SUBSYSTEMS OF SET THEORY AND ANALYSIS

by

Harvey Martin Friedman

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF
PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF
TECHNOLOGY
August, 1967

Signature of Author..
Department of Mathematics
August , 1967

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Thesis Supervisor

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Committee on Graduate
Students

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Abstract

Usual set theory is formulated in terms of closure conditions. A typical example is the power set axiom, which asserts closure under the operation of power set.

In Chapter I we consider set theory based on closure conditions applied only to definable sets. We formalize this set theory and call it ZF*. Our principal result of Chapter I is that, provably in first-order arithmetic, ZF is consistent if ZF* is. Our proof of this theorem uses a Skolem hull construction and a syntactic transformation.

In Chapter II, we consider three theories of hyperarithmetic analysis, Δ_1^1-CA, Σ_1^1-AC, and Σ_1^1-DC. These are called theories of hyperarithmetic analysis primarily because the hyperarithmetic sets form a minimum ω-model for each of them. We first show that Σ_1^1-DC is a conservative extension of Δ_1^1-CA for purely Π_2^1 sentences. The proof is by means of an inner model construction. Careful attention has to be paid to limit the axioms we use to prove relevant sentences about hyperarithmetic sets. We then show that there are theorems of Σ_1^1-DC which are not theorems of Σ_1^1-AC. This is done by first finding a suitable sentence S and considering the auxiliary

theories Σ_1^1-AC + S and Σ_1^1-DC + S. We then obtain our independence result via Gödel's Theorem, by showing that Con(Σ_1^1-AC + S) is provable in Σ_1^1-DC + S. Last, we show that Σ_1^1-AC is a conservative extension, for purely Π_2^1 sentences, of T, a natural subsystem of predicative analysis. The proof uses an inner model construction on certain auxiliary theories. Thus, a model for each finite subsystem of Σ_1^1-AC is obtained as an inner model of a model of an extension, by the negation of an instance of induction, of a corresponding finite subsystem of T. Thus non-standard models are implicit in the construction.

Chapter III is concerned with hierarchies (based on the jump operator) on recursive linear orderings. Let X be the set of recursive linear orderings which have no hyperarithmetic descending chains. Joseph Harrison showed that there are elements of X, which are not well-orderings, on which there are hierarchies. We first show that under certain weak conditions on a recursive linear ordering, that if there is a hierarchy on it, then it must be in X. Finally, we establish the existence of a recursive linear ordering which is in X, yet on which there are no hierarchies. The proofs of these assertions use certain Lemmas which are proved in the following indirect way: one assumes the Lemma is false, and then forms a theory consisting of the negation of the Lemma together with certain true sentences; then one shows that the resulting theory proves its own consistency.

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35

TABLE OF CONTENTS

BIOGRAPHICAL NOTE

The author was born in Chicago, Illinois in September, 1948. The author attended Highland Park High School in Highland Park, Illinois from September, 1961 to June, 1964, and Massachusetts Institute of Technology from September, 1964 through August, 1967.

ACKNOWLEDGMENTS

The author wishes to express thanks to Professors G. Sacks, S. Feferman, and G. Kreisel for their valuable conversations; to Professors Kreisel and Feferman for their valuable correspondence; and to Professor Sacks for his remarkable patience and encouragement.

BIBLIOGRAPHY

1. S. Feferman, Systems of Predicative Analysis, Journal of Symbolic Logic, vol. 29 (1964), pp. 1-27.
2. Joseph Harrison, Dissertation, Some Applications of Recursive Pseudo-Well Orderings, August 1966, pp. 4-7, 31-33.
3. S. C. Kleene, Introduction to Metamathematics, 1950.
4. G. Kreisel, Survey of Proof Theory, to appear.
5. W. W. Tait, Normal Derivability in Classical Logic, to appear.

INTRODUCTION

Set theory is usually formulated in terms of closure conditions. A typical example is the power set axiom, which asserts closure under the operation of power set. In Chapter I, we consider set theory based on closure conditions applied only to definable sets. We formalize this set theory and call it ZF*, and we give a consistency proof of ZF relative to ZF*.

A direct method presents itself for obtaining this relative consistency result; namely, to use a constructible set construction, and prove within ZF* the relativized to the constructible sets of each instance of ZF. This is, of course, in analogy with the method of proof for the consistency of ZF + AxC relative to ZF. However, an examination of the basic principles needed for such a constructible set construction to go through reveals the need for the least counterexample principle for ordinals to be provable in ZF*. By the least counterexample principle for ordinals, we mean the schema $(\exists\alpha)P\alpha \longrightarrow (\exists\mu\alpha)P\alpha$, where P is any formula. It does not appear that this schema is derivable in ZF*, even if P is restricted to have only one free variable, α. Of course, in ZF, the schema is derivable by means of a closure condition applied to all sets as follows: assume $(\exists\alpha)P\alpha$, and fix such an α. Then form $\{\beta|\beta\epsilon\alpha \;\&\; P\beta\}$, and use Foundation to obtain $(\mu\alpha)P\alpha$. This illustrates the basic difference between ZF and ZF*, in that the closure condition, $\exists\{\beta|\beta\epsilon\alpha \;\&\; P\beta\}$, is necessarily provable in ZF* only when α is given a definition.

Such a direct attack seems hopeless. An outline of our proof can be found in Section 4 of Chapter I.

Towards the end of Chapter I, we show how to add elements "on top of" a model of ZF, to obtain nonstandard models of ZF*. By this means, we obtain results concerning independence from ZF*.

In Chapter II, we consider three theories of hyperarithmetic analysis. These are Δ_1^1-CA, Σ_1^1-AC, and Σ_1^1-DC. Here CA refers to comprehension axiom; AC, to axiom of choice; DC to dependent choice. The hyperarithmetic sets form a minimum ω-model for each of these theories. We first prove that Σ_1^1-DC is a conservative extension of Δ_1^1-CA for purely Π_2^1 sentences. The proof is by means of an inner model construction. We show that given a model of Δ_1^1-CA, if we then take the submodel of all sets hyperarithmetic in a fixed set, this submodel satisfies Σ_1^1-DC. Careful attention has to be paid to the way in which the notion of relative hyperarithmeticity is formalized. The formulation in terms of hierarchies seems to be the correct one here (not Δ_1^1). The usual proof that the sets hyperarithmetic in a fixed set always form an ω-model of Σ_1^1-DC is too crude for our purposes. It uses the comparability of all recursive well-orderings, which is a principle too strong to be provable in Δ_1^1-CA. However, if we know, in Δ_1^1-CA, that given orderings have hierarchies (based on the jump operator) on them, we can then conclude, in Δ_1^1-CA, their comparability. Thus, the key point of our proof of the conservative extension result is the judicious use of hierarchies on orderings.

Our inner model construction can, in the standard way, be transformed into a finitary consistency proof of Σ^1_1-DC relative to Δ^1_1-CA.

The second result of Chapter II is the independence of Σ^1_1-DC from Σ^1_1-AC. It would seem to be the case that a Cohen type argument would be not only useful here, but perhaps necessary. We found that quite the contrary is true. Our proof does not use a Cohen type argument, and Cohen type arguments do not seem to be helpful here, since it seems difficult to find Cohen type models (starting with the standard model, the hyperarithmetic sets, of Σ^1_1-AC) which do not have the following property: any arithmetical predicate (in x) satisfied to have a solution in the new model has a solution hyperarithmetic (in x) in the new model. This property can be seen, from our Theorem 1 of Chapter II, to imply that the new model satisfies Σ^1_1-DC.

Instead, we use Gödel's theorem. We choose a sentence S and consider the auxiliary theories Σ^1_1-AC + S and Σ^1_1-DC + S. We show that Σ^1_1-DC + S proves the consistency of Σ^1_1-AC + S. So if Σ^1_1-DC + S = Σ^1_1-AC + S, then Σ^1_1-DC + S is inconsistent. But S is chosen to be a true sentence; so Σ^1_1-DC $\neq$ Σ^1_1-AC.

Notice that the assumption Con(Σ^1_1-DC + S) is needed for the independence. We do not know if there is a finitary independence proof (i.e., a finitary proof of consistency of Σ^1_1-AC + $\sim$F relative to Σ^1_1-AC, for some F that is provable in Σ^1_1-DC).

The key property of Σ_1^1-DC is that for any Π_2^1 sentence, B, one can construct, in Σ_1^1-DC + B, an ω-model for B. The key property of the sentence S is that the theory Σ_1^1-AC + S is equivalent, in Σ_1^1-DC, to an extension of induction by a Π_2^1 sentence. S intuitively says that for all sets x, every recursive well-ordering in x has a hierarchy starting from x.

More information may be obtained than independence. We see that Σ_1^1-DC + S proves Con(Σ_1^1-AC + S); hence Σ_1^1-DC proves S $\longrightarrow$ Con(Σ_1^1-AC + S), which is a purely Σ_2^1 sentence. Furthermore, an examination of our proof yields that Σ_1^1-AC together with only a finite number of instances of no parameter Σ_1^1-DC is needed to prove S $\longrightarrow$ Con(Σ_1^1-AC + S). Finally, Σ_1^1-DC + S proves $\exists$ an ω-model for Σ_1^1-AC + S. From all this we can conclude that there is a purely Σ_2^1 sentence which is provable in Σ_1^1-AC together with a finite number of instances of no parameter Σ_1^1-DC, but which is not an ω-consequence of Σ_1^1-AC. (An instance of no parameter Σ_1^1-DC is the same as Σ_1^1-DC, except the hypothesis, $(f)(\exists g)A(f,g)$, must have A arithmetical with no free variables other than f and g.)

The last result in Chapter II is concerned with the relation between Σ_1^1-AC and a certain natural subsystem of predicative analysis, T. T represents the first ϵ_0 levels of predicative analysis. We show that Σ_1^1-AC is a conservative extension of T for purely Π_2^1 sentences. This result, together with the known characterization of the provable ordinals of T, gives the provable ordinals of Σ_1^1-AC. Furthermore, our proof of conservative extension uses an inner model construction, which can be transformed into a finitary proof of consistency of Σ_1^1-AC relative to T. Now T is known to have a predicative consistency proof; and so, then, must Σ_1^1-AC.

The relative consistency result is somewhat surprising, since the minimum ω-model of Σ_1^1-AC is so much larger than the minimum ω-model of T. With this in mind, it is not surprising that nonstandard models (i.e., non-ω-models) must be essential in our proof. In the proof, we obtain a model of each finite subsystem of Σ_1^1-AC as an inner model of a model of an extension, by the negation of an instance of induction, of a corresponding finite subsystem of T.

In Chapter III, we consider which elements of W^* have hierarchies. We generalize W^* to include recursive linear orderings whose field is not necessarily ω. We also generalize hierarchies, so that at successor we merely have a set in which the jump of the set at the predecessor is recursive; at limits, we have a set in which the effective union of the previous sets is recursive. Harrison proved that $\exists\, n \in W^* - W$ on which there are hierarchies, in the less general sense W^*, W and hierarchies. He left open whether every $n \in W^*$ has a hierarchy. We answer it in the negative for our general notion of hierarchy and the less general notion of W^*.

We also show, under weak conditions, that if n has a hierarchy, then $n \in W^*$.

The proofs use certain Lemmas which are proved in the following highly indirect way: we assume the Lemma is false, and form a theory by adding true sentences to the negation of the Lemma; we then show that the resulting theory proves its own consistency.

CHAPTER I

1. General Situation. Suppose we are given a comprehension axiom $(x_1)\ldots(x_n)(\exists y)Rx_1\ldots x_n y$. We are interested here in forming the derived schema consisting of the axioms

$$[(\exists! x_1)(\exists! x_2)\ldots(\exists! x_n)(F_1 x_1 \;\&\ldots\&\; F_n x_n)] \longrightarrow$$
$$(\exists x_1)(\exists x_2)\ldots(\exists x_n)(F_1 x_1 \;\&\; F_2 x_2 \;\&\ldots\&\; F_n x_n \;\&\; (\exists y)Rx_1\ldots x_n y),$$

where the F_i are formulae with only the free variable x_i.

We are purposely vague about the general situation (what is a comprehension axiom?), since we have only looked at this derived schema when the original axiom is drawn from a natural set of axioms, such as ZF, or analysis, or other naturally occurring systems.

More specifically, we will look at the schema of schema formed by taking the union of all the schema defined above corresponding to each of the comprehension axioms of ZF. We will not perturb the other (non-comprehension) axioms of ZF, except in minor ways. We call this derived theory ZF*. (We inessentially modify the Replacement schema in ZF for convenience, so that each instance is appropriately placed in the form $(x_1)\ldots(x_n)(\exists y)Rxy$, so that we may pass to the derived schema in the manner above.) The axioms of ZF and ZF* are spelled out in detail, in an elegant form, in the next section.

We are interested in the relation between ZF and ZF* as axiomatic theories. Our main result is that $\vdash_{ENT}$ ConZF* $\longrightarrow$ ConZF. It is obvious that ZF* is a subsystem of ZF.

2. Remarks on Terminology and Notation. The only (standard) symbols that can occur in a formula of ZF (or ZF*) are the 2 2-ary relation symbols "=", "ϵ"; the 2 quantifiers $(\exists x)$ and (x); the propositional connectives; and variables x_i, y_i, z_i, u_i, v_i, etc. Everything else is nonstandard; when nonstandard symbols occur in a formula, they are meant to be expanded out in such a way that the mere occurrence of a non-standard symbol implies existence of the corresponding set. For example, $x = \bigcup y$ is an abbreviation for $(\exists z)[(w)(w \epsilon z \equiv (\exists u)(w \epsilon u \ \& \ u \epsilon z)) \ \& \ x = z]$. Also, say, $\emptyset \epsilon x$ would be $(\exists y)[(z)(z \notin y) \ \& \ y \epsilon x]$.

3. Axioms. For ZF, we have

0. Axioms for predicate calculus with equality.
1. Extensionality. $(x_0 = x_1) \equiv (x_2)(x_2 \epsilon x_0 \equiv x_2 \epsilon x_1)$.
2. Infinity. $(\exists x_0)(\emptyset \epsilon x_0 \ \& \ (x_1)(x_1 \epsilon x_0 \longrightarrow x_1 \cup \{x_1\} \epsilon x_0))$.
3. Power set. $(x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 \equiv (x_3)(x_3 \epsilon x_2 \longrightarrow x_3 \epsilon x_0))$.
4. Sum set. $(x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 \equiv (\exists x_3)(x_3 \epsilon x_0 \ \& \ x_2 \epsilon x_3))$.
5. Replacement schema. Let Axy be a formula with the free variables x and y and possibly more free variables $x_1,..,x_n$. Then

$$(x_1)\ldots(x_n)(y)[(\exists y_1)(y_2)(y_2 \in y_1 \equiv (\exists y_3)(Ay_3y_2 \ \& \ (y_4)$$
$$(Ay_3y_4 \longrightarrow y_4 = y_2) \ \& \ y_3 \in y))]$$

is an instance. (The domain is y and the axiom asserts the range of the partial function, $A'y_3y_2 = Ay_3y_2 \ \& \ (z)$ $(Ay_3z \longrightarrow z = y_2)$, on the domain y, exists.)

6. Foundation. $(x_0)(x_0 \neq \emptyset \longrightarrow (\exists x_1)(x_1 \in x_0 \ \& \ (x_2)(x_2 \in x_1 \longrightarrow x_2 \notin x_0)))$.

It is clear that by the usual process of making partial functions into total functions axiom schema 5 is the same as the usual formulation in the present context.

Now let Ax be a formula of 1 free variable. Then the formula $(\exists y)(x)(x \in y \equiv Ax)$ is abbreviated as C_A.

For ZF* we have

0. Same as ZF.

1. Same as ZF.

2. Infinity. $(\exists x_0)(\emptyset \in x_0 \ \& \ (x_1)(x_1 \in x_0 \longrightarrow x_1 \cup \{x_1\} \in x_0) \ \& \ (y)(y \in x_0 \equiv \mathrm{Fin}(y))$. Fin(y) will be defined later. Intuitively it means y is a finite ordinal.

