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Case Study of Controller Design Limitations of an Ill-Conditioned System Using l1 and m Design Concepts

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Case Study of Controller Design Limitations of an Ill-Conditioned System Using ℓ_1 and μ Design Concepts

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Abstract

The focus of this paper is a comprehensive case study of the fundamental limitations of and controller design for the ill-conditioned high purity distillation column problem. As a result of the work on this problem, general ideas are presented concerning the design of robust controllers using norm-based methods.

The case study of the distillation column problem includes three main components. The first component is a study of the physical plant including performance objectives, singular value analysis, and uncertainty description. It is shown that the relationship between the plant singular value directions and the input uncertainty makes this control problem difficult. The second component is the application of μ-synthesis, a design methodology for robust performance incorporating the concept of structured uncertainty in an H_∞ setting, on the distillation column. The results of the designs are analyzed and conclusions are made concerning tradeoffs for controller design and frequency domain characteristics of the resulting controllers. The third component of the study is the application of the ℓ_1 design methodology. Since only within the past year was a computationally stable algorithm for ℓ_1 designs developed and implemented, a study of minimizing the weighted sensitivity and mixed sensitivity/KS problem is completed. The results of this preliminary work are used to solve ℓ_1 structured uncertainty problems for the distillation column. The results of the designs are used to analyze the system. Comparisons are made with discrete time μ designs generated in this paper.

The design work on the distillation column problem resulted in an increased understanding of the general procedure of designing robust controllers using H_∞ and ℓ_1 design methodologies. First the effect of varying parameters of a first order frequency domain performance weight is studied to determine the effect on the time domain behavior of the resulting system. Second, directional information about the plant is included in the design procedure to increase the level of robust performance. Third, comparisons of the advantages of each methodology are made.

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1 Introduction

Recently, the high purity distillation column has been a benchmark problem for many control design methodologies. A series of papers presented at the 1991 Conference on Decision and Control was devoted to controller designs for this system [15], [11], [21]. It is the goal of this paper to study the limits to controller design of this system using current state of the art norm-based robust control methodologies. The goal is not simply to design controllers, but to analyze the inherent limitations of the system and the characteristics of the resulting controllers.

Contributions are made in this paper toward better understanding the ℓ_1 structured uncertainty and μ design methodologies and the controller design limitations of this ill-conditioned plant. First, the concept of structured uncertainty for robust performance design and analysis is applied to the distillation column problem in the ℓ_1 setting. While the concept of structured uncertainty applied in an H_∞ setting (μ) has been studied previously for this an other non-academic problems, the results of this paper provide one of the first applications of this concept in an ℓ_1 setting. Second, information is presented on the selection of weights for performance specifications in norm-based controller designs. The effect of varying parameters in a first order performance weight on the sensitivity is investigated and related to time domain characteristics such as rise time and overshoot. Also, it is shown that adding directional information to the performance weight in both the ℓ_1 structured uncertainty and μ design procedures increases the level of robust performance. Third, a comprehensive study of the distillation column control problem is presented. One aspect of the study is an analysis of the difficulties in controller design for the system including a concise review of previous work on the problem. The second aspect of the study is the analysis and comparison of the results of the ℓ_1 structured uncertainty and μ designs for the system in terms of robust performance in the time domain and controller characteristics in the frequency domain.

2 Background

2.1 Distillation Column

The goal of the distillation column (see Figure 1) is to separate the feed mixture into its light and heavy components with a desired degree of purity. For more detailed description of column operation see [17] and [18]. The system inputs are the reflux (L) and the boilup (V). The

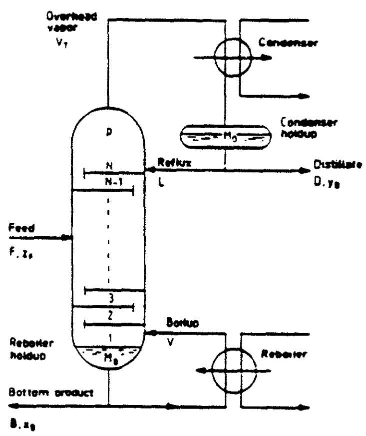


Figure 1: High Purity Distillation Column Model

Gain	Input Direction: $\begin{matrix} \Delta L \\ \Delta V \end{matrix}$	Output Direction: $\begin{matrix} \Delta y_D \\ \Delta x_B \end{matrix}$
0.0139	$\begin{bmatrix} 0.7075 \\ 0.7067 \end{bmatrix}$	$\begin{bmatrix} 0.7744 \\ -0.6328 \end{bmatrix}$
1.9712	$\begin{bmatrix} 0.7067 \\ -0.7075 \end{bmatrix}$	$\begin{bmatrix} 0.6328 \\ 0.7744 \end{bmatrix}$

Table 1: Singular Value Analysis of LV Configuration

measured outputs are the composition of light component in the top and bottom products. There are two sources of input uncertainty which are present. The first is the up to 20% uncertainty in actuator gain. The second is the up to 1 second time delay due to the flow dynamics. See Appendix A and B for a description of the plant and uncertainty models.

There are three main requirements which should be met by the controller. The first is robust stability for uncertainty mentioned above. The second is robust performance with respect to the time domain specifications of [15]. (These can also be found in Appendix B). Finally, as described in [19] the system should achieve robust disturbance rejection.

The high degree of plant directionality and the resulting difficulties of controller design have a direct physical interpretation which can be inferred from the singular value decomposition shown in Table 1. The nominal high gain direction corresponds to increasing the light component composition in both top and bottom products and results from increasing reflux while decreasing boilup. The low gain direction corresponds to increasing the light component composition in the top component while decreasing the light component in the bottom product and results from increasing both the reflux and boilup. With this physical interpretation, we can now see how difficulties would arise in controlling this system. If the goal was to make both products more pure (a move in the low gain direction), we would attempt to greatly increase the reflux and boilup equally. As a result of actuator gain uncertainty, reflux may be raised more than boilup. This difference results in a command with a component in the high gain direction. The effect of this high gain direction error will overshadow the intended command component in the low gain direction. As a numerical example, assume, for the sake of simplicity, that $v_{max_{LV}} = [1 \quad -1]'/\sqrt{2}$ and $v_{min_{LV}} = [1 \quad 1]'/\sqrt{2}$. Assume that the command $v = \alpha v_{max_X} + \beta v_{min_X}$ is applied to the system. The actual command entering the nominal plant, after passing through the actuator with the uncertainty given in Appendix A can be shown to be $v_{act} = v + \Delta v$ with

$$\Delta v = \left[\left(\frac{\delta_1+\delta_2}{2}\right)\alpha + \left(\frac{\delta_1-\delta_2}{2}\right)\beta\right] v_{max_{LV}} + \left[\left(\frac{\delta_1+\delta_2}{2}\right)\beta + \left(\frac{\delta_1-\delta_2}{2}\right)\alpha\right] v_{min_{LV}} \tag{1}$$

Thus it is possible (when $\delta_2 = -\delta_1$) that a command with a large component in the low plant gain direction will have an error term with a large component in the high gain direction,

$$\Delta v = \delta_1 \beta v_{max_{LV}} + \delta_1 \alpha v_{min_{LV}} \quad \text{for} \quad \delta_2 = -\delta_1 \tag{2}$$

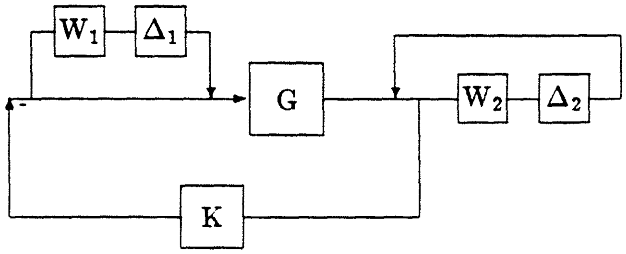
Note that if each gain direction was associated with an independent set of input variables, the problems described above would not occur.

2.2 Previous Research

Recently, several design methodologies have been applied to this problem in an attempt to provide robust stability and robust performance for the strict time domain specifications; however, little effort has been made to quantitatively state the limits to performance based on the model uncertainty. Also, several of these methods do not explicitly incorporate known model uncertainty and performance constraints.

In [19] Skogestad, Morari, and Doyle study an inverse based controller (high condition number) and a diagonal controller (low condition number) for the high purity distillation column. A slightly different set of specifications, model, and uncertainty than that found in Appendix A were used in this paper. Nominal performance is found to suffer with the use of the diagonal controller since due to plant directionality more gain is needed in low plant gain directions which this controller cannot provide. Robust performance is found to suffer with the use of the inverse controller since there is uncertainty as to the actual direction of low plant gain.

Skogestad, Morari, and Doyle also apply μ synthesis (D-K iteration) to the distillation col-



W_1: Model Uncertainty Weight
W_2: Performance Weight

Figure 2: Δ Block Structure used in μ synthesis

umn. μ analysis and synthesis techniques were developed to address the issues of robust performance. More detail on D-K iteration and μ can be found in [10], [8], [9], [20]. The Δ block structure for the uncertainty and performance blocks can be seen in Figure 2 with Δ_1 and Δ_2 linear, time-invariant, $\| \Delta \|_\infty \leq 1$. Note that W_2 is not unique in its representation of the desired time domain performance specifications. In fact W_2 contains little information to limit overshoot. The effect of this lack of information becomes apparent when observing that a value very close to one (1.06) is achieved for μ, yet the overshoot in the time domain simulations does not meet the robust performance specifications.

In [11] Hoyle, Hyde, and Limbeer design a two degree of freedom controller for the distillation column using H_∞ theory. The presence of uncertainty is modeled using coprime factorization; however, bounds on the actual plant uncertainty are not included in the design procedure. Instead, they are used as a robust stability check after the design is complete. The time domain specifications are also handled indirectly. The quantity $\| R_{yb} - M_0 \|_\infty$ is minimized over all stabilizing compensators where R_{yb} is the weighted closed loop transfer function and M_0 is a transfer function with a step response that meets the desired specifications. Note that due to the reliance of performance on the prefilter, the good robust performance characteristics do not apply to output disturbance rejection; the need for which was discussed in [19]. Also, it is very difficult to make any general statements on the system limitations since little information on the plant or specifications were used directly in the design procedure.

In [21] Zhou and Kimura apply the robust stability degree assignment procedure to the distillation column. A robustness function, which is nothing more than a weighting matrix, is used to model uncertainty. From the analysis of [19] Zhou and Kimura hypothesize that by matching controller singular values in the middle frequency range (10^{-1} - 10^{0} rad/sec) robust performance will be achieved. Very complicated, high order, rational weighting functions with directional properties consistent with plant directionality are introduced into the robustness function. While the general concept used to choose the weights is clear, the procedure used to choose these specific values is not. The time domain specification of rise time is met by forcing all closed loop poles to be less than some $-\beta$ where $\beta > 0$. This forcing of closed loop pole location is accomplished by performing the design in a space with a shifted $j\omega$ axis. Overshoot specifications are treated by iterating over β until an acceptable level is found. Time domain and frequency domain simulations show that the singular value matching does produce consistency in the level of performance over possible plants in the model set; however, this level of performance is not very good with respect to the specifications (Appendix A). Concerning the ability to determine system limitations, some information can be obtained from this procedure through the tradeoffs between overshoot and achievable β.

3 Analysis and Synthesis Using μ

The complex structured singular value, μ, provides a good theoretical framework for analyzing and synthesizing systems for robust performance ([16], [8], [9], [20]). The following sections contain the results of applying μ to the distillation column, the limitations of the system based on these results, and finally the application of μ for the distillation column in discrete time.

3.1 μ-Synthesis for the Distillation Column Problem

The following subsections describe the rationale for the specific weighting functions which are used in this design. The structure shown in Figure 2, with Δ_1 and Δ_2 linear, time-invariant, $\| \Delta \|_\infty \leq 1$ is used in these designs; however the choice of weighting functions is unique to this paper. Both of the weights chosen to model the performance objectives and uncertainty were first order. Large order weights always add to controller order and can obscure the intuition behind the choice; therefore, since the complexity of the model uncertainty did not warrant a high order weight for accurate modeling, first order weights will be used.

3.1.1 Uncertainty Modeling

A technique similar to that used in [11] is used to bound the uncertainty; however, unlike the design procedure in [11] this bound will be used in the actual design method. The uncertainty described in Appendix A will be modeled as a multiplicative norm bounded input perturbation.

$$G_{LV}(s) = G_{LV}^{nom}(s)(I + \Delta) \tag{3}$$

$$\Delta = \begin{bmatrix} (1+\delta_1)e^{-\tau_1 s} - 1 & 0 \\ 0 & (1+\delta_2)e^{-\tau_2 s} - 1 \end{bmatrix} \tag{4}$$

Through algebraic manipulations similar to those in [11] a bound on the maximum singular value of this perturbation is obtained. Also, a first order transfer function is found to bound this function.

$$\sigma_{max}(\Delta(j\omega)) \leq \sqrt{2.44 - 2.4\cos\omega} \leq \sigma_{max}\left(\frac{2.3123\,(s + .09)}{s+1}\right) = w_1(s) \tag{5}$$

It can be shown analytically that the magnitude of this function does indeed bound the uncertainty. Therefore in Figure 2 $W_1(s) = w_1(s)I$. Obviously this model will produce a conservative design since the physical uncertainty is known to be diagonal and a full Δ block is used to model it. The degree of conservatism, however, is not great [19]. Note that the norm bounded weighted perturbation used in [19] does not bound Equation 4.

3.1.2 Performance Weight

In this paper a more systematic approach to choosing a performance weight is used than in the previous work discussed in Section 2.2. Qualitatively, the time domain design specifications that were given in Appendix B require good nominal and robust disturbance rejection and command following. It is a difficult task to directly translate time-domain specifications into frequency-domain specifications on the sensitivity of the form

$$\| W_2(s)S(s) \|_\infty < 1 \tag{6}$$

through the choice of W_2. Some time-domain specifications are easier to translate than others. For example, the specification of the absolute value of steady state error to step inputs ≤ 0.01 can be translated directly into a specification on the sensitivity: $S(j0) \leq 0.01 = -40dB$. The performance constraints in Appendix B require for each input given that $| \; e(30) \; | = | \; r(30) - y_g(30) \; | \leq 0.1$, where e is the error, r is the reference command and y_g is the measured output of the plant, G. This implies a time constant of $\frac{1}{0.07676}$, since $e^{-.07676\times 30} = 0.09998$. This indicates that most dominant poles of the sensitivity should be less than -0.07676. The overshoot constraints are the most difficult to quantify in the frequency-domain in terms of Equation 6. The high frequency gain of the sensitivity should be limited to decrease overshoot somewhat. There is really no qualitative way of choosing a weight that incorporates time-domain overshoot specifications.

From the results of the previous paragraph, loop shaping ideas in H_∞ designs were used to arrive at the performance weight of

$$W_2(s) = 0.8\frac{s+.072}{s+.0005}I_{2\times 2} = w_2(s)I_{2\times 2} \tag{7}$$

where $W_2^{-1}(s)$ has characteristics discussed in the previous paragraph. Note that the zero of $W_2(s)$ is greater that -0.07676. By performing several designs in which this and other parameters were modified, it was found that this weight produced the best time response results with respect to the given specifications. Therefore the numbers calculated in the previous paragraph are only starting points and not necessarily hard constraints on weighting function parameters. It will be shown in Section 3.1.3 that adding directionality to this performance weight can improve robust performance.

