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Differential Invariant Signatures and Flows in Computer Vision: A Symmetry Group Approach

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Abstract

Computer vision deals with image understanding at various levels. At the low level, it addresses issues such us planar shape recognition and analysis. Some classical results on differential invariants associated to planar curves are relevant to planar object recognition under different views and partial occlusion, and recent results concerning the evolution of planar shapes under curvature controlled diffusion have found applications in geometric shape decomposition, smoothing, and analysis, as well as in other image processing applications. In this work we first give a modern approach to the theory of differential invariants, describing concepts like Lie theory, jets, and prolongations. Based on this and the theory of symmetry groups, we present a high level way of defining invariant geometric flows for a given Lie group. We then analyze in detail different subgroups of the projective group, which are of special interest for computer vision. We classify the corresponding invariant flows and show that the geometric heat flow is the simplest possible one. This uniqueness result, together with previously reported results which we review in this paper, confirms the importance of this class of flows.

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1 Introduction

Invariant theory has recently become a major topic of study in computer vision (see [47] and references therein). Indeed, since the same object may be seen from a number of points of view, one is motivated to look for shape invariants under various transformations.

Indeed, the problem of recognizing and locating a partially visible planar object, whose shape underwent a geometric viewing transformation, often arises in machine vision tasks. Attempts to address such shape recognition problems raise the question of invariants under viewing transformations [7, 22, 47].

Work in model based shape analysis and recognition has already resulted in many useful products, such as optical character recognizers, handwriting recognizer interfaces to computers, printed-circuit board inspection systems and quality control devices. In spite of such successes many low-level problems remain to be addressed. Efficient ways for analyzing, recognizing and understanding planar shapes, when they do not come from a well-defined and documented catalogue of shapes, when they are distorted by a geometric viewing transformation, such as perspective projection or when they are partially occluded, must still be developed.

Another topic that has been receiving much attention from the image analysis community is the theory of scale-spaces or multiscale representation. This was introduced by Witkin in [69] and developed after that by several authors in different frameworks [2, 5, 9, 14, 20, 21, 32, 33, 36, 37, 38, 39, 41, 42, 45, 46, 55, 56, 59, 70, 71]. Initially, most of the work was devoted to linear scale-spaces derived via linear filtering. In the last years, a number of non-linear and geometric scale-spaces have been investigated as well.

The combination of invariant theory with geometric multiscale analysis was first investigated in [57, 58, 59]. There, the authors introduced an affine invariant geometric scale-space, and extended part of the work to other groups as well in [61, 62, 63]. Related work was also carried out in [1, 11]. As we will see in future sections, this kind of multiscale analysis replaces for some applications the originally used linear ones. The obtained representations allows for example to compute invariant signatures at different scales and in a robust way. These flows are already being used with satisfactory results in different applications [21, 56, 59, 60]

In this paper, we would like first of all to set out the basic theory of differential invariants for computer vision. It will therefore have a tutorial nature. We will sketch enough of the language of differential geometry in order to make precise the notion of infinitesimal invariance. While the theory may be stated in classical terms (it was developed after all by Sophus Lie in the previous century), we believe that there is a strong advantage to working with the more modern machinery if only to avoid the classical proliferation of multi-indices. Moreover, this will give the interested reader a guide to the modern literature on the subject. Other approaches to the formulation of differential invariants, as those based on Cartan moving frames, can be found in [8, 18, 19, 29, 49], as well as in some of the papers in [47].

Our main application of this theory, will be to the new theory of geometric-invariant scale-spaces, based on invariant geometric diffusion equations. These give geometric multi-resolution representations of shape which are invariant to a number of the typical viewing transformations in vision: Euclidean, affine, similarity, projective. The theory of differential invariants allows a unification of all these scale-spaces. Moreover, using this theory, we classify the flows and show that the geometric heat flows are the simplest possible amongst all invariant equations.

We now briefly summarize the contents of this paper. In Section 2, we give an outline of a modern treatment of the theory of differential invariants. In particular, we discuss manifolds, vector fields, Lie groups and Lie algebras and their actions on spaces, and the relevant notions of jets, prolongations, and symmetry groups. These will allow us to give a rigorous definition of "differential invariant." The reader familiar with this topic, or interested in having a less formal presentation of the invariant geometric flows, can skip this section in a first reading, since the following sections are almost self contained. In Section 3, we give the theory of invariant diffusion equations as applied to the affine, Euclidean, projective, and similarity groups. In particular, using the invariant theory developed in Section 2, we prove that for any of the preceding subgroups of the projective group $\mathrm{SL}(\mathbf{R}, 3)$, say G, the flows we have defined give the unique G-invariant evolution equation of lowest order (up to constant factor). See Theorem 8 and Corollary 1 below. Next in Section 4, we show how our flows may be made area or length preserving which effectively solves the problem of shrinkage in computer vision. Finally, in Section 5, we give some concluding remarks.

2 Basic Invariant Theory

In this section, we review the classical theory of differential invariants. In order to do this in a rigorous manner, we first sketch some relevant facts from differential geometry and the theory of Lie groups. The material here is based on the books by Olver [49, 50] to which we refer the interested reader for all the details. See also the classical work of Sophus Lie on the theory of differential invariants [40] as well as [8, 19, 49, 50, 67].

We will assume a certain mathematical background, i.e., the reader should be familiar with the basic definitions of "manifold" and "smooth function." Accordingly, all the manifolds and mappings we consider below are C^∞. (This type of foundational material may be found in [30, 67].) We should add that in this section no proofs will be given, and the results will just be stated.

2.1 Vector Fields and One-Forms

Since we will be considering the theory of differential invariants, we will first review the infinitesimal (differential) structure of manifolds. Accordingly, a *tangent vector* to a manifold M at a point $x \in M$ is geometrically given as the tangent to a (smooth) curve passing through x. The collection of all such tangent vectors gives the tangent space $TM|_x$ to M at x, which is a vector space of the same dimension m as M. In local coordinates, a curve is parametrized by $x = \phi(t) = (\phi^1(t), \ldots \phi^m(t))$, and has tangent vector $\mathbf{v} = \xi^1 \frac{\partial}{\partial x^1} + \cdots + \xi^m \frac{\partial}{\partial x^m}$ at $x = \phi(t)$, with components $\xi^i = \frac{d\phi^i}{dt}$ given by the components of the derivative $\phi'(t)$. Here, the tangent vectors to the coordinate axes are denoted by $\frac{\partial}{\partial x^i} = \partial_{x^i}$, and form a basis for the tangent space $TM|_x$. If $f : M \to \mathbf{R}$ is any smooth function, then its directional derivative along the curve is

$$\frac{d}{dt} f[\phi(t)] = \mathbf{v}(f)(\phi(t)) = \sum_{i=1}^{m} \xi^i(\phi(t)) \frac{\partial f}{\partial x^i}(\phi(t)),$$

which provides one motivation for using a derivational notation for tangent vectors. The tangent spaces are patched together to form the tangent bundle $TM = \bigcup_{x \in M} TM|_x$ of the manifold, which is an m-dimensional vector bundle over the m-dimensional manifold M. A *vector field* $\mathbf{v}$ is a smoothly (or analytically) varying assignment of tangent vector $\mathbf{v}|_x \in TM|_x$. In local coordinates, a vector field has the form

$$\mathbf{v} = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i},$$

where the coefficients $\xi^i(x)$ are smooth (analytic) functions.

A parametrized curve $\phi : \mathbf{R} \to M$ is called an *integral curve* of the vector field $\mathbf{v}$ if its tangent vector agrees with the vector field $\mathbf{v}$ at each point; this requires that $x = \phi(t)$ satisfy the first order system of ordinary differential equations

$$\frac{dx^i}{dt} = \xi^i(t), \qquad 1 \leq i \leq m.$$

Standard existence and uniqueness theorems for systems of ordinary differential equations imply that through each $x \in M$ there passes a unique, maximal integral curve. We use the notation $\exp(t\mathbf{v})x$ to denote the maximal integral curve passing through $x = \exp(0\mathbf{v})x$ at $t = 0$, which may or may not be defined for all t. The family of (locally defined) maps $\exp(t\mathbf{v})$ is called the *flow* generated by the vector field $\mathbf{v}$, and obeys the usual exponential rules:

$$\begin{aligned} \exp(t\mathbf{v})\exp(s\mathbf{v})x &= \exp((t+s)\mathbf{v})x, \quad t, s \in \mathbf{R}, \\ \exp(0\mathbf{v})x &= x, \\ \exp(-t\mathbf{v})x &= \exp(t\mathbf{v})^{-1}x, \end{aligned}$$

the equations holding where defined. Conversely, given a flow obeying the latter equalities, we can reconstruct a generating vector field by differentiation:

$$\mathbf{v}|_x = \frac{d}{dt}\exp(t\mathbf{v})|_{t=0}\, x, \qquad x \in M.$$

Applying the vector field $\mathbf{v}$ to a function $f : M \to \mathbf{R}$ determines the infinitesimal change in f under the flow induced by $\mathbf{v}$:

$$\mathbf{v}(f) = \sum_{i=1}^{n} \xi^i(x)\frac{\partial f}{\partial x^i} = \left.\frac{d}{dt} f(\exp(t\mathbf{v})x)\right|_{t=0},$$

so that

$$f(\exp(t\mathbf{v})x) = f(x) + t\mathbf{v}(f)(x) + \frac{1}{2}t^2\mathbf{v}(\mathbf{v}(f)) + \dots.$$

Next given a (smooth) mapping $F : M \to N$, we define the *differential* $dF : TM|_x \to TN|_{F(x)}$ by

$$[(dF)(\mathbf{v})](f)(F(x)) := \mathbf{v}(f \circ F)(x),$$

where $f : N \to \mathbf{R}$ is a smooth function, $\mathbf{v}$ is a vector field.

In general, given a point $x \in M$, a *one-form at* x is a real-valued linear map on the tangent space

$$\omega : TM|_x \to \mathbf{R}.$$

In local coordinates $x = (x^1, \dots, x^m)$, the differentials dx^i are characterized by $dx^i(\partial_{x^j}) = \delta_{ij}$ (the Kronecker delta), where $\partial_{x^1}, \dots, \partial_{x^m}$ denotes the standard basis of $TM|_x$. Then locally,

$$\omega = \sum_{i=1}^{m} h_i(x)dx^i.$$

In particular, for $f : M \to \mathbf{R}$, we get the one-form df given by its differential, so

$$df(\mathbf{v}) := \mathbf{v}(f).$$

The (vector space) of one-forms at x is denoted by $T^*M|_x$ and is called the *cotangent space.* (It can be regarded as the dual space of $TM|_x$.) As for the tangent bundle, the cotangent spaces can be patched together to form the cotangent bundle T^*M over M.

2.2 Lie Groups

In this section, we collect together the basic necessary facts from the theory of Lie groups which will be used below. Recall first that a *group* is a set, together with an associative multiplication operation. The group must also contain an identity element, denoted e and each group element g has an inverse g^{-1} satisfying $g \cdot g^{-1} = g^{-1} \cdot g = e$. Historically, it was Galois who made the fundamental observation that the set of symmetries of an object forms a group (this was in his work on the roots of polynomials). However, the groups of Galois were discrete; in this paper we study the continuous groups first investigated by Sophus Lie.

Definition. A *Lie group* is a group G which also carries the structure of a smooth manifold so that the operations of group multiplication $(g, h) \mapsto g \cdot h$ and inversion $g \mapsto g^{-1}$ are smooth maps.

Example. The basic example of a real Lie group is the general linear group $\mathrm{GL}(\mathbf{R}, n)$ consisting of all real invertible $n \times n$ matrices with matrix multiplication as the group operation; it is an n^2-dimensional manifold, the structure arising simply because it is an open subset (namely, where the determinant is nonzero) of the space of all $n \times n$ matrices which is itself isomorphic to $\mathbf{R}^{n^2}$.

A subset $H \subset G$ of a group is a subgroup if and only if it is closed under multiplication and inversion; if G is a Lie group, then a subgroup H is a *Lie subgroup* if it is also a submanifold. Most Lie groups can be realized as Lie subgroups of $\mathrm{GL}(\mathbf{R}, n)$; these are the so-called "matrix Lie groups", and, in this paper, we will assume that all Lie groups are of this type. One can also define a notion of *local Lie group* in the obvious way (see e.g., [30]).

Example. We list here some of the key "classical groups". The *special linear group* $\mathrm{SL}(\mathbf{R}, n) = \{A \in \mathrm{GL}(\mathbf{R}, n) : \det A = 1\}$ is the group of volume-preserving linear transformations. The group is connected and has dimension $n^2 - 1$. The *orthogonal group* $\mathrm{O}(n) = \{A \in \mathrm{GL}(\mathbf{R}, n) : A^T A = I\}$ is the group of norm-preserving linear transformations — rotations and reflections — and has two connected components. The *special orthogonal group* $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(\mathbf{R}, n)$ consisting of just the rotations is the component containing the identity. This is also called the *rotation group.*

2.2.1 Transformation Groups

In many cases in vision (and physical) problems, groups are presented to us as a family of transformations acting on a space. In the case of Lie groups, the most natural setting is as groups of transformations acting smoothly on a manifold. More precisely, we have the following:

Definition. Let M be a smooth manifold. A *group of transformations* acting on M is given by a Lie group G and smooth map $\Phi : G \times M \to M$, denoted by $\Phi(g, x) = g \cdot x$, which satisfies

$$e \cdot x = x, g \cdot (h \cdot x) = (g \cdot h) \cdot x, \text{for all } x \in M,\ g \in G.$$

One can also define in the obvious way the notion of a *local Lie group action.*

Example. The key example is the usual linear action of the group $\mathrm{GL}(\mathbf{R}, n)$ of $n \times n$ matrices acting by matrix multiplication on column vectors $x \in \mathbf{R}^n$. This action includes linear actions (representations) of the subgroups of $\mathrm{GL}(\mathbf{R}, n)$ on $\mathbf{R}^n$. Since linear transformations map lines to lines, there is an induced action of $\mathrm{GL}(\mathbf{R}, n)$ on the projective space $\mathbf{RP}^{n-1}$. The diagonal matrices λI (I denotes the identity matrix) act trivially, so the action reduces effectively to one of the projective linear group $\mathrm{PSL}(\mathbf{R}, n) = \mathrm{GL}(\mathbf{R}, n)/\{\lambda I\}$. If n is odd, $\mathrm{PSL}(\mathbf{R}, n) = \mathrm{SL}(\mathbf{R}, n)$ can be identified with the special linear group, while for n even, since $-I \in \mathrm{SL}(\mathbf{R}, n)$ has the same effect as the identity, the projective group is a quotient $\mathrm{PSL}(\mathbf{R}, n) = \mathrm{SL}(\mathbf{R}, n)/\{\pm I\}$.