3. Power set. The instances are $C_A \longrightarrow (\exists x_0)(x_1)(x_1 \in x_0 \equiv (x_2)(x_2 \in x_1 \longrightarrow Ax_2))$, A with 1 free variable.

4. Sum set. The instances are $C_A \longrightarrow (\exists x_0)(x_1)(x_1 \in x_0 \equiv (\exists x_2)(x_1 \in x_2 \ \& \ Ax_2))$, A with 1 free variable.

5. Replacement schema. The instances are $C_A \longrightarrow (\exists x_0)(x_1)(x_1 \in x_0 \equiv (\exists x_2)(Ax_2 \ \& \ Bx_2x_1 \ \& \ (x_3)(Bx_2x_3 \longrightarrow x_3 = x_1)))$, for A with 1 free variable, B with exactly 2 free variables.

6. Same as ZF.

Remarks: Our formulation of Power set in ZF* is seen to be equivalent to the derived schema of Power set in ZF (as given in <u>General Situation</u>) by noticing 1) that if $(\exists! y)Fy$, then C_A, where A is $(\exists y)(Fy \mathbin{\&} x \epsilon y)$ and 2) that if C_A, then $(\exists! x)(y)(y \epsilon x \equiv Ay)$. These latter are obtained by axiom 1 of ZF*, Extensionality. The same remark applies to Sum Set.

Essentially the same idea yields that the union of the derived schema of the instances of Replacement in ZF is equivalent, in the present context, to the schema

$$(C_{A_1} \mathbin{\&} \ldots C_{A_n} \mathbin{\&} C_A) \longrightarrow (\exists x_0)(x_1 \epsilon x_0 \equiv (\exists x_2)(Ax_2 \mathbin{\&} B(x_2, x_1, \{x|A_1x\}, \{x|A_2x\}, \ldots, \{x|A_nx\}) \mathbin{\&} (x_3)(B(x_2, x_3, \{x|A_1x\}, \ldots \{x|A_nx\}) \longrightarrow x_3 = x_1)))),$$

where B has n + 2 free variables, A and the A_i have 1 free variable. We want to show this schema is contained (in the present context) in Replacement in ZF*. But the above is easily seen to follow from that instance of 5 of ZF*, setting A as A, Bxy as $(C_{A_1} \mathbin{\&} \ldots \mathbin{\&} C_{A_n}) \longrightarrow B(x_2, x_1, \{x|A_1x\}, \{x|A_2x\}, \ldots, \{x|A_nx\})$. (That the above schema contains Replacement in ZF* is obvious.) NOTE: "In the present context" means "using the other axioms of ZF*."

4. <u>Outline of Proof of Main Theorem</u>. The main theorem is $\vdash_{ENT} \mathrm{Con}ZF^* \longrightarrow \mathrm{Con}ZF$. The first step in proving this is developing in ZF* an adequate definition of ordinals, which

turns out to be a much more delicate matter than for ZF, due to the lack of certain key instances of Replacement in ZF*. By an adequate definition of ordinals in ZF*, we mean a definition of ordinals such that provably in ZF*, members of ordinals are ordinals, and (for a natural definition in ZF* of ω) ω is an ordinal, and ordinals are comparable by ε, and the ε-relation on the ordinals is transitive, and antisymmetric, and antireflexive, and every ordinal has a (natural) successor, except possibly the greatest ordinal. (It even turns out that in ZF* we can prove there is no greatest ordinal for our definition of ordinal given later.) The definition is made and Lemma 1 establishes the above properties for it in the next section; we even obtain more: that, provably in ZF*, the new definition of ordinal coincides with the usual definition given in ZF, on <u>definable</u> sets. (This is made precise in Lemma 1, f).)

Next we develop an adequate definition of L within ZF*. Among the properties of the predicate $x \in L$ needed, we must have, provable in ZF*, every member of an $x \in L$ is $\in L$, $\omega \in L$, the new definition of $x \in L$ coincides with the usual definition of constructibility for definable x, and a definable well-ordering of L.

With this machinery, an apparently straightforward "proof" of our main result comes to mind. Namely, just to prove the relativized to L of each instance of ZF in ZF* by taking least counterexamples of various things in ZF*, as Gödel established the relativization of the axioms of ZF to

L within ZF. But a moment's reflection will reveal that one can hardly expect that ZF* will prove any general least counterexample principles; i.e., one may well be able to prove in ZF* that $(\exists x)(x \in L \ \& \ Px)$, yet not be able to prove $(\exists \mu x)(x \in L \ \& \ Px)$ in ZF*, where μ is defined in terms of the definable well-ordering of L. In fact, we do not even see how to prove each instance of the relativized of ZF* to L, within ZF*!

In order to get our main result, we form an auxiliary system $ZF^{*\prime} \subsetneq ZF^*$, whose definition depends on a certain crucial transformation on sentences, T. This subsystem has the property that each instance of ZF*' semi-relativized (semi-relativization is a certain modification of relativization) to L, is provable in ZF*. We form another auxiliary theory ZF' which is related to ZF*' about as ZF + V = L is to ZF*. It turns out that $ZF' \supseteq ZF$. It also turns out that in the theory obtained by semi-relativizing each comprehension axiom of ZF*' and retaining the other axioms, one can give a Skolem hull argument for each finite subsystem of ZF' that proves the existence of a (suitably definable) model of this finite subsystem, and hence its consistency.

Putting all this together we get a finitary proof that (n) $\vdash_{ZF^*} Con(ZF_n)$. So $\vdash_{ENT} Con ZF^* \longrightarrow Con ZF$.

5. <u>Development of Ordinals</u>. We define $\mathrm{Ord}x = \mathrm{Trans}(x)$ & ε-Conn(x) & (x is semi-closed under succession) & there are no 3-chains in x, i.e., $\mathrm{Ord}(x) = (y)(z)((y \in x \ \& \ z \in y) \longrightarrow z \in x)$ & $(y)(z)((y \in x \ \& \ z \in x) \longrightarrow y \in z \vee z \in y \vee y = z))$ & $(y)(y \in x \longrightarrow (\exists z)$ $[(w)(w \in z \equiv (w \in y \vee w = y))$ & $(z \in x \vee z = x)])$ & $(y)(z)(w)(\sim[y \in z$ & $z \in w$ & $(w \in y \vee w = y)])$. We define $\mathrm{Ord}'(x) = \mathrm{Ord}x$ & $(y)((\mathrm{Ord}y$ & $y \subseteq x) \longrightarrow (y \in x \vee y = x))$. $\mathrm{Ord}''(x) = \mathrm{Ord}'(x)$ & $(y)(\mathrm{Ord}'y \longrightarrow$ $(x \subseteq y \vee y \subseteq x))$. Ord" will be our notion of ordinal in ZF*. It is obvious that $(x)(y)(\mathrm{Ord}''x$ & $\mathrm{Ord}''y \longrightarrow (x \in y \vee y \in x \vee y = x))$. For Ord(x), we define y <u>is successor of</u> x if and only if $y = x \cup \{x\}$.

<u>Lemma 1</u>: The following are Theorems of ZF*, where Ax has 1 free variable:

a) If $\mathrm{Ord}x$ & $y \in x$, then $\mathrm{Ord}y$.

b) If $\mathrm{Ord}'x$ and $y \in x$ then $\mathrm{Ord}'y$.

c) For x with Ord(x), and x not a successor, $\bigcup x = x$.
For Ord(x), x a successor, $x = y \cup \{y\}$, we have $\bigcup x = y$.

d) If $\mathrm{Ord}''(x)$ and $y \in x$ then $\mathrm{Ord}''(y)$.

e) If $y = \{x | Ax\}$, $\mathrm{Ord}'(y)$, then $\mathrm{Ord}''(y)$.

f) If $y = \{x | Ax\}$, $\mathrm{Ord}(y)$, then $\mathrm{Ord}''(y)$.

g) $\mathrm{Ord}''(\omega)$. (Explained below.)

h) If $y = \{x | Ax\}$, and $Ax \longrightarrow \mathrm{Ord}''(x)$, then $\mathrm{Ord}''(\bigcup y)$.

Proof: a) Claim Trans(y). Let $z \in y$, $w \in z$. Then by Trans(x), we have $w \in x$ and $y \in x$. By ε-Conn(x), $w \in y \vee y \in w$ $\vee y = w$. We can't have $y \in w \vee y = w$ because we would have a 3-chain. So $w \in y$. To see ε-Conn(y), let $z \in y$, $w \in y$. Then $z \in x$ and $w \in x$, and so $z \in w \vee w \in z \vee w = z$ by ε-Conn(x).

Towards showing y semi-closed, let $z \epsilon y$. Hence $z \epsilon x$, and $z \cup \{z\} \epsilon x$ or $z \cup \{z\} = x$. By ε-Conn(x), we have $z \cup \{z\} \epsilon y$ or $z \cup \{z\} = y$ or $y \epsilon z \cup \{z\}$. But if $y \epsilon z \cup \{z\}$, then $y \epsilon z \vee y=z$. The first yields the 3-chain $z \epsilon y$, $y \epsilon z$, $z=x$; the second yields $y \epsilon y$, which yields the 3-chain $y \epsilon y$, $y \epsilon y$, $y \epsilon y$. So $y \notin z \cup \{z\}$.

Now suppose $a \epsilon b$, $b \epsilon c$, $(c \epsilon a \vee c=a)$, $a,b,c, \epsilon y$. Then $a,b,c \epsilon x$, and we have a 3-chain in x.

b) By a), we have Ord(y). Towards showing Ord'(y), let $z \subseteq y$, Ord(z). By Trans(x), $z \subseteq x$. Hence $z \epsilon x \vee z=x$. But $z \neq x$, for if $z=x$, then $x \subseteq y$, and hence $y \epsilon y$. So $z \epsilon x$. By ε-Conn(x), $z \epsilon y \vee z=y \vee y \epsilon z$. But $y \notin z$, since if $y \epsilon z$, then $y \epsilon y$.

c) Let Ord(x), x with $(y)(y \cup \{y\} \neq x)$. By Trans(x), every member of a member of x is a member of x. So $\cup x \subseteq x$, if $\cup x$ exists. Now let $y \epsilon x$. By semi-closure of x, $y \cup \{y\} \epsilon x$. But $y \epsilon y \cup \{y\}$, and so $x = \cup x$, since also every member of x is a member of a member of x. If $x = z \cup \{z\}$, then again $\cup x \subseteq x$, if $\cup x$ exists. But $z \notin \cup x$, since x has no 3-chains. So $\cup x \subseteq z$, if it exists. But every member of z is a member of x, since if $w \epsilon z$, then w is a member of a member of x. So $\cup x$ exists and is z.

d) By c), Ord'(y). Let Ord'(z). Then $x \subseteq z \vee z \subseteq x$. If $x \subseteq z$, then $y \subseteq z$. If $z \subseteq x$, then $z \epsilon x \vee z=x$. If $z=x$, then $y \subseteq x$. If $z \epsilon x$, then $y \epsilon z \vee z \epsilon y \vee z=y$. Hence $x \subseteq y \vee y \subseteq z$.

e) Let Ord'(y), $y = \{x|Ax\}$. Then either

1) All elements of y are Ord". Then let z be any Ord'. For every $x \in y$ we have $x \subseteq z \vee z \subseteq x$. Either all members of y are $\subseteq z$ or some member of $y \supseteq z$. Suppose the first holds. Then if y is not a successor, by c) we have $\bigcup y = y$, and so $y \subseteq z$. If $y = u \cup \{u\}$, then $u \subseteq z$. Since Ord'(z), we have $u \in z \vee u=z$. If $u=z$, then $z \subseteq x$. If $u \in z$, then $y \subseteq z$.

Suppose the second holds, i.e., some $x \in y$ contains z. Then clearly $z \subseteq y$. So y is ord".

2) Some element of y is not Ord". We have just proved that any Ord' such that every member is Ord" is Ord". The (unique) ε-least element of y which is not Ord" is definable, and has every member an Ord". (There is a least by suitable use of Replacement schema of ZF*, and Foundation.) Just apply 1) to obtain a contradiction.

f) We merely have to show all definable Ords are Ord'. As in e) we go down to a definable Ordinal all of whose members are Ord". Let y be such an ordinal, $y = \{x|Bx\}$. Let $z \subseteq y$, Ordz. We wish to show $z \in y \vee z=y$. Suppose $z \notin y$ & $y \notin z$. Then $\exists\, w \in y$ with $w \notin z$. But since Ord"w, we have $w \subseteq z \vee z \subseteq w$. If $z \subseteq w$ then $z=w \vee z \in w$ and so $z \in y$. So $w \subseteq z$, $w \notin z$. Now either $w \cup \{w\} \in y \vee w \cup \{w\} = y$. If $w \cup \{w\} = y$ then since $z \subseteq y$, $z=w$, and so $z \in y$. So $w \cup \{w\} \in y$. Now then Ord" $(w \cup \{w\})$. Hence $w \cup \{w\} \subseteq z \vee z \subseteq w \cup \{w\}$. The first is out, so $z \subseteq w \cup \{w\}$, and since Ord'$(w \cup \{w\})$ we have $z = w \cup \{w\} \vee z \in w \cup \{w\}$, either one implying $z \in y$.

g) We now explain Axiom 3. $\mathrm{Fin}(x) = \mathrm{Ord}x \ \& \ (\exists y)(x=y \cup \{y\})$ and $(z)(z\epsilon x \longrightarrow (\exists w)(z=w \cup \{w\}))$. x is a successor Ord = $\mathrm{Ord}x \ \& \ (\exists y)\ (y=x \cup \{x\})$. We let ω be the x_0 in Ax. 2 of ZF*.

Now clearly ω is definable, and so we merely have to show $\mathrm{Ord}(\omega)$.

1) $\mathrm{Trans}(\omega)$. Let $n\epsilon\omega$. Let $x\epsilon n$. Then x is a successor Ord or $\emptyset$. Since n is transitive, x is a successor Ord or $\emptyset$ and every member of x is a successor Ord or $\emptyset$, and so x is a finite Ord (i.e., $\mathrm{Fin}(x)$), and so $x\epsilon\omega$.

2) ϵ-$\mathrm{Conn}(\omega)$. Suppose $\exists\ n\epsilon\omega$ such that for some $m\epsilon\omega$, $n\not\epsilon m \ \& \ m\not\epsilon n \ \& \ n\neq m$. Take any ϵ-least such n and call it k. There is an ϵ-least by Replacement in ZF* and Foundation. Now $k \neq \emptyset$, since $\emptyset\epsilon m \vee \emptyset=m$, for any $m\epsilon\omega$, by Foundation and $\mathrm{Trans}(m)$. Hence k is a successor Ord, and $k=1 \cup \{1\}$, and by $\mathrm{Trans}(\omega)$, $1\epsilon\omega$. We must have, for some $m\epsilon\omega$, $k\not\epsilon m \ \&$ $m\not\epsilon k \ \& \ k\neq m \ \& \ (1\epsilon m \vee m\epsilon 1 \vee m=1)$. But if $m=1$ then $m\epsilon k$. If $m\epsilon 1$, then $m\epsilon k$. If $1\epsilon m$ then $1 \cup \{1\} = m \vee 1 \cup \{1\}\epsilon m$, i.e., $k=m \vee k\epsilon m$. Contradiction, by the lack of 3-chains in Ords, and the comparability of 1 and m.

3) Semi-closure under succession. This is insured directly from Axiom 2 of ZF*.

4) No 3-chains. Suppose $n\epsilon m$, $m\epsilon r$, $r\epsilon n \vee r=n$, where $n,m,r\epsilon\omega$. By $\mathrm{Trans}(r)$, we have $n\epsilon r$. By $\mathrm{Trans}(r)$ again, we have $r\epsilon r$, if $r\epsilon n$. If $r=n$, we also have $r\epsilon r$. These contradict $\mathrm{Ord}(r)$.

h) We have only to show $\mathrm{Ord}(\bigcup y)$.

1) $\mathrm{Trans}(\bigcup y)$. If $z \in \bigcup y$, then $z \in w$ for some $\mathrm{Ord}''(w)$, $w \in y$; hence any $u \in z$ has $u \in w$, $w \in y$. So $u \in \bigcup y$.

