Quantum Field Theory of Scalar Cosmological Perturbations

by

Svetlin Valentinov Tassev

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Bachelor of Science in Physics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2005

Author ....
Department of Physics
May 16, 2005

Certified by ....
Edmund Bertschinger
Professor of Physics
Thesis Supervisor

Accepted by ....
David Pritchard
Senior Thesis Coordinator, Department of Physics

Quantum Field Theory of Scalar Cosmological Perturbations

by

Svetlin Valentinov Tassev

Submitted to the Department of Physics
on May 16, 2005, in partial fulfillment of the
requirements for the degree of
Bachelor of Science in Physics

Abstract

Using canonical quantization we show that the spectrum of the scalar cosmological fluctuations as calculated until now is not correct. We derive the correct expression for the spectrum, and show that our correct treatment alleviates the fine-tuning problem in inflation.

Thesis Supervisor: Edmund Bertschinger
Title: Professor of Physics

Acknowledgments

I want to thank Prof. Edmund Bertschinger for his guidance throughout this project.

Contents

Chapter 1

Introduction

One of the greatest successes of modern inflationary cosmology is to provide a natural resolution of some of the greatest problems of cosmology. However, observational constraints lead to the appearance of another problem - the fine tuning problem in inflation. In this Chapter we will introduce the reasons inflation was first proposed, and then introduce the fine tuning problem in inflation.

Big Bang cosmology is based on the fact that on scales larger than about 300Mpc the universe is homogeneous and isotropic. The first description of the expanding universe was presented in the framework of general relativity (GR), and the metric that describes the expansion is the Friedmann-Robertson-Walker (FRW, or RW) metric.

There are three main reasons why inflation was first developed: the horizon problem, the flatness problem, and the unwanted relics problem.

1.1 Problems With Pre-Inflationary Cosmology

1.1.1 Horizon Problem

The Cosmic Background Explorer (COBE) observations showed that the universe is indeed isotropic and homogeneous. However, the data showed that this is true on scales much larger than standard Big Bang cosmology could explain. COBE observed the Cosmic Microwave Background (CMB), coming from the surface of last scattering (sls). The sls corresponds to a redshift of about 1100, and an age of the universe of about 300,000 yrs. This is the time when electrons and protons recombined to form neutral atoms, which allowed for radiation to decouple from matter, and free-stream from the sls to the present. The observed temperature fluctuations in the CMB are $\delta T/T \approx 10^{-5}$. However, a simple calculation based on standard Big Bang cosmology shows that the sls consists of about 10^4 causally disconnected regions. This means that on angles larger than about $\theta \approx 1^\circ$, which is the approximate size for one causally connected region, the sls should show large temperature fluctuations, which contradicts the observations.

1.1.2 Flatness Problem

The density of the universe, combined with an equation of state for all particle species, describes completely the expansion history of the universe. One can define a critical density, $\rho_{\rm crit} \equiv 8\pi G\rho/3H^2$, where H is the Hubble parameter. If the universe has a density, ρ, greater than $\rho_{\rm crit}$, then it has a closed geometry (unbounded but finite); if $\rho < \rho_{\rm crit}$, then the geometry is open (unbounded and infinite); and $\rho = \rho_{\rm crit}$ is the marginal case, corresponding to a flat geometry (unbounded and infinite). The ratio $\Omega \equiv \rho/\rho_{\rm crit}$ is determined from observations (e.g. from the WMAP data of the CMB) to be equal to 1, within several percent. Such a coincidence is highly unlikely. To demonstrate this, if we extrapolate back in time, this corresponds to $|\Omega - 1| \approx 10^{-16}$ at the nucleosynthesis epoch. This fine tuning problem was finally resolved when the inflationary scenario was proposed.

1.1.3 Unwanted Relics Problem

As we go back in the history of the universe, the temperature rises dramatically. In the standard scenario, this leads to a high density of relics predicted by almost any theory dealing with high energies. Such relics can include magnetic monopoles, topological defects, gravitinos, and a myriad of others. However, observations exclude any significant amount of such relics. The only way to explain this is if we somehow rarify their density.

1.2 Overview of Inflation

Inflation, developed by A. Guth in late 1979 [4], postulates a period of exponential expansion of the universe lasting for a very short period of time which varies between 10^{-40}s and 10^{-30}s, depending on the model of inflation. In this way, regions that were once causally connected, appear to become disconnected after inflation. During this exponential expansion, the radius of curvature of the universe became so large that locally (within the observable universe at least) it seems flat to within a few percent. This, therefore, solves all of the above problems in one sweep, given that inflation lasted long enough so that the scale factor could increase by a factor of e^N, where $N \gtrsim 60$ to account for the observed CMB *isotropy*. New research in this area shows that there exists probably an upper bound on N, as well (e.g. [1]).

To get an exponential expansion using the field equations, one needs a cosmological constant, or equivalently a particle species or a field with equation of state $w \equiv \rho/p$, such that $-1 \leq w < -1/3$. This means that a "substance" with negative pressure has to be introduced. The easiest way to get that is by introducing a scalar field, $\phi(\underline{x}, t)$, dubbed the inflaton scalar field. Such a field is characterized by a potential $V(\phi(\underline{x}, t))$, and its equation of motion is entirely determined by the Lagrangian density for the field (2.1).

However, this does not answer the question, how such a field could arise in a classical description. Indeed, a quantum mechanical approach is needed to describe correctly this problem, since such a description would be able to account for the quantum fluctuations in the field (see next section). Quantum field theory (QFT) gives the necessary machinery to tackle the problem in finest detail. However, many authors refrain from using QFT, and rather try to use classical or semi-classical approximations (e.g. [3]). Although such approximations are sometimes perfectly valid for a homogeneous scalar field, and can give intuition of the evolution of the perturbations, there is no way that they can describe the perturbations to the field rigorously.

1.3 Inflaton Field Fluctuations and Cosmological Perturbations

The above discussion gives rise to the question: Why do we care about perturbations to the scalar field, when a homogeneous field is completely enough to solve all of the above quoted problems? The answer to this question is that quantum fluctuations in the field provided the seeds for the future structure formation in the universe. To see how this goes, we give an outline of the successive steps required to propagate the quantum fluctuations to cosmological perturbations.

1.3.1 Inflation

The inflaton field starts out with a large value of $V(\phi)$. The field fluctuates around its mean value due to quantum fluctuations, which qualitatively are due to the Heisenberg uncertainty principle. These fluctuations can be decomposed as a sum over modes with definite momentum. In the ground state, the fluctuations in each mode are then the same as those of the ground state of the harmonic oscillator in ordinary quantum mechanics. This means that there is an amplitude of the vacuum fluctuations, given by the variance. When plugged into the field equations, these inflaton field fluctuations are translated to metric perturbations. In such case, we can canonically quantize the perturbations to the gravitational potential by treating it as a field on a given background (Minkowski or RW in our case). Therefore, according to the Copenhagen interpretation, we can think of the perturbation to the gravitational potential as a distribution with zero mean, and some variance.

As the universe expands, the characteristic wavelength of each mode is redshifted in analogy with the cosmological redshift of photons. When the characteristic wavelength of a mode is within the Hubble horizon, the fluctuations oscillate around the mean. However, once the exponential expansion drives the mode outside the Hubble horizon, the mode stops evolving and the value of the potential effectively freezes at some value drawn from the distribution (see discussion in section 1.4).

1.3.2 Radiation Domination

During radiation and matter domination, the modes that were once superhorizon, reenter the Hubble horizon. If this happens during radiation domination, a lot of effects come into play connected with the physics of the coupled matter-radiation fluid. So, the amplitudes of the fluctuations evolve with time and the spectrum of the perturbations changes from the original inflaton fluctuations spectrum. However, the spectrum of the modes that were superhorizon at matter-radiation decoupling, stays practically unchanged and directly probes the inflationary era.

How do we measure the spectrum of the fluctuations? At the surface of last scattering, photons are emitted with a given frequency. However, since the gravitational potential varies from place to place, these photons are gravitationally blue- or red-shifted depending on the local value of the potential. Thus, there are temperature fluctuations in the CMB, which is exactly what COBE and WMAP measured: the CMB anisotropies.

1.3.3 Matter Domination

In the matter domination era, about 85% of the composition of the universe is Cold Dark Matter (CDM). The fluctuations in the potential translate into fluctuations in the density of the CDM. To gain more insight into how this interaction takes place, we should discuss the origin of CDM.

CDM has several established properties: it consists of elementary particles which interact only weakly (more weakly than neutrinos) and gravitationally. The model for CDM is that these relic particles decoupled very early in the expanding universe. Therefore, their momenta were severely redshifted, which yields practically zero thermal velocities, hence the name Cold.

The two major candidates for these particles are the Weakly Interacting Massive

Particles (WIMPs) and the axion. We consider the axion in our discussion, since it has a number of desirable similarities with the inflaton field, as we will see below. In short, matter domination can be thought of as axion domination. The axion is a Goldstone boson, arising from breaking the Peccei-Quinn symmetry, proposed to explain the strong CP problem in Quantum Chromodynamics (QCD). Since the momenta of the axions are redshifted to practically zero, they can be considered to be very well in the state of zero energy (apart from the fluctuations which we will discuss), thus comprising a Bose-Einstein condensate. In such case, individual particle wavefunctions overlap almost completely, and we can no longer separate individual particles. Thus, naturally, the Bose-Einstein condensate is described by a field, in this case - the axion field. The axion field has a potential which deviates from quadratic only at very high energies, and so we can consider it as just a simple quadratic potential. In such case the only difference between the inflaton and the axion field in our discussion will be the difference in the equation of state. The axions have (almost) zero thermal velocities, therefore, the axion field is pressureless as should be expected for matter. Thus, its equation of state is $w = 0$.