3.1.3 Directionally Weighted Design

While the robust performance of the μ design with the weights described in the previous paragraph was good, improvements can be made. The directionality of this plant has been discussed in detail. It has been shown that system performance is strongly dependent on command input direction. It is therefore unrealistic to believe that any control system can provide identical levels of performance to inputs in different directions; however, it may be possible that the responses resulting from inputs in different directions all meet a given set of specifications. The goal is to translate this concept into μ design format.

Consider the singular value decomposition for the sensitivity resulting from the μ design with diagonal performance weight at low frequencies. The singular value directions are complex and the magnitude of the directions (and accompanying sign) associated with the maximum and minimum singular values at low frequencies are shown in Table 2. Note that the complex directions change slightly as a function of frequency; however, the change in magnitude is not great. Note that the input directions correspond to the maximum and minimum plant gain

Gain	Input Direction:	Output Direction:
High	$\begin{bmatrix} 0.7745 \\ -0.6326 \end{bmatrix}$	$\begin{bmatrix} 0.7955 \\ -0.6060 \end{bmatrix}$
Low	$\begin{bmatrix} -0.6326 \\ -0.7745 \end{bmatrix}$	$\begin{bmatrix} -0.6060 \\ -0.7955 \end{bmatrix}$

Table 2: Low Frequency Directionality of the Sensitivity

output directions. If the high gain direction of the sensitivity were weighted more than the low gain direction then the resulting closed loop sensitivity condition number should be lower. This would result in less deviation among time responses. Also, it was shown in [19] that achieving robust performance for commands in the low plant gain direction is more difficult than for commands in the high plant gain direction. Thus this type of directional weighting makes sense.

From the preceding analysis, a new performance weight of

$$W_{2_{new}}(s) = w_2(s) \begin{bmatrix} 0.9036 & -0.1512 \\ -0.0920 & 0.8454 \end{bmatrix} \tag{8}$$

where

$$\begin{bmatrix} 0.9036 & -0.1512 \\ -0.0920 & 0.8454 \end{bmatrix} = \begin{bmatrix} 0.7955 & -0.6060 \\ -0.6060 & -0.7955 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.7500 \end{bmatrix} \begin{bmatrix} 0.7745 & -0.6326 \\ -0.6326 & -0.7745 \end{bmatrix}' \tag{9}$$

Several variations on this weight were tried; however, this particular weight produced the best results in terms of the time-domain specifications on the plant.

3.2 Analysis of Design Results and System Limitations

In the following subsections, the results of the previous designs will be used to make statements about the limitations on control design for robust performance for this plant. Step command inputs will be used to analyze the system in the time-domain. The singular value plots of various input/output maps will be examined in the frequency-domain. Finally the structured singular value will be used to analyze the robustness and conservatism of the design.

3.2.1 Time-Domain Analysis

The response of the system to step inputs in three directions will be studied. These input directions were given in the specifications in Appendix B. These commands can be broken into components based on the high and low plant gain output directions given in Table 1. The commands $\Delta y_D = 1$, $\Delta x_B = 0$ and $\Delta y_D = 0$, $\Delta x_B = 1$ have a large component in the low plant gain direction, with the former having a larger component than the latter. Robust performance should be more difficult to achieve for commands in these directions. The input direction given by $\Delta y_D = 0.4$, $\Delta x_B = 0.6$ is almost entirely in the high plant gain direction and therefore should have better robust performance; this is in fact the case.

Time responses of systems (nominal and perturbed) resulting from the μ design with directional performance weighting for inputs in the directions $\Delta y_D = 1$, $\Delta x_B = 0$ and $\Delta y_D = 0$, $\Delta x_B = 1$ are shown in Figures 3 and 4. It can be shown that the responses of all gain uncertainty combinations lie within the bounds of the responses shown. Finally, it can also be shown that responses to a step in the direction $\Delta y_D = 0.4$, $\Delta x_B = 0.6$, which as previously discussed is in the high plant gain direction, for all combinations of uncertainty are very good. The robust performance of this design (directional performance weight), with respect to the specifications in Appendix B, is better than the robust performance of the μ design with a diagonal performance weight.

Analyzing the μ design with a directional performance weight, specifications are met for virtually all uncertainty combinations; however, as was mentioned in the discussion of the design procedure, there are some input direction/uncertainty configurations for which the time responses push the limits of some of the design specifications while satisfying the rest of the specifications easily. To illustrate this point consider the command entering the nominal plant as a function of the actual command and the uncertainty present. Thus if the command was $\Delta y_D = 1$, $\Delta x_B = 0$ and the actual plant was $G_{LV}^{nom}(I + \Delta)$ where

$$I + \Delta = \begin{bmatrix} 1+\delta_1 & 0 \\ 0 & 1+\delta_2 \end{bmatrix} = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.8 \end{bmatrix} \tag{10}$$

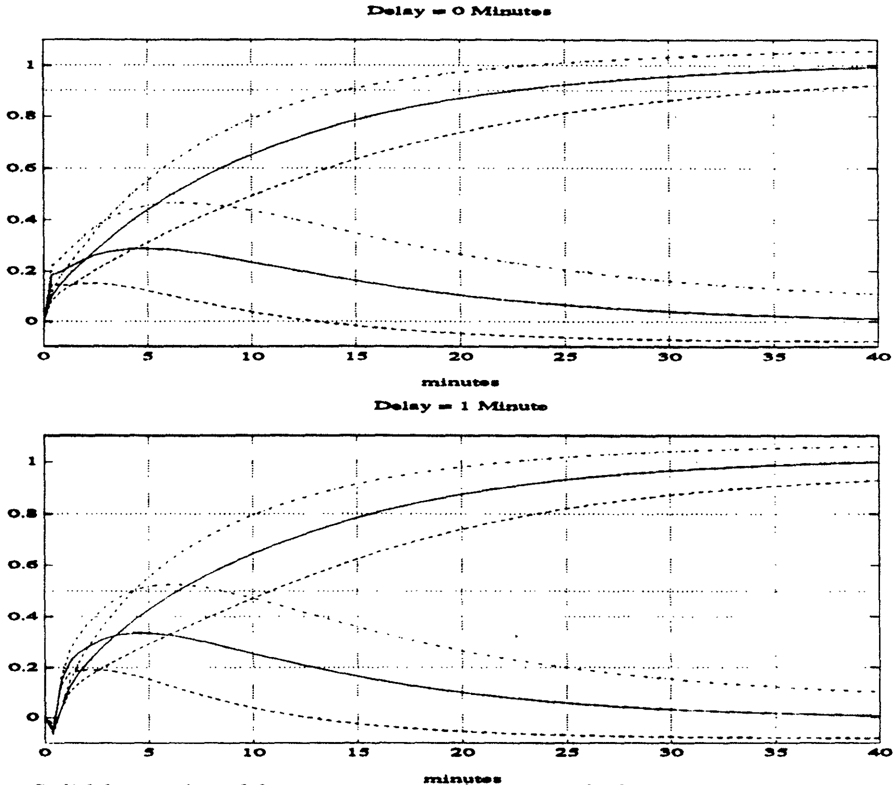
then the command entering the nominal plant would be $\Delta y_D = 1.2$, $\Delta x_B = 0$. Compare the time responses shown in Figures 3 and 4. For the example just presented the magnitude of the resulting command entering the nominal plant is increased $(1.2 > 1)$. The rise time specification $(\Delta y_D > 0.9 \quad \forall \quad t \geq 30)$ is met without any problem, but the overshoot increases to push the limits of $\Delta y_D \leq 1.1 \quad \forall \quad t$ and $\Delta x_B \leq 0.5 \quad \forall \quad t$. Similar time responses occur for all combinations of uncertainty and nominal commands for which the magnitude of the nominal command is increased. The opposite occurs when the magnitude of the command entering the nominal plant is smaller than the magnitude of the original command. For example when $\delta_1 = -0.2$ and $\delta_2 = 0.2$ and the nominal command is $\Delta y_D = 1$ and $\Delta x_B = 0$ as before, then the command entering the nominal plant is $\Delta y_D = 0.8$, $\Delta x_B = 0$. In this case the overshoot constraints (same as before) are met easily, but the rise time is much slower and the specification of $\Delta y_D > 0.9 \quad \forall \quad t \geq 30$ is missed slightly (actual rise time is 35 seconds). Better robust performance can be achieved by acknowledging the directionality of the plant, i.e. not requiring that inputs in all directions for all plants in the model set produce identical responses. The responses will be different, but they may all lie within a given band of acceptable responses.

Notice that in Figures 3 and 4 there is an initial "jump" in the high plant gain direction which tends to increase the composition of light component in both products. This initial jump seems to improve the robust performance. This will be related to the high frequency behavior of the closed loop system in the next section.

3.2.2 Frequency-Domain Analysis

The closed loop frequency responses for the design with directional performance weighting are shown in Figure 5. These responses are very similar to those from resulting from the design with diagonal performance weighting; however, there are a few differences which should be pointed out. The singular values of the controller are matched better around 1 rad/min for the directionally weighted design than for the design with diagonal performance weighting. Also, the condition number of the sensitivity at low frequency is slightly smaller for the directionally weighted design. This indicates that the closed loop system performance should be slightly more consistent for different inputs.

The shape and directionality of the singular values of the transfer function from the disturbances to the controls (KS) provide information into the behavior of the system and are shown in more detail in Figure 6. The dotted and dash dotted lines are the gain with respect to the directions corresponding to the plant low gain and high gain direction respectively. For frequencies less than about 0.05 rad/min, the transfer function from disturbances to controls is the inverse of the plant. At approximately w = 1.474 rad/min the gain of KS in the low and high plant gain directions cross. The transfer function from disturbances to controls has high



Solid line: closed loop system with nominal plant
Dashed line: closed loop system with perturbed plant $\delta_1 = -0.2$, $\delta_2 = 0.2$
Dash-Dot line: closed loop system with perturbed plant $\delta_1 = 0.2$, $\delta_2 = -0.2$
Figure 3: Directional μ Design Response to a Step in Direction $\Delta y_d = 1$ $\Delta x_B = 0$

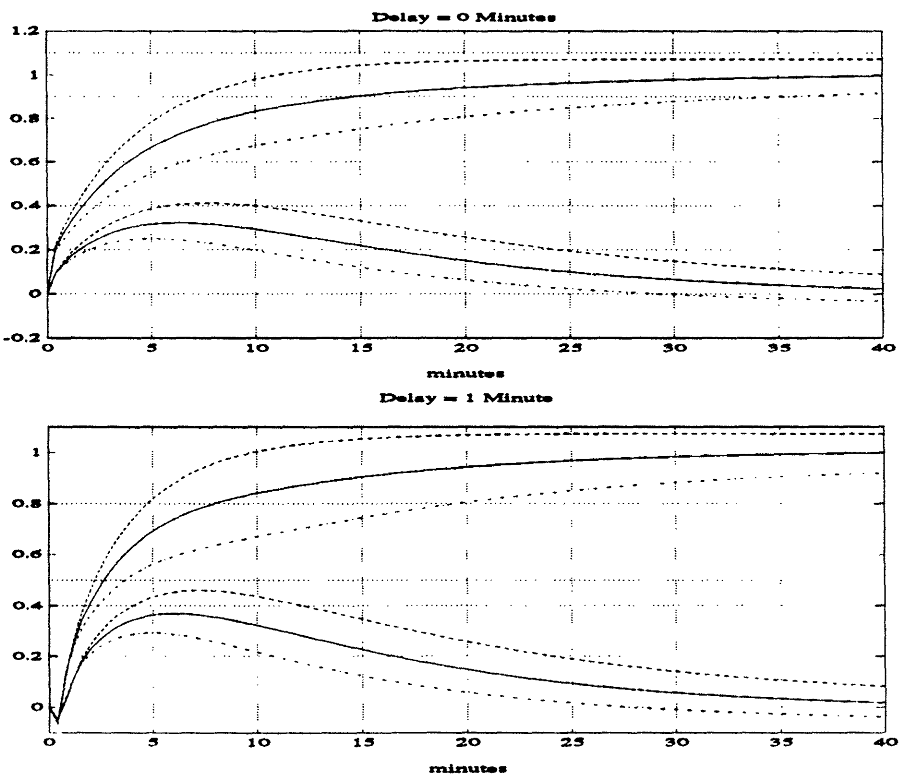


Figure 4: Directional μ Design Response to a Step in Direction $\Delta y_d = 0$ $\Delta x_B = 1$

gain in the direction corresponding to the high plant gain direction for frequencies greater than the crossover frequency. In [19] inverse and diagonal controllers were studied and it was found that neither produced good robust performance. The results of this μ design and the analysis of Figure 6 indicate that a combination of an inverse and a diagonal controller, at low and middle frequency ranges is needed. At high frequencies, however, the directions corresponding to the maximum and minimum controller singular values must be the opposite of what they were at low frequencies. Note that due to the high frequency gain in the design, the resulting system bandwidth is much higher than the bandwidth required to produce an adequate rise time. Zhou and Kimura predicted the need for this controller singular value matching in the middle frequency range [21]; however, they conjectured that high frequency behavior would not affect robust performance greatly. This design indicates otherwise. The high frequency behavior could explain the nominal performance degradation seen in their design. Note the high frequency gain in the large plant gain direction produces the initial transient jumps seen in the time responses in Figures 3 and 4. One could argue that the initial jump resulting from this high frequency signal is used by the controller to test the system for uncertainty. The results of this high frequency transient shape the resulting response.

3.2.3 μ Analysis

Analyzing the D-K iterations of the μ design showed that the increase in high frequency gain which resulted from small values of $d(j\omega)$ at high frequencies decreased the robust performance curve but increased the robust stability curve. An increase in the delay would further increase the robust stability curve thus limiting robust performance. This indicates limitations based on the tradeoffs between robust stability and robust performance. If the delays were not present then the uncertainty weighting would be a constant. This would allow smaller values of $d(j\omega)$ at high frequencies which would increase controller gain and produce better robust performance with respect to the gain uncertainty.

3.3 Discrete-Time μ

All of the analysis and design work in the previous sections has been done in continuous time. Since most controllers are implemented digitally, and since the ℓ_1 methodology which will be investigated in Section 4 is based on discrete time systems, this section will deal with discrete-time versions of the previous work.