2) $\varepsilon\text{-Conn}(\bigcup y)$. If $z, w \in \bigcup y$, then $z \in z' \in y, w \in w' \in y, z', w'$ both Ord". So $z \subseteq w'$ or $w \subseteq z'$. Without loss of generality, assume $z' \subseteq w'$. Then $z \in w'$, $w \in w'$. Hence $z \in w \vee w \in z \vee w = z$.

3) Semi-closure. Let $z \in \bigcup y$. Then $z \in w \in y$, $\mathrm{Ord}''(w)$. So $z \cup \{z\} \in w$ or $z \cup \{z\} = w$. In the first case, $z \cup \{z\} \in \bigcup y$. In the second suppose

A) some $u \in y$ has $w \subseteq u$, but $w \neq u$. Hence $w \in u$, and so $z \cup \{z\} = w \in \bigcup y$.

B) Every $u \in y$ has either $w = u$ or $u \subseteq w$. Then clearly $\bigcup y = w = z \cup \{z\}$.

A) and B) are exhaustive.

4) No 3-chains. Suppose $a \in b$, $b \in c$, $(c \in a \vee c = a)$, where $a, b, c \in \bigcup y$. Then a, b, c are Ord", and $a \in c$. If $c \in a$, then $c \in c$. If $c = a$, $a \in c$, then $c \in c$. Contradicts $\mathrm{Ord}(c)$.

6. <u>Development of L</u>. We wish to define a class of sets, L, which has a definable well-ordering, and provably so in ZF*. L, of course, will not be an object. We are interested in the predicate $x \in L$.

We let n, m, r, p, q be special variables for elements of ω. We let $\alpha, \beta, \gamma, \ldots$ be special variables for sets x with $\mathrm{Ord}''x$. We write $\alpha+1$ for $\alpha \cup \{\alpha\}$. We let λ be a special variable for limit (non-successor and non-null) Ord"'s.

We say <u>x=M(α)</u> = $(\exists f)(\mathrm{Dom} f = \alpha+1\ \&\ f(\emptyset) = \emptyset\ \&\ (\lambda)(\lambda\epsilon\alpha+1 \longrightarrow \bigcup_{x\epsilon\lambda} f(x) = f(\lambda))\ \&\ (\beta)(\beta\epsilon\alpha+1 \longrightarrow f(\beta+1) = \mathrm{Fodo}(f(\beta))\ \&\ x=f(\alpha))$, where $\mathrm{Fodo}(y) = \{x \mid (\exists x_0)(\exists n)(x=\{z \mid z\epsilon y\ \&\ \langle y,\varepsilon_y\rangle \models n(z)[x_0]\}\ \&\ \mathrm{FinSeq}(x_0,y))\}$, where $\langle y,\varepsilon_y\rangle \models n(z)[x_0]$ means the structure $\langle y,\varepsilon_y\rangle$ satisfies the formula with Gödel number n at the sequence of elements of the domain y, (z,x_0), which is the sequence starting with z, followed by the sequence x_0.

Remark on formalization: We formalize the satisfaction relation in ZF* the same way we do in ZF. Also see 2. <u>Remarks on Terminology and Notation</u>.

<u>Lemma 2</u>: For each Ax with 1 free variable, and for each $Czy_1\ldots y_n$, and Bzw with only free variables shown, the following is provable in ZF*: If $y = \{x \mid Ax\}$ and if Bzw is a well-ordering of $y \cup \{y\}$, and $y_1,\ldots,y_n \epsilon y$, then $\{z \mid z\epsilon y\ \&\ \langle y,\varepsilon_y\rangle \models \bar{n}(z)[y_1,\ldots,y_n]\} = \{z \mid z\epsilon y\ \&\ Czy_1\ldots y_n\}$, where n is the Gödel number of C. Thus Fodo(x) means the set of all sets first-order definable over x, for definable sets x.

Proof: Suppose Czy_1 is $(\exists w)(w\epsilon z\ \&\ w\not\epsilon y_1)$. We note that both $\{z \mid z\epsilon y\ \&\ Czy_1\}$.. and $\{z \mid z\epsilon y\ \&\ \langle y,\varepsilon_y\rangle \models \bar{n}(z)[y_1]\}$ exist by Replacement on the definable y. We want to show $Czy_1 \equiv \langle y,\varepsilon_y\rangle \models \bar{n}[z,y_1]$, for $z\epsilon y$. The proof in the case of ZF is routine. What complicates it in the case of ZF* is that certain sets definable in terms of the members of y may not provably exist in ZF*, and also that the theory of Gödel numbering may not be formalizable in ZF*. The latter is not the case, since ω is definable, and hence by Replacement and Foundation, induction on ω provably holds in ZF*, and also we may define + and x, and prove the relevant properties. What comprehension axioms are needed to establish our equivalence? Apparently, what's involved is just that the theory of finitely hereditary sequences of elements of y and natural numbers provably in ZF* have the intended interpretation. For instance, we must verify provability in ZF*, for sentences like "for every $y_1,y_2,y_3 \epsilon y$, $n,m \epsilon \omega$, there exists the sequence $<\{y_1,n\},\{\{n,m\}\},\{y_2,y_3,\{n\}\}>$." Such sentences can clearly be proved by suitable instances of Replacement in ZF* for definable sets y_1,y_2,y_3, and n,m. One assumes in ZF* that such a sentence in false, and goes to definable counterexamples y_1,y_2,y_3 and n,m via the definable well-ordering of $y \cup \{y\}$, Bzw.

<u>Lemma 3</u>: Each instance of the following is provable in ZF*:

a) $[\alpha = \{x|Ax\}$ & $(\beta=\alpha \vee \beta\epsilon\alpha)$ & $(\exists x)(\underline{x=M(\beta)})] \longrightarrow \exists! f$ satisfying the conditions given in the definition of $\underline{x=M(\beta)}$.

b) $\alpha = \{x|Ax\} \longrightarrow (\exists! x_0)(\underline{x_0=M(\alpha)})$. Also $\beta\epsilon\alpha = \{x|Ax\} \longrightarrow (\exists! x_0)(\underline{x_0=M(\beta)})$.

Proof: Assume $\alpha = \{x|Ax\}$ & $(\exists x)(\underline{x=M(\alpha)})$. Suppose we have 2 functions f,g satisfying conclusion, and $f \neq g$. We take, using Replacement and Foundation in ZF* in the usual way, β to be the ϵ-least element of $\alpha + 1$ with the property that $\exists f$ and g, $f \neq g$ satisfying definition of $\underline{x=M(\alpha)}$, with $f(\beta) \neq g(\beta)$. Then β is definable. Suppose $Lim(\beta)$. Then use Replacement on β to get the $\{y|(\exists\gamma)(\gamma\epsilon\beta$ & for all f

satisfying definition of $\underline{x=M(\alpha)}$, $f(\gamma) = y)\}$. We can apply sum set in ZF* to get a union, $\cup$. In any f satisfying the definition of $\underline{x=M(\alpha)}$, clearly $f(\beta) = \cup$. But this is contrary to hypothesis, $f(\beta) \neq g(\beta)$.

Now suppose $\beta = \delta+1$. By hypothesis, $f(\delta)$ is fixed when f varies over the functions satisfying the definition of $\underline{x=M(\alpha)}$. Clearly $f(\delta+1) = \mathbf{Fodo}(f(\delta))$ for any f satisfying the definition of $\underline{x=M(\alpha)}$, and so $f(\delta+1)$ is also independent of f, again contradicting the definition of β.

Clearly $\beta \neq \emptyset$.

So no such β exists.

Now suppose for some $\beta\epsilon\alpha$, part a) false. Take least such β, and apply above, since least such β is definable. This concludes part a).

b) Now suppose $(\exists! x_0)(\underline{x_0=M(\beta)})$ for all $\beta\epsilon\alpha$, but not for $\beta = \alpha$. We may assume this without loss of generality, by taking least counterexamples. We conclude the proof of Lemma 3 by obtaining a contradiction. Suppose $\mathrm{Lim}(\alpha)$. Using Replacement in ZF* on α we can get the set of all $M(\beta)$'s, $\beta\epsilon\alpha$. (We write $M(\beta)$ for that x_0 with $\underline{x_0=M(\beta)}$ if it is unique.) Thus we may take the union by sum set in ZF* and call this $\cup$. Now it is not hard to see, under our hypothesis that using replacement there is an f consisting of only $\langle\beta,M(\beta)\rangle$'s, $\beta\epsilon\alpha$, and $\langle\alpha,\cup\rangle$, and that this is the required f in the definition of $\underline{\cup=M(\alpha)}$. So $(\exists x_0)(\underline{x_0=M(\alpha)})$. Uniqueness comes from a). Suppose $\alpha = \gamma+1$. It is easy to see that γ is definable and by our hypotheses, $f(\gamma)$ is definable. It is obvious that elements of Fodo(f(γ)) are elements of P(f(γ)) and that P(f(γ)) exists by power set in ZF*. Furthermore, Fodo(f(γ)) exists since it can be gotten by replacement on P(f(γ)). Proceed as above to get an appropriate f to give Fodo(f(γ)) = M(α). Uniqueness follows from a). The case $\alpha = \emptyset$ is trivial.

We want to insure in ZF* there being a definable well-ordering of L (among other things). This insurance is easily obtained by a natural definition of L in ZF, but not in ZF*. We have no choice but to complicate the definition of ϵL by adding on conditions.

We define xϵL, approximately as $(\exists\alpha)(\exists y)(y=M(\alpha)\ \&\ x\epsilon y)$. But this is not good enough for our purposes. We define 5 extra conditions on this α and y:

1) $(\beta)(z)([z=M(\beta)\ \&\ x\epsilon z)] \longrightarrow (\beta\epsilon\alpha \vee \beta=\alpha))$, and $(z)(z=M(\alpha) \longrightarrow z=y)$. Whenever x and α have such a y, we say $O(x) = \alpha$.

2) For all $\beta\epsilon\alpha$, there is a unique corresponding $y=M(\beta)$, and if $\beta,\gamma \leq \alpha$, then $M(\beta) \not\subseteq M(\gamma)$.

3) The M(β)'s, $\beta\epsilon\alpha$, and M(α) are transitive sets.

4) For every $z\epsilon M(\alpha)$, we have $(\exists\beta)(O(z) = \beta)$ and $O(z) \leq \alpha$. Also if $z,w \epsilon M(\alpha)$, then $[(z\epsilon M(O(w)) \longrightarrow O(z)\leq O(w)]$.

5) Now, there is a usual definable mapping F in full set theory (identity this with the 2-ary relation $F(x) = y$) mapping the constructible sets 1-1 into ordinals. Of course, ZF* may well not be able to prove $(x)(\exists y)(F(x) = y)$. Condition 5) will be that (the 2-ary relation) F is a 1-1

function when restricted to domain $M(\alpha)$, and $(x)(x\epsilon M(\alpha) \longrightarrow \text{Ord}''(F(x)))$. (This is the F which, in full set theory, assigns (Gödel numbers in the form of ordinals) to each constructible set a sequence of ordinals, the first being the rank of the set in the constructible hierarchy, $\alpha+1$; the rest of the sequence codes in, via F on the sets in $M(\alpha)$, how the constructible set in question is first-order defined over $M(\alpha)$.)

We define $x\epsilon L \equiv (\exists\alpha)(\exists y)(\underline{y=M(\alpha)}\ \&\ x\epsilon y\ \&\ \alpha$ and y satisfy conditions 1)-5) above).

We define $x < y \equiv x\epsilon L\ \&\ y\epsilon L\ \&\ F(x)\epsilon F(y)$, for F as in 5).

<u>Lemma 4</u>: The following are provable in ZF*: if $x\epsilon L$ then $(y)(y\epsilon M(O(x)) \longrightarrow y\epsilon L)$. Also $(y)(y\epsilon x \longrightarrow y\epsilon L)$, if $x\epsilon L$.

Proof: Let $y\epsilon M(O(x))$. By 4) in the definition of $x\epsilon L$, we have y, $M(O(y))$, $O(y)$ satisfy 1), 2), and 3) in the definition of $y\epsilon L$, since $O(y) \leq O(x)$. Towards verifying 4) in the definition of $y\epsilon L$, let $z\epsilon M(O(y))$. Then $z\epsilon M(O(x))$. Then $(\exists\alpha)(O(z) = \alpha)$. $O(z) \leq O(x)$, for suppose not. Then we get a contradiction via condition 2) in the definition of $x\epsilon L$. The rest of condition 4) follows for y because of condition 4) being satisfied for y, and that $M(O(y)) \subseteq M(O(x))$.

y satisfies 5) (i.e., y, together with $O(y)$, $M(O(y))$ since x does, and $M(O(y)) \subseteq M(O(x))$.

To show $(y)(y\epsilon x \longrightarrow y\epsilon L)$, notice by $\text{Trans}(M(O(x))$ we have (assuming $y\epsilon x$) that $y\epsilon M(O(x))$, and so by the first part of Lemma 4, $y\epsilon L$.

Lemma 5: For each Ax, 1 free variable, $\beta = \{x|Ax\} \longrightarrow (x)(x\epsilon M(\beta) \longrightarrow x\epsilon L)$, is provable in ZF*. (Note that $M(\beta)$ exists unambiguously by Lemma 3.) Also $\omega\epsilon L$.

Proof of Lemma 5: Form $\{\alpha|\alpha\epsilon\beta$ & $M(\alpha)$ does not satisfy conclusion$\}$. Take ϵ-least member, and call it γ. Case 1. γ is a limit. Now γ is definable in ZF*. Let $x\epsilon M(\gamma)$. Then $x\epsilon M(\alpha)$, for some $\alpha\epsilon\gamma$, and hence by the definition of γ, $x\epsilon L$. Case 2. $\gamma = \delta + 1$. Let $x\epsilon M(\delta+1)$. There is a definable well ordering on $M(\delta)$, $<$, and we may use this to definably well-order, in ZF*, the finite sequences of elements of $M(\delta)$ in the natural way, proving in ZF* that it is a well-ordering. With this well-ordering of $M(\delta+1)$, we take a least, in $M(\delta+1)$, x such that there is no α, $M(\alpha)$ satisfying condition 1, assuming there is an x. This least x is definable in ZF*, and so we consequently can form $\{\alpha|\alpha\epsilon\delta+1$ & $x\epsilon M(\alpha)\}$ and take the ϵ-least member, thereby obtaining a contradiction.

So every $x\epsilon M(\delta+1)$ possesses a (unique) $O(x)$.

Conditions 2)-4) are treated similarly, taking definable counterexamples and using definable well-orderings. The proof of 5), after taking least counterexamples, is much like our indication of construction of a definable well-ordering of $M(\delta+1)$ on the basis of one for $M(\delta)$, above.

To show $\omega\epsilon L$, it suffices to prove $\omega\epsilon M(\omega+1)$. The proof is like the proof of this fact in ZF.

Lemma 6: $\vdash_{ZF*}(x)(x=\omega \equiv (x=\omega)')$, where A' is A relativized to the predicate ϵL.

Proof: Left to the reader.

7. The System ZF*'. We define a transformation mapping formula in prenex form in the standard notation (described in 2. Remarks on Terminology and Notation) into formulae which contain the '<' symbol. If B is in prenex form define B- to be the usual prenex form for ~B. Take T to be the identity on formulae with no quantifiers, and take $T((\exists x_i)Bx_i)$ to be $(\exists x_i)(T(Bx_i) \& (x_j)(x_j < x_i \longrightarrow T(B\text{-}x_j)))$. $T((x_i)Bx_i)$ is $(x_i)(T(Bx_i) \vee (\exists x_j)(x_j < x_i \& T(B\text{-}x_j)))$. It is easily proved by induction that T(B-) and ~T(B) are equivalent for any prenex B. Recall that the interpretation of $x_j < x_i$ is $x_j \epsilon L \& x_i \epsilon L \& F(x_j) \epsilon F(x_i)$.

We form ZF*' as follows: Extensionality & Foundation & Infinity & Power Set & Sum Set & Modified Replacement. The latter is the only difference between ZF*' and ZF*. The other axioms are the same. Replacement in ZF*' is as follows: Any instance of Replacement in ZF* is an instance in ZF*' provided that the Axy be of the form T(Bxy) & $(z)(z < y \longrightarrow T(Bxz))$. Bxy having only 2 free variables. It is obvious that ZF*' $\subseteq$ ZF*.

