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Local Tests for Consistency of Support Hyperplane Data

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Abstract

Support functions and samples of convex bodies in R^n are studied with regard to conditions for their validity or consistency. Necessary and sufficient conditions for a function to be a support function are reviewed in a general setting. An apparently little known classical such result for the planar case due to Rademacher and based on a determinantal inequality is presented and a generalization to arbitrary dimensions is developed. These conditions are *global* in that they involve values of the support function at all points. The corresponding *discrete* problem of determining the validity of a set of samples of a support function is treated. Conditions similar to the continuous inequality results are given for the consistency of a set of discrete support observations. These conditions are in terms of a series of *local* inequality tests involving only neighboring support samples. Our results serve to generalize existing planar conditions to arbitrary dimensions by providing a generalization of the notion of nearest neighbor for plane vectors which utilizes a simple positive cone condition on the respective support sample normals.

1 Introduction

This paper addresses the problem of identifying the validity or consistency of a support function or its samples. Support samples result from measurements of the extent of an object or set in a

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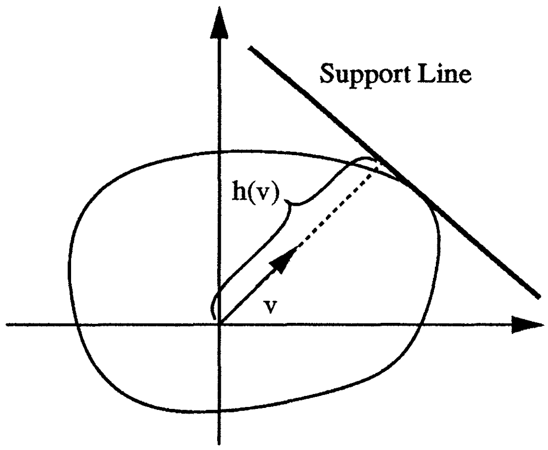

Figure 1: Illustration of a support measurement.

particular direction and provide samples of the support function of the object in the given direction, as shown in Figure 1. The **support function** $H(v)$ [1,2] of an object $\mathcal{O} \subset R^n$ is given by:

$$H(v) \equiv \sup_{x \in \mathcal{O}} x^T v \tag{1}$$

where $v \in R^n$. The **reduced support function** $h(v)$ is given by

$$h(v) \equiv H(v/||v||) \tag{2}$$

and precisely represents the distance from the origin of the corresponding supporting hyperplane to $\mathcal{O}$ in direction v. Thus, it is usually the *reduced* support function $h(v)$ or its samples that is actually obtained from physical measurements.

Due to the presence of noise, a group of discrete such observations will not, in general, be consistent, i.e. there might be *no* object that could have all the observations as support measurements. Such a situation is shown in Figure 2, where support measurements h_1, h_2, and h_3 are mutually consistent (e.g. for the object shown) but h_4 is not. No object could have all these lines as support measurements. The possibility of such inconsistent observations leads to the examination of what constraints are required on a set of support observations for consistency. This work is an extension and generalization of the approach taken in [3–5], where the planar case was treated.

Support measurements, such as we consider in this work, arise in many ways in object re-

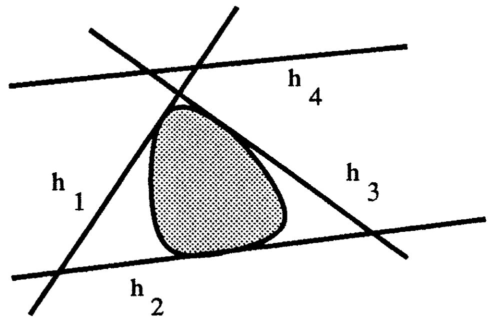

Figure 2: Illustration of inconsistent, noisy support measurements.

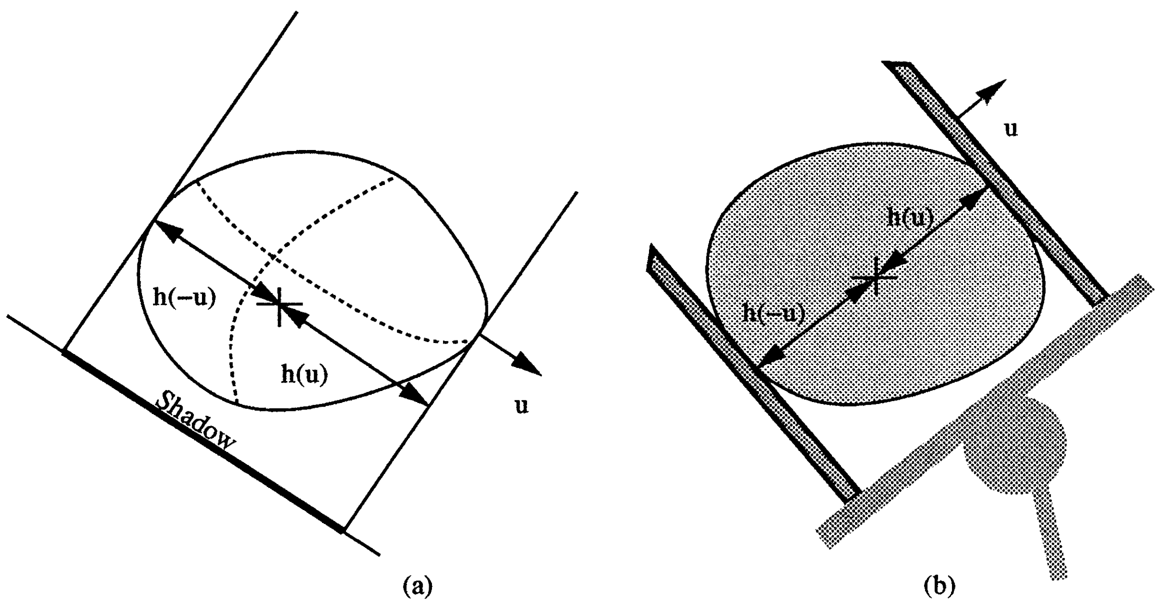

Figure 3: Applications of support measurements.

construction problems. A silhouette may be viewed as a set of support observations [6–8], where the directions of observation v are confined to a particular subspace, as illustrated in Figure 3a. One-dimensional shadows or projections correspond precisely to a pair of support observations in opposite directions. In the realm of robotics, these support type measurements can arise from repeated grasps or probes by a gripper, as shown in Figure 3b [9, 10]. Finally, in low dose tomography the line integral observations may yield little more than shadow information [3, 4, 11], thus fitting into the silhouette framework above. Even when this is not the case, a preliminary step of projection support extraction coupled with object boundary estimation may be useful or desirable [3, 12]. This approach has proven particularly helpful in reflection tomography arising in laser range data [13].

These problems all share the common goal of *set reconstruction* from support measurements [14–17]. Besides being of general interest to computational geometers, set reconstruction from support data is also fundamental to geometric probing [18–20], robot vision [21] and chemical component analysis [17,22–24]. The explicit statement of support consistency constraints which we provide allows their use in estimation and optimization algorithms. For example, we may find the set of consistent support values that is, in some sense, closest to the given observations, thus projecting our noisy observations onto the set of consistent support observations.

Outline

Conditions for a function to be a support function are reviewed in a general setting, with a summary of classical results in Section 2. An apparently little known result due to Rademacher [25] for the planar case is given. This result, based on a determinantal inequality, is interpreted geometrically as a set of *global* tests for consistency. An apparently new extension of this planar result to the general dimensional case is presented. These classically based results and our interpretation of them are used in Section 3 to guide our examination, interpretation, and treatment of the conditions for discrete support *sample* consistency. Such sampling appears because of the inherently discrete nature of support measurements in applications. *Local* tests for the consistency of such a set of support samples in arbitrary dimensions are provided. These tests are simple linear inequality tests and are thus simple to perform. In this framework, the local nature of the test is reflected in a banded structure of a corresponding matrix-vector inequality, yielding an efficient test.

2 Support Functions: The Continuous Case

2.1 Characterization of Support Functions

The support function $H(v)$ of a set, as defined in (1), is a scalar function of the vector v and hence a map from R^n to R. A natural question is which functions $H(v)$ could be support functions. Indeed, the problem is classical and the answer is provided by the following result, again classical:

Result 1 (Support Function Conditions) *A function $H(v)$ is the support function of a convex object if and only if it is defined for all vectors v and has the following properties:*

1. $H(0) = 0$.