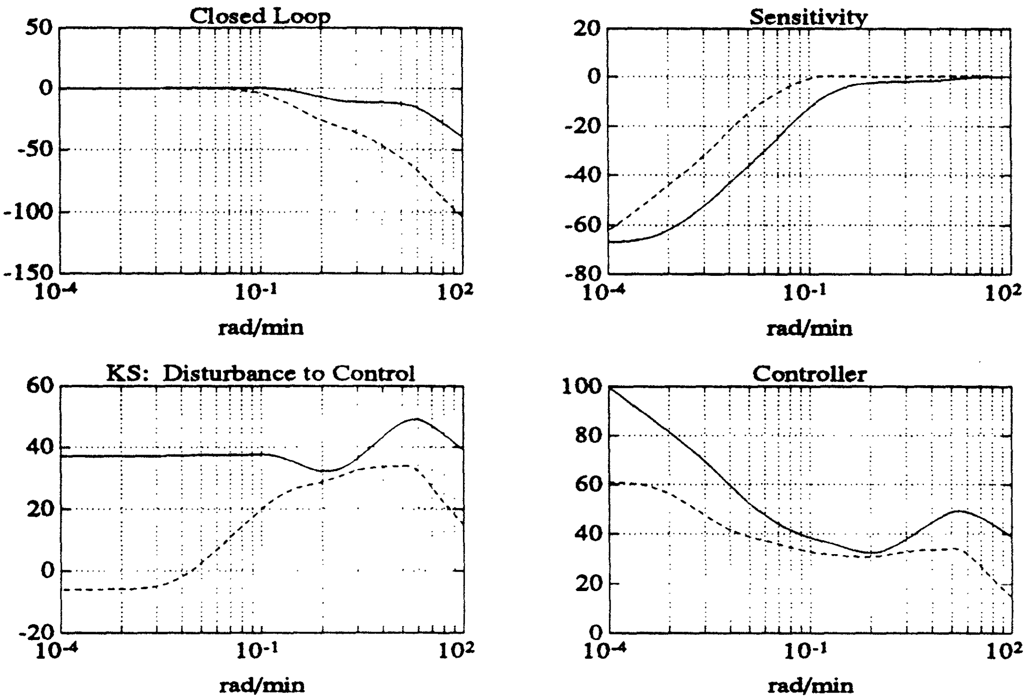


Figure 5: Max and Min Singular Values for Directionally Weighted Design

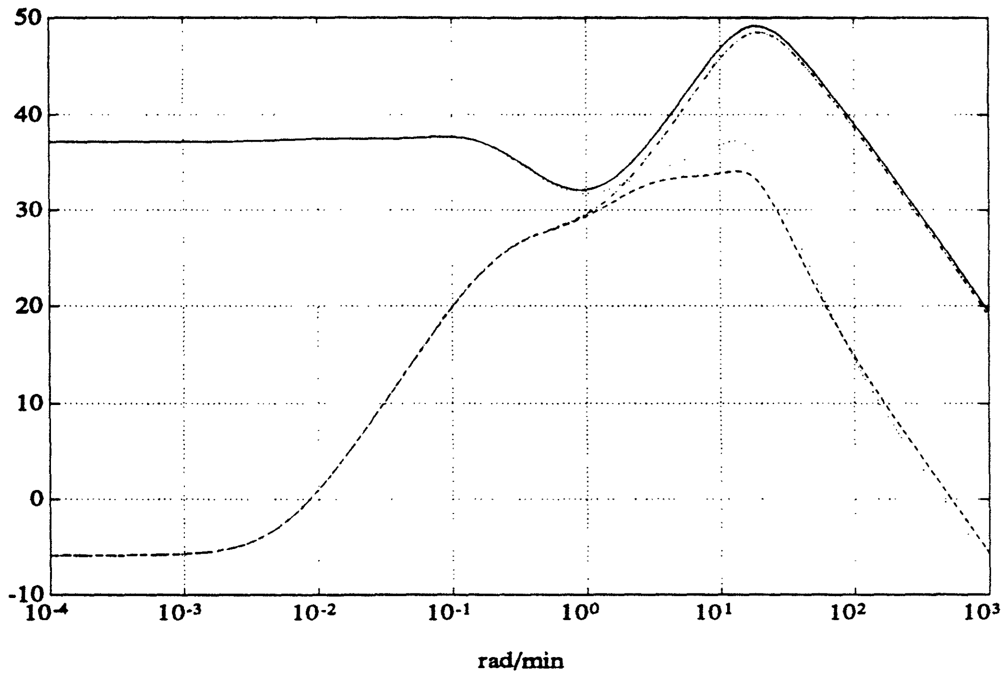


Figure 6: Disturbance to Control (KS) Maximum and Minimum Singular Values and Gains along Maximum and Minimum Plant Directions

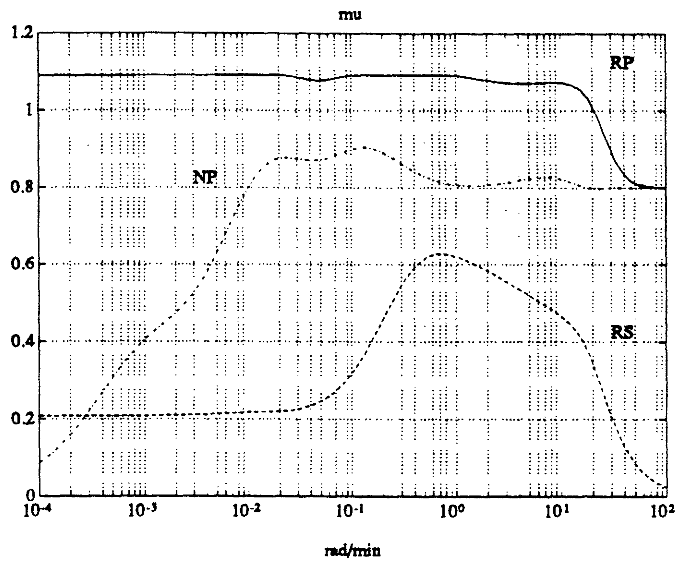


Figure 7: μ —— , Nominal Performance –. – . – . , Robust Stability – – –

Consider first discretizing the controller obtained from the continuous time design in Section 3.1.3. The choice of discretization method is very important for obtaining good performance, especially for long sampling periods. A standard sampled data system is implemented by augmenting the plant with a sampler at the output and a zero order hold at the input. An antialiasing filter of $\frac{a}{s+a}$ was used to filter the measurements where $a = \frac{\pi}{1.4T}$ and T is the sampling period. Two sampling periods were considered. The first was $T = 0.01$ minutes. With this small sampling period, the behavior of the continuous time controller was recovered. The next sampling period was $T = 1$ minutes. The performance of the sampled data system is expected to degrade with this longer sampling period. The sampling rate $T = 1$ minute was used in the ℓ_1 designs; therefore, the results for this sampling will be discussed further in this section.

The best and most obvious discretization method to use is the standard bilinear transformation $s = \frac{\frac{z}{T}-1}{\frac{z}{T}+1}$ in the sampled data system. When the controller was discretized for a sampled data system operating with $T = 1$ minutes, as expected, the level of performance degraded from the performance of the continuous time design. Figures 8 and 9 show step responses for plants with delays of 0 and 0.5 minutes. Note that these and all other time responses for sampled data systems are the output of the continuous time plant in the sampled data system. Discussion and comparison of discrete time μ responses to ℓ_1 designs can be found in Section 4.4.4.

Now consider designing a discrete-time controller directly, i.e. performing a discrete time μ

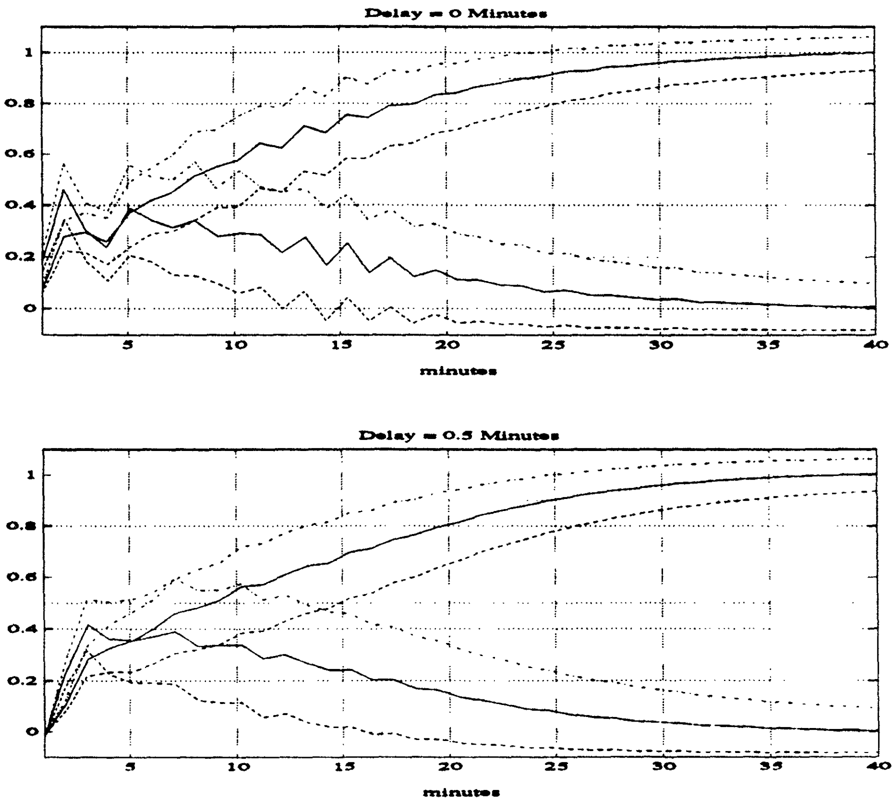


Figure 8: Discretization of Continuous-Time Design (Bilinear Transform) $T = 1$ Response to a Step in Direction $\Delta y_d = 1$ $\Delta x_B = 0$

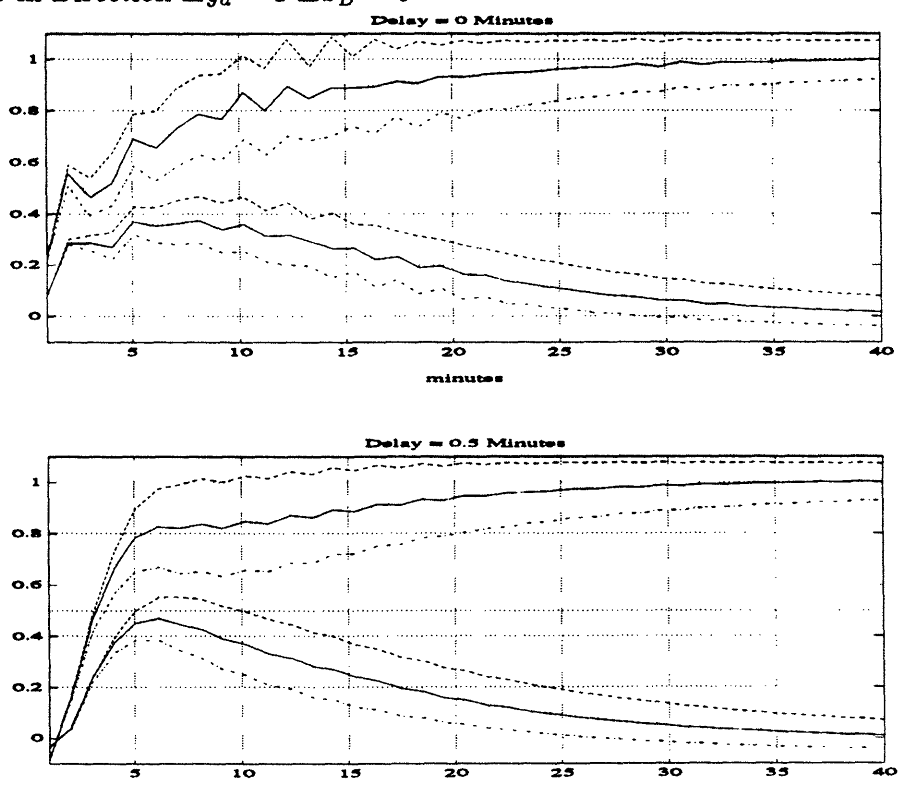


Figure 9: Discretization of Continuous-Time Design (Bilinear Transform) $T = 1$ Response to a Step in Direction $\Delta y_d = 0$ $\Delta x_B = 1$

design. This is accomplished by discretizing the design plant model which includes the proper continuous time weighting functions with a sampler and zero-order hold as described above (see Figure 14). The weights used in these designs are the same as for the directionally weighted design of Section 3.1.3; however, just because these weights produced good results for the continuous time system does not mean that other weights do not exist which will produce better responses for the sampled data system. The actual H_∞ design can be performed in continuous time by transforming the discrete time system back to continuous time using the bilinear transformation $z = \frac{s-1}{s+1}$. Once the continuous time μ design is completed, the discrete time controller is recovered by taking the bilinear transformation of the resulting continuous time controller.

Consider the effect of performing the design with the sampling period, $T = 1$ minute. Since the Nyquist rate corresponding to this sampling period, 3.142 rad/min occurs before the controller singular values roll off in the continuous time design, one would expect performance degradation. This is apparent in the achievable μ for this design configuration of 1.4931 (see Figure 13) as compared to 1.0914 for the continuous time design. There is a marked degradation in the step responses shown in Figures 10 and 11. Looking at the frequency domain, it appears that for low frequencies the resulting design is very similar to the continuous-time design. The bandwidth constraint imposed by the low sampling frequency forces the controller to roll off much sooner than in the continuous-time design. The lack of high frequency activity constrains the resulting performance.

4 Analysis and Design Using the ℓ_1 Norm

In recent years the use of the ℓ_1 system norm for control system design has been studied in detail, [2], [3], [5], [12], [7], [6]. Recently, through the work of Diaz-Bobillo and Dahleh [7], [6], a computational method for solving the ℓ_1 multiblock problem, the Delay Augmentation Algorithm was developed. The ℓ_1 minimization of the weighted sensitivity of the distillation column will be studied to learn more about nominal performance specifications. Comparisons will be made with the results of discrete time H_∞ minimizations of the weighted sensitivity. Next, robustness will be added by solving the ℓ_1 problem for the mixed sensitivity, KS problem. Finally, the ℓ_1 structured uncertainty problem, which contains the two previous problems as subproblems will be solved to obtain robust performance. It must be noted that due to the very recent development of the Delay Augmentation Algorithm, its use is limited by the computational

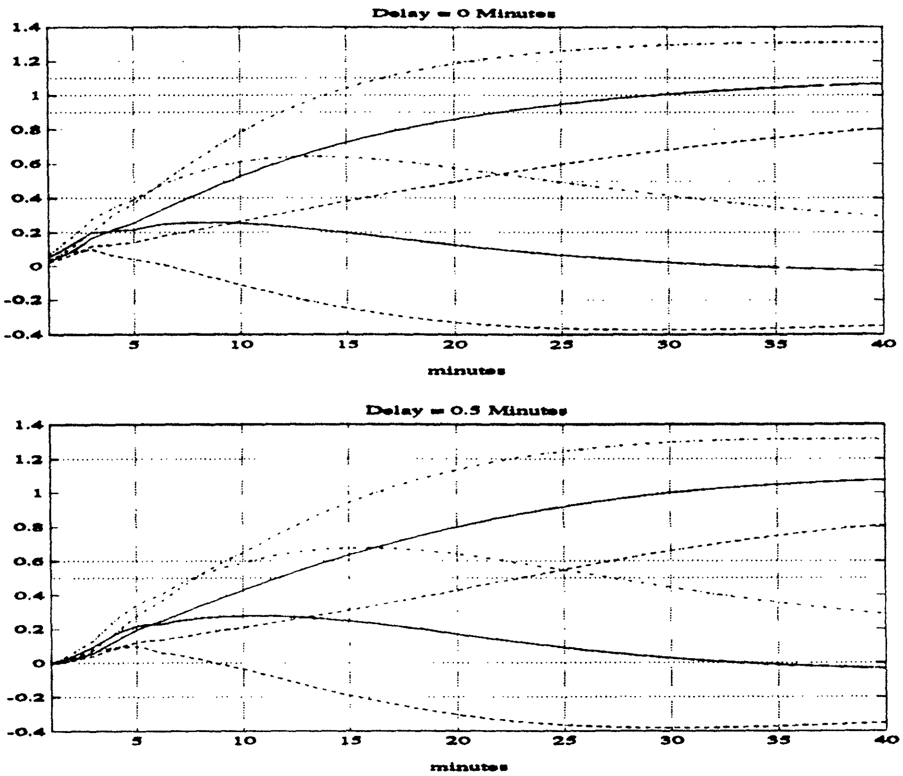


Figure 10: Discrete Time μ Design T = 1 min., Response to a Step in Direction $\Delta y_d = 1$ $\Delta x_B = 0$

