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Stability of Switched and Hybrid Systems*

Michael S. Branicky†

Center for Intelligent Control Systems
and
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, MA 02139

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Abstract

This paper outlines work on the stability analysis of hybrid systems. Particularly, we concentrate on the continuous dynamics and model the finite dynamics as switching among finitely many continuous systems. We introduce multiple Lyapunov functions as a tool for analyzing Lyapunov stability. We use IFS theory as a tool for Lagrange stability. By enforcing the conditions of our theorems, one can also synthesize hybrid systems with desired stability properties.

1 Introduction

We have in mind the following model as a prototypical example of a *switching system*:

$$\dot{x}(t) = F_i(x(t)), \qquad x(0) = x_0 \tag{1}$$

where $x(\cdot) \in R^n$ and $i = 1, \ldots, N$. Such systems are of "variable structure" or "multi-modal"; they are a simple model of (the continuous portion) of hybrid systems. Hybrid systems are those that inherently combine logical and continuous processes, *e.g.*, coupled finite automata and ODEs [5, 7, 2]. For instance, the particular i at any given time may be chosen by some "higher process," such as a controller, computer, or human operator. It may also be a function of time or state or both. In the latter case, we may really just arrive at a single (albeit complicated) nonlinear time-varying equation. However, one might gain some leverage in the analysis of such systems by considering them to be amalgams of simpler systems. We add the assumptions that (1) each F_i is globally Lipschitz continuous and (2) the i's are picked in such a way that we have finite switches in finite time. Models like Equation (1) have been studied for stability [4, 8]. We use some of their notation. However, those papers concentrated on the special case where the F_i are linear.

We also discuss difference equations:

$$x[k+1] = F_i(x[k+1]), \qquad x[0] = x_0$$

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†Dept. of Electrical Engineering and Computer Science. Direct correspondence to: PO Box 397205, Cambridge, MA 02139-7205. E-mail: branicky@lids.mit.edu

2 Multiple Lyapunov Functions

In this section, we discuss stability of switching systems via multiple Lyapunov functions (MLFs). We assume the reader is familiar with basic Lyapunov theory (continuous and discrete time), say, at the level of [6]. The level of rigor of the proofs is similar to those in that book. Let $S(r) = \{x \in R^n | x^T x = r^2\}$, $B(r) = \{x \in R^n | x^T x < r^2\}$, and $\overline{B}(r) = \{x \in R^n | x^T x \leq r^2\}$ represent the sphere, ball, and

Below, we will be dealing with systems that switch among vector fields (resp. difference equations), over time or regions of state-space. One can associate with such a system the following *switching sequence*, indexed by an initial time, t_0 and an initial state, x_0: $s(x_0, t_0) = (i_0, t_0), (i_1, t_1), \ldots, (i_N, t_N), \ldots$. The sequence may or may not be infinite. The switching sequence, along with Equation (1), completely describes the system according to the following rule: (i_k, t_k) means that the system evolves according to $\dot{x} = F_{i_k}(x(t), t)$ for $t_k \leq t < t_{k+1}$. We can take projections of this sequence onto its first and second coordinates, yielding the sequence of indices, $\pi_1(s(x_0, t_0)) = i_0, i_1, \ldots, i_N, \ldots$, and the sequence times, $\pi_2(s(x_0, t_0)) = t_0, t_1, \ldots, t_N, \ldots$, respectively.

Using this notation, when we say that V is a Lyapunov function for $\dot{x} = F_i(x)$ (resp. $x[k+1] = F_i(x[k])$, we mean that V is a continuous, positive definite function (about the origin) such that $\dot{V} \leq 0$ (resp. $V(x[k+1]) \leq V(x[k])$) *whenever the vector field (resp. difference equation) F_i is active*, that is, for all intervals $\{[t_j, t_{j+1}) \mid i_j = i\}$ (resp. indices $\{k_j \mid i_j = i\}$).

Remark 1 *Suppose we have a finite number of Lyapunov functions V_i, $i = 1, \ldots, N$, corresponding to the continuous-time vector fields $\dot{x} = f_i(x)$. Let s_k be the switching times of the system. If, whenever we switch in mode (or region) i, with corresponding Lyapunov function V_i, we have $V_i(x(s_k)) \leq V_i(x(s_j))$, where $s_j < s_k$ is the last time we switched out of mode (or region) i, then the system is stable in the sense of Lyapunov. Initially, we set $s_0 = t_0$ and $V_j(x(s_0)) = \infty$, for $j \neq i_0$, the starting mode.*

Proof We will do the proof for the case $N = 2$. Let $R > 0$ be arbitrary. Let $m_i(\alpha)$ denote the minimum value of V_i on $S(\alpha)$. Pick $r_i < R$ such that in $B(r_i)$ we have $V_i < m_i(R)$. This choice is possible via the continuity of V_i. Let $r = \min(r_i)$. With this choice, if we start in $B(r)$, either vector field alone will stay within $B(R)$.

Now, pick $\rho_i < r$ such that in $B(\rho_i)$ we have $V_i < m_i(r)$. Set $\rho = \min(\rho_i)$. Thus, if we start in $B(\rho)$, either vector field alone will stay in $B(r)$. Therefore, whenever the other is first switched on we will have $V_i(x(s_1)) < m_i(R)$, so that we will stay within $B(R)$.