In vision, of particular importance is the case of $\mathrm{GL}(\mathbf{R}, 2)$, so we discuss this in some detail. The linear action of $\mathrm{GL}(\mathbf{R}, 2)$ on $\mathbf{R}^2$ is given by

$$(x, y) \longmapsto (\alpha x + \beta y, \gamma x + \delta y), \qquad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}(\mathbf{R}, 2).$$

As above, we can identify the projective line $\mathbf{RP}^1$ with a circle S^1. If we use the projective coordinate $p = x/y$, the induced action is given by the *linear fractional* or Möbius transformations:

$$p \longmapsto \frac{\alpha p + \beta}{\gamma p + \delta}, \qquad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}(\mathbf{R}, 2).$$

In this coordinate chart, the x–axis $\{(x, 0)\}$ in $\mathbf{R}^2$ is identified with the point $p = \infty$ in $\mathbf{RP}^1$, and the linear fractional transformations have a well-defined to include the point at ∞.

Example. Let $\mathbf{v}$ be a vector field on the manifold M. Then the flow $\exp(t\mathbf{v})$ is a (local) action of the one-parameter group $\mathbf{R}$, parametrized by the "time" t, on the manifold M.

Example. In general, if G is a Lie group which acts as a group of transformations on another Lie group H, we define the *semi-direct product* $G \times_s H$ to be the Lie group which, as a manifold just looks like the Cartesian product $G \times H$, but whose multiplication is given by $(g, h) \cdot (\tilde{g}, \tilde{h}) = (g \cdot \tilde{g}, h \cdot (g \cdot \tilde{h}))$, and hence is different from the Cartesian product Lie group, which has multiplication $(g, h) \cdot (\tilde{g}, \tilde{h}) = (g \cdot \tilde{g}, h \cdot \tilde{h})$.

The *(full) affine group* $\mathrm{A}(n)$ is defined as the group of affine transformations $x \mapsto Ax + a$ in $\mathbf{R}^n$, parametrized by a pair (A, a) consisting of an invertible matrix A and a vector $a \in \mathbf{R}^n$. The group multiplication law is given by $(A, a) \cdot (B, b) = (AB, a + Ab)$, and hence can be identified with the semi-direct product $\mathrm{GL}(\mathbf{R}, n) \times_s \mathbf{R}^n$. The affine group can be realized as a subgroup of $\mathrm{GL}(\mathbf{R}, n + 1)$ by identifying the pair (A, a) with the $(n + 1) \times (n + 1)$ matrix

$$\begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}.$$

Let $\mathrm{GL}_+(\mathbf{R}, n)$ denote the subgroup of $\mathrm{GL}(\mathbf{R}, n)$ with positive determinant. Then the *group of proper affine motions* of $\mathbf{R}^n$ is the semidirect product of $\mathrm{GL}_+(\mathbf{R}, n)$ and the translations. Similarly, the *special affine group* is given by the semidirect product of $\mathrm{SL}(\mathbf{R}, n)$ and $\mathbf{R}^n$.

We may also define the *Euclidean group* $\mathrm{E}(n)$ as the semi-direct product of $\mathrm{O}(n)$ and translations in $\mathbf{R}^n$, and the *group of Euclidean motions* as the semidirect product of the rotation group $\mathrm{SO}(n)$ and $\mathbf{R}^n$. The *similarity group* in $\mathbf{R}^n$, $\mathrm{Sm}(n)$, is generated by rotations, translations, and isotropic scalings.

In the sequel, we will usually not differentiate between the real affine group and the group of proper affine motions, and the Euclidean group and the group of Euclidean motions.

Example. In what follows, we will consider all the above subgroups for $n = 2$, i.e., acting on the the plane $\mathbf{R}^2$. In this case, they are all subgroups of $\mathrm{SL}(\mathbf{R}, 3)$, the so-called *group of projective transformations on* $\mathbf{R}^2$. More precisely, $\mathrm{SL}(\mathbf{R}, 3)$ acts on $\mathbf{R}^2$ as follows: for $A \in SL(\mathbf{R}, 3)$

$$(\tilde{x}, \tilde{y}) = \left(\frac{a_{11}x + a_{21}y + a_{31}}{a_{13}x + a_{23}y + a_{33}}, \frac{a_{12}x + a_{22}y + a_{32}}{a_{13}x + a_{23}y + a_{33}} \right)$$

where

$$A = [a_{ij}]_{1 \le i,j \le 3}.$$

2.2.2 Representations

Linear actions of Lie groups, that is, "representations" of the group, play an essential role in applications. Formally, a *representation* of a group G is defined by a group homomorphism $\rho : G \to \mathrm{GL}(V)$ from G to the space of invertible linear operators on a vector space V. This means that ρ satisfies the properties: $\rho(e) = I$, $\rho(g \cdot h) = \rho(g)\rho(h)$, $\rho(g^{-1}) = \rho(g)^{-1}$.

One important method to turn a nonlinear group action into a linear representation is to look at its induced action on functions on the manifold. Given any action of a Lie group G on a manifold M, there is a naturally induced representation of G on the space $\mathcal{F} = \mathcal{F}(M)$ of real-valued functions $F : M \to \mathbf{R}$, which maps the function F to $\bar{F} := g \cdot F$ defined by

$$\bar{F}(\bar{x}) = F(g^{-1} \cdot \bar{x}),$$

or equivalently,

$$(g \cdot F)(g \cdot x) = F(x).$$

The introduction of the inverse g^{-1} in this formula ensures that the action of G on $\mathcal{F}$ is a group homomorphism: $g \cdot (h \cdot F) = (g \cdot h) \cdot F$ for all $g, h \in G$, $F \in \mathcal{F}$.

The representation of G on the function space $\mathcal{F}$ will usually decompose into a wide variety of important subrepresentations, e.g., representations on spaces of polynomial functions, representations on spaces of smooth (C^∞) functions, or L^2 functions, etc. In general, representations of a group containing (nontrivial) subrepresentations are called reducible. An *irreducible representation*, then, is a representation $\rho : G \mapsto \mathrm{GL}(V)$ which contains no (non-trivial) sub-representations, i.e., there are no subspaces $W \subset V$ which are invariant under the representation, $\rho(g)W \subset W$ for all $g \in G$, other than $W = \{0\}$ and $W = V$. The classification of irreducible representations of Lie groups is a major subject of research in this century.

Example. Consider the action of the group $\mathrm{GL}(\mathbf{R}, 2)$ on the space $\mathbf{R}^2$ acting via matrix multiplication. This induces a representation on the space of functions

$$\bar{F}(\bar{x}, \bar{y}) = \bar{F}(\alpha x + \beta y, \gamma x + \delta y) = F(x, y), \text{ where } A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}(\mathbf{R}, 2).$$

Note that is F is a homogeneous polynomial of degree n, so is $\bar{F}$, so that this representation includes the finite-dimensional irreducible representations $\rho_{n,0}$ of $\mathrm{GL}(\mathbf{R}, 2)$ on $\mathcal{P}^{(n)}$, the space of homogeneous polynomials of degree n. For example, on the space $\mathcal{P}^{(1)}$ of linear polynomials, the coefficients of general linear polynomial $F(x, y) = ax + by$ will transform according to

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}(\mathbf{R}, 2),$$

so that the representation $\rho_{1,0}(A) = A^{-T}$ can be identified with the inverse transpose representation.

2.2.3 Orbits and Invariant Functions

Let G be a group of transformations acting on the manifold M. A subset $S \subset M$ is called G–*invariant* if it is unchanged by the group transformations, meaning $g \cdot x \in S$ whenever $g \in G$ and $x \in S$ (provided $g \cdot x$ is defined if the action is only local). An *orbit* of the transformation group is a minimal (nonempty) invariant subset. For a global action, the orbit through a point $x \in M$ is just the set of all images of x under arbitrary group transformations $\mathcal{O}_x = \{g \cdot x : g \in G\}$.

Clearly, a subset $S \subset M$ is G-invariant if and only if it is the union of orbits. If G is connected. its orbits are connected. The action is called *transitive* if there is only one orbit, so (assuming the group acts globally), for every $x, y \in M$ there exists at least one $g \in G$ such that $g \cdot x = y$.

A group action is called *semi-regular* if its orbits are all submanifolds having the same dimension. The action is called *regular* if, in addition, for any $x \in M$, there exist arbitrarily small neighborhoods U of x with the property that each orbit intersects U in a pathwise connected subset. In particular, each orbit is a regular submanifold, but this condition is not sufficient to guarantee regularity; for instance, the one-parameter group $(r, \theta) \mapsto (e^t(r-1)+1, \theta + t)$, written in polar coordinates, is semi-regular on $\mathbf{R}^2 \setminus \{0\}$, and has regular orbits, but is not regular on the unit circle.

Example. Consider the two-dimensional torus $T = S^1 \times S^1$, with angular coordinates (θ, φ), $0 \le \theta, \varphi < 2\pi$. Let α be a nonzero real number, and consider the one-parameter group action $(\theta, \varphi) \mapsto (\theta + t, \varphi + \alpha t) \bmod 2\pi$, $t \in \mathbf{R}$. If α/π is a rational number, then the orbits of this action are closed curves, diffeomorphic to the circle S^1, and the action is regular. On the other hand, if α/π is an irrational number, then the orbits of this action never close, and, in fact, each orbit is a dense subset of T. Therefore, the action in the latter case is semi-regular, but not regular.

The *quotient space* M/G is defined as the space of orbits of the group action, endowed with a topology induced from that of M. As the irrational flow on the torus makes clear, the quotient space can be a very complicated topological space. However, regularity of the group action will ensure that the quotient space is a smooth manifold.

Given a group of transformations acting on a manifold M, by a *canonical form* of an element $x \in M$ we just mean a distinguished, simple representative x_0 of the orbit containing x. Of course, there is not necessarily a uniquely determined canonical form, and some choice, usually based on one's æsthetic sense of "simplicity", must be employed for such forms.

Now orbits and canonical forms of group actions are characterized by the invariants, which are defined as real-valued functions whose values are unaffected by the group transformations.

Definition An *invariant* for the transformation group G is a function $I : M \to \mathbf{R}$ which satisfies $I(g \cdot x) = I(x)$ for all $g \in G$, $x \in M$.

Proposition 1 *Let $I : M \to \mathbf{R}$ be a function. Then the following three conditions are equivalent:*

1. *I is a G-invariant function.*

2. *I is constant on the orbits of G.*

3. *The level sets $\{I(x) = c\}$ of I are G-invariant subsets of M.*

For example, in the case of the orthogonal group $O(n)$ acting on $\mathbf{R}^n$, the orbits are spheres $|x| =$ constant, and hence any orthogonal invariant is a function of the radius $I = F(r)$, $r = |x|$. Invariants are essentially classified by their "quotient representatives": every invariant of the group action induces a function $\tilde{I} : M/G \to \mathbf{R}$ on the quotient space, and conversely. The canonical form x_0 of any element $x \in M$ must have the same invariants: $I(x_0) = I(x)$; this condition is also sufficient if there are enough invariants to distinguish the orbits, i.e., x and y lie in the same orbit if and only if $I(x) = I(y)$ for every invariant I which according to the next theorem is the case for regular group actions.

An important problem is the determination of *all* the invariants of a group of transformations. Note that if $I_1(x), \ldots, I_k(x)$ are invariant functions, and $\Phi(y_1, \ldots, y_k)$ is any function, then $\hat{I} = \Phi(I_1(x), \ldots, I_k(x))$ is also invariant. Therefore, to classify invariants, we need only determine all different functionally independent invariants. Many times globally defined invariants are difficult to find, and so one must be satisfied with the description of locally defined invariants of a group action.

Theorem 1 *Let G be a Lie group acting regularly on the m-dimensional manifold M with r-dimensional orbits. Then, locally, near any $x \in M$ there exist exactly $m-r$ functionally independent invariants $I_1, \ldots, I_{m-r}$ with the property that any other invariant can be written as a function of the fundamental invariants: $I = \Phi(I_1, \ldots, I_{m-r})$. Moreover, two points x and y in the coordinate chart lie in the same orbit of G if and only if the invariants all have the same value, $I_\nu(x) = I_\nu(y)$, $\nu = 1, \ldots, m-r$.*

This theorem provides a complete answer to the question of local invariants of group actions. Global and irregular considerations are more delicate; for example, consider the one-parameter isotropy group $(x, y) \mapsto (\lambda x, \lambda y)$, $\lambda \in \mathbf{R}^+$. Locally, away from the origin, x/y or y/x any function thereof (e.g., $\theta = \tan^{-1}(y/x)$) provides the only invariant. However, if we include the origin, then there are no non-constant invariants. On the other hand, the scaling group

$$(x, y) \mapsto (\lambda x, \lambda^{-1} y), \quad \lambda \neq 0,$$

has the global invariant xy. In general, if G acts transitively on the manifold M, then the only invariants are constants, which are completely trivial invariants. More generally, if G acts transitively on a dense subset $M_0 \subset M$, then the only *continuous* invariants are constants. For example, the only continuous invariants of the irrational flow on the torus are the constants, since every orbit is dense in this case. Similarly, the only continuous invariants of the standard action of $\mathrm{GL}(\mathbf{R}, n)$ on $\mathbf{R}^n$ are the constant functions, since the group acts transitively on $\mathbf{R}^n \setminus \{0\}$. (A discontinuous invariant is provided by the function which is 1 at the origin and 0 elsewhere.)

2.2.4 Lie Algebras

Besides invariant functions, there are other important invariant objects associated with a transformation group, including vector fields, differential forms, differential operators, etc. We begin by considering the case of an invariant vector field, which will, in the particular case of a group acting on itself by right (or left) multiplication, lead to the crucially important concept of a Lie algebra or "infinitesimal" Lie group. A basic feature of (connected) Lie groups is the ability work infinitesimally, thereby effectively linearizing complicated invariance criteria.

Definition. Let G act on the manifold M. A vector field $\mathbf{v}$ on M is called *G–invariant* if it is unchanged by the action of any group element: $dg(\mathbf{v}|_x) = \mathbf{v}|_{g \cdot x}$ for all $g \in G$, $x \in M$.

In particular, if we consider the action of G on itself by right multiplication, the space of all invariant vector fields forms the Lie algebra of the group. Given $g \in G$, let $R_g : h \mapsto h \cdot g$ denote the associated right multiplication map. A vector field $\mathbf{v}$ on G is *right-invariant* if it satisfies $dR_g(\mathbf{v}) = \mathbf{v}$ for all $g \in G$.

Definition. The *Lie algebra* $\mathcal{G}$ of a Lie group G is the space of all right-invariant vector fields.

Every right invariant vector field $\mathbf{v}$ is uniquely determined by its value at the identity e, because $\mathbf{v}|_g = dR_g(\mathbf{v}|_e)$. Therefore, we can identify $\mathcal{G}$ with $TG|_e$, the tangent space to the manifold G at the identity, and hence $\mathcal{G}$ is a finite-dimensional vector space having the same dimension as G.

The Lie algebra associated with a Lie group comes equipped with a natural multiplication, defined by the *Lie bracket* of vector fields given by:

$$[\mathbf{v}, \mathbf{w}](f) := \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f)).$$

By the invariance of the Lie bracket under diffeomorphisms, if both $\mathbf{v}$ and $\mathbf{w}$ are right invariant, so is $[\mathbf{v}, \mathbf{w}]$. Note that the bracket satisfies the *Jacobi identity*

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] = 0.$$

The basic properties of the Lie bracket translate into the defining properties of an (abstract) Lie algebra.

