
- (b) If the 3rd component problem is nontrivial, then

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3,$$

where  $\mathcal{C}$  is the set of maximal cardinality assignments of the original problem, and  $\mathcal{C}_k$  is the set of feasible assignments of the  $k$ th component problem. Otherwise,

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2.$$

- (c) If the 3rd component problem is nontrivial, then

$$\mathcal{C}^* = \mathcal{C}_1^* \times \mathcal{C}_2^* \times \mathcal{C}_3^*,$$

where  $\mathcal{C}^*$  is the set of maximal benefit/maximal cardinality assignments of the original problem, and  $\mathcal{C}_k^*$  is the set of maximal benefit/feasible assignments of the  $k$ th component problem. Otherwise,

$$\mathcal{C}^* = \mathcal{C}_1^* \times \mathcal{C}_2^*.$$

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- (d) For  $k = 1, 2, 3$ , the  $k$ th component problem is independent of the maximal cardinality assignment  $S$ ; that is, for any other maximal cardinality assignment  $S'$ , the corresponding sets  $I'_1, J'_1, I'_2, J'_2, I'_3, J'_3$ , as per Eqs. (1)-(6) are the same as the sets  $I_1, J_1, I_2, J_2, I_3, J_3$  corresponding to  $S$ .

**Proof:** (a) Follows from parts (b) and (d) of Prop. 1.

(b) We will assume that the 3rd component problem is nontrivial (the proof is similar and simpler when this is not so). We have

$$\mathcal{C} \supset \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3,$$

since for  $k = 1, 2, 3$ , all assignments in  $\mathcal{C}_k$  have cardinality equal to  $\text{card}(S_k)$ , and  $\text{card}(S)$  is maximal and equal to  $\sum_{k=1}^3 \text{card}(S_k)$ .

Let  $S' \in \mathcal{C}$ . We will prove that

$$S' = S'_1 \cup S'_2 \cup S'_3,$$

where for  $k = 1, 2, 3$ ,  $S'_k$  is an assignment from  $\mathcal{C}_k$ . Let

$$S'_1 = \{(i, j) \in S' \mid j \in J_1\},$$

and

$$\overline{S}'_1 = \{(i, j) \in S' \mid (i, j) \notin S'_1\}.$$

We have

$$\text{card}(\overline{S}'_1) \leq \text{card}(J_2) + \text{card}(J_3), \tag{9}$$

while from Prop. 1(c), there is no pair  $(i, j) \in \mathcal{A}$  such that  $i \notin I_1$  and  $j \in J_1$ , implying that

$$\text{card}(S'_1) \leq \text{card}(I_1). \tag{10}$$

Since

$$\text{card}(S'_1) + \text{card}(\overline{S}'_1) = \text{card}(S') = \text{card}(S) = \text{card}(I_1) + \text{card}(J_2) + \text{card}(J_3),$$

it follows that equality holds in Eqs. (9) and (10), so that  $\text{card}(S'_1) = \text{card}(I_1)$ . Since there is no pair  $(i, j) \in \mathcal{A}$  such that  $i \notin I_1$  and  $j \in J_1$ , we see that all persons in  $I_1$  are assigned to objects in  $J_1$ , implying that  $S'_1 \in \mathcal{C}_1$ .

Let us also define

$$S'_2 = \{(i, j) \in S' \mid i \in I_2\},$$

$$S'_3 = \{(i, j) \in S' \mid (i, j) \notin S'_1 \cup S'_2\}.$$

### 3. A Modified Auction Algorithm

We can similarly prove that  $\text{card}(S'_2) = \text{card}(J_2)$  and that  $S'_2 \in \mathcal{C}_2$ . Since all persons in  $I_1$  are assigned to objects in  $J_1$  under  $S'$ , and no person in  $I_2$  can be assigned to an object not in  $J_2$  [by Prop. 1(c)], it follows that for all objects  $j$  such that  $(i, j) \in S'_3$  we have  $i \in I_3$ . Similarly, we can prove that for all persons  $i$  such that  $(i, j) \in S'_3$  we have  $j \in J_3$ , proving that  $S'_3 \in \mathcal{C}_3$ , and completing the proof.

(c) In view of the decomposition proved in part (b), each assignment  $S^* \in \mathcal{C}^*$  must be of the form

$$S^* = S_1^* \cup S_2^* \cup S_3^*,$$

where for  $k = 1, 2, 3$ ,  $S_k^*$  belongs to  $\mathcal{C}_k$ . Furthermore,  $S^*$  must offer maximal benefit over all assignments of  $\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ . For this to be true,  $S_k^*$  must be optimal for the  $k$ th component problem, that is, we must have  $S_k^* \in \mathcal{C}_k^*$  for  $k = 1, 2, 3$ .

(d) From part (b) we have that  $I_1 = I'_1$ , and from Prop. 1(b), there are no pairs  $(i, j)$  where  $i \notin I_1$  and  $j \in J_1$  or  $j \in J'_1$ . Therefore  $J_1 = J'_1$ . Similarly we obtain that  $I_2 = I'_2$  and  $J_2 = J'_2$ , which in view of the definition (5) and (6) of  $I_3$  and  $J_3$ , also implies that  $I_3 = I'_3$  and  $J_3 = J'_3$ . **Q.E.D.**

In view of Prop. 2(d), one can refer to the sets  $I_1, J_1, I_2, J_2, I_3$ , and  $J_3$  without making reference to a corresponding maximal cardinality assignment  $S$ .

