Twisted Stable Homotopy Theory

by

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Abstract

There are two natural interpretations of a twist of stable homotopy theory. The first interpretation of a twist is as a nontrivial bundle whose fibre is the stable homotopy category. This kind of radical global twist forms the basis for twisted parametrized stable homotopy theory, which is introduced and explored in Part I of this thesis. The second interpretation of a twist is as a nontrivial bundle whose fibre is a particular element in the stable homotopy category. This milder notion of twisting leads to twisted generalized homology and cohomology and is central to the well established field of parametrized stable homotopy theory. Part II of this thesis concerns a computational problem in parametrized stable homotopy, namely the determination of the twisted K-homology of the simple Lie groups. In more detail, the contents of the two parts of the thesis are as follows.

Part I: I describe a general framework for twisted forms of parametrized stable homotopy theory. An ordinary parametrized spectrum over a space X is a map from X into the category *Spec* of spectra; in other words, it is a section of the trivial *Spec*-bundle over X. A twisted parametrized spectrum over X is a section of an arbitrary bundle whose fibre is the category of spectra. I present various ways of characterizing and classifying these twisted parametrized spectra in terms of invertible sheaves and local systems of categories of spectra. I then define homotopy-theoretic invariants of twisted parametrized spectra and describe a spectral sequence for computing these invariants. In a more geometric vein, I show how a polarized infinite-dimensional manifold gives rise to a twisted form of parametrized stable homotopy, and I discuss how this association should be realized explicitly in terms of semi-infinitely indexed spectra. This connection with polarized manifolds provides a foundation for applications of twisted parametrized stable homotopy to problems in symplectic Floer and Seiberg-Witten-Floer homotopy theory.

Part II: I prove that the twisted K-homology of a simply connected simple Lie group G of rank n is an exterior algebra on $n-1$ generators tensor a cyclic group. I give a detailed description of the order of this cyclic group in terms of the dimensions of irreducible representations of G and show that the congruences determining this cyclic order lift along the twisted index map to relations in the twisted $Spin^c$ bordism group of G.

Thesis Supervisor: Michael J. Hopkins
Title: Professor of Mathematics

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I feel blessed to have had such inspiring and caring teachers and family and friends, and I hope that they do not need to read it in my thesis acknowledgments to know how grateful I am and how much I have treasured their presence in my life.

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Part I

Twisted Parametrized Stable Homotopy Theory

Chapter 1

Introduction

Despite its widespread use and compelling application to problems in symplectic topology and gauge theory, Floer homology remains rather a mystery. The very existence of the Floer homology of an infinite-dimensional manifold depends on delicate and haphazard properties of the manifold and of the flow associated to a Floer function; moreover, it is completely unknown how the Floer homology depends on the choice of Floer function. Confronted with this situation, Cohen, Jones, and Segal [8] asked whether it would be possible to build a Floer homotopy type encoding the relevant data from the manifold and the function in such a way that a homology functor would recover Floer homology. Besides elucidating the structure of Floer theory and clarifying its dependence on the Floer function, such a Floer homotopy type would immediately provide other invariants such as Floer K-theory and Floer bordism. Cohen, Jones, and Segal suggested that prospectra might encode some of the Floer data; though this thought proved useful, in retrospect it is clear that prospectra can only account for the Floer homotopy types of trivially polarized manifolds—this restriction on the polarization accounts for the difficulty Cohen, Jones, and Segal had finding examples of Floer prospectra.

The purpose of this paper is not to answer the Floer homotopy question, but to introduce a framework, namely twisted parametrized stable homotopy theory, that is a necessarily component of any description of Floer homotopy. That some twisted form of homotopy theory was needed to account for nontrivial polarizations was first realized by Furuta [21], and it will turn out that the twisted space he wrote down (as a conjectural model for the Seiberg-Witten-Floer homotopy type of T^3) is a very specialized example of our twisted parametrized spectra. A twisted parametrized spectrum is a section of a bundle whose fibre is the category of spectra, and as such it has the same relationship to an ordinary parametrized spectrum that a section of a line bundle has to a function. A polarized infinite-dimensional manifold has a naturally associated bundle of categories of spectra, and the fundamental ansatz is that geometric information about such a manifold (its Floer homotopy type, for example) involves this bundle and its sections. In this paper, we present the theory of twisted parametrized spectra, including various definitions, characterizations, and classifications of them, a thorough description of their homotopy-theoretic invariants, and an overview of their relationship to infinite-dimensional polarized manifolds. The specific association of a twisted parametrized homotopy type to a manifold with Floer function is the subject of ongoing work with Mike Hopkins and Ciprian Manolescu and will appear elsewhere [9, 10].

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Chapter 2

Invertible Sheaves of Categories of Spectra

Let X be an algebraic variety over a ring R. The structure sheaf $\mathcal{O}_X$ can be described as the sheaf of R-valued functions on X. The most fundamental $\mathcal{O}_X$-modules are the invertible or locally free rank-one modules. These modules, which we will often think of as line bundles, are classified by the first cohomology group of X with coefficients in the units of R; denote by $\mathcal{L}(c)$ the line bundle associated to the cohomology class $c \in H^1(X; R^\times)$. A global section of $\mathcal{L}(c)$ is determined by a 0-cochain on X whose coboundary is c, that is by an element $f \in C^0(X; R)$ such that $\delta f = c$.

Twisted parametrized stable homotopy theory is a precise analog of these algebraic concepts: the ring R is replaced by the category Sp of spectra, a "categorical semi-ring" under the wedge and smash products. The set of Sp-valued functions on a space X is naturally interpreted as the category of parametrized spectra on X, and the structure "sheaf" is therefore the structure stack $\mathcal{O}_X$ of parametrized spectra. There is a notion of locally free rank-one module over the structure stack of parametrized spectra and we refer to such modules, briefly, as *haunts.* Haunts are classified by the first cohomology group of X with coefficients in the so-called Picard category $\mathrm{Pic}(S^0)$ of units of Sp. The fundamental objects of twisted parametrized stable homotopy theory are the global sections of a haunt; these global sections are the twisted parametrized spectra or *specters* for short. Thus, a specter has the same relationship to a parametrized spectrum as a section of a line bundle has to a function. Moreover, a specter for the haunt $\mathcal{L}(c)$ associated to a class $c \in H^1(X; \mathrm{Pic}(S^0))$ is determined by a 0-cochain with coboundary c, that is by an element $f \in C^0(X; Sp)$ together with an identification $\delta f \cong c$.

This fundamental analogy is summarized in table 2.1 and is explained in detail in the following sections.

2.1 The Structure Stack of Parametrized Spectra

We begin by describing the category of parametrized spectra and its associated homotopy theory. A spectrum E is, most naively, a series E_i of based spaces equipped with structure maps $\Sigma E_i \to E_{i+1}$ from the suspension of one space to the next. Similarly, we can describe a parametrized spectrum over X by giving a series E_i of based spaces over X together with structure maps $\Sigma_X E_i \to E_{i+1}$ from the fibrewise suspension of one space to the next. (A based space over X is a space together with a projection map to X and a section of this projection.) This naive viewpoint is sufficient for many purposes, including taking the homology and cohomology of a base space X with coefficients in a parametrized spectrum, but it fails to provide a foundation for a good smash product on the category of parametrized spectra. As we are interested in considering this category to be a semi-ring, it is essential that we have a highly associative and commutative smash product. We therefore work with the category of orthogonal parametrized spectra. An orthogonal parametrized spectrum on X is a diagram spectrum in the category of based spaces over X, where the diagram category is finite dimensional inner product spaces and their isometries. See Mandell, et al. [34] for a description of diagram spectra and May-Sigurdsson [37] for an extensive discussion of orthogonal parametrized spectra. In keeping with our sheaf-theoretic philosophy, will we let $\mathcal{O}_X(U)$ denote the category of (orthogonal) parametrized spectra on the open set $U \subset X$.

We have selected a model for parametrized spectra because we need to ensure that we have a well behaved smash product, but we are not of course interested

	In Algebraic Geometry	In Twisted Parametrized Stable Homotopy Theory
The basic ring R is	an ordinary ring	the category Sp of spectra
The structure sheaf $\mathcal{O}_X$ is given by	$\mathcal{O}_X(U) = R$-valued functions on U	$\mathcal{O}_X(U) =$ the category of parametrized spectra on U
A line bundle $\mathcal{L}$ is	a sheaf that is an invertible $\mathcal{O}_X$-module	a stack that is a locally free rank-one $\mathcal{O}_X$-module
The set of units in the ring R is	the multiplicative group $R^\times$	the monoidal category $\mathrm{Pic}(S^0)$ of invertible S^0-modules
Line bundles $\mathcal{L}(c)$ are classified by	elements $c \in H^1(X; R^\times)$	elements $c \in H^1(X; \mathrm{Pic}(S^0))$ or maps $X \to B\,\mathrm{Pic}(S^0)$
A global section of a line bundle is	an ordinary section	*a twisted parametrized spectrum*
Such a section is presented by	an element $f \in C^0(X; R)$ such that $\delta f = c$	an element $f \in C^0(X; Sp)$ together with $\delta f \cong c$

Table 2.1: The analogy with basic algebraic geometry

in the point-set-level peculiarities of this particular model. As such, we need to keep in mind a homotopy theory, that is a model structure or at least a notion of weak equivalences, on the category $\mathcal{O}_X(U)$ of parametrized spectra; we should consider a category $\mathcal{C}$ equivalent to $\mathcal{O}_X(U)$ not if there is an equivalence of categories between them, but if there is a functor inducing an equivalence of homotopy theories. We will focus on the homotopy theory on $\mathcal{O}_X(U)$ associated to the stable model structure defined in [37]. Suffice it to say that the fibrant objects in the stable model structure are, in particular, quasi-fibrations of spectra over X, and the weak equivalences between fibrant objects are maps that induce weak equivalences on each fibre. We can think of this as a homotopy theory of quasi-fibrations of spectra, rather than of all parametrized spectra; this reduction will be important in our consideration of local systems of categories of spectra in section 3. As an aside, we note that there may be other interesting model structures on parametrized spectra, for example ones in which there is a much larger class of fibrant objects; the formulation of twisted parametrized spectra in section 2.2 will work for these alternate model structures, producing a very different and perhaps even more intriguing theory.

Remark 2.1.1. We have fixed a notion of homotopy theory on the category of parametrized spectra, namely the one coming from the stable model structure. The concept of an ∞-category conveniently encodes the notion of a category together with an associated homotopy theory. An ∞-category is, roughly speaking, a category together with 2-morphisms, 3-morphisms, and so on, such that all the n-morphisms are invertible for $n > 1$. Though ∞-categories are as yet little utilized, many familiar structures, including simplicial categories, Segal categories, Segal spaces, and quasi-categories, give models for ∞-categories; see Lurie [31, 32] for a thorough treatment. The pairing of a category and a notion of homotopy theory will be so pervasive that in this section and in section 2.2 we will frequently use "category" to mean "∞-category" and implicitly take associated notions, such as equivalence, monoidal structure, module, and so forth, to refer to their ∞-categorical analogs. The reader who is bothered by the resulting inexplicitness should defer to section 3 where haunts and specters are recharacterized in more traditional terms. (Model-theoretically inclined readers may want to take "category" to mean "model category" and this will be perfectly suitable by way of understanding, but we will be utilizing categories of categories (read ∞-categories of ∞-categories) and the category of model categories is not known to have a model structure.)

∼∼∼∼∼

The association to an open set $U \subset X$ of the category $\mathcal{O}_X(U)$ of parametrized spectra over U is meant to function as a "sheaf of rings" analogous to the structure sheaf of R-valued functions on an algebraic variety. First we consider the ring-like structure on $\mathcal{O}_X(U)$ and then proceed to the sheaf- or stack-like properties of $\mathcal{O}_X$.

The category $\mathcal{O}_X(U)$ of parametrized spectra is a symmetric bimonoidal category in the sense of Laplaza [28]; that is, it comes equipped with two symmetric monoidal functors (wedge and smash) and natural distributivity isomorphisms satisfying various coherence relations. In fact, the category is better behaved that the average symmetric bimonoidal category because the wedge product is the categorical coproduct; the additive associativity isomorphisms and the distributivity isomorphisms are therefore canonically defined. Of course, there is a rigidification functor [18, 12] that replaces a symmetric bimonoidal category with an equivalent bipermutative category (where the associativity isomorphisms are identity transformations). We will frequently and implicitly use the bipermutative category associated to $\mathcal{O}_X(U)$, particularly when discussing modules in the next section. This symmetric bimonoidal or bipermutative structure makes $\mathcal{O}_X(U)$, for all intents and purposes, into a semi-ring.

Philosophically, a stack $\mathcal{C}$ is a presheaf of categories satisfying descent up to equivalence of categories; (see, for example, Moerdijk's treatment in [40]). That is, the category $\mathcal{C}(U)$ living over a large open set U is determined, up to equivalence, by the categories $\mathcal{C}(V)$ living over small open subsets $V \subset U$, in the same way that the value of a sheaf is determined by its local behavior. Note that these categories $\mathcal{C}(U)$ need not be groupoids (as is usually assumed) and that both "presheaf" and "equivalence" can be freely interpreted. For example, "presheaf" might mean literal presheaf, presheaf up to coherent natural isomorphism, or presheaf up to coherent natural homotopy equivalence; similarly, "equivalence of categories" might mean ordinary equivalence of categories, Quillen equivalence of model categories, homotopical equivalence of ∞-categories, or something analogous. In general, we take presheaf to mean presheaf up to coherent natural isomorphism, and equivalence of categories to mean homotopical equivalence of ∞-categories.

As the notation suggests, we have restriction functors $i^* : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ associated to inclusions $i : V \subset U$. Because these restrictions of parametrized spectra boil down to literal restriction functors in categories of topological spaces over the base X, these functors give $\mathcal{O}_X$ the structure of a presheaf of categories on X. This presheaf $\mathcal{O}_X$ is a stack. Though it is a stack in the usual, literal sense that it satisfies descent up to equivalence of categories (as can be checked using the prestack gluing condition given in [40, p.11]), we are only concerned with the fact that it is a stack in the sense that it satisfies descent up to homotopical equivalence of categories.

Summary 2.1.2. The association $\mathcal{O}_X$ to an open subset $U \subset X$ of the category $\mathcal{O}_X(U)$ of orthogonal parametrized spectra on U is a stack of symmetric bimonoidal (∞-)categories.

2.2 Modules over the Structure Stack

We have replaced an ordinary ring R by the semi-ring category *Sp* of spectra and we are investigating invertible sheaves in this new context. We have introduced our basic "sheaf of rings" $\mathcal{O}_X$, namely the structure stack of parametrized spectra. In this section, we describe and classify locally free rank-one modules over this structure stack—we refer to these modules, briefly, as "haunts"—and we study their categories of global sections. These global sections are the fundamental objects of twisted parametrized stable homotopy theory and we call them "twisted parametrized spectra" or "specters" for short.

2.2.1 Haunts

A module over a stack $\mathcal{R}$ of symmetric bimonoidal categories is a stack $\mathcal{M}$ of symmetric monoidal categories together with an action $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ appropriately compatible (by analogy with a module over a ring) with the monoidal structures. We do not spell out this compatibility; see Dunn [12] for an extensive discussion of modules over semi-ring categories, Lurie [32] for some of the technicalities involved in semi-ring ∞-categories and their modules, and remark 2.2.2 below for an explanation of why we do not attend to the details of these compatibility relations. Such a module $\mathcal{M}$ over a symmetric bimonoidal stack $\mathcal{R}$ is locally free of rank one if for all points $x \in X$ there exists an open set $U \subset X$ containing x and an object $S \in \mathcal{M}(U)$ such that the map $\mathcal{R}|_U \to \mathcal{M}|_U$ determined for $V \subset U$ by

$$\begin{aligned} \mathcal{R}(V) &\to \mathcal{M}(V) \\ A &\mapsto A \cdot S|_V \end{aligned}$$

is an equivalence of symmetric monoidal stacks.

Definition 2.2.1. A *haunt* on a space X is a locally free rank-one module over the structure stack $\mathcal{O}_X$ of parametrized spectra on X.

Remark 2.2.2. Because the additive monoidal structure in the category $\mathcal{O}_X$ of parametrized spectra is given by the categorical coproduct, most of the compatibility conditions [12] for $\mathcal{O}_X$-modules are automatically satisfied, provided the monoidal structure on the module is also the coproduct. Indeed, it is generally sufficient to treat $\mathcal{O}_X$ as a multiplicative monoid and study stacks with an action of this monoid. As a point of philosophy, though, it is important to keep in mind that we are really dealing with modules over *ring* stacks.

Remark 2.2.3. We limit our attention to locally free rank-one modules over parametrized spectra, but we imagine that there may be quite interesting and intricate homotopy-theoretic information in the structure of higher rank modules.

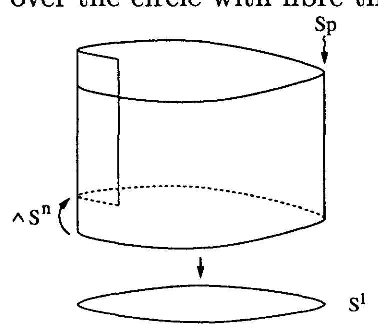
In doing geometry over an ordinary ring R, we think of invertible sheaves as line bundles. Such a line bundle is most easily and explicitly described by taking trivial R-bundles over an open cover $\{U_i\}$ and specifying appropriate gluing data $r_{ij} : R \times U_{ij} \to R \times U_{ij}$ on the two-fold intersections $U_{ij} = U_i \cap U_j$. By analogy, we think of haunts as bundles whose fibre is the category Sp of spectra. To specify such a bundle, we can take a trivial Sp-bundle over an open cover and give gluing data $r_{ij} : Sp \times U_{ij} \to Sp \times U_{ij}$ on the intersections; more precisely, this gluing data amounts to automorphisms $r_{ij} : \mathcal{O}_X(U_{ij}) \to \mathcal{O}_X(U_{ij})$. Before formalizing this viewpoint, we give two examples.

Example 2.2.4. Cover the base space $X = \mathrm{S}^1$ by two open semicircles U_0 and U_1 and denote by V and W the two components of the intersection U_{01}. Glue $\mathcal{O}_X|_{U_0}$ and $\mathcal{O}_X|_{U_1}$ together along V by the identity map on $\mathcal{O}_X(V)$ and along W by the map

$$\begin{aligned} \mathcal{O}_X(W) &\to \mathcal{O}_X(W) \\ T &\mapsto T \wedge_W (\mathrm{S}^n \times W) \end{aligned}$$

In other words, the monodromy around the circle is the map $Sp \to Sp$ given by suspension by S^n. Schematically the resulting haunt appears as in figure 2-1. As we will see, every haunt over S^1 is equivalent to this suspension haunt for some integer n.

Figure 2-1: A bundle over the circle with fibre the category of spectra



Example 2.2.5. Now take the base space X to be S^3 with its usual hemispherical cover by two open sets D^3_0 and D^3_1; the intersection of these open sets is the equatorial band $\mathrm{S}^2 \times (-\epsilon, \epsilon)$. Let $\mathrm{S}^2 \rtimes \mathrm{S}^2$ denote the nontrivial S^2 bundle over S^2. We can define a haunt over X by gluing the trivial bundles $Sp \times D^3_0$ and $Sp \times D^3_1$ as follows:

$$\begin{aligned} \mathcal{O}_X(\mathrm{S}^2 \times (-\epsilon, \epsilon)) &\to \mathcal{O}_X(\mathrm{S}^2 \times (-\epsilon, \epsilon)) \\ &T \mapsto T \wedge_{\mathrm{S}^2 \times (-\epsilon,\epsilon)} ((\mathrm{S}^2 \rtimes \mathrm{S}^2) \times (-\epsilon, \epsilon)) \end{aligned}$$

This gluing produces the only nontrivial haunt over S^3.

Line bundles are built by gluing together trivial R-bundles over intersections. At a point, this gluing is determined by an R-module automorphism of the ring R, that is by an element of $\mathrm{Aut}_R R$. Of course, any such automorphism is multiplication by a unit of R; in other words, $\mathrm{Aut}_R R \cong R^\times$. The gluing data for a line bundle is therefore a 1-cocycle with values in $R^\times$ or more precisely, with values in the sheaf $\mathcal{O}_X^\times$ of $R^\times$-valued functions. Up to equivalence, these bundles are classified by the cohomology group $H^1(X; \mathcal{O}_X^\times)$.

Appropriately interpreted, all these facts remain true when R is replaced by the category *Sp* of spectra—see table 2.1. Any automorphism of *Sp* as a module over itself is given by smashing with an invertible spectrum; moreover, there is an equivalence of categories $\mathrm{Aut}_{Sp}\, Sp \cong Sp^\times$. Here the objects of $Sp^\times$ are the invertible spectra and the morphisms are weak equivalences of spectra. This category is denoted $\mathrm{Pic}(S^0)$ in the literature and we will use that notation to refer to both the category and its realization. The gluing data for a haunt is a 1-cocycle with values in $\mathrm{Pic}(S^0)$, by which we mean a cocycle with values in the associated sheaf $\mathcal{P}ic(S^0)$ of "$\mathrm{Pic}(S^0)$-valued functions". This "sheaf" is a stack of monoidal categories; the category $\mathcal{P}ic(S^0)(U)$ has objects invertible parametrized spectra on U and morphisms weak equivalences. The classification is as expected:

Proposition 2.2.6. *Haunts on X are classified up to equivalence by the cohomology group $H^1(X; \mathcal{P}ic(S^0))$ or equivalently by the group of homotopy classes of maps $[X, B\,\mathrm{Pic}(S^0)]$.*

Remark 2.2.7. The reader who (sensibly enough) grimaces at the prospect of cohomology with coefficients in a stack of monoidal categories can happily focus on the latter characterization in terms of homotopy classes of maps, which will be established in detail in section 3.2.1.

Any invertible spectrum is weakly equivalent to a sphere of some integer dimension, so $B\operatorname{Pic}(S^0)$ has the homotopy type of $B(\mathbb{Z} \times BGL_1(S^0))$ where $GL_1(S^0)$ denotes the module automorphisms of the sphere spectrum, also known as the group of stable self equivalences of the sphere. In particular, $\pi_1(B\operatorname{Pic}(S^0)) = \mathbb{Z}$ and $\pi_3(B\operatorname{Pic}(S^0)) = \mathbb{Z}/2$, explaining the classifications mentioned in examples 2.2.4 and 2.2.5. We will discuss the homotopy groups of $B\operatorname{Pic}(S^0)$ in more detail in the context of specter invariants in section 4.

2.2.2 Specters

As previously mentioned, specters generalize parametrized spectra in the same way that sections of line bundles generalize functions:

Definition 2.2.8. A *twisted parametrized spectrum* or *specter* on X is a global section of a haunt over X. That is, it is an object $S \in \mathcal{M}(X)$ of the category of global sections of a locally free rank-one module $\mathcal{M}$ over the structure stack $\mathcal{O}_X$ of parametrized spectra on X.

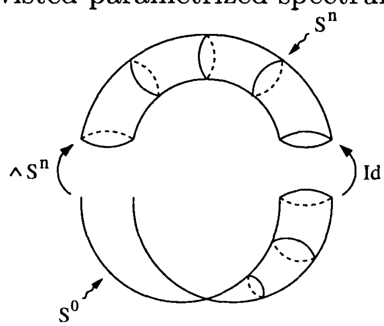
An ordinary R-valued function on X is determined by a 0-cocycle with values in R, that is by a 0-cochain $f \in C^0(X; R) := C^0(X; \mathcal{O}_X)$ such that the coboundary vanishes: $\delta f = 0$. Suppose $c \in Z^1(X; R^\times) := Z^1(X; \mathcal{O}_X^\times)$ is a 1-cocycle defining a line bundle $L(c)$ with fibre R. A section of $L(c)$ is presented by a 0-cochain $f \in C^0(X; \mathcal{O}_X)$ cobounding the cocycle c, which is to say such that $\delta f = c$. This section need not trivialize the line bundle because it is allowed to take non-invertible values, unlike the defining cocycle for the bundle.

Analogously, a parametrized spectrum on X can be described by a 0-cocycle with values in the stack of parametrized spectra. This amounts to giving a 0-cochain $f \in C^0(X; Sp) := C^0(X; \mathcal{O}_X)$, namely a parametrized spectrum on each open set of a cover, together with a compatible system of equivalences on intersections; this system of equivalences is concisely encoded in the equation $\delta f \cong 0$. Let $c \in Z^1(X; Sp^\times) :=$

$Z^1(X; \mathcal{P}ic(S^0))$ denote a 1-cocycle defining a haunt $L(c)$; concretely, this means that for a cover $\{U_i\}$ we have invertible parametrized spectra $c_{ij} \in \mathcal{P}ic(S^0)(U_{ij})$ on two-fold intersections together with fixed weak equivalences $\phi_{ijk} : c_{ij} \wedge_{U_{ijk}} c_{jk} \longrightarrow c_{ik}$ satisfying the obvious coherence relation on four-fold intersections. A specter for this haunt, that is a section of $L(c)$, is most easily presented by a 0-cochain $f \in C^0(X; \mathcal{O}_X)$ together with an identification $\delta f \cong c$ of the coboundary of f with the cocycle c. What this means is that on two-fold intersections compatible equivalences are given between $f_i \wedge_{U_{ij}} c_{ij}$ and f_j. This cochain presentation is well suited to giving explicit examples of specters.

Example 2.2.9. Let L_n denote the haunt over S^1 whose monodromy is suspension by S^n. This haunt is depicted in figure 2-1 and can be presented, roughly speaking, as follows: take two copies of the stack of parametrized spectra on an interval, that is of $\mathcal{O}_{D^1}$, and glue them together at the two pairs of endpoints by the maps $Sp \xrightarrow{\text{id}} Sp$ and $Sp \xrightarrow{\wedge S^n} Sp$ respectively. We now define a specter T for L_n. Over the first interval D^1_0 the specter is a trivial parametrized spectrum with fibre S^n; that is $T|_{D^1_0} = S^n \times D^1_0$. Over the second interval D^1_1 the specter is a cone on $S^n \sqcup S^0$; that is, writing $D^1_1 = C(* \sqcup *)$, we have $T|_{D^1_1} = C(S^n \sqcup S^0)$. See figure 2-2.

Figure 2-2: A twisted parametrized spectrum over the circle



Example 2.2.10. Let L_{S^3} denote the nontrivial haunt over S^3 described in example 2.2.5. Roughly speaking, L_{S^3} is constructed using the equatorial gluing function $\psi : \mathcal{O}_{S^2} \longrightarrow \mathcal{O}_{S^2}$ given by $\psi(P) = P \wedge_{S^2} (S^2 \rtimes S^2)$. Define a specter T for L_{S^3} by $T = (S^0 \times D^3) \cup_\psi C(S^2 \rtimes S^2)$; that is, on one hemisphere the specter is a trivial parametrized spectrum with fibre S^0 and on the other hemisphere (thought of as the cone $C(S^2)$) the specter is the cone on the bundle $S^2 \rtimes S^2$.

Remark 2.2.11. The procedure (illustrated in these examples) of piecing together parametrized spaces to give a global geometric object has also appeared in a preprint by Furuta [21] under the rubric "prespectra with parametrized universe". Furuta's viewpoint is similar to ours in spirit, but his 'parametrized prespectra' are substantially more rigid than specters; in particular, few specters can be realized as 'parametrized prespectra'. It would therefore seem to be difficult to develop a reasonable homotopy theory of twisted spectra (which is essential for applications to Floer homotopy) using the rigid geometry of 'parametrized prespectra'.

In section 2.1 we emphasized the fact that the homotopy theory of parametrized spectra should be conceived as a homotopy theory of fibrations of spectra. The above examples of specters have singularities and are therefore not locally fibrations of spectra. There is a fibrant replacement functor which takes such a singular specter and returns a locally fibrant specter with the same "global homotopy type"; (see section 4 for a discussion of global homotopy invariants of specters). On the one hand, it is easier to explicitly describe and compute invariants of singular specters; on the other hand, it is easier to characterize and classify specters using fibrant presentations.

The following model for fibrant replacement of specters bears close resemblance to the one May and Sigurdsson use for parametrized spectra; a more thorough technical treatment of the replacement functor can be found in their manuscript [37]. Suppose S is a specter over the base X. Let PX denote the path fibration on X and s and t the source and target maps $PX \to X$. A fibrant model $F(S)$ for the specter S is very roughly given by $s_!(t^*(S))$—here $s_!$ denotes integration over the fibre in a sense analogous to that given for parametrized spectra in [37]. In other words, the points of the fibre $F(S)_x$ at a point $x \in X$ are pairs consisting of a path in X from x to y and a point of the fibre S_y. We proceed to some examples of fibrant specters.