2. $H(\alpha v) = \alpha H(v)$ *for* $\alpha > 0$.

3. $H(v + w) \leq H(v) + H(w)$, $\forall v, w \in R^n$.

A proof was given by Minkowski for the 3-dimensional case with other refinements and generalizations provided by Rademacher and others (see e.g. [1]). Thus, only positively homogeneous, convex functions are support functions and vice versa. It is condition 3 of subadditivity that is the interesting one, as will be seen later. Note that these conditions are *global*, in the sense that they must hold for *all* vectors v and w and thus combine values of the support function over its entire range.

Note that the support function $H(v)$ is easily obtained from the reduced support function $h(v)$ due to the positive homogeneity of $H(v)$ ($H(\lambda v) = \lambda H(v)$ for $\lambda > 0$). In fact, the support function $H(v)$ is completely determined by its values on the unit sphere $||v|| = 1$, and thus by the function $h(v)$. In particular, for $v \neq 0$, $H(v) = ||v||\, h(v/||v||)$, so that if u is a unit vector $H(u) = h(u)$. As a result, conditions on the support function are actually often phrased in terms of the more physically based *reduced* support function, an approach we will take in what follows. If a valid reduced support function can be found then it may easily be extended to yield a corresponding (full) support function.

For example, in the planar case, a differential condition in terms of $h(v)$ is often used in place of Result 1. Since in the planar case $h(v)$ is only a function of the direction of v we may parameterize $h(v)$ by the polar angle θ of v. A twice differentiable function $h(\theta)$ of θ is then a (reduced) support function if and only if $h_{\theta\theta}(\theta) + h(\theta) > 0$, where $h_{\theta\theta}(\theta)$ is the second derivative of $h(\theta)$ with respect to θ. Note that this differential condition is a local constraint, in the sense that it only involves value properties of the function at the point θ. In particular, $h_{\theta\theta}(\theta) + h(\theta)$ is equal to the reciprocal of the curvature of the object. In Result 4 we will provide a similar such local result for the discrete case in arbitrary dimensions. Finally, note that Result 1 is more fundamental than the commonly used planar differential inequality in that it does not require differentiability of $h(\theta)$.

Determinantal Condition for the Planar Case

Rademacher has shown that it is possible in the planar case to replace the subadditivity condition 3 of Result 1 by a determinantal condition on $h(u)$ over *unit* vectors u. In particular, he showed that under conditions 1 and 2 of Result 1, condition 3 holds if and only if

$$\begin{vmatrix} h(u_1) & u_1^T \\ h(u_2) & u_2^T \\ h(u_3) & u_3^T \end{vmatrix} \begin{vmatrix} 1 & u_1^T \\ 1 & u_2^T \\ 1 & u_3^T \end{vmatrix} \geq 0 \tag{3}$$

for all *unit* vectors u_1, u_2, and u_3, where $|*|$ denotes the determinate of the argument [1,25]. Note that there is no requirement on the differentiability of $h(u)$. Thus we now have a condition directly in terms of the *physically measured* quantity $h(u)$. This condition is of interest for its geometric interpretation. Assume that u_1, u_2, and u_3 are distinct and that u_3 is in the positive or negative cone of $\{u_1, u_2\}$ (which may always be done for three vectors in the plane through relabeling). Using a determinantal equality (see Appendix A) (3) may then be rewritten as

$$\beta(u_1, u_2, u_3) \left(u_3^T \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}^{-1} \begin{bmatrix} h(u_1) \\ h(u_2) \end{bmatrix} - h(u_3) \right) \geq 0 \tag{4}$$

where $\beta(u_1, u_2, u_3)$ is a scalar that depends on the u_i. In particular, if u_3 is in the positive cone of $\{u_1, u_2\}$ then $\beta(u_1, u_2, u_3) \geq 0$ and if u_3 is in the negative cone of $\{u_1, u_2\}$ then $\beta(u_1, u_2, u_3) < 0$ The term in parentheses in (4), which we call ρ, is the signed distance along the direction u_3 from the support line with normal u_3 to the intersection point of the support lines with normals u_1 and u_2, as shown in Figure 4 for the positive cone case.

In the plane then, the determinantal condition (3), and thus condition 3 of Result 1, requires that support functions satisfy an intuitive notion of consistency (as illustrated in Figure 2 or 4) for *all* triples of values of the function. This intuition provides a fundamentally geometric condition for a function to be a support function in the plane, but it is still a *global* condition, in the sense

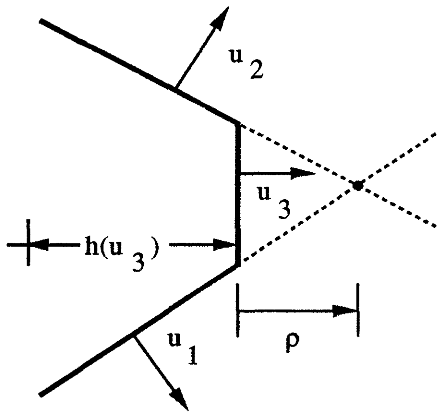

Figure 4: Illustration of 2-dimensional determinantal condition.

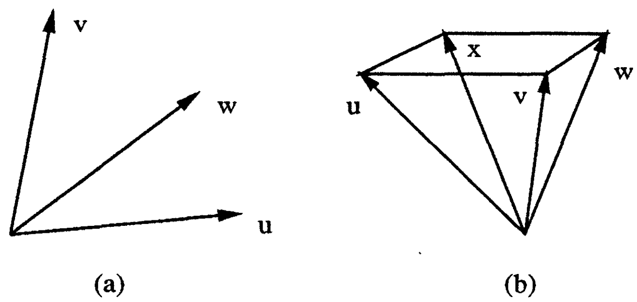

Figure 5: Difference between 2- and 3-dimensional situation.

that all possible combinations of samples must be checked.

Higher Dimensions

We now turn our attention to finding an equivalent of the geometrically interpretable determinantal inequality condition (3) for the higher dimensional case. Unfortunately, in three and higher dimensions the exactly analogous condition (i.e. validity of such a determinant inequality for *all* vectors u_i) has been shown by Rademacher to be satisfied only by the support functions of balls [25]. We identify the difficulty in directly extending this result and present a natural generalization of condition (3) that is valid for all dimensions. The result appears to be new.

The difference between the two and higher dimensional cases is that in the plane, given three vectors, one of the vectors is *always* in the positive or negative cone of the remaining two, as shown in Figure 5a where w is in the positive cone of u and v. In higher dimensions this is not necessarily true, as illustrated by the combination of normals in Figure 5b. If we consider the geometric interpretation of the test, it seems reasonable to suppose that by restricting our attention to groups of vectors for which this cone condition *is* satisfied, we might obtain the desired result. This is precisely what we do, yielding the following result which is proved in Appendix B:

Result 2 (General Inequality Condition) *A function $H(v)$ is a support function if and only if it is defined for all v and has the following properties:*

1′. $H(0) = 0$.

2′. $H(\alpha v) = \alpha H(v)$ *for* $\alpha > 0$.

3′. *The following determinantal inequality is satisfied for all $(n+1)$-tuples of unit vectors u_i with one in the full positive cone of the others:*

$$\begin{vmatrix} h(u_1) & u_1^T \\ h(u_2) & u_2^T \\ \vdots & \vdots \\ h(u_{n+1}) & u_{n+1}^T \end{vmatrix} \begin{vmatrix} 1 & u_1^T \\ 1 & u_2^T \\ \vdots & \vdots \\ 1 & u_{n+1}^T \end{vmatrix} \geq 0 \tag{5}$$

Recall that for unit vectors u, $H(u) = h(u)$. In a finite dimensional space a cone is said to be *full* if it cannot be contained in a proper subspace (the implication being that the set $\{u_i\}$ is independent). As before, there is no requirement on the differentiability of $h(u)$. Note that the condition requires testing of only positive cone n-tuples. This is a refinement of Rademacher's result for the planar case. It is easy to also include tests of negative cone n-tuples in Result 2, since this simply adds *additional* tests which are not really needed.