Definition. A *Lie algebra* $\mathcal{G}$ is a vector space equipped with a bracket operation $[\,\cdot\,,\,\cdot\,] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ which is bilinear, anti-symmetric, and satisfies the Jacobi identity.

Theorem 2 *Let G be a connected Lie group with Lie algebra $\mathcal{G}$. Every group element can be written as a product of exponentials: $g = \exp(\mathbf{v}_1)\exp(\mathbf{v}_2)\cdots\exp(\mathbf{v}_k)$, for $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathcal{G}$.*

Example. The Lie algebra $\mathcal{GL}_n$ of $\mathrm{GL}(\mathbf{R}, n)$ can be identified with the space of all $n \times n$ matrices. Coordinates on $\mathrm{GL}(\mathbf{R}, n)$ are given by the matrix entries $X = (x_{ij})$. The right-invariant vector field associated with a matrix $A \in \mathcal{GL}_n$ is given by $\mathbf{v}_A = \sum_{i,j,k} a_{ij} x_{jk} \partial_{x^k}$. The exponential map is the usual matrix exponential $\exp(t\mathbf{v}_A) = e^{tA}$. The Lie bracket of two such vector fields is found to be $[\mathbf{v}_A, \mathbf{v}_B] = \mathbf{v}_C$, where $C = BA - AB$. Thus the Lie bracket on $\mathcal{GL}_n$ is identified with the *negative* of the matrix commutator $[A, B] = AB - BA$.

The formula $\det\exp(tA) = \exp(t\ \mathrm{tr}A)$ proves that the Lie algebra $\mathcal{SL}_n$ of the unimodular subgroup $\mathrm{SL}(\mathbf{R}, n)$ consists of all matrices with trace 0. The subgroups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ have the same Lie algebra, $\mathcal{SO}(n)$, consisting of all skew-symmetric $n \times n$ matrices.

Finally, we want to define the key concept of an *invariant one-form*. In order to do this, we will first have to define the *pullback of a one-form*. Let $F : M \to N$ be a smooth mapping of manifolds, and let η denote a one-form in $T^*N|_{y=F(x)}$. Then $F^*(\eta) \in T^*M|_x$ is the one-form given by

$$F^*(\eta)(\mathbf{v}) := \eta(dF(\mathbf{v})),$$

where $\mathbf{v} \in TM|_x$.

Definition. Let G act on the manifold M. A one-form ω on M is called *G–invariant* if it is unchanged by the pull-back action of any group element

$$g^*(\omega|_{g\cdot x}) = \omega|_x, \quad \forall g \in G, \ \ x \in M.$$

Dual to the right-invariant vector fields forming the Lie algebra of a Lie group are the right-invariant one-forms known as the *Maurer-Cartan forms*. See [50, 67] for details.

The following result follows from the definitions:

Lemma 1 *Let G be a transformation group acting on M.. Then:*

1. *If I is an invariant function, then dI is an invariant one-form.*

2. *If I is an invariant function, and ω an invariant one-form, then $I\omega$ is an invariant one-form.*

2.2.5 Infinitesimal Group Actions

Just as a one-parameter group of transformations is generated as the flow of a vector field, so a general Lie group of transformations G acting on the manifold M will be generated by a set of vector fields on M, known as the *infinitesimal generators* of the group action, whose flows coincide with the action of the corresponding one-parameter subgroups of G. More precisely, if $\mathbf{v}$ generates the one-parameter subgroup $\{\exp(t\mathbf{v}) : t \in \mathbf{R}\} \subset G$, then we identify $\mathbf{v}$ with the infinitesimal generator $\widehat{\mathbf{v}}$ of the one-parameter group of transformations (or flow) $x \mapsto \exp(t\mathbf{v}) \cdot x$. Note that the infinitesimal generators of the group action are found by differentiating the various one-parameter subgroups:

$$\widehat{\mathbf{v}}|_x = \frac{d}{dt} \exp(t\mathbf{v})|_{t=0}\, x, \qquad x \in M, \quad \mathbf{v} \in \mathcal{G}. \tag{1}$$

If $\Phi_x : G \to M$ is given by $\Phi_x(g) = g \cdot x$ (where defined), so $\widehat{\mathbf{v}}|_x = d\Phi_x(\mathbf{v}|_e)$, and hence $d\Phi_x(\mathbf{v}|_g) = \widehat{\mathbf{v}}|_{g \cdot x}$. Therefore, resulting vector fields satisfy the same commutation relations as the Lie algebra of G, forming a finite-dimensional Lie algebra of vector fields on the manifold M isomorphic to the Lie algebra of G. Conversely, given a finite-dimensional Lie algebra of vector fields on a manifold M, we can reconstruct a (local) action of the corresponding Lie group via the exponentiation process.

Theorem 3 *If G is a Lie group acting on a manifold M, then its infinitesimal generators form a Lie algebra of vector fields on M isomorphic to the Lie algebra $\mathcal{G}$ of G. Conversely, any Lie algebra of vector fields on M which is isomorphic to $\mathcal{G}$ will generate a local action of the group G on M.*

Consequently, for a fixed group action, the associated infinitesimal generators will, somewhat imprecisely, be identified with the Lie algebra $\mathcal{G}$ itself, so that we will not distinguish between an element $\mathbf{v} \in \mathcal{G}$ and the associated infinitesimal generator of the action of G, which we also denote as $\mathbf{v}$ from now on.

Given a group action of a Lie group G, the infinitesimal generators also determine the tangent space to, and hence the dimension of the orbits.

Proposition 2 *Let G be a Lie group with Lie algebra $\mathcal{G}$ acting on a manifold M. Then, for each $x \in M$, the tangent space to the orbit through x is the subspace $\mathcal{G}|_x \subset TM|_x$ spanned by the infinitesimal generators $\mathbf{v}|_x$, $\mathbf{v} \in \mathcal{G}$. In particular, the dimension of the orbit equals the dimension of $\mathcal{G}|_x$.*

2.2.6 Infinitesimal Invariance

As alluded to above, the invariants of a connected Lie group of transformations can be effectively computed using purely infinitesimal techniques. Indeed, the practical applications of Lie groups ultimately rely on this basic observation.

Theorem 4 *Let G be a connected group of transformations acting on a manifold M. A function $F : M \to \mathbf{R}$ is invariant under G if and only if*

$$\mathbf{v}[F] = 0, \tag{2}$$

for all $x \in M$ and every infinitesimal generator $\mathbf{v} \in \mathcal{G}$ of G.

Thus, according to (4) the invariants of a one-parameter group with infinitesimal generator $\mathbf{v} = \sum_i \xi^i(x)\partial_{x^i}$ satisfy the first order, linear, homogeneous partial differential equation

$$\sum_{i=1}^{m} \xi^i(x)\frac{\partial F}{\partial x^i} = 0. \tag{3}$$

The solutions of (3) can be computed by the *method of characteristics*. We replace the partial differential equation by the characteristic system of ordinary differential equations

$$\frac{dx^1}{\xi^1(x)} = \frac{dx^2}{\xi^2(x)} = \cdots = \frac{dx^m}{\xi^m(x)}. \tag{4}$$

The general solution to (4) can be written in the form $I_1(x) = c_1, \ldots, I_{m-1}(x) = c_{m-1}$, where the c_i are constants of integration. It is not hard to prove that the resulting functions $I_1, \ldots, I_{m-1}$ form a complete set of functionally independent invariants of the one-parameter group generated by $\mathbf{v}$.

Example. We consider the (local) one-parameter group generated by the vector field

$$\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1+z^2)\frac{\partial}{\partial z}.$$

The group transformations are

$$(x,y,z) \longmapsto \left(x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon, \frac{\sin\varepsilon + z\cos\varepsilon}{\cos\varepsilon - z\sin\varepsilon}\right)$$

The characteristic system (4) for this vector field is

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1+z^2}.$$

The first equation reduces to a simple separable ordinary differential equation $\frac{dy}{dx} = -x/y$, with general solution $x^2 + y^2 = c_1$, for c_1 a constant of integration; therefore the cylindrical radius $r = \sqrt{x^2+y^2}$ is one invariant. To solve the second characteristic equation, we replace x by $\sqrt{r^2 - y^2}$, and treat r as constant; the solution is $\tan^{-1} z - \sin^{-1}(y/r) = \tan^{-1} z - \tan^{-1}(y/x) = c_2$, where c_2 is a second constant of integration. Therefore $\tan^{-1} z - \tan^{-1}(y/x)$ is a second invariant; a more convenient choice is found by taking the tangent of this invariant, and hence we deduce that $r = \sqrt{x^2+y^2}$, $w = (xz - y)/(yz + x)$ form a complete system of functionally independent invariants, provided $yz + x \neq 0$.

2.2.7 Invariant Equations

In addition to the classification of invariant functions of group actions, it is also important to characterize invariant systems of equations. A group G is called a *symmetry group* of a system of equations

$$F_1(x) = \cdots = F_k(x) = 0, \tag{5}$$

defined on an m-dimensional manifold M, if it maps solutions to other solutions, i.e., if $x \in M$ satisfies the system and $g \in G$ is any group element such that $g \cdot x$ is defined, then we require that $g \cdot x$ is also a solution to the system. Knowledge of a symmetry group of a system of equations allows us to construct new solutions from old ones, a fact that is particularly useful when applying these methods to systems of differential equations; [49, 50]. Let $\mathcal{S}_{\mathcal{F}}$ denote the subvariety defined by the functions $\mathcal{F} = \{F_1, \ldots, F_k\}$, meaning the set of all solutions x to the system (5). (Note that G is a symmetry group of the system if and only if $\mathcal{S}_{\mathcal{F}}$ is a G–invariant subset.) Recall that the system is *regular* if the Jacobian matrix $(\frac{\partial F_i}{\partial x^k})$ has constant rank n in a neighborhood of $\mathcal{S}_{\mathcal{F}}$, which implies (via the implicit function theorem), that the solution set $\mathcal{S}_{\mathcal{F}}$ is a submanifold of dimension $m - n$. In particular, if the rank is maximal, equaling k, on $\mathcal{S}_{\mathcal{F}}$, the system is regular.

Proposition 3 *Let $F_1(x) = \cdots = F_k(x) = 0$ be a regular system of equations. A connected Lie group G is a symmetry group of the system if and only if*

$$\mathbf{v}[F_\nu(x)] = 0, \qquad \textit{whenever} \quad F_1(x) = \cdots = F_k(x) = 0, \qquad 1 \le \nu \le k,$$

for every infinitesimal generator $\mathbf{v} \in \mathcal{G}$ of G.

Example. The equation $x^2 + y^2 = 1$ defines a circle, which is rotationally invariant. To check the infinitesimal condition, we apply the generator $\mathbf{v} = -y\partial_x + x\partial_y$ to the defining function $F(x, y) = x^2 + y^2 - 1$. We find $\mathbf{v}(F) = 0$ everywhere (since F is an invariant). Since dF is nonzero on the circle, the solution set is rotationally invariant.

2.3 Prolongations

In this section, we review the theory of jets and prolongations, to formalize the notion of differential invariants.

2.3.1 Point Transformations

We have reviewed linear actions of Lie groups on functions. While of great importance, such actions are not the most general, and we will have to consider more general nonlinear group actions. Such transformation groups figure prominently in Lie's theory of symmetry groups of differential equations, and appear naturally in the geometrically invariant diffusion equations of computer vision that we consider below. The transformation groups will act on the basic space coordinatized by the independent and dependent variables relevant to the system of differential equations under consideration. Since we want to treat *differential equations*, we must be able to handle the derivatives of the dependent variables on the same footing as the independent and dependent variables themselves. In this section, we describe a suitable geometric space for this purpose — the so-called "jet space." We then discuss how group transformations are "prolonged" so that the derivative coordinates are appropriately acted upon, and, in the case of infinitesimal generators, state the fundamental prolongation formula that explicitly determines the prolonged action.

A general system of (partial) differential equations involves p independent variables $x = (x^1, \ldots, x^p)$, which we can view as local coordinates on the space $X \simeq \mathbf{R}^p$, and q dependent variables $u = (u^1, \ldots, u^q)$, coordinates on $U \simeq \mathbf{R}^q$. The total space will be an open subset $M \subset X \times U \simeq \mathbf{R}^{p+q}$.

A solution to the system of differential equations will be described by a smooth function $u = f(x)$. The graph of a function, $\Gamma_f = \{(x, f(x))\}$, is a p-dimensional submanifold of M which is *transverse*, meaning that it has no vertical tangent directions. A vector field is *vertical* if it is tangent to the vertical fiber $U_{x_0} \equiv \{x_0\} \times U$, so the transversality condition is $T\Gamma_f|_{z_0} \cap TU_{x_0}|_{z_0} = \{0\}$ for each $z_0 = (x_0, f(x_0))$ with x_0 in the domain of f. Conversely, the Implicit Function Theorem implies that any p-dimensional submanifold $\Gamma \subset M$ which is transverse at a point $z_0 = (x_0, u_0) \in \Gamma$ locally, coincides with the graph of a single-valued smooth function $u = f(x)$.

The most basic type of symmetry we will discuss is provided by a (locally defined) smooth, invertible map on the space of independent and dependent variables:

$$(\bar{x}, \bar{u}) = g \cdot (x, u) = (\varphi(x, u), \psi(x, u)). \tag{6}$$

The general type of transformations defined by (6), are often referred to as *point transformations* since they act pointwise on the independent and dependent variables. Point transformations act on functions $u = f(x)$ by pointwise transforming their graphs; in other words if $\Gamma_f = \{(x, f(x))\}$ denotes the graph of f, then the transformed function $\bar{f} = g \cdot f$ will have graph

$$\Gamma_{\bar{f}} = \{(\bar{x}, \bar{f}(\bar{x}))\} = g \cdot \Gamma_f = \{g \cdot x(x, f(x))\}. \tag{7}$$

In general, we can only assert that the transformed graph is another p-dimensional submanifold of M, and so the transformed function will not be well-defined unless $g \cdot \Gamma_f$ is (at least) transverse to the vertical space at each point. This will be guaranteed if the transformation g is sufficiently close to the identity transformation, and the domain of f is compact.

Example. Let

$$g_t \cdot (x, u) = (x \cos t - u \sin t,\ x \sin t + u \cos t),$$

be the one-parameter group of rotations acting on the space $M \simeq \mathbf{R}^2$ consisting of one independent and one dependent variable. Such a rotation transforms a function $u = f(x)$ by rotating its graph; therefore, the transformed graph $g_t \cdot \Gamma_f$ will be the graph of a well-defined function only if the rotation angle t is not too large. The equation for the transformed function $\bar{f} = g_t \cdot f$ is given in implicit form

$$\begin{aligned} \bar{x} &= x \cos t - f(x) \sin t, \\ \bar{u} &= x \sin t + f(x) \cos t, \end{aligned} \tag{8}$$

and $\bar{u} = \bar{f}(\bar{x})$ is found by eliminating x from these two equations. For example, if $u = ax + b$ is affine, then the transformed function is also affine, and given explicitly by

$$\bar{u} = \frac{\sin t + a \cos t}{\cos t - a \sin t} \bar{x} + \frac{b}{\cos t - a \sin t},$$

which is defined provided $\cot t \neq a$, i.e., provided the graph of f has not been rotated to be vertical.