Example 2.2.12. As before, let L_n denote the haunt over S^1 whose monodromy is suspension by S^n. Define a specter T for L_n as follows: the fibre T_x of T at every point $x \in S^1$ is $\bigvee_{i\in\mathbb{Z}} S^{n\cdot i}$ and the monodromy operator is the natural equivalence $\Sigma^n(\bigvee S^{n\cdot i}) \simeq \bigvee S^{n\cdot i}$. This is a model for the specter in example 2.2.9 which is locally a fibration of spectra. More generally, any specter for the haunt L_n on S^1 can be described by giving a spectrum A together with an equivalence of A with its n-th suspension:

$$\{\text{Specters}/(L_n, S^1)\} \leftrightsquigarrow \{\text{Spectra } A \text{ with } \phi : \Sigma^n(A) \simeq A\}$$

Example 2.2.13. There is only one nontrivial haunt over S^2; call it L_{S^2}. Let U and V denote the two hemispheres of S^2. The haunt L_{S^2} is constructed, roughly speaking, by gluing $\mathcal{O}_U$ and $\mathcal{O}_V$ along the equatorial S^1 using the function $\psi : \mathcal{O}_{S^1} \to \mathcal{O}_{S^1}$ given by $\psi(P) = P \wedge_{S^1} (S^0 \rtimes S^1)$; here $S^0 \rtimes S^1$ denotes the nontrivial S^0 bundle over S^1. Suppose we want to construct a specter T for L_{S^2} that is locally a fibration of spectra. The restriction $T|_U$ of the specter to one hemisphere U will be equivalent to the parametrized spectrum $A \times U$, for some spectrum A. On the boundary of U, the specter is $A \times \partial U = A \times S^1$, and therefore on the boundary of V the specter must be $\psi(A \times S^1) = (A \times S^1) \wedge_{S^1} (S^0 \rtimes S^1)$—let us denote this last parametrized spectrum by $\mathbf{A}^{\mathrm{tw}(-1)}$. As $\mathbf{A}^{\mathrm{tw}(-1)}$ is a fibrant parametrized spectrum over S^1 with fibre A, it is constructed by gluing together two copies of $A \times D^1$ along the boundaries $A \times S^0$. Such a gluing is determined by a map $S^0 \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ denotes the homotopy automorphisms of the spectrum A. In the case of $\mathbf{A}^{\mathrm{tw}(-1)}$ this gluing is the map $-1_A : S^0 \to \mathrm{Aut}(A)$ taking one point to id_A and the other point to $-\mathrm{id}_A$. The specter T restricts on the hemisphere V to a fibrant parametrized spectrum $T|_V$ with boundary $\mathbf{A}^{\mathrm{tw}(-1)}$—this parametrized spectrum defines a nullhomotopy of the gluing map -1_A of $\mathbf{A}^{\mathrm{tw}(-1)}$. In summary:

$$\{\text{Specters}/L_{S^2}\} \leftrightsquigarrow \{\text{Spectra } A \text{ with } \phi : D^1 \to \mathrm{Aut}(A) \text{ s.t. } \phi_0 = \mathrm{id}_A, \phi_1 = -\mathrm{id}_A\}$$

Example 2.2.14. Recall the nontrivial haunt L_{S^3} over S^3 determined by the equatorial gluing function $\psi(P) = P \wedge_{S^2} (S^2 \rtimes S^2)$. Note that this haunt is isomorphic to the haunt determined by the gluing function $\psi'(P) = P \wedge_{S^2} (S^0 \rtimes S^2)$, where $S^0 \rtimes S^2$ is the unique nontrivial S^0-spectrum bundle over S^2. Suppose T is a specter for L_{S^3} that is locally a fibration of spectra. Then, as in the previous example, T restricted to one hemisphere U is a trivial bundle with fibre spectrum A. The boundary $T|_{\partial U}$ of this restriction is $A \times \partial U = A \times S^2$ and the boundary $T|_{\partial V}$ of the other restriction must therefore be $\psi'(A \times S^2) = (A \times S^2) \wedge_{S^2} (S^0 \rtimes S^2)$—denote this last parametrized spectrum by $\mathbf{A}^{\mathrm{tw}(\eta)}$. The fibrant parametrized spectrum $\mathbf{A}^{\mathrm{tw}(\eta)}$ over S^2 is determined by the gluing function $\eta_A : S^1 \to \mathrm{Aut}(A)$; here η_A is the function $\eta \wedge A$ where $\eta : S^1 \to \mathrm{Aut}(S^0)$ is the nontrivial element of the first stable stem. The restriction $T|_V$ of the specter to the second hemisphere constitutes a nullhomotopy of the gluing function η_A for the boundary parametrized spectrum $\mathbf{A}^{\mathrm{tw}(\eta)}$. Again we have a classification:

$$\{\text{Specters}/L_{S^3}\} \leftrightsquigarrow \{\text{Spectra } A \text{ with } \phi : D^2 \to \mathrm{Aut}(A) \text{ s.t. } \phi_{S^1} = \eta_A\}$$

∼∼∼∼∼

We have seen a variety of examples of specters and have described all the specters associated to a few particular haunts. We now systematically investigate the equivalence classification of specters, giving a homotopy-theoretic description of specters that naturally parallels the characterization of haunts in proposition 2.2.6.

Ordinary line bundles with flat connection on X are classified by maps $X \to BR^\times$. There is a universal $R^\times$-bundle $ER^\times$ over $BR^\times$ and an associated R-bundle $P(R) := ER^\times \times_{R^\times} R$. A section s of the line bundle $L(c)$ associated to a map $c : X \to BR^\times$ is determined by a lift of c to a map $s : X \to P(R)$:

$$\begin{tikzcd} R \arrow[r] & P(R) \arrow[d] \\ X \arrow[ur, dashed, "s"] \arrow[r, "c"'] & BR^\times \end{tikzcd}$$

We describe the parametrized stable homotopy analog of this description. Haunts (the line bundles over the structure stack) on X are classified by maps $X \to B\,\mathrm{Pic}(S^0)$; as before $\mathrm{Pic}(S^0)$ denotes the realization of the category of invertible spectra. Let Sp_w denote the realization of the subcategory of weak equivalences of the category of spectra, and note that $\mathrm{Pic}(S^0)$ acts on Sp_w. There is a universal $\mathrm{Pic}(S^0)$-bundle $E\,\mathrm{Pic}(S^0)$ over $B\,\mathrm{Pic}(S^0)$ and an associated Sp_w-bundle $P(Sp_w) := E\,\mathrm{Pic}(S^0) \times_{\mathrm{Pic}(S^0)} Sp_w$. We can consider lifts of the classifying map $c : X \to B\,\mathrm{Pic}(S^0)$ to $P(Sp_w)$:

$$\begin{tikzcd} & Sp_w \arrow[r] & P(Sp_w) \arrow[d] \\ & X \arrow[ur, dashed] \arrow[r, "c"'] & B\,\mathrm{Pic}(S^0) \end{tikzcd}$$

Indeed, such lifts classify specters:

Proposition 2.2.15. *Let $c : X \to B\,\mathrm{Pic}(S^0)$ be the classifying map for a haunt $L(c)$. Weak equivalence classes of specters for the haunt $L(c)$ are in one-to-one correspondence with homotopy classes of lifts of c to maps $X \to P(Sp_w)$.*

That there should be such a homotopy-theoretic classification of specters was suggested to us by Bill Dwyer. A proof of a slightly stronger result will appear in section 3.2.2.

We conclude our discussion of haunts and specters with a few remarks about products. Given two ordinary invertible sheaves (line bundles) L and L' over X we can form their tensor product $L \otimes_{\mathcal{O}_X} L'$. Analogously, given two haunts we should be able to form their tensor product. Making sense of tensoring two modules over a stack of semi-ring categories would require a bit of doing; (see Dunn [12] for a definition of the tensor product of modules over a bipermutative category). Using the classification of haunts in proposition 2.2.6 we can side step this categorical tensor construction: define the product $L \otimes L'$ of two haunts to be the haunt classified by the product (in the group structure on $B\,\mathrm{Pic}(S^0)$) of the classifying maps $c, c' : X \to B\,\mathrm{Pic}(S^0)$. There is a product of specters covering this tensor product of haunts. The classifying projection $P(Sp_w) \to B\,\mathrm{Pic}(S^0)$ for specters is a map of multiplicative monoids. Given two specters T and T' classified by lifts $s, s' : X \to P(Sp_w)$ of the haunt maps $c, c' : X \to B\,\mathrm{Pic}(S^0)$, we simply define the product $T \wedge T'$ to be the specter (for the haunt $L \otimes L'$) classified by the product lift $s \cdot s' : X \to P(Sp_w)$.

Chapter 3

Local Systems of Categories of Spectra

In section 2, we described haunts as invertible sheaves, or more specifically as locally free rank-one modules over the structure stack of parametrized spectra; specters, the twisted parametrized spectra, were global sections of these modules. In this section, we leave behind that sheaf-theoretic approach and reformulate haunts as local systems of categories of spectra. Such a local system is determined by assigning to each open set in a cover a category equivalent to the category of ordinary, not parametrized, spectra and to the two-fold intersections in the cover compatible equivalences of these categories of spectra.

In order to formalize this local systems viewpoint, we need a clear notion of equivalence between two categories of spectra, which is to say we need a "category of categories of spectra"—such a category is typically referred to as a homotopy theory of homotopy theories. We chose to work with the model category of simplicial categories, which contains in particular the subcategory of simplicial categories weakly equivalent to the simplicial localization of the category of spectra. When the base space B has the structure of a simplicial complex, and the open sets of the cover of B are the stars of the simplicies, then the data of a local system amounts to a functor from the diagram category of simplicies of B into (a subcategory of weak equivalences in) the category of simplicial categories. This diagram functor approach has the disadvantage that it clouds the conceptual simplicity and the geometry of haunts and specters, but it has the advantage that it avoids the technicalities of stacks of ∞-categories and thereby eases the proofs of propositions 2.2.6 and 2.2.15.

After proving those propositions, we give *another* reformulation of haunts and specters: haunts are precisely A_∞ ring spectra arising as Thom spectra of multiplicative stable spherical fibrations on loop spaces, and the associated specters are simply modules over the ring spectrum. This last description has the advantage of being entirely elementary—it avoids both modules over stacks of parametrized spectra and diagrams in the model category of simplicial categories—but it thoroughly obscures various constructions with and applications of specters, and so in section 4 we return to our original sheaf-theoretic perspective.

3.1 The Homotopy Theory of Homotopy Theories

As it will play a central role in our discussion of local systems in section 3.2, we recall what is known about the homotopy theory of homotopy theories. In recent decades, model categories have been the predominant notion of abstract homotopy theory. However, there is not known to be a model structure on the category of model categories, and this is a huge impediment to constructing (as we are doing in this paper) bundles of homotopy theories. We must consider a weaker notion of abstract homotopy theory in order to have a decent homotopy theory of homotopy theories. There are various options, including the model category of Segal categories (due to Hirschowitz-Simpson [23]), the model category of complete Segal spaces (due to Rezk [42]), and the model category of simplicial categories (due to Dwyer-Hirschhorn-Kan [13] and Bergner [4]). Which we pick does not matter because all three model categories are Quillen equivalent (a result due to Bergner [4]); we work with the model category of simplicial categories, as this is the simplest to describe.

By a 'simplicial category' we will mean a category enriched over simplicial sets. Dwyer and Kan [15] realized that to a model category M there is canonically associated a simplicial category $L^H M$, the hammock localization of M, which encodes all of the homotopy-theoretic information contained in M. This is the sense in which simplicial categories are a faithful representation of abstract homotopy theories. In particular, the homotopy category $\mathrm{Ho}(M)$ of M is recovered as the category of components $\pi_0(L^H M)$ of the hammock localization; here the category of components $\pi_0(C)$ of a simplicial category C (also called the homotopy category of C) has the same objects as C, but has morphisms $\mathrm{Hom}_{\pi_0(C)}(x,y) = \pi_0(\mathrm{Hom}_C(x,y))$. A morphism $f : x \to y$ in a simplicial category C is called a homotopy equivalence if it becomes an isomorphism in $\pi_0(C)$.

The model structure on the category of simplicial categories is as follows. A map $\phi : C \to D$ of simplicial categories is a weak equivalence if it is a Dwyer-Kan equivalence, namely if ϕ is a weak equivalence on Hom sets and an equivalence on homotopy categories; that is, ϕ is a weak equivalence if $\phi : \mathrm{Hom}_C(x,y) \to \mathrm{Hom}_D(\phi(x),\phi(y))$ is a weak equivalence of simplicial sets for all objects $x, y \in C$, and if $\phi : \pi_0(C) \to \pi_0(D)$ is an equivalence of categories. A map $\phi : C \to D$ is a fibration if it is a fibration on Hom sets and if all homotopy equivalences in D lift to C; that is, ϕ is a fibration if $\phi : \mathrm{Hom}_C(x,y) \to \mathrm{Hom}_D(\phi(x),\phi(y))$ is a fibration of simplicial sets for all $x, y \in C$, and if for all objects $x \in C$ and all homotopy equivalences $h : \phi(x) \to z$ in D, there exists a homotopy equivalence $\widetilde{h} : x \to y$ in C such that $\phi(\widetilde{h}) = h$. Cofibrations of simplicial categories are determined, as usual, by the left lifting property. These define a model structure on the category of simplicial categories [4]. Given an object in a model category, there is a good notion of the space of automorphisms of that object, and in the following we will be focused on the automorphisms of the category of spectra (thought of as a simplicial category via its hammock localization).

Remark 3.1.1. Considering that our objection to model categories as a representation of homotopy theory was that there is no obvious model category of model categories, it seems odd to have insisted on having a model category of simplicial categories rather than merely a simplicial category of simplicial categories. Of course, we can recover a simplicial category of simplicial categories as the hammock localization of the model category of simplicial categories, but it would be better not to have to rely on the crutch of a model structure. The real solution to this and many other problems is to work directly in the ∞-category of ∞-categories. We do not do this because the details of such an ∞-categorical theory are not yet fully in place; see Lurie [31, 32], though, for substantial progress in that direction.

3.2 Diagrams of Simplicial Categories

We described haunts as locally free rank-one modules over the structure stack of parametrized spectra. We now reinterpret this notion in terms of diagrams of simplicial categories weakly equivalent to the hammock localization of the category of spectra. We then express the category of specters for such a haunt as a homotopy limit in the model category of simplicial categories.

3.2.1 Haunts

Let H be a haunt on the space X. By definition, if U is a sufficiently small open set in X, then the category $H(U)$ is homotopy equivalent to the category $\mathcal{O}_X(U)$ of parametrized spectra on U. If the subspace U is moreover contractible, then $\mathcal{O}_X(U)$ is homotopy (indeed Quillen) equivalent to the category Sp of spectra. (Roughly speaking, this follows because a fibrant parametrized spectrum is a quasifibration and any quasifibration over a contractible space is trivializable—see [37].) Let $\{U_i\}$ denote a contractible cover of X; to specify a haunt on X it will be sufficient, by the above remarks, to assign to the subsets U_i categories C_i that are each appropriately equivalent to the category of spectra and to the two-fold intersections $\{U_{ij}\}$ a collection of compatible equivalences $C_i \cong C_j$. We formalize these compatibilities using diagram functors into the category of simplicial categories.

First we fix some notation. The space X is homotopy equivalent to the realization of a simplicial set B; let $s(B)$ denote the category of simplicies of B, that is the category whose objects are the simplicies of B and whose morphisms are the face and degeneracy maps. Let $sCat$ denote the model category of simplicial categories, and let Sp now denote the object of $sCat$ given by the hammock localization of the category of spectra. Let $w(sCat, Sp)$ denote the weak equivalence component of $sCat$

containing Sp; that is, the objects of $w(sCat, Sp)$ are simplicial categories that can be connected to Sp by a zig-zag of weak equivalences, and the morphisms are weak equivalences of simplicial categories.

Remark 3.2.1. We pause to consider a few set-theoretic issues. The category $sCat$ is really the category of small simplicial categories. The hammock localization of the category of spectra is not only not small, it need not even have small Hom sets; we therefore chose a small simplicial category homotopically equivalent to that hammock localization—by the notation Sp we will implicitly refer to that small replacement. In a similar vein, the weak equivalence component of Sp in $sCat$ is not a small category; we will need to use it in constructions that only apply to small categories, so we chose a small subcategory of this weak equivalence component that is homotopically equivalent to the full component—we implicitly refer to that small replacement by the notation $w(sCat, Sp)$. We will not henceforth distinguish between such (simplicial) categories and their small replacements.

Recharacterization 3.2.2. A *haunt* over a simplicial set B is a functor from $s(B)$, the category of simplicies of B, to $w(sCat, Sp)$, the weak equivalence component of the category of simplicial categories containing the category of spectra. The *category of haunts* over B, denoted Haunt_B, is the full diagram category $w(sCat, Sp)^{s(B)}$.

We immediately have a notion of the *space of haunts*, namely the realization $|N.\,\mathrm{Haunt}_B\,|$ of the nerve of this diagram category. Note that we will not in general distinguish between simplicial sets and their realizations. There is a natural candidate for a classifying space for haunts, namely $N.w(sCat, Sp)$. The idea that a weak equivalence component of an object of a model category can function as a classifying complex is of course due to Dwyer and Kan [16]. Indeed, there is a suggestive homotopy equivalence $N.w(sCat, Sp) \simeq B\,\mathrm{haut}(Sp)$. Here $B\,\mathrm{haut}(Sp)$ is the nerve of the simplicial category with one object and with morphisms the simplicial monoid $\mathrm{haut}(Sp)$ of homotopy automorphisms of the category of spectra; (this simplicial monoid is defined to be the sub-simplicial monoid of $\mathrm{Hom}_{L^H(sCat)}(Sp, Sp)$ consisting of the components projecting to isomorphisms in $\mathrm{Hom}_{\pi_0(L^H(sCat))}(Sp, Sp)$). The classification of haunts can therefore be expressed as follows.

Theorem 3.2.3. *The space $N.\mathrm{Haunt}_B$ of haunts over a simplicial set B is weakly homotopy equivalent to the (derived) mapping space $\mathrm{Hom}(B, B\,\mathrm{haut}(Sp))$. In other words, $B\,\mathrm{haut}(Sp)$ is a classifying space for haunts.*

Remark 3.2.4. We adopt the convention that all mapping spaces are implicitly derived unless otherwise noted. We will also take "holim" and "hocolim" to refer to the homotopically invariant homotopy limit and colimit functors; these are sometimes referred to as the corrected homotopy limit and colimit and can be defined respectively by composing functorial objectwise fibrant or cofibrant replacement with the Bousfield-Kan holim or hocolim functor.

Before proving the theorem, we state one lemma:

Lemma 3.2.5. *Suppose M is a model category that is Quillen equivalent to a cofibrantly generated simplicial model category. Let B be a simplicial set and let $s(B)$ denote the category of simplicies of B. For any object X of M there is a weak homotopy equivalence*

$$N.(w(M,X)^{s(B)}) \simeq \operatorname*{holim}_{s(B)} N.w(M,X)$$

Dwyer and Kan prove this for M equal to the category of simplicial sets [16, Thm 3.4], but their proof works for any cofibrantly generated simplicial model category. Moreover, both sides of the equivalence are weakly homotopy invariant under Quillen equivalence between not-necessarily-simplicial model categories; this follows using various results from [14, 15].

Proof of Theorem 3.2.3. We have the chain of equivalences:

$$\begin{aligned} N.\mathrm{Haunt}_B \equiv N.(w(sCat, Sp)^{s(B)}) &\simeq \operatorname*{holim}_{s(B)} N.w(sCat, Sp) \\ &\simeq \mathrm{Hom}(N.s(B), N.w(sCat, Sp)) \\ &\simeq \mathrm{Hom}(B, B\,\mathrm{haut}(Sp)) \end{aligned}$$

The category of simplicial categories is Quillen equivalent to the category of complete Segal spaces [4] which is a cofibrantly generated simplicial model category [42]; the first equivalence therefore follows from the above lemma. The second equivalence is a consequence of [22, Prop 18.2.6], and the third is immediate. □

3.2.2 Specters

We now discuss specters in this new context of diagrams of simplicial categories. In the sheaf-theoretic framework of section 2 a specter was a global section of a haunt. A haunt is now a functor H from the category of simplicies $s(B)$ into the category $sCat$ of simplicial categories. Naturally enough, a "section" of such a diagram H of simplicial categories should be some appropriately consistent choice of objects $\{x_b \in H(b)\}_{b\in s(B)}$ of the simplicial categories $H(b)$ in the diagram—we can think of an object $x_b \in H(b)$ as a locally constant section of the haunt H restricted to the simplex b. It would be too much to ask that the collection of objects x_b be strictly compatible with the morphisms in the diagram H. Instead, we merely demand that there be chosen homotopies to "glue the objects together"—this gluing data is formally encoded in a homotopy limit.

Recharacterization 3.2.6. Let H be a haunt over B, that is a functor from the category of simplicies $s(B)$ to the category of simplicial categories that lands in the weak equivalence component $w(sCat, Sp)$ of the category of spectra. The (simplicial) *category of specters* for the haunt H, denoted Specter_H, is defined to be the homotopy limit $\operatorname*{holim}_{s(B)} H$.

Of course, this homotopy limit is only defined because we have a model structure on the category of simplicial categories, and as usual we mean the homotopically invariant homotopy limit. We also have an associated *space of specters* for the haunt H, namely $N.w(\text{Specter}_H)$. Here $w(C)$ denotes the sub-simplicial category of the simplicial category C whose objects are the same as those of C but whose morphisms $\text{Hom}_{w(C)}(a, b)$ are the components of $\text{Hom}_C(a, b)$ projecting to isomorphisms in $\text{Hom}_{\pi_0(C)}(a, b)$; note that $N.w(C)$ is, a priori, a bisimplicial set and we implicitly take its diagonal.

There isn't a classifying space for specters, per se, but there is a classifying fibration; that is, there is a fibration $\psi : U\mathrm{haut}(Sp) \to B\,\mathrm{haut}(Sp)$ such that the space of specters for a fixed haunt $H : B \to B\,\mathrm{haut}(Sp)$ is homotopy equivalent to the space of lifts of H along ψ. The fibre of ψ should be the "space of spectra" and we think of the total space $U\mathrm{haut}(Sp)$ as the "universal haunt". In fact, this classifying fibration comes from a fibration of simplicial categories, which is defined as the diagonal map in the following diagram:

$$\begin{tikzcd} \operatorname*{hocolim}_{w(sCat,Sp)}(-) \arrow[r, hook, "\sim"] \arrow[d] & \overline{\operatorname*{hocolim}_{w(sCat,Sp)}(-)} \arrow[dl, "\Psi"] \\ \operatorname*{hocolim}_{w(sCat,Sp)} * & \end{tikzcd}$$

That is, we factor the left hand map by a weak equivalence followed by a fibration Ψ; this fibration is the desired specter classifying fibration of simplicial categories. Here $\operatorname*{hocolim}_{w(sCat,Sp)}(-)$ refers to the homotopy colimit of the inclusion $w(sCat, Sp) \to sCat$.

The analogous specter classifying fibration of spaces is the right hand vertical arrow in the diagram

$$\begin{tikzcd} N.w\left(\overline{\operatorname*{hocolim}_{w(sCat,Sp)}(-)}\right) \arrow[rr, hook, "\sim"] \arrow[d] & & U\,\mathrm{haut}(Sp) \arrow[d, "\psi"] \\ N.w\left(\operatorname*{hocolim}_{w(sCat,Sp)} *\right) \arrow[r, leftrightarrow, "\simeq"] & N.w(sCat, Sp) \arrow[r, leftrightarrow, "\simeq"] & B\,\mathrm{haut}(Sp) \end{tikzcd}$$

The fibration factorization here defines the space $U\,\mathrm{haut}(Sp)$. Note that the fibre of ψ over the point $X \in w(sCat, Sp)$ is weakly equivalent to $N.w(X)$ which is in turn weakly equivalent to the space of spectra $N.w(Sp)$. Morally speaking, the bundle $U\mathrm{haut}(Sp)$ is the bundle $E\,\mathrm{haut}(Sp) \times_{\mathrm{haut}(Sp)} N.w(Sp)$ associated to the tautological bundle $E\,\mathrm{haut}(Sp) \to B\,\mathrm{haut}(Sp)$. However, on its face the action of the simplicial monoid $\mathrm{haut}(Sp)$ on $N.w(Sp)$ is only defined up to weak homotopy, which is insufficient for defining the associated bundle; presumably the action can be made strict, but we do not pursue that here.

We now state the simplicial-category-level classification result for specters:

Theorem 3.2.7. *Let $H : s(B) \longrightarrow w(sCat, Sp)$ be a functor defining a fixed haunt on the simplicial set B. This functor determines an associated map $h : \operatorname*{hocolim}_{s(B)} * \longrightarrow \operatorname*{hocolim}_{w(sCat,Sp)} *$ classifying the haunt. The category of specters* $\mathrm{Specter}_H$ *for this haunt is weakly equivalent, as a simplicial category, to the category of lifts of h along the specter classifying fibration:*

$$\begin{tikzcd} & \overline{\operatorname*{hocolim}_{w(sCat,Sp)} (-)} \arrow[d, "\Psi"] \\ \operatorname*{hocolim}_{s(B)} * \arrow[r, "h"'] \arrow[ur, dashed] & \operatorname*{hocolim}_{w(sCat,Sp)} * \end{tikzcd}$$

In other words the category of specters is weakly equivalent to a (derived) mapping space in the overcategory of $\operatorname*{hocolim}_{w(sCat,Sp)} *$, namely

$$\mathrm{Specter}_H \simeq \mathrm{Hom}_{\operatorname*{hocolim}\limits_{w(sCat,Sp)} *}\left(\operatorname*{hocolim}_{s(B)} *, \overline{\operatorname*{hocolim}_{w(sCat,Sp)} (-)}\right)$$

Proof. The first step in the proof is a consequence of the following general lemma:

Lemma 3.2.8. *Let M be either an ∞-topos (such as simplicial sets or spaces) or a model category of homotopy theories (such as simplicial categories, complete Segal spaces, or quasi-categories). Let D be a small category and $F : D \longrightarrow M$ a functor. Denote by $\phi : \operatorname*{hocolim}_D F \longrightarrow \operatorname*{hocolim}_D *$ the natural projection. Provided F takes all morphisms in D to weak equivalences in M, the homotopy limit of F is weakly equivalent to the object of derived sections of the map ϕ; that is*

$$\operatorname*{holim}_D F \simeq \mathrm{Hom}_{\operatorname*{hocolim}\limits_D *}\left(\operatorname*{hocolim}_D *, \operatorname*{hocolim}_D F\right)$$

In the case of specters, we therefore have the equivalence

$$\mathrm{Specter}_H \equiv \operatorname*{holim}_{s(B)} H \simeq \mathrm{Hom}_{\operatorname*{hocolim}\limits_{s(B)} *}\left(\operatorname*{hocolim}_{s(B)} *, \operatorname*{hocolim}_{s(B)} H\right)$$

We want to translate this mapping space into a space of lifts of the classifying map $h : \underset{s(B)}{\operatorname{hocolim}} * \to \underset{w(sCat,Sp)}{\operatorname{hocolim}} *$. We begin by rewriting one of the homotopy colimits in terms of a larger indexing category:

$$\underset{s(B)}{\operatorname{hocolim}} H \simeq h^* \left(\underset{w(sCat,Sp)}{\operatorname{hocolim}} (-) \right)$$

Here h^* denotes the derived pullback from the overcategory in $sCat$ of $\underset{w(sCat,Sp)}{\operatorname{hocolim}} *$ to the overcategory of $\underset{s(B)}{\operatorname{hocolim}} *$. Next, we have a Quillen adjunction

$$sCat / \left(\underset{s(B)}{\operatorname{hocolim}} * \right) \underset{h_!}{\overset{h^*}{\rightleftarrows}} sCat / \left(\underset{w(sCat,Sp)}{\operatorname{hocolim}} * \right)$$

where the pushforward $h_!$ is given by precomposition with the map h. This adjunction leads to an equivalence of function complexes

$$\operatorname{Hom}_{\underset{s(B)}{\operatorname{hocolim}} *} \left(\underset{s(B)}{\operatorname{hocolim}} *, h^* \left(\underset{w(sCat,Sp)}{\operatorname{hocolim}} (-) \right) \right) \simeq \operatorname{Hom}_{\underset{w(sCat,Sp)}{\operatorname{hocolim}} *} \left(\underset{s(B)}{\operatorname{hocolim}} *, \underset{w(sCat,Sp)}{\operatorname{hocolim}} (-) \right)$$

as desired. □

Not surprisingly, the analogous result at the level of spaces is the following.

Corollary 3.2.9. *Let $h : B \to B\operatorname{haut}(Sp)$ denote the classifying map for a fixed haunt H over the simplicial set B. The associated space of specters $N.w\operatorname{Specter}_H$ is weakly homotopy equivalent to the space $\operatorname{Hom}_{B\operatorname{haut}(Sp)}(B, U\operatorname{haut}(Sp))$ of maps from B to the universal haunt $U\operatorname{haut}(Sp)$ that commute with projection to $B\operatorname{haut}(Sp)$. In other words, the space of specters for the haunt H is weakly equivalent to the space of lifts in the diagram*

$$\begin{tikzcd} & U\operatorname{haut}(Sp) \arrow[d, "\psi"] & N.w(Sp) \arrow[l] \\ B \arrow[r, "h"'] \arrow[ur, dashed] & B\operatorname{haut}(Sp) & \end{tikzcd}$$

Proof. The chain of equivalences is

$$
\begin{aligned}
N.w\,\mathrm{Specter}_H = N.w \operatorname*{holim}_{s(B)} H &\simeq N.w\,\mathrm{Hom}_{\operatorname*{hocolim}_{w(sCat,Sp)} *}\left(\operatorname*{hocolim}_{s(B)} *, \operatorname*{hocolim}_{w(sCat,Sp)}(-)\right) \\
&= N.w\,\mathrm{Hom}_{\operatorname*{hocolim}_{w(sCat,Sp)} *}\left(\operatorname*{hocolim}_{s(B)} *, \overline{\operatorname*{hocolim}_{w(sCat,Sp)}(-)}\right) \\
&\simeq N.\,\mathrm{Hom}_{\operatorname*{hocolim}_{w(sCat,Sp)} *}\left(\operatorname*{hocolim}_{s(B)} *, w\left(\overline{\operatorname*{hocolim}_{w(sCat,Sp)}(-)}\right)\right) \\
&\simeq \mathrm{Hom}_{N.w(sCat,Sp)}\left(N.s(B), N.w\left(\overline{\operatorname*{hocolim}_{w(sCat,Sp)}(-)}\right)\right) \\
&\simeq \mathrm{Hom}_{B\,\mathrm{haut}(Sp)}(B, U\,\mathrm{haut}(Sp))
\end{aligned}
$$

.

The first homotopy equivalence is a consequence of the theorem, and the second line follows from the definition of the Hom set as a derived mapping space. Next note that the functor $w : sCat \to hGpd$ from simplicial categories to homotopy groupoids is right adjoint to the inclusion—the homotopy equivalence in the third line follows because $\operatorname*{hocolim}_{s(B)} *$ and $\operatorname*{hocolim}_{w(sCat,Sp)} *$ are already homotopy groupoids. The category of homotopy groupoids is in fact Quillen equivalent to the category of simplicial sets; the equivalence in the fourth line follows, and the fifth line is immediate. □

3.3 A_∞ Thom Spectra on Loop Spaces

We begin this section by identifying the monoid $\mathrm{haut}(Sp)$ of automorphisms of the category of spectra with a classifying space $\mathbb{Z} \times BG$ for stable spherical fibrations. Using this identification, we can associate to a haunt over X an A_∞ ring spectrum arising as a Thom spectrum over the loop space of X. This in turn allows us to recharacterize the category of specters for the haunt as a category of modules over that ring spectrum. This recharacterization on the one hand obscures the intrinsic symmetry of specters and breaks the natural connection with parametrized homotopy theory, which is essential to the definition of specter invariants in section 4; on the other hand, because module spectra are familiar objects, the change in perspective demystifies specters and will be important in applications to symplectic Floer homotopy [9].

3.3.1 Automorphisms of the Category of Spectra

Haunts over a space X are classified by maps from X to the space $B\,\mathrm{haut}(Sp)$, a deloop of the simplicial monoid of homotopy automorphisms of the category of spectra. We investigate the homotopy type of this classifying space. The category of spectra has a natural monoidal structure, the smash product; given an invertible spectrum J, the functor $J \wedge - : Sp \to Sp$ smashing with J determines a self homotopy equivalence of the category of spectra. Roughly speaking, this association determines a map from the category of invertible spectra (which we called $\mathrm{Pic}(S^0)$ in section 2) to the space of self equivalences $\mathrm{haut}(Sp)$. That this map is a weak equivalence is well known to experts, but we are not aware of a statement or a proof in the literature:

Theorem 3.3.1. *Let* $\mathrm{Pic}(S^0)$*, the Picard category, denote the subcategory of the category of spectra whose objects are invertible spectra and whose morphisms are weak equivalences. There is a weak equivalence*

$$\mathrm{Pic}(S^0) \simeq \mathrm{haut}(Sp)$$

from the nerve of the Picard category to the simplicial set of self homotopy equivalences of the category Sp *of spectra.*

Proof. We merely sketch the proof. The category of spectra is a model category representing a particular homotopy theory, and we can work in any of a number of equivalent categories of homotopy theories. We have primarily utilized $sCat$, the category of simplicial categories, but for this theorem it is more convenient to work in $qCat$, the category of quasi-categories—there is a Quillen equivalence between $sCat$ and $qCat$ [32]. Recall that a quasi-category is a simplicial set that satisfies a weak Kan condition, namely that a horn $\partial\Delta^n \backslash \Delta_i^{n-1}$ fills in provided the missing face Δ_i^{n-1} is internal, that is $0 < i < n$; this weak Kan condition reflects the idea that morphisms (edges) in a category are composable, but need not be invertible up to homotopy. By $qCat$ we refer, in fact, to the category of simplicial sets equipped with a model structure in which quasicategories are precisely the fibrant objects.