Now let us interpret this test. Assume u_{n+1} is in the positive cone of the remaining $\{u_i\}$. By using Lemma 2 of Appendix A, we may rewrite (5) as

$$u_{n+1}^T \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}^{-1} \begin{bmatrix} h(u_1) \\ h(u_2) \\ \vdots \\ h(u_n) \end{bmatrix} - h(u_{n+1}) \geq 0 \tag{6}$$

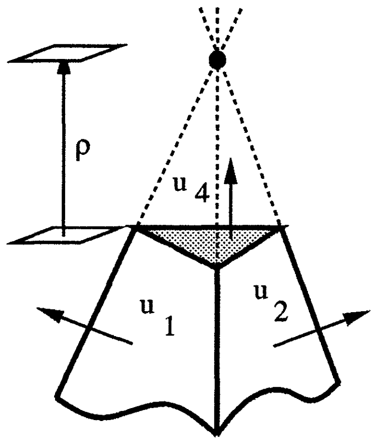

Figure 6: Illustration of 3-dimensional determinantal condition.

Similar to the planar case, the left hand side may be naturally interpreted as the signed distance ρ, positive in the direction of u_{n+1}, from the support hyperplane with normal u_{n+1} to the point determined by the intersection of the hyperplanes with normals given by u_i, $i = 1, \ldots, n$, as shown for the $n = 3$ case in Figure 6.

Condition $3'$ of our Result 2 thus generalizes the intuition of the planar case to arbitrary dimensions. As in the planar case, this condition is still a global one, in the sense that *all* $(n+1)$-tuples of vectors satisfying a positive cone condition must be checked. In the following sections we use the intuitions obtained in the continuous case to develop conditions that characterize the consistency of a given set of support *samples.* An equivalent *local* result is given in Result 4, where only $(n+1)$-tuples that are neighbors (defined in an appropriate way) need be checked for consistency.

3 Consistency of Support Samples

In the previous section, we dealt with continuous support functions defined for all directions. The main condition for validity of a support function is a consistency check (the analytic condition $3'$ of Result 2) on all positive cone $(n+1)$-tuples. In this section the discrete case arising from the sampling of a support function is treated. Due to the presence of noise, a group of such discrete observations will not, in general, be consistent, i.e. there might be no object that could have all the observations as support measurements. An example of such a situation was given in Figure 2. To be precise, we term a set of support samples *consistent* if there exists a valid support function whose values at the sample points match the given values. Thus we have the problem of determining when there exists a valid support function $h(u)$ (equivalently $H(u)$) such that $H(u_i) = h(u_i) = h_i$ for a given a set of m samples $\{h_i\}$ in (unit) directions $\{u_i\}$.

One obvious approach we could take to identifying inconsistency is to attempt to explicitly find offending hyperplanes, such as h_4 in Figure 2. Essentially what we are doing when we say that h_4 is the "inconsistent" sample is implicitly intersecting the directed halfspaces provided by the (h_i, u_i) pairs to obtain a convex polyhedral region and then attempting to identify those hyperplanes that do not contribute to this region, i.e. that are active constraints. This problem is equivalent to finding the non-binding constraints in a linear programming (LP) problem. This task is computationally expensive, essentially necessitating the solution of a dual LP problem itself (details may be found in [26,27]). Further, it is not really desirable. Such an approach assumes that all the error resides in the inconsistent support measurements, such as h_4 of the figure, while the rest are perfect. From an estimation theoretic perspective, the corresponding noise model does not seem reasonable. It is more realistic to assume that *all* the measurements are corrupted. Hence we instead develop tests or constraints which simply tell of the *existence* of inconsistency. These constraints essentially serve to define the set of all consistent support samples for a given fixed set of measurement directions u_i. This set will in fact be seen to define a polygonal cone in the space of support samples. We are then free to use the conditions as a constraint in the reconstruction of a consistent set as we see fit. For example, one could use these conditions to project onto the consistent support set.

Our first result shows that a set of samples is consistent if and only if a certain geometric condition (essentially each sample hyperplane being an active constraint in the set definition) is satisfied for *all* $(n+1)$-tuples of sample normals. We then show that under a certain set of assumptions (non-emptiness of intersection) we need only check the $(n+1)$-tuples satisfying a positive cone condition. For such $(n+1)$-tuples the geometric condition is identical to the analytic determinantal condition ($3'$ of Result 2) of the continuous case. Finally, under our assumption of nonempty intersection, we do not even require consistency of all these positive cone $(n+1)$-tuples, but rather a particular subset corresponding to a natural notion of the $(n+1)$-tuples being *local* or neighbors of each other.

3.1 Identifying Consistency

First we present a general test for consistency of a set of support samples. This result states that a set of support samples is (globally) consistent (i.e., there is a valid support function which agrees with all the samples) if and only if *every* $(n+1)$-tuple of the set is consistent.

Result 3 (Discrete Consistency) *A set of support samples h_i with associated (unit) direction vectors u_i is consistent if and only if every $(n+1)$-tuple of samples satisfies the following geometric condition:*

> *For every sample in the $(n+1)$-tuple the hyperplane corresponding to the sample has nonempty intersection with the resulting intersection of all the corresponding $(n+1)$ halfspaces.* (7)

Result 3 is proved using Helly's theorem in Appendix C. Now consider the condition (7). For any $(n+1)$-tuple with associated unit direction vectors u_i, one of the following situations must hold: 1) one u_i is in the positive cone of the others, 2) one u_i is in the negative cone of the others, or 3) none of the u_i is in the positive or negative cone of the others. We will term $(n+1)$-tuples in class 1) positive cone $(n+1)$-tuples and those in class 2) negative cone $(n+1)$-tuples. The $(n+1)$-tuples in class 3) (neither positive or negative cone) always satisfy condition (7), and therefore are really unimportant in determining consistency. In addition, if we assume that the intersection of all the support sample halfspaces corresponding to the (h_i, u_i) pairs is nonempty (a gross type of consistency, since it is clearly a necessary condition for the consistency of samples of nontrivial support functions), then the $(n+1)$-tuples in class 2) comprising the negative cone tests also satisfy condition (7). Thus, under a nonempty intersection assumption, it is actually sufficient to check the condition (7) for just the $(n+1)$-tuples satisfying 1) — i.e., the positive cone tests.

Now, if in such a positive cone $(n+1)$-tuple the cone is *full* (i.e. nondegenerate), then the geometrical condition (7) is equivalent to our determinantal one (5). This can be seen by considering the geometrical interpretation of the condition (5) provided through Lemma 2 and comparing it to

(7). Thus these conditions are interchangeable for full positive-cone tests (actually this equivalence is true for full negative-cone tests also, though we will not use this fact in what follows since we will assume nonempty intersection instead). We use these insights to obtain the following corollary to Result 3 involving the determinantal condition (5).

Corollary 1 (Positive Cone Consistency) *Given a set of support samples h_i with associated (unit) direction vectors u_i in R^n, assume that the intersection of all the support sample halfspaces is nonempty and assume that for every positive cone $(n+1)$-tuple of samplc directions the cone is full. The set of samples is then consistent if and only if condition $3'$ of Result 2 is satisfied by the samples of the set, i.e. if and only if (5) or (7) is satisfied for every positive cone $(n+1)$-tuple of the set.*

We actually believe that both the assumptions of the corollary are not truly essential. In particular, we believe the first assumption of nonemptiness may actually be replaced by some type of sampling rate constraint, i.e. that if we sample the support function densely enough the satisfaction of the positive cone tests will *imply* satisfaction of the negative cone tests. Indeed, precisely such a requirement was used in obtaining a similar result for the planar case in [3,4]. As it stands, the negative cone tests evidently just assure nonemptiness, and under dense enough sampling we believe that satisfaction of the positive cone tests will also assure this. The second assumption of fullness of the positive cones of $(n+1)$-tuples is is a degeneracy condition, assuring independence of the u_i, $i = 1, \dots, n$, which is necessary for (5) to be well defined. We could, alternatively define our tests from this point forward solely in terms of the condition (7), which is insensitive to this degeneracy. We prefer to work with condition (5) however, because of its connection to the continuous condition (5) and its straightforward implementability.