Let us go back to our discussion of the perturbations for superhorizon modes that reenter the Hubble horizon during the axion domination era. There is a relation connecting directly the scalar field and the gravitational potential. Once in the axion domination era, the gravitational potential operator is translated back to a scalar field operator. Thus, the modes that were once frozen, continue their evolution by following the equations of motion for the axion scalar field. So, before horizon crossing, the modes are described by the inflaton scalar field evolution, and after reentering they again follow scalar field evolution (with different w, corresponding to the axion field).

The pressureless axion field evolution leads to the formation of large-scale structure in the universe during the axion domination era. The transition from the quantum to the classical regime is explained in the next section.

1.4 Field Value Measurement Problem

In the outline above, there is one missing piece of physics. Namely: How do we draw a value for the gravitational potential from its distribution at the end of inflation (or beginning of radiation domination)? There is enormous amount of effort put into this quantum measurement problem. The universe should somehow "perform a measurement" of the amplitude of the gravitational field for each mode. This should happen either at horizon crossing or while the mode is a frozen superhorizon mode. According to the contemporary interpretation, a quantum measurement occurs in two stages. The quantum state is first "squeezed", and then a particular value is selected from the distribution of the observable in a process called decoherence [9].

The properties of a given quantum state can be investigated in the phase space of two non-commuting operator observables, such as position and momentum. The phase space trajectory of the state will be traced out by the expectation values of the operators in this state, while at the same time we can visualize the uncertainty in these observables as a fuzzy cloud around the expectation values. Due to the Heisenberg uncertainty principle, the area of this fuzzy cloud can be finite, but nonzero. If we choose to precisely determine the value of one of the observables, the cloud in phase space reduces to a line, meaning that the other observable will be completely undetermined. For coherent states, we choose to allow for some uncertainty in both observables, which is represented by a (roughly) spherical cloud in phase space. For squeezed states, the cloud becomes a long skinny ellipse. In the case of the state of the universe, this may correspond to squeezing the uncertainty in the gravitational potential. After allowing for decoherence to take place, the state effectively reduces to a classical state with a precise value of the gravitational potential.

The good news is that we have some intuition of how the first process comes about due to terms in the Hamiltonian describing the gravitational potential and the scalar field. So, after the state is squeezed, even without further knowledge of how decoherence takes place, we already know that a certain value of the gravitational potential is picked up with very small uncertainty. In one case it was shown [7] that the squeezing is so strong, that the variance, which during inflation falls off exponentially, is practically zero. Thus, as these authors call it, we can have "decoherence without decoherence", just by following the evolution of the squeezing of the vacuum state. However, even if in general this is not the case, decoherence probably takes place during reheating (the end of inflation, when photons, quarks, etc., are created in the universe), when the inflaton field interacts with the extremely large number of degrees of freedom of the particles just created.

1.5 Fine Tuning Problem in Inflationary Cosmology

Fine tuning problems in physics deal with the fact that a given quantity, not restricted by a known physical symmetry, or process, etc., should better have a value of 0, (order of magnitude) 1, or ∞ in natural units, in order to sound "natural" at first glance. However, small or large dimensionless numbers are unnatural and call for explanation. There is a proposition in physics that if a process is not forbidden by nature, then it must occur. Therefore, in the absence of a mechanism which can explain why a parameter is found (by measurements/observations) to be almost exactly equal to 0, then such a mechanism is definitely needed to complete the theory. This requires some new physics, as was the case with the invention of the Peccei-Quinn symmetry to solve the strong CP problem, for example. In the other extreme, when some calculation blows up, as is the case with the vacuum energy in QFT, then one should devise a method to renormalize the theory. In either case, if a parameter deviates largely from the above values, then something important is missing from the physical description (and as a result sometimes the anthropic principle is used until a good theoretical resolution is found).

The fine tuning problem in inflation is connected with the inflaton potential, and more specifically with the values of the parameters involved in the potential. There are two constraints to the inflaton potential:

1. It should be such that the scale factor increases with the needed $\gtrsim 60$ e-foldings during inflation (but probably should not exceed this value by too much, as discussed in section 1.2).

2. CMB observations give a small value for the amplitude of the gravitational field fluctuations during inflation, which in turn constrains the allowed parameter space.

For example, for the simplest case of a quadratic inflaton potential $V = m^2\phi^2$, the above restrictions require that

$$m \lesssim 10^{-5} m_{\mathrm{Pl}} \tag{1.1}$$

where m_{Pl} is the Planck mass.

1.6 Proposed Solution

There have been numerous suggestions for solving the fine tuning problem in inflation. These include topological defects, 5-dimensional assisted inflation, stochastic inflation, etc.

Our method is to check the calculations done so far. The fine tuning problem comes about in a treatment called the slow-roll approximation. Indeed this approximation allows for a straightforward treatment of inflation. The other approximation made in the classical inflationary scenario, is that in the canonical quantization of the scalar field, the creation and annihilation operators are treated as being constant operators.

Many people have calculated the spectrum of primordial fluctuations and get a result which is simply *wrong*, because it is based on wrong equations of motion. Our correct treatment proves to weaken (and in some cases, solve) the fine tuning problem, without reverting to more exotic theories than inflation itself.

In Chapter 2, we start our discussion in Minkowski spacetime. We introduce the concept of Fock space, and show how to perform canonical quantization in a continuous and discrete treatments. Then we work out two examples of QFT calculations, calculating the expectation values of the Hamiltonian and the canonical stress tensor in the vacuum state. By working out these examples we give the basics needed to handle the more complicated calculations in a RW background metric.

In Chapter 3, we investigate the quantization of fields in a RW background. We start by calculating the classical solutions to the spatially homogeneous equations. Then we introduce the second order perturbation Lagrangian, and the Hamilton equations, from which we solve for the equation of motion. After quantizing the field, we show how the gravitational potential can be quantized in a similar manner. Finally, we get the *correct* power spectrum of the inflationary perturbations.

In Chapter 4 we compare our results with the COBE results and we show that we weaken the fine tuning problem in inflationary cosmology. We work out several examples and show how our correct treatment of inflation alleviates the the fine-tuning problem for certain classical potentials.

In Chapter 5 we make our concluding remarks, and outline possible future continuation of this thesis.

Chapter 2

Minkowski Spacetime

Throughout the thesis we use natural units ($\hbar = c = G = 1$), unless otherwise noted.

2.1 Basics

There exist numerous possible choices for the inflaton potential which have been investigated in detail in the literature. However, since we want to keep things as simple as possible in this Chapter, we will use the quadratic potential, $V(\phi) = m^2\phi^2/2$.

The Lagrangian density for a scalar field, $\phi(\underline{x}, t)$, is

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) = \frac{1}{2}\left(\dot{\phi}^2 - |\underline{\nabla}\phi|^2 - m^2\phi^2\right) \tag{2.1}$$

Here $|\underline{\nabla}\phi|^2 \equiv \partial^i\phi\partial_i\phi = \delta^{ij}\partial_i\phi\partial_j\phi$ and the dot denotes time derivative. Note that for a scalar field the partial derivative, ∂_μ, equals the covariant derivative.

We use perturbation theory by substituting $\phi \rightarrow \phi_0(t) + \phi(\underline{x}, t)$, where $\phi_0(t)$ is the spatially homogeneous part of the field, and $\phi(\underline{x}, t)$ is the perturbation to the field. Substituting this into the Lagrangian we see that the zeroth and first order Lagrangians give identical equation of motion, namely $\ddot{\phi}_0 = -m^2\phi_0$. The dynamics of the perturbations is contained in the second order Lagrangian, which takes exactly the same form as above, with ϕ now being the perturbation to the field. From now on, ϕ will represent the perturbation to the field, and not the total field value, unless defined otherwise.

The conjugate momentum to the perturbation is then given by

$$\pi = \frac{\mathcal{L}}{\partial\phi_{,0}} = -\partial^0\phi = \partial_0\phi = \dot{\phi} \tag{2.2}$$

where $\phi_{,\mu} \equiv \partial\phi/\partial x^\mu$.

The Hamiltonian density is given by

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left(\pi^2 + |\underline{\nabla}\phi|^2 + m^2\phi^2\right) \tag{2.3}$$

The canonical stress tensor (ST) is given by

$$\Theta^\mu{}_\nu = \mathcal{L}\delta^\mu{}_\nu - \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\phi_{,\nu} \tag{2.4}$$

Fortunately, for the scalar field in Minkowski spacetime the canonical stress tensor is symmetric and is conserved, which means that we can use it as the energy-momentum tensor in Minkowski spacetime. For this case, it is given by

$$\Theta^\mu_\nu = -\frac{1}{2}\delta^\mu_\nu\left((\partial\phi)^2 + m^2\phi^2\right) + \partial^\mu\phi\partial_\nu\phi \tag{2.5}$$

where $(\partial\phi)^2 \equiv \partial^\mu\phi\partial_\mu\phi$.

The energy-momentum tensor (EMT) in curved spacetime is given by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}} \tag{2.6}$$

where S_M is the action of matter (or field, in our case); $g = \det g_{\mu\nu}$; and $\delta/\delta\chi$ is the functional derivative with respect to the field χ defined for one variable as

$$\frac{\delta F[f(x)]}{\delta f(y)} \equiv \lim_{\epsilon\to 0}\frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon} \tag{2.7}$$

The difference between the canonical ST and the EMT defined above is that in general the canonical ST is a pseudotensor, while the EMT is a tensor. A pseudotensor behaves like a tensor under linear coordinate transformations with constant coefficients, but is not transformed as a tensor under general coordinate transformations. As a consequence, in some cases $\partial_\mu \Theta^\mu_\nu = 0$, while $\partial_\mu T^\mu_\nu = -\Gamma^\mu_{\mu\alpha} T^\alpha_\nu + \Gamma^\alpha_{\mu\nu} T^\mu_\alpha \neq 0$.