The proof for general N requires N concentric circles constructed as the two were above. □

The stability theorem of [8] is a special case of the above. Specifically, it requires that $V_{i_{j+1}}(x(s_{j+2})) < V_{i_j}(x(s_{j+1}))$, a stronger condition. Moreover, the proof of asymptotic stability in [8] is flawed since it only proves state convergence and not state convergence plus stability, as required. It can be fixed using our theorem.

Remark 2 *Suppose we have a finite number of Lyapunov functions V_i, $i = 1, \ldots, N$, with the same point of global minimum, corresponding to the discrete-time difference equations $x[k+1] = f_i(x[k])$. Let s_k be the switching times of the system. If, whenever we switch in mode (or region) i, with corresponding Lyapunov function V_i, we have $V_i(x(s_k)) \leq V_i(x(s_j))$, where $s_j < s_k$ is the last time we switched out of mode (or* *region) i, then the system is stable in the sense of Lyapunov. Initially, we set $s_0 = t_0$ and $V_j(x(s_0)) = \infty$, for $j \neq i_0$, the starting mode.*

Proof We will do the proof for the case $N = 2$. Let $R > 0$ be arbitrary. Let $m_i(\alpha, \beta)$ denote the minimum value of V_i on the closed annulus $\overline{B}(\beta) - B(\alpha)$. Pick $R_0 < R$ so that none of the f_i can jump out of $B(R)$ in one step. Pick $r_i < R_0$ such that in $B(r_i)$ we have $V_i < m_i(R_0, R)$. This choice is possible via the continuity of V_i. Let $r = \min(r_i)$. With this choice, if we start in $B(r)$, either equation alone will stay within $B(R)$.

Pick $r_0 < r$ so that none of the f_i can jump out of $B(r)$ in one step. Now, pick $\rho_i < r_0$ such that in $B(\rho_i)$ we have $V_i < m_i(r_0, r)$. Set $\rho = \min(\rho_i)$. Thus, if we start in $B(\rho)$, either equation alone will stay in $B(r_0)$, and hence $B(r)$. Therefore, whenever the other is first switched on we will have $V_i(x(s_1)) < m_i(R_0, R)$, so that we will stay within $B(R_0)$, and hence $B(R)$.

The proof for general N requires N sets of concentric circles constructed as the two were above. □

Both proofs also work when the F_i are time-varying.

3 Iterated Function Systems

We begin with some background [1, 9, 3].

Definition 3 (IFS) *An* IFS (iterated function system) *is a complete metric space and a set $\{f_i\}_{i \in I}$ of contractive functions such that I is a compact space and the map $(x, i) \mapsto f_i(x)$ is continuous.*

Definition 4 *A* contractive function *f is one such that there exists $s < 1$ where $d(f(x), f(y)) \leq s d(x, y)$, for all x, y.*

The image of a set X under an IFS is the set $Y = \bigcup_{i \in I} f_i(X)$. It is compact. Now suppose W is an IFS. Let $S(W)$ be the semi-group generated by W under composition. For example, $W = \{f, g\}$; $S(W) = f,\ g,\ f \circ f,\ f \circ g,\ g \circ f,\ g \circ g, \ldots$. Now, define A_W to be the closure of the fixed points of $S(W)$. We have

Theorem 5 *Suppose $W = \{w_i\}_{i \in I}$ is an IFS on X. Then A_W is compact and*

1. *$A_W = \bigcup_{i \in I} w_i(A_W)$.*
2. *$A_W = \bigcup_\sigma \{\lim_{n \to \infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x)\}$, for all $x \in X$, where $\sigma = (\sigma_1, \sigma_2, \ldots)$, $\sigma_i \in I$.*

The relevance of this theorem is that (i) A_W is an invariant set under the maps $\{w_i\}_{i \in I}$ and (ii) all points approach A_W under iterated composition of the maps $\{w_i\}_{i \in I}$.

Clearly,this theory can be applied in the case of a set of contractive discrete maps indexed by a compact set (usually finite). But, to obtain contractive maps while switching among differential equations requires a little thought .... Assume there is some lower limit on switching time, T. Then we can convert this into an IFS as follows: Let $I = \bigcup_{j=1,\ldots,N} j \times [T, 2T]$. Notice that for any switching time $r \geq T$, there is a decomposition into smaller intervals as follows:

$$r = \sum_{i=1}^{M} t_i, \qquad t_i \in [T, 2T]$$

Proof Let $k = \lfloor r/(2T) \rfloor$ and $q = r - 2Tk$. Now, $2T > q \geq 0$. If $q = 0$, the decomposition is $t_i = 2T$, $i = 1, \ldots, k$. If $2T > q \geq T$, the decomposition is $t_i = 2T$, $i = 1, \ldots, k$; $t_{k+1} = q$; the first equation not applying if $k = 0$. Finally, if $T > q > 0$, then (we must have $k \geq 1$ since $r \geq T$) and $2T > q + T > T$, so the decomposition is $t_i = 2T$, $i = 1, \ldots, k-1$; $t_k = T$; $t_{k+1} = q$; the first equation not applying if $k = 1$. □

Now, we see that for each i, if it is active for a time $r \geq T$, we can write the solution in that interval as $\phi_r^i(x) = (\circ_{j=1}^M \phi_{t_j}^i)(x)$, where ϕ_t^i is the fundamental solution for F_i acting for time t. Thus the switching sequence can be converted to an iterated composition of maps indexed by the compact set I.

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