While the inequality tests resulting from Corollary 1 are conveniently computable, in that they are simple linear functions of the support measurements h_i, the procedure is problematic in that *all* positive cone combinations must be checked for consistency. The number of such tests grows combinatorially with the number of observations. For example, in the planar case with support values equally spaced in angle, if m is the number of observations then the number of tests grows as $m^2/8$. This growth becomes worse in higher dimensions because of the increased number of

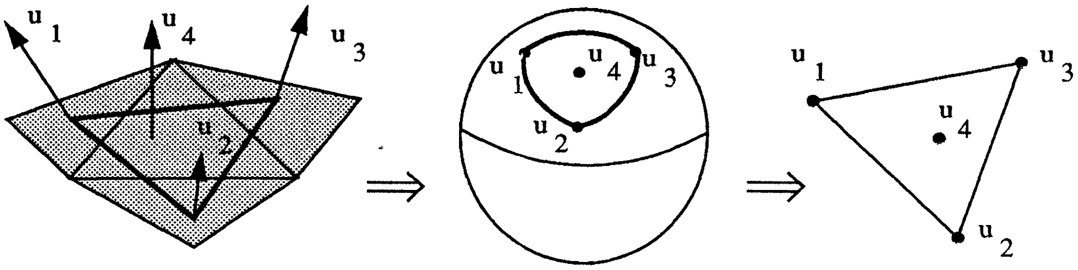

Figure 7: Graphical representation scheme for normal relationships.

degrees of freedom. As a result we seek a *local* test, utilizing only local support information in its application. This local approach may be viewed as a discrete version of the curvature constraint $h_{\theta\theta}(\theta) + h(\theta) > 0$ discussed in Section 2 (though our result will not require differentiability of the underlying support function).

3.2 Local Tests

We now develop a general local test for consistency of a set of support samples. Such a test (but without this interpretation) was given for the planar, equal angle case in [3–5], and our results serve to generalize this work. Since we already have a global consistency result in Result 3 or Corollary 1 (in the sense that positive cone $(n+1)$-tuples over the *entire* range of directions must be checked), our work reduces to showing that local consistency implies global consistency.

To this end, a result is first given that allows satisfaction of the determinant test (5) over given sub-domains of sample orientations to be extended to satisfaction over a larger domain. We term this result a *consistency merging* result. Before presenting the result we provide a geometrical description of it. Consider the situation shown in Figure 7 for the 3-dimensional case. The normals to support planes are mapped to points representing their tips on the unit (Gaussian) sphere, as shown. A spherical triangle connects these points on the sphere. We represent this spherical triangle by a corresponding planar triangle. Any point in the positive cone of the vertex direction normals is a point in the triangle and vice versa. For example, in Figure 7 u_4 is in the positive cone of u_1, u_2, and u_3.

With this graphical scheme, our merging result is illustrated for the three-dimensional case in

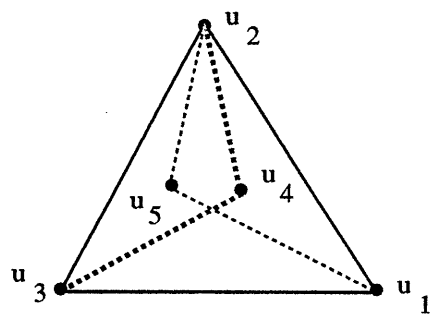

Figure 8: Illustration of meaning of Lemma 1.

Figure 8. Here u_4 is in the positive cone of $\{u_1, u_2, u_5\}$ and conversely u_5 is in the positive cone of $\{u_2, u_3, u_4\}$. The result states that, given the above inclusions, if $\{u_1, u_2, u_4, u_5\}$ form a consistent $(n+1)$-tuple and $\{u_2, u_3, u_4, u_5\}$ form a consistent $(n+1)$-tuple then the enlarged set $\{u_1, u_2, u_3, u_4\}$ also forms a consistent $(n+1)$-tuple (and by symmetry so does $\{u_1, u_2, u_3, u_5\}$). Thus, consistency over the smaller triangles (positive cone $(n+1)$-tuples) implies consistency over the larger triangle (positive cone $(n+1)$-tuple). In the higher-dimensional case the triangle of Figure 8 becomes an $(n-1)$-dimensional *simplex* and the interior triangles sub-simplices. The full result is as follows:

Lemma 1 (Consistency Merging) *Given a set of $(n+2)$ support samples h_i with associated unit sample directions u_i in R^n, suppose that $u_{n+1} \in \mathsf{cone}^+\{u_1, \ldots, u_{n-1}, u_{n+2}\}$, $u_{n+2} \in \mathsf{cone}^+\{u_2, \ldots, u_n, u_{n+1}\}$, $\{u_{n+1}, u_{n+2}\} \in \mathsf{cone}^+\{u_1, \ldots, u_{n-1}, u_n\}$ and that these three cones are full. If both the sets of support samples $\{h_1, \ldots, h_{n-1}, h_{n+1}, h_{n+2}\}$ and $\{h_2, \ldots, h_n, h_{n+1}, h_{n+2}\}$ are consistent then so are the enlarged sets $\{h_1, \ldots, h_n, h_{n+1}\}$ and $\{h_1, \ldots, h_n, h_{n+2}\}$.*

In the above, cone^+ denotes the positive cone of a set and by consistency of a set we mean satisfaction of the determinantal inequality (5) or, equivalently, the condition (7) by the set. The proof of this result is in Appendix D.

A suitable notion of "local" now needs to be defined for the general case, or, in the context of Lemma 1, we need to know the minimal domain over which consistency must be satisfied. For the planar case, as studied in [3–5], this notion of locality is straightforward, depending on normal ordering and adjacency. In higher dimensions, however, the situation is not so clear. Adjacent faces do not necessarily correspond to nearest normals anymore. A natural notion of locality is suggested

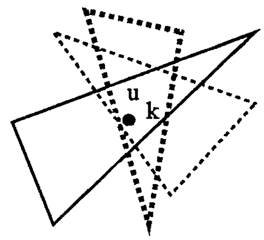

Figure 9: Illustration of a local family

both by condition 3′ of Result 2 and by Lemma 1 with their emphasis on a positive cone condition on the unit sample normals. Given a sample normal, we define what we mean to be "local" to that sample normal in the following:

Definition 1 (Local Family) *Given a set $\mathcal{S}$ of m distinct unit vectors in R^n and a member from this set u_k, we define the local family corresponding to u_k to be the set of all distinct $(n+1)$-tuples of vectors from $\mathcal{S}$ such that u_k is one element of the $(n+1)$-tuple and the remaining n vectors of the $(n+1)$-tuple contain only themselves and u_k from $\mathcal{S}$ in their full positive cone.*

Thus, the local family corresponding to the element u_k is a *set of $(n+1)$-tuples*, each containing u_k and with the property that the only nontrivial element of the parent set contained in the positive cone of the remaining n-tuple is the generating element u_k. In terms of the paradigm of Figure 7, a local family is defined by the set of all (spherical) triangles (simplices in higher dimensions) containing the given normal u_k but no others, as shown in Figure 9 for the $n = 3$ case. In contrast to the planar case, where there is just a single local neighbor, this notion of locality implies a *family* of tests associated to each normal, one for each $(n+1)$-tuple in the corresponding local family.

Local Constraint

With these ideas of locality defined we are prepared to present our main result showing that local consistency and global consistency are equivalent.

Result 4 (Local Consistency $\Longleftrightarrow$ Global Consistency) *Given a set of support samples h_i with associated (unit) direction vectors u_i in R^n, assume that the intersection of all the support sample halfspaces is nonempty and assume that for every positive cone $(n+1)$-tuple of sample directions the cone is full. Then the overall set of samples is consistent if and only if for each* *sample normal u_k, all elements of the corresponding local family are consistent.*

In other words, the overall set is consistent if and only if all elements of all local families are consistent. Again, consistency of a $(n+1)$-tuple means satisfaction of (5) or (equivalently) (7). Thus, we have that a set is globally consistent if and only if it is locally consistent, where locality is defined in the sense of the local family of a sample normal. The proof of the result is in Appendix E.

Note that each of the tests (5) required in Result 4 is linear in the support samples h_i. As a result, given a set of m samples and t such tests (where t is the total number tests to be performed), we may write the corresponding set of tests as

$$Q\mathbf{h} \geq \mathbf{0} \tag{8}$$

where $\mathbf{h} = [h_1, h_2 \cdots, h_m]^T$ is the vector of support samples (termed the support vector), Q is a $t \times m$ sparse matrix guaranteed to have only $n+1$ non-zero entries in each row, and $\mathbf{0}$ is a t vector of zeros. Since the definition of the local families depends only on the sample normals u_i and not on the support samples themselves, the matrix Q also depends only on the normals u_i. Consequently, once these directions are fixed the matrix Q may be *precomputed* and then applied to many different sets of measurements. Note, since the inequality constraints are linear and finite, the set of all consistent support samples is defined by a polygonal cone in the m-dimensional space of support samples with fixed direction.