2.2 Canonical Quantization. Continuous Treatment

The field $\phi(\underline{x}, t)$ acts on states that can be represented in Fock space as a linear combination of basis states, characterized with a given number of particles (occupation number), $n_{\underline{k}}$, for each value of the momentum vector, $\underline{k}$. The basis states in Fock space are $\left|n_{\underline{k}_1}, n_{\underline{k}_2}, n_{\underline{k}_3}, \cdots, n_{\underline{k}_m}, \cdots\right\rangle$, where the labels in the ket are infinite and uncountable, since the components of $\underline{k}_m$ can take any real value. The basis states can be expressed as a tensor product of the occupation state for each $\underline{k}$:

$$\left|n_{\underline{k}_1}, n_{\underline{k}_2}, n_{\underline{k}_3}, \cdots, n_{\underline{k}_m}, \cdots\right\rangle = \left|n_{\underline{k}_1}\right\rangle \otimes \left|n_{\underline{k}_2}\right\rangle \otimes \left|n_{\underline{k}_3}\right\rangle \otimes \cdots \otimes \left|n_{\underline{k}_m}\right\rangle \otimes \cdots \tag{2.8}$$

It is important to note that all states in the tensor product are normalized to one in both the continuous and in the discrete treatment. This means that the basis states themselves are also normalized to 1.

$$\left\langle n_{\underline{k}_m} \middle| n_{\underline{k}_m}\right\rangle = 1 \tag{2.9}$$

In quantum mechanics, the position and momentum of a simple harmonic oscillator can be decomposed as a sum of creation and annihilation operators. By analogy, we can introduce creation, $a^\dagger(\underline{k})$, and annihilation, $a(\underline{k})$, operators which when acting on a Fock state, increase or decrease the number of particles with momentum $\underline{k}$ by 1

$$a(\underline{k}) \left|\cdots, n_{\underline{k}}, \cdots\right\rangle = \sqrt{\delta^3(0)}\sqrt{n_{\underline{k}}} \left|\cdots, n_{\underline{k}} - 1, \cdots\right\rangle \tag{2.10}$$

$$a^\dagger(\underline{k}) \left|\cdots, n_{\underline{k}}, \cdots\right\rangle = \sqrt{\delta^3(0)}\sqrt{n_{\underline{k}} + 1} \left|\cdots, n_{\underline{k}} + 1, \cdots\right\rangle \tag{2.11}$$

The 0 in the delta function is actually $\underline{k} - \underline{k}$, which means that the delta function itself has units of k^{-3} (the inverse of its argument raised to the power D, where D is the number of dimensions of the Dirac delta function - 3 in this case.). The presence of $\delta^3(0)$ is a bit surprising, but it is easily explained in light of equation (2.17), from which it follows, that this is just the volume of space, which clearly diverges. This normalization is chosen so that the following commutation relations hold

$$[a(\underline{k}), a^\dagger(\underline{k}')] = \delta^3(\underline{k} - \underline{k}') \tag{2.12}$$

$$\left[a^\dagger(\underline{k}'), a^\dagger(\underline{k}')\right] = [a(\underline{k}), a(\underline{k}')] = 0 \tag{2.13}$$

We can introduce the number operator,

$$\begin{aligned} N(\underline{k}) \left|\cdots, n_{\underline{k}}, \cdots\right\rangle &\equiv a^\dagger(\underline{k}) a(\underline{k}) \left|\cdots, n_{\underline{k}}, \cdots\right\rangle \\ &= \delta^3(0) n_{\underline{k}} \left|\cdots, n_{\underline{k}}, \cdots\right\rangle \end{aligned} \tag{2.14}$$

which gives the number of particles with momentum $\underline{k}$.

Again by analogy with the simple harmonic oscillator, the scalar field $\phi(\underline{x}, t)$ and its conjugate momentum, $\pi(\underline{x}, t)$, can be decomposed as a series of modes in k-space, with angular frequency ω_k, associate with momentum $\underline{k}$. This expansion is called the canonical quantization, since we can obtain the classical equations of motion in terms of Poisson brackets after we make the canonical substitution $\{\} \longrightarrow -\imath[]$.

$$\phi(\underline{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left[e^{\imath(\underline{k}\cdot\underline{x} - \omega_k t)} a(\underline{k}) + \text{h.c.}\right] \tag{2.15}$$

$$\pi(\underline{x}, t) = \dot{\phi} = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} \left[-\imath e^{\imath(\underline{k}\cdot\underline{x} - \omega_k t)} a(\underline{k}) + \text{h.c.}\right] \tag{2.16}$$

where "+h.c." means "plus the hermitian conjugate of the rest of the terms in the brackets". By this construction, we see that the fact that $a^\dagger(\underline{k})$ is the Hermitian conjugate of $a(\underline{k})$ and not of some other operator, makes the measured value of the field and its conjugate momentum real numbers, since the operators are Hermitian.

Using the relation

$$\int \frac{d^3x}{(2\pi)^3} e^{\imath \underline{q}\cdot\underline{x}} = \delta^3(\underline{q}) \tag{2.17}$$

where $\delta^3(\underline{q})$ is the Dirac delta function, we can calculate the equal time Heisenberg commutation relation, imposed by causality

$$[\phi(\underline{x},t), \pi(\underline{y},t)] = \imath\delta^3(\underline{x}-\underline{y}) \tag{2.18}$$

With these expressions we can calculate the integrals over all space of π^2, ϕ^2 and $|\underline{\nabla}\phi|^2$ to find the Hamiltonian. As an example:

$$\int d^3x\phi^2 = \int d^3x \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} \frac{1}{2\sqrt{\omega_k\omega_{k'}}} \left[e^{\imath(\underline{k}\cdot\underline{x}-\omega_k t)}a(\underline{k}) + \text{h.c.}\right] \times \left[e^{\imath(\underline{k}'\cdot\underline{x}-\omega_{k'} t)}a(\underline{k}') + \text{h.c.}\right] \tag{2.19}$$

Using the relation (2.17) for the delta function we can perform first the integral on x in (2.19). We obtain

$$\int d^3x\phi^2 = \int \frac{d^3k}{2\omega_k}[e^{-2\imath\omega_k t}a(\underline{k})a(-\underline{k}) + e^{2\imath\omega_k t}a^\dagger(\underline{k})a^\dagger(-\underline{k}) + a^\dagger(\underline{k})a(\underline{k}) + a(\underline{k})a^\dagger(\underline{k})] \tag{2.20}$$

Calculating in the same way the rest of the terms, and using $\omega_k^2 = k^2 + m^2$, we get for the Hamiltonian

$$H = \int d^3x\mathcal{H} = \int d^3k\frac{\omega_k}{2}\left[a^\dagger(\underline{k})a(\underline{k}) + a(\underline{k})a^\dagger(\underline{k})\right] \tag{2.21}$$

When the contributions from all modes are summed up, we get a total energy which clearly diverges. However, one can measure (not counting GR effects, of course) only energy differences, so this zero-point energy should be subtracted away in any calculation (a procedure called regularization, renormalization or normal ordering, depending on the details of the particular calculation). One may think that such zero-point fluctuations of the vacuum are unobservable, however, there are situations in which such vacuum fluctuations give rise to macroscopical effects. Examples of such effects include the Casimir effect (example: two plane parallel conducting planes attract in vacuum), the Unruh effect (accelerated observers in vacuum detect particles with a distribution characteristic of a thermal bath), Hawking radiation (black holes emit radiation, due to vacuum fluctuations near the event horizon). In this thesis we will calculate one more such effect — inflationary fluctuations in the early universe.

2.3 Canonical Quantization. Discrete Treatment

Many calculations in QFT are done most easily by discretizing space and then substituting the integrals over x- and k-space by infinite sums over infinitesimal elements in x- and k-space. To do that we must first divide the x-space in a lattice with lattice spacing Δx. Furthermore, we will restrict ourselves in a cube with side, L, imposing periodic boundary conditions. Then the number of lattice points will be N^3, where $N = L/\Delta x$. At the end of the day, we will go back to the continuous limit, by letting $L \rightarrow \infty$ and $\Delta x \rightarrow 0$. By the discrete lattice representation of Fourier integrals, the periodic boundary conditions lead to discretization of k-space with lattice spacing, $\Delta k = 2\pi/L$. By analogy, the discretization of x-space imposes periodic boundary conditions in k-space on a box with side $K = 2\pi N/L$. We can assign a triplet of numbers to each x-space lattice point by $\underline{x} = \Delta x(n_1, n_2, n_3)$, $n_i = 0, 1, 2, ..., N-1$. The same can be done for the lattice points in k-space, $\underline{k} = \Delta k(m_1, m_2, m_3)$, where $m_i = -\frac{N}{2}, -\frac{N}{2}+1, -\frac{N}{2}+2, ..., \frac{N}{2}-1$. In this case the basis states are infinite (since we let $N \rightarrow \infty$) but countable.