Lemma 7: Extensionality & Foundation & Infinity are theorems of ZF* when relativized to L.

Proof: For (Infinity)' take x_0 to be ω. For (Extensionality)' and (Foundation)' just note from Lemma 4 that $(x \epsilon L \& y \epsilon x) \longrightarrow y \epsilon L$.

Lemma 8: The Power set and Sum set axioms (of ZF*) are theorems of ZF* when relativized to L.

Proof: In power set, we have $x = \{y | Ay\}$. The relativized to L will be equivalent to $x = \{y | y \in L \& (Ay)'\} \& x \in L$. Now observe that the relation $(z \subseteq w)'$ is equivalent to $z \subseteq w \&$ $z \in L \& w \in L$. So we have to verify that if $x \in L \& x = \{y | y \in L \&$ $(Ay)'\}$, then there is a set $x_0 \in L$ with x_0 the set of all subsets z of x such that $z \in L$. Now the hypothesis tells us that x is definable, and so x has a definable power set, $|P(x)$. We use replacement on $|P(x)$ to get the set of all $O(y)$'s with $y \in L$ and $y \in |P(x)$. This is a definable set of Ord", and so it has a union, $\cup$. $\cup$ is definable, and so, all $y \subseteq x$ with $y \in L$ have $y \in M(\cup)$, because if $\beta \in \alpha$, then $M(\beta) \subseteq m(\alpha)$. The required set x_0 of all subsets y of x with $y \in L$ is in $M(\cup + 1)$, by Lemma 2; hence $x_0 \in L$, by Lemma 5.

The relativized of Sum Set is checked similarly.

Lemma 9: If $(\exists y)(y < z \& Ayx_1 \ldots x_n)$, and $O(z), O(x_1), \ldots, O(x_n)$ all $\in \alpha = \{x | Bx\}$, then $\exists \mu\, O(y)$, with $y < z \& Ayx_1 \ldots x_n$. (That is, if A any formula with the free variables shown, B any formula with 1 free variable, the above is provable in ZF*).

Proof: One just assumes there are counterexamples $z, x_1, \ldots, x_n \in M(\alpha)$ to this lemma, and then go to definable counterexamples. But we obtain a contradiction, since there is provably a $\mu O(y)$ for these supposed definable counterexamples.

Lemma 10: The semi-relativized of each of Replacement in ZF*' is a theorem of ZF*.

Proof: We take a particular instance of Replacement in ZF*', e.g., that one whose Axy is $T((\exists z)(w)Czwxy)$ & $(u)(u < y \longrightarrow \sim T((\exists z)(w)Czwxu))$, where C is quantifier-free. We let D be a definable domain, $D \epsilon L$. We wish to show in ZF* that there is a set $S \epsilon L$ of all $y \epsilon L$ such that for some $x \epsilon L$ with $x \epsilon D$, y is the unique y with $(Axy)'$. This is easily seen to be equivalent to finding a set $S \epsilon L$ of all $y \epsilon L$ such that for some $x \epsilon D$,

1) $x \epsilon L$ & $y \epsilon L$ & $(\exists z)_L[(w)_L(Czwxy \vee (\exists w')_L(w' < w$ & $\sim Czw'xy))$ & $(z')_L(z' < z \longrightarrow (\exists w)_L(\sim Cz'wxy$ & $(w')_L(w' < w \longrightarrow Cz'w'xy))$ & $(u)_L\{u < y \longrightarrow (z)_L[(\exists w)_L(\sim Czwxu$ & $(w')_L(w' < w \longrightarrow Czw'xu)) \vee (\exists z')_L(z' < z$ & $(w)_L(Cz'wxu \vee (\exists w')_L(w' < w$ & $\sim Cz'w'xu))]\}$.

Convenient Notation: If X and Y are expressions occurring in 1), then let [X,Y] be the subformula of 1) beginning with X and ending with Y.

Let $\cup$ = union of the O(y)'s such that 1) holds for some $x \epsilon D$. We proceed to place definable bounds on the quantifiers above in such a way that the new formula is equivalent to 1) for $x \epsilon D$, $y \epsilon M(\cup)$. (Note that for each $x \epsilon D$ there is at most 1 y satisfying 1).)

Let $f_1(\langle x,y \rangle)$ be undefined if 1 is false: be O(z) for $z \epsilon L$ with $[(w)_L, Cz'w'xy]$ otherwise. Define $\cup_1$ = union of the range of f_1 on $D \times M(\cup)$.

Let $f_2(\langle x,y,z \rangle)$ be undefined if $[(w)_L, \sim Czw'xy]$; be O(w) for $w \epsilon L$ with $\sim[Czwxy, \sim Czw'xy]$, otherwise. Define

U_2 = union of the range of f_2 on $D \times M(U) \times M(U_1)$.

Let $f_3(\langle x,y,z,w\rangle)$ be undefined if $(w')_L(w' < w \longrightarrow Czw'xy)$; otherwise be $\mu O(w')$ for w' with $w' < w$ & $\sim Czw'xy$. (See Lemma 9.) Define U_3 = union of the range of f_3 on $D \times M(U) \times M(U_1) \times M(U_2)$.

Let $f_4(\langle x,y,z\rangle)$ be undefined if $[(z')_L, Cz'w'xy]$; be $\mu O(z')$ with $\sim[z' < z, Cz'w'xy]$, otherwise. Define U_4 = union of the range of f_4 on $D \times M(U) \times M(U_1)$.

Let $f_5(\langle x,y,z,z'\rangle)$ be undefined if $\sim[(\exists w)_L, Cz'w'xy]$; be $O(w)$ for $w \epsilon L$ with $[\sim Cz'wxy, Cz'w'xy]$ otherwise. Define U_5 = union of the range of f_5 on $D \times M(U) \times M(U_1) \times M(U_4)$.

Let $f_6(\langle x,y,z,z',w\rangle)$ be undefined if $[(w')_L, Cz'w'xy]$; otherwise be $\mu O(w')$ with $w' < w$ & $\sim Cz'w'xy$. Define U_6 = union of the range of f_6 on $D \times M(U) \times M(U_1) \times M(U_4) \times M(U_5)$.

Let $f_7(\langle x,y\rangle)$ be undefined if $[(u)_L, \sim Cz'w'xu]$; otherwise be $\mu O(u)$ with $\sim[u < y, \sim Cz'w'xu]$. Define U_7 = union of the range of f_7 on $D \times M(U)$.

Let $f_8(\langle x,y,u\rangle)$ be undefined if $[(z)_L, \sim Cz'w'xu]$; otherwise be $O(z)$ for $z \epsilon L$ with $\sim[(\exists w)_L, \sim Cz'w'xu]$. Define U_8 = union of the range of f_8 on $D \times M(U) \times M(U_7)$.

Let $f_9(\langle x,y,u,z\rangle)$ be undefined if $\sim[(\exists w)_L, Czw'xu]$; otherwise be $O(w)$ with $w \epsilon L$ and $[\sim Czwxu, Czw'xu]$. Define U_9 = union of the range of f_9 on $D \times M(U) \times M(U_7) \times M(U_8)$.

Let $f_{10}(\langle x,y,u,z,w\rangle)$ be undefined if $[(w')_L, Czw'xu]$; otherwise be $\mu O(w')$ with $w' < w$ & $\sim Czw'xu$. Define U_{10} = union of the range of f_{10} on $D \times M(U) \times M(U_7) \times M(U_8) \times M(U_9)$.

Let $f_{11}(\langle x,y,u,z \rangle)$ be undefined if $\sim[(\exists z')_L, \sim Cz'w'xu]$; otherwise be $\mu O(z')$ with $[z' < z, \sim Cz'w'xu]$. Define $\mathcal{U}_{11}$ = union of the range of f_{11} on $D \times M(\mathcal{U}) \times M(\mathcal{U}_7) \times M(\mathcal{U}_8)$.

Let $f_{12}(\langle x,y,u,z,z' \rangle)$ be undefined if $\sim[(w)_L, \sim Cz'w'xu]$; otherwise be $O(w)$ for $w \in L$ with $\sim[Cz'wxu, \sim Cz'w'xu]$. Define U_{12} = union of the range of f_{12} on $D \times M(\mathcal{U}) \times M(\mathcal{U}_7) \times M(\mathcal{U}_8) \times M(\mathcal{U}_{11})$.

Let $f_{13}(\langle x,y,u,z,z',w \rangle)$ be undefined if $\sim[(\exists w')_L, \sim Cz'w'xu]$; otherwise be $\mu O(w')$ with $w' < w \;\&\; \sim Cz'w'xu$. Define $\mathcal{U}_{13}$ = union of the range of f_{13} on $D \times M(\mathcal{U}) \times M(\mathcal{U}_7) \times M(\mathcal{U}_8) \times M(\mathcal{U}_{11}) \times M(\mathcal{U}_{12})$.

Note that by suitable instances of Replacement in ZF*, all of the above are provably well-defined. Note that each $\mathcal{U}_i$ is definable, so that each $M(\mathcal{U}_i) \subseteq L$. It is easily seen that for $x \in D$, $y \in M(\mathcal{U})$, it is the case that Axy is equivalent to the predicate Bxy obtained by placing the bounds $M(\mathcal{U}_i)$, $1 \leq i \leq 13$ on the appropriate quantifiers in 1).

Now each instance of the following is provable in ZF*: If $\alpha = \{x \mid Ax\}$, and α a limit, then for x and $y \in M(\alpha)$, $x < y$ iff $x < y$ holds when the quantifiers in the definition are relativized to $M(\alpha)$, A of 1 free variable. The proof in ZF* of the schema is like the proof in ZF. Use the definable well-ordering of $M(\alpha)$.

Now let $V = \max(\mathcal{U}, \mathcal{U}_i, O(D))$. Then relativizing the quantifiers occurring in the expansions of the "$<$"'s that occur in Bxy, to $M(V+\omega)$, we get the same predicate as Bxy, for $x \in D$. Hence we have shown our S we wanted to show originally ϵL, is first-order definable over $M(V+\omega)$, and hence $\epsilon M(V+\omega+1)$, $V + \omega + 1$ definable.

8. The System ZF'. Making use of the transformation T defined in the previous section, we form ZF' as follows: first, Extensionality & Foundation & Infinity & Power Set and Sum Set axioms of ZF. In addition, we have $(x)(x\epsilon L)$ & $(x)(\mathrm{Ord}x \longrightarrow \mathrm{Ord}''(x))$ & $(x)(\exists y)(z)(y\epsilon x \;\&\; (z\epsilon x \longrightarrow z \nless y))$. Replacement in ZF' will be the following: Let $Bxyy_1\ldots y_n$ be a formula in prenex form with only the free variables shown. Then $(y_1)\ldots(y_n)(x_0)(\exists x_1)(x_2)(x_2\epsilon x_1 \equiv (\exists x_3)(x_3\epsilon x_0 \;\&\; T(Bx_3x_2y_1\ldots y_n) \;\&\; (x_4)(x_4 < x_2 \longrightarrow \sim T(Bx_3x_4y_1\ldots y_n))))$, is an instance.

Lemma 11: ZF' $\supseteq$ ZF.

Proof: First, we wish to show in ZF' each instance of $T(A) \equiv A$. This is trivial for A with no quantifiers.

Suppose $T(A) \equiv A$ is provable in ZF' for all A in prenex form with n quantifiers. We then wish to show that $T(A) \equiv A$ is provable with A having $n + 1$ quantifiers in prenex form. Then we will have shown $T(A) \equiv A$ provable for any A in prenex form in ZF'.

Let B be in prenex form with $n + 1$ quantifiers. Suppose B is $(\exists x_i)(Ax_i)$. Then $T(B)$ is $(\exists x_i)(T(Ax_i) \;\&\; (x_j)(\sim x_j < x_i \vee T(A\text{-}x)))$. Now $\vdash_{ZF'} T(Ax_i) \equiv Ax_i$. Since $T(A\text{-}x_j)$ is equivalent to $\sim T(Ax_j)$ we have $\vdash_{ZF'} T(A\text{-}x_j) \equiv \sim Ax_j$. We have to check that $(\exists x_i)(Ax_i) \equiv (\exists x_i)(Ax_i \;\&\; (x_j)(\sim x_j < x_i \vee \sim Ax_j))$ is provable in ZF'.

Define Cxy to be $y = x \,\&\, T(Ay)$. This is, of course, equivalent with $T(Ay \,\&\, y=x)$, a transformation on a wff of n quantifiers. Now $(x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 \equiv (\exists x)(x \epsilon x_0 \,\&\, Cxx_2 \,\&\, (x_4)(x_4 < x_2 \longrightarrow \sim T(Ay \,\&\, y=x)))))$, is (equivalent to) an axiom of ZF'. But $\vdash_{ZF'} T(Ax_2 \,\&\, x_2 = x_3) \equiv Ax_2 \,\&\, x_2 = x_3$. So $\vdash_{ZF'} (x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 \equiv Ax_2 \,\&\, x_2 \epsilon x_0)$. Now assume $(\exists x_i)Ax_i$. Choose any such x_i. Take $O(x_i)$, and set $x_0 = M(O(x_i))$, and use the above theorem of ZF' to get the set of all elements of $M(O(x_i))$ having the property A. ($M(O(x_i))$ is defined and has required properties since $(x)(x \epsilon L)$ is an axiom of ZF', and $x \epsilon L$ is formalized as in ZF*, previously.) Hence by one of the axioms of ZF', there is a $<$-least member. Hence we have shown by induction the equivalence between $T(A)$ and A, in ZF'. This has the effect of provably in ZF' eliminating the T's in the axioms of ZF', and so $ZF' \supseteq ZF$.

9. <u>The Skolem Argument</u>. We wish to show $(n)\vdash_{ZF*} Con(ZF'_n)$, where ZF'_n is the first n axioms of ZF' in some natural enumeration of them. If we succeed in showing this, then suppose $\sim Con(ZF)$. Then $\sim Con(ZF')$. Then $(\exists n) \sim Con(ZF'_n)$. But then $(\exists n)\vdash_{ENT} Con(ZF'_n)$. Since ENT is formalizable in ZF*, $(\exists n) \vdash_{ZF*} Con(ZF'_n)$. Hence $(\exists n)(\vdash_{ZF*} Con(ZF'_n) \,\&\, \vdash_{ZF*} \sim Con(ZF'_n))$, and so $\sim Con(ZF*)$. Hence $Con ZF* \longrightarrow Con ZF$.

We give, without loss of generality, a Skolem closure argument within ZF* to give, provably in ZF*, a set which is a model for 1) Extensionality in ZF, 2) Foundation in ZF, 3) Infinity in ZF, 4) Power Set in ZF, 5) Sum Set in ZF,

6) $(x)(x \epsilon L)$, 7) $(x)(\text{Ord}x \longrightarrow \text{Ord}''x)$, 8) $(x)(\exists y)(z)(y \epsilon x \ \& \ z \not< y))$, 9) Let $D(xyT)$ be the formula obtained from taking 1) in 7. <u>The System ZF*'</u> and replacing the 4-place quantifier-free predicate C, with some 5-place quantifier-free predicate $E(zwxyT)$. $(D)(T)(\exists S)(y)(y \epsilon S \equiv (\exists x)(x \epsilon D \ \& \ D(x,y,T))$.

The construction, in ZF*, of the model of these 9 sentences will be much like a Skolem construction in which the initial model is $\emptyset$. At each stage n, we throw in some sets $x \epsilon L$, and we take the union as n ranges over ω.

We simultaneously define α_n and S_n. We are interested in $\bigcup_{n \epsilon \omega} S_n$.

$S_n = M(\alpha_n)$. $S_0 = M(\emptyset) = \emptyset$. $\alpha_0 = \emptyset$.

Consider, for each $x \epsilon S_n$, the $<$-least $y \epsilon x$ with $(z)(z \epsilon y \longrightarrow z \not\epsilon x)$.

Consider, for each $x \epsilon S_n$, $P_L(x)$ = set of all $y \subseteq x$ with $y \epsilon L$.

Consider, for each $x, y \epsilon S_n$, with $x \not\subseteq y$, the $<$-least element of x not in y.