This form of constraint is particularly convenient for constrained support reconstruction. For example, suppose that we are given a set of noisy support observations in the vector $\mathbf{y}$ taken in corresponding known directions u_i and that we wish to reconstruct the least square error estimate of $\mathbf{h}$ from these observations subject to consistency. The resulting problem combines (8) with a least squares criteria to yield the following linear inequality constrained least squares problem, which is straightforward to solve:

$$\widehat{\mathbf{h}} = \arg\min_{Q\mathbf{h}\geq\mathbf{0}} \|\mathbf{h} - \mathbf{y}\|_2 \tag{9}$$

This model of known u_i but noisy h_i is reasonable for many problems, particularly medical and non-destructive evaluation tomography problems, where the user may exercise great control over the geometry of the data acquisition. The situation is obviously more complicated if we consider the, perhaps more realistic, situation of noisy measurements h_i coupled with imperfectly known geometry u_i. Such situations arise in geophysical problems and target tracking [13,28].

As an example of the potential savings involved in using a local test instead of the general global result of Corollary 1, we consider the situation resulting from 20 regularly spaced support samples in R^3. This is the largest number of normals possible with completely regular spacing, corresponding to the face normals of an icosahedron. Applying Corollary 1 to this case with 20 uniformly spaced normals would yield a total of 1620 tests of the type (5). In other words, the support vector would have 20 elements and the corresponding matrix Q of the matrix inequality (8) would be 1620×20. In contrast, using the local test given by Result 4 results in only 320 inequality tests, or a 320×20 matrix Q. This is still large, but over a factor of 5 better than before. And we would expect this ratio to increase as we increase dimension.

Identifying the local families in practice is laborious but straightforward. For each sample normal u_k one may exhaustively test all possible remaining n-tuples to see if u_k is in the resulting positive cone. We may test if u_k is in the positive cone of a given n-tuple by checking the coefficients of the vector $[u_1|u_2|\cdots|u_n]^{-1}u_k$, for positivity, where the columns of the matrix $[u_1|u_2|\cdots|u_n]$ are composed of the sample vectors of the n-tuple. As discussed above, this need only be done once for a given set of sample directions.

3.3 The Two-Dimensional Case

Since working with the three-dimensional case is notationally cumbersome, let us consider the two-dimensional case ($n = 2$) shown in Figure 10 for illustration. In the plane we may parameterize the unit direction vectors u_i by their angle θ_i so that $u_i = [\cos(\theta_i),\ \sin(\theta_i)]^T$. Suppose the m sample angles θ_i are arranged in increasing order so that $\theta_{i+1} \geq \theta_i$ and are chosen so that $\theta_{i+2} - \theta_i < \pi/2$ (this is the sampling rate constraint alluded to in the discussion following Corollary 1). This sampling condition will ensure that no local family is empty. Applying Result 4 to this planar case, we obtain the following consistency condition for a planar set of support samples which must hold

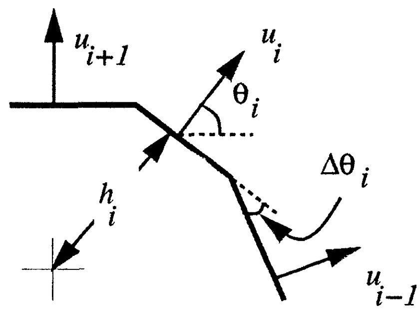

Figure 10: Two-dimensional case.

for all $1 \leq i \leq m$:

$$\begin{bmatrix} \sin(\Delta\theta_{i+1}) & -\sin(\Delta\theta_i + \Delta\theta_{i+1}) & \sin(\Delta\theta_i) \end{bmatrix} \begin{bmatrix} h_{i-1} \\ h_i \\ h_{i+1} \end{bmatrix} \geq 0 \tag{10}$$

where $\Delta\theta_i = \theta_i - \theta_{i-1}$ is the angular difference between normal i and normal $i-1$, $\theta_0 \equiv \theta_m$ $\theta_{m+1} \equiv \theta_1$, and similarly for h_i. Note that under the conditions above, the normal associated with θ_i is always in the positive cone of the adjacent two normals.

In terms of the support vector $\mathbf{h} = [h_1, h_2 \cdots, h_m]^T$ we may write this condition as the vector-matrix inequality $Q\mathbf{h} \geq \mathbf{0}$, as we did in (8), where the matrix Q is now given by:

$$Q = \begin{bmatrix} -\sin(\Delta\theta_1 + \Delta\theta_2) & \sin(\Delta\theta_1) & 0 & & \sin(\Delta\theta_2) \\ \sin(\Delta\theta_3) & -\sin(\Delta\theta_2 + \Delta\theta_3) & \sin(\Delta\theta_2) & \cdots & 0 \\ 0 & \sin(\Delta\theta_4) & -\sin(\Delta\theta_3 + \Delta\theta_4) & & \vdots \\ \vdots & 0 & \sin(\Delta\theta_5) & \cdots & 0 \\ 0 & \vdots & & & \sin(\Delta\theta_{m-1}) \\ \sin(\Delta\theta_{m+1}) & 0 & 0 & \cdots & -\sin(\Delta\theta_m + \Delta\theta_{m+1}) \end{bmatrix}$$

Note in this planar case that Q is square. Such a test was given in [3–5] for this planar case, but restricted to the equal-angle situation where $\Delta\theta_i = \Delta\theta$ and its significance as a *local* test (all positive cone 4-tuples are not tested) was not brought out. The planar test (10) for non-uniformly spaced angles may also be found in [13].

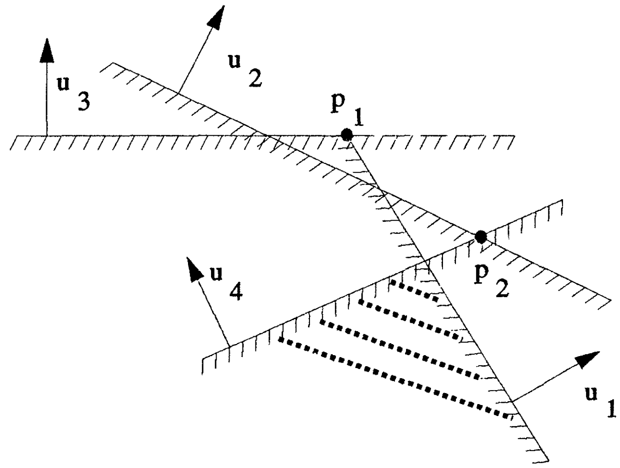

Figure 11: Illustration of non-specificity of local tests.

Note that these types of tests do *not* identify which constraints are inconsistent. To see this, consider the situation shown in Figure 11, where the intersections of the adjacent support lines used in the local tests are shown as the points p_i and the object is assumed contained in the darker, shaded region at the bottom. The support measurements with normals u_2, u_3, u_4, would fail the local test at point p_2, since the line associated with u_3 is behind p_2 (in the direction given by u_3). While this failure does confirm the existence of inconsistency, note that the set of samples associated with u_1, u_2, and u_3, would pass their local test at p_1. The distance from the u_2 support line to p_1 is positive in the direction given by u_2 so the local test is satisfied. Thus, while the sample with normal u_2 is also inconsistent, it is not identified by the local tests.

4 Conclusions

In this work we have presented a unified and general treatment of the consistency requirements for both a support function and a set of support samples corresponding to a fixed set of directions. We extended a classical determinantal condition for the existence and uniqueness of a support function and then used the resulting insights to develop a general dimensional inequality test for consistency of a set of support samples. We subsequently developed *local* test for sample consistency, in the sense that each inequality only involved local support information. Such a result may be viewed as the general dimensional, discrete equivalent of the well known planar support curvature constraint.