### 3. A MODIFIED AUCTION ALGORITHM

We now briefly discuss how to use the theory of the preceding section to modify the auction algorithm so that it solves efficiently the maximum benefit/maximum cardinality assignment problem. Descriptions of various auction algorithms can be found in several sources, including the textbook [Ber91], and the tutorial papers [Ber90] and [Ber92]. For our purposes it is important to mention that there are auction algorithms specially tailored to the asymmetric assignment problem that are computationally very efficient and admit  $\epsilon$ -scaling. These algorithms were introduced in [BCT91] and are based on the idea of forward/reverse auction.

Briefly, in the forward/reverse version of the auction algorithm, we maintain an infeasible assignment  $S$  and a profit-price pair  $(\pi, p)$  satisfying for some  $\epsilon > 0$ , the following two conditions

$$\pi_i + p_j \geq a_{ij} - \epsilon, \quad \forall (i, j) \in \mathcal{A}, \quad (11)$$

$$\pi_i + p_j = a_{ij}, \quad \forall (i, j) \in S, \quad (12)$$

known as  $\epsilon$ -complementary slackness (or  $\epsilon$ -CS for short). A key fact is that for an asymmetric assignment problem with  $m \leq n$  and integer data, a feasible assignment  $S$  satisfying  $\epsilon$ -CS together

with a pair  $(\pi, p)$ , and such that

$$p_j \leq \min_{k: \text{assigned under } S} p_k, \quad \forall j : \text{unassigned under } S, \quad (13)$$

is within  $m\epsilon$  of being optimal; see [BCT91], [Ber91].

There are two types of iterations, *forward* and *reverse*. At each forward iteration, an unassigned person  $i$  is selected, who finds a “best” object  $j$  [one for which  $a_{ij} - p_j$  is maximal over all other objects in  $A(i)$ ], increases  $p_j$  by the maximum increment for which  $\epsilon$ -CS will not be violated, and gets assigned to  $j$ , displacing any person who may have been assigned to  $j$ .

Reverse iterations are really the same as forward iterations; the roles of persons and objects (and also profits and prices) are just reversed. In particular, in reverse auction unassigned objects compete for persons by lowering their prices, thereby increasing the profits of persons. In the algorithm of [BCT91] for strictly asymmetric problems ( $m < n$ ), which is also given together with a code implementation in [Ber91], reverse auction is used only after a feasible assignment is obtained, in order to reduce the prices of the unassigned objects so that the condition (13) is eventually satisfied. (Another forward/reverse algorithm given in [BeC92] mixes forward and reverse iterations more liberally, and it turns out that this is advantageous in some circumstances. This algorithm can also be modified along the lines given in this section to solve the maximal benefit/maximal cardinality problem, although we will not discuss the modification.)

Since termination in the auction algorithm can only occur with a feasible assignment, when the problem is infeasible, the auction algorithm will just keep on iterating. However, it can be seen that in this case, the auction algorithm will find an assignment of maximal cardinality in a finite number of iterations, provided that unassigned persons are selected for bidding in an order that is either cyclic, or else ensures that each person will get a chance to “bid” at least once within some fixed number of iterations. We do not prove this fact, which is intuitively quite clear given an understanding of the auction algorithm ideas; for a proof, see [Ber92].

We now propose the following modification of the auction algorithm, assuming that  $m \leq n$ . During the first scaling phase of the algorithm we periodically check whether there exists an augmenting path from some unassigned person to some unassigned object (a simple search of the breadth-first type can be used for this; see, e.g., [Ber91], p. 26). Once the cardinality of the current assignment becomes maximal but is less than  $m$ , this check will establish that the problem is infeasible. Hence, with this modification, the auction algorithm will either find a feasible assignment and a set of prices satisfying  $\epsilon$ -CS, or it will establish that the problem is infeasible and simultaneously obtain an assignment of maximal cardinality. In the former case, the algorithm will proceed with subsequent scaling phases of the algorithm, but with the breadth-first feature suppressed. In the latter case, we can use the maximal cardinality assignment obtained to decompose the problem into a feasible

symmetric problem and two feasible asymmetric problems as discussed in Section 2. We may then eliminate the unnecessary arcs  $(i, j)$  with  $i \in I_1$  and  $j \notin J_1$ , or  $j \in J_2$  and  $i \notin I_2$ , and proceed with a separate auction algorithm for each of the three component problems (starting with the profit-price pair already obtained).

Note some nice features of the approach just described: In the case of a feasible problem, it involves little additional computation (the breadth-first searches of the first scaling phase) over the unmodified algorithm. In the case of an infeasible problem, the computation of the first scaling phase is not wasted, as it provides good starting prices for the subsequent scaling phases. Furthermore, computational savings in the subsequent phases result from the elimination of some of the arcs. In addition, because the dimension of the three component problems are smaller than the dimension of the original problem, the threshold value of  $\epsilon$  to attain optimality is larger than  $1/m$  for integer data, thereby saving some scaling phases.

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