Let Sp denote a quasicategory modeling the category of spectra; we presume that Sp is equipped with a monoidal structure modeling the smash product. In this context $\mathrm{Pic}(S^0)$ is a subquasicategory of Sp which is described as follows. The vertices of $\mathrm{Pic}(S^0)$ are the invertible objects in Sp, that is the vertices $v \in Sp$ such that there exists a $w \in Sp$ with $v \wedge w$ weakly equivalent to $S^0 \in Sp$; the k-simplicies of $\mathrm{Pic}(S^0)$ are the k-simplicies of Sp all of whose vertices are invertible and all of whose edges are weak equivalences. By definition, the simplicial monoid $\mathrm{haut}(Sp)$ has k-simplicies the set of weak equivalences $\Delta^k \times Sp \xrightarrow{\simeq} Sp$. There is now a natural map

$$\mu : \mathrm{Pic}(S^0) \to \mathrm{haut}(Sp)$$

which takes a k-simplex P in $\mathrm{Pic}(S^0)_k$ to the composite $\Delta^k \times Sp \xrightarrow{P\times \mathrm{id}} \mathrm{Pic}(S^0) \times Sp \to Sp \times Sp \to Sp$. This composite is an equivalence and is therefore a k-simplex in $\mathrm{haut}(Sp)$.

We would like μ to be an equivalence. It suffices to show that any map $F : (\Delta^k, \partial\Delta^k) \to (\mathrm{haut}(Sp), \mathrm{Pic}(S^0))$ is homotopic, relative to its boundary, to a map $F' : (\Delta^k, \partial\Delta^k) \to (\mathrm{Pic}(S^0), \mathrm{Pic}(S^0))$. Suppose $k = 0$, so F is simply an equivalence $Sp \to Sp$; we take F' to be the map $Sp \to Sp$ given by smashing with $F(S^0)$, that is, $F' = F(S^0) \in \mathrm{Pic}(S^0)$. One extremely convenient feature of quasicategories (as distinguished from, for example, model categories) is that one can naturally take homotopy colimits over any simplicial set, not only over a category; this feature is helpful in defining a comparison map between F and F'. The quasicategory Sp is in particular a simplicial set, and given an object $X \in Sp$, let $(-/X)$ denote the "overcategory" of X, that is the subsimplicial set of Sp whose 0-simplicies are maps $Y \to X$. Moreover, denote by $(-/X)_{\mathrm{sph}}$ the corresponding "spherical subcategory", that is the full subcategory whose 0-simplicies are the maps $S^i \to X$. We have the comparison map

$$F(S^0) \wedge X \simeq \operatorname*{hocolim}_{(-/X)_{\mathrm{sph}}} F(S^i) \to \operatorname*{hocolim}_{(-/X)} F(Y) \simeq F(X)$$

This map is an equivalence when $X = S^0$, and the left hand side preserves homotopy colimits. It is a consequence of Lurie's extensive work on quasicategories [32] that because F is an equivalence, it preserves homotopy colimits. The comparison map is therefore a natural weak equivalence, as desired. The cases of higher k could be handled similarly. □

Corollary 3.3.2. *The simplicial set* $\mathrm{haut}(Sp)$ *of self equivalences of the category of spectra has the homotopy type* $\mathbb{Z} \times BG$ *where* G *is the space of stable self homotopy equivalences of the sphere, that is* $G = \mathrm{haut}(S^0) \simeq \operatorname*{colim}_n \mathrm{haut}(S^n)$ *where in the last expression* S^n *denotes the ordinary n-sphere.*

Proof. Given the theorem, this is a consequence of the weak equivalence $\mathrm{Pic}(S^0) \simeq \mathbb{Z} \times BG$. To see that equivalence, first note that any invertible spectrum is weakly equivalent to some shift S^n of the sphere spectrum. Thus the category $\mathrm{Pic}(S^0)$ has $\mathbb{Z}$ components; the n-th component is all spectra weakly equivalent to S^n together with all weak equivalences between them. By Dwyer and Kan's classification theorem [16], this component has the homotopy type $B\,\mathrm{haut}(S^n) \simeq B\,\mathrm{haut}(S^0)$. □

3.3.2 Specters as Module Spectra

Armed with the identification of the classifying space $B\,\mathrm{haut}(Sp)$ with $B(\mathbb{Z} \times BG)$, we can describe the A_∞ ring spectrum corresponding to a haunt. Let $h : X \to B\,\mathrm{haut}(Sp) \simeq B(\mathbb{Z} \times BG)$ be the classifying map for a haunt over the space X. The map $\Omega h : \Omega X \to \mathbb{Z} \times BG$ defines a stable spherical fibration, which we will denote $\eta(h)$, over ΩX. Because Ωh is a loop map, the spherical fibration $\eta(h)$ is multiplicative and the associated Thom spectrum $\mathrm{Th}(\eta(h))$ is therefore an A_∞ ring spectrum [33]. The fibration $\eta(h)$ can be thought of more geometrically as follows. The haunt is a local system or bundle over X whose fibre is the category of spectra, and the monodromy of the haunt around a loop $\ell \in \Omega X$ is an invertible spectrum, namely the fibre $\eta(h)_\ell$; the multiplicative structure of the spherical fibration corresponds, naturally enough, to the composition of the loop monodromies.

If the Thom spectrum $\mathrm{Th}(\eta(h))$ encodes the structure of the haunt h, we might expect to be able to describe the associated category of specters in terms of this Thom spectrum. Let $\mathbb{T}$ be a specter for the haunt h and suppose $\mathbb{T}$ is fibrant in the sense that it is locally isomorphic to a quasifibration of spectra. Given a loop $\ell : S^1 \to X$ in X, we can pull $\mathbb{T}$ back to a specter $\ell^*\mathbb{T}$ on S^1. From example 2.2.12 we know that this specter is determined by giving the spectrum T at the basepoint of S^1 together with an equivalence $\phi_\ell : \eta(h)_\ell \wedge T \simeq T$; in other words, we need to glue T back to itself, but shifted by the monodromy sphere $\eta(h)_\ell$ along the given loop. The family of compatible equivalences $\{\phi_\ell\}_{\ell \in \Omega X}$ amounts precisely to an action of $\mathrm{Th}(\eta(h))$ on T. To a specter for the haunt h we can therefore associate a module over the ring spectrum $\mathrm{Th}(\eta(h))$; indeed there is an equivalence of categories:

Proposition 3.3.3. *Let H denote a fixed haunt over X with classifying map $h : X \to B(\mathbb{Z} \times BG)$. The loop map Ωh determines a multiplicative stable spherical fibration $\eta(h)$ with associated Thom spectrum $\mathrm{Th}(\eta(h))$ an A_∞ ring spectrum. There is a weak equivalence of simplicial categories*

$$\mathrm{Specter}_H \simeq L^H(\mathrm{Th}(\eta(h))\text{-}mod)$$

between the category of specters for H and the hammock localization of the category of module spectra over the Thom spectrum $\mathrm{Th}(\eta(h))$.

That specters can be thought of as modules over a ring spectrum was also realized by Mike Hopkins and Jeff Smith, and the formulation here owes various details to discussions with them.

Sketch of proof. We have already done most of the work in establishing, in theorem 3.2.7, that

$$\mathrm{Specter}_H \simeq \mathrm{Hom}_{\underset{w(sCat,Sp)}{\mathrm{hocolim}} *}\left(\underset{s(B)}{\mathrm{hocolim}} *, \underset{w(sCat,Sp)}{\mathrm{hocolim}}(-)\right)$$

Here B is a simplicial set with the homotopy type of X. By theorem 3.3.1 the base space $\underset{w(sCat,Sp)}{\mathrm{hocolim}} * \simeq N.w(sCat, Sp)$ has the homotopy type $B\,\mathrm{Pic}(S^0)$. In particular, we can model this space by the simplicial category, also denoted $B\,\mathrm{Pic}(S^0)$, with just one object $*$ and with morphism space the (simplicial) monoid $\mathrm{Pic}(S^0)$. Similarly, the total space $\underset{w(sCat,Sp)}{\mathrm{hocolim}}(-)$ of the classifying fibration can be explicitly modeled by a simplicial category, denoted $U\,\mathrm{Pic}(S^0)$, as follows. The objects of $U\,\mathrm{Pic}(S^0)$ are ordinary cofibrant and fibrant spectra. The morphism space $\mathrm{Hom}_{U\,\mathrm{Pic}(S^0)}(T, S)$ of $U\,\mathrm{Pic}(S^0)$ is the homotopy colimit of the functor $\mathrm{Hom}_{Sp}(- \wedge T, S) : \mathrm{Pic}(S^0)_{cf} \to sSet$. Here $\mathrm{Pic}(S^0)_{cf}$ denotes the category of cofibrant and fibrant invertible spectra. The idea behind this construction is that roughly speaking a morphism from T to S in $U\,\mathrm{Pic}(S^0)$ should consist of a pair (γ, ϕ) of an invertible spectrum $\gamma \in \mathrm{Pic}(S^0)$ and a morphism of spectra $\phi : \gamma \wedge T \to S$. Because $\underset{\mathrm{Pic}(S^0)_{cf}}{\mathrm{hocolim}} * \simeq \mathrm{Pic}(S^0)_{cf}$, the projection $\mathrm{Hom}_{Sp}(- \wedge T, S) \to *$ induces a map $U\,\mathrm{Pic}(S^0) \to B\,\mathrm{Pic}(S^0)$. Thinking of $X \simeq \underset{s(B)}{\mathrm{hocolim}} *$ as a simplicial category, we therefore have the reformulation:

$$\mathrm{Hom}_{\underset{w(sCat,Sp)}{\mathrm{hocolim}} *}\left(\underset{s(B)}{\mathrm{hocolim}} *, \underset{w(sCat,Sp)}{\mathrm{hocolim}}(-)\right) \simeq \mathrm{Hom}_{B\,\mathrm{Pic}(S^0)}(X, U\,\mathrm{Pic}(S^0))$$

The final equivalence is

$$\mathrm{Hom}_{B\,\mathrm{Pic}(S^0)}(X, U\,\mathrm{Pic}(S^0)) \simeq L^H(\mathrm{Th}(\eta(h))\text{-mod})$$

On this count we merely indicate the map from the right to the left hand side. We can model X by the simplicial category with one object and with morphism space ΩX. The map $h : X \to B\,\mathrm{Pic}(S^0)$ corresponds to the map of morphism spaces $\Omega X \to \mathrm{Pic}(S^0)$ classifying the spherical fibration $\eta(h)$. Given a $\mathrm{Th}(\eta(h))$-module T, one immediately has, for any k-simplex $s : \Delta^k \to \Omega X$, a morphism $\mathrm{Th}(\eta(h)|_s) \wedge T \to T$. By a shift in perspective, the stable spherical bundle $\eta(h)|_s$

on Δ^k naturally corresponds to a k-simplex in the morphism category $\mathrm{Pic}(S^0)$ of $B\,\mathrm{Pic}(S^0)$, and the map $\mathrm{Th}(\eta(h)|_s) \wedge T \to T$ then provides a lift of this morphism k-simplex to $U\,\mathrm{Pic}(S^0)$, as desired. $\square$

Example 3.3.4. Consider again the haunt L_n over S^1 whose monodromy is suspension by S^n. This haunt is classified by the map $S^1 \to B(\mathbb{Z} \times BG)$ representing $n \in \mathbb{Z} = \pi_1(B(\mathbb{Z} \times BG))$. The loop of this map, $\Omega S^1 \simeq \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z} \times BG$, classifies the fibration over $\mathbb{Z}$ whose fibre at i is $S^{n \cdot i}$. The associated Thom spectrum is $\bigvee_{i \in \mathbb{Z}} S^{n \cdot i}$, which we considered as a specter already in example 2.2.12. The set of specters for L_n is, as we saw in that example, the same as the set of module spectra over this distinguished specter $\bigvee_{i \in \mathbb{Z}} S^{n \cdot i}$.

Chapter 4

Invariants of Specters

A parametrized spectrum P over a space X has two naturally associated spectra, namely the total spectrum P/X representing the homotopy type of P and the spectrum of sections $\Gamma(P)$ representing the cohomotopy type of P. The generalized homology groups of these associated spectra provide invariants of the parametrized spectrum. A specter, that is a twisted parametrized spectrum, has no globally defined homotopy or cohomotopy type analogous to the total spectrum P/X or the spectrum of sections $\Gamma(P)$. Nevertheless, it is frequently possible to define invariants associated to a specter by first applying a generalized homology functor and *then* taking an associated global spectrum, rather than vice versa as is typical in parametrized homotopy theory.

We return to the stack-theoretic perspective of section 2. There, we treated the category of spectra as a ring and defined haunts to be locally free rank-one modules over a parametrized version of this ring; specters were global sections of these modules. We begin this section by describing an analogous construction where the basic ring is the category of R-modules for a commutative ring spectrum R. This leads to a notion of R-haunt and R-specter. We then see how to associate to a haunt H an R-haunt H_R and to a specter T for H an R-specter T_R for H_R; this base change is the aforementioned "generalized homology functor". When the R-haunt H_R is trivializable, the R-specter T_R has the form of a parametrized R-module and therefore has a global homotopy type T_R/X. The homotopy groups of T_R/X are the R-homology invariants of the original specter T. We describe a few examples of these specter invariants and discuss a spectral sequence for computing them.

4.1 R-Haunts and R-Specters

The category of spectra, or equivalently the category of modules over the sphere spectrum S^0, has a smash product which, roughly speaking, gives it the structure of a commutative ring. The stack of parametrized spectra on a space X therefore functions as a sheaf of rings, and we characterized haunts as locally free rank-one modules over this stack. There are natural subcategories of the category of spectra that have their own commutative products, and we study modules over the stacks associated to these categorical rings. Specifically, if R is an A_∞ ring spectrum, then we have the category R-mod of modules over R. If R is moreover E_∞, then there is a natural product $\wedge_R$ which, roughly speaking, gives R-mod the structure of a commutative ring; see Dunn [12] for a detailed discussion of the monoidal structures on such module categories. We can associate to an open set $U \subset X$ the category $\mathcal{O}_X^R(U)$ of parametrized R-modules on U. These parametrized R-modules form a stack, which again has a monoidal structure coming from $\wedge_R$. Morally, an R-haunt is a locally free rank-one module over this structure stack $\mathcal{O}_X^R$ of parametrized R-modules. The stack $\mathcal{O}_X^R$ is naturally a stack of ∞-categories, and by a module over this stack we refer to a stack of ∞-categories with an appropriate action of $\mathcal{O}_X^R$. A more complete definition of R-haunt is therefore as follows:

Definition 4.1.1. An *R-haunt* is a stack $\mathcal{M}$ of ∞-categories on a space X with an action of the monoidal stack of ∞-categories $\mathcal{O}_X^R$ of parametrized R-modules satisfying the following condition: the stack $\mathcal{M}$ is locally free of rank one in the sense that for all points $x \in X$ there exists an open set $U \subset X$ containing x and an object $Q \in \mathcal{M}(U)$ such that the map $\mathcal{O}_X^R|_U \to \mathcal{M}|_U$ given by $A \mapsto A \cdot Q$ is an equivalence of stacks of ∞-categories. A twisted parametrized R-module or *R-specter* is a global section of an R-haunt.

In order to make this precise, one must give a thorough treatment of stacks of

∞-categories and of monoidal stacks of ∞-categories; we do not do this, but note that Lurie's work [32] provides key elements of such a treatment. Also note that, as in remark 2.2.2, the additive structure on these stacks is given by the categorical coproduct and so is justifiably ignored—this saves us the horror of contemplating symmetric bimonoidal stacks of ∞-categories.

An R-haunt is locally equivalent to the stack of parametrized R-modules and as such can be specified concretely in terms of gluing functions. Suppose $\{U_i\}$ is an open cover of X; on an intersection U_{ij} a gluing function is a self equivalence of $\mathcal{O}_X^R|_{U_{ij}}$ as a module over itself. Such an equivalence is given by smashing with an invertible parametrized R-module. Let $\mathrm{Pic}(R)$ denote the category of invertible R-modules together with their homotopy equivalences, and let $\mathcal{P}ic(R)$ denote the corresponding sheaf of invertible parametrized R-modules. The gluing data for an R-haunt is therefore a 1-cocycle c with values in $\mathcal{P}ic(R)$, which is to say a compatible system c_{ij} of R-module gluing functions on the one-fold intersections of the cover. Sensibly enough, R-haunts on X are classified by homotopy classes of maps from X to $B\,\mathrm{Pic}(R)$.

An R-specter looks locally like a parametrized R-module, but has a global twist determined by the R-haunt. Such a twisted parametrized R-module can be presented as follows: to describe an R-specter for the R-haunt associated to a gluing function c for the cover $\{U_i\}$ it suffices to give a parametrized R-module f_i on each open set U_i together with compatible equivalences between $f_i \wedge_{(R,U_{ij})} c_{ij}$ and f_j. Compare section 2.2.2. There is a classification of R-specters analogous to that of ordinary specters in proposition 2.2.15, but we do not go into detail.

∼∼∼∼∼

In section 3.2.1 we formulated haunts not as stacks of ∞-categories, but as diagrams in the category of ∞-categories—or more precisely in the category $sCat$ of simplicial categories. The diagram had the homotopy type of the base space X, and the functor to $sCat$ took objects to simplicial categories weakly equivalent to the category of spectra, and morphisms to weak equivalences. A similar approach is possible for R-haunts but it requires more work. Specifically, let R-mod denote the ∞-category of R-modules; this is an element of the category ∞-*Cat* of ∞-categories. Now define (R-mod)-mod to be the subcategory of ∞-*Cat* of modules over R-mod; in turn $w(\text{(R-mod)-mod}, \text{R-mod})$ denotes the subcategory of (R-mod)-mod of objects weakly equivalent to R-mod, together with the (R-mod)-module weak equivalences between them. Finally, an R-haunt would be a functor from an appropriate diagram homotopy equivalent to X into $w(\text{(R-mod)-mod}, \text{R-mod})$. The associated ∞-category of R-specters would be the homotopy limit of this functor.

There is also a Thom spectrum approach to R-haunts and R-specters. Suppose an R-haunt H_R is classified by the map $h_R : X \to B\,\mathrm{Pic}(R)$. The loop of this map $\Omega h_R : \Omega X \to \mathrm{Pic}(R)$ classifies a fibration $\eta(h_R)$ of invertible R-modules over ΩX; (indeed, the homotopy type of $\mathrm{Pic}(R)$ is $\mathrm{Pic}^0(R) \times BGL_1(R)$ where $\mathrm{Pic}^0(R)$ denotes the equivalence classes of invertible R-modules, and $GL_1(R)$ denotes the R-module self-equivalences of R). This fibration $\eta(h_R)$ is in particular a parametrized spectrum and has an associated total spectrum $\eta(h_R)/X$ which we denote suggestively $\mathrm{Th}(\eta(h_R))$. This total spectrum is an associative R-algebra and it encodes the structure of the R-haunt. An associated R-specter is simply a $\mathrm{Th}(\eta(h_R))$-module, by which we mean an R-module M together with an appropriately compatible action $\mathrm{Th}(\eta(h_R)) \wedge_R M \to M$.

4.2 R-Homology of Specters

We now describe how to associate to haunts and specters respectively R-haunts and R-specters and we discuss the basic construction of specter invariants. There is a natural map S^0-mod $\to$ R-mod from the category of spectra to the category of R-modules, given by $T \mapsto T \wedge_{S^0} R$. This is a map of rings and it underlies the fundamental base-change operation from specters to R-specters. Given a haunt, that is a locally free rank-one module $\mathcal{M}$ over the stack of parametrized spectra $\mathcal{O}_X$, we can form the tensor stack $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X^R$ where $\mathcal{O}_X^R$ is the stack of parametrized R-modules. This tensor will be an R-haunt. (See Dunn [12] for a definition of tensoring over ring categories.) Any global section $P \in \mathcal{M}(X)$ of $\mathcal{M}$ transforms to the section $P \otimes R$; we therefore have an R-specter associated to any ordinary specter.

In more down-to-earth terms, a haunt is presented by gluing together trivial bundles using invertible parametrized S^0-modules c_{ij}; the gluing functions for the associated R-haunt are simply $c_{ij} \wedge R$. Similarly, a specter is locally given by parametrized spectra f_i and the associated R-specter is presented by the parametrized R-modules $f_i \wedge R$. Implicit here is the fact that the map S^0-mod $\to$ R-mod restricts to a map $\mathrm{Pic}(S^0) \to \mathrm{Pic}(R)$ of Picard groups and this latter map deloops to a map $B\,\mathrm{Pic}(S^0) \to B\,\mathrm{Pic}(R)$ of classifying spaces. Thus we also have the purely homotopy-theoretic characterization of the haunt transformation, namely that the R-haunt associated to the haunt $X \to B\,\mathrm{Pic}(S^0)$ is classified by the composite $X \to B\,\mathrm{Pic}(S^0) \to B\,\mathrm{Pic}(R)$.

Remark 4.2.1. The map S^0-mod $\to$ R-mod given by smashing with R makes sense for any A_∞ ring spectrum R. This map moreover induces a map $GL_1(S^0) \to GL_1(R)$ and even a map $BGL_1(S^0) \to BGL_1(R)$. However, in order to build the R-haunt associated to an ordinary haunt, we need moreover a map $B(\mathbb{Z} \times BGL_1(S^0)) \to B(\mathrm{Pic}^0(R) \times BGL_1(R))$. Even barring the issue of what $\mathrm{Pic}^0(R)$ should mean, the space $BGL_1(R)$ cannot deloop unless R is commutative. This provides another indication that in order to have R-homology invariants of specters, R must be a commutative ring spectrum. This commutativity requirement on the generalized homology invariants might appear surprising and like a fluke of the formulation, but in fact it reflects an essential aspect of the mathematical structure; later on we will see that the slogan is 'semi-infinite homotopy types only have commutative generalized homology invariants'.

Given a specter P for the haunt H, we have "taken its R-homology" and produced an R-specter P_R for the R-haunt H_R. The essential idea behind specter invariants is that the structure of this R-haunt H_R might be substantially simpler than that of H. In particular, if H_R is trivializable, that is if the composite $X \to B\,\mathrm{Pic}(S^0) \to B\,\mathrm{Pic}(R)$ is null homotopic, then any trivialization $\tau : H_R \xrightarrow{\sim} (X \times \text{R-mod})$ transforms the R-specter P_R into a parametrized R-module $\tau(P_R)$. The associated total spectrum $\tau(P_R)/X$ represents a global R-homotopy type for the specter P, even though P

does not itself have a global homotopy type. The homotopy groups of $\tau(P_R)/X$ are what we might call the R-homology groups of P, denoted $R^\tau_*(P)$; these are the most straightforward and most easily computable invariants of specters. Note that these groups definitely do depend on the trivialization τ, but this ambiguity can be identified and controlled.

Summary 4.2.2. Let P be a specter for the haunt H and suppose that τ is a trivialization of the associated R-haunt H_R. Then the *R-homology groups of the specter* P are defined to be the homotopy groups of the total spectrum of the trivialization of the associated R-specter P_R:

$$R^\tau_i(P) := \pi_i(\tau(P_R)/X)$$

The potential ambiguity in the trivialization of a trivializable R-haunt H_R is governed by the space of automorphisms of the trivial R-haunt $X \times \text{R-mod}$. More specifically, the space of trivializations of H_R is a torsor for $\mathrm{Aut}(X \times \text{R-mod}) \simeq \mathrm{Hom}(X, \mathrm{Pic}(R))$. Homotopic trivializations τ and τ' determine the same invariants $R^\tau_*(P) \simeq R^{\tau'}_*(P)$, and so we need only consider the set of components of the space of trivializations of H_R—this set of components is a torsor for $[X, \mathrm{Pic}(R)] = [X, \mathrm{Pic}^0(R) \times BGL_1R]$. It often happens that $[X, BGL_1R]$ has only one element; if X is connected, the set of components of the space of trivializations is then a torsor for $\mathrm{Pic}^0(R)$. In this situation, which is to say when we have a specter P for a haunt H with H_R trivializable and $[X, \mathrm{Pic}(R)] = \mathrm{Pic}^0(R)$, we can describe how the R-homology of P is affected by a change in trivialization as follows. A given trivialization τ can be modified by an invertible R-module $M \in \mathrm{Pic}^0(R)$ to the trivialization $M \cdot \tau$, and we have

$$R^{M\cdot\tau}_*(P) = \pi_*((M \cdot \tau)(P_R)/X) = \pi_*((\tau(P_R) \wedge_R M)/X) = \pi_*(\tau(P_R)/X \wedge_R M).$$

If the ring spectrum R is such that the homotopy $\pi_*(M)$ of an invertible module M is projective over R_*, then we conclude that $R^{M\cdot\tau}_*(P) = R^\tau_*(P) \otimes_{R_*} \pi_*(M)$—in other words, changing the trivialization shifts the homology of P by the homotopy of an invertible R-module. This shift ambiguity is always present; we think of a homology group determined up to such a shift as uniquely determined and we ignore the τ-dependency, writing simply $R_*(P)$.

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To have homology invariants of a specter P over a haunt H, we need to find a ring spectrum R such that the composite $X \to B\operatorname{Pic}(S^0) \to B\operatorname{Pic}(R)$ is nullhomotopic. As such, we need a thorough understanding of the homotopy types of various $B\operatorname{Pic}(R)$ and of the transformations $B\operatorname{Pic}(S^0) \to B\operatorname{Pic}(R)$. We already noted that $B\operatorname{Pic}(R) \simeq B(\operatorname{Pic}^0(R) \times BGL_1(R))$. By definition, $GL_1(R)$ consists of the unit components of the zero space of the spectrum R, so its zero-th homotopy group is $\pi_0(R)^\times$ and its higher homotopy agrees with that of R. Barring the issue of computing $\operatorname{Pic}^0(R)$, which in general is a difficult problem, this allows us to write down the homotopy groups of $B\operatorname{Pic}(R)$ in terms of those of R. These groups are listed for a few common ring spectra in table 4.1.

Classifying Space	Homotopy Groups							
	$i=1$	2	3	4	5	6	7	8
$B\operatorname{Pic}(S^0)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
$B\operatorname{Pic}(\mathrm{H}\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	0	0	0	0
$B\operatorname{Pic}(K)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$B\operatorname{Pic}(MU)$	$(\mathbb{Z})$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}^3$

Table 4.1: The homotopy groups $\pi_i(B\operatorname{Pic}(R))$ of the classifying spaces for R-haunts

The parenthetical group $\pi_1(B\operatorname{Pic}(MU))$ is conjectural. We now describe three simple examples of specters and their potential homology invariants. The first specter has no global homology type, the second has a uniquely determined global homology type, and the third has a global homology type that depends on the choice of trivialization.

Example 4.2.3. Consider the specter described in example 2.2.9: the base space is S^1, the haunt L_n has monodromy S^n, and the specter T is $S^n \times D^1 \to D^1$ on one semicircle and the cone $C(S^n \sqcup S^0) \to C(* \sqcup *) = D^1$ on the other semicircle. The classifying map of L_n represents $n \in \mathbb{Z} = \pi_1(B\operatorname{Pic}(S^0))$; in particular there is no global parametrized spectrum corresponding to T and therefore no ordinary homotopy type. Moreover, the map $B(\operatorname{Pic}(S^0)) \to B(\operatorname{Pic}(\mathrm{H}\mathbb{Z}))$ is an isomorphism on π_1, so the homology haunt $(L_n)_{\mathrm{H}\mathbb{Z}}$ is still nontrivial; correspondingly, the homology specter $T_{\mathrm{H}\mathbb{Z}}$ is not a parametrized $\mathrm{H}\mathbb{Z}$-module and so T does not have homology invariants.

Example 4.2.4. Next consider the specter from example 2.2.10: the base space is S^3, the haunt L is the haunt determined by the equatorial transition function $-\wedge_{S^2} (S^2 \rtimes S^2)$, and the specter P is $S^0 \times D^3$ on one hemisphere and $C(S^2 \rtimes S^2)$ on the other. On the one hand, the classifying map $S^3 \to B\operatorname{Pic}(S^0)$ for L is nontrivial, representing $1 \in \mathbb{Z}/2 = \pi_3(B\operatorname{Pic}(S^0))$; on the other hand, there are no nontrivial $\mathrm{H}\mathbb{Z}$-haunts on S^3 and so $L_{\mathrm{H}\mathbb{Z}}$ is trivializable. Thus, even though the specter P has no global homotopy type, it does have homology invariants. Moreover, $\operatorname{Pic}(\mathrm{H}\mathbb{Z})$ has homotopy only in degrees 0 and 1, so the set of trivializations of $L_{\mathrm{H}\mathbb{Z}}$, namely $[S^3, \operatorname{Pic}(\mathrm{H}\mathbb{Z})] \cong \operatorname{Pic}^0(\mathrm{H}\mathbb{Z}) = \{\Sigma^n \mathrm{H}\mathbb{Z}\}$, is as small as possible. The homology invariants of P are therefore uniquely determined up to degree shift, that is up to tensoring with $\pi_*(\Sigma^n \mathrm{H}\mathbb{Z})$.

Let us calculate the homology $H_*(P)$ of the specter P. Roughly speaking, the spectrum $\mathrm{H}\mathbb{Z}$ cannot see the difference between the transition functions $-\wedge_{S^2} (S^2 \rtimes S^2)$ and $-\wedge_{S^2} (S^2 \times S^2)$. As a result, the parametrized $\mathrm{H}\mathbb{Z}$-module $P_{\mathrm{H}\mathbb{Z}}$ is equivalent to the one obtained by gluing together $\mathrm{H}\mathbb{Z} \times \mathrm{D}^3$ and $C(\Sigma^2 \mathrm{H}\mathbb{Z} \times \mathrm{S}^2)$ using the map $-\wedge_{S^2} (S^2 \times S^2)$. By desuspending the second hemisphere, this is in turn equivalent to the parametrized $\mathrm{H}\mathbb{Z}$-module $(\mathrm{H}\mathbb{Z} \times \mathrm{D}^3) \cup_{\mathrm{S}^2} (\mathrm{C}(\mathrm{H}\mathbb{Z} \times \mathrm{S}^2))$. This last $\mathrm{H}\mathbb{Z}$-module is simply the reduced homology of S^3; thus $H_*(P)$ is $\mathbb{Z}$ in a single degree and zero in all other degrees. We will formalize this sort of computation in a moment.