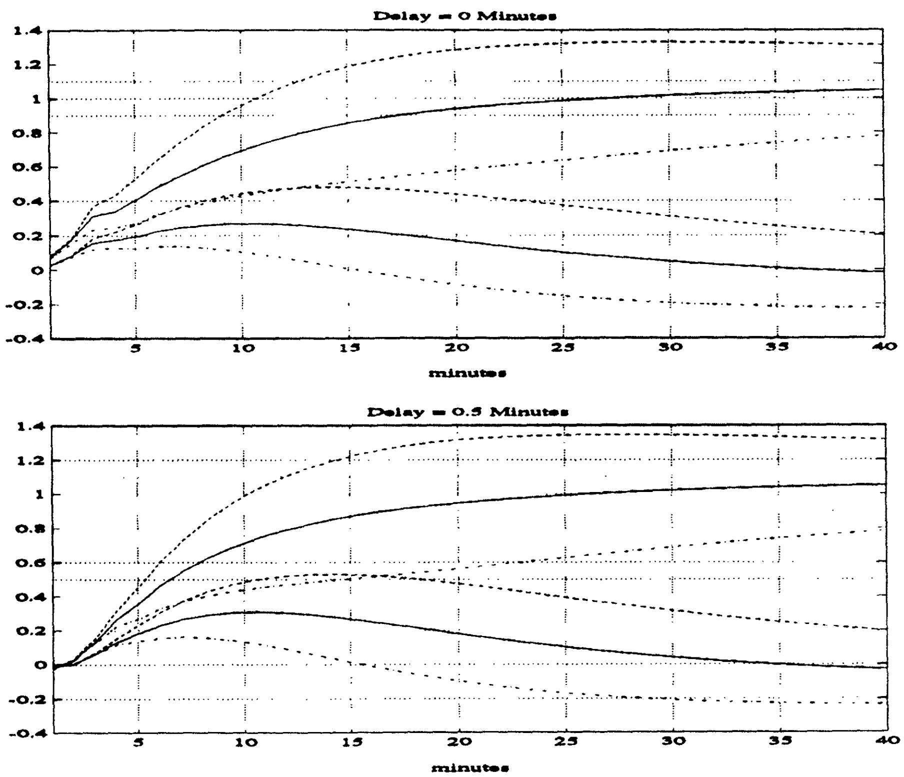


Figure 11: Discrete Time μ Design T = 1 min., Response to a Step in Direction $\Delta y_d = 0$ $\Delta x_B = 1$

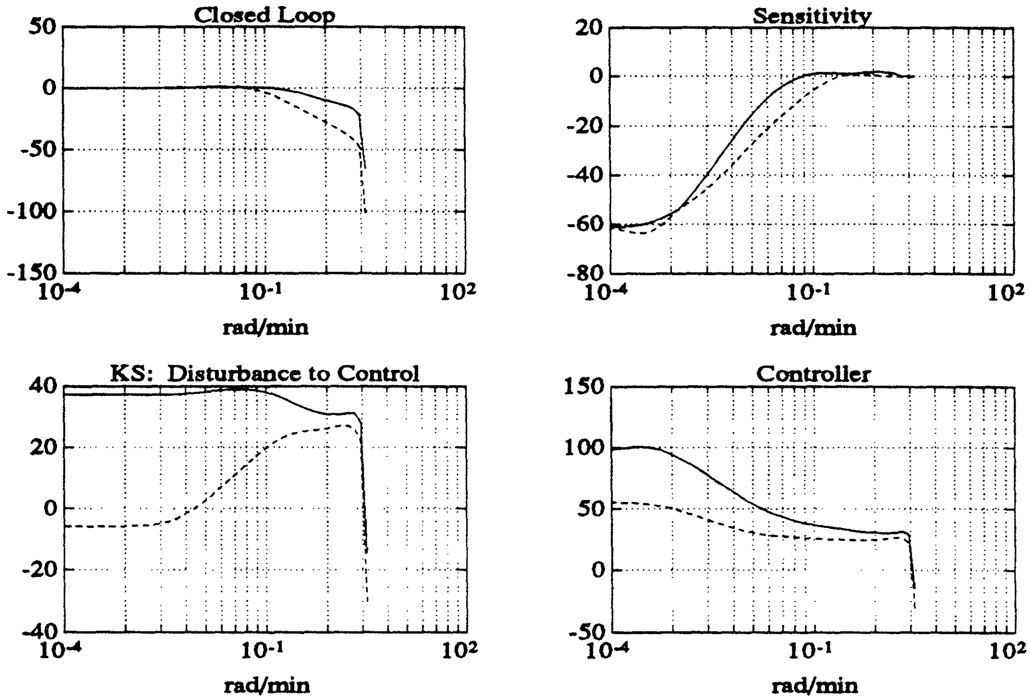


Figure 12: Max and Min Singular Values for Discrete Time μ Design, $T = 1$ minute

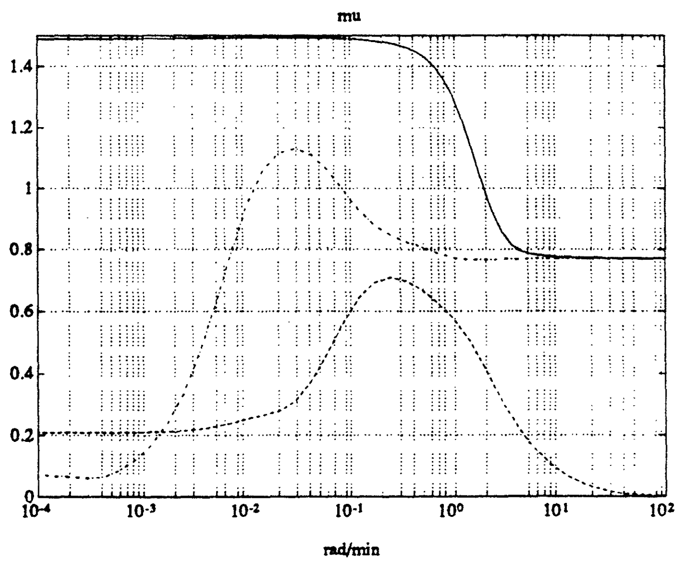
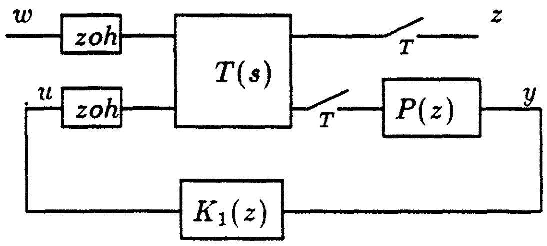


Figure 13: μ — , Nominal Performance –. – . – . , Robust Stability – – – Discrete Time μ Design for $T = 1$ minute



$$P(z) = \frac{z-0.0115}{z-1.0015} I_{2\times 2}$$

Figure 14: Discretized Design Plant Model

difficulty of the problem.

4.1 Formulation of Design Problems

The computational algorithm for solving the ℓ_1 problem requires the problem to be in discrete time. Since most physical systems operate in continuous time, the control problem must be translated into a sampled data system. This is accomplished by posing the control problem in the standard form which should include (in continuous time) all performance and norm bounded perturbation weights. The choice of weights for ℓ_1 designs will be discussed in subsequent sections. This formulation should also contain any necessary prefilters to model computational delays and anti-aliasing filters. Once this system is formed, the entire system is transformed into discrete time using a standard zero order hold continuous to discrete time transformation. Other transformations, depending on the method of implementation, may also be used.

A sampling period, T, must be chosen to perform the transformation. It is important to choose T small enough to capture the desired closed loop bandwidth; however, choosing T too small may result in poor convergence of the Delay Augmentation Algorithm. Choosing a very small T results in pushing the poles and zeros of the resulting discrete time system very close to the unit circle. Thus the resulting interpolation conditions will die out very slowly resulting in an optimal solution with large support. It was also conjectured in [7], [6] that the rate of decay of stable zeros of $\hat{U}(\lambda)$ (where $\Phi = \hat{H} - \hat{U}\hat{Q}\hat{V}$ and the objective is to minimize $\| \Phi \|_1$) can determine the rate of convergence of the upper bound, $\overline{\nu}_N$, to the optimal ν° in the Delay Augmentation Algorithm.

All discrete time designs performed on the distillation column plant were performed at a sampling rate of 1 minute with prefilter used to model delay effects of $\frac{s-2.2}{s+2.2}$. It is desired to add a precompensator to increase the low frequency gain of the plant so as to meet the steady state error specifications. Note that existence of the ℓ_1 solution is not guaranteed when interpolations occur on the unit circle. The specifications of this problem do not call for zero steady state error, so an exact integrator is not required. The precompensator $P(z) = \frac{z-0.0115}{z-1.0015} I_{2\times 2}$ is included as shown in Figure 14. This figure also shows the discretization procedure. Note that as was discussed in the previous paragraph, the steady state error specifications could have been met by including a stable pole very close to $z = 1$ in the weight on the sensitivity; however, the use of this method on the distillation column example seemed to produce slower convergence of $\underline{\nu}$ and $\overline{\nu}$ for multiblock problems using the Delay Augmentation Algorithm. After the design is completed, the factor $P(z)$ is lumped with the resulting controller, i.e. let $K_1(z)$ be the controller resulting from the ℓ_1 minimization problem, then the actual controller to be implemented will be $K(z) = K_1(z)P(z)$.

4.2 ℓ_1 Minimization of the Weighted Sensitivity, The Nominal Performance Problem

The results of minimizing the H_∞ norm for a one block problem like the weighted sensitivity are well known. The solution is all pass in the frequency domain. Thus frequency loop shaping ideas are very compatible with H_∞ minimizations. What is more difficult is translating time domain specifications into frequency domain loop shaping ideas. When using the ℓ_1 design procedure, little is known about how the choice of weighting functions affects the design. It is the goal of this section to investigate the effect of varying a parameter in a first order weighting function on the time domain and frequency domain characteristics of minimizing $\| WS \|_1$. Comparisons will be made with H_∞ minimizations as well. The numerical results of the H_∞ and ℓ_1 designs performed are summarized in Table 3.

Before the specific results of the distillation column are discussed consider what can be inferred about the solution of the weighted sensitivity problem from the fact that the solution of the one-block problem is polynomial in λ, [7] [6]. Let the finite pulse response of the optimal closed loop in the SISO case be given by $\Phi^o(\lambda) = W(\lambda)S(\lambda) = a_0 + a_1\lambda + \ldots + a_n\lambda^n$. This implies that the standard z-transform of Φ where $z = \lambda^{-1}$ is $\Phi^o(z) = W(z)S(z) = \frac{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}{z^n}$ Suppose

$$W(z) = W_0 \sum_{i=1}^{m} \frac{z_i z - 1}{p_i z - 1} \tag{11}$$

then

$$S(z) = (W(z))^{-1}\,\Phi^o(z) = \frac{1}{W_0}\left(\sum_{i=1}^{m} \frac{p_i z - 1}{z_i z - 1}\right) \frac{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}{z^n} \tag{12}$$

Thus the poles of $W(z)$ will appear as zeros of the sensitivity and unless zeros of $W(z)$ occur naturally as zeros of the optimal FIR solution, they will appear as poles in the optimal sensitivity. The optimal sensitivity contains the inverse of the weight; however, unlike the solution of the H_∞ problem, it is multiplied by the optimal FIR system which can strongly affect the characteristics of the optimal sensitivity.

Consider now the results of the example problems given in Table 3, specifically, W_2^1. Note that this weight is nearly the identity. (Due to computational difficulties W_3^1 was used as an approximation of an identity weight.) Aside from the discrepancies due to the discretization process, the H_∞ solution tries to invert the weight. Note that minimizing $\| S \|_\infty$ does not produce a reasonable sensitivity function. In fact, including the multiplicative input perturbation with $\delta_1 = 0.2$ and $\delta_2 = -0.2$ results in an unstable system. Minimizing $\| S \|_1$, however does produce a reasonable sensitivity function. The results of this minimization can best be interpreted in the time domain using the fact the ℓ_1 norm is the induced ℓ_∞ norm. It is possible to show that

$$\| S \|_1 \geq 2 + 2e_1 + 2e_2 \tag{13}$$

where $e_1 > 0$ and $e_2 > 0$ are the overshoot and undershoot of the step response of the closed loop system to a unit step. As a result of this, minimizing the ℓ_1 norm of the sensitivity will tend to minimize the overshoot and undershoot of the resulting system. At the same time, the integrator is trying to force the output of the system to a step command to its final value as fast as possible.

Consider next the results of the minimizations for the weight W_1^1, a low pass function. As expected, the result of the H_∞ design is a sensitivity function that inverts the weight. The results of the ℓ_1 designs were dramatically different from the H_∞ design. The response is much quicker and overshoot is now present. This can be explained via frequency domain and time domain arguments. Consider the frequency content of the weight. The larger magnitude of the weight at low frequencies results in a sensitivity with lower magnitude at these frequencies and therefore a closed loop with a larger bandwidth. This produces the fast rise time. Extending the time domain interpretation of minimizing the ℓ_1 norm of the sensitivity discussed in the previous paragraph, one can explain the results of this minimization. Weighting the sensitivity can be interpreted as filtering the output. The filters (weights) being considered here are low pass and will therefore attenuate overshoot. It is therefore reasonable to expect that the step

Minimization of $\| WS \|$								
Weight	H_∞ Design			ℓ_1 Design				
	$\| WS \|$		ord(K)	$\| WS \|$		ord(K)	max supp	
	H_∞	ℓ_1		H_∞	ℓ_1			
$W_1^1(s) = \frac{s+0.08}{s+0.0012}$	1.0373	4.2327	10	1.8760	2.0748	10	4	
$W_2^1(s) = \frac{s+0.78}{s+0.8}$	1.0080	5.5262	10	1.8369	2.0040	10	4	
$W_3^1(s) = \frac{s+0.08}{s+1.8}$	1.6682	8.4548	10	1.6861	2.0009	10	4	