Next we introduce creation and annihilation operators, $\tilde{a}^\dagger_{\underline{k}}$ and $\tilde{a}_{\underline{k}}$, for this discrete k-space, which will be connected with the continuous ones by a simple relation. We want the new $\tilde{a}^\dagger_{\underline{k}}$ and $\tilde{a}_{\underline{k}}$ to satisfy the standard relations

$$\tilde{a}_{\underline{k}}\,|\cdots,n_{\underline{k}},\cdots\rangle = \sqrt{n_{\underline{k}}}\,|\cdots,n_{\underline{k}}-1,\cdots\rangle \tag{2.22}$$

$$\tilde{a}^\dagger_{\underline{k}}\,|\cdots,n_{\underline{k}},\cdots\rangle = \sqrt{n_{\underline{k}}+1}\,|\cdots,n_{\underline{k}}+1,\cdots\rangle \tag{2.23}$$

$$[\tilde{a}_{\underline{k}},\tilde{a}^\dagger_{\underline{k}'}] = \delta_{\underline{k}\underline{k}'} \tag{2.24}$$

$$[\tilde{a}_{\underline{k}},\tilde{a}_{\underline{k}'}] = [\tilde{a}^\dagger_{\underline{k}},\tilde{a}^\dagger_{\underline{k}'}] = 0 \tag{2.25}$$

$$\tilde{N}_{\underline{k}}\,|\cdots,n_{\underline{k}},\cdots\rangle \equiv \tilde{a}^\dagger_{\underline{k}}\tilde{a}_{\underline{k}}\,|\cdots,n_{\underline{k}},\cdots\rangle = n_{\underline{k}}\,|\cdots,n_{\underline{k}},\cdots\rangle \tag{2.26}$$

where $\delta_{\underline{k}\underline{k}'}$ is the Kronecker delta. Note that despite appearances, each Fock state has finite number (N^3) of labels, since there are finite number of $\underline{k}$ values. Note also that the units of $\tilde{a}_{\underline{k}}$ and $a(\underline{k})$ are different. $\tilde{a}_{\underline{k}}$ and $\tilde{a}^\dagger_{\underline{k}}$ are both dimensionless, which follows directly from their definitions (2.22) and (2.23). However, the Dirac delta function has the units of the inverse of its argument, therefore $[a(\underline{k})] = [a^\dagger(\underline{k})] = k^{-3/2}$.

In the discrete case, the integral goes to a sum as $\int d^3k \longrightarrow (\Delta k)^3 \sum_{\underline{k}_j}$. This is actually a triple sum — one for each dimension in k-space. To be more concise from now on we will suppress the j in $\underline{k}_j$ appearing in the sum whenever this subscript labels different lattice points in k-space. So, now we can finally write

$$\begin{aligned}\phi_j(t) \equiv \phi(\underline{x}_j,t) &\approx \frac{(\Delta k)^3}{(2\pi)^{3/2}}\sum_{\underline{k}}\frac{1}{\sqrt{2\omega_k}}\left[e^{\imath(\underline{k}\cdot\underline{x}-\omega_k t)}a(\underline{k}) + \text{h.c.}\right] \\ &= A\sum_{\underline{k}}\frac{1}{\sqrt{2\omega_k}}\left[e^{\imath(\underline{k}\cdot\underline{x}-\omega_k t)}\tilde{a}_{\underline{k}} + \text{h.c.}\right]\end{aligned} \tag{2.27}$$

where A is some constant to be determined. In the discrete case π_j has a somewhat different meaning, since it is no longer the conjugate momentum to ϕ_j. It is just defined to be $\pi_j \equiv \dot{\phi}_j$ and how it relates to the actual conjugate momenta will be shown below. From (2.27) we have the relation

$$A\tilde{a}_{\underline{k}} = \frac{(\Delta k)^3}{(2\pi)^{3/2}}a(\underline{k}) \tag{2.28}$$

To find the normalization constant A, we need to write down the Lagrangian for the discrete case

$$L = (\Delta x)^3 \sum_{\underline{x}_j} \mathcal{L}_j \tag{2.29}$$

where

$$\mathcal{L}_j = \frac{1}{2}\left(\dot{\phi_j}^2 - |\underline{\nabla}\phi_j|^2 - m^2\phi_j^2\right) \tag{2.30}$$

From here we can find the discrete momenta p_j conjugate to the generalized coordinates ϕ_j. These conjugate momenta, p_j, are not the same as the conjugate momentum, π, to the field, ϕ:

$$p_j = \frac{\partial L}{\partial \dot{\phi}_j} = (\Delta x)^3 \dot{\phi}_j \equiv (\Delta x)^3 \pi_j \tag{2.31}$$

In the continuous limit, π_j becomes the conjugate momentum to the field, and p_j has no longer a meaning. From this equation and from the canonical quantization (replacing the Poisson bracket with $-\imath$ times the commutator), and the value of the Poisson bracket of ϕ_j and p_j, we require that

$$[\phi_j, p_{j'}] = \imath\delta_{jj'} \tag{2.32}$$

which means that

$$[\phi_j, \pi_{j'}] = \imath\delta_{jj'}/(\Delta x)^3 \tag{2.33}$$

Let us calculate this commutator using the expansion of ϕ_j and π_j.

$$[\phi_j, \pi_{j'}] = \frac{A^2}{2} \sum_{\underline{k},\underline{k}'} \left(\frac{\omega_{k'}}{\omega_k}\right)^{1/2} \Big\{ -\imath e^{\imath(\underline{k}\cdot\underline{x}_j + \underline{k}'\cdot\underline{x}_{j'} - \omega t - \omega' t)} [\tilde{a}_{\underline{k}}, \tilde{a}_{\underline{k}'}]$$
$$+\imath e^{-\imath(\underline{k}\cdot\underline{x}_j + \underline{k}'\cdot\underline{x}_{j'} - \omega t - \omega' t)} [\tilde{a}^\dagger_{\underline{k}}, \tilde{a}^\dagger_{\underline{k}'}]$$
$$+\imath e^{\imath(\underline{k}\cdot\underline{x}_j - \underline{k}'\cdot\underline{x}_{j'} - \omega t + \omega' t)} [\tilde{a}_{\underline{k}}, \tilde{a}^\dagger_{\underline{k}'}]$$
$$-\imath e^{-\imath(\underline{k}\cdot\underline{x}_j - \underline{k}'\cdot\underline{x}_{j'} - \omega t + \omega' t)} [\tilde{a}^\dagger_{\underline{k}}, \tilde{a}_{\underline{k}'}] \Big\} \tag{2.34}$$

Using the commutators of $\tilde{a}^\dagger_{\underline{k}}$ and $\tilde{a}_{\underline{k}'}$, we get

$$[\phi_j, \pi_{j'}] = \imath \frac{A^2}{2} \sum_{\underline{k}} [e^{\imath \underline{k}\cdot(\underline{x}_j - \underline{x}'_j)} + e^{-\imath \underline{k}\cdot(\underline{x}_j - \underline{x}'_j)}] = \imath A^2 N^3 \delta_{jj'} \tag{2.35}$$

We get the last equation using the expression for the Kronecker delta $\sum_{\underline{k}} e^{\imath \underline{k}\cdot(\underline{x}_j - \underline{x}'_j)} = N^3 \delta_{jj'}$, in which the 3 comes from the triple sum in the 3 dimensional k-space. Comparing (2.35) and (2.33), we conclude that

$$A = \frac{1}{L^{3/2}} \tag{2.36}$$

From here it follows that

$$\tilde{a}_{\underline{k}} = (\Delta k)^{3/2} a(\underline{k}) \tag{2.37}$$
$$\tilde{a}^\dagger_{\underline{k}} = (\Delta k)^{3/2} a^\dagger(\underline{k}) \tag{2.38}$$

which indeed makes $\tilde{a}_{\underline{k}}$ and $\tilde{a}^\dagger_{\underline{k}}$ dimensionless. As we can see $a^\dagger(\underline{k})$ and $a(\underline{k})$ get a factor of $(\Delta k)^{-3/2} = (L/2\pi)^{3/2}$ which in the continuous case becomes exactly $\sqrt{\delta^3(0)}$. Some authors use $L = 1$ (dimensionless) and then they do not get the factor of $\sqrt{\delta^3(0)}$ in $a^\dagger(\underline{k})$ and $a(\underline{k})$. However, we keep this factor, so that we can keep track of the dimensionality of the results which allows us to easily check the consistency of our calculations.

The Hamiltonian in this discrete case becomes

$$H = \sum_{\underline{x}_j} p_j \dot{\phi}_j - L = (\Delta x)^3 \sum_{\underline{x}_j} \frac{1}{2} \left(\pi_j^2 + |\underline{\nabla} \phi_j|^2 + m^2 \phi_j^2 \right) \tag{2.39}$$

Using the mode expansions for ϕ_j and π_j, we see that with the normalization constant that we found, all terms such as $\tilde{a}_{\underline{k}} \tilde{a}_{-\underline{k}}$ cancel out, leaving

$$H = \sum_{\underline{k}} \omega_k (\tilde{a}^\dagger_{\underline{k}} \tilde{a}_{\underline{k}} + \frac{1}{2}) \tag{2.40}$$

which is exactly the discrete representation of the Hamiltonian found in the continuous treatment.