Example 4.2.5. Let P be the parametrized spectrum $S^0 \times (S^2 \times S^1)$ over $S^2 \times S^1$. This spectrum defines a specter for the trivial haunt H and as such P has homology invariants. However, there are two distinct trivialization τ_0 and τ_1 of $H_{\mathrm{H}\mathbb{Z}}$. They yield respectively the parametrized $\mathrm{H}\mathbb{Z}$-modules $\tau_0(P_{\mathrm{H}\mathbb{Z}}) = \mathrm{H}\mathbb{Z} \times (\mathrm{S}^2 \times \mathrm{S}^1)$ and $\tau_1(P_{\mathrm{H}\mathbb{Z}}) = \mathrm{H}\mathbb{Z} \rtimes (\mathrm{S}^2 \times \mathrm{S}^1)$; this last module exhibits a mobius transformation of the fibre $\mathrm{H}\mathbb{Z}$ along the S^1 factor of the base. The homology groups of these two trivializations are

$H_*^{\tau_0}(P) \cong \{\mathbb{Z}\colon \mathbb{Z}\colon \mathbb{Z}\colon \mathbb{Z}\}$ and $H_*^{\tau_1}(P) \cong \{\mathbb{Z}/2\colon 0\colon \mathbb{Z}/2\colon 0\}$. The difference between these two groups is characteristic of the ambiguity involved in H$\mathbb{Z}$-specters on non-simply connected base spaces. This dependence on the trivialization indicates that H$\mathbb{Z}$-specter invariants are not naturally graded abelian groups but rather objects in a more subtle algebraic category.

Typically specters are constructed by specifying parametrized spectra over the open sets of a cover; these parametrized spectra do not agree on the intersections but instead are glued together by the transition functions of the haunt. This explicit local presentation suggests a method for computing specter invariants in terms of the local homology invariants of the defining parametrized spectra:

Proposition 4.2.6. *Suppose H is a haunt, on a connected space X, whose associated R-haunt H_R admits a trivialization $\tau : H_R \xrightarrow{\sim} (X \times R\text{-mod})$. Let P be a specter for H and let P_R denote the associated R-specter. Then there is a "Mayer-Vietoris" spectral sequence*

$$E^2_{pq} = H_p(X; \underline{\pi_q(\tau(P_R))}) \Rightarrow R^\tau_{p+q}(P).$$

Here $\underline{\pi_q(\tau(P_R))}$ denotes the cosheaf $U \mapsto \pi_q((\tau(P_R)|_U)/U)$.

Suppose $\{U_i\}$ is a fixed contractible cover of X, and let $- \wedge_{U_{ij}} \rho_{ij}$ be transition functions defining H. Suppose the specter P is presented by parametrized spectra P_i on U_i together with identifications $\gamma_{ij} : P_i|_{U_{ij}} \wedge_{U_{ij}} \rho_{ij} \xrightarrow{\sim} P_j|_{U_{ij}}$. Then the above spectral sequence has the form

$$E^1_{pq} = \bigoplus_{i_1 < \ldots < i_p} R_q(P_{i_1}|_{U_{i_1 \ldots i_p}}) \Rightarrow R^\tau_{p+q}(P).$$

If the trivialization is given by automorphisms $\tau_i : (U_i \times R\text{-mod}) \xrightarrow{\sim} (U_i \times R\text{-mod})$ appropriately compatible with the R-haunt transition functions $(\rho_{ij})_R = \rho_{ij} \wedge_{U_{ij}} R$, then the d^1 differential is given by the maps

$$R_q(P_{i_1}|_{U_{i_1 \ldots i_p}}) \xrightarrow{(\iota_k)_*} R_q(P_{i_1}|_{U_{i_1 \ldots \widehat{i_k} \ldots i_p}})$$
$$R_q(P_{i_1}|_{U_{i_1 \ldots i_p}}) \xrightarrow{(\iota_1)_* \circ (\tau_{i_2})_*^{-1} \circ (\tau_{i_1})_*} R_q(P_{i_2}|_{U_{i_2 \ldots i_p}})$$

Here $\iota_k : U_{i_1 \dots \widehat{i_k} \dots i_p} \to U_{i_1 \dots i_p}$ *and* $\iota_1 : U_{i_2 \dots i_p} \to U_{i_1 \dots i_p}$ *denote the inclusions.*

The first half of this proposition is just a statement, in ordinary parametrized homotopy theory, about the parametrized spectrum $\tau(P_R)$; it is not in itself particularly useful because one must expressly identify $\tau(P_R)$ in terms of the original specter P in order to compute the cosheaf homology. The second half is more explicit and addresses the situation that actually arises with twisted parametrized spectra. In particular, the above E^1 term does not depend on the trivialization τ and can be immediately computed in any given case.

Chapter 5

Polarized Hilbert Manifolds and Semi-Infinite Spectra

Thusfar our discussion has been purely topological: the category of spectra has a complicated space of automorphisms and it is natural to study bundles of categories of spectra and their associated sections, namely twisted parametrized spectra or "specters". These bundles are, however, intimately connected to the geometry of infinite-dimensional manifolds, and homotopy-theoretic invariants of such manifolds often take the form of twisted parametrized spectra. In the first part of this section, we describe the relevant geometry, namely symplectic polarizations of real Hilbert bundles, and we show how a manifold equipped with this structure gives rise to a bundle of categories of spectra, that is to a haunt. We also discuss a related structure, a "unitary" polarization and investigate the invariants of specters for haunts associated to unitary polarizations; the resulting description of these invariants provides an extensive generalization of the Cohen-Jones-Segal complex-oriented Floer invariants [8]. In the second part of this section, we introduce a conjectural construction of the category of specters, for a given polarized bundle, in terms of parametrized semi-infinitely indexed spectra. Specifically, instead of indexing spectra on finite-dimensional subspaces of a countably infinite-dimensional vector space, we introduce spectra indexed on the semi-infinite subspaces of a Hilbert space that are compatible with a fixed polarization. A parametrized version of these semi-infinite spectra provides an explicit geometric viewpoint on the category of specters.

5.1 The Homotopy Theory of Polarized Bundles

We describe four types of polarizations, namely symplectic and complex on a real Hilbert space and symplectic and complex on a complex Hilbert space, and discuss the homotopy types of the corresponding classifying spaces—it turns out that there is no distinction between the two notions of polarization on a complex Hilbert space. We then describe polarized bundles and note that a symplectic polarization on a real Hilbert bundle gives rise to a haunt. We conclude by investigating the special class of "unitary" polarized bundles, namely symplectic polarizations of a real bundle that lift to polarizations of a complex bundle. In particular we show that under mild conditions, a specter for a unitary polarization admits $\mathrm{H}\mathbb{Z}$-, K-, and MU-homology invariants.

5.1.1 Polarizations of Hilbert Space

A finite dimensional vector bundle on a space X is classified by a map from X to the classifying space $BO(n)$. The topology of the classifying space is governed by the (non-trivial) topology of the orthogonal group $O(n)$ of automorphisms of $\mathbb{R}^n$. By contrast, the orthogonal group $O(\mathcal{H})$ of Hilbert space is contractible [27] and therefore the classifying space $BO(\mathcal{H})$ carries no topological information; indeed, all Hilbert bundles on a given space are isomorphic. In particular, if X is a Hilbert manifold, that is an infinite-dimensional manifold whose tangent bundle is a Hilbert bundle, then the tangent bundle of X carries no information at all about the topology of X. The situation is not as bad as it might seem, however, because many naturally occurring infinite-dimensional manifolds come equipped with a polarization. This polarization is a reduction of the structure group of X from the orthogonal group $O(\mathcal{H})$ to the so-called restricted orthogonal group $O_{\mathrm{res}}(\mathcal{H})$. This latter group does have an interesting topology, and so we can recover information about such a polarized Hilbert manifold X from its polarized tangent bundle.

There are various notions that go under the name "polarization" and we spend a moment describing and distinguishing them; references include [41, 44, 8] but the reader is warned that the terminology and definitions in those papers disagree with one another and at points with our treatment. We fix an infinite-dimensional separable real Hilbert space $\mathcal{H}$. Morally, a polarization of $\mathcal{H}$ is an equivalence class of decompositions $V \oplus W$ of $\mathcal{H}$ or $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \otimes \mathbb{C}$ arising from an eigenvalue decomposition of an appropriate operator $J : \mathcal{H} \to \mathcal{H}$. The subspaces V and W are sums of collections of eigenspaces of J; that the polarization is an equivalence class of decompositions rather than a single decomposition reflects an ambiguity over whether to assign certain eigenspaces to V or to W. For example, if J is a self-adjoint Fredholm operator, then the associated decompositions $V \oplus W$ are roughly those in which V contains almost all the eigenspaces for negative eigenvalues of J and W contains almost all the eigenspaces for positive eigenvalues of J. If on the other hand J is a skew-adjoint Fredholm operator, then the decompositions are those in which V contains almost all the eigenspaces for positive imaginary eigenvalues and W contains almost all the eigenspaces for negative imaginary eigenvalues.

In practice many polarizations arise from self- and skew-adjoint Fredholm operators as above, but we can simplify the definitions of polarizations if we restrict attention to self- and skew-adjoint orthogonal isomorphisms. That is, suppose $J : \mathcal{H} \to \mathcal{H}$ is a self-adjoint orthogonal isomorphism; in this case, $J^2 = 1$ and so $\mathcal{H}$ is split into the $+1$ and -1 eigenspaces V and $V^{\perp}$. Of course, any orthogonal decomposition arises as the eigenvalue decomposition of such an operator, and so decompositions $\mathcal{H} = V \oplus V^{\perp}$ are in one-to-one correspondence with orthogonal operators J with $J^2 = 1$. An equivalence class of such decompositions defines a "symplectic" polarization, as follows.

Definition 5.1.1. A *symplectic polarization* on a real Hilbert space $\mathcal{H}$ is a collection of orthogonal decompositions $\{\mathcal{H} = V \oplus V^{\perp}\}$ satisfying the following conditions:

- both V and $V^{\perp}$ are infinite dimensional,
- for any two decompositions $V \oplus V^{\perp}$ and $W \oplus W^{\perp}$ in the collection, the projec-

tions $V \to W$ and $V^\perp \to W^\perp$ are Fredholm and the projections $V \to W^\perp$ and $V^\perp \to W$ are Hilbert-Schmidt,

- any decomposition $W \oplus W^\perp$ satisfying the second property with respect to a decomposition $V \oplus V^\perp$ in the collection is in the collection.

From now on, whenever we mention a decomposition of a Hilbert space, we implicitly assume that both factors of the decomposition are infinite dimensional. Corresponding to the above definition we have a restricted orthogonal group:

Definition 5.1.2. Let $V \oplus V^\perp$ be a fixed decomposition of the real Hilbert space $\mathcal{H}$. The *symplectic restricted orthogonal group* $O^s_{\text{res}}(\mathcal{H})$ is the subgroup of the orthogonal group $O(\mathcal{H})$ of operators ϕ such that $\phi(V) \oplus \phi(V^\perp)$ is in the same symplectic polarization class as $V \oplus V^\perp$.

Note that this is *not* the group that Pressley and Segal [41] refer to as the restricted orthogonal group. Indeed we will see later that it has a radically different homotopy type than their $O_{\text{res}}(\mathcal{H})$.

Note 5.1.3. The space of symplectic polarizations of a real Hilbert space is the quotient $O(\mathcal{H})/O^s_{\text{res}}(\mathcal{H})$.

Now by contrast, suppose we had begun with a skew-adjoint orthogonal isomorphism $J : \mathcal{H} \to \mathcal{H}$; in this case, $J^2 = -1$ and so $\mathcal{H}_\mathbb{C}$ is decomposed into the $+i$ and $-i$ eigenspaces W and $\overline{W}$. Indeed, for any decomposition $\mathcal{H}_\mathbb{C} = W \oplus \overline{W}$ of $\mathcal{H}_\mathbb{C}$ into a subspace W and its conjugate $\overline{W}$, the following three conditions are equivalent:

- the decomposition is orthogonal with respect to the Hermitian metric $\langle -, - \rangle$ on $\mathcal{H}_\mathbb{C}$ extending the inner product on $\mathcal{H}$,
- the subspaces W and $\overline{W}$ are isotropic with respect to the bilinear form $(a, b) = \langle a, \overline{b} \rangle$ on $\mathcal{H}_\mathbb{C}$,
- there is an orthogonal operator $J : \mathcal{H} \to \mathcal{H}$ with $J^2 = -1$ having W and $\overline{W}$ as its $+i$ and $-i$ eigenspaces respectively.

We refer to such decompositions as "orthogonal". An equivalence class of these orthogonal decompositions defines a "complex" polarization:

Definition 5.1.4. A *complex polarization* on a real Hilbert space $\mathcal{H}$ is a collection of orthogonal decompositions $\{\mathcal{H}_{\mathbb{C}} = W \oplus \overline{W}\}$ satisfying the same conditions as in definition 5.1.1. Given a fixed orthogonal decomposition $W \oplus \overline{W}$, the *complex restricted orthogonal group* $O^c_{\text{res}}(\mathcal{H})$ on the real Hilbert space $\mathcal{H}$ is the subgroup of $O(\mathcal{H})$ of operators ϕ such that $\phi(W) \oplus \phi(\overline{W})$ is in the same complex polarization class as $W \oplus \overline{W}$.

This complex restricted orthogonal group is what Pressley and Segal [41] refer to as $O_{\text{res}}(\mathcal{H})$.

Note 5.1.5. The space of complex polarizations of a real Hilbert space is the quotient $O(\mathcal{H})/O^c_{\text{res}}(\mathcal{H})$.

There is yet another notion that goes under the name polarization. Let $\mathcal{H}$ now be a complex Hilbert space. Suppose $J : \mathcal{H} \to \mathcal{H}$ is a self-adjoint unitary operator; then $J^2 = 1$ and $\mathcal{H}$ is decomposed into $+1$ and -1 eigenspaces. Two such decompositions $V \oplus V^{\perp}$ and $W \oplus W^{\perp}$ are considered equivalent if, as in definition 5.1.1, the projections $V \to W$ and $V^{\perp} \to W^{\perp}$ are Fredholm and the other two projections are Hilbert-Schmidt. An equivalence class of these decompositions defines a polarization of the complex Hilbert space $\mathcal{H}$; the group of unitary operators preserving such a polarization is called $U_{\text{res}}(\mathcal{H})$ and the space of such polarizations is $U(\mathcal{H})/U_{\text{res}}(\mathcal{H})$. Similarly if J is skew-adjoint unitary, then $J^2 = -1$ and $\mathcal{H}$ is decomposed into $+i$ and $-i$ eigenspaces. Equivalence classes of these decompositions also give a notion of polarization, but because $\mathcal{H}$ is complex there is a one-to-one correspondence between self- and skew-adjoint unitary operators and the two notions of polarization coincide. This correspondence, which in a sense encodes the two-fold complex Bott periodicity, can obscure the distinction between the two notions in the real case.

We briefly discuss the homotopy types of these various spaces of polarizations. As above, the symplectic restricted orthogonal group $O^s_{\text{res}}(\mathcal{H})$ of a real Hilbert space is the space of orthogonal operators $\phi : \mathcal{H} \to \mathcal{H}$ such that the projections $\phi(V) \to V$

and $\phi(V^\perp) \to V^\perp$ are Fredholm and the projections $\phi(V) \to V^\perp$ and $\phi(V^\perp) \to V$ are Hilbert-Schmidt, for a fixed decomposition $V \oplus V^\perp$. Suppose that we have chosen $\phi|_V$ such that $\phi(V) \to V$ is Fredholm and $\phi(V) \to V^\perp$ is Hilbert-Schmidt. Then the subspace $\phi(V^\perp)$ is necessarily $\phi(V)^\perp$ and we can chose $\phi|_{V^\perp}$ to be any orthogonal isomorphism $V^\perp \xrightarrow{\cong} \phi(V)^\perp$—the space of such choices is of course contractible. Specifying $\phi|_V$ amounts to choosing a Fredholm map $V \to V$ and a Hilbert-Schmidt map $V \to V^\perp$. The space of Hilbert-Schmidt operators is contractible and so $O^s_{\text{res}}(\mathcal{H})$ has the homotopy type of the space of Fredholm operators, namely $\mathbb{Z} \times BO$. The associated space of polarizations $O(\mathcal{H})/O^s_{\text{res}}(\mathcal{H})$ therefore has the homotopy type $B(\mathbb{Z} \times BO) \simeq U/O$. By contrast, Pressley and Segal [41] show that $O^c_{\text{res}}(\mathcal{H})$ has the homotopy type $O/U \simeq \Omega O$ and so the space of polarizations $O(\mathcal{H})/O^c_{\text{res}}(\mathcal{H})$ has the homotopy type $B(O/U) \simeq B(\Omega O) \simeq O$. When $\mathcal{H}$ is a complex Hilbert space, the restricted group $U_{\text{res}}(\mathcal{H})$ has the homotopy type $\mathbb{Z} \times BU$ and the space of polarizations is $U(\mathcal{H})/U_{\text{res}}(\mathcal{H}) \simeq B(\mathbb{Z} \times BU) \simeq B(\Omega U) \simeq U$. The homotopy types of these various spaces of polarizations are summarized in table 5.1.

Polarization Type	Structure Group	Classifying Space
symplectic on $\mathcal{H}_\mathbb{R}$	$O^s_{\text{res}} \simeq \mathbb{Z} \times BO$	$B(\mathbb{Z} \times BO) \simeq U/O$
complex on $\mathcal{H}_\mathbb{R}$	$O^c_{\text{res}} \simeq \Omega O$	$B(\Omega O) \simeq O$
symplectic on $\mathcal{H}_\mathbb{C}$	$U_{\text{res}} \simeq \mathbb{Z} \times BU$	$B(\mathbb{Z} \times BU) \simeq U$
complex on $\mathcal{H}_\mathbb{C}$	$U_{\text{res}} \simeq \Omega U$	$B(\Omega U) \simeq U$

Table 5.1: The homotopy types of the classifying spaces for polarizations

We will be primarily concerned with symplectic polarizations of real Hilbert spaces and unless otherwise indicated, "polarization" will refer to this notion.

5.1.2 Symplectic Polarizations and Haunts

A priori a Hilbert bundle E on a space X is classified by a map $X \to BO(\mathcal{H})$—of course this map contains no topological information. To give a polarization of this bundle is to specify, continuously in X, a polarization on each fibre of E:

Definition 5.1.6. A polarization of the Hilbert bundle E on the space X is a reduction of the structure group of E from $O(\mathcal{H})$ to $O^s_{\text{res}}(\mathcal{H})$. In other words it is a lift of the classifying map $X \to BO(\mathcal{H})$ to a map $X \to BO^s_{\text{res}}(\mathcal{H})$.

As there is no harm in doing so, we usually think of a polarization on the Hilbert bundle E simply as a map $X \to BO^s_{\text{res}}(\mathcal{H}) \simeq B(\mathbb{Z} \times BO)$. A polarization of a Hilbert manifold is simply a polarization of its (trivial) tangent bundle. The fundamental link between the geometry of polarizations and twisted parametrized stable homotopy theory is the association

$$\left\{ \begin{array}{c} \text{Polarized Hilbert} \\ \text{bundles on } X \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \text{Haunts} \\ \text{on } X \end{array} \right\}$$

This association is determined by composing the classifying map $X \to B(\mathbb{Z} \times BO)$ of the polarized bundle with the deloop of the J-homomorphism $B(\mathbb{Z} \times BO) \xrightarrow{BJ} B(\mathbb{Z} \times BG) = B(\mathbb{Z} \times BGL_1(S^0))$.

The basic philosophy behind this correspondence is that geometric structures on infinite dimensional manifolds are intimately connected with polarizations and that homotopy-theoretic information about these structures can be encoded in twisted parametrized spectra for the haunt associated to the polarization. The specific nature of this connection will be the subject of future work with Michael Hopkins [9] and Ciprian Manolescu [10]. Here we record a few illustrative examples of polarized manifolds and their associated haunts.

There are two widely utilized sources of polarized manifolds: the first is loop spaces of symplectic and almost complex manifolds, and the second is moduli spaces of connections in gauge theory—see for example [8]. We discuss the first source of examples. Given a symplectic manifold M, a choice of metric determines an almost complex structure $M \xrightarrow{c} BU(n)$ on the tangent bundle of M. The loop of this classifying map, or indeed of the classifying map for any almost complex manifold, can be used to determine a polarization on LM:

$$LM \xrightarrow{Lc} LBU(n) \to \Omega BU(n) \to \Omega BU \simeq U \to U/O \simeq B(\mathbb{Z} \times BO)$$

This polarization and its associated haunt can be highly nontrivial.

Example 5.1.7. Let S^6 have its usual almost complex structure. The haunt associated to the resulting polarization of the loop space LS^6 is nontrivial. Indeed, the classifying map

$$LS^6 \to U \to U/O \to B(\mathbb{Z} \times BG)$$

for this haunt restricts on $S^5 \hookrightarrow LS^6$ to a generator of $\pi_5(B(\mathbb{Z} \times BG)) = \pi_3(S^0) = \mathbb{Z}/24$. To see this, note that because $\pi_4(S^3) = \mathbb{Z}/2$, the Hurewicz map $\pi_6(BSU(3)) \to H_6(BSU(3))$ is multiplication by 2. Because the Euler characteristic of S^6 is 2, this implies that the almost complex structure $S^6 \to BSU(3)$ is a generator of $\pi_6(BSU(3)) \cong \pi_6(BSU)$. The loop $LS^6 \to LBU$ of this almost complex structure therefore induces an isomorphism on π_5 and the claim follows:

$$\begin{tikzcd}
\pi_5(LS^6) \arrow[r] \arrow[d, equal] & \pi_5(LBU) \arrow[r] \arrow[d, equal] & \pi_5(U) \arrow[r] \arrow[d, equal] & \pi_5(U/O) \arrow[r] \arrow[d, equal] & \pi_5(B(\mathbb{Z} \times BG)) \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r, two heads] & \mathbb{Z}/24
\end{tikzcd}$$

A specter for the resulting canonical haunt on LS^6 will have no global homotopy type. We will see in a moment though that any such specter has H$\mathbb{Z}$-, K-, and MU-homology invariants.

5.1.3 Unitary Polarizations and Specter Invariants

Many examples of polarized manifolds have the property that the polarization map $X \to U/O$ factors through a map $X \to U$; (this is true for instance of example 5.1.7 above). As we saw earlier, the stable unitary group U classifies polarizations on complex Hilbert bundles E. The projection map $U \to U/O$ corresponds to viewing a polarization of E as a symplectic polarization of the underlying real Hilbert bundle $E_{\mathbb{R}}$; (similarly, the inclusion map $U \to O$ corresponds to viewing the polarization of E as a complex polarization of the underlying real Hilbert bundle $E_{\mathbb{R}}$). For lack of better terminology, we say that a symplectic polarization $X \to U/O$ of a real Hilbert bundle is *unitary* if it lifts to a polarization $X \to U$ of a complex Hilbert bundle.

Haunts associated to unitary polarizations are much better behaved than arbitrary haunts in a sense we now describe, and as a result their specters have a much simplified invariant theory. Suppose $X \to U \to U/O$ is the classifying map for a unitary polarization. The associated haunt H has a corresponding H$\mathbb{Z}$-haunt $H_{\mathrm{H}\mathbb{Z}}$ classified by the composite

$$X \to U \to U/O \to B(\mathbb{Z} \times BG) \to B(\mathbb{Z} \times B\mathbb{Z}/2) \simeq B\mathbb{Z} \times B^2\mathbb{Z}/2.$$

The homology group $H^2(U; \mathbb{Z}/2)$ is zero, so this composite factors through $B\mathbb{Z}$. Let $q \in \mathbb{Z}$ denote the smallest nonzero integer in the image of $H_1(X; \mathbb{Z}) \to H_1(B\mathbb{Z}; \mathbb{Z})$, and suppose T is a specter for the haunt H on X. The haunt $H_{\mathrm{H}\mathbb{Z}}$ is nontrivial; thus $T_{\mathrm{H}\mathbb{Z}}$ does not have the form of a parametrized H$\mathbb{Z}$-module and so has no associated global H$\mathbb{Z}$-module, therefore no corresponding chain complex and no homology groups, per se. There is nevertheless a *$\mathbb{Z}/q$-graded* chain complex associated to $T_{\mathrm{H}\mathbb{Z}}$ and therefore T has $\mathbb{Z}/q$-graded homology groups for invariants. This fact nicely explains the idea (by now prevalent in the literature) that semi-infinite and Floer homology theories are naturally graded not by the integers but by finite cyclic groups.

If the unitary polarization $X \to U \to U/O$ is trivial on H^1, we have a more complete description of the corresponding haunt and its associated invariants:

Proposition 5.1.8. *Let X be a space having the homotopy type of a finite CW complex and let E be a Hilbert bundle on X equipped with a unitary polarization $X \to U \to U/O$. Suppose the induced map $H^1(U; \mathbb{Z}) \to H^1(X; \mathbb{Z})$ is zero. Then any specter for the haunt H associated to the polarized bundle E admits global* H$\mathbb{Z}$-, *K-, and MU-homology invariants.*

Proof. Because of the H^1 condition, the classifying map for the polarization factors through SU, and because X is homotopy finite, this map in turn factors through some $SU(n)$. The haunt H is therefore classified by a map $X \to SU(n) \to B(\mathbb{Z} \times BG)$. Let R denote one of the spectra H$\mathbb{Z}$, K, or MU. The R-haunt H_R is classified by the composition $X \to SU(n) \to B(\mathbb{Z} \times BG) = B\,\mathrm{Pic}(S^0) \to B\,\mathrm{Pic}(R)$. It is not of course the case that the map $B\,\mathrm{Pic}(S^0) \to B\,\mathrm{Pic}(R)$ is null, but the composition $SU(n) \to B\,\mathrm{Pic}(S^0) \to B\,\mathrm{Pic}(R)$ will be null for the spectra R in question. This we can see by considering the map from the Atiyah-Hirzebruch spectral sequence for $B\,\mathrm{Pic}(S^0)^*(SU(n))$ to the Atiyah-Hirzebruch spectral sequence for $B\,\mathrm{Pic}(R)^*(SU(n))$. If for example $R = K$ and $n = 3$ the map of E_2 terms is as follows:

$$
\begin{array}{c|ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
2 & \mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 \\
3 & \mathbb{Z}/2 & 0 & 0 & \boxed{\mathbb{Z}/2} & 0 & \mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 \\
4 & \mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 \\
5 & \mathbb{Z}/24 & 0 & 0 & \mathbb{Z}/24 & 0 & \boxed{\mathbb{Z}/24} & 0 & 0 & \mathbb{Z}/24 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & \mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & 0 & \boxed{\mathbb{Z}/2}
\end{array}
\longrightarrow
\begin{array}{|ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 \\
\mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & 0 & \mathbb{Z}/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & \boxed{\mathbb{Z}}
\end{array}
$$

Total degree zero terms are boxed. The map is necessarily zero at E_2 in total degree zero and therefore $B\,\mathrm{Pic}(S^0)^0(SU(3)) \to B\,\mathrm{Pic}(K)^0(SU(3))$ is zero. The cases of $H\mathbb{Z}$ and MU and of other n are similar. We therefore conclude that the R-haunt $X \to B\,\mathrm{Pic}(R)$ is trivializable and so any corresponding specter has R-homology invariants. $\square$

The proposition says that for a large class of polarized manifolds, the associated semi-infinite homotopy types, namely the specters, will have homology, K-theory, and complex bordism invariants. This provides an explanation of and a substantial generalization of the remark in Cohen-Jones-Segal [8] that trivially polarized manifolds should have semi-infinite homology, K-theory, and complex bordism invariants.

5.2 Semi-Infinitely Indexed Spectra

In section 5.1.2 we saw that the classifying space for polarizations maps to the classifying space for haunts; any polarized bundle therefore has an associated haunt and a corresponding category of specters. In this section we will sketch, using a notion of parametrized semi-infinitely indexed spectra, a conjectural realization of this category of specters in terms of the geometry of the polarized bundle.

We begin by defining (non-parametrized) semi-infinitely indexed spectra. Classically a prespectrum E is presented by specifying a space $E(\mathbb{R}^n)$ for each integer n

together with appropriate structure maps. This notion naturally evolved (in work of May and company [36, 30]) into that of coordinate free prespectra; a coordinate free prespectrum is presented by giving a space $E(V)$ for every finite dimensional subspace V of a fixed countably-infinite dimensional inner product space $\mathbb{R}^\infty$, together with appropriate structure maps $\Sigma^{W-V}E(V) \to E(W)$ for each inclusion $V \subset W$. A prespectrum is a spectrum if the adjoint structure maps $E(V) \to \Omega^{W-V}E(W)$ are homeomorphisms. Such a spectrum has the property that if V and W have the same dimension, then $E(V)$ and $E(W)$ are homeomorphic, and moreover $E(V)$ varies continuously with V. This can be seen by noting that for any compact family $\{V_t\}$ of finite-dimensional subspaces of $\mathbb{R}^\infty$, there is a finite-dimensional subspace W such that W contains all the subspaces of the family; the spaces $E(V_t)$ are therefore determined as $\Omega^{W-V_t}E(W)$, which is evidently a continuous family. Later on, Elmendorf [17] included as part of the definition of a spectrum the requirement that the spaces $E(V)$ vary continuously in V.

The fundamental idea behind semi-infinitely indexed spectra is that the natural subspaces of a polarized Hilbert space $\mathcal{H}$ are not the finite dimensional subspaces but the "negative energy" or "semi-infinite" subspaces, that is the subspaces V such that $V \oplus V^\perp$ is a decomposition of $\mathcal{H}$ in the given polarization class. A semi-infinitely indexed spectrum is then roughly an assignment of a space $E(V)$ to each such semi-infinite V, together with appropriate structure maps. It is not the case that given a compact family $\{V_t\}$ of semi-infinite subspaces, there exists a semi-infinite subspace W containing all the V_t; indeed, there exist decompositions $V \oplus V^\perp$ and $W \oplus W^\perp$ in the same polarization class such that V and W span the whole Hilbert space $\mathcal{H}$. As such, it is important that we impose a continuity condition on the spaces $E(V)$—we do so roughly along the lines of [17] and we thank Mike Hopkins for bringing that reference to our attention.

Definition 5.2.1. Let X be a space together with a distinguished subset $X^{(2)}$ of X^2 and a finite dimensional vector bundle γ on $X^{(2)}$. Define $X^{(3)}$ to be $\{(a,b,c) \in X^3 | (a,b),(a,c),(b,c) \in X^{(2)}\}$. Let p_{12}, p_{13}, and p_{23} denote the projections $X^{(3)} \to X^{(2)}$ to the indicated factors, and similarly denote by p_1, p_2, and p_3 the projections

$X^{(3)} \to X$. Suppose there is an identification $p_{13}^*\gamma = p_{12}^*\gamma \oplus p_{23}^*\gamma$. Then an X-*prespectrum* is a bundle T on X together with a map $\sigma : S^\gamma \wedge_{X^{(2)}} p_1^* T \to p_2^* T$ such that the diagram

$$\begin{tikzcd}
S^{p_{23}^*\gamma} \wedge_{X^{(3)}} S^{p_{12}^*\gamma} \wedge_{X^{(3)}} p_1^*T \arrow[r, equal] \arrow[d] & S^{p_{13}^*\gamma} \wedge_{X^{(3)}} p_1^*T \arrow[d] \\
S^{p_{23}^*\gamma} \wedge_{X^{(3)}} p_2^*T \arrow[r] & p_3^*T
\end{tikzcd}$$

commutes. This data forms an *X-spectrum* if the adjoint of σ is a homeomorphism.