A Derivation of Geometric Lemma

In this appendix we prove a result we repeatedly use, and which we state as the lemma:

Lemma 2 *Suppose the unit vectors $\{u_i\}$, $i = 1, \ldots, n$ are independent and that the unit vector u_{n+1} is in the (full) positive or negative cone of the set $\{u_i\}$, $i = 1, \ldots, n$. Then the following equality holds for some $\beta(u_1, \ldots, u_{n+1})$, with $\beta \geq 0$ if u_{n+1} is in the positive cone and $\beta < 0$ if u_{n+1} is in the negative cone, and $\beta = 0$ if and only if $u_{n+1} = u_i$ for some $i = 1, \ldots, n$:*

$$\begin{vmatrix} H(u_1) & u_1^T \\ H(u_2) & u_2^T \\ \vdots & \vdots \\ H(u_{n+1}) & u_{n+1}^T \end{vmatrix} \begin{vmatrix} 1 & u_1^T \\ 1 & u_2^T \\ \vdots & \vdots \\ 1 & u_{n+1}^T \end{vmatrix} = \beta(u_1, \ldots, u_{n+1}) \left(u_{n+1}^T \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}^{-1} \begin{bmatrix} H(u_1) \\ H(u_2) \\ \vdots \\ H(u_n) \end{bmatrix} - H(u_{n+1}) \right) \tag{11}$$

Note that if the u_i are not independent the determinantal condition is trivially zero and our expression is not well defined since the matrix of the u_i is not invertible. Also, note that if one of the u_i is in a cone formed by the others and it happens not to be u_{n+1}, we need only interchange rows and relabel. Such operations do not change the sign of the result because the row exchanges will take place in both the determinant terms on the left hand side of (11).

Proof of lemma: First note that since u_{n+1} is in the cone formed by the set $\{u_i\}$, $i = 1, \ldots, n$, we may write it as the following linear combination:

$$u_{n+1} = [u_1 | \cdots | u_n] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{12}$$

with $\alpha_i \geq 0$ for the positive cone case and $\alpha_i \leq 0$ for the negative cone case.

Now apply the following determinantal identity to each term of the left hand side of (11):

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B|$$

Doing this to the first term yields:

$$\begin{vmatrix} H(u_1) & u_1^T \\ H(u_2) & u_2^T \\ \vdots & \vdots \\ H(u_{n+1}) & u_{n+1}^T \end{vmatrix} = (-1)^n \left|\begin{array}{c|c} u_1^T & H(u_1) \\ u_2^T & H(u_2) \\ \vdots & \vdots \\ u_n^T & H(u_n) \\ \hline u_{n+1}^T & H(u_{n+1}) \end{array}\right| \tag{13}$$

$$= (-1)^{n+1} \begin{vmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{vmatrix} \left(u_{n+1}^T \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}^{-1} \begin{bmatrix} H(u_1) \\ H(u_2) \\ \vdots \\ H(u_n) \end{bmatrix} - H(u_{n+1}) \right) \tag{14}$$

Similarly applying the determinantal identity to the second term yields for it:

$$\begin{vmatrix} 1 & u_1^T \\ 1 & u_2^T \\ \vdots & \vdots \\ 1 & u_{n+1}^T \end{vmatrix} = (-1)^{n+1} \begin{vmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{vmatrix} \left(u_{n+1}^T \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - 1 \right) \tag{15}$$

Now combining the two expressions and equating terms with the expression on the right hand side of the lemma shows that the scalar β is given by:

$$\beta(u_1, \ldots, u_{n+1}) = \begin{vmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{vmatrix}^2 \left(u_{n+1}^T \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - 1 \right) \tag{16}$$

$$= (-1)^{n+1} \begin{vmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{vmatrix} \begin{vmatrix} 1 & u_1^T \\ 1 & u_2^T \\ \vdots & \vdots \\ 1 & u_{n+1}^T \end{vmatrix} \tag{17}$$

Substituting the expression (12) for u_{i+1} into the second term shows that it is equal to $(\sum_{i=1}^{n} \alpha_i - 1)$ thus:

$$\beta(u_1, \ldots, u_{n+1}) = \begin{vmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{vmatrix}^2 \left(\sum_{i=1}^{n} \alpha_i - 1 \right) \tag{18}$$

To show β is of the appropriate sign, let us separately consider the two cases of u_{n+1} contained in the positive or negative cone of the remaining u_i. Note that the first term of β in (18) is clearly positive for either case, since the u_i, $i = 1, \ldots, n$ are independent (they form a *full* cone) by assumption.

Case 1: Positive Cone For this case we have that $\alpha_i \geq 0$, for each i. Since u_{n+1} is a unit vector and $u_i^T u_j \leq 1$ for all i, j we have:

$$1 = \|u_{n+1}\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \, u_i^T u_j \; \leq \; \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j = \left(\sum_{i=1}^{n} \alpha_i \right)^2 \tag{19}$$

Since $\alpha_i \geq 0$ this implies that $\sum \alpha_i \geq 1$ so that the second term in (18) is nonnegative. This shows that $\beta \geq 0$.

Now from (18) $\beta = 0$ if and only if $\sum_{i=1}^{n} \alpha_i = 1$. From (19) this implies that $\beta = 0$ if and only if:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \, u_i^T u_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \tag{20}$$

Each term on the left hand side of (20) is less than or equal to the corresponding term on the right hand side. In particular, equality for a term is achieved if and only if either $u_i^T u_j = 1$ or $\alpha_i \alpha_j = 0$. Since $u_i^T u_j < 1$ if $i \neq j$, we can have equality in (20) (equivalently, $\beta = 0$) if and only if

$$\alpha_i \alpha_j = 0, \ \forall i, j, \ i \neq j \tag{21}$$

Since $\sum_{i=1}^{n} \alpha_i = 1$ this can only be if $\alpha_i = 1$ for some i and $\alpha_j = 0$ for all j such that $j \neq i$, so that $u_{n+1} = u_i$ for some $i = 1, \ldots, n$. Thus $\beta = 0$ if and only if $u_{n+1} = u_i$ for some $i = 1, \ldots, n$, and the positive cone case is shown.

For a more geometrical understanding of the case when $\beta = 0$, note that the condition that $\sum_{i=1}^{n} \alpha_i = 1$ coupled with (12) implies that $\beta = 0$ if and only if u_{n+1} is in the hyperplane defined by $\{u_1, \ldots, u_n\}$ (we can also arrive at this conclusion by considering that $\beta = 0$ implies that the last term in (17) is identically zero, which can only be if all the vectors $\{u_1, \ldots, u_{n+1}\}$ lie in a hyperplane). Since the u_i are unit vectors, this hyperplane intersects the unit sphereoid in an $(n-1)$-dimensional hypersphere containing the vectors $\{u_1, \ldots, u_n\}$, as shown by the circle through u_1, u_2, and u_3 for the 3-dimensional case in Figure 12. Now since u_{n+1} itself is a unit vector, it must lie somewhere in this intersection hypersphere. The only points on this intersection hypersphere that are also in the positive cone of the $\{u_1, \ldots, u_n\}$ (denoted by the triangle in Figure 12) are the u_i themselves.

Case 2: Negative Cone For this case we have that $\alpha_i \leq 0$, for each i. From (18) clearly $\beta < 0$.

B Proof of Result 2

To prove the result we need only show that condition $3'$ of Result 2 implies and is implied by condition 3 of Result 1. First, we show that condition 3 implies $3'$. Consider an arbitrary $(n+1)$-

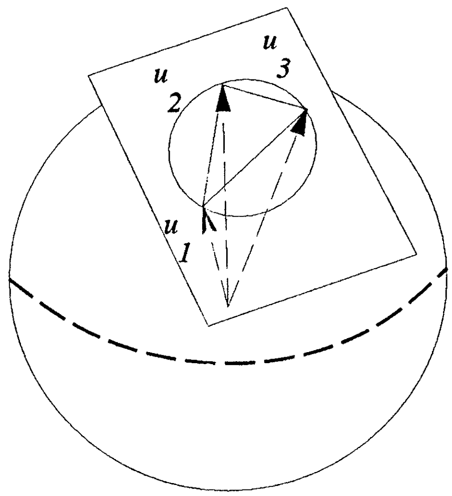

Figure 12: Illustration of 2-dimensional intersection.