Consider, for each $x \epsilon S_n$, $\bigcup_L(x)$ = set of all $y \epsilon L$ such that $(\exists z)(z \epsilon x \ \& \ y \epsilon z)$.

Consider, for each $T \epsilon S_n$, the unique $S \epsilon L$ satisfying the semi-relativized of 9) to L. (Call this 9)").

We continue "considering" through 9)", closing S_n, in effect, under the "Skolem functions" for 9)", in such a way that, as in <u>7. The System ZF*</u>, we have that the Skolem functions produce values definable in terms of the arguments.

We take α_{n+1} = (union of the O(z)'s for the z's considered above) + ω. Take $S_{n+1} = M(\alpha_{n+1})$.

We can then use appropriate instances of Replacement in ZF* in combination with the definable well-ordering $<$, to show that if the S_n and α_n are not well-defined for each n, then there are definable counterexamples to our construction in the following sense: for some specifically definable sets, the sets corresponding to them that we considered above do not exist. But this is impossible by Lemmas 7, 8, 9, and 10. So our construction is well-defined in ZF*.

Now our model $\bigcup_{n\in\omega} S_n$ is an $M(\alpha)$, α definable, α a limit. In particular, it is transitive. It also contains ω. Due to the absoluteness of the definition of L and of the definition of ω in $M(\alpha)$'s, α a limit, it is easily seen in ZF*, putting all this together, that the sentences 1)-9) are true when the quantifiers range over $M(\alpha)$. Furthermore, since $M(\alpha)$ is definable, the definition of satisfaction and the induction on α are easily developable in ZF*, in order to prove, in ZF*, that Con(1)-9)).

From the remarks at the beginning of this section, we immediately have

<u>Theorem 1</u>: $\vdash_{ENT}$ Con ZF* $\longrightarrow$ Con ZF.

10. <u>ZF, Parameterless ZF, and ZF*</u>. This section is devoted partially to further consideration of the system ZF* of Chapter 1, and partially to some other subsystems of ZF.

We define a sentence of set theory to be <u>arithmetical</u> if it is the relativized of some sentence of set theory to ω.

<u>Corollary 1</u>: ZF is a conservative extension of ZF* for arithmetical sentences.

Proof: Let A be arithmetical, and $\vdash_{ZF} A$. We can show $Con(ZF^* + \sim A) \longrightarrow Con(ZF + \sim A)$ by modifying the proof of Theorem 1 slightly; just redefine the systems $ZF^{*\prime}$, ZF' as $ZF^{*\prime} + \sim A$, $ZF' + \sim A$, respectively. Due to Lemma 6, all of our Lemmas carry over. Now since $\sim Con(ZF + \sim A)$, we have $\sim Con(ZF^* + \sim A)$, and so $\vdash_{ZF^*} A$.

Our next Theorem concerns sentences of the form $(x)(\exists! y)Axy$, A arbitrary, with only 2 free variables, that are provable in ZF*. Now in ZF there are many such sentences which define, provably, in ZF, a Skolem function which moves everything and which is 1-1. An example is $(x)(\exists y)(y = |P(x))$. Another is $(x)(\exists y)(y = \{x\})$. Not so in ZF*. Thus,

<u>Theorem 2</u>: Let Axy be any formula with only free variables shown, and let $C = (x)(\exists! y)(Axy \;\&\; y \neq x) \;\&\; (x)(y)(z)((Axy \;\&\; Axz) \longrightarrow y = z)$. Then C is not provable in ZF*.

Proof: We let C be a sentence of the above form, and we construct a model for $ZF^* + \sim C$, given an arbitrary model for ZF, $\mathfrak{A} = \langle X,R\rangle$, where R is a 2-ary relation on X, $X \neq \emptyset$. (All models are assumed to be equality models. Note that ZF* is a first-order theory with equality.)

We define $\mathfrak{B}$ as follows: The domain is to be $X \cup Q$, where Q is the rationals. The 2-ary relation, Sxy, is defined as Rxy if $x,y \in X$; $x < y$ if $x,y \in Q$; false, if $x \in Q$, $y \in X$; true if $x \in X$, $y \in Q$.

We claim $\mathcal{B}$ satisfies ZF* $+\sim$C.

First, we show that the elements of Q in $\mathcal{B}$ are indistinguishable in the sense that if $Ax_1 \ldots x_n y_1 \ldots y_m$ holds in $\mathcal{B}$ for $x_i \epsilon Q$, $y_j \epsilon X$, then so does $Az_1 \ldots z_n y_1 \ldots y_m$ for $z_i \epsilon Q$ if the two sequences of rationals, x_i, y_i have the same order relations in Q, (i.e., there is a 1-1 order preserving map). To see this, it suffices to show that, given such a pair of similar sequences of rationals, there is an automorphism of $\mathcal{B}$ which keeps the elements of X fixed, and which maps, in an order-preserving way, the sequence x_i onto z_i. And such an automorphism is easily given by any map which fixes the elements of X and maps the rationals 1-1 onto itself, which maps the x_i into the y_j.

Now suppose $\mathcal{B} \models (x)(\exists ! y)(Axy)$. Then by indistinguishability, it is clear that for $x \epsilon Q$, we have $\mathcal{B} \models Axy$ for some $y \epsilon X$, for otherwise we would have Axy for $x,y \epsilon Q$, and hence Axz for $z = y + 1$. But now I claim that $\mathcal{B}$ satisfies $A(x+1,y)$, since $y \epsilon X$, by indistinguishability, assuming $x \epsilon Q$. So $\mathcal{B}$ does not satisfy C.

Clearly $\mathcal{B}$ satisfies axiom 0 of ZF*.

To verify 1 of ZF*, suppose $\mathcal{B} \models (x_2)(x_2 \epsilon x_0 \equiv x_2 \epsilon x_1)$, for x_2, x_1 in the domain. Then either x_1 and $x_0 \epsilon X$, or x_1, $x_0 \epsilon Q$. In the first case, we can conclude that $\mathfrak{A} \models (x_2)(x_2 \epsilon x_0 \equiv x_2 \epsilon x_1)$, and so since $\mathfrak{A} \models$ ZF, we have $x_0 = x_1$. In the second case, we have $1/2(x_0 + x_1) \leq x_0$ iff $1/2(x_0 + x_1) \leq x_1$. But then $x_0 = x_1$.

To verify 2 of ZF*, set $x_0 = \omega$ of $\mathfrak{A}$. It is easy to see that $\mathfrak{B} \models \emptyset \in x_0$, since $\emptyset$ in $\mathfrak{B}$ is same as $\emptyset$ in $\mathfrak{A}$. Also, $x \cup \{x\}$ remains unchanged for $x \in X$, when we pass from $\mathfrak{A}$ to $\mathfrak{B}$. Also, the members in $\mathfrak{A}$ of x_0 are the same as the members of x_0 in $\mathfrak{B}$. Also, the subsets in $\mathfrak{A}$ of x_0, or any of its members, are identical with the corresponding elements in $\mathfrak{B}$. Putting this together, we see that axiom 2 of ZF* is satisfied in $\mathfrak{A}$ "in the same way" as it is in $\mathfrak{B}$.

To see that 3 of ZF* is satisfied by $\mathfrak{B}$, suppose $\mathfrak{B}$ satisfies C_A. Then the unique element defined in C_A must be $\in X$, by indistinguishability. We let this element be x. What we are looking for is a power set of x in the model $\mathfrak{B}$. We claim that $y = P(x)$ in the model $\mathfrak{A}$ does the trick. We have to show that $\mathfrak{B}$ satisfies $y = P(x)$. But this is obvious, since the only members of y in $\mathfrak{B}$ are the members of y in $\mathfrak{A}$, and the only subsets of x in $\mathfrak{B}$ are the subsets of x in $\mathfrak{A}$.

Axiom 4 of ZF* is checked similarly.

To see that axiom 6 of ZF* is satisfied in $\mathfrak{B}$, let $x \in X \cup Q$. If $x \in X$, take x_0 in foundation as an R-least member of x in $\mathfrak{A}$. If $x \in Q$, take x_0 to be $\emptyset$ in $\mathfrak{A}$ (or $\mathfrak{B}$). Obviously $\mathfrak{B} \models \emptyset \in x$.

Axiom 5 of ZF* is the most complicated. The set that C_A defines in $\mathfrak{B}$ is again $\in X$. Call it D. Then we are interested in the range in $\mathfrak{B}$ of the partial function in $\mathfrak{B}$,

Bx_2x_1 & $(x_3)(Bx_2x_3 \rightarrow x_3=x_1)$, on the domain D. Now every x_2 with $S(x_2,D)$ has $x_2 \epsilon X$, and so, by indistinguishability, if $S(x_2,D)$, and Bx_2x_1 & $(x_3)(Bx_2x_3 \rightarrow x_3=x_1)$, then $x_1 \epsilon X$. Now suppose there is a formula Cx_2x_1 such that, for $x_2 \epsilon X$, $x_1 \epsilon X$, Cx_2x_1 holds in $\mathfrak{A}$ iff Cx_2x_1 holds in $\mathfrak{B}$. Then by Replacement in the model $\mathfrak{A}$, we would have (this instance of) Replacement in $\mathfrak{B}$, and we would be done. It remains to show that for each formula $Ax_1 \ldots x_n$, with the free variables shown, there is a formula $Bx_1 \ldots x_n$ which holds in $\mathfrak{A}$ iff $Ax_1 \ldots x_n$ holds in $\mathfrak{B}$, when $x_i \epsilon X$.

It suffices to prove by induction that for any formula $Ax_1 \ldots x_n$, and for any <u>partial</u> function f from $\{i \mid 1 \leq i \leq n\}$ into Q, there is a formula $Bx_{i_1} \ldots x_{i_k}$, $\{i_1, \ldots, i_k\} = \{i \mid 1 \leq i \leq n \ \& \ i \notin Dom(f)\}$, such that for any sequence $x_1 \ldots x_n$ with $x_i \epsilon X$ iff $i \notin Dom(f)$, $x_i = f(i)$ if $i \epsilon Dom(f)$, we have $\mathfrak{B} \models Ax_1 \ldots x_n$ iff $\mathfrak{A} \models Bx_{i_1} \ldots x_{i_k}$.

To see this for $Ax_1 \ldots x_n$ quantifier-free, take B to be the formula obtained from A by 1) replacing all instances of $x_i \epsilon x_j$, $i \notin Dom(f)$, $j \epsilon Dom(f)$, by $x_i = x_i$, 2) replacing all instances of $x_i \epsilon x_j$, or $x_i = x_j$, or $x_j = x_i$, $i \epsilon Dom(f)$, $j \notin Dom(f)$, by $x_j \neq x_j$, 3) replacing all instances of $x_i \epsilon x_j$, or $x_i = x_j$, $i, j \epsilon Dom(f)$, by $(\exists v)(v=v)$ if $f(i) < f(j)$, $\sim(\exists v)(v=v)$ if not; or $(\exists v)(v=v)$ if $f(i) = f(j)$, $\sim(\exists v)(v=v)$ if not, respectively.

Put $Ax_1 \ldots x_n$ in prenex form, and suppose our claim is true for all formulae with less quantifiers.

We may assume that $Ax_1 \ldots x_n$ is $(x_0)Cx_0x_1 \ldots x_n$, since the existential case follows from this case by taking negations. Let f be a partial function from $\{i \mid 1 \leq i \leq n\}$ into Q. Now 2 finite partial functions g,h from $\omega \longrightarrow Q$ are said to be of the same type if 1) they have the same domain D, 2) $g(x) < g(y)$ iff $h(x) < h(y)$.

Consider the set of partial functions on $\{i \mid 0 \leq i \leq n\}$ which are identical to f on $\{i \mid 1 \leq i \leq n\}$. There are only a finite # of types represented in this set. Pick a representative from each type, and call this set $\{f_1, f_2, \ldots, f_k\}$, $f_1 = f$. Let D_i, $1 \leq i \leq k$, be the formula given by the inductive hypothesis for $Cx_0x_1\ldots x_n$ for f_i; i.e., each D_i has exactly the free variables x_ℓ for $\ell \notin \mathrm{Dom}(f_i)$, $0 \leq \ell \leq n$, and for any $x_0\ldots x_n$ with $x_\ell \epsilon X$ for $\ell \notin \mathrm{Dom}(f_i)$, $x_\ell = f_i(\ell)$ for $\ell \epsilon \mathrm{Dom}(f_i)$, we have $\mathfrak{B} \models Cx_0x_1\ldots x_n$ iff $\mathfrak{A} \models D_i x_{p_1}\ldots x_{p_q}$, $\{p_1\ldots p_q\} = \{r \mid 0 \leq r \leq n \ \& \ r \notin \mathrm{Dom}(f_i)\}$.

Then we take B to be $(x_0)D_1 \ \& \bigwedge_{2 \leq i \leq k} D_i$. By indistinguishability, it is easily seen that B and A satisfies the conclusion of our claim for the function f. This concludes the proof of Theorem 2.

We now define some new subsystems of ZF. ZF^n is to be the same as ZF except for the Replacement schema. ZF^n, instead, only allows the Axy in the Replacement schema at most to have $n + 2$ free variables; x and y and possibly n other ones. It is easy to see that ZF* is a subsystem of ZF^0. We also have

<u>Corollary 2</u>: $ZF^* \neq ZF^0$.

Proof: Consider the model of ZF* constructed in the proof of Theorem 2. Consider the sentence $A = (x)(\exists y)(\emptyset \epsilon x \longrightarrow (\emptyset \notin y \;\&\; (z)(z \neq \emptyset \longrightarrow (z \epsilon x \longleftrightarrow z \epsilon y))))$. A is obviously provable in ZF^0. But our model of ZF^* does not satisfy A.

We believe strongly that $ZF^0 \neq ZF$, and in fact in the stronger conjecture that $ZF^n \neq ZF^{n+1}$, for all n. Although the details of a proof of $ZF^0 \neq ZF$ have not yet been carried out, we can give the definition of a very promising model of $ZF^0 + (\exists x)(y)(\exists z)(\sim(z \epsilon y \longleftrightarrow z = x))$.

We let $M(\alpha)$ be the minimal model of ZF. We let S be any Cohen generic set of natural numbers over $M(\alpha)$. We let $M^S(\alpha)$ be the corresponding Cohen model for $ZF + V \neq L$.

We let F_S be the set of all sets of natural numbers which are finitely different from S (i.e., whose symmetric difference from S is finite).

Recall that $R(\beta+1) = |P(R(\beta))$, $R(\lambda) = \bigcup_{\beta < \lambda} R(\beta)$.

Consider the sets $x \epsilon M^S(\alpha)$ such that for any formula $Ayzx_1$, we have, in $M^S(\alpha)$, $\{z \mid (\exists y)(y \epsilon x \;\&\; A(x,y,F_S))\} \neq \{x\}$. We let X be the set of all such x. We let Y be the set of all $x \epsilon X$ such that any finite combination of union and power set on x in $M^S(\alpha)$ gives a set in X.

For each β, let $R'(\beta)$ be the unique rank in the cumulative hierarchy up to β in the model $M^S(\alpha)$.

Define, for each β, a function f_β whose domain is $R'(\beta) \cap Y$, and by the equations $f_{\beta+1}(x) = \{y \mid y \epsilon Y \;\&\; (\exists z)(z \epsilon x \;\&\; f_\beta(z) = y)\}$, $f_\lambda = \bigcup_{\beta < \lambda} f_\beta$.

We define $x \epsilon Z$ as $[(\exists \beta)(\beta < \alpha \;\&\; x \in \text{Range}(f_\beta)) \;\&\; x \epsilon Y]$.

We conjecture that $\langle Z, \varepsilon \rangle$ is the desired model.

We feel that a detailed analysis of the relations between the theories ZF^n, hopefully by finding natural sentences to distinguish each ZF^n from ZF^{n+1}, would involve non-trivial applications of the notion of forcing. In particular, careful attention seems to be required as to the model-theoretic properties of models of theories obtained by forcing; e.g., the definability or indistinguishability of various elements of the models constructed.

CHAPTER II

1. Definitions of Systems. We will have for our language, number variables, n, m, p, q, r; set variables, $x, y, z, w, u, v, \ldots$; the relation $n \epsilon x$ between numbers and sets; and the constant number "0" which can hold only between variables of the same type.