Suppose $\mathcal{H}$ is equipped with a fixed polarization. Let $Gr_{\text{res}}(\mathcal{H})$ denote the grassmannian of decompositions $V \oplus V^\perp$ in the polarization class. The space $Gr_{\text{res}}^{(2)}(\mathcal{H})$ is the set of pairs of decompositions $(V \oplus V^\perp, W \oplus W^\perp)$ such that $V \subset W$, and the vector bundle γ is the orthogonal complement $V^{\perp W}$.

Definition 5.2.2. A *semi-infinitely indexed (pre)spectrum*, or "semi-infinite (pre)spectrum" for short, is a $Gr_{\text{res}}(\mathcal{H})$-(pre)spectrum.

We bother with the abstract definition 5.2.1 because it facilitates comparisons between X-spectra as X varies. In particular, fix a decomposition $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$ in the polarization class and let $Gr(\mathcal{H}^+)$ denote the grassmannian of finite dimensional subspaces of $\mathcal{H}^+$. We will refer to $Gr(\mathcal{H}^+)$-spectra as Hilbert spectra. Furthermore, denote by $\mathbb{R}^\infty$ a countably-infinite-dimensional dense subspace of $\mathcal{H}^+$ and let $Gr(\mathbb{R}^\infty)$ be the grassmannian of finite dimensional subspaces of $\mathbb{R}^\infty$. We now have natural restriction maps

$$\begin{gathered}
Gr_{\text{res}}(\mathcal{H})\text{-spectra} \to Gr(\mathcal{H}^+)\text{-spectra} \\
(V \subset \mathcal{H} \rightsquigarrow E(V)) \mapsto (W \subset \mathcal{H}^+ \rightsquigarrow E(\mathcal{H}^- \oplus W)) \\
Gr(\mathcal{H}^+)\text{-spectra} \to Gr(\mathbb{R}^\infty)\text{-spectra} \\
(W \subset \mathcal{H}^+ \rightsquigarrow F(W)) \mapsto (U \subset \mathbb{R}^\infty \rightsquigarrow F(U))
\end{gathered}$$

A detailed treatment of the theory of semi-infinitely indexed spectra and their relationship to ordinary spectra will appear elsewhere—in particular one must construct a semi-infinite spectrification functor and semi-infinite sphere spectra (leading to a semi-infinite notion of stable weak equivalence) and one must build left adjoints to the two restriction maps above. For now we leave as a conjecture the following:

Conjecture 5.2.3. There is a notion of weak equivalence of semi-infinite spectra and a notion of weak equivalence of Hilbert spectra such that the above restriction map from $Gr_{\text{res}}(\mathcal{H})$-spectra to $Gr(\mathcal{H}^+)$-spectra induces an equivalence of (simplicial) homotopy categories. Similarly the restriction map from $Gr(\mathcal{H}^+)$-spectra to $Gr(\mathbb{R}^\infty)$-spectra induces an equivalence of homotopy categories.

∼∼∼∼∼

Parametrized semi-infinite spectra are no more difficult to define than semi-infinite spectra. The idea of parametrized universes (of the countably-infinite variety) for spectra first appeared in Elmendorf [17]. Though we were not aware of this reference during our development of parametrized semi-infinite spectra, it is a very clean presentation of the classical case and we follow it in spirit. Elmendorf's real insight was not so much the use of parametrized universes, per se, as the realization that one could build a category of spectra on all (parametrized) universes at once and that this larger category was substantially better than the category of spectra indexed on a single universe. Unfortunately, basic facts about countably-infinite universes that made this possible fail to be true of semi-infinite universes, and so we necessarily shy aware from this aspect of Elmendorf's treatment.

Let E be a polarized Hilbert bundle on a space X. The restricted grassmannian $Gr_{\text{res}}(E)$ of this bundle is the space of semi-infinite subspaces V of fibres E_p, $p \in X$, of E; that is, the subspaces V are such that $V \oplus V^\perp$ is a decomposition in the polarization class of E_p. The space of pairs $Gr^{(2)}_{\text{res}}(E)$ and the finite-dimensional bundle γ are defined as before. We immediately have

Definition 5.2.4. A *parametrized semi-infinite spectrum* for the polarized bundle E is a $Gr_{\text{res}}(E)$-spectrum.

Granting conjecture 5.2.3, the claim is quite simply that the category of parametrized semi-infinite spectra for the polarized Hilbert bundle E on the space X is homotopically equivalent to the category of specters (twisted parametrized spectra) for the haunt on X associated to the polarized bundle. Thereby, semi-infinite spectra provide a geometric realization of the homotopy-theoretic correspondence, via the classifying map $B(\mathbb{Z} \times BO) \to B(\mathbb{Z} \times BG)$, between polarizations and haunts.

Part II

On the Twisted K-Homology of Simple Lie Groups

Chapter 6

Introduction

By way of motivation we present six interpretations of twisted K-theory. These interpretations inform the methods and perspectives adopted in the paper but are otherwise unnecessary for what follows. We then summarize our results on the twisted K-homology of simple Lie groups and overview our main techniques, namely the twisted Rothenberg-Steenrod spectral sequence, Tate resolutions, Bott generating varieties, and twisted $Spin^c$ bordism.

6.1 Six Interpretations of Twisted K-Theory

6.1.1 1-Dimensional Elements in Elliptic Cohomology

A twisting on a space X of a cohomology theory represented by a spectrum R is a bundle of spectra on X with fibre R and the associated twisted cohomology of X is given by the homotopy classes of sections of this bundle. Such twistings are classified by maps from X to the classifying space $B\,\mathrm{Aut}\,R$ of homotopy automorphisms of the spectrum R. If R is an A_∞ ring spectrum, the classifying space BGL_1R of homotopy units in R maps to $B\,\mathrm{Aut}\,R$ and thereby classifies a subset of the twistings—we refer to these twistings as elementary.

The classifying space $BGL_1\mathrm{H}\mathbb{C}$ for elementary twistings of ordinary cohomology with complex coefficients is $B\mathbb{C}^*$; (here $\mathrm{H}\mathbb{C}$ denotes the Eilenberg-MacLane spectrum for $\mathbb{C}$). There is a map $B\mathbb{C}^* \to \mathbb{Z} \times BU$ of this classifying space into the representing space for K-theory; any twisting $X \to BGL_1\mathrm{H}\mathbb{C}$ for the ordinary cohomology of X therefore determines a K-theory class on X. Of course, there is a natural geometric interpretation of the K-theory classes arising in this way, namely as the classes represented by flat line bundles on X. The twisted cohomology of X is simply the cohomology of X with coefficients in the line bundle, reinterpreted as the homotopy classes of sections of an associated $\mathrm{H}\mathbb{C}$ bundle.

The classifying space BGL_1K for elementary twistings of complex K-theory splits, as an infinite loop space, as $T \times S$. The factor T is a $K(\mathbb{Z}, 3)$ bundle over $K(\mathbb{Z}/2, 1)$ which splits as a space but has nontrivial infinite loop structure classified by $\beta Sq^2 \in H^3(\mathrm{H}(\mathbb{Z}/2); \mathbb{Z})$. There is a natural infinite loop map $T \to \mathrm{TMF}$ from T to the representing space for topological modular forms, and so by projecting through T a map $BGL_1K \to \mathrm{TMF}$. In particular an elementary twisting of K-theory for X determines a TMF-class on X. (Notice that TMF is the analog of real K-theory, that is of KO, and so the map $BGL_1K \to \mathrm{TMF}$ corresponds to the composite $BGL_1\mathrm{H}\mathbb{C} \to \mathbb{Z} \times BU \to \mathbb{Z} \times BO$; it is not known whether there exists an appropriate factorization $BGL_1K \to E \to \mathrm{TMF}$ for every elliptic spectrum E.) The geometric interpretation of these TMF classes is simplified if we restrict our attention to those classes coming from twistings involving only the $K(\mathbb{Z}, 3)$ factor of T. Such a twisting is determined by a map $X \to K(\mathbb{Z}, 3)$ or equivalently by a BS^1 bundle on X. We think of this bundle as a stack locally isomorphic to the sheaf of line bundles on X and as such as a 1-dimensional 2-vector bundle on X. In this sense we imagine the TMF classes coming from K-theory twistings as 1-dimensional elliptic elements and twisted K-theory as K-theory with coefficients in this "elliptic line bundle".

6.1.2 Projective Hilbert Space Bundles

There is a very simple and well-known reformulation of twistings of K-theory as projective Hilbert space bundles and of the corresponding twisted K-theory groups as families of Fredholm operators on these bundles. Indeed, the space of unitary operators on Hilbert space is contractible, so the group of projective unitary operators has the homotopy type of BS^1. As such a twisting $\alpha : X \to K(\mathbb{Z}, 3)$ of K-theory determines a projective bundle $\mathcal{H}(\alpha)$ of Hilbert spaces on X. The space of Fredholm operators on a Hilbert space has the homotopy type of $\mathbb{Z} \times BU$ and depends only on the projectivization of the Hilbert space. Sections of the $\mathbb{Z} \times BU$ bundle associated to the twisting α can therefore be thought of as Fredholm operators on the projective bundle $\mathcal{H}(\alpha)$. It remains to develop a general index theory for elliptic operators on these projective bundles, but substantial progress has been made by Mathai, Melrose, and Singer [35], who prove an index theorem in the case that the twisting α is a torsion class in $H^3(X; \mathbb{Z})$.

6.1.3 $K(\mathbb{Z}, 2)$-Equivariant K-Theory

We would like to discuss an algebro-geometric model for twisted K-theory, and the proper formulation is suggested by reinterpreting twisted K-theory as a $K(\mathbb{Z}, 2)$-equivariant theory; this formulation will also hint at connections with the representation theory of loop groups. As before, a twisting is a map $\alpha : X \to BK(\mathbb{Z}, 2)$ defining a principal $K(\mathbb{Z}, 2)$-bundle $P(\alpha)$ on X. The set of sections of the associated bundle $P(\alpha) \times_{K(\mathbb{Z},2)} (\mathbb{Z} \times BU)$ is the same as the set of $K(\mathbb{Z}, 2)$-equivariant maps from $P(\alpha)$ to $\mathbb{Z} \times BU$; that is, the twisted K-theory of X is the "$K(\mathbb{Z}, 2)$-equivariant" K-theory of $P(\alpha)$. In particular, elements of the twisted K-theory of X are represented by virtual vector bundles on the total space $P(\alpha)$ of the $K(\mathbb{Z}, 2)$-principal bundle associated to the twisting; these vector bundles V are required to be $K(\mathbb{Z}, 2)$-equivariant in the sense that for a line $L \in K(\mathbb{Z}, 2)$, the virtual vector space $V_{L \cdot x}$ at the point $L \cdot x \in P(\alpha)$ is equal to $L \otimes V_x$, for all points $x \in P(\alpha)$.

6.1.4 Perfect Complexes of α-Twisted Sheaves

Our 'space' X will now be a scheme, and a twisting of K-theory is a $\mathbb{G}_m$-gerbe on X. These gerbes are classified by $H^2(X; \mathbb{G}_m)$ and can be thought of as stacks locally isomorphic to the category of invertible sheaves. Elements of the twisted K-theory of X for a twisting gerbe α should be virtual sheaves of locally free $\mathcal{O}_\alpha$-modules on α

that are $B\mathbb{G}_m$-equivariant in an appropriate sense. More precisely an element of the twisted K-theory of X is a perfect complex of α-twisted sheaves on the gerbe α, that is a complex of α-twisted sheaves locally quasiisomorphic to a finite length complex of free finite rank $\mathcal{O}_\alpha$-modules. In the topological situation the analogue of the perfect complex on α is a two term complex of bundles on $P(\alpha)$, each of countably infinite rank, with a differential that is locally an isomorphism off of a finite rank subbundle. We would like to emphasize that this notion of α-twisted K-theory elements on the scheme X does not depend on the class $\alpha \in H^2(X; \mathbb{G}_m)$ being torsion.

6.1.5 Central Extensions of Loop Groups

We now specialize to the case (which indeed will be our primary focus in this paper) that our space is a connected simply connected compact Lie group G. A twisting map $\alpha : G \to K(\mathbb{Z}, 3)$ gives a map from the free loop space LG to the classifying space BS^1 by the composition $LG \to LK(\mathbb{Z}, 3) \to \Omega K(\mathbb{Z}, 3) \simeq BS^1$, and thereby gives a principal S^1-bundle on LG. The total space $\widetilde{LG}$ of this principal bundle can be given a group structure as an S^1-central extension of LG. The classifying space $B\widetilde{\Omega G}$ of the based loop central extension $\widetilde{\Omega G} \subset \widetilde{LG}$ is precisely the total space $P(\alpha)$ of the principal $K(\mathbb{Z}, 2)$ bundle over G. Moreover, to an irreducible highest-weight representation of $\widetilde{LG}$ one can associate an equivariant map from $P(\alpha)$ to $\mathbb{Z} \times BU$ and thereby an element of the twisted K-theory of G [38]. The precise relation between the representation theory of loop groups and twisted K-theory is described by Freed, Hopkins, and Teleman [19]—they prove that the group of positive energy unitary representations of $\widetilde{LG}$ is the twisted G-equivariant K-theory of G.

6.1.6 B-Fields and D-Branes

A great deal of the limelight focused on twisted K-theory has come from the widespread realization that certain boundary conditions in string theory naturally represent elements in the twisted K-theory of spacetime. In this context the twistings are represented by nontrivial Neveu-Schwarz B-fields; the elements of twisted K-theory are

D-branes, submanifolds of spacetime with a twisted $Spin^c$ structure on their normal bundles. More generally, such a submanifold M may be equipped with a vector bundle V and the class represented by the pair (M, V) is the pushforward of V to the twisted K-theory of the ambient spacetime X. When the space X is a Lie group, as in this paper, the twisted K-theory can be thought of as a topological model for the space of D-branes in a Wess-Zumino-Witten model for conformal field theory. Frequently the spacetime X is itself a $Spin^c$ manifold; the D-branes are then twisted $Spin^c$ submanifolds and represent elements in the twisted K-homology of X. In this case, a D-brane M naturally represents a class in a more refined group, the twisted $Spin^c$ bordism of X, and there is a twisted index map that recovers the twisted K-homology class of M. This perspective guides the discussion of the twisted $Spin^c$ bordism of Lie groups in the last section of this paper.

6.2 Results

We prove that the twisted K-homology ring of a simple Lie group is an exterior algebra tensor a cyclic group, we give a detailed description of the orders of these cyclic groups in terms of the dimensions of irreducible representations of related groups, and we show that these orders originate, via a twisted index map, from relations in the twisted $Spin^c$ bordism group.

Theorem 6.2.1. *Let G be a compact, connected, simply connected, simple Lie group of rank n. The twisted K-homology ring of G with nonzero twisting class $k \in \mathrm{H}^3(G;\mathbb{Z}) \cong \mathbb{Z}$ is an exterior algebra of rank $n-1$ tensor a cyclic group:*

$$K_{\cdot}^{\tau(k)}(G) \cong \Lambda[x_1, \ldots, x_{n-1}] \otimes \mathbb{Z}/c(G,k).$$

Here $c(G,k)$ is an integer depending on the group and the twisting.

This fact was first noticed in the case of $SU(n)$ by Hopkins. The proof is in section 8 for groups other than $Spin(n)$, and in section 9.4 for $Spin(n)$.

Theorem 6.2.2. *For the classical groups, the cyclic orders $c(G,k)$, $k>0$, of the twisted K-homology groups of G are:*

$$
\begin{aligned}
c(SU(n+1),k) &= \gcd\left\{\tbinom{k+i}{i}-1 : 1\le i\le n\right\}\\
c(Sp(n),k) &= \gcd\left\{\sum_{-k\le j\le -1}\tbinom{2j+2(i-1)}{2(i-1)} : 1\le i\le n\right\}\\
c(Spin(4n-1),k) &= \gcd\Big\{\left\{\tbinom{k}{i} : 1\le i\le 2n-2\right\}\\
&\qquad \cup\left\{2\tbinom{k}{2n-1}\right\}\cup\left\{2\tbinom{k}{2i+1}+\tbinom{k}{2i} : n\le i\le 2n-2\right\}\Big\}\\
c(Spin(4n+1),k) &= \gcd\left\{\left\{\tbinom{k}{i} : 1\le i\le 2n-1\right\}\cup\left\{2\tbinom{k}{2i+1}+\tbinom{k}{2i} : n\le i\le 2n-1\right\}\right\}\\
c(Spin(4n+2),k) &= \gcd\Big\{\left\{\tbinom{k}{i} : 1\le i\le 2n\right\}\\
&\qquad \cup\left\{2\tbinom{k}{2n+1}\right\}\cup\left\{2\tbinom{k}{2i+1}+\tbinom{k}{2i} : n+1\le i\le 2n-1\right\}\Big\}\\
c(Spin(4n),k) &= \gcd\left\{\left\{\tbinom{k}{i} : 1\le i\le 2n-1\right\}\cup\left\{2\tbinom{k}{2i+1}+\tbinom{k}{2i} : n\le i\le 2n-2\right\}\right\}.
\end{aligned}
$$

(Note that $c(G,-k)=c(G,k)$. The formulas for $c(Spin(4n-1),k)$ and $c(Spin(4n),k)$ exclude the degenerate case $n=1$.) The proofs for $SU(n)$, $Sp(n)$, and $Spin(n)$ occur respectively in sections 9.2, 9.3, and 9.4. A general method for computation, applicable to the exceptional groups, is discussed in section 9.5, and the cyclic order for G_2 is given in section 9.2.

Proposition 6.2.3. *Let G be as in Theorem 6.2.1. Suppose M_i is a collection of $Spin^c$ manifolds over ΩG whose fundamental classes generate $K.\Omega G$ as an algebra. Then there are twisted $Spin^c$ structures on the bordisms $W_i = M_i\times I$ such that the cyclic order of the twisted K-homology of G is* $\gcd(\mathrm{ind}(\partial W_1),\ldots,\mathrm{ind}(\partial W_n))$, *where* $\mathrm{ind}: MSpin^c_.* \longrightarrow K.*$ *is the index map from $Spin^c$ bordism to K-homology.*

The proof of this proposition is the focus of section 10.2.

6.3 Techniques and Overview

The primary tool for calculating twisted K-homology rings is the twisted Rothenberg-Steenrod spectral sequence; this is the original method used by Hopkins in the case $G = SU(n)$. The spectral sequence is:

$$E^2 = \mathrm{Tor}^{K.\Omega G}(\mathbb{Z}, \mathbb{Z}_{\tau(k)}) \Rightarrow K^{\tau(k)}_{\cdot}(G),$$

where $\mathbb{Z}_{\tau(k)}$ is the integers with a twisted $K.\Omega G$-module structure depending on k. In section 7.1 we present various generalities about twisted homology theories; then in section 7.2 we use a method of Segal [43] to construct this Rothenberg-Steenrod spectral sequence in twisted K-homology.

As the K-homology rings of loop spaces of simple Lie groups are known, our primary task is computing the Tor groups over these rings. Remarkably, for $G \neq Spin(n)$ this can be done without identifying the twisted $K.\Omega G$-module structure on $\mathbb{Z}$. These Tor groups are calculated in section 8 by an iterated series of filtration spectral sequences applied to a judiciously chosen Tate resolution. The spectral sequences are seen to collapse and to be extension-free, completing the proof of Theorem 6.2.1 for $G \neq Spin(n)$.

The Tor computation for $Spin(n)$ requires a detailed knowledge of the twisted module structure on $\mathbb{Z}$; this module structure is also precisely what is needed to identify the cyclic orders of the twisted K-homology groups. The best way to identify this module structure is via generating varieties for the loop space of the group, and this is the subject of section 9. Sections 9.2, 9.3, and 9.5 describe generating varieties for various groups, compute the cyclic orders in the corresponding cases, and discuss a general method for determining the cyclic order. Section 9.4 describes the twisted module structure for $Spin(n)$ and presents the belated Tor calculation for this group.

The computation in section 9 of the cyclic order in terms of the dimensions of irreducible representations does not give much geometric insight into these torsion groups. We give, in section 10, an interpretation of these orders in terms of relations in the twisted $Spin^c$ bordism group of G. The main tool, presented in section 10.1, is a cocycle model for twisted $Spin^c$ bordism. This model allows explicit descriptions of nullbordisms of particular $Spin^c$ manifolds over G corresponding to relations in the twisted K-homology of G—see section 10.2. We conclude in section 10.3 by discussing potential representatives in $MSpin^{c,\tau}_{\cdot}(G)$ for the exterior generators of $K^{\tau}_{\cdot}(G)$.

Chapter 7

Twisted K-Theory and the Rothenberg-Steenrod Spectral Sequence

7.1 Twisted Homology Theories

We review the definitions and basic properties of twisted homology and cohomology theories. There are by now various models for these theories, but the following perspective owes as much to Goodwillie as to folklore.

For a spectrum F, the cohomology of a space X with coefficients in F can be defined as

$$F^n(X) := \operatorname{colim} \Gamma_h(X, X \times \Omega^i F_{i+n});$$

here $\Gamma_h(X, E)$ refers to homotopy classes of sections of the (here trivial) bundle E on X. The maps in the colimit are induced by applying the usual structure maps $\Omega^i F_{i+n} \to \Omega^{i+1}\Sigma F_{i+n} \to \Omega^{i+1} F_{i+1+n}$ fibrewise to the bundle $X \times \Omega^i F_{i+n} \to X$. Now let E be a bundle of based spectra over X, with fibre spectrum F; this means in particular that for each i we have a fibration $E_i \to X$, a section $X \to E_i$, and a fibrewise structure map $\Sigma_X E_i \to E_{i+1}$. (Note that Σ_X denotes fibrewise suspension and Ω_X will denote the fibrewise loops.) The cohomology of X with coefficients in E

is defined to be

$$E^n(X) := \operatorname{colim} \Gamma_h(X, \Omega^i_X E_{i+n})$$

where the colimit maps are, as expected, induced by $\Omega^i_X E_{i+n} \to \Omega^{i+1}_X \Sigma_X E_{i+n} \to \Omega^{i+1}_X E_{i+1+n}$.

The parallel in homology is similar. The homology of X with coefficients in F is

$$F_n(X) := \operatorname{colim}[S^{i+n}, (X \times F_i)/X],$$

with maps induced by $\Sigma((X \times F_i)/X) = (X \times \Sigma F_i)/X \to (X \times F_{i+1})/X$. As above, when E is a bundle of based spectra, we have a 'base point' section $X \to E_i$ for all i. The homology of X with coefficients in E is

$$E_n(X) := \operatorname{colim}[S^{i+n}, E_i/X];$$

the colimit maps are induced by $\Sigma(E_i/X) = (\Sigma_X E_i)/X \to E_{i+1}/X$.

For completeness we also mention the reduced analogs of homology and cohomology with coefficients in a bundle of spectra. The reduced cohomology with coefficients in a trivial F bundle can be given as

$$\tilde{F}^n(X) := \operatorname{colim} \Gamma^b_h(X, X \times \Omega^i F_{i+n}),$$

that is as the colimit of homotopy classes of sections taking the base point of X to the basepoint of $\Omega^i F_{i+n}$. The reduced cohomology with coefficients in E is then

$$\tilde{E}^n(X) := \operatorname{colim} \Gamma^b_h(X, \Omega^i_X E_{i+n});$$

the maps are induced as before. Similarly, the reduced homology with coefficients in a trivial bundle is

$$\tilde{F}_n(X) := \operatorname{colim}[S^{i+n}, (X \times F_i)/(X \vee F_i)].$$

The twisted reduced homology is finally

$$\tilde{E}_n(X) := \text{colim}[S^{i+n}, E_i/(X \vee F_i)];$$

the maps are induced by $\Sigma(E_i/(X \vee F_i)) = (\Sigma_X E_i)/(X \vee \Sigma F_i) \to E_{i+1}/(X \vee F_{i+1})$. Of course, these reduced groups are special cases of the relative groups:

$$E^n(X, A) := \text{colim}\, \Gamma_h(X, A; \Omega^i_X E_{i+n}, s(A)),$$

where s is the distinguished base point section; similarly,

$$E_n(X, A) := \text{colim}[S^{i+n}, (E_i/X)/((E_i|_A)/A)].$$

The most important fact about twisted homology theories is that they are honest homology theories in an appropriate category. Indeed, consider the category of pairs (X, A) of spaces, where A is a closed subspace of X and X is equipped with a bundle E of based spectra with fibre spectrum F. From the above description of the homology $E_n(X, A)$, it is immediate that twisted homology on this category of pairs is a homology theory in the classical sense.

∼∼∼∼∼

In this paper we will only be concerned with bundles of spectra associated to principal $K(\mathbb{Z}, 2)$ bundles over our space X. As usual, we fix a model for $K(\mathbb{Z}, 3)$ and select a particular universal $K(\mathbb{Z}, 2)$ bundle on it. A map $\alpha : X \to K(\mathbb{Z}, 3)$ gives a principal $K(\mathbb{Z}, 2)$ bundle $P(\alpha)$ on X, classified up to isomorphism by the homotopy class of the map. For any basepoint-preserving action of $K(\mathbb{Z}, 2)$ on a spectrum F, we can form the associated F bundle to $P(\alpha)$. The resulting bundle $P(\alpha) \times_{K(\mathbb{Z},2)} F$ is a bundle of based spectra on X, as above. Note that on the level of spaces, the action of $K(\mathbb{Z}, 2)$ on F is given by maps $K(\mathbb{Z}, 2)_+ \wedge F_i = (K(\mathbb{Z}, 2) \times F_i)/(K(\mathbb{Z}, 2) \times *) \to F_i$, and we often denote the spectrum action simply by a map $K(\mathbb{Z}, 2)_+ \wedge F \to F$.

Our primary examples are twisted $Spin^c$-bordism and twisted K-theory. The $K(\mathbb{Z},2)$ bundle

$$K(\mathbb{Z},2) = BU(1) \to BSpin^c \to BSO$$

is principal, with classifying map $BSO \xrightarrow{\beta w_2} BBU(1) = K(\mathbb{Z},3)$ classifying the integral Bockstein of the second Stiefel-Whitney class. In particular we have an action $K(\mathbb{Z},2) \times BSpin^c \to BSpin^c$; on Thom spaces this action is $K(\mathbb{Z},2)_+ \wedge MSpin^c \to MSpin^c$, that is, a based action of $K(\mathbb{Z},2)$ on the $Spin^c$ Thom spectrum. The α-twisted $Spin^c$-bordism groups are then, of course, the stable homotopy groups $\pi_i((P(\alpha) \times_{K(\mathbb{Z},2)} MSpin^c)/X)$.

The K-theory spectrum K is a module over $Spin^c$-bordism by the usual index map $MSpin^c \xrightarrow{\text{ind}} K$. Taking the above based action $K(\mathbb{Z},2)_+ \wedge MSpin^c \xrightarrow{\phi} MSpin^c$ and smashing over $MSpin^c$ with K, we have a compatible based action on K-theory:

$$\begin{tikzcd}
K(\mathbb{Z},2)_+ \wedge MSpin^c \arrow[r, "\phi"'] \arrow[d, "\mathrm{id}\wedge\mathrm{ind}"'] & MSpin^c \arrow[d, "\mathrm{ind}"] \\
K(\mathbb{Z},2)_+ \wedge K \arrow[r, "\phi\wedge_{MSpin^c}(\mathrm{id})"'] & K
\end{tikzcd}$$

The corresponding map on associated principal bundles $P(\alpha) \times_{K(\mathbb{Z},2)} MSpin^c \to P(\alpha) \times_{K(\mathbb{Z},2)} K$ induces a map from twisted $Spin^c$-bordism to twisted K-theory which we call the twisted index map. This map will be important in section 10.

Twisted K-theory can be defined more directly by choosing an explicit model for $\mathbb{Z} \times BU$ (typically the space of Fredholm operators on a fixed Hilbert space $\mathcal{H}$) that admits an explicit action by some model for $BU(1)$ (typically the space of projective unitary operators on $\mathcal{H}$); see, for example, Atiyah [3]. Whatever the formal definition, the geometric action being modeled is the following: a complex line L (representing a point in $BU(1)$) acts on a virtual-dimension-zero (or stable) vector space V (representing a point in BU) by tensor product, that is, $V \mapsto L \otimes V$.

It is worth noting, though, that this heuristic action of tensoring a vector bundle with a line can be misleading if we pay insufficient attention to the virtual dimension zero condition. It is tempting to think of elements of α-twisted K-cohomology as sections of an α-twisted gerbe of rank n, for some sufficiently large n; (such a section is locally a rank-n vector bundle, twisted globally by α). However, in this paper we are dealing with non-torsion twistings, and therefore no nontrivial element of twisted K-cohomology is representable by a section of any finite rank gerbe. We are inescapably in either a virtual-dimension-zero or an infinite-dimensional situation—which would seem to be a matter of personal penchant.

7.2 The Twisted Rothenberg-Steenrod Spectral Sequence

The "twisted" Rothenberg-Steenrod spectral sequence computing the twisted K-homology of a space is in fact the ordinary Rothenberg-Steenrod (a.k.a. homology Eilenberg-Moore) spectral sequence in an appropriate category, and as such requires little comment. We briefly recall the spectral sequence in generality, then describe its application to the geometric bar complex on the loop space of a simple Lie group.

We work in the category $\mathcal{K}$ of pairs $(X; E)$, where X is a space and E is a bundle of based spectra on X with fibre the K-theory spectrum; the morphisms are those bundle maps that are homotopy equivalences on each fibre. Similarly, we have a category of triples $(X, A; E)$ where A is a closed subspace of X and E is again a bundle on X. As mentioned in the last section, the functors

$$(X, A; E) \mapsto E_n(X, A) = \operatorname{colim}[S^{i+n}, (E_i/X)/((E_i|_A)/A)]$$

form a homology theory in the classical sense. In particular, for any simplicial object $S.$ in $\mathcal{K}$, there is a spectral sequence a la Segal [43] with E^2 term $H_p(E_q(S.))$ converging to the homology of the realization $E_{p+q}(|S.|)$.

Let G be a simple, simply connected Lie group and $k \in H^2(\Omega G; \mathbb{Z}) = \mathbb{Z}$ an integer describing a line bundle L^{-k} on the loop space ΩG. On the one hand there is the trivial projection map in $\mathcal{K}$ from $(\Omega G; \Omega G \times K)$ to $(*; K)$. On the other hand, there is a twisted map $\tau(k) : (\Omega G; \Omega G \times K) \to (*; K)$ given by $\Omega G \times K \xrightarrow{k \times \mathrm{id}} K(\mathbb{Z}, 2) \times K \to K$, where the last map is the $K(\mathbb{Z},2)$ action on the spectrum K described in section 7.1. The geometric bar construction $B_\tau\Omega G = B.(*,\Omega G, *_\tau)$ is a simplicial object in $\mathcal{K}$. To describe the corresponding spectral sequence we need only compute the effect of $\tau(k)$ in homology and identify the realization $|B_\tau\Omega G|$.