2.3.2 Jets and Prolongations

Since we are interested in symmetries of differential equations, we need to know not only how the group transformations act on the independent and dependent variables, but also how they act on the derivatives of the dependent variables. In the last century, this was done automatically, without worrying about the precise mathematical foundations of the method; in modern times, geometers have defined the "jet space" (or bundle) associated with the space of independent and dependent variables, whose coordinates will represent the derivatives of the dependent variables with respect to the independent variables. This gives a rigorous, cleaner, and more geometric interpretation of this theory.

Given a smooth, scalar-valued function $f(x_1, \ldots, x_p)$ of p independent variables, there are $p_k = \begin{pmatrix} p+k-1 \\ k \end{pmatrix}$ different k–th order partial derivatives

$$\partial_J f(x) = \frac{\partial^k f}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}},$$

indexed by all unordered (symmetric) multi-indices $J = (j_1, \ldots, j_k)$, $1 \le j_k \le p$, of order $k = \#J$. Therefore, if we have q dependent variables $(u^1, \ldots, u^q)$, we require $q_k = qp_k$ different coordinates u_J^α, $1 \le \alpha \le q$, $\#J = k$ to represent all the k-th order derivatives $u_J^\alpha = \partial_J f^\alpha(x)$ of a function $u = f(x)$. For the space $M = X \times U \simeq \mathbf{R}^p \times \mathbf{R}^q$, the n–th jet space $\mathrm{J}^n = \mathrm{J}^n M = X \times U^n$ is the Euclidean space of dimension $p + q \begin{pmatrix} p+n \\ n \end{pmatrix}$, whose coordinates consist of the p independent variables x^i, the q dependent variables u^α, and the derivative coordinates u_J^α, $1 \le \alpha \le q$, of orders $1 \le \#J \le n$. The points in the vertical space $U^{(n)}$ are denoted by $u^{(n)}$, and consist of all the dependent variables and their derivatives up to order n; thus a point in J^n has coordinates $(x, u^{(n)})$.

A smooth function $u = f(x)$ from X to U has n–th *prolongation* $u^{(n)} = \mathrm{pr}^{(n)} f(x)$ (also known as the *n-jet*), which is a function from X to $U^{(n)}$, given by evaluating all the partial derivatives of f up to order n; thus the individual coordinate functions of $\mathrm{pr}^{(n)} f$ are $u_J^\alpha = \partial_J f^\alpha(x)$. Note that the graph of the prolonged function $\mathrm{pr}^{(n)} f$, namely $\Gamma_f^{(n)} = \{(x, \mathrm{pr}^{(n)} f(x))\}$, will be a p-dimensional submanifold of J^n. At a point $x \in X$, two functions have the same n–th order prolongation, and so determine the same point of J^n, if and only if they have n–th order contact, meaning that they and their first n derivatives agree at the point. (This is the same as requiring that they have the same n–th order Taylor polynomial at the point x.) Thus, a more intrinsic way of defining the jet space J^n is to consider it as the set of equivalence classes of smooth functions using the equivalence relation of n–th order contact. If g is a (local) point transformation (6), then g acts on functions by transforming their graphs, and hence also acts on the derivatives of the functions in a natural manner. This allows us to naturally define an induced "prolonged transformation" $(\bar{x}, \bar{u}^{(n)}) = \mathrm{pr}^{(n)} g \cdot (x, u^{(n)})$ on the jet space J^n, given directly by the chain rule. More precisely, for any point $(x_0, u_0^{(n)}) = (x_0, \mathrm{pr}^{(n)} f(x_0)) \in \mathrm{J}^n$, the transformed point $(\bar{x}_0, \bar{u}_0^{(n)}) = \mathrm{pr}^{(n)} g \cdot (x_0, u_0^{(n)}) = (\bar{x}_0, \mathrm{pr}^{(n)} \bar{f}(\bar{x}_0))$ is found by evaluating the derivatives of the transformed function $\bar{f} = g \cdot f$ at the image point $\bar{x}_0$, defined so that $(\bar{x}_0, \bar{u}_0) = (\bar{x}_0, \bar{f}(\bar{x}_0)) = g \cdot (x_0, f(x_0))$. This definition assumes that $\bar{f}$ is smooth at $\bar{x}_0$ — otherwise the prolonged transformation is not defined at $(x_0, u_0^{(n)})$. It is not hard to see that the construction does not depend on the particular function f used to represent the point of J^n; in particular, using the identification of the points in J^n with Taylor polynomials of order n, it suffices to determine how the point transformations act on polynomials of degree at most n in order to compute their prolongation.

Example. For the one-parameter rotation group considered above, the first prolongation $\mathrm{pr}^{(1)} g_t$ will act on the space coordinatized by (x, u, p) where p represents the derivative coordinate u_x. Given a point (x_0, u_0, p_0), we choose the linear polynomial $u = f(x) = p_0(x - x_0) + u_0$ to represent it, so $f(x_0) = u_0$, $f'(x_0) = p_0$. The transformed function is given by

$$\bar{f}(\bar{x}) = \frac{\sin t + p_0 \cos t}{\cos t - p_0 \sin t}\, \bar{x} + \frac{u_0 - p_0 x_0}{\cos t - p_0 \sin t}.$$

Then, by (8), $\bar{x}_0 = x_0 \cos t - u_0 \sin t$, so $\bar{f}(\bar{x}_0) = \bar{u}_0 = x_0 \sin t + u_0 \cos t$, and $\bar{p}_0 = \bar{f}'(\bar{x}_0) =$

$(\sin t + p_0 \cos t)/(\cos t - p_0 \sin t)$, which is defined provided $p_0 \neq \cot t$. Therefore, dropping the 0 subscripts, the prolonged group action is

$$\mathrm{pr}^{(1)} g_t \cdot (x, u, p) = \left(x \cos t - u \sin t, x \sin t + u \cos t, \frac{\sin t + p \cos t}{\cos t - p \sin t} \right), \tag{9}$$

defined for $p \neq \cot t$. Note that even though the original group action is globally defined, the prolonged group action is only locally given.

2.3.3 Total Derivatives

The chain rule computations used to compute prolongations are notationally simplified if we introduce the concept of a total derivative. The total derivative of a function of x, u and derivatives of u is found by differentiating the function treating the u's as functions of the x's.

Formally, let $F(x, u^{(n)})$ be a function on J^n. Then the *total derivative* $D_i F$ of F with respect to x^i is the function on $\mathrm{J}^{(n+1)}$ defined by

$$D_i(x, \mathrm{pr}^{(n+1)} f(x)) = \frac{\partial F(x, \mathrm{pr}^{(n)} f(x))}{\partial x^i}.$$

For example, in the case of one independent variable x and one dependent variable u, the total derivative D_x with respect to x has the general formula

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \cdots.$$

In general, the total derivative with respect to the i–th independent variable is the first order differential operator

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u_{J,i}^{\alpha} \frac{\partial}{\partial u_J^{\alpha}},$$

where $u_{J,i}^{\alpha} = D_i(u_J^{\alpha}) = u_{j_1 \ldots j_k i}^{\alpha}$. The latter sum is over all multi-indices J of arbitrary order. Even though D_i involves an infinite summation, when applying the total derivative to any function $F(x, u^{(n)})$ defined on the n–th jet space, only finitely many terms (namely, those for $\#J \leq n$) are needed. Higher order total derivatives are defined in the obvious manner, with $D_J = D_{j_1} \cdot \ldots \cdot D_{j_k}$ for any multi-index $J = (j_1, \ldots, j_k)$, $1 \leq j_\nu \leq p$.

2.3.4 Prolongation of Vector Fields

Given a vector field $\mathbf{v}$ generating a one-parameter group of transformations $\exp(t\mathbf{v})$ on $M \subset X \times U$, the associated n–th order prolonged vector field $\mathrm{pr}^{(n)}\mathbf{v}$ is the vector field on the jet space J^n which is the infinitesimal generator of the prolonged one-parameter group $\mathrm{pr}^{(n)} \exp(t\mathbf{v})$. Thus,

$$\mathrm{pr}^{(n)}\mathbf{v}|_{(x,u^{(n)})} = \frac{d}{dt} \mathrm{pr}^{(n)}[\exp(t\mathbf{v})]|_{t=0} \cdot (x, u^{(n)}). \tag{10}$$

The explicit formula for the prolonged vector field is given by the following very important "prolongation formula" (see [50], Theorem 2.36 for the proof):

Theorem 5 *The n–th prolongation of the vector field*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

is given explicitly by

$$pr^{(n)}\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{j=\#J=0}^{n} \varphi_J^\alpha(x,u^{(j)})\frac{\partial}{\partial u_J^\alpha}, \tag{11}$$

with coefficients

$$\varphi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha, \tag{12}$$

where the "characteristics" of $\mathbf{v}$ *are given by*

$$Q^\alpha(x,u^{(1)}) := \varphi^\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial u^\alpha}{\partial x^i}, \quad \alpha = 1,\ldots,q. \tag{13}$$

Remark. One can easily prove [49, 50] that a function $u = f(x)$ is invariant under the group generated by $\mathbf{v}$ if and only if it satisfies the characteristic equations

$$Q^\alpha(x, \mathrm{pr}^{(1)}f(x)) = 0, \quad \alpha = 1,\ldots,q.$$

Example. Suppose we have just one independent and dependent variable. Consider a general vector field $\mathbf{v} = \xi(x,u)\partial_x + \varphi(x,u)\partial_u$ on $M = \mathbf{R}^2$. The characteristic (13) of $\mathbf{v}$ is the function

$$Q(x,u,u_x) = \varphi(x,u) - \xi(x,u)u_x.$$

From the above remark, we see that a function $u = f(x)$ is invariant under the one-parameter group generated by $\mathbf{v}$ if and only if it satisfies the ordinary differential equation $\xi(x,u)u_x = \varphi(x,u)$. The second prolongation $\mathbf{v}$ is a vector field

$$\mathrm{pr}^{(2)}\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \varphi(x,u)\frac{\partial}{\partial u} + \varphi^x(x,u^{(1)})\frac{\partial}{\partial u_x} + \varphi^{xx}(x,u^{(2)})\frac{\partial}{\partial u_{xx}},$$

on J^2, whose coefficients φ^x, φ^{xx} are given by

$$\begin{aligned} \varphi^x &= D_x Q + \xi u_{xx} = \varphi_x + (\varphi_u - \xi_x)u_x - \xi_u u_x^2, \\ \varphi^{xx} &= D_x^2 Q + \xi u_{xxx} \\ &= \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x + (\varphi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 + (\varphi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx}. \end{aligned}$$

For example, the second prolongation of the infinitesimal generator $\mathbf{v} = -u\partial_x + x\partial_u$ of the rotation group is given by

$$\mathrm{pr}^{(2)}\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x} + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}},$$

where the coefficients are computed as

$$\varphi^x = D_x Q + u_{xx}\xi = D_x(x+uu_x) - uu_{xx} = 1+u_x^2, \varphi^{xx} = D_x^2 Q + u_{xxx}\xi = D_x^2(x+uu_x) - uu_{xxx} = 3\,u_x u_{xx}.$$

The group transformations can then be readily recovered by integrating the system of ordinary differential equations

$$\frac{dx}{dt} = -u, \quad \frac{du}{dt} = x, \quad \frac{dp}{dt} = 1+p^2, \quad \frac{dq}{dt} = 3pq,$$

where we have used p and q to stand for u_x and u_{xx} to avoid confusing derivatives with jet space coordinates. We find the second prolongation of the rotation group to be

$$\left(x\cos t - u\sin t, x\sin t + u\cos t, \frac{\sin t + p\cos t}{\cos t - p\sin t}, \frac{q}{(\cos t - p\sin t)^3} \right),$$

as could be computed directly.

2.4 Differential Invariants

At long last, we can precisely define the notion of "differential invariant." Indeed, recall that an invariant of a group G acting on a manifold M is just a function $I : M \to \mathbf{R}$ which is not affected by the group transformations. A *differential invariant* is an invariant in the standard sense for a prolonged group of transformations acting on the jet space J^n. Just as the ordinary invariants of a group action serve to characterize invariant equations, so differential invariants will completely characterize invariant systems of differential equations for the group, as well as invariant variational principles. As such they form the basic building block of many physical theories, where one often begins by postulating the invariance of the equations or the variational principle under an underlying symmetry group. In particular, they are essential in understanding the invariant heat-type flows presented below.

Suppose G is a local Lie group of point transformations acting on an open subset $M \subset X \times U$ of the space of independent and dependent variables, and let $\mathrm{pr}^{(n)}G$ be the n-th prolongation of the group action on the n-th jet space $\mathrm{J}^n = \mathrm{J}^n M$. A *differential invariant* is a real-valued function $I : \mathrm{J}^n \to \mathbf{R}$ which satisfies $I(\mathrm{pr}^{(n)}g \cdot (x, u^{(n)})) = I(x, u^{(n)})$ for all $g \in G$ and all $(x, u^{(n)}) \in \mathrm{J}^n$ where $\mathrm{pr}^{(n)}g \cdot (x, u^{(n)})$ is defined. Note that I may only be locally defined.

The following gives a characterization of differential invariants:

Proposition 4 *A function $I : J^n \to \mathbf{R}$ is a differential invariant for a connected group G if and only if*

$$pr^{(n)}\mathbf{v}(I) = 0,$$

for all $\mathbf{v} \in \mathcal{G}$ where $\mathcal{G}$ denotes the Lie algebra of G.

A basic problem is to classify the differential invariants of a given group action. Note first that if the prolonged group $\mathrm{pr}^{(n)}G$ acts regularly on (an open subset of) J^n with r_n– dimensional orbits, then, locally, there are $p + q^{(n)} - r_n = p + q \begin{pmatrix} p+n \\ n \end{pmatrix} - r_n$ functionally independent n-th order differential invariants. Furthermore, any lower order differential invariant $I(x, u^{(k)})$, $k < n$ is automatically an n–th differential invariant, and will be included in the preceding count. (Here we are identifying $I : \mathrm{J}^k \to \mathbf{R}$ and its composition $I \circ \pi_k^n$ with the standard projection $\pi_k^n : \mathrm{J}^n \to \mathrm{J}^k$.)

If $\mathcal{O}^{(n)} \subset \mathrm{J}^n$ is an orbit of $\mathrm{pr}^{(n)}G$, then, for any $0 \leq k < n$ its projection $\pi_k^n(\mathcal{O}) \subset \mathrm{J}^n$ is an orbit of the k-th prolongation $\mathrm{pr}^{(k)}G$. Therefore, the generic orbit dimension r_n of $\mathrm{pr}^{(n)}G$ is a *nondecreasing* function of n, bounded by r, the dimension of G itself. This implies that the orbit dimension eventually stabilizes, $r_n = r^*$ for all $n \geq n_0$. We call r^* the *stable orbit dimension*, and the minimal order n_0 for which $r_{n_0} = r^*$, the *order of stabilization* of the group.