We introduce function variables in the usual way by defining them in terms of sets of natural numbers in the usual way. We will have full number theory at our disposal, since all systems considered here will have the unrestricted induction axiom schema (called I) $[A0 \; \& \; (n)(An \longrightarrow An')] \longrightarrow (n)An$, where A is any formula with possibly free variables of both kinds, and can have both number and set quantification.

Also, all systems considered here will have the axiom of extensionality (that any two sets with the same members are equal), and we will tacitly assume that the system we will call I includes this axiom. Thus I will be the unrestricted axiom schema of induction plus the axiom of extensionality.

In addition, all systems considered will have the recursively enumerable comprehension axiom schema (called ReCA),

$$(x_1)\ldots(x_l)(e)(\exists y)(n)(n \epsilon y \equiv (\exists k)T_l(e, n, k, x_1, \ldots, x_l)).$$

The predicates T_l are understood to be written out in the usual way with bounded quantifiers. (Or, we may have instead introduced them as primitive, and defined them by adding axioms of primitive recursion.) The predicates T_l

above are what Kleene would call $T^{f_1,\ldots,f_\ell}$, where f_i is the characteristic function for the set, x_i. See Kleene [3], p. 291.

Before we get into the mathematics of the systems we will be considering, we will state, informally, some propositions concerning what can be done in the system I + ReCA.

<u>Proposition 1</u>: We can, in I + ReCA, justify all uses of coding normally found in the development of hierarchy theory. Thus, we may justify the use of such symbols as $\langle x,y\rangle$ and $(x)_n$ (respectively $\{2^{n+1} \cdot 3^{m+1} | n \epsilon x \ \& \ m \epsilon y\}$, and $\{m | p_n^{m+1} \epsilon x\}$, where p_n is the nth prime).

We define the relation between functions and sets alluded to above, as $f(n) = m$ iff $\langle n,m\rangle \epsilon f$, where $\langle n,m\rangle = 2^{n+1} \cdot 3^{m+1}$.

<u>Proposition 2</u>: In ReCA + I, we may provably perform "collapse of like quantifiers". In other words, the usual way of collapsing 2 successive universal set quantifiers (or function quantifiers) into one can be completely justified on the basis of ReCA + I.

<u>Def. 1</u>: A predicate $A(n,x_1,\ldots,x_p)$ of n is said to be essentially Π_1^1 if it is in prenex form followed by a matrix $T(\bar{e},n,m,x_1,\ldots,x_p,f_1,\ldots,f_\ell,n_1,\ldots,n_k)$, where there are no set quantifiers in A, and the f_i occur as universal function quantifiers, in any order, mixed together with possibly number quantifiers (n_i) and $(\exists n_j)$ and $(\exists m)$. (No existential function quantifiers in A.) An essentially Σ_1^1 formula is just the prenex form of the negation of an essentially Π_1^1

formula.

Proposition 3: For every arithmetical predicate (with parameters) there is a predicate $(f)(\exists m)T(\bar{e},n,m,x_1,\ldots,x_\ell, n_1,\ldots,n_p)$ which is, provably in I + ReCA, equivalent. There is also a predicate $(\exists f)(m)\sim T(\bar{e},n,m,x_1,\ldots,x_\ell,n_1,\ldots,n_p)$ which is, provably in I + ReCA, equivalent.

Proposition 4: Define $(x)^{(\ell)}$ as the ℓ-th jump of x. $y = (x)^{(\ell)} \equiv \exists$ a sequence $(x_0,\ldots,x_\ell)$ such that $x_0 = x$ and x_{i+1} = jump of x_i. (Thus, $y = (x)^{(\ell)}$ is a predicate of 3 variables, defined in the usual way.) Then in I + ReCA we may prove $(\ell)(\exists! y)(u = (x)^{(\ell)})$.

This is proved by induction on ℓ.

Consider

1) $(x)(p)(e)\{(n)[(f)(\exists m)T_2(e,n,m,f,x) \equiv (\exists g)(r)\sim T_2(p,n,r,g,x)] \longrightarrow (\exists y)(n)(n\epsilon y \equiv (f)(\exists m)T_2(e,n,m,f,x))\}$.

2) The schema, $(x)\{(n)[Anx \equiv Bnx] \longrightarrow (\exists y)(n)(n\epsilon y \equiv Anx)\}$, where A is essentially Π^1_1, B is essentially Σ^1_1.

3) $(x)(e)((n)(\exists f)(m)\sim T_{2,1}(e,0,m,n,f,x) \longrightarrow (\exists y)(n)(y)_n$ is a function & $(m)\sim T_{2,1}(e,0,m,n,y_n,x)))$.

4) $(x)((n)(\exists f)A(n,f,x) \longrightarrow (\exists y)(n)((y_n)$ is a function & $A(n,y_n,x)))$, where A is essentially Σ^1_1.

5) $(x)(e)((f)(\exists g)(m)\sim T_3(e,0,m,f,g,x) \longrightarrow (f)(\exists y)(n)(y_0 = f$ & $(m)\sim T_3(e,0,m,y_n,y_{n+1},x)))$.

6) $(x)((f)(\exists g)A(f,g,x) \longrightarrow (f)(\exists y)(n)(y_0 = f$ & $A(y_n,y_{n+1},x)))$, where A is essentially Σ^1_1 having only the free variables f,g,x.

We call I + ReCA + 1), the pure Δ^1_1-CA; I + ReCA + 2), essentially Δ_1^1-CA; I + ReCA + 3), pure Σ_1^1-AC; I + ReCA + 4), essentially Σ_1^1-AC; I + ReCA + 5), pure Σ_1^1-DC; I + ReCA + 6), essentially Σ_1^1-DC.

NOTE: "CA" is supposed to mean "comprehension axiom"; "AC", axiom of choice; "DC", dependent choices.

<u>Proposition 5</u>: In pure Σ_1^1-AC, for every essentially Π_1^1 predicate, there is a pure Π_1^1 predicate $(f)(\exists n)T(\bar{e},m,n,\ldots)$ which is provably equivalent.

To see this, first use Proposition 2 to collapse adjacent like quantifiers. Then use pure Σ_1^1-AC to interchange a number and function quantifier, and then use Proposition 2 and then pure Σ_1^1-AC, etc.

<u>Proposition 6</u>: Pure Σ_1^1-AC $\supseteq$ pure Δ_1^1-CA.

The idea is that, in pure Δ_1^1-CA, one has that for each m there is a solution to one of two Π_1^0 predicates, and one uses pure Σ_1^1-AC to form a Skolem function. Then the required set in pure Δ_1^1-CA is obtained, in pure Σ_1^1-AC, recursively in the jump of the Skolem function.

<u>Proposition 7</u>: Pure Σ_1^1-AC = essentially Σ_1^1-AC.

By Proposition 5, one has only to consider the case $(n)(\exists f)(\exists g)(m)\sim T \longrightarrow \exists$ Skolem function. But we may collapse the f and g quantifiers in the usual way, and apply pure Σ_1^1-AC to get a Skolem function, which will have recursive in it the Skolem function wanted in the implication above.

A similar argument shows

<u>Proposition 8</u>: Pure Σ_1^1-AC $\supseteq$ essentially Δ_1^1-CA.

Conjecture: We do not know whether pure Δ_1^1-CA = essentially Δ_1^1-CA, or whether pure Δ_1^1-CA = Σ_1^1-AC, or whether Σ_1^1-AC = essentially Δ_1^1-CA, and we conjecture that none of the three statements is correct.

Proposition 9: Pure Σ_1^1-DC $\longrightarrow$ Σ_1^1-AC. (By Prop. 5, we call essentially Σ_1^1-AC, and pure Σ_1^1-AC, just Σ_1^1-AC.)

Hence also

Proposition 10: Essentially Σ_1^1-DC = pure Σ_1^1-DC.

Proposition 11: We can prove in ReCA + I the recursion theorem (with parameters).

2. Preliminary Lemmas. We will eventually show that Σ_1^1-DC is a conservative extension for purely Π_2^1 sentences of pure Δ_1^1-CA.

We define, in I + ReCA, several notions.

Def. 1: P(n,m) is defined as the Gödel number (in the usual 1-1 onto Gödel numbering of pairs of natural numbers) of the pair $\langle n,m\rangle$.

Def. 2: φ_n^x is defined in the usual way as the nth partial recursive function in x. Note that in I + ReCA we can prove $(n)(x)(\exists\, \varphi_n^x)$.

Def. 3: $RLO^x(n)$ is defined as "Range $(\varphi_n^x) \subseteq \{0,1\}$ & $(m)(m \in Dom(\varphi_n^x))$ & φ_n^x defines a linear ordering," where φ_n^x defines a linear ordering means that 1) $(m)(\varphi_n^x(P(m,m)) = 0)$, 2) $(p)(q)(r)([\varphi_n^x(P(p,q)) = 1 \;\&\; \varphi_n^x(P(q,r)) = 1] \longrightarrow \varphi_n^x(P(p,r)) = 1))$. 3) $(p)(q)(\varphi_n^x(P(p,q)) = 1 \longrightarrow \varphi_n^x(P(q,p)) = 0$. 4) $(p)(q)(p = q \vee \varphi_n^x(P(p,q)) = 1 \vee \varphi_n^x(P(q,p)) = 1)$. We define $RLO(n) = RLO^{\emptyset}(n)$.

Def. 4: We define $p <_{n^x} q$ = $RLO^x(n)$ & $\varphi_n^x(P(p,q)) = 1$. We define $p \leq_{n^x} q$ = $RLO^x(n)$ & $(\varphi_n^x(P(p,q)) = 1 \vee p = q)$. When $x = \emptyset$, we use the subscript n instead of $n^\emptyset$. ($\emptyset$ denotes the empty set.)

Def. 5: We define $W^x(n)$ as $RLO^x(n)$ & $(y)(\exists m)(y=\emptyset \vee (m \epsilon y$ & $(r)(r \epsilon y \longrightarrow m \leq_{n^x} r)))$.

Def. 6: $Suc_{n^x}(p,q)$ = $RLO^x(n)$ & $q <_{n^x} p$ & $(r)(r <_{n^x} p \longrightarrow r \leq_{n^x} q)$. $Suc_{n^x}(p) = (\exists q) Suc_{n^x}(p,q)$. $Lim_{n^x}(p)$ = $RLO^x(n)$ & $(\exists q)(q <_{n^x} p$ & $\sim Suc_{n^x}(p))$. $O_{n^x}(p)$ = $RLO^x(n)$ & $(q)(p \leq_{n^x} q)$.

Def. 7: $H^y_{n^x}(z)$ = $RLO(n^x)$ & $(p)(m)(Suc_{n^x}(m,p) \longrightarrow ((z)_p)^{(1)} = (z)_m)$ & $(p)(lim_{n^x}(p) \longrightarrow (x)_p = \{P(r,s) | s <_{n^x} p$ & $r \epsilon (z)_s\})$ & $(p)(O_{n^x}(P) \longrightarrow (x)_p = y)$. Thus $H^y_{n^x}(z)$ is the predicate of 4 variables asserting that z is a hierarchy on the RLO, n^x, starting from y. It will be useful later on to include the condition $(k)(k \epsilon z \longrightarrow (\exists p)(k \epsilon (z)_p))$, in the definition above.

Def. 8: $H^y_{n^x_l}(z)$ is the predicate of 5 variables asserting that z is a hierarchy on the RLO, n^x, up to but not including ℓ, starting from y. Thus if $p \geq_{n^x} \ell$, then $(z)_p = \emptyset$.

Def. 9: $x \leq_T y$ is defined as "x is recursive in y" in the usual way.

Def. 10: $|n^x| < |m^x|$ means $RLO^x(n)$ & $RLO^x(m)$ & $(\exists f)(f$ is a proper imbedding of n^x into m^x, i.e., $(p)(q)\{(p \leq_{n^x} q \longleftrightarrow$

$f(p) \leq_{m^x} f(q))$ & $([p <_{m^x} q$ & $q \in \text{Range } (f)] \longrightarrow p \in \text{Range } (f))\}$.

$|n^x| = |m^x|$ means that the range of the f above is ω.

$|n^x| \leq |m^x| \equiv |n^x| = |m^x| \vee |n^x| < |m^x|$.

Def. 11: $\text{Hyp}^x(y) \equiv (\exists n)(\exists z)(\text{RLO}^x(n)$ & $(H^x_{n^x}(z))$ & $W^x(n)$ & $y \leq_T z)$.

Lemma 1: $(x)(y)(z)([W^z(n)$ & $H^z_{n^z}(x)$ & $H^z_{n^z}(y)] \longrightarrow y=x)$ is provable in I + ReCA. Also, $[W^z(n)$ & $H^z_{n^z_p}(x)$ & $H^z_{n^z_p}(y)] \longrightarrow y = x$ is also provable.

Proof: Given n,x,y,z with $W^z(n)$, $H^z_{n^z}(y)$. Form $\{p | (x)_p \neq (y)_p\}$ in I + ReCA. By $W^x(n)$, we have a $<_{n^x}$-least element q. Clearly we must not have $0_{n^x}(q)$. If $\lim_{n^x}(q)$, then, since $(s)(s <_{n^x} q \longrightarrow (x)_s = (y)_s)$, we have $(x)_q = (y)_q$. If $\text{Suc}_{n^x}(q,s)$, then since $(x)_s = (y)_s$, we have $(x)_q = (y)_q$.

Lemma 2: The following is provable in pure Δ^1_1-CA: if $W^z(n)$ & $W^z(m)$ & $(\exists x)(H^z_n(x))$, then exactly one of the following three holds: a) there is a unique imbedding of the ordering n^z onto a proper initial segment of the ordering m^z, and there is no imbedding of the ording m^z onto an initial segment of the ordering n^z.

b) there is a unique imbedding of the ordering n^z onto the whole ordering m^z, and vice versa, and there are no imbeddings onto proper initial segments in either direction.

c) the interchanging of n with m in a).

Furthermore, denote $x^{(l)}$ for the result of applying ℓ iterations of the jump operator to x, $\ell \in \omega$. Then all the above maps may be found $\leq_T x^{(\ell)}$, for some ℓ depending on n and m.

Proof: In a), if there is an imbedding, f, of n^z onto a proper initial segment of m^z, then it must be unique, since if g is another, form, in I + ReCA, $\{k|f(k) \neq g(k)\}$, and take an n-least member, p. Running through the 3 cases, $0_{n^z}(p)$, $Suc_{n^z}(p)$, and $\lim_{n^z}(p)$ in a straightforward way, using the linearity of n^z and m^z, we get a contradiction. Same with c).

If there is an imbedding, g, of m onto an initial segment of n, take g^{-1}, and note that g^{-1} must disagree with f somewhere. Then follow the procedure above.

If there is an imbedding of n onto the whole ordering m, then there is one from m to n by taking inverse. (Note that we can form inverses by ReCA.) And there are no proper imbeddings, using least counterexample argument above.

So the main thing is the existence of these imbeddings.

We define a function $f_{n^z}(p)$ = greatest k such that

$(\exists q)(Suc^k_{n^z}(p,q))$, where Suc^k is k iterations of Suc. (This is well defined, since $(p)(q)(r)([Suc_{n^z}(p,r) \& Suc_{n^z}(p,q)] \longrightarrow r = q))$. There is always a greatest k, since otherwise we would have an arithmetically defined chain through n^z, and finitely many iterations of ReCA (+ I) would realize this claim as an object, and would contradict $W^z(n)$.

Fix n,m, with $W^z(n)$, $W^z(m)$, $(\exists x)(H^z_{n^z}(x))$. Let $f_{n^z}(p)$ be abbreviated $f(p)$. We claim that, for each p, either there is an imbedding of the ordering m onto an initial segment S of the ordering n, so that $k \in S \longrightarrow k <_{n^z} p$, or there is an imbedding of $\{k | k <_{n^z} p\}$ of n, onto an initial segment of m, and these imbeddings are to be found $\leq_T (x)_p^{(10(f(p)+1))}$. (This is the result of applying $10(f(p)+1)$ iterations of the jump operator to $(x)_p$.)