Given a class ϕ in the K-homology of ΩG the image of ϕ under $\tau(k)$ is evidently equal to the evaluation $\langle\tau(k)^*(1),\phi\rangle$, where $\langle -,-\rangle$ denotes the Kronecker pairing. The pullback $\tau(k)^*(1)$ is L^k, and the resulting map $K.\Omega G \xrightarrow{\langle L^k,-\rangle} K.*$ defines a module structure on $K.*$ which we denote $(K.*)_\tau$. The E^2 term of our spectral sequence is therefore $\mathrm{Tor}^{K.\Omega G}(K.*,(K.*)_\tau)$.

As a space the realization of $B_\tau\Omega G$ is evidently $B\Omega G \simeq G$; we identify the K-bundle. The K-bundle on the realization is defined by a 1-cocycle $\tau(k)$ with values in $K(\mathbb{Z},2)$ and as such is classified by the image of $\tau(k)$ in $H^3(B\Omega G;\mathbb{Z})$. We have $H^3(B\Omega G) \cong H^3(\Sigma\Omega G)$ and it is enough to identify the restriction of $\tau(k)$ to the 1-skeleton $\Sigma\Omega G$ of $B\Omega G$. It is, however, immediate that this cocycle on the 1-skeleton of the geometric bar construction $B_\tau\Omega G$ has homology invariant $k \in H^3(\Sigma\Omega G)$. In summary:

Proposition 7.2.1. *There is a spectral sequence of algebras with E^2 term*

$$E^2_{pq} = \mathrm{Tor}^{K.\Omega G}_{p,q}(K.*,(K.*)_\tau)$$

converging as an algebra to the twisted K-homology $K^\tau_{p+q}(G)$.

Chapter 8

Tate Resolutions and $\mathrm{Tor}^{K.\Omega G}(\mathbb{Z}, \mathbb{Z}_T)$ for $G \neq Spin(n)$

For each group G, we describe the K-homology of the loop space of G, give an appropriate Tate resolution of $K.* = \mathbb{Z}$ over $K.\Omega G$, and compute the torsion group using a series of filtration spectral sequences.

We recall Tate's main result on algebra resolutions over a commutative Noetherian ring R. An ideal $I \subset R$ is said to be generated by the regular sequence $a_1, \ldots, a_r \in R$ if $I = (a_1, \ldots, a_r)$ and a_i is not a zero-divisor in $R/(a_1, \ldots, a_{i-1})$ for all i.

Theorem 8.0.2 (Tate [46]). *Let $A \subset B$ be ideals of R generated respectively by the regular sequences $(s_1, \ldots, s_m)$ and $(t_1, \ldots, t_n)$. For any choice of constants $c_{ji} \in R$ such that $s_j = \sum_{i=1}^{n} c_{ji} t_i$, the differential graded algebra*

$$D = (R/A\langle T_1, \ldots, T_n\rangle\{S_1, \ldots, S_m\}; d(T_i) = [t_i], d(S_j) = \sum_{i=1}^{n}[c_{ji}]T_i)$$

is a resolution of R/B as an R/A-module. Here the T_i are strictly skew commutative generators of degree 1, and the S_j are divided power algebra generators of degree 2.

In particular, $\mathrm{Tor}_{R/A}(R/B, Q)$ will be given as the homology $H(D \otimes_{R/A} Q)$. In our applications, R will be a polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, the ideal A will depend on the group, the ideal B will be $(x_1, \dots, x_n)$, and Q will be an R/A-module $\mathbb{Z}_\tau$ on which x_i acts by an integer c_i depending on the group and the twisting class.

8.1 Tor for $SU(n+1)$ and $Sp(n)$

Elementary calculation shows that the integral cohomology rings of $SU(n+1)$ and $Sp(n)$ are exterior algebras on n generators. Application of the spectral sequence $\mathrm{Ext}^{H^\cdot(G;k)}(k,k) \Rightarrow H_.(\Omega G;k)$, k a field, then implies that the integral Pontryagin rings $H_.(\Omega SU(n+1))$ and $H_.(\Omega Sp(n))$ are both polynomial on n generators, all in even degree. In each case the Atiyah-Hirzebruch spectral sequence for K-theory then collapses, and the K-theory Pontryagin ring is again polynomial.

The Tate resolution in this case is especially simple, as the ideal A is trivial. Let G denote either $SU(n+1)$ or $Sp(n)$ and $k \in \mathbb{Z} \cong H^3(G;\mathbb{Z})$ the twisting class. Choose reduced generators x_i of $K_.\Omega G$, so that $K_.\Omega G \cong \mathbb{Z}[x_1, \dots, x_n]$. (Note that, unless otherwise noted, we treat K-theory as $\mathbb{Z}/2$-graded.) The $K_.\Omega G$ module structure on $\mathbb{Z}_\tau$ is given, as in section 7.2, by the map $K_.\Omega G \to K_.*$ sending a class x to $\langle L^k, x\rangle$, where L is a generating line bundle. We defer the explicit evaluation of these maps to section 9. For now, we denote by c_i the image of x_i in $\mathbb{Z}_\tau$; of course this constant depends on both the group and the twisting, but we tend to omit both dependencies from the notation. By Tate's theorem,

$$\begin{aligned}\mathrm{Tor}^{K_.\Omega G}(\mathbb{Z}, \mathbb{Z}_\tau) &= H(\mathbb{Z}[x_1, \dots, x_n]\langle T_1, \dots, T_n\rangle \otimes_{\mathbb{Z}[x_1,\dots,x_n]} \mathbb{Z}_\tau; d) \\ &= H(\mathbb{Z}\langle T_1, \dots, T_n\rangle; dT_i = c_i).\end{aligned}$$

To evaluate this homology group we employ the following general procedure. Suppose we know the homology of the subalgebra generated by $T_1, \dots, T_i$. We filter the subalgebra generated by $T_1, \dots, T_{i+1}$ by powers of T_{i+1} and look at the associated spectral sequence. The only differential is d^1, which is given by multiplication by c_{i+1}, and by induction we can thus compute the homology of the original algebra.

We assume for now that c_1 is not zero; this is indeed the case (see sections 9.2

and 9.3). The homology of $(\mathbb{Z}\langle T_1\rangle, d)$ is $\mathbb{Z}/c_1$. The, quite degenerate, spectral sequence of the filtration of $(\mathbb{Z}\langle T_1, T_2\rangle, d)$ by T_2 is therefore

$$\mathbb{Z}/c_1 \xleftarrow{c_2} \mathbb{Z}/c_1.$$

The homology is $\mathbb{Z}/g_{12}\langle y_2\rangle$, where $g_{12} = \gcd\{c_1, c_2\}$ and y_2 is an exterior class. More generally we will denote by $g_{1..i}$ the greatest common divisor $\gcd\{c_1, c_2, \ldots, c_i\}$. The induction step is, as expected, the homology of

$$\mathbb{Z}/g_{1..i}\langle y_2, \ldots, y_i\rangle \xleftarrow{c_{i+1}} \mathbb{Z}/g_{1..i}\langle y_2, \ldots, y_i\rangle,$$

and the Tor groups are given by

$$\begin{aligned}\mathrm{Tor}^{K.\Omega SU(n+1)}(\mathbb{Z}, \mathbb{Z}_\tau) &= \mathbb{Z}/(g_{1..n}(SU(n+1), k))\langle y_2, \ldots, y_{n-1}\rangle \\ \mathrm{Tor}^{K.\Omega Sp(n)}(\mathbb{Z}, \mathbb{Z}_\tau) &= \mathbb{Z}/(g_{1..n}(Sp(n), k))\langle y_2, \ldots, y_{n-1}\rangle.\end{aligned}$$

We belabor this calculation only because, when we come to more complicated examples, especially $Spin(n)$, it will help to have a clear model.

8.2 Tor for the Exceptional Groups

The exceptional Lie groups are nature's best attempts to make a finite dimensional Lie group out of $K(\mathbb{Z}, 3)$. In particular they are homotopy equivalent to $K(\mathbb{Z}, 3)$ through a range of dimensions, and so their loop spaces are homotopy equivalent to $K(\mathbb{Z}, 2)$ through a similar range. The K-homology of $K(\mathbb{Z}, 2)$ is the subalgebra of $\mathbb{Q}[a]$ generated by $\{a, \binom{a}{2}, \binom{a}{3}, \ldots\}$; see [1]. Extensive computations by Duckworth [11] show that for G exceptional, the K-homology $K.\Omega G$ differs from a polynomial ring only in the aforementioned low-dimensional flirtation with $K(\mathbb{Z}, 2)$. For example, Duckworth proves that $K.\Omega E_8$ is a polynomial ring on seven generators tensor the subalgebra of $\mathbb{Q}[a]$ generated by the elements $\{a, \binom{a}{2}, \binom{a}{3}, \binom{a}{4}, \binom{a}{5}\}$. In order to use Tate resolutions, we must give explicit algebra presentations of these K-homology rings:

Proposition 8.2.1. *The K-homology rings of the loop spaces of the exceptional Lie groups are given by*

$$\begin{aligned}
K.\Omega G_2 &= \frac{\mathbb{Z}[a,b,x_3]}{(a(a-1)-2b)}\\
K.\Omega F_4 &= \frac{\mathbb{Z}[a,b,c,x_4,x_5,x_6]}{(a(a-1)-2b,b(a-2)-3c)}\\
K.\Omega E_6 &= \frac{\mathbb{Z}[a,b,c,x_4,x_5,x_6,x_7,x_8]}{(a(a-1)-2b,b(a-2)-3c)}\\
K.\Omega E_7 &= \frac{\mathbb{Z}[a,b,c,d,x_5,x_6,x_7,x_8,x_9,x_{10}]}{(a(a-1)-2b,b(a-2)-3c,b(b+1)-a(b+c)-2d)}\\
K.\Omega E_8 &= \frac{\mathbb{Z}[a,b,c,d,e,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12}]}{(a(a-1)-2b,b(a-2)-3c,b(b+1)-a(b+c)-2d,d(a-4)-5e)}
\end{aligned}$$

Note that the unsightly third relation in the rings for E_7 and E_8 is essential and cannot be replaced by the more sensible relation $c(a-3)-4d$. We remark that, because the 'lettered' generators in these K-homology rings come from corresponding generators in $K.(K(\mathbb{Z},2))$, the twisted pushforwards of these elements are easily computed. In particular, the twisted pushforward of a, denoted again by c_1, is just k, the twisted pushforward of b is $c_2 = \binom{k}{2}$, of c is $c_3 = \binom{k}{3}$, and so on, with each generator mapping to its respective binomial coefficient.

As always, our starting point is the Tate resolution:

$$\mathrm{Tor}^{K.\Omega G_2}(\mathbb{Z},\mathbb{Z}_\tau) = H(\mathbb{Z}\langle T_1,T_2,T_3\rangle\{S_1\}; dT_i = c_i, dS_1 = (c_1-1)T_1 - 2T_2).$$

Consider the subalgebra generated by T_1, T_2, and S_1. If k is even, we can rewrite this DGA as

$$(\mathbb{Z}\langle T_1',T_2'\rangle\{S_1\}; dT_1' = 0, dT_2' = \frac{k}{2}, dS_1 = T_1'),$$

where $T_1' = (k-1)T_1 - 2T_2$ and $T_2' = \frac{k}{2}T_1 - T_2$. The Kunneth theorem immediately shows that the homology of this DGA is $\mathbb{Z}/(\frac{k}{2})$. If k is odd, we instead change the basis to $T_1' = \frac{k-1}{2}T_1 - T_2$ and $T_2' = kT_1 - 2T_2$. The algebra then takes the form

$$(\mathbb{Z}\langle T_1', T_2'\rangle\{S_1\}; dT_1' = 0, dT_2' = k, dS_1 = 2T_1'),$$

and by the Kunneth theorem its homology is $\mathbb{Z}/k$. In other words, the homology of the subalgebra in question is, in any case, $\mathbb{Z}/g_{12}$, where as before $g_{12} = \gcd\{c_1, c_2\}$. Filtering as in section 8.1 we see that the full Tor group is $\mathbb{Z}/g_{123}\langle y_3\rangle$.

The Tate resolution for F_4 gives

$$\begin{aligned}\mathrm{Tor}^{K.\Omega F_4}(\mathbb{Z}, \mathbb{Z}_\tau) = H(&\mathbb{Z}\langle T_1, T_2, T_3, T_4, T_5, T_6\rangle\{S_1, S_2\};\\ &dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2, dS_2 = (c_1 - 2)T_2 - 3T_3).\end{aligned}$$

We focus on the subalgebra generated by $\{T_1, T_2, T_3, S_1, S_2\}$. The method used for G_2, of changing basis to split the algebra into simpler pieces, works here as well; the basis change now depends on k modulo 6. We spell out only the case $k = 1 \pmod 6$. As basis change for the T_i's we take

$$\begin{pmatrix} \frac{k-1}{2} & -1 & 0 \\ -\frac{k-1}{6} & \frac{k-1}{3} & -1 \\ \frac{3k-1}{2} & -1-k & 3 \end{pmatrix}.$$

The algebra then has the form

$$(\mathbb{Z}\langle T_1', T_2', T_3'\rangle\{S_1, S_2\}; dT_1 = dT_2 = 0, dT_3 = k, dS_1 = 2T_1', dS_2 = 3T_2' + T_1').$$

The spectral sequence associated to the filtration of the $\{T_1', T_2', S_1, S_2\}$ subalgebra by powers of S_2 is

$$
\begin{array}{l|ccccccc}
 & \underset{(S_1^{(2)}T_1'T_2')}{\mathbb{Z}/2} & \longleftarrow & \underset{(S_2S_1^{(2)}T_1')}{\mathbb{Z}/2} & & \underset{(S_2^{(2)}S_1T_1'T_2')}{\mathbb{Z}/2} & \longleftarrow & \cdots \\
 & \underset{(S_1^{(2)}T_1')}{\mathbb{Z}/2} & & \underset{(S_2S_1T_1'T_2')}{\mathbb{Z}/2} & \longleftarrow & \underset{(S_2^{(2)}S_1T_1')}{\mathbb{Z}/2} & & \cdots \\
 & \underset{(S_1T_1'T_2')}{\mathbb{Z}/2} & \longleftarrow & \underset{(S_2S_1T_1')}{\mathbb{Z}/2} & & \underset{(S_2^{(2)}T_1'T_2')}{\mathbb{Z}/2} & \longleftarrow & \cdots \\
 & \underset{(S_1T_1')}{\mathbb{Z}/2} & & \underset{(S_2T_1'T_2')}{\mathbb{Z}/2} & \xleftarrow{(1,1)} & \underset{(S_2^{(2)}T_1',S_2^{(2)}T_2')}{\mathbb{Z}/2\oplus\mathbb{Z}} & \longleftarrow & \cdots \\
 & \underset{(T_1'T_2')}{\mathbb{Z}/2} & \xleftarrow{(1,1)} & \underset{(S_2T_1',S_2T_2')}{\mathbb{Z}/2\oplus\mathbb{Z}} & \xleftarrow{(1,3)} & \underset{(S_2^{(2)})}{\mathbb{Z}} & & \\
 & \underset{(T_1',T_2')}{\mathbb{Z}/2\oplus\mathbb{Z}} & \xleftarrow{(1,3)} & \underset{(S_2)}{\mathbb{Z}} & & & & \\
 & \underset{(1)}{\mathbb{Z}} & & & & & & \\
\hline
\end{array}
$$

There are, of course, no differentials beyond d^1 and the homology of the $\{T_1', T_2', S_1, S_2\}$ subalgebra is therefore $\mathbb{Z}/6$ in odd degree, 0 in positive even degree, and $\mathbb{Z}$ in degree zero; consequently the homology of the $\{T_1', T_2', T_3', S_1, S_2\}$ subalgebra is $\mathbb{Z}/k$ concentrated in degree zero. In general, ie for k not necessarily congruent to 1 modulo 6, this $\mathbb{Z}/k$ is replaced by $\mathbb{Z}/g_{123}$ and the full Tor group for F_4 is $\mathbb{Z}/g_{1..6}\langle y_4, y_5, y_6\rangle$. The computation for E_6 is identical, but for two additional exterior generators in the final Tor group.

This basis change approach quickly becomes impractical: for E_8 the congruence of k modulo 60 determines the structure of the basis change and of the subsequent homology computation. If we are willing to give up our ability to write down explicit generators for the Tor groups, we can do the computation without such a case by case analysis. We briefly reconsider the groups G_2 and F_4. For G_2 the main step was computing the homology of the DGA

$$D = (\mathbb{Z}\langle T_1, T_2\rangle\{S_1\}; dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2);$$

recall that $c_1 = k$ and $c_2 = \binom{k}{2}$. The homology of the $\{T_1, T_2\}$ subalgebra is $\mathbb{Z}/g_{12}\langle y_2\rangle$, where the generator y_2 can be taken to be $-(c_1/g_{12})T_2$ modulo terms involving T_1; (we will refer to terms with lower indices, sensibly enough, as 'lower terms' and so say, for example, that "y_2 is $-(c_1/g_{12})T_2$ modulo lower terms"). Thus, when we filter D by powers of S_1, the homology of D becomes the homology of

$$(\mathbb{Z}/g_{12}\langle y_2\rangle\{S_1\}; dS_1 = (2g_{12}/c_1)y_2).$$

Note that $2g_{12}/c_1$ is an integer, so this expression makes sense. We observe that $2g_{12}/c_1$ is actually a unit in $\mathbb{Z}/g_{12}$; indeed $g_{12} = c_1/\gcd(2, c_1)$ so

$$\gcd(2g_{12}/c_1, g_{12}) = \gcd(2/\gcd(2, c_1), c_1/\gcd(2, c_1)) = 1.$$

The homology of D is therefore simply $\mathbb{Z}/g_{12}$, as previously noted, and thus the full Tor group is again an exterior algebra tensor a cyclic group.

The case of F_4 (and therefore of E_6) is again similar. The main step is the computation of the homology of the DGA

$$D = (\mathbb{Z}\langle T_1, T_2, T_3\rangle\{S_1, S_2\}; dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2, dS_2 = (c_1 - 2)T_2 - 3T_3).$$

(Here again $c_i = \binom{k}{i}$.) Using the G_2 result we see that the homology of the $\{T_1, T_2, T_3, S_1\}$ subalgebra is $\mathbb{Z}/g_{123}\langle y_3\rangle$ where y_3 is $-(g_{12}/g_{123})T_3$ modulo lower terms. As above the homology of D is thereby reduced to the homology of

$$(\mathbb{Z}/g_{123}\langle y_3\rangle\{S_2\}; dS_2 = (3g_{123}/g_{12})y_3).$$

Again, this differential is an isomorphism, ie $3g_{123}/g_{12}$ is a unit in g_{123}. The trick is the same: observe that $g_{123} = g_{12}/\gcd(3, g_{12}) = c_1/(\gcd(3, c_1)\gcd(2, c_1))$; from this we have

$$\begin{aligned}\gcd(3g_{123}/g_{12}, g_{123}) &= \gcd(3/\gcd(3, c_1), c_1/(\gcd(3, c_1)\gcd(2, c_1))) \\ &= \gcd(3/\gcd(3, c_1), c_1/\gcd(3, c_1)) = 1.\end{aligned}$$

The homology of D is thus, again, $\mathbb{Z}/g_{123}$. The full Tor group follows.

Despite the increased complexity of the K-homology rings of ΩE_7 and ΩE_8, the Tor calculations in these cases are no more elaborate than for the other exceptional groups. The presentation in Proposition 8.2.1 suggests an appropriate Tate resolution and the Tor group over $K.\Omega E_7$ is given by the homology of the DGA

$$(\mathbb{Z}\langle T_1, \dots, T_{10}\rangle\{S_1, S_2, S_3\}; dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2,$$
$$dS_2 = (c_1 - 2)T_2 - 3T_3, dS_3 = (c_2 + 1)T_2 - (c_2 + c_3)T_1 - 2T_4).$$

Using the F_4 computation, we see that the homology of the $\{T_1, T_2, T_3, T_4, S_1, S_2\}$ subalgebra is $\mathbb{Z}/g_{1234}\langle y_4\rangle$, where y_4 is $-(g_{123}/g_{1234})T_4$ modulo lower terms. The homology of the $\{T_1, T_2, T_3, T_4, S_1, S_2, S_3\}$ subalgebra is therefore the homology of

$$(\mathbb{Z}/g_{1234}\langle y_4\rangle\{S_3\}; dS_3 = (2g_{1234}/g_{123})y_4).$$

We observe that $2g_{1234}/g_{123}$ is a unit in $\mathbb{Z}/g_{1234}$ and so the homology of this subalgebra is $\mathbb{Z}/g_{1234}$ concentrated in degree zero. The full Tor group is finally $\mathbb{Z}/g_{1..10}\langle y_5, y_6, \dots, y_{10}\rangle$. In this calculation it is critical that the third relation in the presentation of $K.\Omega E_7$ is $b(b+1) - a(b+c) - 2d$ and not the expected $c(a-3) - 4d$. The latter relation would produce a differential $dS_3 = (4g_{1234}/g_{123})y_4$ and thereby (because $4g_{1234}/g_{123}$ is not always a unit in $\mathbb{Z}/g_{1234}$) a plethora of nontrivial higher torsion.

The Tor computation for E_8 is entirely analogous. The Tate resolution is dictated by the presentation in Proposition 8.2.1 and the necessary combinatorial fact is that $5g_{12345}/g_{1234}$ is a unit in $\mathbb{Z}/g_{12345}$.

8.3 Proof of Theorem 6.2.1

We can now establish the bulk of our main theorem. We assume the computation of the torsion group for $Spin(n)$, which is carried out in section 9.4:

$$\mathrm{Tor}^{K.\Omega Spin(n)}(K.*, (K.*)_\tau) = \Lambda[x_1, \dots, x_{n-1}] \otimes \mathbb{Z}/c(Spin(n), k).$$

Though we have treated K-homology as $\mathbb{Z}/2$-graded in our Tor computations, properly it is $\mathbb{Z}$-graded, and the E^2 term of the Rothenberg-Steenrod spectral sequence has the appearance:

$$\left| \begin{array}{cccc} \mathrm{Tor}_0^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \mathrm{Tor}_1^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \mathrm{Tor}_2^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \cdots \\ 0 & 0 & 0 & \\ \mathrm{Tor}_0^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \mathrm{Tor}_1^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \mathrm{Tor}_2^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \cdots \\ 0 & 0 & 0 & \\ \mathrm{Tor}_0^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \mathrm{Tor}_1^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \mathrm{Tor}_2^{K_0\Omega G}(\mathbb{Z},\mathbb{Z}_\tau) & \cdots \end{array} \right.$$

In our cases these torsion groups are generated in Tor-degree 1; the (homological) differentials vanish on the generators and thus the spectral sequence collapses at the E^2 term.

We show that there are no additive extensions. We have established that the E^∞ term of the spectral sequence is a cyclic group, say $\mathbb{Z}/c$, tensor an exterior algebra. The filtration is homological, so the subgroup $(\mathbb{Z}/c)\{1\} \subset E^\infty$ generated by the identity element of the torsion group $\mathrm{Tor}^{K.\Omega G} = E^2 = E^\infty$ is actually a subgroup of the K-homology $K^\tau(G)$. The construction of the spectral sequence shows that this identity element in the torsion group corresponds to the identity element in the K-homology. The identity element $1 \in K^\tau(G)$ is therefore killed by multiplication by c, and so the entire K-homology ring is c-torsion, as desired.

For degree reasons, the only possible multiplicative extension is $y_i^2 = d \in K^\tau(G)$; that is the square of the K-homology class represented by an exterior generator $y_i \in \mathrm{Tor}_1$ could be a constant integer, an element of Tor_0. However, by construction the exterior classes y_i are represented by *reduced* classes in $K^\tau(G)$ and so their squares are also certainly reduced, eliminating the possibility of multiplicative extensions. $\square$

Chapter 9

Generating Varieties, the Cyclic Order of $K^\tau G$, and $\mathrm{Tor}^{K.\Omega Spin(n)}(\mathbb{Z}, \mathbb{Z}_\tau)$

The twisted K-homology of a simple Lie group is an exterior algebra tensor a cyclic group. The order of this cyclic group depends on the twisting class and is, as yet, determined by a mysterious set of constants. We will see that this cyclic order of the twisted K-homology $K^\tau G$ is the greatest common divisor of the dimensions of a particular set of representations of G. The main ingredient in computing the cyclic order is a detailed understanding of the twisted module structure of $\mathbb{Z}_\tau$, that is of the twisting map $K.\Omega G \overset{\tau(k)}{\rightarrow} K.*$. Bott's theory of generating varieties allows us to produce explicit representatives of classes in $K.\Omega G$, as fundamental classes of complex algebraic varieties, and thereby to describe the twisting map.

9.1 Generating Varieties and Holomorphic Induction

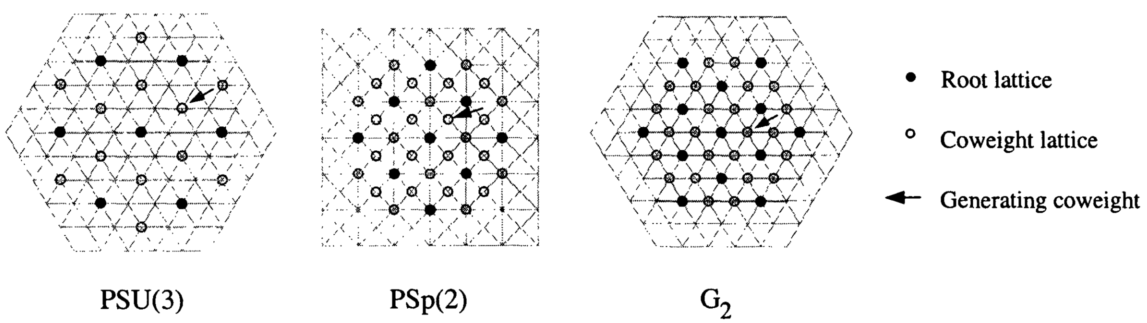
9.1.1 Bott Generating Varieties

A generating variety for ΩG is, for us, a space V and a map $i : V \rightarrow \Omega G$ such that the images $i_*(H.V)$ and $i_*(K.V)$ of the homology and K-theory of V generate

$H.\Omega'G$ and $K.\Omega'G$, respectively, as algebras, where $\Omega'G$ is the identity component of ΩG. In [6], Bott produced a beautiful, systematic family of generating varieties of the form G/H, as we now describe; these particular homogeneous spaces are better known as coadjoint orbits and as such are smooth complex algebraic varieties with an even-dimensional cell decomposition.

We briefly review Bott's construction. Let G be a compact and connected but not necessarily simply connected Lie group. Denote by $\Gamma_G = \ker(\exp : \mathfrak{t} \to T)$ the coweight lattice of G; we do not distinguish between a coweight and the corresponding circle in G. A coweight $\ell \in \Gamma_G$ is called generating if for every root $r \in \mathfrak{t}^*$ of G, there is an element w of the Weyl group such that $r(w \cdot \ell) = 1$. Note that the coweight lattice Γ_G of the group is contained in the coweight lattice Γ_W of the Lie algebra, which is also the coweight lattice of the adjoint form of G. A coweight is generating if, roughly, it is as short as possible in Γ_W; thus, even if a group does not have a generating coweight, its adjoint form will. The simple rank 2 groups with generating coweights, namely $PSU(3)$, $PSp(2)$, and G_2, are illustrated in Figure 9-1.

Figure 9-1: Generating Coweights for Rank 2 Lie Groups



Suppose $\ell \in \Gamma_G$ is a generating coweight for G, and let $C(\ell) \subset G$ denote the pointwise centralizer of the corresponding circle; note that $C(\ell)$ can also be described as the image under the exponential map of the subalgebra of $\mathfrak{g}$ generated by the root spaces associated to roots r perpendicular to ℓ, that is to roots where $r(\ell) = 0$. The map

$$G \to \Omega G$$
$$g \mapsto g\ell g^{-1}\ell^{-1}$$

descends to a map on cosets $G/C(\ell) \to \Omega G$. The main theorem, which is due to Bott in homology and to Clarke [7] in K-theory, is that $G/C(\ell)$ is a generating variety for ΩG.

Suppose G is simply connected and ℓ is a generating circle for PG, the adjoint form of G. Then $PG/C_{PG}(\ell) = G/C_G(\widetilde{\ell})$ where $\widetilde{\ell}$ denotes a loop in G covering ℓ. The composite

$$G/C_G(\widetilde{\ell}) = PG/C_{PG}(\ell) \to \Omega' PG = \Omega G$$

is therefore a generating variety for ΩG. For example, the generating varieties corresponding to the marked coweights in Figure 9-1 are $SU(3)/U(2)$, $Sp(2)/U(2)$, and $G_2/U(2)$ respectively. In general there may be more than one Bott generating variety for a group; we list a Bott generating variety for each of the classical groups in the following table:

Group	Generating Variety
$SU(n+1)$	$SU(n+1)/U(n)$
$Spin(2n+1)$	$Spin(2n+1)/(Spin(2n-1) \times_{\mathbb{Z}/2} Spin(2))$
$Sp(n)$	$Sp(n)/U(n)$
$Spin(2n)$	$Spin(2n)/(Spin(2n-2) \times_{\mathbb{Z}/2} Spin(2))$

Here the $\mathbb{Z}/2$ action on $Spin(n)$ is the one whose quotient is $SO(n)$.

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We need to compute the twisted map $K.\Omega G \xrightarrow{\tau(k)} K.*$. To this end we want to represent the algebra generators of $K.\Omega G$ in a way that allows us to compute their twisted images. In our computations we utilize generating varieties to represent algebra generators in three independent ways; we refer to these briefly as representing them via subvarieties, via an evaluation dual basis, and via a Poincare dual basis.

In some cases we have a sufficiently explicit handle on the generating variety V for ΩG that we can describe a collection of maps $W_i \to V \to \Omega G$ such that W_i is a K-oriented manifold and the images in $K.\Omega G$ of the K-homology fundamental classes of the W_i are the desired algebra generators; frequently, though not always, the W_i are subvarieties of the generating variety V. Variants of this 'subvariety' representation are used for $SU(n)$, G_2, and $Sp(n)$ in sections 9.2 and 9.3.

The K-cohomology of the Bott generating variety V is easily determined from the representation theory of G. Specifically, if the Bott generating variety V is the quotient G/H with G simply connected, then $K^{\cdot}V = R[H]/i^*(I[G])$, where $i : H \to G$ is the inclusion and $I[G]$ is the augmentation ideal of the representation ring $R[G]$. If there is a minor miracle and we can write down a clean basis for this ring, then we can take an evaluation dual basis for the K-homology $K.V$; the image of this basis in $K.\Omega G$ will generate as an algebra and the twisting map will be easily computable. This is the approach taken for $Spin(n)$ in section 9.4.

More commonly, any apparent basis for the K-cohomology of the generating variety is quite haphazard. In this case we consider the Poincare dual basis (to some chosen basis) for the K-homology $K.V$. Again the images of these classes in $K.\Omega G$ will generate, but computing their twisted images requires a bit more work. Specifically, we use holomorphic induction to write these images in terms of the dimensions of irreducible representations of G, as described in detail in the next section. This Poincare dual approach is the one that provides a general procedure and is the subject of section 9.5.