Now a transformation group G acts *effectively* on a space M if

$$g \cdot x = h \cdot x, \quad \forall\, x \in M,$$

if and only if $g = h$. Define the *global isotropy group*

$$G_M := \{g : g \cdot x = x \ \ \forall\, x \in M\}.$$

Then G acts effectively if and only if G_M is trivial. Moreover, G acts *locally effectively* if the global isotropy group G_M is a discrete subgroup of G in which case G/G_M has the same dimension and the same Lie algebra as G. We can now state the following remarkable result [54]:

Theorem 6 *The transformation group G acts locally effectively if and only if its dimension is the same as its stable orbit dimension, so that*

$$r_n = r^* = \dim G,$$

for all n sufficiently large.

There are a number of important results on the stabilization dimensions, maximal orbit dimensions, and their relationship to invariants; see [49, 51]. We will suffice with the following theorem which is very useful for counting the number of independent differential invariants of large order:

Theorem 7 *Suppose, for each $k \geq n$, the (generic) orbits of $pr^{(n)}G$ have the same dimension $r_k = r_n$. Then for every $k > n$ there are precisely $q_k = q \begin{pmatrix} p+k-1 \\ k \end{pmatrix}$ independent k-th order differential invariants which are not given by lower order differential invariants.*

Next we note that the basic method for constructing a complete system of differential invariants of a given transformation group is to use invariant differential operators [49, 50, 51]. A *differential operator* is said to be G-invariant if it maps differential invariants to higher order differential invariants, and thus, by iteration, produces hierarchies of differential invariants of arbitrarily large order. For sufficiently high orders, one can guarantee the existence of sufficiently many such invariant operators in order to completely generate all the higher order independent differential invariants of the group by successively differentiating lower order differential invariants. See [49, 50, 51] for details. Hence, a complete description of all the differential invariants is obtained by a set of low order *fundamental* differential invariants along with the requisite invariant differential operators.

In our case (one independent variable), the following theorem is fundamental:

Theorem 8 *Suppose that G is a group of point transformations acting on a space M having one independent variable and q dependent variables. Then there exist (locally) a G-invariant one-form $dr = g dx$ of lowest order, and q fundamental, independent differential invariants $J_1, \ldots, J_q$ such that every differential invariant can be written as a function of these differential invariants and their derivatives $\mathcal{D}^m J_\nu$, where*

$$\mathcal{D} := \frac{d}{dr} = \frac{1}{g}\frac{d}{dx},$$

is the invariant differential operator associated with dr. The parameter r gives an invariant parametrization of the curve and is called arc-length.

Remark. A version of Theorem 8 is true more generally. See [49, 51].

With this, we have completed our sketch of the theory of differential invariants. Once again, we refer the reader to the texts [49, 50, 51] for a full modern treatment of the subject, including methods for constructing and counting differential invariants.

With the above background, we are ready to turn to our main topic, namely invariant flows in vision.

3 Invariant Flows

In this section, a general approach for formulating invariant flows is described. In particular, we will consider the uniqueness of our models (see Theorem 8). Thus, given a certain transformation (Lie) group G, we show how to obtain the corresponding invariant geometric heat flow. We also show how to formulate this flow just in terms of Euclidean parameters such as the Euclidean curvature. This formulation permits us to employ already existing results and techniques for the analysis of such flows. This topic was first presented in [61, 62]. Here we emphasize a novel classification and uniqueness result.

3.1 Special Differential Invariants

In order to separate the geometric concept of a plane curve from its parametric description, it is useful to consider the *image* (or *trace*) of $\mathcal{C}(p)$, denoted by $\mathrm{Img}[\mathcal{C}(p)]$. Therefore, if the curve $\mathcal{C}(p)$ is parametrized by a new parameter w such that $w = w(p)$, $\frac{\partial w}{\partial p} > 0$, we obtain

$$\mathrm{Img}[\mathcal{C}(p)] = \mathrm{Img}[\mathcal{C}(w)].$$

In general, the parametrization gives the "velocity" of the trajectory. Given a transformation group G, the curve can be parametrized using what is called the *group arc-length*, dr, which is a non-trivial G-invariant one-form of minimal order (see Theorem 7). This parametrization, which is an invariant of the group, is useful for defining differential invariant descriptors [30, 49, 50]. In order to perform this re-parametrization, the *group metric* g is defined by the equality

$$dr = g dp,$$

for any parametrization p. Then r is obtained via the relation (after fixing an arbitrary initial point)

$$r(p) := \int_0^p g(\xi) d\xi, \tag{14}$$

and the re-parametrization is given by $\mathcal{C} \circ r$. We have of course,

$$\mathrm{Img}[\mathcal{C}(p)] = \mathrm{Img}[\mathcal{C}(r)].$$

For example, in the Euclidean case we have

$$g_{euc} := \| \frac{\partial \mathcal{C}}{\partial p} \|, \tag{15}$$

and the Euclidean arc-length is given by

$$v := \int_0^p \| \frac{\partial \mathcal{C}}{\partial \xi} \| d\xi.$$

This parametrization is Euclidean invariant (since the norm is invariant), and implies that the curve $\mathcal{C}(s)$ is traversed with constant velocity ($\| \frac{\partial \mathcal{C}}{\partial v} \| \equiv 1$). For examples of other groups, see Sections 3.5, 3.6, 3.7 and [30, 49, 50, 68].

Based on the group metric and arc-length, the *group curvature* χ, is computed. (Note that g, r, and χ can be computed using Lie theory as well as Cartan moving frames method [8, 19, 30, 49].) The group curvature, as a function of the arc-length, is defined as the simplest non-trivial differential invariant of the group (see Theorem 7).

For example, in the Euclidean case, since

$$\| \frac{\partial \mathcal{C}}{\partial v} \| \equiv 1.$$

we have that $\mathcal{C}_v \perp \mathcal{C}_{vv}$, and the Euclidean curvature is defined as

$$\kappa := \| \mathcal{C}_{vv} \| .$$

κ is also the rate of change of the angle between the tangent to the curve and a fixed direction. The corresponding invariants of the affine group will be presented below in Section 3.5.1.

3.2 Geometric Invariant Flow Formulation

We are now ready to describe the type of evolution that we want to deal with. First let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ be a family of smooth curves, where p parametrizes the curve and t the family. (In this case, we take p to be independent of t.) Assume that this family evolves according to the following evolution equation:

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \frac{\partial^2 \mathcal{C}(p,t)}{\partial p^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p), \end{aligned} \tag{16}$$

which is the classical heat equation. If $\mathcal{C}(p,t) = [x(p,t), y(p,t)]^T$, then $[x(p,t), y(p,t)]$ satisfying (16) can also be obtained by convolution of $[x_0(p), y_0(p)]$ with a Gaussian filter whose variance depends on t. Equation (16) has been studied by the computer vision community, and is used for the definition of a linear scale-space for planar shapes [5, 14, 20, 36, 37, 38, 39, 41, 42, 69, 71].

The Gaussian kernel, being one of the most used in image analysis, has several undesirable properties, principally when applied to planar curves. One of these is that the filter is not intrinsic to the curve (see [63] for a detailed description of this problem). This can be remedied by replacing the linear heat equation by *geometric heat flows* [26, 27, 57, 58, 59, 61, 62, 63]. In particular, if the *Euclidean* geometric heat flow [23, 24, 26, 27, 28] is used, a scale-space invariant to rotations and translations is obtained. If the *affine* one is used [57, 58, 62], an affine invariant multi-scale representation is obtained [59]. This and other geometric heat flows are presented below.

Another problem with the Gaussian kernel is that the smoothed curve shrinks when the Gaussian variance (or the time) increases. Several approaches, briefly discussed in Section 4, have been proposed in order to partially solve this problem for Gaussian-type kernels (or linear filters). These approaches violate basic scale-space properties. In [63], the authors showed that this problem can be completely solved using a variation of the geometric heat flow methodology, which keeps the area enclosed by the curve constant. The flows obtained, precisely satisfy all the basic scale-space requirements. In the Euclidean case, the flow is local as well. The same approach can be used for deriving length preserving heat flows. In this case, the similarity flow exhibits locality. In short, *we can get geometric smoothing without shrinkage.* In order to give a complete picture of invariant geometric flows, basic results of this area/length preserving approach are given in Section 4 as well.

Assume that we want to formulate an intrinsic *geometric heat flow* for plane curves which is invariant under certain transformation group G. Let r denote the group arc-length (Theorem 7). Then, the *invariant geometric heat flow* is given by [61, 62, 63]

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p). \end{aligned} \tag{17}$$

If G acts linearly, it is easy to see that since dr is an invariant of the group, so is $\mathcal{C}_{rr}$. $\mathcal{C}_{rr}$ is called the *group normal.* For nonlinear actions, the flow (17) is still G-invariant, since as pointed out in Theorem 7, $\frac{\partial}{\partial r}$ is the *invariant derivative.* See [49] and our discussion in Section 3.3 below. In fact, as we will see the evolution given by (17) is in a certain precise sense the simplest possible non-trivial G-invariant flow.

We have just formulated the invariant geometric heat flow in terms of concepts intrinsic to the group itself, i.e., based on the group arc-length. For different reasons, which we will explain shortly, it is useful to formulate the group velocity $\mathcal{C}_{rr}$ in terms of Euclidean notions such as the Euclidean normal and Euclidean curvature. In order to do this, we need to calculate

$$< \mathcal{C}_{rr}, \vec{\mathcal{N}} >,$$

where $\vec{\mathcal{N}}$ is the Euclidean unit (inward) normal, and $< \cdot, \cdot >$ is the standard inner product in $\mathbf{R}^2$. In this way, we will be able to decompose the group normal $\mathcal{C}_{rr}$ into its Euclidean unit normal $\vec{\mathcal{N}}$ and Euclidean unit tangential $\vec{\mathcal{T}}$ components, and to re-write the flow (17) as

$$\frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}}. \tag{18}$$

In order to calculate α and β, assume for the moment that the curve $\mathcal{C}$ is parametrized by the Euclidean arc-length v. Then,

$$\frac{\partial^2 \mathcal{C}}{\partial r^2} = \frac{1}{g^2}\frac{\partial^2 \mathcal{C}}{\partial v^2} - \frac{g_v}{g^3}\frac{\partial \mathcal{C}}{\partial v}, \tag{19}$$

where g is the group metric defined in Section 3.1. (In this case, g is a function of v.) Now, using the relations

$$C_{vv} = \kappa \vec{\mathcal{N}}, \quad C_v = \vec{\mathcal{T}},$$

we obtain

$$\alpha = -\frac{g_v}{g^3}, \quad \beta = \frac{\kappa}{g^2}. \tag{20}$$

In general, $g(v)$ in (20) is written as a function of κ and its derivatives (see Section 3.5).

The importance of the formulation (18) can be seen from the following:

Lemma 2 ([15]) *Let β be a geometric quantity for a curve, i.e., a function whose definition is independent of a particular parametrization. Then a family of curves which evolves according to*

$$\mathcal{C}_t = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}}$$

can be converted into the solution of

$$\mathcal{C}_t = \bar{\alpha} \vec{\mathcal{T}} + \beta \vec{\mathcal{N}}$$

for any continuous function $\bar{\alpha}$, by changing the space parametrization of the original solution. Since β is a geometric function, $\beta = \bar{\beta}$ when the same point in the (geometric) curve is considered.

In particular, the above lemma shows that $\mathrm{Img}[\mathcal{C}(p,t)] = \mathrm{Img}[\hat{\mathcal{C}}(w,t)]$, where $\mathcal{C}(p,t)$ and $\hat{\mathcal{C}}(w,t)$ are the solutions of

$$\mathcal{C}_t = \alpha\vec{\mathcal{T}} + \beta\vec{\mathcal{N}},$$

and

$$\hat{\mathcal{C}}_t = \bar{\beta}\vec{\mathcal{N}},$$

respectively. For proofs of the lemma, see [15, 59].

Therefore, assuming that the normal component β of $\vec{v}$ (the curve evolution velocity) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial \mathcal{C}}{\partial t} = \beta\vec{\mathcal{N}}, \tag{21}$$

where $\beta = <\vec{v}, \vec{\mathcal{N}}>$, i.e., the projection of the velocity vector in the Euclidean normal direction. Since C_{rr} is a geometric quantity, equation (18) can be reduced to (21).

The formulation (17) gives a very intuitive formulation of the invariant geometric heat flow. On the other hand, the formulation given by equation (18), together with (20), gives an Euclidean-type flow which also allows us to simplify the flow using the result of Lemma 2. This type of analysis is crucial, since it allows one to understand the partial differential equation underlying the flow and to study its essential properties (such as short and long term existence, convergence, etc.). This will be a key technique when we study affine invariant flows in Section 3.5. Finally, reduction of equation (17) to (21) allows one to numerically implement the flow on computer. In fact, there is now available an efficient numerical algorithm due to Osher and Sethian [53, 65] to do this.

The flow given by (17) is non-linear, since the group arc-length r is a function of time. This flow gives the invariant geometric heat-type flow of the group, and provides the invariant direction of the deformation. For subgroups of the full projective group $\mathrm{SL}(\mathbf{R},3)$, we show in Theorem 8 below that the most general invariant evolutions are obtained if the group curvature χ and its derivatives (with respect to arc-length) are incorporated into the flow:

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \Psi(\chi, \frac{\partial \chi}{\partial r}, \ldots, \frac{\partial^n \chi}{\partial r^n})\frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p), \end{aligned} \tag{22}$$

where $\Psi(\cdot)$ is a given function. (We discuss the existence of possible solutions of (22) in [62].) Since the group arc-length and group curvature are the basic invariants of the group transformations, it is natural to formulate (22) as the most general geometric invariant flow.

Since we have expressed the flow (17) in terms of Euclidean properties (equations (18), (20)), we can do the same for the general flow (22). All what we have to do is to express χ as a function of κ and it derivatives. This is done by expressing the curve components $x(p)$ and $y(p)$ as a function of κ, and then computing χ.

3.3 Uniqueness of Invariant Heat Flows

In this section, we give a fundamental result, which elucidates in what sense our invariant heat-type equations (17) are unique. We use here the action of the projective group $\mathrm{SL}(\mathbf{R},3)$ on $\mathbf{R}^2$ as defined in Section 2.2.1. We will first note that locally, we may express a solution of (17) as the graph of a function $y = u(x,t)$.

Lemma 3 *Locally, the evolution (17) is equivalent to*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2},$$

where g is the G–invariant metric ($g = dr/dx$).

Proof. Indeed, locally the equation

$$C_t = C_{rr},$$

becomes

$$x_t = x_{rr}, \quad y_t = y_{rr}.$$

Now $y(r,t) = u(x(r,t),t)$, so

$$y_t = u_x x_t + u_t, \quad y_{rr} = u_{xx}x_r^2 + u_x x_{rr}.$$

Thus,

$$u_t = y_t - u_x x_t = y_{rr} - u_x x_{rr} = x_r^2 u_{xx}.$$

Therefore the evolution equation (17) reduces to

$$u_t = g^{-2}u_{xx},$$

since $dr = gdx$. □

We can now state the following fundamental result:

Theorem 9 *Let G be a subgroup of the projective group $SL(\mathbf{R},3)$. Let $dr = gdp$ denote the G-invariant arc-length and χ the G-invariant curvature. Then*

1. *Every differential invariant of G is a function*

$$I\left(\chi, \frac{d\chi}{dr}, \frac{d^2\chi}{dr^2}, \ldots, \frac{d^n\chi}{dr^n}\right)$$

of χ and its derivatives with respect to arc length.