2.4 Vacuum Expectation Value of the Canonical Stress Tensor

Let us denote the ground state in Fock space by $|0\rangle$, indicating that the occupation number is zero for all $\underline{k}$. This state is called the vacuum state since no particles are present. We want to calculate the vacuum expectation value (vev) of the canonical stress tensor using the discrete treatment.

$$\langle 0| \Theta_{\mu\nu}(\underline{x}) |0\rangle = \langle 0| \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \left((\partial \phi)^2 + m^2 \phi^2 \right) |0\rangle \tag{2.41}$$

Here the subscript showing that x, ϕ and π are discrete is dropped. So, we need to calculate $\langle 0| \partial_\mu \phi \partial_\nu \phi |0\rangle$ and $\langle 0| \phi^2 |0\rangle$, and we can use $\langle 0| (\partial \phi)^2 |0\rangle = \eta^{\mu\nu} \langle 0| \partial_\mu \phi \partial_\nu \phi |0\rangle$. To simplify the calculation, let's define the momentum four vector $k^\mu = (\omega_k, \underline{k})$, which means that $k_\mu = (-\omega_k, \underline{k})$. So, we can write $\underline{k} \cdot \underline{x} - \omega_k t = k_\mu x^\mu$ in the expressions for ϕ_j and π_j. Also, note that since we are acting on the ground state with the canonical stress tensor (ST), the only quadratic combination of operators that will give a non-zero element is $\tilde{a}_{\underline{k}}\tilde{a}^{\dagger}_{\underline{k}'} = \delta_{kk'}$. As an example

$$\begin{aligned} \langle 0|\, \partial_\mu\phi\partial_\nu\phi\, |0\rangle &= \tfrac{1}{L^3}\sum_{\underline{k}}\sum_{\underline{k}'}\tfrac{1}{2\sqrt{\omega_{\underline{k}}\omega_{\underline{k}'}}}k_\mu k'_\nu e^{\imath(k_\alpha - k'_\alpha)x^\alpha}\,\langle 0|\,\tilde{a}_{\underline{k}}\tilde{a}^{\dagger}_{\underline{k}'}\,|0\rangle & (2.42)\\ &= \tfrac{1}{L^3}\sum_{\underline{k}} k_\mu k_\nu \tfrac{1}{2\omega_k} & (2.43) \end{aligned}$$

The last line is obtained by using the Kronecker delta to eliminate one of the sums. By analogy, we obtain

$$m^2\,\langle 0|\,\phi^2\,|0\rangle = \frac{m^2}{L^3}\sum_{\underline{k}}\frac{1}{2\omega_k} \tag{2.44}$$

$$\langle 0|\,(\partial\phi)^2\,|0\rangle = \frac{-m^2}{L^3}\sum_{\underline{k}}\frac{1}{2\omega_k} \tag{2.45}$$

where we used $k^\mu k_\mu = -\omega_k^2 + \underline{k}^2 = -m^2$ which is Lorentz invariant. As we can see, the last two terms cancel in the canonical stress tensor, and so we finally get

$$\langle 0|\,\Theta_{\mu\nu}(\underline{x})\,|0\rangle = \frac{1}{L^3}\sum_{\underline{k}} k_\mu k_\nu \frac{1}{2\omega_k} \tag{2.46}$$

This means that the vacuum expectation value of the density is

$$\langle 0|\,\Theta_{00}(\underline{x})\,|0\rangle = \frac{1}{L^3}\sum_{\underline{k}}\frac{\omega_k}{2} \tag{2.47}$$

Taking the limit of $L \to \infty$, which means that $\Delta k \to 0$, we can go to the continuous case, obtaining

$$\langle 0|\,\Theta_{\mu\nu}(\underline{x})\,|0\rangle = \int \frac{d^3k}{(2\pi)^3} k_\mu k_\nu \frac{1}{2\omega_k} \tag{2.48}$$

$$\langle 0|\,\Theta_{00}(\underline{x})\,|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2} \tag{2.49}$$

If we stare at the equations a little longer, we will see that for a bit longer calculations, there is a shortcut. Let's introduce a notation: $x^\mu_{(i)}$, $\phi_{(i)}$, $\partial^{(i)}_\mu \equiv \partial/\partial x^\mu_{(i)}$. The index $i = 1, 2$ denotes different copies of x and ϕ as will be explained below.

Now, pretend that in each term in the ST which contains two ϕ's, the two ϕ's are two different fields, i.e. $\phi_{(1)}$ and $\phi_{(2)}$. We introduce this so that each derivative, $\partial_\mu^{(i)}$, in the ST, acts on its respective $\phi_{(i)}$. Therefore, the first term in the ST becomes $\langle 0| \partial_\mu^{(1)} \phi_{(1)} \partial_\nu^{(2)} \phi_{(2)} |0\rangle$. In this way we can pull out $\langle 0| \phi_{(1)}\phi_{(2)} |0\rangle$ after the canonical ST, and act with the ST on this expectation value. After we are done, we effectively set $(1) = (2)$, meaning $x^\mu_{(1)} = x^\mu_{(2)}$, $\phi_{(1)} = \phi_{(2)}$, and we get the same answer as if we did all expectation values separately. $\langle 0| \phi_{(1)}\phi_{(2)} |0\rangle$ is known as the propagator in quantum field theory.

Chapter 3

Scalar Field in Flat RW Spacetime

We will be dealing with flat RW spacetimes exclusively to the end of the thesis.

3.1 Spatially Homogeneous Classical Solution

From now on, t denotes conformal time, defined as $t = \int dt_{\text{proper}}/a(t_{\text{proper}})$. We can then linearize the RW metric with respect to the metric perturbations, Φ and Ψ, and obtain

$$ds^2 = a^2(t)\left[-dt^2(1+2\Phi) + (1-2\Psi)(dx^2+dy^2+dz^2)\right] \tag{3.1}$$

which is called the conformal Newtonian gauge [5]. In second order perturbation theory, the anisotropic stress for a scalar field vanishes, and therefore we have

$$\Phi = \Psi \tag{3.2}$$

and Φ in this case corresponds to the classical Newtonian gravitational potential.

In curved spacetime the Lagrangian density for a scalar field, $\phi(\underline{x}, t)$, is given by

$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{3.3}$$

This means that to zeroth order (expanding to first order gives the same equation of motion)

$$^{(0)}\mathcal{L} = a^2 \left[\frac{\dot{\phi}_0^2}{2} - a^2 V(\phi_0) \right] \tag{3.4}$$

Using the Lagrangian equation of motion we get the equation of motion for the spatially homogeneous part of the field

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V_{,\phi_0} = 0 \tag{3.5}$$

The EMT for a perfect fluid can be written as

$$T^\mu_\nu = (p+\rho)U^\mu U_\nu + p\delta^\mu_\nu \tag{3.6}$$

where $U^\mu = (a^{-1}, 0, 0, 0)$ for the uniformaly epxanding matter in a RW universe, satisfying the normalization condition $U^\mu U_\mu = -1$.

We want to express ρ_0 and p_0 in terms of the homogeneous part of the field, ϕ_0. However, we cannot use the canonical ST in this case, since it not only includes the energy and pressure of matter, but also the energy "stored" in the gravitational field. The canonical ST is used in GR mainly for calculating the energy carried away by gravitational fields. In this case, we want to use the EMT (2.6) which yields

$$T^\mu_\nu = g^\alpha_\mu \partial_\alpha \phi_0 \partial_\nu \phi_0 - \delta^\mu_\nu \left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi_0 \partial_\beta \phi_0 + V(\phi) \right] \tag{3.7}$$

We can equate (3.6) and (3.7) and we get

$$\rho_0 = \frac{1}{2a^2}\dot{\phi}_0^2 + V(\phi_0) \tag{3.8}$$

$$p_0 = \frac{1}{2a^2}\dot{\phi}_0^2 - V(\phi_0) \tag{3.9}$$

where ρ_0 and p_0 are the density and pressure associated with ϕ_0.

The Friedmann equation in conformal time (after restoring G) is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^2 \tag{3.10}$$

which gives the relation between $a(t)$ and $\phi_0(t)$.

If we assume an equation of state, instead of specifying $V(\phi_0)$ in advance, we can solve for the time evolution of the scalar field, ϕ_0. For the equation of state we will use

$$w = p/\rho \tag{3.11}$$

which takes values $w = -1$ during deSitter inflation, and $w = 0$ for a matter dominated universe. If we define $\lambda \equiv 2/(1+3w)$, then for $w = const$ the solution to the Friedmann equation and the equation of state gives [2] (after restoring G):

$$\phi_0(t) = \sqrt{\frac{\lambda(\lambda+1)}{4\pi G}}\log(t/t_0) \tag{3.12}$$

$$a(t) = (t/t_0)^\lambda \tag{3.13}$$

$$V(\phi_0) = \frac{\lambda(2\lambda-1)}{8\pi G t_0^2}e^{-\phi_0\varsigma} \tag{3.14}$$

$$\varsigma = \mathrm{sign}(\lambda)\sqrt{16\pi G(\lambda+1)/\lambda} \tag{3.15}$$

which is called power-law inflation (for $\lambda \leqslant -1$).

During inflation $\lambda \leq -1$ and $t_0 < 0$ since the solution assumes t going from negative values ($|t| \gg 1$) to zero, when the scale factor blows up as needed. Then, during radiation and matter domination t starts from zero and grows monotonically. So, we can consider the conformal time to be continuous from $-\infty$ to 0^- for inflation. Then, at the end of inflation, w changes quickly from -1 to $1/3$ as we enter the radiation domination era. During this transition w and is no longer constant, so the validity of the above solution breaks down. However, we can bootstrap the solutions for inflation to the solution for radiation and matter domination for which the conformal time starts from 0^+ and goes to t_{present}.

3.2 Perturbations of the Scalar Field, Treated as a Quantum Operator

3.2.1 The Lagrangian in Curved Spacetime

In order to quantize the gravitational potential, we need to obtain a Hamiltonian, and then perform canonical quantization. However, the Hamiltonian formulation separates the time and spatial components, and the equations are no longer manifestly covariant. However, the obtained equations of motion are identical to those obtained from the Lagrangian formulation in any coordinate system. For one particle, the choice of a time coordinate is easy - this would be the proper time. In our case, we choose our time variable to be the conformal time.

Let's make the following field definition [6]

$$\chi = a\left(\phi + \frac{a\dot{\phi}_0}{\dot{a}}\Phi\right) \tag{3.16}$$

where ϕ is the perturbation to the field, a is the scale factor, and Φ is the gravitational potential. χ is usually referred to as the Mukhanov variable.

The second order Lagrangian in curved spacetime is given by [6]

$$^{(2)}\mathcal{L} = \frac{1}{2}\left(\dot{\chi}^2 - \left|\underline{\nabla}\chi\right|^2 - \mu^2(t)\chi^2\right) \tag{3.17}$$

Here, $\mu^2(t) \equiv -\ddot{z}/z$, where $z \equiv a^2\dot{\phi}_0/\dot{a}$. Here, $z(t)$ is a definite function that depends only on $\phi_0(t)$, since $a(t)$ depends on $\phi_0(t)$ through the Friedmann equation (3.10).