9.1.2 The Twisting Map via Holomorphic Induction

In section 7.2 we described the $K.\Omega G$-module structure on $\mathbb{Z}_\tau$ by the twisting map

$$K.\Omega G \xrightarrow{\tau(k)} K.*$$
$$x \mapsto \langle L^k, x\rangle,$$

where $L \in K^{\cdot}\Omega G$ is a generating line bundle. The purpose of this section is to outline the computation of the twisted image $\langle L^k, x\rangle$ when x is represented as the image of the Poincare dual of a bundle on some K-oriented manifold.

Let $i : W \to \Omega G$ be a map from a K-oriented manifold W to ΩG and let $\eta \in K^{\cdot}W$ be a bundle on W such that $i_*(\mathrm{D}\eta) = x \in K.\Omega G$; here, D denotes the Poincare duality map. We first translate the evaluation $\langle L^k, x\rangle$ into a pushforward on W:

$$\langle L^k, i_*(\mathrm{D}\eta)\rangle = \langle i^*(L^k), \mathrm{D}\eta\rangle = \langle i^*(L^k), \eta \cap [W]\rangle = \langle i^*(L^k) \cup \eta, [W]\rangle.$$

The third equality is simply

$$\langle i^*(L^k), \eta \cap [W]\rangle = \pi_*(i^*(L^k) \cap (\eta \cap [W])) = \pi_*((i^*(L^k) \cup \eta) \cap [W]) = \langle i^*(L^k) \cup \eta, [W]\rangle,$$

where $\pi : W \to *$ denotes projection. We are thereby reduced to computing K-theory pushforwards $\langle \mu, [W]\rangle = \pi_!(\mu)$.

We can translate this K-theory pushforward to a cohomology pushforward using the Chern character. Multiplicative transformations of cohomology theories do not commute with pushforward, and the discrepancy in the case of the Chern character is the Todd genus. The Chern character is the identity on a point, and so we have

$$\langle \mu, [W]_K\rangle = \mathrm{ch}(\pi_!^K(\mu)) = \pi_!^H(\mathrm{ch}(\mu) \cup \mathrm{Td}(W)) = \langle \mathrm{ch}(\mu) \cup \mathrm{Td}(W), [W]_H\rangle.$$

Next we reduce this cohomology pushforward to a calculation in sheaf cohomology. We always work with manifolds W that are smooth projective complex algebraic varieties, and as such the Hirzebruch-Riemann-Roch theorem applies:

$$\langle \mathrm{ch}(\mu) \cup \mathrm{Td}(W), [W]_H \rangle = \Sigma(-1)^i H^i(W; \mu) =: \chi(\mu).$$

Here of course $H^i(W;\mu)$ denotes the cohomology of the sheaf of holomorphic sections of the bundle μ.

In some cases, this reduction is sufficient, as we can use Serre duality and related techniques to compute the sheaf cohomology groups. If, as in our computations it always is, W is a homogeneous space G/H which is a Kähler manifold, and μ is pulled back along $G \to BH$ from an irreducible representation of H, then we can take advantage of Bott's theory of holomorphic induction [5]. (If our original μ is not irreducible, we simply decompose it into irreducible components and apply holomorphic induction to each component.) To avoid reviewing the whole of Bott's theory, we describe holomorphic induction procedurally as it will arise in our computations.

Let G be a compact connected simply connected Lie group and H a centralizer of a circle in G; in particular H and G share a maximal torus and their weight lattices coincide. In this situation, as remarked earlier, the K-theory of the quotient G/H is simply the quotient of representation rings: $K^{\cdot}(G/H) = R[H]/i^*(I[G])$. We may further assume that we have chosen an order on the roots of G such that H is generated by a subset of the simple roots of G; in particular this determines standard Weyl chambers for G and H. (That there is such a choice of order is the content of Wang's theorem and depends on H being the centralizer of a torus in G; see Bott [5].) Let μ denote simultaneously a weight in the Weyl chamber of H, the corresponding irreducible representation of H, and the associated bundle on G/H. Let ρ denote half the sum of the positive roots of G, and let S denote the union of the hyperplanes perpendicular to the roots of G. Further, for a weight ω of G, let $T(\omega)$ denote the unique weight in the Weyl chamber of G that is the image of ω under an element of the Weyl group. The index $\mathrm{ind}(\omega)$ of a weight ω not in S is the number of hyperplanes of S intersecting a straight line connecting ω and $T(\omega)$. Bott's theorem is the following:

Theorem 9.1.1 (Bott [5]). *In the above situation,*

- *if* $\mu + \rho \in S$ *then* $H^{\cdot}(G/H, \mu) = 0$;
- *if* $\mu + \rho \notin S$ *then* $H^{\cdot}(G/H, \mu)$ *is nonzero only in dimension* $\operatorname{ind}(\mu + \rho)$;
- *in this case,* $H^{\operatorname{ind}(\mu+\rho)}(G/H, \mu)$ *is isomorphic to the irreducible representation of* G *with highest weight* $T(\mu + \rho) - \rho$.

When $\mu + \rho \in S$ we say that μ is singular. Thus, when μ is singular, $\chi(\mu) = 0$, and otherwise

$$\chi(\mu) = (-1)^{\operatorname{ind}(\mu+\rho)} \dim([T(\mu + \rho) - \rho]_G),$$

where $[-]_G$ denotes the irreducible representation of G with the specified highest weight.

In any given case, the dimension of this irreducible representation is easily computed using the Weyl character formula, and so the preceding method provides a systematic approach to computing the twisting map on a class represented by the image of the Poincare dual to a K-cohomology class of an appropriate homogeneous space. We proceed to specific examples.

9.2 Subvarieties of $\Omega SU(n+1)$ and ΩG_2

A generating variety for $\Omega SU(n+1)$ is $SU(n+1)/U(n) = \mathbb{C}\mathrm{P}^n \xrightarrow{i} \Omega SU(n+1)$, and the induced map in homology is

$$\widetilde{H}.\mathbb{C}\mathrm{P}^n = \mathbb{Z}\{z_1, \ldots, z_n\} \to \mathbb{Z}[w_1, \ldots, w_n] = H.\Omega SU(n+1)$$
$$z_i \mapsto w_i.$$

Here the classes z_i are represented by the fundamental homology classes of the subvarieties $\mathbb{C}\mathrm{P}^i \subset \mathbb{C}\mathrm{P}^n$. The Atiyah-Hirzebruch spectral sequence collapses for both $\Omega SU(n+1)$ and $\mathbb{C}\mathrm{P}^n$ and there are no extensions. In particular $K.\Omega SU(n+1)$ is polynomial on n generators, as previously noted, and $\widetilde{K}.\mathbb{C}\mathrm{P}^n$ is free abelian of rank n.

Lemma 9.2.1. *The set* $\{[\mathbb{C}\mathrm{P}^i]\}_{i=1}^n$ *of fundamental* K*-homology classes of the subvarieties* $\mathbb{C}\mathrm{P}^i \subset \mathbb{C}\mathrm{P}^n$ *forms a basis for* $\widetilde{K}.\mathbb{C}\mathrm{P}^n$.

Proof. By induction it is enough to show that under the projection $K.\mathbb{C}\mathrm{P}^i \to K.(\mathbb{C}\mathrm{P}^i, \mathbb{C}\mathrm{P}^{i-1}) = \mathbb{Z}$ the fundamental class of $\mathbb{C}\mathrm{P}^i$ maps to a generator. This follows immediately from the naturality of Poincare duality,

$$\begin{tikzcd} K.\mathbb{C}\mathrm{P}^i \arrow[r] \arrow[d, equal] & K.(\mathbb{C}\mathrm{P}^i, \mathbb{C}\mathrm{P}^{i-1}) \arrow[d, equal] \\ K^{\cdot}\mathbb{C}\mathrm{P}^i \arrow[r] & K^{\cdot}(\mathbb{C}\mathrm{P}^i - \mathbb{C}\mathrm{P}^{i-1}), \end{tikzcd}$$

because the unit in $K^{\cdot}\mathbb{C}\mathrm{P}^i$ certainly maps to a generator of $K^{\cdot}(\mathbb{C}\mathrm{P}^i - \mathbb{C}\mathrm{P}^{i-1})$. □

The images $i_*([\mathbb{C}\mathrm{P}^i])$ generate $K.\Omega SU(n+1)$ as an algebra and we may therefore take $\{x_i = i_*([\mathbb{C}\mathrm{P}^i]) - 1\}$ to be the reduced polynomial generators.

We now have to evaluate the pushforward $\langle L^k, [\mathbb{C}\mathrm{P}^i]\rangle$, where we use L to denote the generating line bundle on $\mathbb{C}\mathrm{P}^i = SU(i+1)/U(i)$; this L is of course the pullback of the generating line bundle on $\Omega SU(n+1)$. The bundle L corresponds to an irreducible representation of $U(i)$, thus to a weight of $U(i)$ and so a weight, also denoted L, of $SU(i+1)$; this weight L is in the Weyl chamber of $SU(i+1)$. The irreducible representation of $SU(i+1)$ corresponding to L is the dual of the standard representation; (that it is the dual of the standard representation and not the standard representation is the effect of a sign choice—see the remark at the end of this section). It happens that the k-fold symmetric power of this representation is irreducible, and so the dimension of the irreducible representation corresponding to L^k is $\binom{k+i}{i}$. The image of $x_i = i_*([\mathbb{C}\mathrm{P}^i]) - 1 \in K.\Omega SU(n+1)$ in $\mathbb{Z}_\tau$ is therefore $c_i = \binom{k+i}{i} - 1$, and the cyclic order of $K^\tau(SU(n+1))$ is

$$c(SU(n+1), k) = \gcd\left\{\binom{k+1}{1} - 1, \binom{k+2}{2} - 1, \dots, \binom{k+n}{n} - 1\right\}.$$

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The procedure for calculating the cyclic order of $K^\tau G_2$ is similar: we find fundamental class representatives for algebra generators of the homology of ΩG_2 and then show that the corresponding K-homology fundamental classes also generate. The map $\Omega G_2 \to \mathbb{C}\mathrm{P}^\infty$ classifying the generating line bundle is a homology equivalence through degree 4. Using this, the Serre spectral sequence for $\Omega SU(3) \to \Omega G_2 \to \Omega S^6$ shows that

$$H.\Omega G_2 \cong \mathbb{Z}[a_2, a_4, a_{10}]/a_2^2 = 2a_4.$$

The composition

$$\mathbb{C}\mathrm{P}^2 \to \Omega SU(3) \to \Omega G_2 \to \mathbb{C}\mathrm{P}^\infty$$

is simply the inclusion and as such, a_2 and a_4 in $H.\Omega G_2$ are represented respectively by the fundamental classes $[\mathbb{C}\mathrm{P}^1]$ and $[\mathbb{C}\mathrm{P}^2]$. The Bott generating variety for ΩG_2 is $G_2/U(2)$, where the $U(2)$ in question is included in G_2 along a pair of complex-conjugate short roots. The manifold $G_2/U(2)$ has dimension 10 and the image of its homology generates $H.\Omega G_2$; we may therefore choose a_{10} to be the image of the fundamental homology class $[G_2/U(2)]$.

The Atiyah-Hirzebruch spectral sequence for ΩG_2 collapses, and the low-degree equivalence between ΩG_2 and $\mathbb{C}\mathrm{P}^\infty$ resolves the extension. The K-homology $K.\Omega G_2$ is thereby isomorphic to $\mathbb{Z}[a, b, x_3]/(a^2 + 3a = 2b)$.

Lemma 9.2.2. *The reduced algebra generators of $K.\Omega G_2 \cong \mathbb{Z}[a, b, x_3]/(a^2 + 3a = 2b)$ may be taken to be the reduced fundamental K-homology classes $[\mathbb{C}\mathrm{P}^1] - 1$, $[\mathbb{C}\mathrm{P}^2] - 1$, and $[G_2/U(2)] - 1$ respectively.*

Proof. Let $(G_2/U(2))_8$ denote the 8-skeleton of the generating variety, that is everything except the top cell. As in Lemma 9.2.1, the fundamental K-homology class of $G_2/U(2)$ maps to a generator of $K.(G_2/U(2), (G_2/U(2))_8)$. Comparing the Atiyah-Hirzebruch spectral sequences for $G_2/U(2)$ and ΩG_2 we see that $[G_2/U(2)]$ lives in filtration 10 in $K.\Omega G_2$ and projects to the generator a_{10} in $H_{10}\Omega G_2$. The fundamental K-homology classes $[\mathbb{C}\mathrm{P}^1]$ and $[\mathbb{C}\mathrm{P}^2]$ certainly project to the generators a_2 and a_4 respectively in the appropriate filtration quotients, and this completes the proof. □

We remark that these algebra generators differ by a change of basis from those implicitly chosen in section 8.2 and this explains the difference in the relation; the Tor computation and the cyclic order is not affected by the change.

We need only compute the pushforward $\langle L^k, [G_2/U(2)]\rangle$. The bundle L corresponds to the shortest weight μ perpendicular to the roots of $U(2)$; as a weight of G_2, μ is the long root of G_2 in the Weyl chamber. The pushforward is therefore the dimension of the irreducible representation of G_2 with highest weight $k\mu$. By the Weyl character formula (see for example [20]) this dimension is

$$\dim([k\mu]_{G_2}) = \frac{(k+1)(k+2)(2k+3)(3k+4)(3k+5)}{120}.$$

The cyclic order of $K^\tau G_2$ is finally

$$c(G_2,k) = \gcd\left\{\binom{k+1}{1} - 1, \binom{k+2}{2} - 1, \frac{(k+1)(k+2)(2k+3)(3k+4)(3k+5)}{120} - 1\right\}.$$

∾∾∾∾∾

A remark on signs is in order. If we have chosen a generating line bundle L on ΩG_2 a priori, the weight corresponding to L may be $-\mu$ instead of μ as claimed above. The dimension resulting from holomorphic induction on the weight $k(-\mu)$ is wildly different from the dimension associated to $k(\mu)$, and this might be cause for worry. However, the greatest common divisor is in all cases unaffected by the change. The easiest way around this ambiguity is to chose L such that the corresponding weight is μ and not $-\mu$; we must then pick the generating variety $\mathbb{C}\mathrm{P}^2$ for $\Omega SU(3)$ in such a way that the given L corresponds there to the dual of the standard representation (and not to the standard representation) as described in the discussion of $SU(n+1)$ above—this is easily accomplished. Similar remarks apply to all our computations and we make convenient sign choices without comment.

9.3 Generating Varieties for $\Omega Sp(n)$

The homology and K-homology of $\Omega Sp(n)$ are polynomial in n generators. The natural Bott generating variety for $\Omega Sp(n)$ is $Sp(n)/U(n)$, which has homology and

K-homology of rank $n^2 + n$. Identifying the n elements which generate therefore requires more doing—we return to this question later. Luckily, $\Omega Sp(n)$ has smaller generating varieties—see [24, 39]; in particular we work with $(\mathbb{C}P^{2n-2})^{L^2}$, the Thom complex of the square of the tautological bundle.

Let $P_i(V)$ or $P(V)$ denote the projectivization of the bundle V on $\mathbb{C}P^i$; note that we can rewrite our generating variety $V(n) = (\mathbb{C}P^{2n-2})^{L^2}$ as $P_{2n-2}(L^2+1)/P_{2n-2}(L^2)$. We think of the quotient map $P(L^2 + 1) \to P(L^2 + 1)/P(L^2)$ as a resolution of our (quite singular) generating variety, and we represent homology and K-homology classes in $V(n)$ (and thus in $\Omega Sp(n)$) as the images of fundamental classes of subvarieties of $P(L^2 + 1)$. The reduced homology of $V(n)$ is free of rank one in each even degree between 2 and $4n - 2$, and the degree $2i$ group is generated by the image of the fundamental class $[P_{i-1}(L^2 + 1)]$. In particular, the algebra generators $\{a_{4i-2}\}$ of $H.\Omega Sp(n) = \mathbb{Z}[a_2, a_6, a_{10}, \ldots, a_{4n-2}]$ are represented by the fundamental classes $[P_{2(i-1)}(L^2 + 1)]$, for $1 \leq i \leq n$.

The K-homology situation is the same.

Lemma 9.3.1. *The reduced polynomial generators of the K-homology $K.\Omega Sp(n) \cong \mathbb{Z}[x_1, \ldots, x_n]$ can be taken to be the reduced K-homology fundamental classes $f_*[P_{2(i-1)}(L^2+1)] - 1$, $1 \leq i \leq n$; here f is the composite*

$$P_{2(i-1)}(L^2 + 1) \to P_{2(n-2)}(L^2 + 1) \to P_{2(n-2)}(L^2 + 1)/P_{2(n-2)}(L^2) \to \Omega Sp(n).$$

The K-homology fundamental classes map, in the appropriate filtration quotients, to the homology fundamental classes; the proof is the same as for $SU(n + 1)$ and G_2.

To evaluate the twisting map, specifically to calculate $\langle L^k, f_*[P_{2(i-1)}(L^2 + 1)]\rangle$, we need to identify the bundle $f^*(L^k)$. We do this by writing down a bundle on $P(L^2 + 1) = P_{2(i-1)}(L^2 + 1)$ that is trivial on $P(L^2) = P_{2(i-1)}(L^2)$, and show that the corresponding bundle on the quotient $V(i)$ is the pullback $f'^*(L)$ where f' is the inclusion $V(i) \to V(n) \to \Omega Sp(n)$. Let γ be the tautological bundle on the total space $P(L^2 + 1)$ and let $\pi : P(L^2 + 1) \to \mathbb{C}P^{2(i-1)}$ be the bundle projection. The subspace $P(L^2)$ is of course just the base $\mathbb{C}\mathrm{P}^{2(i-1)}$ and so γ restricts to $\pi^*(L^2)|_{P(L^2)}$ on $P(L^2)$. In particular then, the bundle $\gamma \otimes \pi^*(L^{-2})$ is trivial on the subspace $P(L^2)$ and so pulls back from a bundle ϕ on $V(i)$. To see that ϕ is equal to $f'^*(L)$ (up to our usual sign ambiguity), and therefore that $\gamma \otimes \pi^*(L^{-2}) = f^*(L)$, it is enough to check that the first Chern class of $\gamma \otimes \pi^*(L^{-2})$ is a module generator of $H^2(P(L^2+1)) = \mathbb{Z}\{c_1(\gamma), \pi^*(c_1(L))\}$; this much is clear.

We now compute the pushforward

$$\langle L^k, f_*[P_{2(i-1)}(L^2+1)]\rangle = \langle (\gamma \otimes \pi^*(L^{-2}))^k, [P_{2(i-1)}(L^2+1)]\rangle.$$

First pushforward along the fibres:

$$\langle \gamma^k \otimes \pi^*(L^{-2k}), [P_{2(i-1)}(L^2+1)]\rangle = \langle \mathrm{Sym}^k(L^2+1) \otimes L^{-2k}, [\mathbb{C}\mathrm{P}^{2(i-1)}]\rangle.$$

This is a parameterized version of the pushforward

$$\langle \gamma^k_{\mathrm{taut}}, [\mathbb{C}\mathrm{P}^n]\rangle = \langle \gamma^k_{\mathrm{taut}}, [P(\mathbb{C}^{n+1})]\rangle = \dim(\mathrm{Sym}^k(\mathbb{C}^{n+1})) = \binom{k+n}{k}$$

used in the preceding section. Next

$$\mathrm{Sym}^k(L^2+1) \otimes L^{-2k} = L^{-2k} + L^{-2k+2} + \ldots + 1$$

and so

$$\langle \mathrm{Sym}^k(L^2+1)\otimes L^{-2k}, [\mathbb{C}\mathrm{P}^{2(i-1)}]\rangle = \binom{-2k+2(i-1)}{2(i-1)} + \binom{-2k+2+2(i-1)}{2(i-1)} + \ldots + 1.$$

Here we use $\binom{a+b}{b}$ to denote $\frac{(a+b)(a+b-1)\ldots(a)}{(b)(b-1)\ldots(1)}$ even when $a+b$ is negative; we implicitly observe that this expression does give the correct pushforward even when the bundle, as in the case of L^{-2k+2l}, corresponds to a weight that is not in the Weyl chamber of

$SU(2(i-1)+1)$. This finishes our calculation of the cyclic order of $K^{\tau}Sp(n)$:

$$c(Sp(n),k) = \gcd\left\{ \sum_{-k\leq j\leq -1} \binom{2j+2(i-1)}{2(i-1)} : 1 \leq i \leq n \right\}.$$

∼∼∼∼∼

It would be more natural to express the cyclic order of $K^{\tau}Sp(n)$ in terms of the dimensions of irreducible representations of symplectic groups. This is possible if we work with subvarieties of the Bott generating variety $V = Sp(n)/U(n)$. There is a natural collection of n subvarieties of V, namely $\{Sp(i)/U(i)\}$. It is not the case that the fundamental homology classes of these subvarieties represent algebra generators for $H.\Omega Sp(n)$; indeed, the algebra generators are in dimensions $\{4i-2\}$, while these subvarieties have dimensions $\{i^2+2\}$. It is therefore remarkable that the K-homology fundamental classes of these subvarieties do appear to generate the K-homology of $\Omega Sp(n)$.

Conjecture. The K-homology ring $K.\Omega Sp(n)$ is polynomial on the classes represented by the reduced K-homology fundamental classes $[Sp(i)/U(i)] - 1$, for $1 \leq i \leq n$.

Using the Weyl character formula, this immediately gives a description of the cyclic order:

$$c(Sp(n),k) = \gcd\left\{ \frac{\left(\prod_{1\leq j<l\leq i} (l-j)(2k+2i+2-(j+l))\right)\left(\prod_{1\leq j\leq i} (k+i+1-j)\right)}{(2i-1)!(2i-3)!\ldots 3!1!} : 1 \leq i \leq n \right\}.$$

These gcd's agree with those determined using the generating variety $(\mathbb{C}\mathrm{P}^{2n-2})^{L^2}$.

9.4 The Tor Calculation for $Spin(2n+1)$

We now pay our debt to the proof of Theorem 6.2.1 by calculating $\mathrm{Tor}^{K.\Omega Spin(2n+1)}(\mathbb{Z}, \mathbb{Z}_{\tau})$; in the process we determine the cyclic order of $K^{\tau}Spin(2n+1)$. (The computations for the even $Spin$ groups are similar and the resulting cyclic order is recorded in

Theorem 6.2.2.) For the other simple groups, we were able to calculate the Tor group without knowing the map $K.\Omega G \to (K.*)_\tau$ and we determined, after the fact, the structure of this twisting map. The ring $K.\Omega Spin(2n+1)$ is too complicated to permit this a-priori Tor calculation; we must first identify algebra generators of $K.\Omega Spin(2n+1)$ and compute the twisting map. It happens that the K-cohomology of the Bott generating variety $V = Spin(2n+1)/(Spin(2n-1) \times_{\mathbb{Z}/2} Spin(2))$ admits a particularly simple representation-theoretic basis, and an evaluation dual basis maps to a set of algebra generators in $K.\Omega Spin(2n+1)$. Once we know the twisted pushforwards of these algebra generators, the Tor computation becomes tractable.

The structure of the K-homology ring of $\Omega Spin(2n+1)$ was described by Clarke [7]:

$$K.\Omega Spin(2n+1) = \frac{\mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, 2\sigma_n, 2\sigma_{n+1} + \sigma_n, \ldots, 2\sigma_{2n-1} + \sigma_{2n-2}]}{(\rho_1, \ldots, \rho_{n-1})},$$

$$\rho_k = \sigma_k^2 + \sum_{i=0}^{k-1} (-1)^{k-i} \sigma_i \sum_{j=k}^{2k-i-1} \binom{k-i-1}{j-k} (2\sigma_{j+1} + \sigma_j).$$

One can see why the a-priori Tor calculation is unlikely to be fruitful. The K-cohomology of the Bott generating variety is simply the quotient of the representation ring of $Spin(2n-1) \times_{\mathbb{Z}/2} Spin(2)$ by the image of the augmentation ideal of the representation ring of $Spin(2n+1)$. Clarke writes this quotient in a convenient form:

$$K^{\cdot} V = \mathbb{Z}[\mu, \gamma]/(\mu^n - 2\gamma - \mu\gamma, \gamma^2);$$

here $\mu = L - 1$ where L is the generating line bundle whose k-th power determines the twisting. Note that $\mu^{2n} = 0$ in this ring, and so $(\mu, \mu^2, \ldots, \mu^{2n-1})$ is a basis for $K^{\cdot}(V) \otimes \mathbb{Q}$. Letting $(\sigma_1', \ldots, \sigma_{2n-1}')$ be the evaluation dual basis of $K.(V) \otimes \mathbb{Q}$, we see that

$$(\sigma_1', \sigma_2', \ldots, \sigma_{n-1}', 2\sigma_n', 2\sigma_{n+1}' + \sigma_n', \ldots, 2\sigma_{2n-1}' + \sigma_{2n-2}')$$

is a basis for $K.V$; these elements map, respectively, to the given generators of $K.\Omega Spin(2n+1)$. The twisting map $K.\Omega Spin(2n+1) \to (K.*)_\tau$ takes a generator g to $\langle L^k, g\rangle \in \mathbb{Z}$. Because $\mu^{2n} = 0$, we have

$$\langle L^k, \sigma_i^{'}\rangle = \langle(\mu+1)^k, \sigma_i^{'}\rangle = \binom{k}{i},$$

and the images of our integral generators are respectively

$$\left(\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{n-1}, 2\binom{k}{n}, 2\binom{k}{n+1} + \binom{k}{n}, \ldots, 2\binom{k}{2n-1} + \binom{k}{2n-2}\right).$$

We can now prove that $\mathrm{Tor}^{K.\Omega Spin(2n+1)}(\mathbb{Z}, \mathbb{Z}_\tau)$ is an exterior algebra on $n-1$ generators tensor a cyclic group. We first rewrite the above presentation of $K.\Omega Spin(2n+1)$ in a way that suggests a propitious choice of Tate resolution. Let $(a_1, \ldots, a_{2n-1})$ denote the given generators of $K.\Omega Spin(2n+1)$. For i sufficiently large, the relation ρ_i expresses the generator a_{2i} in lower terms; in particular

$$K.\Omega Spin(4n-1) = \frac{\mathbb{Z}[a_1, a_2, \ldots, a_{2n-2}, a_{2n-1}, a_{2n+1}, a_{2n+3}, \ldots, a_{4n-5}, a_{4n-3}]}{(\rho_1, \rho_2, \ldots, \rho_{n-1})},$$
$$K.\Omega Spin(4n+1) = \frac{\mathbb{Z}[a_1, a_2, \ldots, a_{2n-2}, a_{2n-1}, a_{2n+1}, a_{2n+3}, \ldots, a_{4n-3}, a_{4n-1}]}{(\rho_1, \rho_2, \ldots, \rho_{n-1})}.$$

The remaining relations can be written

$$\rho_i = 2a_{2i} + r_i a_{2i-1} + \ldots,$$

with r_i odd and all unspecified monomials containing some a_j with $j < 2i-1$, except for ρ_1 which is $2a_2 + a_1 - a_1^2$. If we can show that Tor over the subring $R_n = \mathbb{Z}[a_1, \ldots, a_{2n-2}]/(\rho_1, \ldots, \rho_{n-1})$ is exterior on $n-2$ generators, the desired result follows. Rather than presenting the general induction immediately, we discuss the first few cases explicitly.

The case $n = 1$ corresponding to $Spin(3)$ requires no comment. The ring $K.\Omega Spin(7)$ is $\mathbb{Z}[a_1, a_2, a_3, a_5]/(2a_2+a_1-a_1^2)$. This is reminiscent of $K.\Omega G_2$ and indeed the Atiyah-Hirzebruch spectral sequence for the fibration $\Omega G_2 \to \Omega Spin(7) \to \Omega S^7$ collapses; there are no possible multiplicative extensions and so this confirms that $K.\Omega Spin(7)$ is $K.\Omega G_2$ adjoin a generator in degree 6. As in section 8.2, the Tor group in question is

$$\mathrm{Tor}^{R_2}(\mathbb{Z}, \mathbb{Z}_\tau) = \mathrm{Tor}^{\mathbb{Z}[a_1,a_2]/(2a_2+a_1-a_1^2)}(\mathbb{Z}, \mathbb{Z}_\tau) = \mathbb{Z}/g_{12}.$$

(Note that the generator a_i of the subring R_n has image under the twisting map $c_i = \binom{k}{i}$ and as before we abbreviate $\gcd\{c_1, c_2, \ldots, c_i\}$ by $g_{1..i}$.)

The relevant subring of $K.\Omega Spin(11)$ is

$$R_3 = \mathbb{Z}[a_1, a_2, a_3, a_4]/(\rho_1, 2a_4 + 3a_3 + (a_2 + 1)a_2 + (-2a_3 - a_2)a_1).$$

This presentation suggests the Tate resolution

$$\mathrm{Tor}^{R_3} = H(\mathbb{Z}\langle T_1, T_2, T_3, T_4\rangle\{S_1, S_2\};$$
$$dT_i = c_i, dS_1 = 2T_2 + (1 - c_1)T_1, dS_2 = 2T_4 + 3T_3 + (c_2 + 1)T_2 + (-2c_3 - c_2)T_1).$$

The E_1 term of the spectral sequence associated to the filtration of this complex by S_2 is

$$\begin{array}{|llll}
 & \mathbb{Z}/g & \mathbb{Z}/g \oplus \mathbb{Z}/g & \mathbb{Z}/g \\
\mathbb{Z}/g & \mathbb{Z}/g \oplus \mathbb{Z}/g & \mathbb{Z}/g & \\
\mathbb{Z}/g \oplus \mathbb{Z}/g & \mathbb{Z}/g & & \\
\mathbb{Z}/g & & & \\
\hline
\end{array}$$

where $g = g_{1234}$ and the generator in degree $(1, 1)$ is S_2. At first blush the generators in degree $(0, 1)$ have the form $t_3 = (g_{12}/g_{123})T_3 + O(2)$ and $t_4 = (g_{123}/g_{1234})T_4 + O(3)$, where the omitted terms contain only terms involving (T_2 and T_1) and (T_3, T_2, and T_1) respectively. In order to determine the differential on S_2 we need control over the T_3 term in the generator t_4. The basic observation is that if there exists a cocycle t_4' of the form $(g_{123}/g_{1234})T_4 + O(2)$, then some linear combination $t_4 + ct_3$ is cohomologous to t_4' and so we may take the generators in degree $(0, 1)$ to be t_4' and t_3. The existence of this cocycle is ensured by the fact that $(g_{123}/g_{1234})g_4$ is divisible by g_{12}, as is easily checked. The differential on S_2 is therefore $(2g_{1234}/g_{123})t_4' + (3g_{123}/g_{12})t_3$. Because the greatest common divisor of $2g_{1234}/g_{123}$ and $3g_{123}/g_{12}$ is always 1, the torsion group is finally

$$\mathrm{Tor}^{R_3} = \mathbb{Z}/g_{1234}\langle x_4 \rangle;$$

here we can choose the generator x_4 to be $(g_{123}/g_{1234})T_4 + O(3)$.