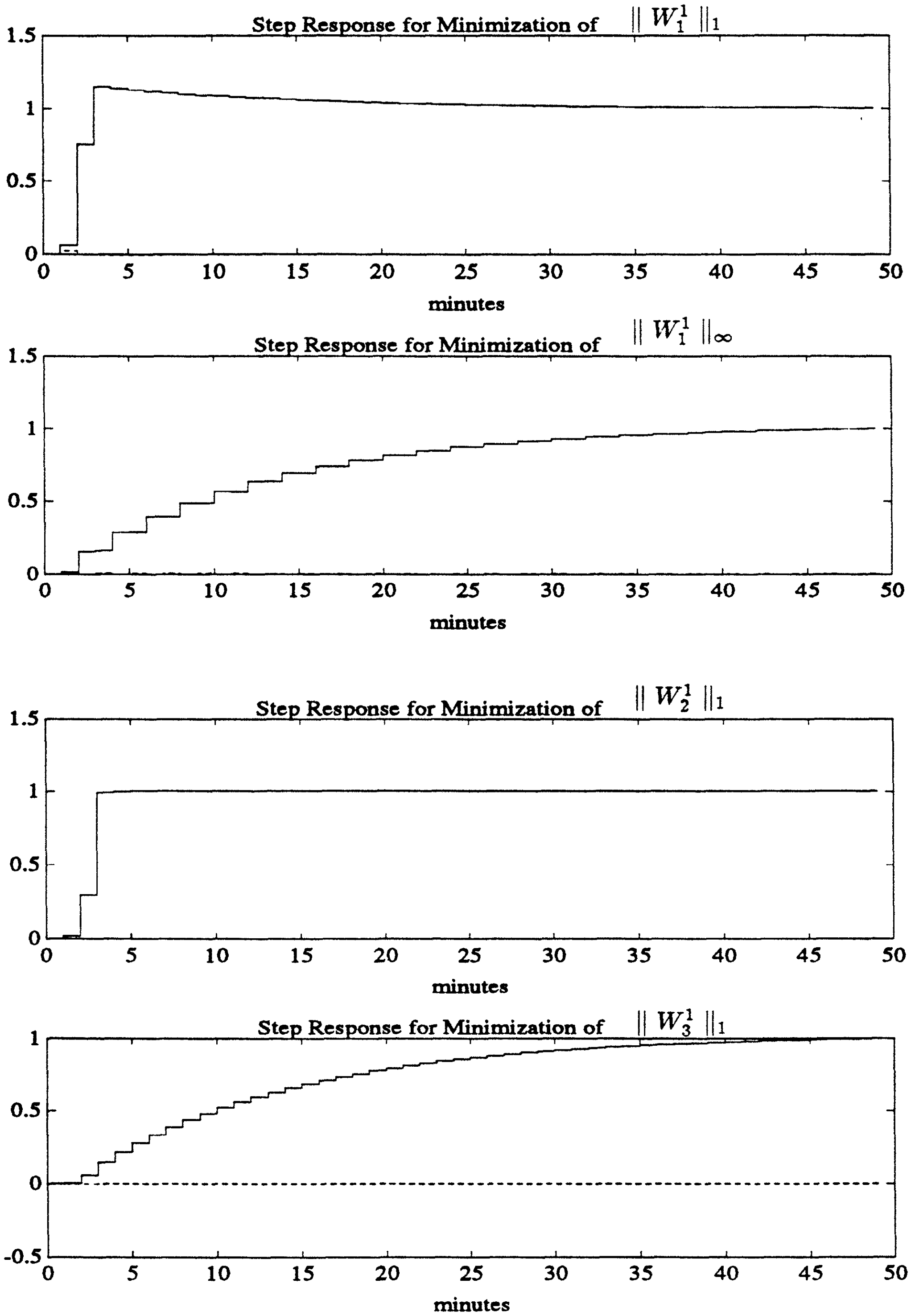
Table 3: Comparison of H_∞ and ℓ_1 Minimizations of $\| WS \|$

response of the actual closed loop systems resulting from these minimizations will have more overshoot and faster rise times since it is the filtered sensitivity on which the minimization is performed. Note that the closed loop sensitivities do contain a pole at $e^{-0.08}$ as a result of the zero of the weights. The effect of this pole can be seen in the decay rate of the overshoot.

Finally consider the results of the minimizations for the weight W_3^1, a high pass function. The result of the H_∞ minimization with this weight does not produce a desirable sensitivity function (see Figure 17). The result of the ℓ_1 minimization shows an effect opposite to that seen for an ℓ_1 minimization with a low pass weight. Rise times increase and overshoot decreases as the magnitude of the pole increases. The increase in rise time is limited by the rise time of a system with a dominant pole at $e^{-0.08}$ resulting from the zero of the weight. Duals of the ideas presented in the previous paragraph for low pass weights as filters can be used to explain this behavior. It is very interesting to compare the similarities of this design with the H_∞ design for the weight W_1^1. Also note the frequency response of the sensitivity for the ℓ_1 design (Figure 16; it looks nothing like the inverse of the weight.

4.3 Minimization of $\left\| \begin{matrix} W_1 KS \\ W_2 S \end{matrix} \right\|_1$

It is desired to add robustness to the weighted sensitivity minimization designs. Recall that by using a high pass weighting function in the sensitivity minimization, the overshoot is decreased and the rise time is increased. While the resulting nominal design was good, the effect on the robustness was to dramatically increase the time to reach steady state for plants other than the nominal in the model set. In this section a high pass weight on the transfer function KS, $\frac{2.3123(s+.09)}{s+1}I$ from Section 3.1.1, will be used to limit high frequency gain and a low pass weighting function, $\left(.5\frac{s+.08}{s+.01}\right)I$ will be used to weight the sensitivity. This low pass function should result in a design with a fast nominal and robust rise and settling times and the weight



Input of $\Delta y_D = 1, \Delta x_B = 0$

Figure 15: Various Time Responses for Minimization of $\| WS \|$

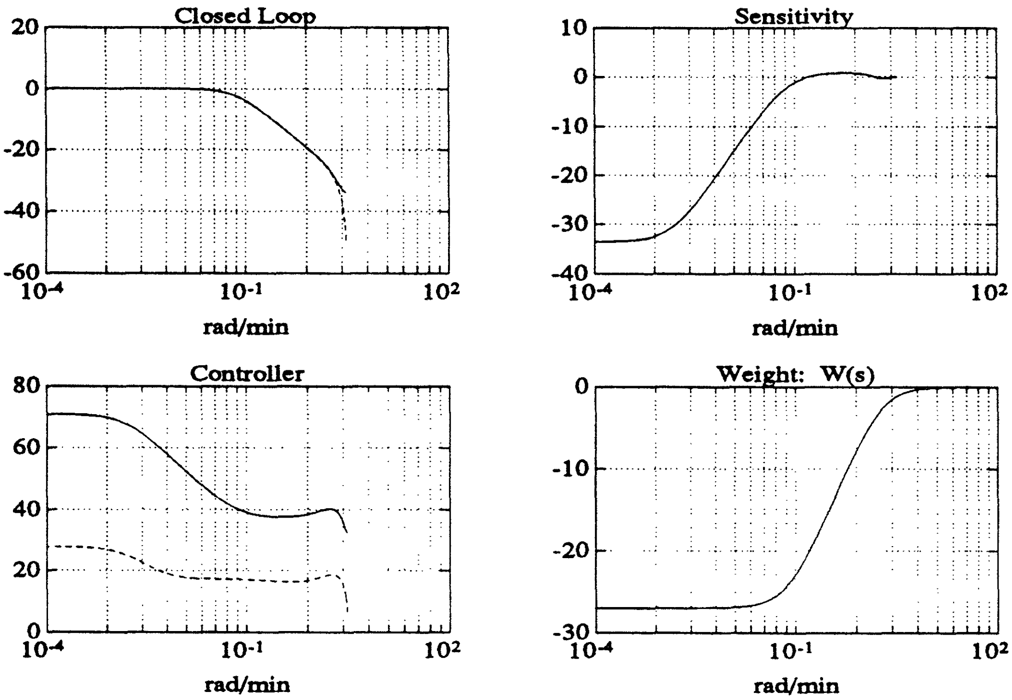


Figure 16: Max and Min Singular Values for Minimization of $\| W_3^1 S \|_1$

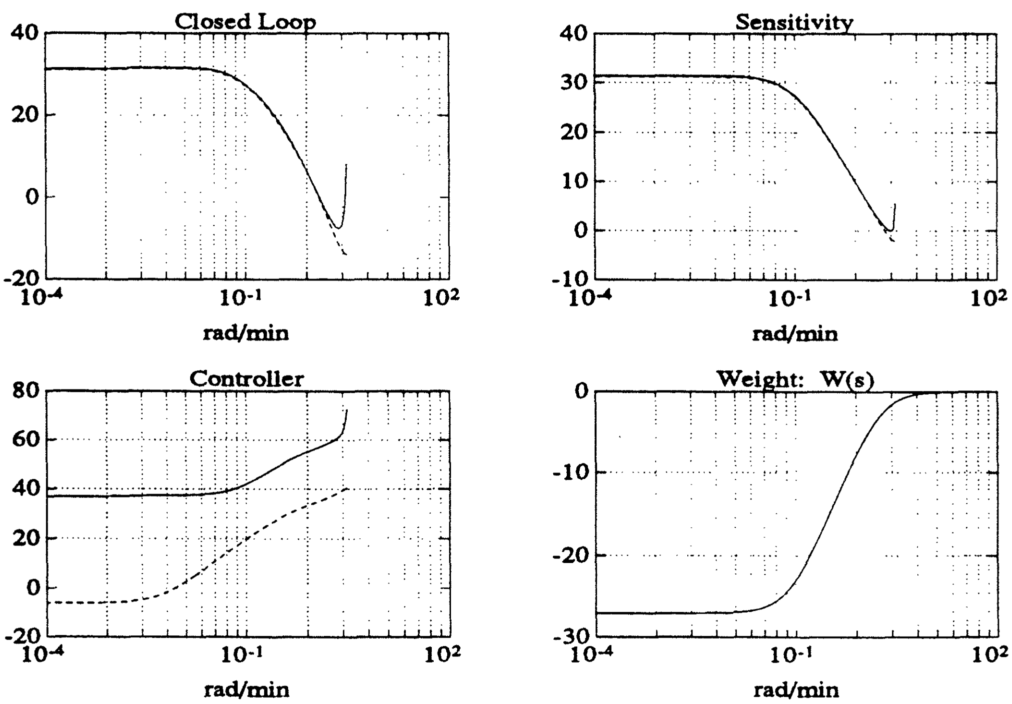


Figure 17: Max and Min Singular Values for Minimization of $\| W_3^1 S \|_\infty$

on KS should reduce the amount of overshoot.

Minimizing $\left\| \begin{matrix} W_1KS \\ W_2S \end{matrix} \right\|_1$ for the choice of weights described in the previous paragraph results in the one-block partition W_1KS being totally dominant as defined in [7], [6]. This means the controller that solves this one block problem, also solves the column problem. The support of W_1KS is small (support of each entry less than 4) and the resulting controller is also of low order (10). Physically, this means that W_1KS is strongly restricted; this also limits the controller action. Since the elements of W_2S do not factor into the minimization at all, the performance of the resulting system is poor with very long modes.

The weight on KS will be scaled so as to obtain more interaction between the partitions. It is hoped that this interaction will result in a system with the nominal performance specified by the choice of W_2 and with increased robustness due to the constraint on KS. Many choices of scaling factors, γ, were tried and the final choice was $\gamma = 0.0007$. Values of γ much smaller than this did not produce significant improvements in robustness with respect to overshoot. Values of γ larger than 0.0015 produced systems with poor robustness with respect to the time for a step response to reach steady state. Figures 18 - 20 show time responses of the resulting closed loop system (nominal and perturbed). Note that the robust performance of the system is better than for the unscaled problem (W_1KS totally dominant) and for weighted sensitivity problem ($\gamma = 0$, W_2S totally dominant). The response of the system to inputs in some directions and uncertainty combinations are better than others.

Consider next the support structure of the problem, Table 4.3. Even though the weight on KS is very small, the structure of the solution of the column problem is very different from the structure resulting from minimizing $\| W_2S \|_1$ or $\| W_1KS \|_1$. The support is much longer. This is because there is not a subproblem whose solution is polynomial with small order. Since Φ_{11} has large support, the controller, which is a function of Φ_{11}, will have large order. This increase in order also increases the numerical difficulty of solving the problem. The difficulty lies in recovering the controller from the closed loop system resulting from the Delay Augmentation iteration. Included in Table 4.3 is a trial with rows $(W_2S)_2$ and $(W_1KS)_1$ interchanged when applying the Delay Augmentation Algorithm. Initial iterations of the Delay Augmentation Algorithm indicated, via the support structure ([7], [6]), that this move should be made to improve convergence; however the row $(W_1KS)_1$ does not retain its small support when moved into Φ_{11}, and the convergence is actually slowed. Later iterations indicate a polynomial structure for $(\Phi_{11})_1 = (W_2S)_1$; however $(\Phi_{11})_2 = (W_2S)_2$ either has a very long polynomial structure or

$\nu^\circ = 1.0527$							
k	N	$\underline{\nu}$	$\overline{\nu}$	Φ_{11}	$\Vert (W_2S)_1 \Vert_1$ $\Vert (W_2S)_2 \Vert_1$ $\Vert (\gamma W_1KS)_1 \Vert_1$ $\Vert (\gamma W_1KS)_2 \Vert_1$	ord K	supp $\begin{pmatrix} W_2S \\ (\gamma W_1KS)_1 \end{pmatrix}$
† 25	22	1.0527	1.0581	W_2S	$\begin{bmatrix} 1.0528 * \\ 1.0549 * \\ 1.0581 \\ 1.0016 \end{bmatrix}$	51	$\begin{bmatrix} 20 & 4 & * \\ 19 & 20 & * \\ 17 & 3 & \end{bmatrix}$
25	22	1.0042	3.2602	$\begin{bmatrix} (W_2S)_1 \\ (\gamma W_1KS)_1 \end{bmatrix}$	$\begin{bmatrix} 1.0167 * \\ 3.2602 \\ 1.0073 * \\ 0.5079 \end{bmatrix}$		$\begin{bmatrix} 6 & 2 & * \\ 13 & 3 & \\ 22 & 22 & * \end{bmatrix}$
50	47	1.0528		W_2S			$\begin{bmatrix} 23 & 4 & * \\ 46 & 45 & * \\ 43 & 43 & \end{bmatrix}$

* indicates this row is an element of Φ_{11}

† indicates the best numerical solution of the problem

Table 4: Summary of Results of Scaled Problem: $\gamma = 0.0007$

does not have polynomial structure. As a result $\Phi_{11} = W_2S$ does not fit the definition of partial dominance; however, convergence is faster when this choice of Φ_{11} is made. The convergence of the Delay Augmentation Algorithm is slower and controller order is higher for problems, such as this one where there appears to be no totally or partially dominant one-block partition.

4.4 Structured Uncertainty

Now the concept of structured uncertainty and its use in the robust performance problem that was introduced in Section 3 in the H_∞ setting will be discussed in the ℓ_1 setting. The basic idea is the same: show the robust performance problem is equivalent to a robust stability problem then solve the robust stability problem. The difference in this section is the class of signals for which robustness is desired, i.e. ℓ_∞ (bounded magnitude) instead of ℓ_2 (bounded energy). The change in objectives results in a change in the condition (from μ) for robust stability of the modified system with respect to the structured uncertainty. Also, to obtain necessity from this condition, the class of allowable perturbations must be enlarged from the class of LTI systems (Section 3) to either the class of linear time varying (LTV) or nonlinear time-invariant (NL) perturbations. The theory behind these ideas is described in detail in [12], [13], [14], [1], and [4].