tuple of unit direction vectors u_i with u_{n+1} in the positive cone of the remaining vectors. If $u_{n+1} = u_i$ for any $i = 1, \ldots, n$ then $3'$ of Result 2 is trivially satisfied. Suppose such is not the case. Using Lemma 2 of Appendix A together with the fact that we may write u_{n+1} as in (12) we obtain:

$$\left| \begin{array}{cc} H(u_1) & u_1^T \\ H(u_2) & u_2^T \\ \vdots & \vdots \\ H(u_{n+1}) & u_{n+1}^T \end{array} \right| \left| \begin{array}{cc} 1 & u_1^T \\ 1 & u_2^T \\ \vdots & \vdots \\ 1 & u_{n+1}^T \end{array} \right| = \beta(u_1, \ldots, u_{n+1}) \left[\sum_{i=1}^{n} \alpha_i H(u_i) - H\left(\sum_{i=1}^{n} \alpha_i u_i \right) \right] \tag{22}$$

$$= \beta(u_1, \ldots, u_{n+1}) \left[\sum_{i=1}^{n} H(\alpha_i u_i) - H\left(\sum_{i=1}^{n} \alpha_i u_i \right) \right] \tag{23}$$

with $\beta(u_1, \ldots, u_{n+1}) > 0$. Now by the subadditivity condition 3 of Result 1, $H(\alpha_i u_i + \alpha_j u_j) \leq H(\alpha_i u_i) + H(\alpha_j u_j)$ for any α_i, α_j, u_i, u_j. It follows that:

$$\sum_{i=1}^{n} H(\alpha_i u_i) - H\left(\sum_{i=1}^{n} \alpha_i u_i \right) \geq 0 \tag{24}$$

Thus (22) must be nonnegative. Since the vectors u_i of the positive cone $(n+1)$-tuple were arbitrary this shows that condition 3 implies condition $3'$.

Now we show that condition $3'$ implies condition 3. Given arbitrary vectors v and w, we will show that if $3'$ is satisfied then $H(v+w) \leq H(v)+H(w)$. If v is a scalar multiple of w this is trivially true from condition 2 or $2'$. Assume such is not the case. In condition $3'$ let $u_1 = v/||v||$, $u_2 = w/||w||$ and choose the remaining u_i, $i = 3, \ldots, n$ arbitrarily to span the subspace perpendicular to v and w. Let $u_{n+1} = (v+w)/||v+w||$, so that u_{n+1} is a unit vector in the full positive cone of the u_i. In particular, we have that:

$$u_{n+1} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \frac{||v||}{||v+w||} \\ \frac{||w||}{||v+w||} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{25}$$

Now by assumption condition $3'$ is satisfied and, using Lemma 2, it follows that:

$$u_{n+1}^T \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}^{-1} \begin{bmatrix} H(u_1) \\ H(u_2) \\ \vdots \\ H(u_n) \end{bmatrix} - H(u_{n+1}) \geq 0 \tag{26}$$

for any $n+1$ unit vectors u_i, with u_{n+1} in the positive cone of the remaining ones but not equal to any of them. Substituting the expressions above for u_1, u_2, and u_{n+1} we obtain

$$\begin{bmatrix} \frac{||v||}{||v+w||} & \frac{||w||}{||v+w||} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}^{-1} \begin{bmatrix} H(u_1) \\ H(u_2) \\ \vdots \\ H(u_n) \end{bmatrix} - H\left(\frac{v+w}{||v+w||}\right) =$$

$$\frac{||v||}{||v+w||} H\left(\frac{v}{||v||}\right) + \frac{||w||}{||v+w||} H\left(\frac{w}{||w||}\right) - H\left(\frac{v+w}{||v+w||}\right) \geq 0.$$

Equivalently, using condition 2,

$$\frac{1}{||v+w||}\left(H(v)+H(w)-H(v+w)\right)\geq 0.$$

Thus

$$H(v+w)\leq H(v)+H(w)$$

and the converse is shown. Together these implications prove the result.

C Proof of Result 3

To prove Result 3 we will make use of the following well known theorem:

Theorem 1 (Helly's Theorem) *A collection of convex sets in R^n has nonempty intersection if and only if every collection of $n+1$ sets at a time has nonempty intersection.*

Given a set of m support samples one can always form a polyhedron (which may possibly be empty) by intersecting the m half-spaces corresponding to the support samples. Call this polyhedron P. That global consistency of a support sample set implies satisfaction of the condition (7) for every $(n+1)$-tuple (i.e. local consistency) is obvious. To show the other direction, we need to show that if every $(n+1)$-tuple of samples satisfies the condition (7), then there exists a valid support function agreeing with the samples or, equivalently, that the intersection of the hyperplane corresponding to each sample with the polygon P is nonempty.

To this end, suppose that every $(n+1)$-tuple of samples satifies the condition (7) (i.e. is locally consistent) and consider the i-th sample. We need to show that the hyperplane corresponding to this sample has nonempty intersection with P. The hyperplane corresponding to the i-th sample consists of the intersection of two halfspaces. The intersection of this hyperplane with P thus consists of the intersection of $m+1$ halfspaces — 2 for the hyperplane of the i-th sample and $m-1$ for the halfspaces of the other $m-1$ samples. Since by assumption every $(n+1)$-tuple of samples satisfies (7), it follows that the intersection of every $(n+1)$-tuple of these halfspaces is nonempty. Hence, by Helly's theorem it follows that the intersection of all $m+1$ halfspaces is nonempty, so that the i-th hyperplane has nonempty intersection with P. Since this is true for each $i = 1, \ldots, m$, the m samples are (globally) consistent, with the support function of P giving one valid support function.

D Proof of Lemma 1

We show that under the hypotheses of the lemma the enlarged set $\{h_1, \ldots, h_n, h_{n+1}\}$ is consistent. Consistency of the set $\{h_1, \ldots, h_n, h_{n+2}\}$ then follows by symmetry. We know that $u_{n+1} \in \mathrm{cone}^+\{u_1, \ldots, u_{n-1}, u_{n+2}\}$ and $u_{n+2} \in \mathrm{cone}^+\{u_2, \ldots, u_n, u_{n+1}\}$, thus we may write u_{n+1} and u_{n+2} as the following linear combinations:

$$u_{n+1} = \begin{bmatrix} u_1 & u_2 & \cdots & u_{n-1} & u_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ 0 \end{bmatrix} + \alpha_{n+2} u_{n+2} \tag{27}$$

$$u_{n+2} = \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_n \end{bmatrix} \begin{bmatrix} 0 \\ \bar{\alpha}_2 \\ \bar{\alpha}_3 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix} + \bar{\alpha}_{n+1} u_{n+1} \tag{28}$$

where $0 \leq \alpha_i$ and $0 \leq \bar{\alpha}_i$. Note that α_n and $\bar{\alpha}_1$ are 0. We may eliminate u_{n+2} from the above two expressions to obtain the following equivalent expression for u_{n+1}:

$$u_{n+1} = \frac{1}{1 - \bar{\alpha}_{n+1}\alpha_{n+2}} \begin{bmatrix} u_1 & u_2 & \cdots & u_{n-1} & u_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 + \bar{\alpha}_2 \alpha_{n+2} \\ \vdots \\ \alpha_{n-1} + \bar{\alpha}_{n-1}\alpha_{n+2} \\ \bar{\alpha}_n \alpha_{n+2} \end{bmatrix} \tag{29}$$

This expression provides the representation of u_{n+1} with respect to the cone defined by $\{u_1, \ldots, u_n\}$. Since u_{n+1} is in the full positive cone of these direction vectors by assumption and this representation is unique, the coefficients of the u_i, $i = 1, \ldots, n$ in the expansion (29) must be nonnegative and finite. In particular, since α_i and $\bar{\alpha}_i$, $i = 1, \ldots, n+2$ are nonnegative and not all zero, we must have that $(1 - \bar{\alpha}_{n+1}\alpha_{n+2}) > 0$.

Now consistency of the set $\{h_1, \ldots, h_{n-1}, h_{n+1}, h_{n+2}\}$ implies through (5) and Lemma 2 that the following inequality is satisfied:

$$u_{n+1}^T \begin{bmatrix} u_1 & u_2 & \cdots & u_{n-1} & u_{n+2} \end{bmatrix}^{-T} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \\ h_{n+2} \end{bmatrix} - h_{n+1} \geq 0$$

Substitution of (27) for u_{n+1} into this inequality and rearrangement yields

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0 & -1 & \alpha_{n+2} \end{bmatrix} \mathbf{h} \geq 0 \tag{30}$$

where $\mathbf{h} = [h_1, h_2, \cdots, h_{n+2}]^T$ is termed the support vector. Similarly, consistency of the set $\{h_2, \ldots, h_n, h_{n+1}, h_{n+2}\}$ together with (28) yields the following inequality, through use of (5) and Lemma 2:

$$\begin{bmatrix} 0 & \bar{\alpha}_2 & \bar{\alpha}_3 & \cdots & \bar{\alpha}_n & \bar{\alpha}_{n+1} & -1 \end{bmatrix} \mathbf{h} \geq 0 \tag{31}$$

Thus, (30) and (31) are true by assumption.