2. *Every G-invariant evolution equation has the form*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2}\, I, \tag{23}$$

where I is a differential invariant for G.

We are particularly interested in the following subgroups of the full projective group: Euclidean, similarity, special affine, affine, full projective. (See our discussion below for the precise results.)

Corollary 1 *Let G be one of the listed subgroups of the projective group $SL(\mathbf{R},3)$. Then there is, up to a constant factor, a unique G-invariant evolution equation of lowest order, namely*

$$\frac{\partial u}{\partial t} = \frac{c}{g^2}\frac{\partial^2 u}{\partial x^2},$$

where c is a constant.

Remark. Part 1 of the Theorem 9 (suitably interpreted) does not require G to be a subgroup of the projective group; however for part 2 and the corollary this is essential. One can, of course, classify the differential invariants, invariant arc-lengths, invariant evolution equations, etc., for any group of transformations in the plane, but the interconnections are more complicated. See Lie [40] and Olver [49] for the details of the complete classification of all groups in the plane and their differential invariants.

Remark. The uniqueness of the Euclidean and affine flows (see the next section), was also proven in [1], using a completely different approach. In contrast with the results here presented, the ones in [1] were proven independently for each group, and when considering a new group, a new analysis had to be carried out. Our result is a general one, and can be applied to any subgroup. Also, with the geometric approach presented here, we believe that the result is clear and intuitive.

Proof of Theorem.
Part 1 follows immediately from Theorem 8, and the definitions of dr and χ. (Note by Theorem 7 for a subgroup of $SL(\mathbf{R}, 3)$ acting on $\mathbf{R}^2$, we have each differential invariant of order k is in fact unique.)

As for part 2, let

$$\mathbf{v} = \xi(x,u)\partial_x + \varphi(x,u)\partial_u$$

be an infinitesimal generator of G, and Let pr $\mathbf{v}$ denote its prolongation to the jet space. Since dr is (by definition) an invariant one-form, we have

$$\mathbf{v}(dr) = [\,\mathrm{pr}\ \mathbf{v}(g) + gD_x\xi\,]dx,$$

which vanishes if and only if

$$\mathrm{pr}\ \mathbf{v}(g) = -gD_x\xi = -g(\xi_x + u_x\xi_u). \tag{24}$$

Applying pr $\mathbf{v}$ to the evolution equation (23), and using condition (24), we have (since ξ and φ do not depend on t)

$$\begin{aligned} \mathrm{pr}\ \mathbf{v}[u_t - g^{-2}u_{xx}I] &= (\varphi_u - u_x\xi_u)u_t - 2g^{-2}(\xi_x + u_x\xi_u)u_{xx}I - \\ &- g^{-2}\mathrm{pr}\ \mathbf{v}[u_{xx}]I - g^{-2}u_{xx}\mathrm{pr}\ \mathbf{v}[I]. \end{aligned} \tag{25}$$

If G is to be a symmetry group, this must vanish on solutions of the equation; thus, in the first term, we replace u_t by $g^{-2}u_{xx}I$. Now, since G was assumed to be a subgroup of the projective group, which is the symmetry group of the second order ordinary differential equation $u_{xx} = 0$, we have pr $\mathbf{v}[u_{xx}]$ is a multiple of u_{xx}; in fact, inspection of the general prolongation formula for pr $\mathbf{v}$ (see Theorem 5) shows that in this case

$$\mathrm{pr}\ \mathbf{v}[u_{xx}] = (\varphi_u - 2\xi_x - 3\xi_u u_x)u_{xx}. \tag{26}$$

(The terms in pr $\mathbf{v}[u_{xx}]$ which do not depend on u_{xx} must add up to zero, owing to our assumption on $\mathbf{v}$.) Substituting (26) into (25) and combining terms, we find

$$\mathrm{pr}\ \mathbf{v}[u_t - g^{-2}u_{xx}I] = g^{-2}u_{xx}\mathrm{pr}\ \mathbf{v}[I],$$

which vanishes if and only if pr $\mathbf{v}[I] = 0$, a condition which must hold for each infinitesimal generator of G. But this is just the infinitesimal condition that I be a differential invariant of G, and the theorem follows. □

The Corollary follows from the fact that, for the listed subgroups, the invariant arc length r depends on lower order derivatives of u than the invariant curvature χ. (This fact holds for most (but *not* all) subgroups of the projective group; one exception is the group consisting of translations in x, u, and isotropic scalings $(x, u) \mapsto (\lambda x, \lambda u)$.) The orders are indicated in the following table:

Group	*Arc Length*	*Curvature*
Euclidean	1	2
Similarity	2	3
Special Affine	2	4
Affine	4	5
Projective	5	7

The explicit formulas are given in the following table:

Group	*ArcLength*	*Curvature*
Euclidean	$\sqrt{1+u_x^2}dx$	$\dfrac{u_{xx}}{(1+u_x^2)^{3/2}}$
Similarity	$\dfrac{u_{xx}dx}{(1+u_x^2)^2}$	$\dfrac{(1+u_x^2)u_{xxx}-3u_xu_{xx}^2}{u_{xx}^2}$
Special Affine	$(u_{xx})^{1/3}dx$	$\dfrac{P_4}{(u_{xx})^{8/3}}$
Affine	$\dfrac{\sqrt{P_4}}{u_{xx}}\,dx$	$\dfrac{P_5}{(P_4)^{3/2}}$
Projective	$\dfrac{(P_5)^{1/3}}{u_{xx}}\,dx$	$\dfrac{P_7}{(P_5)^{8/3}}$

Here

$$\begin{aligned} P_4 &= 3u_{xx}u_{xxxx} - 5u_{xxx}^2, \\ P_5 &= 9u_{xx}^2u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3. \end{aligned}$$

P_7 is a very complicated polynomial depending on seventh and lower order derivatives of u. See [40], page 255 for the precise formula. (The invariant P_7 is called σ there.) Some of this invariants will be specifically derived below and used to present the corresponding invariant flows of the viewing transformations.

3.4 Euclidean Invariant Flows

We now show how to use the general theory presented above, for the computation of the invariant heat flows corresponding to the Euclidean group. In the next section, we will discuss the affine group.

Recall from our discussion in Section 2.2.1 that a general Euclidean transformation in the plane is given by

$$\tilde{\mathcal{X}} = R\mathcal{X} + V,$$

where $\mathcal{X} \in \mathbf{R}^2$, R is a 2×2 rotation matrix, and V is a 2×1 translation vector. The Euclidean transformations constitute a group, and give some of the basic shape deformations which appear in computer vision applications.

We proceed to find, based on the above developed method, an Euclidean invariant geometric heat equation. From (17), we obtain that the *Euclidean geometric heat flow* is given by

$$\begin{aligned} \mathcal{C}_t &= \mathcal{C}_{vv}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p). \end{aligned} \tag{27}$$

(Recall that v is the Euclidean arc-length.) The Euclidean metric is defined by equation (15), and if the curve is already parametrized by arc-length, then of course $g_{euc}(v) \equiv 1$. Therefore, using equation (20) we obtain

$$\alpha_{euc} = 0, \quad \beta_{euc} = \kappa.$$

Then, the "Euclidean type" equation equivalent to (27) is (see equation (18))

$$\mathcal{C}_t = \kappa \vec{\mathcal{N}}. \tag{28}$$

Equation (28) has a large research literature devoted to it. Gage and Hamilton [26] proved that any smooth, embedded convex curve converges to a round point when deforming according to it. Grayson [27] proved that any non-convex embedded curve converges to a convex one. Hence, any simple curve converges to a round point when evolving according to the Euclidean geometric heat equation. The flow is also known as the *Euclidean shortening flow*, since the Euclidean perimeter shrinks as fast as possible when the curve evolves according to (28); see [27]. Equation (28) was also found to be very important for image enhancement applications [2, 60], and was introduced into the theory of shape in computer vision by [33, 34, 35]. In order to proof that the flow indeed smoothes the curve, results from [3, 26, 27] can be used.

3.5 Affine Invariant Flows

In this section, we present the affine flow corresponding to equation (17) as first developed in [57, 58, 62]. We first make some remarks about classical affine differential geometry.

3.5.1 Sketch of Affine Differential Geometry

An affine transformation transforms disks into ellipses, and rectangles into parallelograms. Recall from Section 2.2.1 that a general *affine transformation* in the plane ($\mathbf{R}^2$) is defined by

$$\tilde{\mathcal{X}} = A\mathcal{X} + B, \tag{29}$$

where $\mathcal{X} \in \mathbf{R}^2$ is a vector, $A \in \mathrm{GL}_2^+(\mathbf{R})$ (the group of invertible real 2×2 matrices with positive determinant) is the affine matrix, and $B \in \mathbf{R}^2$ is a translation vector. As we have seen (29) form a real Lie group $\mathrm{A}(2)$, called the *group of proper affine motions.* We will also consider the case of when we restrict $A \in \mathrm{SL}(2, \mathbf{R})$ (i.e., the determinant of A is 1), in which case (29) gives us the *group of special affine motions*, $\mathrm{A}_{sp}(2)$.

In the case of Euclidean motions (in which case A in (29) is a rotation matrix), we have that the Euclidean curvature κ of a given plane curve is a differential invariant of the transformation. In the case of general affine transformations, in order to keep the invariance property, a new definition of curvature is necessary. In this section, this *affine* curvature is presented [6, 30, 57]. See [6] for general properties of affine differential geometry. We should note that when we say "affine invariant" in the sequel, we will mean with respect to $\mathrm{A}_{sp}(2)$ (the group of special affine motions). However, all our "invariants" will be *relative* invariants with respect to the full affine group, in the sense that the transformed quantities will always differ by some function of the determinant of the transforming matrix. See [13, 49] for the precise definition of "relative invariant." We will consider flows which are "absolutely" invariant with respect to the full affine group in Section 3.7.

As above, let $\mathcal{C} : S^1 \to \mathbf{R}^2$ be an embedded curve with parameter p (where S^1 denotes the unit circle). We now make the invariant re-parametrization of $\mathcal{C}(p)$ by defining a new parameter s such that

$$[\mathcal{C}_s, \mathcal{C}_{ss}] = 1, \tag{30}$$

where $[\mathcal{X}, \mathcal{Y}]$ stands for the determinant of the 2×2 matrix whose columns are given by the vectors $\mathcal{X},\ \mathcal{Y} \in \mathbf{R}^2$. This relation is invariant under proper affine transformations, and the parameter s is the *affine arc-length*. Setting

$$g_{aff}(p) := [\mathcal{C}_p, \mathcal{C}_{pp}]^{1/3}, \tag{31}$$

the parameter s is explicitly given by

$$s(p) = \int_0^p g_{aff}(\xi) d\xi. \tag{32}$$

Note, we have assumed (of course) that g_{aff} (the *affine metric*) is different from zero at each point of the curve, i.e., the curve has no inflection points. In general, affine differential geometry is defined just for convex curves [6, 30]. In Section 3.5.2, we will show how to overcome this problem for the evolution of non-convex curves.

By differentiating (30) we obtain that the two vectors $\mathcal{C}_s$ and $\mathcal{C}_{sss}$ are linearly dependent and so there exists μ such that

$$\mathcal{C}_{sss} + \mu \mathcal{C}_s = 0. \tag{33}$$

The last equation and (30) imply

$$\mu = [\mathcal{C}_{ss}, \mathcal{C}_{sss}], \tag{34}$$

and μ is the *affine curvature*, i.e., the simplest non-trivial differential affine invariant function of the curve $\mathcal{C}$; see [6]. Moreover, one can easily show [57] that ds, $\mathcal{C}_s$, $\mathcal{C}_{ss}$, μ, and the area enclosed by a closed curve, are (absolute) invariants of the group $\mathrm{A}_{sp}(2)$ of special affine motions and relative invariants of the group $\mathrm{A}(2)$ of proper affine motions.

3.5.2 Affine Geometric Heat Equation

With s the *affine* arc-length, the affine-invariant geometric heat flow is given by [57, 62]

$$\begin{cases} \mathcal{C}_t = \mathcal{C}_{ss}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p). \end{cases} \tag{35}$$

Since s is only defined for convex curves, the flow (35) is defined *a priori* for such curves only. However, in fact the evolution can be extended to the non-convex case, in the following natural manner. Observe that if $\mathcal{C}$ is parametrized by the Euclidean arc-length, then

$$g_{aff}(v) \;=\; [\mathcal{C}_v, \mathcal{C}_{vv}]^{1/3} = [\vec{\mathcal{T}}, \kappa\vec{\mathcal{N}}]^{1/3} = \kappa^{1/3},$$

and we obtain

$$\alpha_{aff} = -\frac{(\kappa^{1/3})_v}{\kappa}, \quad \beta_{aff} = \kappa^{1/3}.$$

Now one easily compute that

$$\mathcal{C}_{ss} = \kappa^{1/3}\vec{\mathcal{N}} + \text{tangential component.}$$

(See [57, 62].) Hence, using Lemma 2, we obtain that the following flow is geometrically equivalent to the affine invariant flow (35):

$$\mathcal{C}_t = \kappa^{1/3}\vec{\mathcal{N}}. \tag{36}$$

Note that the flow (36) is affine invariant flow, and is also *well-defined for non-convex curves.* (We should also observe here that inflection points are affine invariant.) We should also add that recently Alvarez *et al.* [1] derived (36) using a completely different approach.

In summary, despite the fact that we cannot define the basic differential invariants of affine differential geometry on non-convex curves, nevertheless an affine invariant heat-type flow can be formulated. This is possible due to the possibility to "ignore" the tangential component of the deformation velocity, together with the invariant property of inflection points. One can see that while $\mathcal{C}_{ss}$ contains three derivatives, its normal component contains only two. This allows one to write the geometric affine heat flow as a function of κ.

The key results in this theory are the following [4, 57, 58, 62]:

Theorem 10 **([57])** *A smooth convex curve evolving according to the affine geometric heat flow remains smooth and convex.*

Theorem 11 **([57])** *A convex curve evolving according to the geometric heat flow converges to an elliptical point.*

Theorem 12 **([4, 58, 62])** *Let $\mathcal{C}(\cdot, 0) : S^1 \to \mathbf{R}^2$ be a C^2 embedded curve in the plane. Then there exists a unique one parameter family of C^2 curves $\mathcal{C} : S^1 \times [0, T) \to \mathbf{R}^2$ for some $T > 0$, satisfying the affine heat equation*

$$\mathcal{C}_t = \kappa^{1/3}\mathcal{N}.$$

Moreover, there is a $t_0 < T$ such that $\mathcal{C}(\cdot, t)$ is convex for all $t_0 < t < T$.