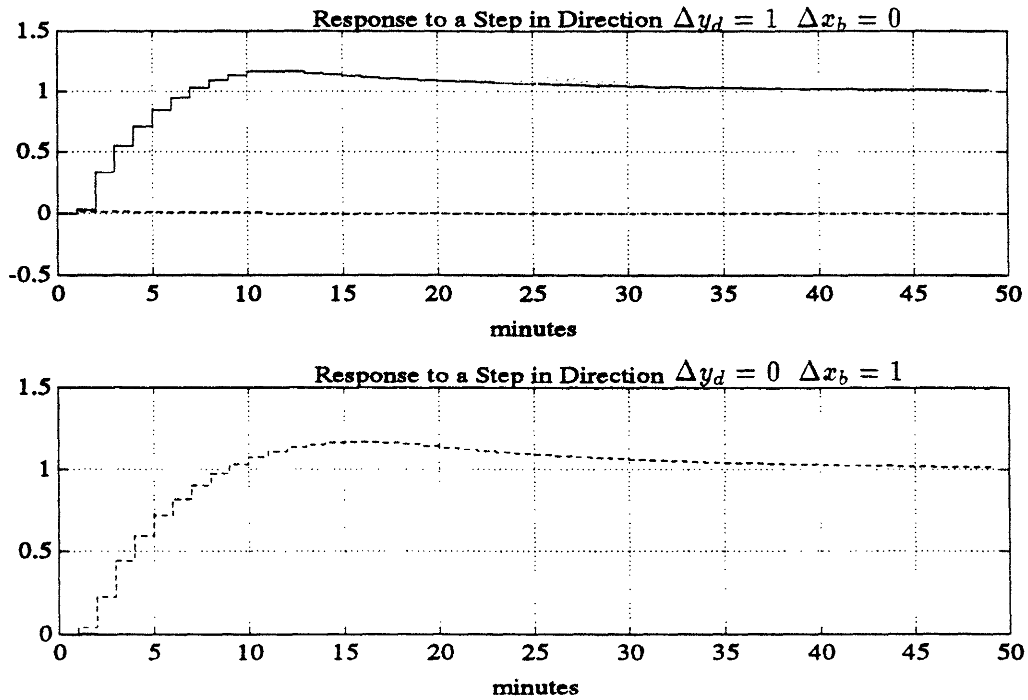


Figure 18: Time Responses for Minimization of $\| \, [\gamma W_1 K S \;\; W_2 S]' \, \|_1$ for the Nominal Plant Model

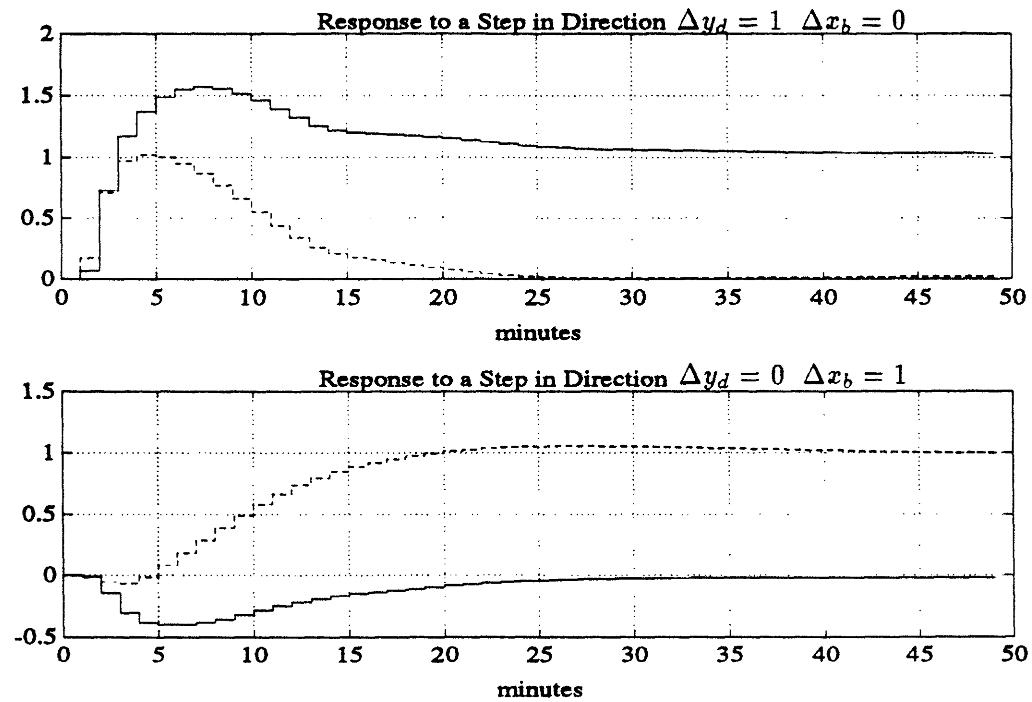


Figure 19: Time Responses for Minimization of $\| \, [\gamma W_1 K S \;\; W_2 S]' \, \|_1$ for the Plant Model $\delta_1 = 0.2$, $\delta_2 = -0.2$, $\tau = 0$

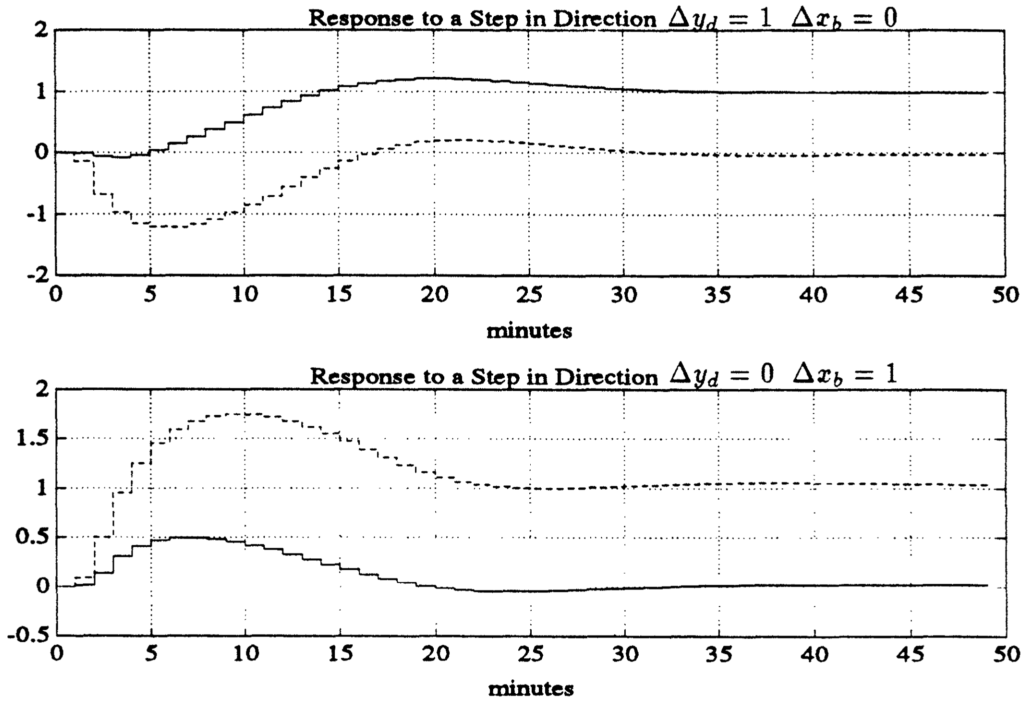


Figure 20: Time Responses for Minimization of $\| \, [\gamma W_1 KS \;\; W_2 S]' \, \|_1$ for the Plant Model $\delta_1 = -0.2$, $\delta_2 = 0.2$, $\tau = 0$

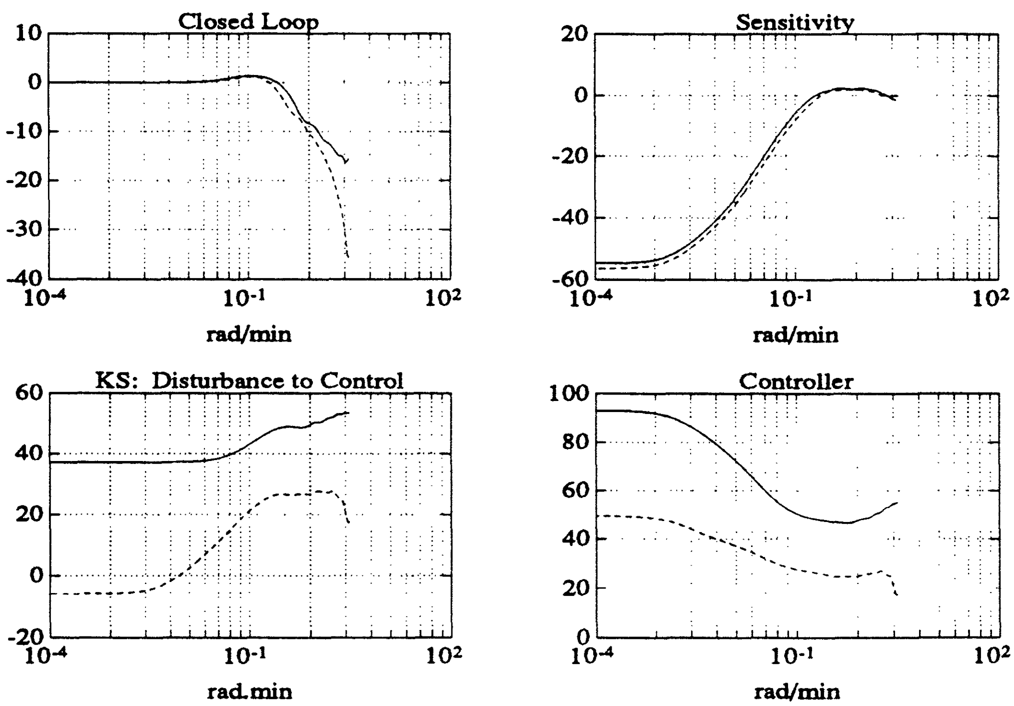


Figure 21: Max and Min Singular Values for Minimization of $\| \, [\gamma W_1 KS \;\; W_2 S]' \, \|_1$

4.4.1 Application to the Distillation Column

It was not possible to produce an acceptable level of robust performance with the weighted sensitivity (Section 4.2) and mixed sensitivity, KS (Section 4.3) designs, thus the need to use structured uncertainty. The procedure used in these sections did not explicitly include a robust performance objective. The Δ structure for this problem (as in Section 3) is shown in Figure 2 with Δ_1 and Δ_2 linear time varying (LTV) and $\| \Delta \|_1 \leq 1$. The perturbation blocks can be pulled out to form the closed loop M matrix,

$$\begin{aligned} M &= \begin{bmatrix} W_1C & W_1KS \\ W_2SG & W_2S \end{bmatrix} \\ &= \begin{bmatrix} -W_1(I+KG)^{-1}KG & W_1K(I+GK)^{-1} \\ -W_2(I+GK)^{-1}G & W_2(I+GK)^{-1} \end{bmatrix} \end{aligned} \tag{14}$$

The weight W_2 used to specify performance was given by $W_2(s) = 0.5\frac{s+0.08}{s+0.007}$. We are concerned with the performance of the closed loop system, specified by $\| W_2S \|_1 < 1$, with respect to step inputs (ℓ_∞ signals). This nominal performance problem is equivalent to the robust stability problem formed by including the perturbation Δ_2, LTV with $\| \Delta_2 \|_1 < 1$ as shown in Figure 2. Note that the discretization of the system is performed as described in Section 4.1. The weight, W_2, was chosen after analyzing the results from Sections 4.2 and 4.3. The zero of the weight was chosen to reflect the slowest desired mode of the closed loop. As a result of the work on the mixed sensitivity-KS problem, W_2 was chosen to be a low pass function and the pole was chosen after iterations of the design procedure to specify the resulting speed of the closed loop step responses as discussed in 4.2. At first the function $W_2(s) = 0.5\frac{s+0.08}{s+0.01}$ was used; however, the rise times of the resulting closed loop systems for $G_{LV} \in \mathcal{G}$ were too slow and reducing the magnitude of the pole decreased the rise time.

The uncertainty will be modeled as an input multiplicative uncertainty as in Section 3.1.1. The choice $W_1 = \frac{2.3123}{2}\frac{s+09}{s+1}$ was made to reflect the frequency content of the uncertainty. The scaling factor of 0.5 was included after several trial designs to reduce the conservatism. Note that the possible perturbations as discussed previously are LTV while the physical uncertainty is LTI. This will also add to the conservatism of the design.

The Delay Augmentation Algorithm was applied to the system $D^{-1}MD$. This minimization problem is a large and difficult problem. First the proper D matrix must be chosen to achieve the best robust performance. Currently there is no good, systematic method to jointly optimize over D and K. The multiplication of $D^{-1}MD$, where $D = \text{diag}(1,\ 1,\ d_2,\ d_2)$ results in the following matrix:

$$D^{-1}MD = \begin{bmatrix} W_1C & d_2W_1KS \\ \frac{1}{d_2}W_2SG & W_2S \end{bmatrix} \tag{15}$$

From the analysis in Section 4.3 and Section 3, it is clear that the optimal d_2 will be much less than one. In fact after varying the value of d_2 the value $d_2 = 0.018$ was finally chosen. Note that a D–K iteration procedure was not used; presumably a better design is possible. The order of the inputs was rearranged so that $\Phi_{11} = KS$ to obtain the best numerical solution with the Delay Augmentation Algorithm.

As previously stated, this is a large problem which poses several computational difficulties. The first difficulty is the sheer size of the problem. $D^{-1}MD$ is a 16 element system. Applying the Delay Augmentation Algorithm to this system for values of k greater than 40 produces linear programs with the number of variables on the order of 10^3 which must be solved to a high degree of accuracy. The second difficulty arises from the effect of the D scalings on the problem. Recall in Section 4.3 that including the scaling factor γ in the mixed sensitivity-KS problem resulted in slower convergence of the Delay Augmentation Algorithm, longer support of the optimal solution, and increased order of the resulting controller; however, the robust performance was improved. The D scalings in the structured uncertainty problem play a role analogous to the scaling factor γ in the mixed sensitivity-KS problem. It is conjectured that given a structured uncertainty problem with a partially dominant subproblem (defined in [7], [6]), including D scalings to improve robust performance can reduce the effect of the partial dominance and result in slower convergence of the algorithm and increased order of the controller. Another possible cause for the slow convergence of the algorithm is the inaccuracies in the computation of the controller. These inaccuracies cause two problems. The first problem is that the upper bound is computed from the ℓ_1 norm of the closed loop system containing this controller; therefore, the upper bound is inaccurate. The second, and possibly more important problem deals with the choice of Φ_{11} in the Delay Augmentation Algorithm. In [7] and [6], it was shown that the choice of the elements of the multiblock problem that become Φ_{11} (of dimension number of controls by number of measurements) plays an important role in the convergence properties of the Delay Augmentation Algorithm; the order of the inputs and outputs should be arranged so that totally or partially dominant one-block partitions of the multiblock problem are located

$W_1(s) = 1.156\frac{s+0.09}{s+1}$ $W_2(s) = 0.5\frac{s+0.08}{s+0.007}$ $D = \mathrm{diag}(1,\ 1,\ 0.018,\ 0.018)$					
k	N	$\underline{\nu}$	$\overline{\nu}$	$\mathrm{supp}(D^{-1}MD)$	$\mathrm{ord}(K)$
10	4	2.0890	309.2424	$\begin{bmatrix} 8 & 8 & 6 & 3 \\ 8 & 8 & 3 & 7 \\ 8 & 8 & 4 & 4 \\ 7 & 7 & 3 & 5 \end{bmatrix}$	39
25	11	2.3207	14.7542	$\begin{bmatrix} 22 & 11 & 10 & 21 \\ 18 & 15 & 11 & 8 \\ 22 & 22 & 10 & 19 \\ 18 & 21 & 7 & 11 \end{bmatrix}$	54
55	25	2.3778	6.0082	$\begin{bmatrix} 42 & 50 & 34 & 19 \\ 40 & 40 & 35 & 26 \\ 47 & 44 & 26 & 25 \\ 29 & 30 & 25 & 24 \end{bmatrix}$	118

Table 5: Numerical Results of ℓ_1 Minimization of $D^{-1}MD$

in the Φ_{11} position. The only choice of Φ_{11} for which the controller could be computed with a reasonable degree of accuracy was $\Phi_{11} = W_1KS$. Thus it is possible that another choice of Φ_{11} could improve convergence.

Numerical results from the minimization of DMD^{-1} using the Delay Augmentation Algorithm are shown in Table 5. The disparity in the upper and lower bounds of the norm in the final iteration is a result of the computational difficulty of the problem discussed in the previous paragraph. One can see, however, that the upper and lower bounds are in fact converging. Note the very large support structure. This indicates that the optimal controller will have large order; however, using model order reduction techniques, the order can be reduced.