To show consistency of the samples $\{h_1, \ldots, h_n, h_{n+1}\}$ we have to show that the following expression is nonnegative

$$\rho = \begin{vmatrix} h_1 & u_1^T \\ h_2 & u_2^T \\ \vdots & \vdots \\ h_{n+1} & u_{n+1}^T \end{vmatrix} \begin{vmatrix} 1 & u_1^T \\ 1 & u_2^T \\ \vdots & \vdots \\ 1 & u_{n+1}^T \end{vmatrix} \tag{32}$$

Applying Lemma 2 again and substituting for u_{n+1} from (29) shows that the expression (32) is equivalent to:

$$\rho = \frac{\beta(u_1, \ldots, u_{n+1})}{1 - \bar{\alpha}_{n+1}\alpha_{n+2}} \Big[\alpha_1, \ (\alpha_2 + \bar{\alpha}_2\alpha_{n+2}), \ (\alpha_3 + \bar{\alpha}_3\alpha_{n+2}), \ \cdots, \\ (\alpha_{n-1} + \bar{\alpha}_{n-1}\alpha_{n+2}), \ \bar{\alpha}_n\alpha_{n+2}, \ (\bar{\alpha}_{n+1}\alpha_{n+2} - 1), \ 0 \Big] \mathbf{h} \tag{33}$$

where β is a nonnegative scalar depending on the $u_1, \ldots, u_{n+1}$. We may equivalently write this as

$$\rho = \frac{\beta(u_1, \ldots, u_{n+1})}{1 - \bar{\alpha}_{n+1}\alpha_{n+2}} \mathbf{h}^T \left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ 0 \\ -1 \\ \alpha_{n+2} \end{bmatrix} + \alpha_{n+2} \begin{bmatrix} 0 \\ \bar{\alpha}_2 \\ \bar{\alpha}_3 \\ \vdots \\ \bar{\alpha}_n \\ \bar{\alpha}_{n+1} \\ -1 \end{bmatrix} \right) \tag{34}$$

which will be recognized as a linear combination of the left hand sides of (30) and (31). Now the terms $\beta/(1 - \bar{\alpha}_{n+1}\alpha_{n+2})$ and α_{n+2} are nonnegative so (33) is equal to a nonnegative linear combination of (30) and (31), which are also nonnegative. Consequently, (33) and thus (32) must also be nonnegative and we have demonstrated consistency of the set $\{h_1, \ldots, h_n, h_{n+1}\}$. The lemma is thus shown.

E Proof of Result 4

That global consistency implies local consistency follows easily, for if a set is globally consistent then by definition the inequality (5) is satisfied for all unit vectors in the full positive cone of an n-tuple of other sample normals.

Thus, we need to show that local sample consistency implies global sample consistency. First note that under the assumptions of the result, global consistency is assured if (5) or (7) are satisfied for *every* positive cone $(n+1)$-tuple due to Corollary 1. Now let us assume that the $(n+1)$-tuples

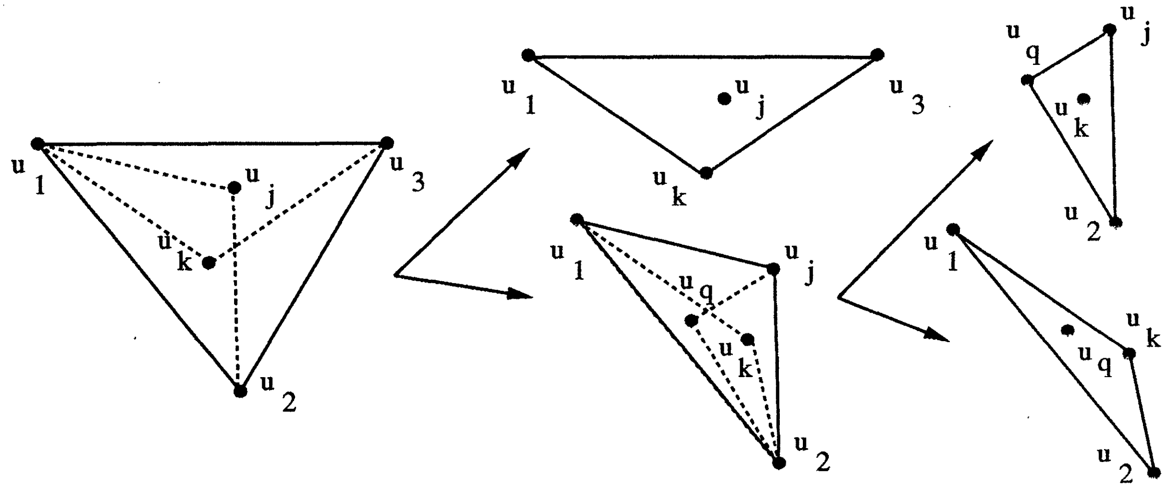

Figure 13: Illustration of geometry on unit (Gaussian) sphere.

corresponding to the local families are consistent and show how this implies that *any* positive cone $(n+1)$-tuple will then be consistent. To this end, consider an arbitrary support sample h_j and its associated unit direction normal u_j, in the positive cone of some (possibility non-local) n-tuple of other sample normals, i.e. an arbitrary positive cone $(n+1)$-tuple. On the surface of the n-dimensional unit (Gaussian) spheroid the unit direction vector u_j (in the positive cone of the n other unit vectors) is a point inside an $(n-1)$-dimensional spherical simplex (generalization of a spherical triangle), as described in association with Figure 7 for the 3-dimensional case. Points in this simplex are points in the positive cone of the vectors at the vertices of the simplex, as is u_4 in Figure 7.

If the point associated with u_j is isolated in the simplex it is consistent by hypothesis, since it is the only vector in the positive cone of the normal directions at the vertex, and hence part of a local family. Suppose instead that there is another point in the simplex with it, say u_k as shown in the leftmost illustration of Figure 13 for the 3-dimensional case. Each of these two interior points (corresponding to direction vectors u_j and u_k) in combination with the n original bounding points (corresponding to an n-tuple of direction vectors) tessellates the original simplex into n disjoint subsimplicies whose union is the original simplex. The two interior points corresponding to u_j and u_k are thus each contained in a subsimplex formed from the other interior point and $n-1$ of the original boundary points. This geometry is shown in the leftmost frame of Figure 13 as two dotted triangles.

Now by the merging result, Lemma 1, if we can show that the samples u_j and u_k are consistent on these smaller subregions then we have shown that they are consistent on the entire region, as desired. Thus we have reduced the problem from showing consistency over the original region to showing consistency over two smaller subregions. This process is shown in the middle illustration of the figure, where we have split the original test into two subtests. We may now repeat the above arguments on each of the subregions, attempting to show the consistency of each. We thus have the following finitely terminating recursive construction:

Show an arbitrary positive cone $(n+1)$-tuple is consistent:

1. If it is isolated in its cone, consistency is shown by hypothesis.

2. If it is not isolated:

   (a) Pick another point in the cone.

   (b) Form two smaller subregions.

   (c) Attempt to show consistency of each subregion.

We keep proceeding in this way until a subregion is found where the interior point is isolated (corresponding to the normal being the only vector in the positive cone of its bounding set) and consistency is satisfied by hypothesis. In Figure 13 we show the next step of this procedure on the right, where we have assumed that the subregion containing u_j is isolated (so that we have reached a leaf of the tree) but the one containing u_k contains another point, u_q and hence must itself be broken into two subregions.

Since at each stage of this procedure another point (sample normal) is removed from the original finite set and since the subregions are nonincreasing at each stage, we must eventually reach the situation where a sample normal is isolated in its simplex and therefore consistent. Using Lemma 1 we may then travel back up the tree we have implicitly created to show consistency of the original sample with respect to the original boundary points. Since the original positive-cone $(n+1)$-tuple we chose was arbitrary, we have shown the result.

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