The case of $Spin(15)$ proceeds similarly. The relevant subring of $K.\Omega Spin(15)$ is

$$R_4 = \mathbb{Z}[a_1, a_2, a_3, a_4, a_5, a_6]/(\rho_1, \rho_2, 2a_6 + 5a_5 + 4a_4 + O(3)),$$

and we take the corresponding Tate resolution. Filtering by S_3 we have the spectral sequence (here condensed)

$$\mathbb{Z}/g\langle x_4, x_5, x_6 \rangle \qquad \mathbb{Z}/g\langle x_4, x_5, x_6 \rangle \qquad \mathbb{Z}/g\langle x_4, x_5, x_6 \rangle \qquad \cdots$$

The torsion g is $g_{1..6}$ and the generators in degree $(0,1)$ are

$$\begin{aligned} x_4 &= (g_{123}/g_{1234})T_4 + O(3) \\ x_5 &= (g_{1234}/g_{1..5})T_5 + O(4) \\ x_6 &= (g_{1..5}/g_{1..6})T_6 + O(5). \end{aligned}$$

It happens that $(g_{1234}/g_{1..5})g_5$ and $(g_{1..5}/g_{1..6})g_6$ are both divisible by g_{123}; we can therefore adjust our generators so that they are

$$\begin{aligned} x_4 &= (g_{123}/g_{1234})T_4 + O(3) \\ x_5 &= (g_{1234}/g_{1..5})T_5 + O(3) \\ x_6 &= (g_{1..5}/g_{1..6})T_6 + O(3). \end{aligned}$$

The differential on S_3 is thus $(2g_{1..6}/g_{1..5})x_6 + (5g_{1..5}/g_{1234})x_5 + (4g_{1234}/g_{123})x_4$. Because $2g_{1..6}/g_{1..5}$ and $5g_{1..5}/g_{1234}$ are relatively prime, there exist constants z_1 and z_2 so that if we set

$$y_6 = x_6 + z_1 x_4 = g_{1..5}/g_{1..6} T_6 + O(4)$$
$$y_5 = x_5 + z_2 x_4 = g_{1234}/g_{1..5} T_5 + O(4),$$

then $\{dS_3, y_6, y_5\}$ forms a basis for the degree $(0,1)$ group. Finally, then, the Tor group is

$$\mathrm{Tor}^{R_4} = \mathbb{Z}/g_{1..6}\langle y_5, y_6\rangle$$

as desired.

The general case is now clear. Suppose we know that

$$\mathrm{Tor}^{R_n} = \mathbb{Z}/g_{1..(2n-2)}\langle x_{n+1}, \ldots, x_{2n-2}\rangle,$$

where $x_i = (g_{1..(i-1)}/g_{1..i})T_i + O(i-1)$. The ring R_{n+1} has two additional generators a_{2n-1} and a_{2n} and one additional relation ρ_n. Filter the appropriate Tate resolution by powers of S_n, then adjust the generators of the degree $(0,1)$ group in the spectral sequence so that the single generator x_{2n} involving T_{2n} does not contain any terms involving T_{2n-1}. This is possible because $g_{1..(2n-2)}$ divides $(g_{1..(2n-1)}/g_{1..(2n)})g_{2n}$. The differential of S_n then has the form

$$dS_n = (2g_{1..(2n)}/g_{1..(2n-1)})x_{2n} + (rg_{1..(2n-1)}/g_{1..(2n-2)})x_{2n-1} + \ldots.$$

As those two leading terms are relatively prime, this ensures that $\mathrm{Tor}^{R_{n+1}}$ again has the desired form. Note that in theory there could be multiplicative extensions in the filtration spectral sequence calculating the Tor group, but the above procedure gives a sufficiently explicit handle on the generating classes as to eliminate this possibility.

This completes the proof of Theorem 6.2.1 for the odd *Spin* groups and also establishes the odd *Spin* cyclic orders given in Theorem 6.2.2. The Tor calculation for the even *Spin* groups is analogous and the resulting cyclic orders are also given in Theorem 6.2.2.

9.5 Poincare-Dual Bases and the Cyclic Order of $K^\tau G$

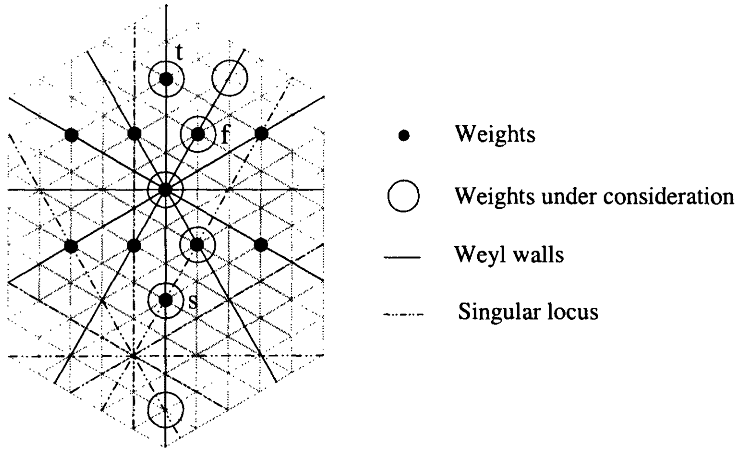
We describe a general procedure for computing the cyclic order of $K^\tau G$ for any simple G and illustrate the method with the group G_2.

Let $V = G/H$ denote the Bott generating variety for ΩG; recall that the K-cohomology of V is $R[H]/i^*I[G]$ where $i : H \to G$ is the inclusion. Pick a module basis $\{w_i\}$ for this ring and consider the Poincare-dual basis $\{Dw_i\}$ of $K.V$. The image of this basis in $K.\Omega G$, which we also denote by $\{Dw_i\}$, is a set of algebra generators for $K.\Omega G$. Note that for any set $\{y_i\}$ of algebra generators for $K.\Omega G$, the cyclic order of $K^\tau G$ is given by $\gcd\{\tau_k(y_i) - \tau_0(y_i)\}$, where τ_k and τ_0 are respectively the twisted and untwisted maps from $K.\Omega G$ to $K.*$. In section 9.1.2 we saw that $\tau_k(Dw_i) = \langle L^k \cup w_i, [V]\rangle$ where L denotes the generating line bundle on V. Decompose w_i into a sum of irreducible representations $\sum v_{ij}$, and let $h(v_{ij})$ denote the highest weight corresponding to v_{ij}. The product $L^k \cup v_{ij}$ is again irreducible, with highest weight $kL+h(v_{ij})$, and Bott's theorem (9.1.1) therefore applies: the pushforward $\langle L^k \cup v_{ij}, [V]\rangle$ is either 0 or is (plus or minus) the dimension of the irreducible representation of G with highest weight $T(kL + h(v_{ij}) + \rho) - \rho$, where T reflects a weight into the fundamental Weyl chamber. This procedure expresses the cyclic order of $K^\tau G$ as the greatest common divisor of a finite set of differences of dimensions of irreducible representations of G.

Recall that the Bott generating variety for G_2 is $G_2/U(2)$ for the short-root inclusion of $U(2)$. Let a and b denote the fundamental weights of G_2 corresponding to the 7 and 14 dimensional representations; in particular $R[G_2] = \mathbb{Z}[a, b]$. Similarly $R[U(2)] = \mathbb{Z}[f, t, t^{-1}]$, where f and t are respectively the standard representation and the determinant representation. The restriction map is

$$
\begin{aligned}
i^*(a) &= f + f^2t^{-1} - 1 + ft^{-1} \\
i^*(b) &= t + f^3t^{-1} - 2f + f^2t^{-1} + f^3t^{-2} - 2ft^{-1} + t^{-1}.
\end{aligned}
$$

Figure 9-2: The Weight Lattice of G_2



Let $s = t^{-1}$; the K-cohomology of the generating variety is then

$$K^{\cdot}V = \mathbb{Z}[f,s]/(f + f^2 s - 1 + fs, 1 + f^3 s^2 - 2fs + f^2 s^2 + f^3 s^3 - 2fs^2 + s^2).$$

(Note that the description of $K^{\cdot}V$ in Clarke [7] omits certain relations, as the ring given there is not finitely generated.) An integral basis for $K^{\cdot}V$ is then $\{1, s, s^2, f, fs, f^2\}$. These representations of $U(2)$ are irreducible except for f^2 which splits as $(f^2 - t) + t$.

Consider the diagram of weights in Figure 9-2. The solid lines are the Weyl walls, the dotted lines describe the set of singular weights, and the seven highest weights h_i under consideration, namely $\{0, s, 2s, f, f+s, 2f, t\}$, are circled. Note that $L = t$ and as such, for $k > 0$, the weight $kL + h_i$ is either singular or is already in the fundamental Weyl chamber. The basis for $K.V$ is of course $\{D1, Ds, D(s^2), Df, D(fs), D(f^2)\}$ and we are interested in the differences $\tau_k(Dw) - \tau_0(Dw)$. Letting $\Gamma_{(n,m)}$ denote the dimension of the irreducible representation of G_2 with highest weight $na + mb$, the six differences are respectively

$$
\begin{aligned}
&\Gamma_{(0,k)} - \Gamma_{(0,0)}\\
&\Gamma_{(0,k-1)} - 0\\
&\Gamma_{(0,k-2)} - 0\\
&\Gamma_{(1,k)} - \Gamma_{(1,0)}\\
&\Gamma_{(1,k-1)} - 0\\
&\Gamma_{(2,k)} + \Gamma_{(0,k+1)} - \Gamma_{(2,0)} - \Gamma_{(0,1)}.
\end{aligned}
$$

Applying the Weyl character formula, we arrive at the cyclic order

$$
\begin{aligned}
c(G_2, k) = \gcd\{&k(422 + 585k + 400k^2 + 135k^3 + 18k^4)/120,\\
&k(2 + 15k + 40k^2 + 45k^3 + 18k^4)/120,\\
&k(2 - 15k + 40k^2 - 45k^3 + 18k^4)/120,\\
&k(601 + 660k + 350k^2 + 90k^3 + 9k^4)/30,\\
&k(16 + 60k + 80k^2 + 45k^3 + 9k^4)/30,\\
&k(2867 + 2550k + 1090k^2 + 225k^3 + 18k^4)/30\}.
\end{aligned}
$$

Indeed, this agrees with the result from section 9.2.

Chapter 10

Twisted $Spin^c$ Bordism and the Twisted Index

The ordinary K-homology of a space X is entirely determined by the $Spin^c$ bordism of X; see [25]. This suggests that much of the structure in twisted K-homology ought to be visible in twisted $Spin^c$ bordism. In section 8 we saw that the cyclic order of the twisted K-homology of a group G is determined by a collection of relations of the form $\tau_k(x) - \tau_0(x) = 0$, where τ_j is the j-twisted map from $K.\Omega G$ to $K.*$. When the class $x \in K.\Omega G$ is represented as the image of the fundamental class of a $Spin^c$ manifold M, there is a natural $Spin^c$ manifold $M(j)$ such that the fundamental class $[M(j)] \in MSpin^c_* $ maps via the index to the element $\tau_j(x) \in K.*$. Moreover, there is an explicitly identifiable twisted $Spin^c$ nullbordism over G of $M(k) - M(0)$. In short, the relations determining the cyclic order of twisted K-homology have realizations in twisted $Spin^c$ bordism. The construction of these nullbordisms is the focus of sections 10.1 and 10.2. Section 10.3 discusses the possibility of representing the exterior generators of the twisted K-homology of G by twisted $Spin^c$ manifolds.

10.1 A Cocycle Model for Twisted $Spin^c$ Bordism

In order to describe twisted $Spin^c$ manifolds explicitly, we need a more geometric, less homotopy-theoretic description of twisted $Spin^c$ structures; in particular we present a cocycle model for twisted $Spin^c$ bordism. This model is presumably well known and in any case takes cues from the Hopkins-Singer philosophy of differential functions [26].

Recall that $Spin^c$ is the total space of a $U(1)$-principal bundle over SO. Correspondingly there is a principal bundle $BU(1) \to BSpin^c \to BSO$ which is classified by $\beta w_2 : BSO \to BBU(1)$, the integral Bockstein of the second Stiefel-Whitney class. A $Spin^c$ structure on an oriented manifold M is a lift to $BSpin^c$ of the classifying map $\nu : M \to BSO$ of the (stable) normal bundle of M. Such a lift is determined by a nullhomotopy of the composite $\beta w_2(\nu) : M \to BSO \to BBU(1)$. Specifying such a nullhomotopy is equivalent to choosing a 2-cochain c on M such that the coboundary of c is $\beta w_2(\nu(M))$; (note that we have chosen once and for all a 3-cocycle g representing the generator of $H^3(BBU(1);\mathbb{Z})$, and the condition on the cochain c is that $\delta c = \nu^*((\beta w_2)^*(g)))$. Ordinary $Spin^c$ bordism of X is therefore equivalent to bordism of oriented manifolds M over X equipped with a 2-cochain c on M such that

$$\delta c = \beta w_2(\nu(M)).$$

The model for twisted $Spin^c$ bordism is similar. We first recall the homotopy-theoretic definition of twisted $Spin^c$ bordism from section 7.1. Given a twisting map $\tau : X \to K(\mathbb{Z}, 3)$, we have a $K(\mathbb{Z}, 2)$-principal bundle P on X and so an associated $BSpin^c$ bundle $Q = P \times_{K(\mathbb{Z},2)} BSpin^c$. More particularly we have a series of bundles

$$Q_n = P \times_{K(\mathbb{Z},2)} BSpin^c(n)$$

and universal vector bundles

$$UQ_n = (P \times_{K(\mathbb{Z},2)} ESpin^c(n)) \times_{Spin^c(n)} \mathbb{R}^n.$$

The corresponding Thom spectrum

$$\mathrm{Th}(UQ) = P_+ \wedge_{K(\mathbb{Z},2)_+} MSpin^c$$

has as its homotopy groups the twisted $Spin^c$ bordism groups of X. The twisted index map to twisted K-homology is induced by the map $\mathrm{id} \wedge \mathrm{ind} : P_+ \wedge_{K(\mathbb{Z},2)_+} MSpin^c \to P_+ \wedge_{K(\mathbb{Z},2)_+} K$.

The principal bundle P and the associated $BSpin^c$ bundle Q are defined by the pullbacks

$$\begin{tikzcd} P \arrow[r] \arrow[d] & EK(\mathbb{Z},2) \arrow[d] \\ X \arrow[r] & K(\mathbb{Z},3) \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r] \arrow[d] & EK(\mathbb{Z},2) \times_{K(\mathbb{Z},2)} BSpin^c \arrow[d] \\ X \arrow[r] & K(\mathbb{Z},3). \end{tikzcd}$$

On the other hand BSO is precisely the quotient $* \times_{K(\mathbb{Z},2)} BSpin^c$, and the diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & BSO \arrow[d, "\beta w_2"] \\ X \arrow[r, "\tau"'] & K(\mathbb{Z},3) \end{tikzcd}$$

is therefore a homotopy pullback. Twisted $Spin^c$ bordism is the homotopy of $\mathrm{Th}(UQ)$; a map from a sphere into $\mathrm{Th}(UQ)$ transverse to the zero section Q determines a manifold M equipped with a map $M \to Q$. This map $M \to Q$ specifies maps $i : M \to X$ and $\nu : M \to BSO$ (classifying the normal bundle of M) and a chosen homotopy between τi and $\beta w_2 \nu$. The choice of this homotopy is equivalent to the choice of a 2-cochain c with coboundary equal to the difference $\nu^*((\beta w_2)^* g) - i^*(\tau^* g)$, where g is as before a 3-cocycle representing the generator of the third cohomology of $K(\mathbb{Z},3)$. In summary, the τ-twisted $Spin^c$ bordism of X is bordism of oriented manifolds M equipped with a map $i : M \to X$ and a 2-cochain c such that

$$\delta c = \beta w_2(\nu(M)) - i^*(\tau),$$

where $\nu(M)$ is the stable normal bundle of M.

10.2 Twisted Nullbordism and the Geometry of the Cyclic Order

In section 8 we saw that the cyclic order of the twisted K-homology of G is the greatest common divisor of the collection of differences $\{\tau_k(x_i) - \tau_0(x_i)\}$, where $\{x_i\}$ is a set of algebra generators for $K.\Omega G$ and τ_j denotes the j-twisted map from $K.\Omega G$ to $K.*$. Frequently, these generators $\{x_i\}$ can be described as the images of the fundamental classes of $Spin^c$ manifolds M_i; (for example, we gave such a description for $SU(n+1)$, $Sp(n)$, and G_2 in sections 9.2 and 9.3). In this case the manifolds M_i admit modified $Spin^c$ structures $M_i(j)$ and the index of $M_i(j)$ is $\tau_j(x_i) \in K.*$. Moreover, there is a twisted $Spin^c$ structure (over G) on $M_i \times I$ cobounding the difference $M_i(k) - M_i(0)$; the relations $\tau_k(x_i) - \tau_0(x_i) = 0$ determining the cyclic order of twisted K-homology therefore have realizations in twisted $Spin^c$ bordism.

Before constructing these twisted $Spin^c$ bordisms, we recall that a $Spin^c$ structure can be altered by a line bundle and we discuss how this alteration affects the pushforward of the fundamental class. A twisted $Spin^c$ manifold is, as before, an oriented manifold M together with a 2-cochain c such that $\delta c = \beta w_2(\nu(M)) - i^*(\tau)$. In the examples we consider, the underlying manifold M is almost complex and so has a canonical ordinary $Spin^c$ structure; in particular M comes equipped with a 2-cochain b such that $\delta b = \beta w_2(\nu(M))$. A twisted structure on M is then given by a choice of 2-cochain d such that $\delta d = -i^*(\tau)$. If the twisting class τ is zero on M, then the 'twisted' $Spin^c$ structure corresponding to the cochain $b+d$ is of course ordinary, but it nevertheless differs from the $Spin^c$ structure determined by the original cochain b. We denote by $M(d)$ this modification of the canonical $Spin^c$ structure on M by the 2-cocycle d; we also refer to the alteration as a modification by the corresponding line bundle $L(d)$. Let $\pi : M \to *$ be the projection to a point; the pushforward in K-theory depends on the $Spin^c$ structure on M as follows:

$$\pi_!^{(M(d))}(1) = \pi_!^{(M)}(L(d)).$$

This relation follows more or less immediately from the fact that the Thom class defined by the $Spin^c$ structure on $M(d)$ is $L(d)$ tensor the Thom class defined by the structure on M; see [29]. In terms of our Rothenberg-Steenrod spectral sequence approach to twisted K-homology, this tells us that the twisted image $\tau_k([M]) = \langle L(k), [M]\rangle$ of the fundamental class $[M]$ is reinterpretable as the ordinary image $\tau_0([M(k)]) = \langle 1, [M(k)]\rangle$ of the fundamental class $[M(k)]$.

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We present twisted $Spin^c$ bordisms realizing the relations in the twisted K-homology of $SU(n+1)$. In section 9.2 we saw that the fundamental classes $\{[\mathbb{C}\mathrm{P}^i]\}$ are algebra generators for $K.\Omega SU(n+1)$, and the relations determining the cyclic order of $K^\tau SU(n+1)$ are

$$0 = \langle L^k, [\mathbb{C}\mathrm{P}^i]\rangle - 1 = \langle L^k, [\mathbb{C}\mathrm{P}^i]\rangle - \langle 1, [\mathbb{C}\mathrm{P}^i]\rangle.$$

(These classes take values in the twisted K-theory of $SU(n+1)$ via the inclusion $\mathbb{C}\mathrm{P}^i \to \Sigma\mathbb{C}\mathrm{P}^i \to \Sigma\mathbb{C}\mathrm{P}^n \to SU(n+1)$, which is of course nullhomotopic.) By the above remarks we can rewrite the relation as

$$0 = \langle 1, [\mathbb{C}\mathrm{P}^i(k)]\rangle - \langle 1, [\mathbb{C}\mathrm{P}^i]\rangle = \langle 1, [\mathbb{C}\mathrm{P}^i(k) - \mathbb{C}\mathrm{P}^i]\rangle.$$

If we can produce a nullbordism of $\mathbb{C}\mathrm{P}^i(k) - \mathbb{C}\mathrm{P}^i$, we will have pulled the given relation back to twisted $Spin^c$ bordism.

Write $\Sigma\mathbb{C}\mathrm{P}^i = ([-2,2] \times \mathbb{C}\mathrm{P}^i)/(\{-2,2\} \times \mathbb{C}\mathrm{P}^i \cup [-2,2] \times *)$ and consider the inclusion $i : \Sigma\mathbb{C}\mathrm{P}^i \to \Sigma\mathbb{C}\mathrm{P}^n \to SU(n+1)$. Choose a 3-cocycle representing the twisting τ on $SU(n+1)$ such that $i^*(\tau)$ is k times the cocycle locally Poincare dual to the submanifold $\mathbb{C}\mathrm{P}^{i-1} \times \{0\}$ in $\Sigma\mathbb{C}\mathrm{P}^i$. The product $\mathbb{C}\mathrm{P}^i \times [-1,1]$ has a canonical $Spin^c$ structure coming from the complex structure of $\mathbb{C}\mathrm{P}^i$. There is a twisted structure on $\mathbb{C}\mathrm{P}^i \times [-1,1]$ defined by the 2-cochain d that is k times the cochain locally Poincare dual to the submanifold $\mathbb{C}\mathrm{P}^{i-1} \times [-1,0]$; denote this twisted structure by $(\mathbb{C}\mathrm{P}^i \times [-1,1])(k|0)$. The coboundary of d is precisely $-i^*(\tau)$. Moreover, the cochain d

restricts to k times the generator of $H^2(\mathbb{C}\mathrm{P}^i \times \{-1\})$ and to zero on $\mathbb{C}\mathrm{P}^i \times \{1\}$. The difference $\mathbb{C}\mathrm{P}^i(k) - \mathbb{C}\mathrm{P}^i$ is therefore zero in $MSpin^{c,\tau}SU(n+1)$, as desired. Notice that the same argument shows that $\mathbb{C}\mathrm{P}^i(l+k) - \mathbb{C}\mathrm{P}^i(l)$ is null for any l, which implies that $\binom{l+k+i}{i} - \binom{l+i}{i}$ is zero in $K^\tau SU(n+1)$. In fact, for any sequence of integers $\{l_i\}$, $1 \leq i \leq n$, the gcd of the set $\{\binom{l_i+k+i}{i} - \binom{l_i+i}{i}\}$ is again the cyclic order of $K^\tau SU(n+1)$.

Whenever algebra generators of $K.\Omega G$ are represented as the fundamental classes of $Spin^c$ manifolds, the same argument produces nullbordisms in $MSpin^{c,\tau}G$ realizing the cyclic order of $K^\tau G$; we forgo details. Note though that in general the twisting cochain d will no longer be locally Poincare dual to a submanifold but merely to an appropriate singular chain.

10.3 Representing the Exterior Generators of Twisted K-Homology

We would like to represent the algebra generators of $K^\tau G$ as the fundamental classes of twisted $Spin^c$ manifolds over G. Here we merely suggest an approach for further investigation, taking clues from the structure of the Rothenberg-Steenrod spectral sequence; in the process we produce a candidate representative for the generator of $K_1^\tau SU(3)$. Finding representatives in general will require a more thorough investigation of $MSpin^{c,\tau}G$ and of the associated map to $K^\tau G$.

The structure of the ordinary $Spin^c$ bordism group is governed by $Spin^c$ characteristic numbers; we briefly recall how to compute these invariants. In section 10.1 we considered the principal bundle $BU(1) \to BSpin^c \to BSO$ classified by $\beta w_2 : BSO \to BBU(1)$. There is another principal bundle $B\mathbb{Z}/2 \to BSpin^c \to BSO \times BU(1)$ classified by $(w_2 \times r) : BSO \times BU(1) \to BB\mathbb{Z}/2$, where r is the nontrivial map $BU(1) \to BB\mathbb{Z}/2$. This latter bundle is usually more convenient for computations of $Spin^c$ characteristic classes. The relationship between the two bundles is encoded in the matrix

$$\begin{tikzcd}
U(1) \arrow[r, "2"] \arrow[d] & U(1) \arrow[r, "r"] \arrow[d] & B\mathbb{Z}/2 \arrow[r, "\beta"] \arrow[d] & BU(1) \arrow[d] \\
Spin^c \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & BSpin^c \arrow[r, "\mathrm{id}"] \arrow[d, "\pi\times\lambda"] & BSpin^c \arrow[d, "\pi"] \\
SO \arrow[r, "0"] \arrow[d, "\Omega\beta w_2"] & BU(1) \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & BSO \times BU(1) \arrow[r, "\pi"] \arrow[d, "w_2 \times r"] & BSO \arrow[d, "\beta w_2"] \\
BU(1) \arrow[r, "2"] & BU(1) \arrow[r, "r"] & BB\mathbb{Z}/2 \arrow[r, "\beta"] & BBU(1).
\end{tikzcd}$$

Indeed this diagram shows that the total spaces of the two fibrations are the same.

Following Anderson, Brown, and Peterson [2], Stong [45] showed that a $Spin^c$ manifold M is zero in $Spin^c$ bordism if and only if all of its rational and mod 2 characteristic numbers vanish. The map

$$(\pi \times \lambda)^* : H^*(BSO \times BU(1); \mathbb{Q}) \to H^*(BSpin^c; \mathbb{Q})$$

is an isomorphism and

$$(\pi \times \lambda)^* : H^*(BSO \times BU(1); \mathbb{Z}/2) \to H^*(BSpin^c; \mathbb{Z}/2)$$

is an epimorphism. In particular a $2n$-dimensional $Spin^c$ manifold M is nullbordant if all the characteristic classes of the underlying oriented manifold vanish and the single $Spin^c$ characteristic number $\langle \lambda(M)^n, [M]_H \rangle$ is zero. The characteristic class λ depends on the $Spin^c$ structure on M as follows. Let $M(d)$ denote as in the last section the modification of the $Spin^c$ structure on M by the line bundle or 2-cocycle d. The class $\lambda(M(d))$ is then $\lambda(M) + 2d$, as is easily checked by noting that the composite $BU(1) \to BSpin^c \xrightarrow{\lambda} BU(1)$ is multiplication by 2.

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We produce a candidate twisted $Spin^c$ representative for the exterior generator of $K^\tau SU(3)$ by investigating the corresponding class in the E^2 term of the Rothenberg-

Steenrod spectral sequence. For simplicity we assume the twisting class k is odd; the even case is entirely analogous.

In section 8.1 we saw that the generator of $K_1^\tau SU(3)$ is represented at the E^2 term of the Rothenberg-Steenrod spectral sequence by $x_2 - \frac{k+3}{2}x_1$; here x_2 and x_1 are elements of $\mathrm{Tor}_1^{K.\Omega SU(3)}(\mathbb{Z}, \mathbb{Z}_\tau)$, therefore of the E^1 term of the spectral sequence, and their differentials are given by

$$d^1 x_2 = \langle 1, [\mathbb{C}\mathrm{P}^2(k)]\rangle - 1 = \langle 1, [\mathbb{C}\mathrm{P}^2(k) - \mathbb{C}\mathrm{P}^2(0)]\rangle$$
$$d^1 x_1 = \langle 1, [\mathbb{C}\mathrm{P}^1(k)]\rangle - 1 = \langle 1, [\mathbb{C}\mathrm{P}^1(k) - \mathbb{C}\mathrm{P}^1(0)]\rangle$$

In section 10.2 we found a twisted $Spin^c$ bordism $X_2 = (\mathbb{C}\mathrm{P}^2 \times I)(k|0)$ whose boundary has index

$$\mathrm{ind}(\partial X_2) = d^1 x_2.$$

Because of this index property, we consider X_2 a geometric representative of the algebraic class x_2. Note that the bordism X_2 is over $\Sigma\mathbb{C}\mathrm{P}^2$ and therefore over $SU(3)$.

Similarly, we have a bordism $\widetilde{X_1} = (\mathbb{C}\mathrm{P}^1 \times I)(k|0)$ whose boundary has index $\mathrm{ind}(\partial\widetilde{X_1}) = d^1 x_1$. Given our selection of X_2, the manifold $\widetilde{X_1}$ is a poor choice for a geometric representative of x_1; we would like to have a five-dimensional bordism X_1, still living over $\Sigma\mathbb{C}\mathrm{P}^2$, with the same index property as $\widetilde{X_1}$. A natural choice for the underlying oriented bordism is $P(\nu+1) \times I$, where $P(\nu+1)$ is the projectivization of the sum of a trivial bundle and the normal bundle of $\mathbb{C}\mathrm{P}^1$ in $\mathbb{C}\mathrm{P}^2$; this projectivization is a resolution of the Thom space of the normal bundle and as such the bordism maps to $\mathbb{C}\mathrm{P}^2 \times I \subset \Sigma\mathbb{C}\mathrm{P}^2$. There is moreover a twisted $Spin^c$ structure on this bordism, denoted $X_1 = (P(\nu+1) \times I)(k|0)$ and produced as in section 10.2, such that

$$\mathrm{ind}(\partial X_1) = d^1 x_1.$$

The linear combination $C = X_2 - \frac{k+3}{2} X_1$ wants to be an element of $MSpin^{c,\tau}\Sigma\mathbb{C}\mathrm{P}^2$ mapping to the exterior generator of $K^\tau SU(3)$. The trouble of course is that C is not a closed manifold and so does not properly represent an element of $MSpin^{c,\tau}\Sigma\mathbb{C}\mathrm{P}^2$.

Note though that the map $\partial C \to \Sigma\mathbb{C}\mathrm{P}^2$ is nullhomotopic by a nullhomotopy on which the twisting class is zero. Suppose there is a nullbordism W of ∂C in $MSpin^c*$; then the union $W \cup_{\partial C} C$ is a closed twisted $Spin^c$ manifold over $\Sigma\mathbb{C}\mathrm{P}^2$, as desired.

The boundary of C is

$$\partial C = (\mathbb{C}\mathrm{P}^2(k) - \mathbb{C}\mathrm{P}^2) - \frac{k+3}{2}(P(\nu+1)(k) - P(\nu+1)).$$

All the SO-characteristic numbers of ∂C certainly vanish. The cohomology ring of $P(\nu+1)$ is $H^{\cdot}(P(\nu+1)) = \mathbb{Z}[y,x]/(y^2, x^2+yx)$, where y is the first Chern class of the tautological bundle on the base $\mathbb{C}\mathrm{P}^1$ and x is the first Chern class of the fibrewise tautological bundle on the total space. The tangential $Spin^c$ characteristic class of $P(\nu+1)(k)$ is

$$\lambda(T(P(\nu+1)(k))) = \lambda(T_{\mathrm{horiz}}) + \lambda(T_{\mathrm{vert}}) = -(2+2k)y - 2x,$$

and the associated characteristic number is

$$\langle \lambda(T(P(\nu+1)(k)))^2, [P(\nu+1)]_H \rangle = 8k+4.$$

Similarly the characteristic number for $\mathbb{C}\mathrm{P}^2(k)$ is

$$\langle \lambda(T(\mathbb{C}\mathrm{P}^2(k)))^2, [\mathbb{C}\mathrm{P}^2]_H \rangle = 4k^2 + 12k + 9.$$

The vanishing of the $Spin^c$ characteristic number for ∂C follows:

$$\langle \lambda(T(\partial C))^2, [\partial C]_H] \rangle = 4k^2 + 12k + 9 - 9 - \frac{k+3}{2}(8k+4-4) = 0.$$

Picking any $Spin^c$ nullbordism W of ∂C, the five-dimensional twisted $Spin^c$ manifold $W \cup_{\partial C} C$ should represent the generator of $K_1^{\tau} SU(3)$.

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