4.4.2 Adding Directional Weights

Many of the time responses of the structured uncertainty design with W_1 and W_2 given in the previous section meet the specifications stated in Appendix B (see Figures 22 and 23. This is an improvement over the designs obtained from the mixed sensitivity - KS problem; however there is room for even more improvement. Consider trying to improve the robust performance by adding directionality to the performance weight as was done in Section 3 for μ designs. Following a similar procedure, choose a weight which has the directionality of the sensitivity found from

$W_1(s) = 1.156\frac{s+0.09}{s+1}$		$W_2(s) = .5W_d\frac{s+0.08}{s+0.007}$		$D = \mathrm{diag}(1,\ 1,\ 0.018,\ 0.018)$	
k	N	$\underline{\nu}$	$\overline{\nu}$	$\mathrm{supp}(D^{-1}MD)$	$\mathrm{ord}(K)$
55	25	2.1491	5.2906	$\begin{bmatrix} 38 & 25 & 46 & 37 \\ 42 & 35 & 35 & 49 \\ 45 & 45 & 26 & 25 \\ 47 & 45 & 25 & 25 \end{bmatrix}$	139

Table 6: Numerical Results of ℓ_1 Minimization of $D^{-1}MD$ with Directional Performance Weight

the structured uncertainty design with diagonal performance weighting. Choose

$$W_2(s) = 0.5\frac{s+.08}{s+.007}W_d \tag{16}$$

where

$$W_d = \begin{bmatrix} 0.9631 & -0.0776 \\ -0.0776 & 0.8369 \end{bmatrix} = \begin{bmatrix} 0.9030 & -0.4296 \\ -0.4296 & -0.9030 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9030 & -0.4296 \\ -0.4296 & -0.9030 \end{bmatrix}' \tag{17}$$

Note that as in Section 3, the effect of this directional weighting is to weight the performance in the low plant gain direction (where robust performance is more difficult to obtain) more heavily than in the high plant gain direction (where robust performance is more easily obtained). One should not expect a plant with a high degree of directionality to have consistent performance in all directions. This should be reflected in W_2 which is used to specify performance.

The Delay Augmentation Algorithm was applied to this new system. Since this problem is similar to the previous problem, it is not surprising that properties of the solution, such as the support structure, and slow convergence have not changed, see Table 6.

4.4.3 Analysis of Results

Time responses for the ℓ_1 design with directional performance weighting are shown in Figures 24 - 25. The inclusion of directionality in the performance weight does improve the robust performance. Comparing individual time responses shown, tradeoffs between the performance for different combinations of uncertainty are apparent. When considering no delay, the overshoot and rise time specifications listed in Appendix B are met for most gain uncertainty combinations and not greatly exceeded when not met. The worst overshoot was $\Delta y_D = .54$ compared to the specification of 0.5 for the command $\Delta y_D = 0$ $\Delta x_B = 1$ with an uncertainty of $\tau = 0$, $\delta_1 = -0.2$,

$\delta_2 = 0.2$. The worst rise time was $t = 44$ minutes compared to the specification of 30 minutes for the command $\Delta y_D = 1$ $\Delta x_B = 0$ and an uncertainty of $\delta_1 = 0.2$, $\delta_2 = -0.2$. The overshoot specification on the output being commanded to 1 to be less than 1.1 has been met for all combinations of gain uncertainty. When a nonzero delay is included, the overshoot increases. As in Section 3 some responses push the overshoot constraints and have no trouble with the rise time constraint, while others behave in the opposite manner. Steady state error of the responses is slightly greater than specifications (see Figure 26); however, it can be adjusted with little or no affect on the rest of the response by making the pole of the precompensator, $P(z)$, closer to 1. Recall that a pure integrator could not be used in the design procedure. The increase in robust performance can also be seen from the value of ρ for each of the resulting designs. Recall that ρ [1], [12], [13], [14], [4], provides a necessary and sufficient test for robust stability and robust performance of systems with LTV perturbations and signals measured by the ℓ_∞ norm. The goal of the designs is to minimize ρ. The value of ρ for the design with a diagonal performance weight was 3.9324, while when directional information was included $\rho = 3.5168$.

The inclusion of directional information in the weight affects the frequency domain characteristics of the solution as well (see Figure 28). The singular values of the controller are more closely matched around 1 rad/min when the directional weight is used. Recall that this was also the case when directional performance weights were used for the μ design. The sampled data systems formed with both ℓ_1 structured uncertainty designs (with and without directional performance weight) were robustly stable to the actuator gain and phase perturbations described in Appendix A. Figure 27 shows the singular values of the discrete time system seen by the perturbation, C (resulting from the design with directional performance weighting) and the inverse of w_1 which bounds the uncertainty in Equation 5. Since $\sigma_{max}\left(\frac{1}{w_1}\right) \geq \sigma_{max}(C)$, the sampled data system will be robustly ℓ_2 stable to the LTI perturbations described in Appendix A. When diagonal performance weighting is used, the stability test is passed, but with a smaller margin. Suppose we wished to consider how the system would behave if the perturbations were actually time varying. For example, suppose that the actuator gain and delay uncertainty were not simply an unknown constant, but varied with time. This would result in a time varying perturbation. By applying the definition of the ℓ_1 norm to Δ defined in Equation 4, it is easy to show that $\| \Delta \|_1 \leq 2.2$, For robust ℓ_∞ stability to LTV perturbations, $\| C \|_1 < \frac{1}{2.2}$ is required; however, for the design with diagonal performance weighting $\| C \|_1 = 2.1434$ and when directional information is included in the performance weight, $\| C \|_1 = 1.9588$. Thus both systems

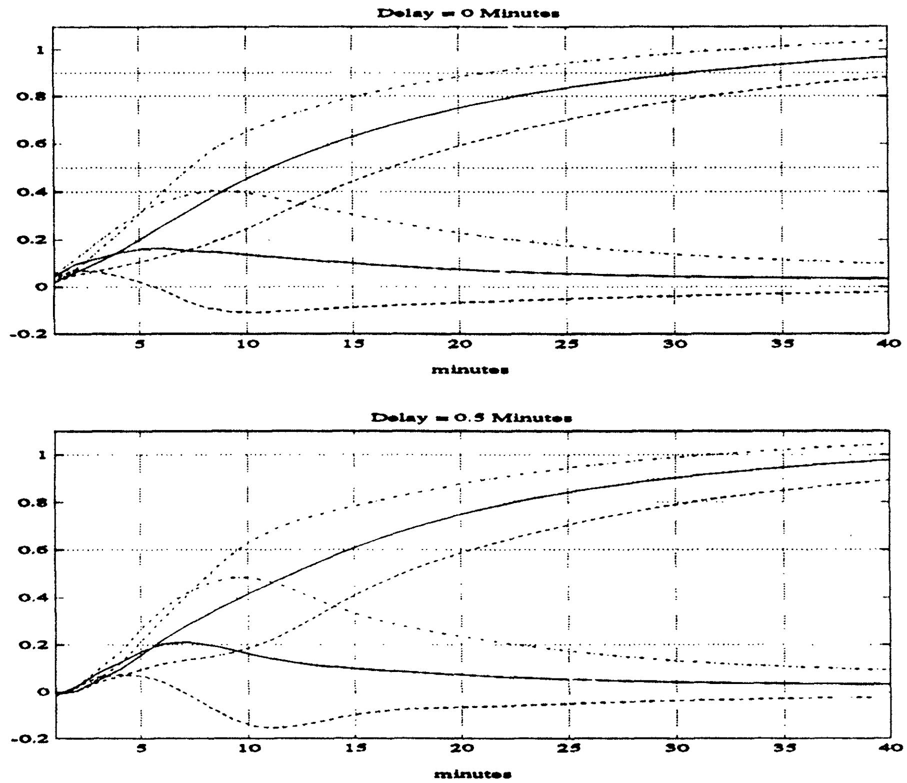


Figure 22: Time Responses for ℓ_1 Structured Uncertainty Design (Diagonal Performance Weight) for a Step Command of $\Delta y_d = 1$ $\Delta x_B = 0$

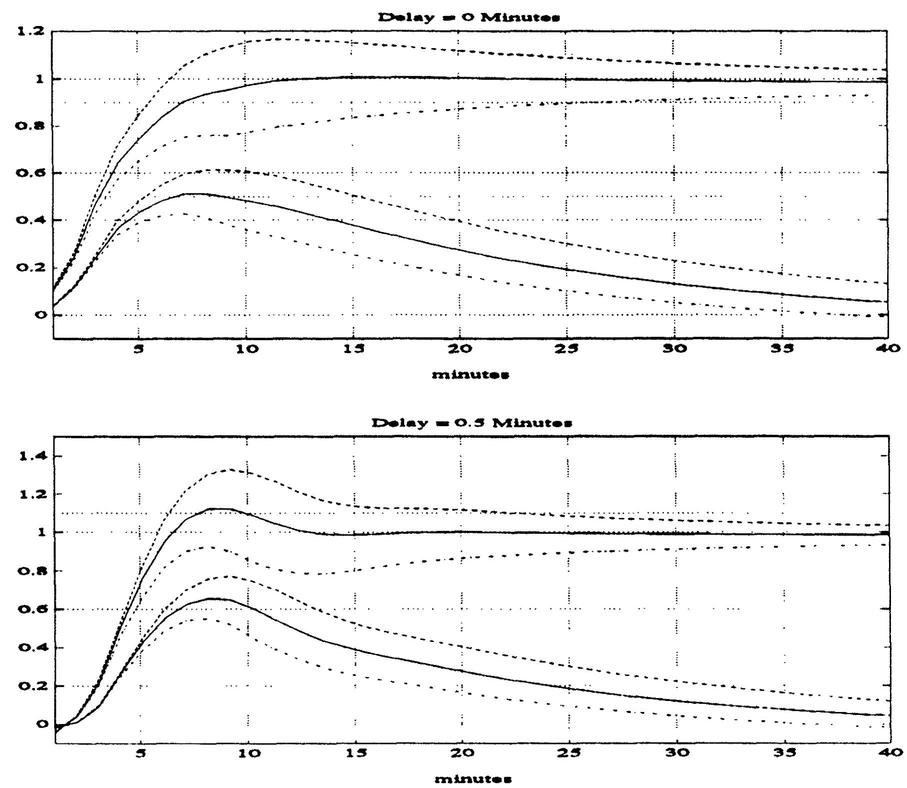


Figure 23: Time Responses for ℓ_1 Structured Uncertainty Design (Diagonal Performance Weight) for a Step Command of $\Delta y_d = 0$ $\Delta x_B = 1$

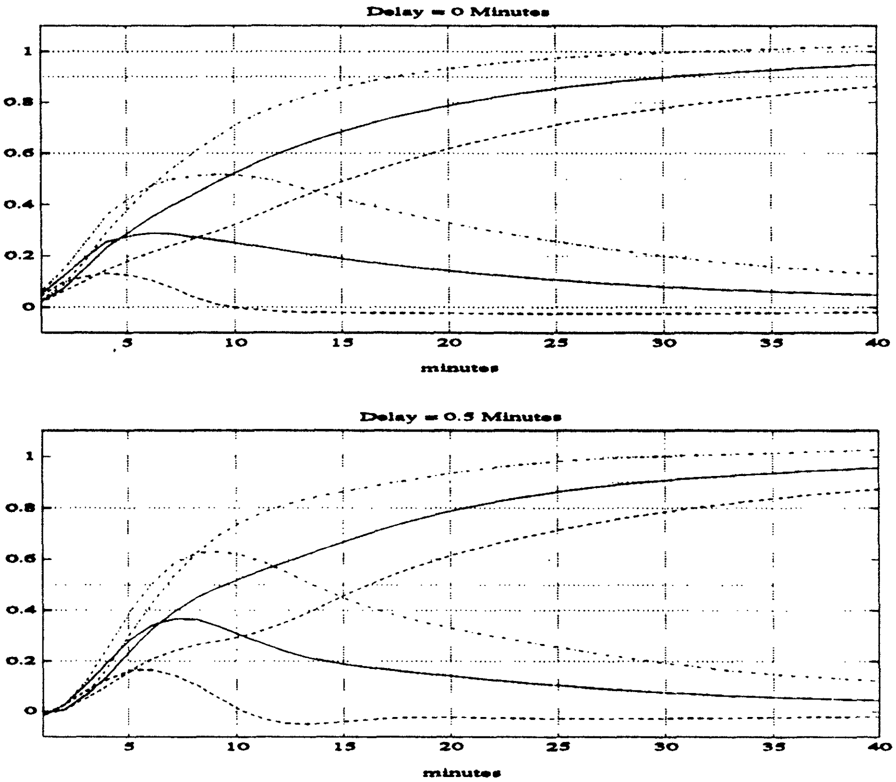


Figure 24: Time Responses for ℓ_1 Structured Uncertainty Design Including Directionality for a Step Command of $\Delta y_d = 1$ $\Delta x_B = 0$

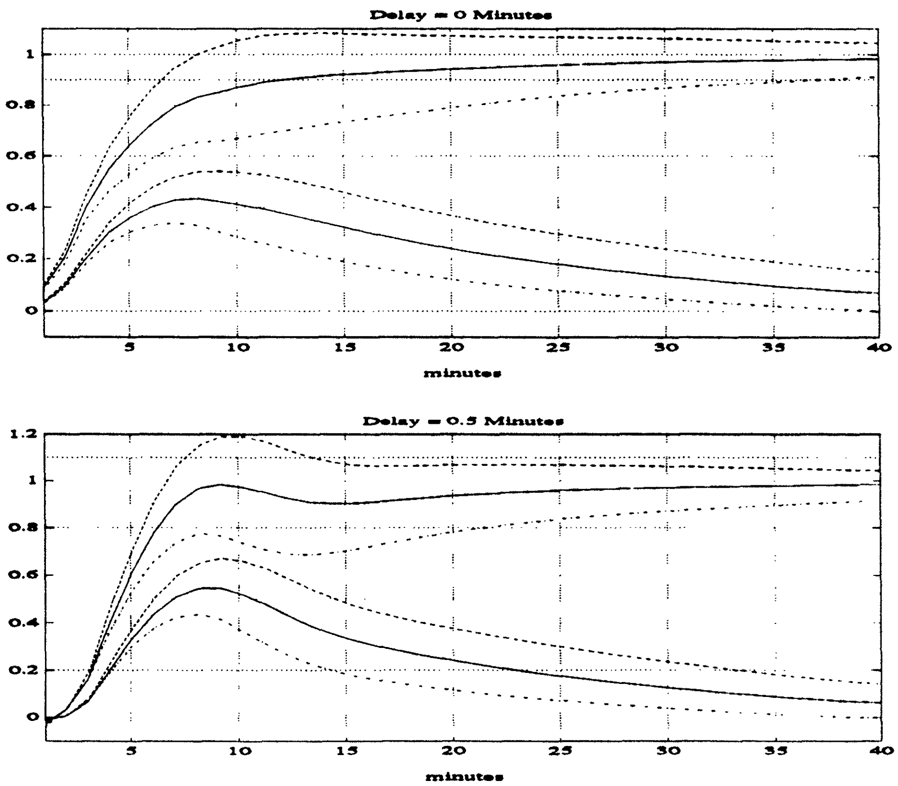


Figure 25: Time Responses for ℓ_1 Structured Uncertainty Design Including Directionality for a Step Command of $\Delta y_d = 0$ $\Delta x_B = 1$