The Hamiltonian [6] is

$$\mathcal{H} = \frac{1}{2}\left(\pi_\chi^2 + \left|\underline{\nabla}\chi\right|^2 + \mu^2(t)\chi^2\right) \tag{3.18}$$

where π_χ is the conjugate momentum of χ.

As you can see, this result, obtained by [6] is fascinating, since it reduces the number of fields from 11 (10 for the metric, and 1 scalar field) to 1. Moreover, there are no cross terms in the Hamiltonian between the field and its conjugate momentum. This can only be achieved in second order perturbation theory when gravity waves and vector perturbations are neglected.

The metric in this problem is that of flat RW with ${}^{(0)}g_{\mu\nu} = a^2\eta_{\mu\nu}$. Gravity and the scalar field are both incorporated in the new field χ, which "lives" on this flat RW background.

3.2.2 The Hamilton Equations

The Hamilton equations are given by

$$\dot{\pi}_\chi = \{\pi_\chi, H\} \tag{3.19}$$

$$\dot{\chi} = \{\chi, H\} \tag{3.20}$$

where $\{\}$ is the Poisson bracket defined as

$$\{A, B\} = \int d^3x \left[\frac{\delta A}{\delta \chi}\frac{\delta B}{\delta \pi_\chi} - \frac{\delta A}{\delta \pi_\chi}\frac{\delta B}{\delta \chi}\right] \tag{3.21}$$

Using Hamilton equations (3.20) and (3.19) we get

$$\dot{\pi}_\chi = -\mu^2(t)\chi + \nabla^2\chi \tag{3.22}$$

$$\dot{\chi} = \pi_\chi \tag{3.23}$$

Eliminating π_χ, we get the following PDE

$$\ddot{\chi} = -\mu^2(t)\chi + \nabla^2\chi \tag{3.24}$$

Taking the spatial Fourier Transform (FT), we can write this equation in k-space

$$\ddot{v}_k = -\left[k^2 + \mu^2(t)\right] v_k \tag{3.25}$$

where $v_k(t) = \mathrm{FT}(\chi(x,t))$.

Approximating $w \approx const$ allows us to use (3.12) and (3.13). We see that $z \propto a$, and thus, $\ddot{z}/z \approx \ddot{a}/a$ for $w \approx$const. This means that $\mu^2(t) = -\lambda(\lambda-1)/t^2$, which gives

$$\mu^2(t) = -\frac{2}{t^2} \tag{3.26}$$

for both matter dominated universe, $\lambda = 2$, and for deSitter inflation, $\lambda = -1$. This is surprising but we should remember that t is conformal time. Plugging in the equations of motion, for $\lambda = -1, 2$ we get that

$$v_+ = \left(\cos kt - \frac{\sin kt}{kt}\right) + \imath\left(\frac{\cos kt}{kt} + \sin kt\right) \xrightarrow{k|t|\gg 1} e^{\imath kt} \tag{3.27}$$

$$v_- = \left(\cos kt - \frac{\sin kt}{kt}\right) - \imath\left(\frac{\cos kt}{kt} + \sin kt\right) \xrightarrow{k|t|\gg 1} e^{-\imath kt} \tag{3.28}$$

Following [2], the choice of normalization will be evident when we calculate the commutator of χ and π_χ. This linear combination of the solutions is necessary in order to get for $k|t| \gg 1$ that $|v_-| = |v_+|$, and recover the Minkowski solutions.

If the Wronskian of two functions is non-zero at a given point, then the two functions are linearly independent at that point. For a second order ODE with a coefficient $P(t)$ multiplying $\dot{v}$, we know that $W'(t)/W(t) = -P(t)$. From here it follows, that the Wronskian is constant for all times, since $P(t) = 0$ in (3.25). The Wronskian equals

$$W(t) = \dot{v}_+ v_- - v_+ \dot{v}_- = 2\imath k \tag{3.29}$$

3.2.3 Canonical Quantization

For arbitrary w, the field χ, and its conjugate momentum π_χ, can be expanded in terms of creation and annihilation operators as

$$\chi(\underline{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k}} \left[e^{\imath \underline{k}\cdot\underline{x}} \upsilon_-(k,t) a(\underline{k},t) + \text{h.c.} \right] \tag{3.30}$$

$$\pi_\chi(\underline{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k}} \left[e^{\imath \underline{k}\cdot\underline{x}} \dot{\upsilon}_-(k,t) a(\underline{k},t) + \text{h.c.} \right] \tag{3.31}$$

since from Hamilton's equations (3.20) we have $\dot{\upsilon}_k = FT(\pi_\chi)$. Our υ_k differs from Mukhanov's [6] by a factor of $\sqrt{k}$. It is important to emphasize that the creation and annihilation operators obey the commutation relations of Minkowski spacetime (2.12) and (2.13). This decomposition indeed yields the commutator (2.18), when we use the value of the Wronskian (3.29).

Integrating the Hamiltonian density (3.18) with respect to spatial variables, we find that the Hamiltonian equals

$$H = \frac{1}{2} \int \frac{d^3k}{2k} \left[f(k,t) a^\dagger_{\underline{k}}(t) a^\dagger_{-\underline{k}}(t) + g(k,t) a_{\underline{k}}(t) a^\dagger_{\underline{k}}(t) + \text{h.c.} \right] \tag{3.32}$$

$$f(k,t) \equiv \dot{\upsilon}_+ \dot{\upsilon}_+ + (k^2 + \mu^2(t)) \upsilon_+ \upsilon_+ \tag{3.33}$$

$$g(k,t) \equiv \dot{\upsilon}_- \dot{\upsilon}_+ + (k^2 + \mu^2(t)) \upsilon_- \upsilon_+ \tag{3.34}$$

From here it is easy to see that, in general, the Hamiltonian does not commute with itself at different times. For $w = -1, 0$, we get

$$f(k,t) = \frac{1 - 2\imath kt}{k^2 t^4} e^{2\imath kt} \tag{3.35}$$

$$g(k,t) = 2k^2 - \frac{1 + 2k^2 t^2}{k^2 t^4} \tag{3.36}$$

Given these expression, for arbitrary w, we can calculate the time dependence of the creation and annihilation operators, using the Heisenberg equation of motion. Canonical quantization gives us operators which are in the Heisenberg picture. In this case the correct Heisenberg operator is not just $a(\underline{k},t)$ but $\upsilon_-(k,t)a(\underline{k},t)$, since the latter has the whole time dependence incorporated into it. So, if we are to use the Heisenberg equation of motion to get the evolution of $a(\underline{k}, t)$, we should do that by plugging the correct Heisenberg operator, $\upsilon_-(k,t)a(\underline{k},t)$, which gives

$$\dot{a}^\dagger_{\underline{k}} = \imath\frac{1}{2k}\left[f^*(k,t)a_{-\underline{k}} + g(k,t)a^\dagger_{\underline{k}}\right] - \frac{\dot{\upsilon}_+(k,t)}{\upsilon_+(k,t)}a^\dagger_{\underline{k}} \tag{3.37}$$

$$\dot{a}_{\underline{k}} = -\imath\frac{1}{2k}\left[f(k,t)a^\dagger_{-\underline{k}} + g(k,t)a_{\underline{k}}\right] - \frac{\dot{\upsilon}_-(k,t)}{\upsilon_-(k,t)}a_{\underline{k}} \tag{3.38}$$

In the $k|t| \gg 1$ limit we get $f(k,t) \to 0$, and $g(k,t) \to 2k^2$ for $w = 0, -1$, which gives constant annihilation and creation operators.

$$a^\dagger_{\underline{k}}(t) = a^\dagger_{\underline{k}}(t_0) \tag{3.39}$$

$$a_{\underline{k}}(t) = a_{\underline{k}}(t_0) \tag{3.40}$$

So, in the limit $k|t| \gg 1$, the eigenstates of the annihilation operator are fixed, an important result which we will use in the next section.

3.2.4 The Initial State Problem for $w = -1, 0$

As we saw in section 3.1, the conformal time increases from $-\infty$ to 0^- during inflation, and then skipping $t = 0$ continues increasing during radiation and axion domination. So, for a given mode before inflation $k|t| \gg 1$ and the scale of the perturbation was inside the Hubble horizon. Then close to the end of inflation, during radiation domination, and in the beginning of matter domination, the Hubble horizon was small (in comoving coordinates), and $k|t| \ll 1$. Then in the late axion domination epoch $k|t| \gg 1$ again, and the perturbation again reentered the Hubble horizon.

Using (3.39) and (3.40) we get that the Hamiltonian for the period before and during the early stages of inflation reduces exactly to the expression we had in the Minkowski spacetime case

$$H = \int d^3k\frac{k}{2}\left[a_{\underline{k}}(t_0)a^\dagger_{\underline{k}}(t_0) + a^\dagger_{\underline{k}}(t_0)a_{\underline{k}}(t_0)\right] \tag{3.41}$$

and it commutes with itself at different times. From this expression we can conclude that the number operator and the Hamiltonian commute for this epoch. This means that we can start our evolution from the Bunch-Davies vacuum, which is the state with zero occupation number, $N\left|0\right\rangle = 0$. This state coincides with the zero-point energy eigenstate for this Hamiltonian, from which we want to start our evolution.

When we calculate the power spectrum of the perturbations in the next section, close to the end of inflation (i.e. $k|t| \ll 1$), we should make an assumption that after the transition epoch ($k|t| \sim 1$) $a_{\underline{k}}(t_0) = a_{\underline{k}}(t)$. The result for the spectrum of the primordial fluctuations obtained using this assumption we show to be the same even without making the assumption. To do that we needed to do a more thorough analysis using the Bogolyubov transformation, which will be presented in a follow-up paper. Our results show that the time dependence of $a_{\underline{k}}(t)$ will affect the state only at horizon crossing, where it causes squeezing of the state, but does not change the spectrum of the perturbations.