We assume this is false,and, as usual, form $\{p|\text{claim is false}\}$. We take an n-least member, q. If $0_{n^z}(q)$, then clearly claim is true for q. If $Suc_{n^z}(q,r)$, then certainly $f(r) + 1 = f(q)$. So there exists a comparison mapping, $\leq_T (x)_r^{(10(f(r)+1))}$, between the segment of n up to r, and m. It is then easy to see that there is a (unique) comparison map between the segment of n up to q, and m, $\leq_T (x)_q^{(10(f(r)+1))}$. The limit case is similarly easy; the uniqueness of these comparison maps is used heavily, and that the property of being a comparison map is low arithmetical, and that the function f is low arithmetical.

<u>Lemma 3</u>: The following is provable in I + ReCA: If $W^y(n)$ and $W^y(m)$, and $|n^y| < |m^y|$, and $(\exists x)(H^y_{n^y}(x))$, then $(\exists z)(z \leq_T (x)_p^{(10(f(f)+1))} \& H^y_{m^y}(z))$, where p is the l.u.b. in n^y, of the range of the imbedding of m^y in n^y, and f is f_{n^y}, defined in the proof of Lemma 2.

Proof: Similar to the proof of Lemma 2. One notes, e.g., that if $H^y_{n^y_p}$ has a solution, it must be unique.

<u>Lemma 4</u>: If $|m^z| < |n^z|$, and p is the l.u.b. in n^z of the image of the imbedding, and $(\exists y)(H^z_{m^z}(y))$, then $(\exists x)(H^z_{n^z_p}(x) \& y \leq_T (x)^{(10(f(p)+1))})$, where f is f_{n^z}. This is provable in I + ReCA.

Proof: One can prove first, as in Lemma 2, that $(\exists x)(H^z_{n^z_p}(x))$. Then, again like Lemma 2, starting with this x with $H^z_{n^z_p}(x)$, one can prove that $y \leq_T (x)^{(10(f(p)+1))}$. (Of course, it is provable in ReCA + I that $(\exists x)(H^z_{n^z_p}(x) \longrightarrow (\exists! x)\ H^z_{n^z_p}(x))$.)

<u>Def. 12</u>: Reas $(n^x) = RLO^x(n) \& (p)(0_{n^x}(p) \vee Lim_{n^x}(p) \vee (\exists k)(\exists q)[Suc^k_{n^x}(p,q) \& lim_{n^x}(q)]) \& (\exists r)(0_{n^x}(r)) \& (p)[\sim(\exists q)(Suc(q,p) \longrightarrow (q)(q \leq_{n^x} p)]$. (Reas(n) reads "n is reasonable.")

<u>Lemma 5:</u> $(z)(n)(x)(y)(m)([Reas(m^z) \& H^z_{n^z}(x) \& H^z_{m^z}(y) \& \sim W^z(m)] \longrightarrow (\exists p)(x \leq_T (y)_p$ & there are infinitely many $q >_{m^z} p))$, is provable in I + ReCA.

Proof: Assume $\mathrm{Reas}(m^z)$, $H_{n^z}^z(x)$, $H_{n^z}^z(y)$, $\sim W^z(m)$. Then there is a set, w, which has no m^z-least member. It is then easy to see, using $I + \mathrm{ReCA}$, that $(\exists g)(k)(g(k+1) <_{n^z} g(k))$. We will show that $(p)(k)((x)_p \leq_T (y)_{g(k)})$. By $\mathrm{Reas}(m^z)$, this is sufficient.

For, we take, in $I + \mathrm{ReCA}$, $\{p \mid (\exists k)(\sim((x)_p \leq_T (y)_{g(k)}))\}$. We can do this, since the predicate we are taking the extension of is arithmetical in x,y. And we take an n^z-least such p, call it q, by $W^z(n)$. We then obtain a contradiction.

1) $0_{n^z}(q)$. To get a contradiction for this case, it suffices to show that $(k)(z \leq_T (y)_{g(k)})$. If $\lim_{m^z}(g(k))$, then clearly $z \leq_T (y)_{g(k)}$. If $0_m(g(k))$, also easy. If $\mathrm{Suc}_{m^z}(g(k))$, then by $\mathrm{Reas}(m^z)$, we have $\mathrm{Suc}^r_{m^z}(g(k),s)$, some r,s with $\lim(s)$, and so $z \leq_T (y)_s$. So clearly $(y)_{g(k)} = (y)_s^{(r)}$, and so $z \leq_T (y)_{g(k)}$.

2) $\mathrm{Suc}_{n^z}(q,r)$. Then $(x)_r \leq_T (y)_{g(k)}$, all k. But then $(x)_r^{(1)} \leq_T (y)_{g(k-1)}$, all $k > 0$, and hence $(x)_q \leq_T (y)_{g(k)}$, all k for the following general reason: if $p <_{m^z} q$, then $(y)_p^{(1)} \leq_T (y)_q$. The case when $\lim_{m^z}(q)$ is easy. When $\mathrm{Suc}_{m^z}(q)$, use $\mathrm{Reas}(m^z)$ as in Case 1) above.

3) $\lim_{n^z}(q)$. Let k be arbitrary. We know $(\ell)\ (\ell <_{n^z} q \longrightarrow (x)_l \leq_T (y)_{g(k+10)})$. Furthermore, the question of whether a set is $(x)_l$, is a question low arithmetical in ℓ, due to the uniqueness of Hierarchies on RLO's, and even hierarchies

on initial segments of RLO's. Hence $(x)_q$ is low arithmetical in $(y)_{g(k+10)}$, and hence $(x)_q$ will re recursive in $(y)^{(10)}_{g(k+10)}$. Hence, by reasonableness of m^z, we have $(x)_q \leq_T (y)_{g(k)}$.

Lemma 6: "The predicate $W^x(n)$ is not Σ^1_1 in x" is provable in I + ReCA.

Proof: We have, in ReCA + I, the Kleene normal form for predicates Σ^1_1 in x, and we use that to formalize the Lemma. So assume A) $(n)(W^x(n) \equiv (\exists f)(m)\sim T(e,n,m,f,x))$. Now there is an explicit recursive function g, for which we can prove in ReCA + I, that B) $(k)(n)[(f)(\exists m)T(k,n,m,f,x) \equiv W^x(g(k,n))]$. Hence $\exists\, \bar{e}$ such that $(n)[(f)(\exists m)T(n,0,m,f,x) \equiv (\exists f)(m)\sim T(\bar{e},n,f,x)]$, is provable in I + ReCA, since the predicate $(f)(\exists m)T(m,0,m,f,x)$ of n is Σ^1_1 in x, by A) and B), and Kleene's normal form Theorem is provable. Now substitute $\bar{e}$ for n to get a contradiction.

3. Conservative Extension Result.

Theorem 1: Given any model M of pure Δ^1_1-CA + S, where S is any purely Σ^1_2 sentence, there exists a model M' of Σ^1_1-DC + S, where S is $(\exists y)(g)(\exists n)T(\bar{e},0,n,y,z)$, for some e.

Proof: A model of pure Δ^1_1-CA + S consists of an interpretation J of the natural numbers, and +,x,0,', together with a set of objects, X, and a binary relation $R(j,x)$, $j \in J$, $x \in X$. We define a new model $Hyp^y(M) = M'$, as 1) having same J,+,x,0,'; 2) having same $R(j,x)$ but restricted to those

$j \epsilon J$, $x \epsilon X$ with $M \models Hyp^y(x)$. Now since $M \models S$, choose y such that $M \models (g)(\exists n)T(\bar{e},0,n,y,g)$. Then clearly $M' \models (g)(\exists n)T(\bar{e},0,n,y,g)$. So $M' \models S$.

We claim $M' \models \Sigma_1^1$-DC. It is sufficient to show that the relativized of each axiom of Σ_1^1-DC to the predicate Hyp^y is a theorem of pure Δ_1^1-CA. (We define the relativized, $T(A)$, of a formula A, to the predicate $Hyp^y(x)$ by $T(A \vee B) = T(A) \vee T(B)$, $T(A \& B) = T(A) \& T(B)$, $T(\sim A) = \sim T(A)$, $T(\exists xA) = (\exists x)(T(A) \& Hyp^y(x))$, $T((x)A) = (x)(Hyp^y(x) \longrightarrow T(A))$, $T(Q) = Q$, Q quantifier free, $T(\exists nA) = (\exists n)T(A)$, $T((n)A) = (n)(T(A))$). In other words, the universal closure, obtained by inserting the universal quantifier (y), of the relativized of each axiom of Σ_1^1-DC to Hyp^y is a Theorem of pure Δ_1^1-CA.

1) Induction. It is clear that the relativized of each instance of induction to Hyp^y is again an instance of induction, and so is provable in pure Δ_1^1-CA.

2) ReCA. To prove the relativized of ReCA, it suffices to prove, in pure Δ_1^1-CA, that $(x)(Hyp^y(x) \longrightarrow Hyp^y(x^{(1)}))$. To see this, let $H^y_{n^y}(z)$, $W^y(n)$, $x \leq_T z$. So $x^{(1)} \leq_T z^{(1)}$. If n^y has a greatest element, i, set $k = f_{n^y}(i) + 2$. If not, set $k = 2$. Define a new ordering m^x by adding on the first $10 \cdot k$ integers on top of n, and make, in the trivial way, the ordering m^y total, so that $RLO^y(m)$ and $|n^y| < |m^y|$. Now clearly by Lemma 4, $(\exists w)(H^y_{m^y_p}(w))$, and so clearly $(\exists w)(H^y_{m^y}(w))$, by I + ReCA. But, also by Lemma 4, $z^{(1)}$

$\leq_T w$. Hence $Hyp^y(z^{(1)})$.

3) Suppose $(f)(Hyp^y(f) \longrightarrow (\exists g)(Hyp^y(g)\ \&\ (m){\sim}T_3$ $(e,0,m,f,g,x)))$, where $Hyp^y(x)$. Let $Hyp^y(f)$. We consider the predicate $Pn \equiv n$ is a member of a finite sequence $n_0,n_1,\ldots,n_k$, $k \geq 0$, such that I) $(i)W^y(n_i)$. II) For any sequence of sets $x_0,x_1,\ldots,x_j$, $j \leq k$, such that $(i)(H^y_{n_i^y}(x_i))$, we have that $\exists$ another sequence z_i such that each $z_i \leq_T x_i$, and $z_0 = f$ and $(i < j)(m){\sim}T_3(e,0,m,z_{i+1},x)$. III) For any sequence $x_0,x_1,\ldots,x_j$, $j < k$, with $(i \leq j)(H^y_{n_i^y}(x_i))$, it is not the case that $(\exists z)(\exists p)[H^y_{(n_{j+1})_p^y}(z)\ \&\ p$ has infinitely many $q >_{n_{j+1}^y} p\ \&\ \exists z_0,z_1,\ldots,z_j,z_{j+1}$, $z_i \leq_T x_i$, $z_{j+1} \leq_T z$, with $(s \leq j)(m){\sim}T_3(e,0,m,z_s,z_{s+1},x)]$.

Also consider $Qn \equiv n$ is a member of a finite sequence $n_0,\ldots,n_k$, $k \geq 0$, $Reas(n_i^y)$, such that $\exists$ sequence of sets $x_0,x_1,\ldots,x_k$ with I) $(i)H^y_{n_i^y}(x_i)$. II) $\exists$ a sequence z_i such that each $z_i \leq_T x_i$ and $z_0 = f$ and $(i)(m){\sim}T_3(e,0,m,z_i,z_{i+1},x)$. III) Let $-1 \leq j < k$. It is not the case that $(\exists z)(\exists p)$ $[z \leq_T x_{j+1}\ \&\ H^y_{(n_{j+1})_p^y}(z)\ \&\ p$ has infinitely many $q >_{n_{j+1}^y} p$ $\&\ \exists\ z_0,z_1,\ldots,z_j,z_{j+1}$, $z_i \leq_T x_i$, $z_{j+1} \leq_T z$, with $(S \leq j)(m)$ ${\sim}T_3(e,0,m,z_s,z_{s+1},x)]$.

We first note that Pn is a Π_1^1 predicate in f,x,y and that Qn is a Σ_1^1 predicate in f,x,y.

We wish to show, in I + ReCA, that $(n)(Pn \equiv Qn)$.

Suppose Pn. Assume that $n_0, n_1, \ldots, n_k$ have the properties mentioned in the definition of Pn. We claim that $(i)(\exists x_i)H^y_{n_i^y}(x_i)$. For, if for some i, $\sim(\exists x_i)H^y_{n_i^y}(x_i)$, then take i to be least such (induction), and then by $(i)W^y(n_i)$ we have that $(q)([W^y(q) \,\&\, (\exists z)(H^y_{q^y}(z))] \longrightarrow |q^y| < |n_i^y|)$, and, in fact, q^y is imbedded into n_i^y with l.u.b. ℓ having infinitely many points $p >_{n_i^y} \ell$. Hence the z with $H^y_{q^y}(z)$ must be $\leq_T$ some w with $H^y_{(n_i)^y_r}(w)$, where r has infinitely many points $s >_{n_i^y} r$. We claim that this violates condition III) of Pn when $j = i-1$. (We allow j to be negative.) For just apply condition 2) of Pn, and that $(f)(\exists g)(m)\sim T$ relativized to Hyp^y, and that $\mathrm{Hyp}^y(f)$. In condition II) of Pn, use that $(s)(s < i \longrightarrow (\exists x_i)H^y_{n_i^y}(x_i))$, and use the x_i.

So we want to show Qn, and we may choose $x_0, x_1, \ldots, x_k$ with $H^y_{n_i^y}(x_i)$. II) in Qn follows from 2) in the definition of Pn. III) in Qn follows from III) in Pn plus the observation that the predicates $H^y_{(n_j^y)_p}$ have unique solutions $\leq_T$ in $(x_j)_p$.

So Pn $\longrightarrow$ Qn.

Suppose Qn. We first to show $(i)W^y(n_i)$. Using Lemma 5, and taking i least with $\sim W^y(n_i)$, we contradict III) of Qn in the same way as the argument above for Pn $\longrightarrow$ Qn. For, one uses II) of Qn, and the relativization of $(f)(\exists g)(m)\sim T$.

So $(i)W^y(n_i)$.

Now II) of Pn follows from II) of Qn by uniqueness of hierarchies on n_i because of $W^y(n_i)$.

III) of Pn follows from III) of Qn for the same reason.

Next, we show that we can eliminate the parameters f and x in Pn and Qn, and have the same meaning and still be, respectively, Π^1_1 and Σ^1_1. For, since $Hyp^y(x)$ and $Hyp^y(f)$, let e_1, e_2 have $W^y(e_1)$ and $W^y(e_2)$ & $(\exists z)(H^y_{e^y_1}(z)$ & $x \leq_T z$ with Gödel number k) & $(\exists w)(H^y_{e^y_2}(w)$ & $f \leq_T w$ with Gödel number l). Then take P'n to be $(x)(f)(w)(z)([H^y_{e^y_1}(z)$ & $H^y_{e^y_2}(w)$ & $x \leq_T$ with Gödel number k & $f \leq_T w$ with Gödel number l] $\longrightarrow P(n,f,x))$. Take Q'n to be $(\exists x)(\exists f)(\exists w)(\exists z)$ $(Q(n,x,f)$ & $H^y_{e^y_1}(w)$ & $x \leq_T z$ with Gödel number k & $f \leq_T w$ with Gödel number ℓ). Here, e_1, e_2, k, and ℓ are <u>constants</u>, not variables.