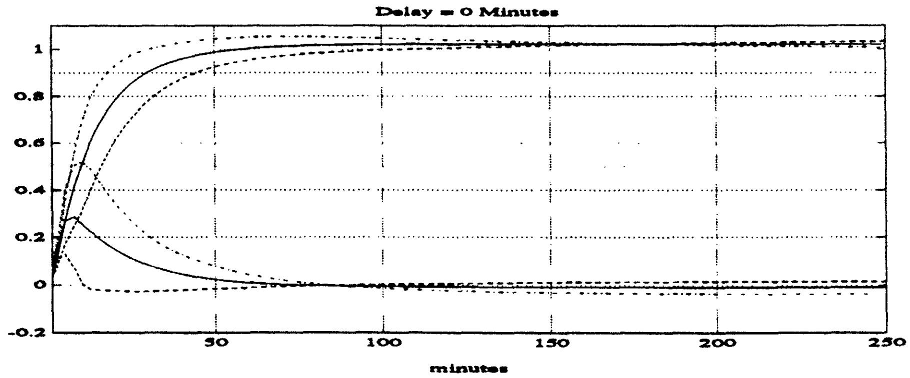


Figure 26: Long Time Responses for ℓ_1 Structured Uncertainty Design (Directional Performance Weight), $P(z) = \frac{z-0.0115}{z-1.0015}$ Step Command of $\Delta y_d = 1$ $\Delta x_B = 0$

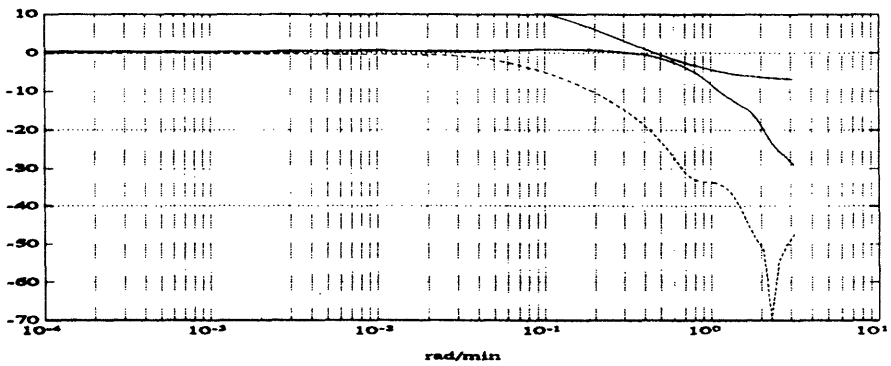


Figure 27: Robust Stability Test for Structured Uncertainty Design with $W_1(s) = 1.156\frac{s+.09}{s+1}W_d$ and $W_2(s) = 0.5\frac{s+.08}{s+.007}$

are not robustly stable to time-varying or nonlinear perturbations of ℓ_1 norm bounded by 2.2. Since the both systems are stable for LTI perturbations, this indicates a sensitivity of the system to time-varying or nonlinear perturbations

4.4.4 Comparison of Results

In this section the results of the discrete time designs, both μ and ℓ_1 structured uncertainty for $T = 1$ minute will be discussed. The ℓ_1 structured uncertainty design with directional weighting produced the best overall time domain responses. Comparing the time responses of this ℓ_1 design (Figures 24 and 25) with those from the discretized continuous time μ controller (Figures 8 and 9), one can see that the maximum level of overshoot of the output being commanded to 0, which should be less than 0.5 for all combinations of commands and uncertainties shown, is slightly greater than 0.6 in both designs. The rise times are slightly better for the discretized μ design; however, the time responses of the discretized μ design are much more jagged than the responses

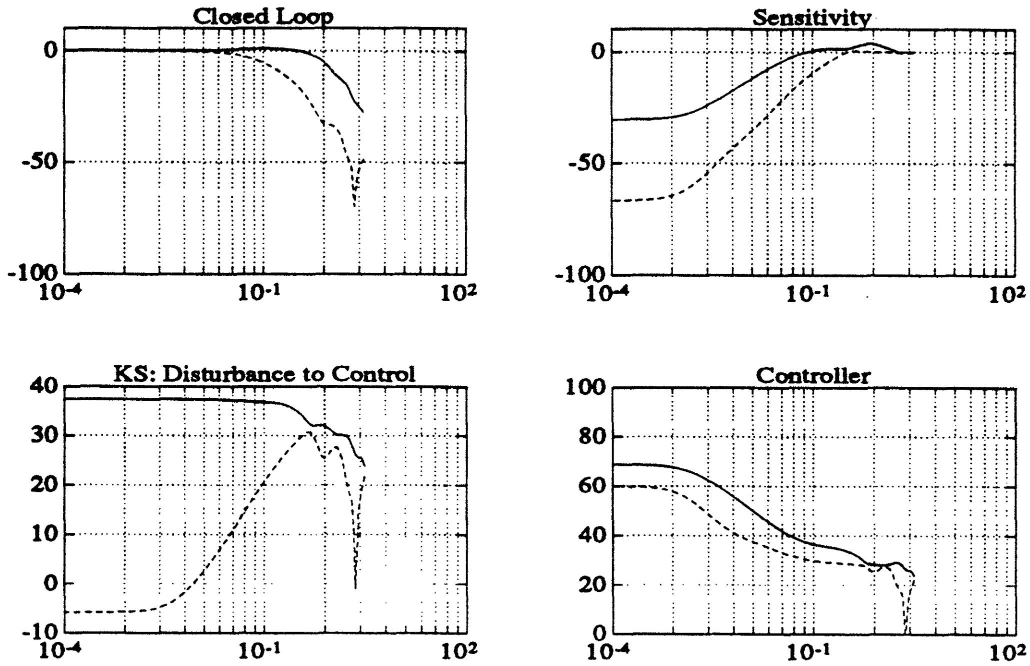


Figure 28: Max and Min Singular Values for Structured Uncertainty Design with $W_1(s) = 1.156\frac{s+.09}{s+1}$ and $W_2(s) = 0.5\frac{s+.08}{s+.007}W_d$

for the ℓ_1 design. This jaggedness is not desirable. The responses of the system resulting from the discrete time μ design (Figures 10 and 11) are much smoother than those of obtained from discretizing the continuous time controller. Comparing the ℓ_1 design to the discrete time μ design one can see that the rise times of the discrete time μ design are 4 to 5 minutes faster than those of the ℓ_1 design; however, the overshoot exceeds specifications by a greater amount and for more combinations of uncertainty for the discrete time μ design than for the ℓ_1 design. For example, the output being commanded to 1 should never exceed 1.1. The only time response shown for the ℓ_1 design which exceeds this specification is the case of $\Delta y_D = 0$, $\Delta x_B = 1$ with the uncertainty combination of $\tau = 0.5$, $\delta_1 = -0.2$, $\delta_2 = 0.2$ for which the maximum value of Δx_B was 1.2. On the other hand, the time responses for the same combinations of inputs and uncertainty are also shown for the discrete time μ design. Four of these time responses exceed this specification with a maximum value greater than 1.35. Note that it may have been possible to produce a better discrete time μ design with different performance weights; however, by doing this, any intuition gained from the loop shaping used in the continuous time designs would probably be lost.

In all discrete time designs with sampling period of $T = 1$ minute, the performance of the continuous time μ design with directional performance weight was never recovered. It is expected that with such a large sampling period, performance will be compromised. The use of this sampling period, however, produces designs that are in compliance with the frequency domain guidelines presented in Appendix B. These guidelines are important for the practical implementation of the designs since if they are not met, the nonlinear plant is not guaranteed to stay in the linear region of operation used for the design procedure.

5 Methodology Analysis

Now that the design work has been completed and the analysis of the distillation column performed, it is time to analyze the methodologies implemented in this paper. In this section, information is brought together concerning how the distillation column problem fits into the framework of the H_∞ and ℓ_1 structured uncertainty problems and how one can intelligently choose performance and uncertainty weights in both problems.

An inherent difficulty of this problem is that it does not fall nicely into the framework of either μ or the structured uncertainty ℓ_1 problem. The plant uncertainty is linear time invariant which suggests an H_∞ stability robustness criterion. The performance specifications are given in terms of templates for the closed loop responses to ℓ_∞ signals which suggests an ℓ_1 performance criterion. A time-varying (or nonlinear) "fictitious" perturbation must be included to transform this performance problem into a stability robustness problem. The robust performance problem is equivalent to the robust stability problem formed by combining this "fictitious" LTV perturbation and the LTI perturbation representing the plant uncertainty. As of today, there does not exist a nonconservative test for the robust stability of a system with respect to this type of mixed LTI/LTV structured uncertainty. The only way to design controllers for a system such as this is to treat the two perturbations as either both LTI or both LTV.

The selection of frequency domain performance weights for this problem is very difficult due to the fact that the performance specifications were given in the time domain. It is possible, however, to make statements concerning the effect on time responses of varying parameters in a stable minimum phase first order weight for both ℓ_1 and H_∞ structured uncertainty designs.

The performance weight for the H_∞ structured uncertainty designs was chosen using frequency domain loop shaping ideas. By decreasing the magnitude of the zero of the weight the rise time of the resulting closed loop system increased. Since the structured uncertainty problem is a multiblock problem, the H_∞ optimal solution is not exactly all-pass; therefore, it is difficult to predict the exact value to use for the zero of the weight for a desired rise-time. Assuming an inversion of the weight and iterating a few times on the initial value of the zero chosen produced desirable rise times for the distillation column. Using the H_∞ norm as a performance measure, there is no direct way for one to specify or change overshoots. This is due to the lack of a direct relationship between overshoot and frequency domain behavior.

The effect of the performance weight in the ℓ_1 structured uncertainty problem is different from that of the H_∞ problem. This was shown in the work on minimizing the weighted sensitivity (Section 4.2) for both the H_∞ and ℓ_1 problem. The trends shown in this section apply also to the structured uncertainty problem. Interestingly, it is the pole of the performance weight which affects the rise time in an ℓ_1 problem, not the zero as was the case in the H_∞ problem. This was apparent in the weighted sensitivity problem as well. Also as in the weighted sensitivity problem, the relationship between the pole and the zero of the weight affects overshoot. As the difference between the pole and the zero becomes more negative, the overshoot increases. Thus the use of the ℓ_1 norm appears to provide a more direct way of affecting the time domain characteristics of the resulting system.

Since the uncertainty present in the distillation column problem is linear time-invariant, it is not surprising that this part of the problem is easiest to model in the H_∞ problem. The frequency domain information can be incorporated into the ℓ_1 problem; however, since the ℓ_1 robustness criteria is only sufficient for LTI systems, the approach taken in this paper was to tradeoff "robustness" for performance until good robust performance was achieved.

It is very difficult to make a judgment on which procedure is actually "best" for controller design for this type of mixed problem. The pros and cons of the methods have been outlined. In both methods a fair amount of iteration on the weights is necessary to obtain the final design. Both methods provide different but useful information about the limitations of the system with respect to various types of uncertainty and the desired performance.

6 Conclusion

A comprehensive study of controller design for the high purity distillation column problem was presented in this thesis using the results from several structured uncertainty designs using the ℓ_1 and H_∞ norm. All designs indicate that including information about the plant directionality in the performance weight can improve robust performance. Also, the singular values of the the resulting transfer function from the disturbances to the controls should match those of the inverted plant at low frequencies and cross in the frequency range $10^{-1} - 2$ rad/min for robust performance. The results of the continuous time μ design show that the high frequency behavior of the system is important, thus the frequency domain specifications and the bandwidth limits imposed by discretizing the system limit the level of robust performance. The results of the ℓ_1 structured uncertainty design with a directional performance weight show, however, that a reasonable level of robust performance can be achieved with a sampling rate of $T = 1$ minute.

The methods used for controller design in this thesis were also analyzed. Translating time domain performance specifications to frequency domain performance weights is a difficult problem for both H_∞ and ℓ_1 designs. The effects on time domain behavior of varying parameters of a first order performance weight using both ℓ_1 and H_∞ methodologies was presented. Both a frequency domain and time domain interpretation of the results of the ℓ_1 designs was presented. While still a difficult problem, it appeared to be easier to directly affect the rise time and overshoot to a step more easily by adjusting a first order weight using the ℓ_1 methodology than using the H_∞ methodology. Since the physical uncertainty was LTI, it was more natural to model the uncertainty as a norm bounded perturbation using the H_∞ norm than using the ℓ_1 norm; however, it was possible to obtain designs with good robustness from the ℓ_1 methodology. The effects on time domain behavior of varying parameters of a first order performance weight for both ℓ_1 and H_∞ designs was presented. It was also found that while this problem does not fit either of the categories for which μ or the ℓ_1 structured uncertainty problem can provide a non-conservative test for robust performance, both methodologies can be used to design controllers for this system.

Appendix

A Distillation Column

A.1 Plant and Uncertainty Models

The following describes the model for the LV (reflux/boilup) configuration of the high purity distillation column. [15]

$$G_{LV}(s) = G_{LV}^{nom}(s) \begin{bmatrix} (1+\delta_1)e^{-\tau_1 s} & 0 \\ 0 & (1+\delta_2)e^{-\tau_2 s} \end{bmatrix} \tag{18}$$

$$0 \leq \tau_1, \tau_2 \leq 1 \qquad -0.2 \leq \delta_1, \delta_2 \leq 0.2$$

where $G_{LV}^{nom}(s) =$

$$\begin{bmatrix} -0.0052 & -0.0018 & 0.0039 & -0.0024 & 0.0714 & -0.0716 \\ -0.0036 & -0.1221 & 0.1693 & 0.2462 & 0.0044 & -0.0451 \\ 0.0047 & 0.1175 & -0.3941 & 0.0332 & -0.0362 & 0.0285 \\ 0.0067 & 0.2222 & -1.1254 & -3.9872 & -0.0534 & 0.0393 \\ 0.0636 & -0.0217 & -0.0040 & 0.0629 & 0 & 0 \\ 0.0786 & 0.0399 & -0.0459 & -0.0209 & 0 & 0 \end{bmatrix} \tag{19}$$

The above is in the standard form: $G = C(sI - A)^{-1}B + D$

$$G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The uncertainty will be handled as a multiplicative input uncertainty:

$$G_{LV}(s) = G_{LV}^{nom}(s)(I + \Delta) \tag{20}$$

$$\Delta = \begin{bmatrix} (1+\delta_1)e^{-\tau_1 s} - 1 & 0 \\ 0 & (1+\delta_2)e^{-\tau_2 s} - 1 \end{bmatrix} \qquad 0 \leq \tau_1, \tau_2 \leq 1 \tag{21}$$

B Performance Specifications

These time domain performance specifications are taken from [15]

The following specifications are given for a step input in the direction $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (the low gain direction).

- $\Delta y_D(t) \leq 1.1 \;\; \forall \; t$ and $\Delta y_B(t) \geq 0.9$ in no more than 30 minutes
- $\Delta x_B(t) \leq 0.5 \;\; \forall \; t$
- $0.99 \leq \Delta y_D(\infty) \leq 1.01$
- $-0.01 \leq \Delta x_B(\infty) \leq 0.01$

The following specifications are given for a step input in the direction $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (the low gain direction).

- $\Delta x_B(t) \leq 1.1 \;\; \forall \; t$ and $\Delta x_B(t) \geq 0.9$ in no more than 30 minutes
- $y_D(t) \leq 0.5 \;\; \forall \; t$
- $0.99 \leq \Delta x_B(\infty) \leq 1.01$
- $-0.01 \leq \Delta y_D(\infty) \leq 0.01$

The following specifications are given for a step input in the direction $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$ (the high gain direction).

- $\Delta y_D(t) \leq 0.5 \;\; \forall \; t$ and $\Delta y_D(t) \geq 0.35$ in no more than 30 minutes
- $x_B(t) \leq 0.7 \;\; \forall \; t$ and $x_B(t) \geq 0.55$ in no more than 30 minutes.
- $0.39 \leq \Delta y_D(\infty) \leq 0.41$
- $0.59 \leq \Delta x_B(\infty) \leq 0.61$

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