3.2.5 Quantization of the Gravitational Potential

The gravitational potential is given by [6]

$$\nabla^2\Phi = 4\pi\frac{a\dot{\phi}_0^2}{\dot{a}}\frac{d}{dt}\left(\frac{\chi}{z}\right) \tag{3.42}$$

From here we get

$$\Phi(x,t) = \frac{\dot{\phi}_0}{a}\int\frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2k}}\left[u_k^*(t)e^{\imath\underline{k}\cdot\underline{x}}a_{\underline{k}} + \text{h.c.}\right] \tag{3.43}$$

where

$$u_k = -4\pi\frac{z}{k^2}\frac{d}{dt}\left(\frac{v_+}{z}\right) \tag{3.44}$$

For $w = -1, 0$ this gives

$$u_k = -\imath 4\pi\frac{e^{\imath kt}}{k^3t^2}\left[k^2t^2 + (1+\lambda)(\imath kt - 1)\right] \tag{3.45}$$

Thus, for inflation we have

$$u_k = -\imath 4\pi \frac{e^{\imath kt}}{k} \tag{3.46}$$

which is also true for axion domination, given that $k|t| \gg 1$.

From (3.8) and (3.9) the prefactor in (3.43) equals

$$\frac{\dot{\phi}_0}{a} = \sqrt{\rho_0 + p_0} \tag{3.47}$$

So, during inflation this prefactor is close to zero.

3.2.6 Power Spectrum of the Perturbations for Nearly de-Sitter Inflation

Given the initial state, we can calculate the correlation function for the potential

$$\langle 0 | \Phi(\underline{x}, t)\Phi(\underline{x} + \underline{r}, t) | 0 \rangle = \frac{\dot{\phi}_0^2}{a^2} \int_0^\infty \frac{dk}{(2\pi)^2} \frac{\sin(kr)}{r} |u_k|^2 \tag{3.48}$$

$$|u_k|^2 = \frac{16\pi^2}{k^6 t^4} \left[k^4 t^4 + k^2 t^2 (\lambda^2 - 1) + (\lambda + 1)^2 \right] \tag{3.49}$$

where the last equation holds for $w = -1, 0$. This should equal the integral over the power spectrum, $\mathrm{P}_\Phi(k, t)$.

$$\langle 0 | \Phi(\underline{x}, t)\Phi(\underline{x} + \underline{r}, t) | 0 \rangle \equiv \int d^3k \mathrm{P}_\Phi(k, t) e^{-\imath \underline{k}\cdot\underline{r}} \tag{3.50}$$

From these two equations we can derive the power spectrum of the fluctuations for different epochs

$$\mathrm{P}_\Phi = \frac{\dot{\phi}_0^2}{a^2} \frac{|u_k|^2}{16\pi^3 k} \tag{3.51}$$

Then, in almost deSitter inflation, we get that

$$\mathrm{P}_\Phi = \frac{1}{\pi k^3}(\rho_0 + p_0) \approx 0 \tag{3.52}$$

3.2.7 Power Spectrum of the Perturbations During Inflation With $-1 \le w < -1/3$

During inflation we have $-1 \le w < -1/3$, hence $\lambda \le -1$. The equation of motion (3.25) gives the following two solutions: $\sqrt{-t}J_{-\lambda+1/2}(kt)$ and $\sqrt{-t}Y_{-\lambda+1/2}(kt)$ for $\upsilon(t)$, where Y and J are the cylindrical Bessel functions. These two solutions can be combined as

$$\upsilon_+ = -\sqrt{-\frac{\pi t}{2}}\left[J_{-\lambda+1/2}(kt) - \imath Y_{-\lambda+1/2}(kt)\right] \tag{3.53}$$

$$\upsilon_- = \upsilon_+^* \tag{3.54}$$

The Wronskian is again time independent and in this case equals

$$W = \dot{\upsilon}_+\upsilon_- - \upsilon_+\dot{\upsilon}_- \equiv 2\imath\varpi \tag{3.55}$$

$$\varpi = \Im[\dot{\upsilon}_+]\Re[\upsilon_+] - \Im[\upsilon_+]\Re[\dot{\upsilon}_+] \tag{3.56}$$

$$= \tfrac{\pi k t}{4}\Re\Big[\Big(J^*_{-\lambda+1/2}(kt) + \imath Y^*_{-\lambda+1/2}(kt)\Big)$$

$$\times\left(\imath J_{-\lambda-1/2}(kt) - \imath J_{-\lambda+3/2}(kt) + Y_{-\lambda-1/2}(kt) - Y_{-\lambda+3/2}(kt)\right)\Big] \tag{3.57}$$

$$= 1 - 2\cos\left[2\pi(\lambda+1)\right] \tag{3.58}$$

where the last equation we found empirically. This gives a canonical quantization given by

$$\chi(\underline{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\varpi}}\left[e^{\imath\underline{k}\cdot\underline{x}}\upsilon_-(k,t)a(\underline{k}) + \mathrm{h.c.}\right] \tag{3.59}$$

$$\pi_\chi(\underline{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\varpi}}\left[e^{\imath\underline{k}\cdot\underline{x}}\dot{\upsilon}_-(k,t)a(\underline{k}) + \mathrm{h.c.}\right] \tag{3.60}$$

which satisfies the commutator between χ and π_χ.

Using (3.53) we get that

$$u_k = -\frac{3l^2 t}{2k}\sqrt{-\frac{\pi}{2t}}\left[J_{-\lambda-1/2}(kt) - \imath Y_{-\lambda-1/2}(kt)\right] \tag{3.61}$$

and the gravitational potential is given by

$$\Phi(x,t) = \frac{\dot{\phi}_0}{a}\int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\varpi}}\left[u_k^*(t)e^{\imath \underline{k}\cdot\underline{x}}a_{\underline{k}} + \text{h.c.}\right] \tag{3.62}$$

Finally we can write the spectrum of the perturbations

$$\mathrm{P}_\Phi = \frac{\dot{\phi}_0^2}{a^2}\frac{|u_k|^2}{16\pi^3|\varpi|} \tag{3.63}$$

which is our general solution which is independent of time for $k|t| \ll 1$. We can take the modulus of ϖ since, we can always exchange q_+ and q_-, resulting in a change of the sign of ϖ. So, we always choose the positive value. By direct substitution, we can see that we recover our expression (3.52) for inflation with $w \approx -1$.

For small values of $\dot{\phi}_0$, we have $\lambda \lesssim -1$. So, we will expand our solution to see how the spectrum of the fluctuations behaves. ϖ has an extremum at $\lambda = -1$, which means that $\epsilon \equiv -1 - \lambda$ makes a second order contribution to ϖ, and a third order contribution to P_Φ.

Since we need $\dot{\phi}_0$ to be real, we require that $\epsilon \geq 0$, which is equivalent to $w \geq -1$. To first order in ϵ we reproduce our previous result for the power spectrum (3.52). Thus, we show that even though the mode functions, $v(k,t)$, change a little for non-zero ϵ, this is a higher order effect.

The spectrum of the curvature perturbations for $w \approx -1$ is given by

$$|\delta_k|^2 \equiv 4\pi k^3 P_\Phi(k) = 4(\rho_0 + p_0) \tag{3.64}$$

which is a constant with respect to k and t. This is the *correct* formula for the curvature perturbations for almost deSitter universe. For an arbitrary value of w, the generalilzation of this is given by (3.63). We will see how it changes our understanding of the spectrum of the perturbations in the next section.

Chapter 4

Comparison With the Wrong Previous Results

In this section we will use ϕ to denote the homogeneous part of the scalar field.

4.1 Comparison With the Wrong Slow-Roll Inflation Solution

Slow-roll inflation assumes large Hubble damping, i.e. small $\dot{\phi}$, and small $\ddot{\phi}$. In this approximation the two slow-roll parameters are given by [8]

$$\varepsilon = -\frac{H'}{H^2} = 4\pi\frac{(\phi')^2}{H^2} = \frac{1}{16\pi}\left(\frac{V_{,\phi}}{V}\right)^2 \tag{4.1}$$

$$\eta = \frac{1}{8\pi}\frac{V_{,\phi\phi}}{V} = \frac{1}{3}\frac{V_{,\phi\phi}}{H^2} \tag{4.2}$$

where a prime denotes proper time derivative. These parameters are both taken to be very small.

The spectrum of the perturbations in the case of the wrong solution is given by [8]

$$|\delta_k|^2_{\mathrm{WS}} = \frac{9}{25\pi\varepsilon}\frac{H^2}{(2\pi)^2} = \frac{32}{9\pi}\frac{V^3}{(V_{,\phi})^2} \tag{4.3}$$

The subscript WS stands for Wrong Solution.

Our correct treatment in the slow-roll approximation gives

$$|\delta_k|^2_{\mathrm{CS}} = \frac{1}{6\pi}\frac{(V_{,\phi})^2}{V} \tag{4.4}$$

The subscript CS stands for Correct Solution.

This means that the wrong result gives a power spectrum which is much bigger in magnitude than what was obtained with the correct treatment. To rewrite everything using the slow roll parameters we get

$$|\delta_k|^2_{\mathrm{WS}} = \frac{6}{25\pi^2}\frac{V}{\varepsilon} \tag{4.5}$$

$$|\delta_k|^2_{\mathrm{CS}} = \frac{8}{3}V\varepsilon \tag{4.6}$$

This means that the energy scale of inflation, E, is given by

$$E_{\mathrm{WS}} \equiv \left(\frac{V}{\varepsilon}\right)^{1/4} \tag{4.7}$$

$$E_{\mathrm{CS}} \equiv (V\varepsilon)^{1/4} \tag{4.8}$$

4.2 Comparison With CMB Results

On large angular scales $\theta \gg 1°$, the CMB anisotropies are caused by the fluctuations of the gravitational potential on the surface of last scattering - an effect called the Sachs-Wolfe effect. On these scales, the CMB anisotropies had superhorizon size at photon decoupling, therefore they directly probe the spectrum of the fluctuations of potential during inflation.