Theorem 13 *Any given C^2 embedded plane curve converges to an elliptical point when evolving according to (36) .*

Moreover, in [4] we show how to extend (36) to Lipschitz initial curves, and in particular, polygons. This eliminates the necessity of the viscosity framework [10, 12, 16] as proposed in [1], being also a stronger result.

In [59], it is formally proven that the affine geometric flow (35) (or its geometric analogue (36)) smoothes the curve. For example, it is shown that the total curvature and the number of inflection points decrease with time (scale-parameter). Figure 1 gives an example of this flow.

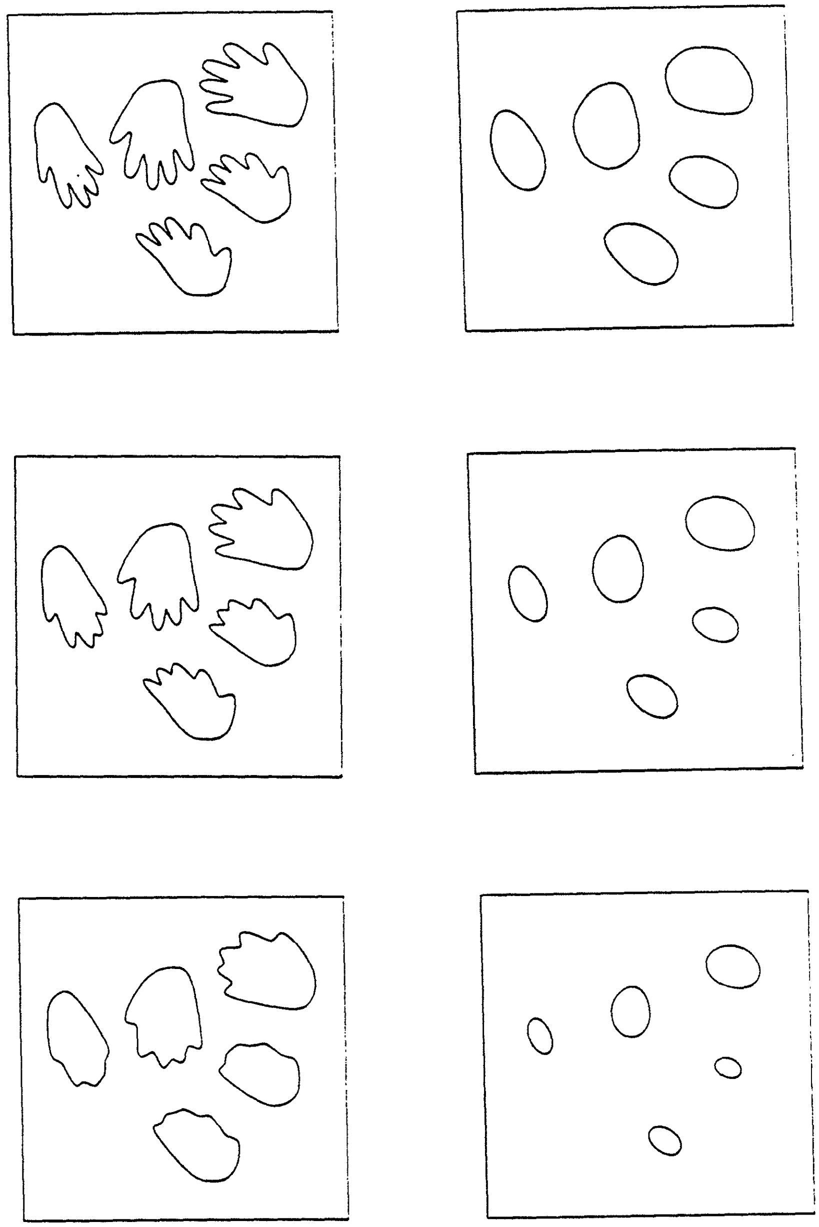

Fig. 1 – Example of the affine geometric heat flow (from [59]). The hands are related during all the evolution by the same affine transformation.

3.6 Projective Invariant Flows

We would like to make some remarks about projective invariant flows. The projective maps constitute the most general geometric transformations on planar shapes (or planar curves). Projective invariant flows have been considered by [17, 18, 60, 61, 62, 64]. Recall from Section 2.2.1 that the projective group acts on $\mathbf{R}^2$ via linear fractional transformations.

Let w denote the projective arc-length. Then

$$\mathcal{C}_t = \mathcal{C}_{ww} \tag{37}$$

is the projective flow. The flow is more complicated than the Euclidean and affine evolutions, because of the higher derivatives involved. Explicitly,

$$dw = g_{pro} dp,$$

where the projective metric is given by,

$$g_{pro}(p) = \left(R(p) - \frac{3}{2}\frac{\partial Q(p)}{\partial p}\right)^{1/3}, \tag{38}$$

where

$$Q(p) = q_2(p) - q_1^2(p) - \frac{\partial q_1(p)}{\partial p}, \quad R(p) = -3q_1(p)q_2(p) + 2q_1^3(p) - \frac{\partial^2 q_1(p)}{\partial p^2},$$

and

$$q_1(p) = \frac{1}{3}\frac{[\mathcal{C}_{ppp}, \mathcal{C}_p]}{[\mathcal{C}_p, \mathcal{C}_{pp}]}, \quad q_2(p) = \frac{1}{3}\frac{[\mathcal{C}_{pp}, \mathcal{C}_{ppp}]}{[\mathcal{C}_p, \mathcal{C}_{pp}]}.$$

Assuming that the curve is parametrized by the Euclidean arc-length v, we obtain

$$q_1(v) = -\frac{1}{3}\frac{\kappa_v}{\kappa}, \quad q_2(v) = \frac{1}{3}\kappa^2,$$

and $Q(v)$, $R(v)$ and $g_{pro}(v)$ can be computed. Clearly, we must assume that g_{pro} is well-defined and non-zero on the curve for the projective flow to be defined.

We would like to discuss why singularities should arise in this flow. First, we must define the real projective space $\mathbf{RP}^2$. This is the space of lines through the origin in $\mathbf{R}^3$. Two nonzero points $p_1, p_2 \in \mathbf{R}^3$ determine the same point $p \in \mathbf{RP}^2$ if and only if they are scalar multiples of each other $p_1 = \lambda p_2$, $\lambda \neq 0$. We will refer to the coordinates of $p_1 \in \mathbf{R}^3$, (x^1, x^2, x^3), as *homogeneous coordinates* of p. Notice this induces a canonical projection

$$\pi : \mathbf{R}^3 \backslash \{0\} \to \mathbf{RP}^2.$$

Coordinate charts on $\mathbf{RP}^2$ are constructed by considering all lines with a given component, say x^i, nonzero; the coordinates are then provided by ratios x^k/x^i, $k \neq i$, which amounts to the choice of canonical representative of such a line given by normalizing its i–th component to unity. Thus we may embed $\mathbf{R}^2$ as an open subset of $\mathbf{RP}^2$ via

$$\iota : \mathbf{R}^2 \hookrightarrow \mathbf{RP}^2, \quad (x, y) \mapsto (x, y, 1).$$

Projective space $\mathbf{RP}^2$ may then be regarded as the completion of $\mathbf{R}^2$ by adjoining all "directions at infinity." Thus we have following diagram:

$$\begin{array}{ccc} & & \mathbf{R}^3\backslash\{0\} \\ & & \downarrow \pi \\ \mathbf{R}^2 & \xrightarrow{\iota} & \mathbf{P}^2 \end{array} .$$

Now suppose, we are given the flow in $\mathbf{R}^2$

$$\mathcal{C}_t = \mathcal{C}_{ww}. \tag{39}$$

Via ι and π we may lift this to a flow in $\mathbf{R}^3$, say

$$\hat{\mathcal{C}}_t = \hat{\mathcal{C}}_{ww}, \tag{40}$$

which is invariant with respect to scaling. Under certain conditions, one can conclude, using results from the theory of reaction-diffusion equations [66], short term existence for this flow [17]. Suppose we consider initial curves which differ by a scaling:

$$\hat{\mathcal{C}}_1(0,w) = \lambda(w)\hat{\mathcal{C}}_2(0,w).$$

Then via (40), the resulting flows will differ by precisely the same scaling:

$$\hat{\mathcal{C}}_1(t,w) = \lambda(w)\hat{\mathcal{C}}_2(t,w).$$

By the uniqueness of the solutions of differential equations, this means that the flow (39) in $\mathbf{R}^2$ is induced by (40).

Given this, even assuming that (40) remains smooth, singularities could develop in (39) in the following two ways: First, since $\mathbf{R}^2 \subset \mathbf{P}^2$ and the flow (39) is induced by the projectively invariant flow via ι, points may go off to infinity, and so the flow in $\mathbf{R}^2$ can be singular. Secondly, the projection π may give cuspidal singular points in (39). Thus the projective invariant case of (17) does not have the nice smoothing properties of the Euclidean and affine models.

3.7 Similarity and Full Affine Invariant Flows

In this section, we consider flows which are invariant relative to the scale-invariant versions of the Euclidean and affine groups, namely the similarity and full affine groups as defined in Section 2.2.1. We begin with the heat flow for the similarity group (rotations, translations, and isotropic scalings). This flow was first presented and analyzed in [63]. We assume for the remainder of this section that our curves are strictly convex ($\kappa > 0$). Accordingly, let $\mathcal{C}$ be a smooth strictly convex plane curve, p the curve parameter, and as above, let $\vec{\mathcal{N}}, \vec{\mathcal{T}}$, and v denote the Euclidean unit normal, unit tangent, and Euclidean arc-length, respectively. Let

$$\sigma := \frac{\partial v}{\partial p}$$

be the speed of parametrization, so that

$$\mathcal{C}_p = \sigma\vec{\mathcal{T}}, \;\; \mathcal{C}_{pp} = \sigma_p\vec{\mathcal{T}} + \sigma^2\kappa\vec{\mathcal{N}}.$$

Then clearly,

$$\mathcal{C}_p \cdot \mathcal{C}_p = \sigma^2,$$

$$[\mathcal{C}_p, \mathcal{C}_{pp}] = \sigma^3 \kappa.$$

For the similarity group (in order to make the Euclidean evolution scale-invariant), we take a parametrization p such that

$$\mathcal{C}_p \cdot \mathcal{C}_p = [\mathcal{C}_p, \mathcal{C}_{pp}],$$

which implies that

$$\sigma = 1/\kappa.$$

Therefore the similarity group invariant arc-length is the standard angle parameter θ, since

$$\frac{d\theta}{dv} = \kappa,$$

where v is the Euclidean arc-length. (Note that $\vec{\mathcal{T}} = [\cos\theta, \sin\theta]^T$.) Thus the ***similarity normal*** is $\mathcal{C}_{\theta\theta}$, and the ***similarity invariant flow*** is

$$\mathcal{C}_t = \mathcal{C}_{\theta\theta}. \tag{41}$$

Projecting the similarity normal into the Euclidean normal direction, the following flow is obtained

$$\mathcal{C}_t = \frac{1}{\kappa}\vec{\mathcal{N}}, \tag{42}$$

and both (41) and (42) are geometrically equivalent flows.

Instead of looking at the flow given by (42) (which may develop singularities), we reverse the direction of the flow, and look at the expanding flow given by

$$\frac{\partial \mathcal{C}}{\partial t} = -\frac{1}{\kappa}\vec{\mathcal{N}}. \tag{43}$$

For completeness, we state the following results for the flow (43) (the proofs are given in [63]; see also [26, 35]):

Lemma 4 ([63]) *The following evolution equations are obtained for a curve evolving according to (43):*

1. *Evolution of Euclidean metric m ($dp = m dv$, where v is the Euclidean arc-length):*

$$m_t = m.$$

2. *Evolution of Euclidean tangent $\vec{\mathcal{T}}$:*

$$\vec{\mathcal{T}}_t = -\frac{\kappa_v}{\kappa^2}\vec{\mathcal{N}}.$$

3. *Evolution of Euclidean normal $\vec{\mathcal{N}}$:*

$$\vec{\mathcal{N}}_t = \frac{\kappa_v}{\kappa^2}\vec{\mathcal{T}}.$$

4. *Evolution of Euclidean perimeter* $\mathbf{P}$:

$$\mathbf{P}_t = \mathbf{P}.$$

5. *Evolution of area* $\mathbf{A}$:

$$\mathbf{A}_t = \oint \frac{1}{\kappa} dv.$$

6. *Evolution of Euclidean curvature* κ:

$$\kappa_t = -\left(\frac{1}{\kappa}\right)_{vv} - \kappa.$$

7. *Evolution of tangential angle* θ:

$$\theta_t = \frac{\kappa_v}{\kappa^2}.$$

Theorem 14 ([63])

1. *A simple convex curve remains simple and convex when evolving according to the similarity invariant flow (43).*

2. *The solution to (43) exists (and is smooth) for all* $0 \le t < \infty$.

Lemma 5 ([63]) *Changing the curve parameter from* p *to* θ, *we obtain that the radius of curvature* r, $r := 1/\kappa$, *evolves according to*

$$r_t = r_{\theta\theta} + r. \tag{44}$$

Theorem 15 ([63]) *A simple (smooth) convex curve converges to a disk when evolving according to (43).*

It is important to note that in contrast with (43), (42) can deform a curve towards singularities. Since (43) can be seen as a smoothing process (heat flow), the inverse equation can be seen as an enhancement process. The importance of this for image processing, as well as the extension of the theory to non-convex curves, is currently under investigation.

Using a similar argument, one may show the invariant equation for the full affine flow ($GL(\mathbf{R}, 2) \times_s \mathbf{R}^2$) is given by

$$\mathcal{C}_t = \frac{\mathcal{C}_{ss}}{\mu}. \tag{45}$$

As for the similarity flow, this will develop singularities. The backwards flow (add a minus sign to the right-hand side of (45)), can be shown to asymptotically converge to an ellipse.

When the heat flow can develop singularities, as in the scale-invariant cases described above, one can use the most general flow given by (22), i.e., to multiply the velocity by functions of the group curvature and its derivatives. We are currently investigating these more general flows and their possible smoothing properties.

4 Geometric Heat Flows Without Shrinkage

In the previous sections, we derived intrinsic geometric versions of the (non-geometric) classical heat flow (or Gaussian filtering). Using this geometric methodology, we now will solve the problem of shrinkage to which we alluded above. This theory is developed in [63], to which we refer the interested reader for all the details and relevant references.

A curve deforming according to the classical heat flow shrinks. This is due to the fact that the Gaussian filter also affects low frequencies of the curve coordinate functions [48]. Oliensis [48] proposed to change the Gaussian kernel to a filter which is closer to the ideal low pass filter. This way, low frequencies are less affected, and less shrinkage is obtained. With this approach, which is also non-intrinsic, the semi-group property holds just approximately. Note that in [5, 71] (see also [1]), it was proved that filtering with a Gaussian kernel is the unique linear operation for which the causality criterion holds. In fact, the approach presented in [48], which is closely related to wavelet approaches [43, 44], violates this important principle.