For adiabatic fluctuations, we have ([8], his equation (212))

$$\frac{l(l+1)C_l}{2\pi} = \frac{A^2}{9} \simeq 10^{-10} \tag{4.9}$$

where the last result quotes the result of COBE; A is defined by $|\delta_k|^2 \equiv 4\pi A^2 \simeq 10^{-8}$.

Using this data we can conclude that the energy scale in the wrong and in the correct treatment of inflation equals

$$E_{\mathrm{WS}} = \left(\frac{V}{\varepsilon}\right)^{1/4} \approx 5 \times 10^{16}\mathrm{GeV} \tag{4.10}$$

$$E_{\mathrm{CS}} = (V\varepsilon)^{1/4} \approx 7 \times 10^{16}\mathrm{GeV} \tag{4.11}$$

This means that the fine-tuning problem is greatly alleviated by the fact that ε which in the slow-roll inflation is $|\varepsilon| \ll 1$ goes in the numerator in the correct expression of the energy scale. This means that the potential in the correct treatment is no longer required to be small in natural units to explain the observed CMB temperature anisotropies.

4.3 Examples of Inflaton Potentials

Now let us consider several "classical" inflaton potential models.

First, we need to express the number of e-foldings, N, in terms of the potential

$$N \equiv \int_{t_i}^{t_f} \frac{\dot{a}}{a^2} dt \tag{4.12}$$

This in the slow-roll limit becomes

$$N \simeq 8\pi \int_{\phi_i}^{\phi_f} \frac{V}{V_{,\phi}} d\phi \tag{4.13}$$

Also, we want to test the slow-roll approximation. To do that we need to show that the kinetic term is much smaller than the potential term in the equation of motion (3.25). This translates into

$$|V_{,\phi}| \ll V \tag{4.14}$$

4.3.1 Quadratic Potential

The potential is given by $V(\phi_0) = m^2\phi_0^2/2$. This potential is nowadays the preferred model of inflation, not only because of its simplicity, but because it does not require special initial conditions (as is the case with "new" inflation, described below).

This means that the number of e-foldings equals

$$N = 2\pi\phi_i^2 \tag{4.15}$$

Using the equations for the spectrum of the perturbations (4.6) and (4.5) and the result for N (4.15) we get

$$|\delta_k|^2_{\mathrm{WS}} \approx 0.3m^2\phi_i^4 \simeq 10^{-8} \tag{4.16}$$

$$|\delta_k|^2_{\mathrm{CS}} \approx 0.2m^2 \simeq 10^{-8} \tag{4.17}$$

Note that this is given in terms of ϕ_i. This is due to the fact that the large angle modes in the CMB that we observe have crossed the Hubble horizon at the early stages of inflation (possibly at few e-foldings). This means that in the slow-roll approximation, our results for the perturbations should be given only in terms of the initial value of the field.

Our result (4.17) is fascinating, since this means that in light of our correct treatment, the CMB spectrum directly measures the inflaton mass. As a comparison, the wrong result depends on the dynamics, and on N in particular. However, in light of the new results on the upper bound on N [1], the wrong and the correct treatments give approximately equal values for the mass of the inflaton. To summarize

$$m_{\mathrm{WS}} \lesssim 10^{-5} \tag{4.18}$$

$$m_{\mathrm{CS}} \approx 10^{-4} \tag{4.19}$$

From the correct treatment combined with the CMB measurements, we get a mass of the inflaton field in the range of the Grand Unified Theory (GUT) energy scale,

$E_{\rm GUT}$, not depending on how long inflation lasted, and how many e-foldings it caused.

4.3.2 ϕ^4 Potential

The potential is given by $V = \lambda\phi^4$. For this case we have

$$N = \pi\phi_i^2 \tag{4.20}$$

from which we get

$$|\delta_k|^2_{\rm WS} \approx \lambda\phi_i^6 \tag{4.21}$$

$$|\delta_k|^2_{\rm CS} \approx \lambda\phi_i^2 \tag{4.22}$$

This gives

$$\lambda_{\rm WS} \lesssim 10^{-11} \tag{4.23}$$

$$\lambda_{\rm CS} \lesssim 10^{-10} \tag{4.24}$$

In this case, the correct model does not give a substantial weakening of the fine-tuning problem.

4.3.3 "New" Inflation

New inflation ("new" as of 1982) uses a potential which arises from spontaneous symmetry-breaking, (e.g. the Peccei-Quinn symmetry breaking). The field starts from an unstable equilibrium at $\phi \approx 0$, and rolls down to the true vacuum. The potential for this case is given by $V = \Lambda^4(1 - (\phi/\mu)^p)$, where $p > 1$. If we require that μ is of order 1 in natural units, we will obtain a constraint on Λ using the CMB result. The requirement (4.14) for slow-roll inflation then is $\phi \ll 1$. The e-foldings constraint is easily satisfied by choosing the value of ϕ_f. For this potential we get

$$|\delta_k|^2_{\rm WS} \approx \frac{\Lambda^4}{\phi_i^{2(p-1)}} \tag{4.25}$$

$$|\delta_k|^2_{\rm CS} \approx \Lambda^4 \phi_i^{2(p-1)} \tag{4.26}$$

Since we require, $p > 1$ and $\phi \ll 1$, we see that in the wrong treatment, there is no way that $\Lambda \sim 1$. In fact $\Lambda \ll 10^{-2}$ for the wrong case. For the correct treatment, the small value of the CMB temperature fluctuations can be entirely ascribed to ϕ_i which is just a dynamical variable, hence there is no problem to get $\Lambda \sim 1$. So, for the case of "new" inflation, the correct treatment completely solves the fine-tuning problem, since we can have all parameters of order unity in natural units.

4.3.4 Power-Law Inflation

This is inflation for which $w = const$. The solution to the dynamics is given by (3.12) and (3.13). From (3.14), the potential is given by $V = V_0 e^{\phi\sqrt{16\pi\epsilon}}$. The number of e-foldings is

$$\phi_i = \frac{NV_0}{2\sqrt{\pi}}\sqrt{\epsilon} \tag{4.27}$$

The slow-roll requirement is $\epsilon \ll 1$, which means $w \approx -1$. From here we get

$$|\delta_k|^2_{\rm WS} \approx 0.02\frac{V_0}{\epsilon} e^{NV_0\epsilon/2} \approx 10^{-8} \tag{4.28}$$

$$|\delta_k|^2_{\rm CS} \approx 0.04 V_0 \epsilon e^{NV_0\epsilon/2} \approx 10^{-8} \tag{4.29}$$

If we choose the natural value $V_0 \approx 1$, we get that

$$\epsilon_{\rm CS} \sim 10^{-7} \tag{4.30}$$

and there are no real solutions for $\epsilon_{\rm WS}$. So, for this potential our correct treatment satisfies all the requirements for slow-roll inflation for natural value of the parameter

$V_0 \sim 1$, for practically arbitrary value of N (in fact we need $N \lesssim 10^7$, which is completely "reasonable"). This means that in this case we completely solve the fine-tuning problem in inflation.

Chapter 5

Conclusions and Outlook

In this thesis we showed that the spectrum of primordial fluctuations as calculated until (e.g. [5]) now is wrong. We find the correct expression for the spectrum using a field variable introduced by Mukhanov. Like most authors, we assume that all modes evolve independently, i.e. we neglect squeezing. In a follow-up paper we will show that the coupling of modes results in squeezing of the state without affecting the power spectrum.

We also show that using our correct treatment, the fine-tuning problem in inflationary cosmology is solved for certain potentials, and is alleviated in others (e.g. "new" inflation and power-law inflation). We also show that in the case of chaotic inflation with quadratic potential, the spectrum of the CMB directly measures the mass of the inflaton field, instead of also depending on the number of e-foldings. Compared with the CMB power spectrum, we get a value for the inflaton mass of 10^{15}GeV which is of the order of the GUT energy scale.

Our discussion of the inflaton field can be continued to encompass the later stages of the evolution of the universe after inflation. As we discussed in Chapter 1, structure formation in the universe may be governed by the dynamics of the axion scalar field. This means that our treatment of the inflaton scalar field can be applied to the axion field, as well. Thus, by extending our analysis, we can rigorously follow the evolution of cosmic perturbations from the primordial vacuum fluctuations to structure formation, without reverting to semi-classical approximations.

References

[1] T. Banks and W. Fischler. An upper bound on the number of e-foldings. *ArXiv Astrophysics e-prints*, July 2003.

[2] E. Bertschinger. Hamiltonian formulation of general relativity. Unpublished notes, May 2005.

[3] P. Coles and K. Spencer. A wave-mechanical approach to cosmic structure formation. *MNRAS*, 342:176–184, June 2003.

[4] A. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23:347–356, January 1981.

[5] A. Liddle and D. Lyth. *Cosmological Inflation and Large-Scale Structure.* Cambridge University Press, 2000.

[6] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger. Theory of cosmological perturbations. *Physics Reports*, 215:203–333, June 1992.

[7] D. Polarski and A. A. Starobinsky. Semiclassicality and decoherence of cosmological perturbations. *Classical and Quantum Gravity*, 13:377–391, March 1996.

[8] A. Riotto. Inflation and the theory of cosmological perturbations. Number 14 in ICTP Lecture Notes, pages 317–414, Trieste, Italy, 2003. The Abdus Salam ICTP Publications.

[9] M. Schlosshauer. Decoherence, the measurement problem, and interpretations of quantum mechanics. *Reviews of Modern Physics*, 76:1267–1305, February 2005.