Lowe [42] proposes to estimate the amount of shrinkage and to compensate for it. The estimate is based on the amount of smoothing (variance/time) and the curvature. This approach, which only reduces the shrinkage problem, is again non-intrinsic, since it is based on Gaussian filtering, and works only for small rates of change. The semi-group property is violated as well.

Horn and Weldon [31] also investigated the shrinkage problem, but only for convex curves. In their approach, the curve is represented by its extended circular image, which is the radius of curvature of the given curve as a function of the curve orientation. The scale-space is obtained by filtering this representation.

We now show how to solve the shrinkage problem with a variation of the geometric flows described above. The resulting flows will keep all the basic properties of scale-spaces, while preserving area (length) and performing geometric smoothing at the same time [63].

4.1 Area Preserving Euclidean Flow

We now solve the shrinkage problem with the Euclidean geometric heat flow following ideas of Gage [25].

Consider the evolution (21) given above. When a closed curve evolves according to (21), it is easy to prove that the enclosed area $\mathbf{A}$ evolves according to

$$\frac{\partial \mathbf{A}}{\partial t} = -\oint \beta dv. \tag{46}$$

Therefore, in the case of the Euclidean geometric heat flow (28) we obtain ($\beta = \kappa$)

$$\frac{\partial \mathbf{A}}{\partial t} = -2\pi, \tag{47}$$

and the area decreases. Moreover

$$\mathbf{A}(t) = \mathbf{A}_0 - 2\pi t,$$

where $\mathbf{A}_0$ is the area enclosed by the initial curve $\mathcal{C}_0$. As pointed out in [25, 26, 27], curves evolving according to (28) can be normalized in order to keep constant area. The normalization process is given by a change of the time scale, from t to τ, such that a new curve is obtained via

$$\tilde{\mathcal{C}}(\tau) := \psi(t)\,\mathcal{C}(t), \tag{48}$$

where $\psi(t)$ represents the normalization factor (time scaling). (The equation can be normalized so that the point $\mathcal{P}$ to which $\mathcal{C}(t)$ shrinks is taken as the origin.) In the Euclidean case, $\psi(t)$ is selected such that

$$\psi^2(t) = \frac{\partial \tau}{\partial t}. \tag{49}$$

The new time scale τ must be chosen to obtain $\tilde{\mathbf{A}}_\tau \equiv 0$. Define the collapse time T, such that $\lim_{t \to T} \mathbf{A}(t) \equiv 0$. Then,

$$T = \frac{\mathbf{A}_0}{2\pi}.$$

Let

$$\tau(t) = -T \ln(T - t). \tag{50}$$

Then, since the area of $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are related by the square of the normalization factor $\psi(t) = \left(\frac{\partial \tau}{\partial t}\right)^{1/2}$, $\tilde{\mathbf{A}}_\tau \equiv 0$ for the time scaling given by (50). The evolution of $\tilde{\mathcal{C}}$ is obtained from the evolution of $\mathcal{C}$ and the time scaling given by (50). Taking partial derivatives in (48) we have

$$\begin{aligned}
\frac{\partial \tilde{\mathcal{C}}}{\partial \tau} &= \frac{\partial t}{\partial \tau}\frac{\partial \tilde{\mathcal{C}}}{\partial t} \\
&= \psi^{-2}(\psi_t \mathcal{C} + \psi \mathcal{C}_t) \\
&= \psi^{-2}\psi_t \mathcal{C} + \psi^{-1}\kappa \vec{\mathcal{N}} \\
&= \psi^{-2}\psi_t \mathcal{C} + \tilde{\kappa} \vec{\mathcal{N}} \\
&= \psi^{-3}\psi_t \tilde{\mathcal{C}} + \tilde{\kappa} \vec{\mathcal{N}}.
\end{aligned}$$

From Lemma 2, the flow above is geometrically equivalent to

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau} = \psi^{-3}\psi_t < \tilde{\mathcal{C}}, \vec{\mathcal{N}} > \vec{\mathcal{N}} + \tilde{\kappa} \vec{\mathcal{N}}. \tag{51}$$

Define the ***support function*** ρ as

$$\rho := - < \mathcal{C}, \vec{\mathcal{N}} > .$$

Then, it is easy to show that

$$\mathbf{A} = \frac{1}{2} \oint \rho dv.$$

Therefore, applying the general area evolution equation (46) to the flow (51), together with the constraint $\tilde{\mathbf{A}}_\tau \equiv 0$, we obtain

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p, \tau) = \tilde{\kappa} \vec{\mathcal{N}} - \frac{\pi \tilde{\rho}}{\tilde{\mathbf{A}}(\tau)} \vec{\mathcal{N}}. \tag{52}$$

Note that the flow (52) exists for all $0 \leq \tau < \infty$. Since $\tilde{\mathbf{A}}_\tau \equiv 0$ when $\tilde{\mathcal{C}}$ evolves according to (52), the enclosed area $\tilde{\mathbf{A}}(\tau)$ in (46) can be replaced by $\mathbf{A}_0$, obtaining

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p, \tau) = \left(\tilde{\kappa} - \frac{\pi \tilde{\rho}}{\mathbf{A}_0}\right) \vec{\mathcal{N}}, \tag{53}$$

which gives a local, area preserving, flow.

Since $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are related by dilations, the flows (28) and (53) have the same geometric properties [25, 26, 27, 63]. The properties of this flow can also be obtained directly from the general results on characterization of evolution equations given in [1]. In particular, since a curve evolving according to the Euclidean heat flow satisfies all the required properties of a multi-scale representation, so does the normalized flow. An example of this flow is presented in Figure 2.

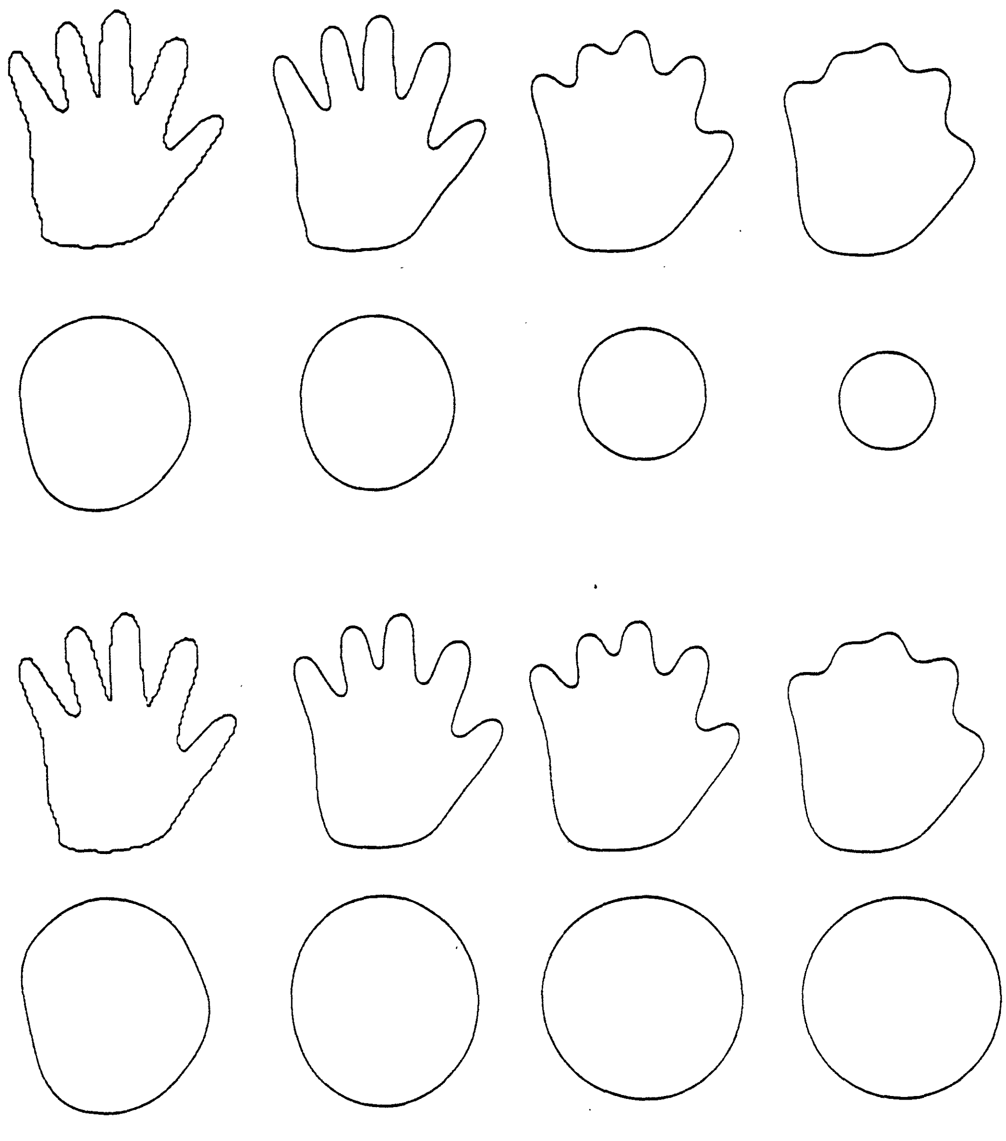

Fig. 2 – Example of the area preserving Euclidean geometric heat flow (from [63]).

4.2 Area Preserving Affine Flow

From the general evolution equation for areas (46), and the flow (36), we have that when a curve evolves according to the affine geometric heat flow, the enclosed area evolves according to

$$\mathbf{A}_t = -\oint \kappa^{1/3} dv. \tag{54}$$

Following [6], we define the *affine perimeter* $\mathbf{L}$ as

$$\mathbf{L} := \oint [\mathcal{C}_p, \mathcal{C}_{pp}]^{1/3} dp.$$

Then it is easy to show that [57]

$$\mathbf{L} = \oint \kappa^{1/3} dv.$$

Therefore,

$$\mathbf{A}_t = -\mathbf{L}. \tag{55}$$

As in the Euclidean case, we define a normalized curve

$$\tilde{\mathcal{C}}(\tau) := \psi(t)\mathcal{C}(t), \tag{56}$$

such that when $\mathcal{C}$ evolves according to (36), $\tilde{\mathcal{C}}$ encloses a constant area. In this case, the time scaling is chosen such that [63]

$$\frac{\partial \tau}{\partial t} = \psi^{4/3}. \tag{57}$$

(We see from the Euclidean and affine examples that in general, the exponent λ in $\frac{\partial \tau}{\partial t} = \psi^{\lambda}$ is chosen such that $\tilde{\beta} = \psi^{1-\lambda}\beta$.) Taking partial derivatives in (56), using the relations (46), (55), and (57), and constraining $\tilde{\mathbf{A}}_\tau \equiv 0$, we obtain the following geometric affine invariant, area preserving, flow:

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau} = \left(\tilde{\kappa}^{1/3} - \frac{\tilde{p}\,\tilde{\mathbf{L}}}{2\tilde{\mathbf{A}}(\tau)}\right)\vec{\mathcal{N}}. \tag{58}$$

Since $\tilde{\mathbf{A}}_\tau \equiv 0$, $\tilde{\mathbf{A}}(\tau)$ in (58) can be replaced by $\mathbf{A}_0$ to obtain

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau} = \left(\tilde{\kappa}^{1/3} - \frac{\tilde{p}\,\tilde{\mathbf{L}}}{2\mathbf{A}_0}\right)\vec{\mathcal{N}}. \tag{59}$$

Note that in contrast with the Euclidean area preserving flow given by equation (53), the affine one is not local. This is due to the fact that the rate of area change in the Euclidean case is constant, but in the affine case it depends on the affine perimeter.

As in the Euclidean case, the flow (59) satisfies the same geometric properties as the affine geometric heat flow (36). Therefore, it defines a geometric, affine invariant, area preserving multi-scale representation.

4.3 Length Preserving Flows

Similar techniques to those presented in previous sections, can be used in order to keep fixed other curve characteristics, e.g., the Euclidean length $\mathbf{P}$ [63]. In this case, when $\mathcal{C}$ evolves according to the general geometric flow

$$\frac{\partial \mathcal{C}}{\partial t} = \beta \vec{\mathcal{N}},$$

and

$$\tilde{\mathcal{C}}(\tau) := \psi(t)\,\mathcal{C}(t), \tag{60}$$

we obtain the following length preserving geometric flow:

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(\tilde{\beta} - \frac{\oint \tilde{\beta}\tilde{\kappa}}{\mathbf{P}_0}\tilde{\rho}\right)\vec{\mathcal{N}}. \tag{61}$$

The computation of (61) is performed again taking partial derivatives in (60), and using the relations (see for example [25])

$$\mathbf{P}_t = -\oint \beta\kappa dv,$$

$$\mathbf{P} = \oint \kappa\rho dv,$$

together with the constraint

$$\tilde{\mathbf{P}}_\tau \equiv 0.$$

The following flows are the corresponding length preserving Euclidean, affine, and similarity heat flows respectively:

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(\tilde{\kappa} - \frac{\oint \tilde{\kappa}^2}{\mathbf{P}_0}\tilde{\rho}\right)\vec{\mathcal{N}}, \tag{62}$$

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(\tilde{\kappa}^{1/3} - \frac{\oint \tilde{\kappa}^{4/3}}{\mathbf{P}_0}\tilde{\rho}\right)\vec{\mathcal{N}}, \tag{63}$$

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(-\tilde{\kappa}^{-1} + \tilde{\rho}\right)\vec{\mathcal{N}}. \tag{64}$$

Note that in the similarity case, the flow is completely local. Another local, length preserving flow may be obtained for the Euclidean constant motion given by

$$\mathcal{C}_t = \vec{\mathcal{N}}. \tag{65}$$

This flow, obtained taking $r \equiv v$ and $\Psi(\chi) = \chi^{-1}$ in (22), models the classical Huygens principle or morphological dilation with a disk [56] (of course, it is not a geometric heat flow of the form (17)). In this case, the rate of change of length is constant and the length preserving flow is given by

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(1 - \frac{2\pi\tilde{\rho}}{\mathbf{P}_0}\right)\vec{\mathcal{N}}. \tag{66}$$

Note that a smooth initial curve evolving according to the Euclidean constant motion (65), as well as to the flow given by (42), can develop singularities [1, 56]. In this case, the physically correct weak solution of the flow is the viscosity (or *entropy* [66]) one [1, 56]. See [56, 63] for examples of this flow.

5 Conclusions

In this paper, we have outlined the theory of differential invariants, and applied them to invariant flows that appear in computer vision. Using this theory, we have defined G-invariant heat-type flows where G is a subgroup of the projective group $SL(\mathbf{R}, 3)$. As we have indicated, these flows, first described in [57, 58, 61, 62, 63], are the simplest possible. In certain cases, such diffusions have been used to define new geometrically invariant scale-spaces. They have also been employed for various problems in image processing and computer vision. See [1, 2, 21, 33, 34, 59, 60, 62] and the references therein. We have also discussed area and length preserving versions of these equations in which there is no shrinkage. See [63] for full details.

In summary, in addition to novel results as the classification and uniqueness of the geometric heat flows, this paper gives a complete picture of the relevant theory of differential invariants and geometric invariant curve flows. Extension of this theory to surface flows can be found in [52].

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