Now, Q'n is Σ^1_1. Also, $(n)(Q'n \longrightarrow W^y(n))$, since $Q'n \equiv P'n$. So define $Rm \equiv (\exists n)(Q'n \ \& \ |m^y| \leq |n^y|)$. So clearly $(m)(Rm \longrightarrow W^y(x))$, and R is Σ^1_1. So by Lemma 6, $(\exists p)(W^y(p) \ \& \sim Rp)$. Fix such a p. So $W^y(p)$ & $(n)(Q'n \longrightarrow$ $|n^y| \leq |p^y|)$. We wish to form $\{\langle r,s\rangle | Q's \ \& \ |s| \leq |$the segment of p^y up to $r|\}$. We can, using pure Δ^1_1-CA, since this is also equivalent to $\{\langle r,s\rangle | P's \ \& \sim($the segment of p^y up to $r|<|s|)\}$, form this (these) set(s). Call this set S. Let $T = \{r | (\exists s)(\langle r,s\rangle \in S\}$. By $W^y(p)$, let p_0 be the $<_{p^y}$-least upper bound of this set. Let q have $W^y(q)$, with $|q| = |$the segment of p^y up to and including $p_0|$. If there is no upper bound for T, take $q = p$. In any case, clearly $(n)(Q'n \longrightarrow |n^y| < |q^y|)$. Also, clearly $(\ell)(\exists n)(Q'n \longrightarrow |n^y| \geq |$the segment of q up to $\ell|)$. Hence, clearly $(\ell)(\exists! z)(H^y_{q_1^y}(z))$.

We wish to show $(\exists w)(H^y_{q^y}(w))$. The only hard case is when q^y has no greatest point. But we can form, in pure Δ^1_1-CA, $\{\langle\ell,j\rangle|(\exists z)(H^y_{q^y_\ell}(z) \;\&\; j\epsilon z)\}$, since it, besides this Σ^1_1 definition, has the Π^1_1 definition $\{\langle\ell,j\rangle|(z)(H^y_{q^y_\ell}(z) \longrightarrow j\epsilon z)\}$, and we may, arithmetically in this set, get a w with $H^y_{q^y}(w)$.

Finally, we claim that the sequence (not necessarily unique) needed to verify the relativized Σ^1_1-DC to Hyp^y, can be defined arithmetically in w with $H^y_{q^y}(w)$, and hence would have what we wanted all along, namely satisfying Hyp^y. To see this, it suffices to show that any finite sequence $n_0,\ldots,n_k$ satisfying the conditions in the definition of Pn can be extended to $n_1,n_1,\ldots,n_{k+1}$, satisfying Pn, and also that there are sequences satisfying the definition of Pn to begin with. To see the latter, let $f \leq_T z$ with $H^y_{e^y}(z)$, $W^y(e)$. Take $\{k|f \leq_T (z)_k\}$, and let k be a l.u.b. Take s with $|s^y| = |$the segment of e^y up to k with an appropriate finite number of points added on top so as to make, by the Lemmas, $(\exists w)(H^y_{s^y}(w) \;\&\; z \leq_T w)|$. If there is no l.u.b., take $s = e$. Take $n_0 = s$. Then the sequence $\langle n_0\rangle$ satisfies the

the definition of Pn. The same trick allows one to see how to extend a sequence $n_0, n_1, \ldots, n_k$ satisfying Pn to an $n_0, n_1, \ldots, n_k, n_{k+1}$ satisfying Pn, for one uses the fact that $(f)(\exists g)(m)\sim T$ relativizes, and use that $\langle n_0, n_1, \ldots, n_k \rangle$ satisfies (the definition of) Qn.

4. Independence Result. We wish to show here that Σ^1_1-DC is independent of Σ^1_1-AC. It suffices to show that $\Sigma^1_1\text{-AC} \nvdash \Sigma^1_1\text{-DC}$. We do this by showing that, for a suitably chosen sentence S with $\mathrm{Con}(S + \Sigma^1_1\text{-DC})$, we can prove in $S + \Sigma^1_1$-DC that $\mathrm{Con}(S + \Sigma^1_1\text{-AC})$. For then, if $\Sigma^1_1\text{-DC} = \Sigma^1_1\text{-AC}$, then $S + \Sigma^1_1$-DC proves $\mathrm{Con}(S + \Sigma^1_1\text{-DC})$, and hence by Gödel's theorem, $S + \Sigma^1_1$-DC would be inconsistent.

Lemma 1: If S is any sentence in prenex form starting with universal number and set quantifiers from left, followed by existential number and set quantifiers, followed by a matrix containing only number quantifiers, then $\mathrm{Pr}(\Sigma^1_1\text{-DC} + S$, "$\exists$ a (coding into natural numbers) ω-model satisfying S").

Proof: Collapse all the universal number quantifiers in the left part of S, and collapse all the existential number quantifiers on the right part of S (before the matrix) so that S becomes $S' = (n)(x)(\exists m)(\exists y)A(n,x,m,y)$. Then push $(\exists m)$ into the matrix, to get $(n)(x)(\exists y)B(n,x,y)$.

Now in set theory, given two ω-models $\mathfrak{A}$ and $\mathfrak{B}$, we can talk of $\mathfrak{B}$ being a closure of $\mathfrak{A}$ under the sentence $(z)(\exists w)C(z,w)$, i.e., $\mathfrak{A} \subseteq \mathfrak{B}$ and $(z)(z \in \mathrm{Dom}(\mathfrak{A}) \longrightarrow (\exists w)(w \in \mathrm{Dom}(\mathfrak{B})\ \&\ C(z,w)))$.

Instead of talking of real ω-models, talk of codings of them into a set of natural numbers in a natural (arithmetical) way. Then we can prove in Σ_1^1-DC + S that for any n and for any finite ω-model, $\exists$ a finite closure of this ω-model under the n sentences $(x)(\exists y)B(\bar{k},x,y)$, $k < n$. This depends heavily on that $(n)(x)(\exists y)B(n,x,y)$ is provable in Σ_1^1-DC + S.

Now, consider the arithmetical predicate $D(z,w)$ = "z is a (coding of) pair (k,x), w is a (coding of) pair $(k+1,y)$ where y is a (coding of) finite ω-model which is a closure of the (coding of) finite ω-model x under the k sentences $(x)(\exists y)B(\bar{p},x,y)$, $p < k+1$." Now applying Σ_1^1-DC to this predicate D, we obtain a sequence of dependent choices in which the desired ω-model of S can be obtained recursively.

Now suppose we found a true sentence S in the form for Lemma 1, such that I + S proves Σ_1^1-AC. Then we would be done, since I + S would be provably consistent in Σ_1^1-DC + S (remember all ω-models provably satisfy I) and Σ_1^1-DC + S is consistent, since S is true.

By well-known techniques, ReCA is finitely axiomatizable, since we need only consider the case of two parameters. We take S to be the conjunction of this finite axiomatization with the sentence $(x)(n)(W^x(n) \longrightarrow (\exists y)(H^x_{n^x}(y)))$. Then S is clearly of the proper form.

We define $(W^*)^x(n) = RLO^x(n) \;\&\; (y)(Hyp^x(y) \longrightarrow \exists$ an n^x-least element of y, provided $y \neq \emptyset)$.

We know that S + I $\supseteq$ ReCA + I, so we can use Lemmas of the previous chapter about provability in ReCA + I.

<u>Lemma 2</u>: The following is provable in $S + I$: for every n with $W^x(n)$, there is a coding of a function $f:\omega \longrightarrow P(\omega)$ such that whenever $(W^*)^x(m)$ & $\sim W^x(m)$, we have $f(m)$ is an imbedding of n^x onto an initial segment of m^x.

Proof: We can prove, like Lemma 2 of the previous section, in $I + ReCA$, that if $(W^*)^x(n)$ & $(\exists z)(H^x_{n^x}(z))$, then the conclusion of the Lemma is true. But S guarantees that $(\exists z)(H^x_{n^x}(z))$ just on the basis of $W^x(n)$.

Now suppose $(n)(\exists f)(m)\sim T(e,0,m,f,n,x)$. The predicate of p, $(\exists z)H^x_{p^x}(z))$ & $Reas^x(p)$ is Σ^1_1, and so, since every p with $W^x(p)$ satisfies it, $\exists$ k such that $(\exists z)(H^x_{k^x}(z))$ & $Reas^x(k)$ & $\sim W^x(k)$. Then by Lemma 5, $(z)(H^x_{k^x}(z) \longrightarrow \sim Hyp^x(z))$. Clearly we have $(n)(\exists f)(\exists z)[H^x_{k^x}(z)$ & $(m)\sim T(e,0,m,f,n,x)]$. Collapsing the quantifiers in the usual way, and putting the result in Kleene Normal Form, we end up with $(n)(\exists h)(m)$ $\sim T(q,0,h,n,x)$. Furthermore, any Skolem function for this sentence would have, recursive in it, a Skolem function for $(n)(\exists f)(m)\sim T(e,0,m,f,n,x)$. So it remains to show, in $S + I$, that $(n)(\exists h)(m)\sim T(q,0,m,h,n,x)$ has a Skolem function.

Now, there is a standard recursive function F such that $(n)(RLO^x(F(n))$ & any set for which there is no $F(n)^x$-least element, has recursive in it a solution h to $(m)\sim T(q,0,m,h,n,x)$ & any solution h has recursive in it a set for which there is no $F(n)^x$-least element). So $(n)((W^*)^x(F(n))$ & $\sim W^x(F(n)))$.

Now consider the Σ^1_1-predicate $Pk = RLO^x(k)$ & $(\exists f)(n)(f$ a coding of a function from ω into $P(\omega)$ such that $f(n)$ is an imbedding of k^x into $F(n)^x$). By Lemma 2 of this section, this Σ_1^1 predicate holds of all k with $W^x(k)$, and so holds for some r with $\sim W^x(r)$, by Lemma 6. Then we have a function f such that for each n, $f(n)$ is an imbedding of r^x into $F(n)^x$. Let X be any non-empty set for which there is no r^x-least member. Then recursively in X, f, we can find a coding g of a function $\omega \longrightarrow P(\omega)$, such that, for each n, $g(n)$ is a non-empty set for which there is no $F(n)^x$-least element. Hence, by the special property of the recursive function F, we may obtain the desired Skolem function for $(n)(\exists h)(m)\sim T(q,0,m,h,n,x)$ recursively in g.

Thus we have verified that $I + S$ proves Σ_1^1-AC, and we immediately have

<u>Theorem 2</u>: Σ_1^1-DC is independent of Σ_1^1-AC.

5. <u>Relation Between Predicative and Hyperarithmetic Analysis</u>.

Σ_1^1-AC (or Δ_1^1-CA or Σ_1^1-DC) are considered reasonable formulations of so-called hyperarithmetic analysis, in view of the fact that they are natural systems whose minimum ω-model consists of exactly the hyperarithmetic sets of natural numbers.

In this section we compare Σ_1^1-AC with a system T which represents the formalization of a small part of predicative analysis (see Feferman, [1]).

The system T is I + ReCA + the infinite list of axioms $(y)(\exists x)(H^y_{k_n}(x))$, where n varies, and k is fixed, and k is the Gödel number of a natural well-ordering of type ε_0.

(Thus the infinite list of sentences is obtained by changing n).

We will show that Σ_1^1-AC is a conservative extension of T with respect to all purely Π_2^1 sentences.

(ReCA is formalized as one sentence, in the standard way, i.e., using only 2 parameters x_0, x_1.)

It is easily seen by well-known techniques that the proof of conservative extension can be made finitary. So we obtain a finitary proof of $\mathrm{Con}(\Sigma_1^1\text{-AC})$ relative to $\mathrm{Con}(T)$. Finitary generally means here, in PRA (primitive recursive arithmetic).

A widely used index of complexity of an axiomatic theory is how large is the least upper bound of its provable ordinals. In the present context, an ordinal α is said to be a provable ordinal of a given fixed theory, if there is an n with $\mathrm{RLO}(n)$ and n has order type α and the theory proves $W(n)$. In view of the conservative extension for Π_2^1 that we will prove, it is clear that the provable ordinals of Σ_1^1-AC are exactly the provable ordinals of T.

It follows from work of Feferman [1] and Tait [5] that the least upper bound on the provable ordinals of T is the ordinal represented by the ε_0-th critical function at 0. (See Feferman [1], pp. 14-16). Furthermore, it also follows from their work that in PRA + rule of primitive recursive induction on the natural RLO corresponding to the ε_0-th critical function at 0, a consistency proof may be given for T. So, by our results, such a consistency proof may be given for Σ_1^1-AC.

By an instance of induction on $\bar{k}_n$ we mean a statement of the form $\{A(\bar{q}) \ \& \ (p)([p <_{\bar{k}} n \ \& \ (m)(m <_{\bar{k}} p \longrightarrow Am)] \longrightarrow Ap)\} \longrightarrow (p)(p <_{\bar{k}} n \longrightarrow Ap)$, where $\bar{q}$ has $O_{\bar{k}}(q)$, and A is any formula (in the language of Analysis, described at the beginning of this chapter), with possibly free variables, and A is in prenex form.

By the complexity of a formula in prenex form, we will mean here the total number of quantifiers occurring. Let Comp(A) be the complexity of A.

For each integer m, there is a natural predicate $T_{\bar{m}}(n)$ on Gödel numbers of formulae of complexity $\leq n$, such that $T_{\bar{m}}(n)$ says that the formula with Gödel number n is true. The formalization of these truth predicates involve placing formulae in a weak Kleene normal form (i.e., no attention is paid to the form of the quantifiers, but only that the matrix be the T-predicate of the appropriate variables).

By the reflection principle, for a theory S, (of this language) of complexity $\leq m$, we mean the single sentence

$(n)([n$ Gödel number of a sentence of complexity $\leq \bar{m}$ & $Pr(S,n)] \longrightarrow T_{\bar{m}}(n))$. We call this $R_m(S)$.

We need some facts from proof theory which follow from work of Tait (see [5]), Feferman (see [1], p. 23) and Kreisel (see [4]).

<u>Fact 1</u>: ∃ primitive recursive functions F, G, and H such that the following are provable in PRA:

1) For each $m,n > 0$, $F(m,n)$ gives the Gödel number of an instance of induction on $\bar{k}_{G(n)}$ applied to a formula of complexity $\leq H(m)$, with no parameters.

2) For each m,n we have that $R_m(I_n)$ is provable in the theory I_n + the formula represented by $F(m,n)$. I_n is defined as the subsystem of I consisting of induction applied only to formulae of complexity $\leq n$.

Let J_p be the sentence "for every purely Σ_1^1 predicate Pn (possibly with parameters), $(\exists \ell)(q)((q <_{\bar{k}} \ell \longrightarrow Pq)$ & $(\ell = p \vee (\ell \leq_{\bar{k}} p$ & $(r)(r \geq_{\bar{k}} \ell \longrightarrow \sim Pr))))$." We let $T' = T$ + the infinite list J_p.

In the theory T, the axioms $(y)(\exists x)H^y_{\bar{k}_n}(x)$, as n varies, have bounded complexity. Choose c' such that the conjunction of any sentence $(y)(\exists x)(H^y_{\bar{k}_n}(x))$ with ReCA and any purely Σ_2^1 sentence and any J_p, has complexity $\leq c'$, some fixed constant c'. We are interested in Fact 1, when $m = c'$. We let $c = H(c')$. We define new theories $T^p_n = I_n$ + $(y)(\exists x)H^y_{\bar{k}_{\bar{p}}}(x)$ + ReCA + $J_{\bar{p}}$. Applying Fact 1 for $m = c'$, we get

Fact 2: $\exists$ primitive recursive functions F and G such that the following are provable in PRA:

1) For each $n > 0$, $F(n)$ gives the Gödel number of an instance of induction on $\bar{k}_{G(n)}$ applied to a formula with no parameters of complexity $\leq c$.

2) For each $n > 0$, we have that $R_{c'}(I_n)$ is provable in the theory I_n + the formula represented by $F(n)$.

Now, since Comp(A) = Comp($\sim$A), we see that $R_{c'}(I_n)$ formally implies $\mathrm{Con}(T^p_n + B)$ within $T^p_n + B$, where p is any integer, and B is any purely Σ_2^1 sentence. So hence the formula represented by F(n) must not be provable in $T_n^p + B$ if $T_n^p + B$ is consistent, by Gödel's Theorem. This is so, no matter what n and p are.

Let $T' = \left(\bigcup_{\substack{n\in\omega \\ p\in\omega}} T_n^p \right)$. Thus

<u>Fact 3</u>: $\exists$ primitive recursive functions F, G, such that, in PRA, under the assumption Con(T' + B), B some fixed Σ_2^1 sentence, we can derive, for some constant c,

(n)(p)Con(T_n^p + B + ~(the sentence "F(n)")) & (n)(F(n) represents an instance of induction on $\bar{k}_{G(n)}$ applied to a formula with no parameters of complexity $\leq$ c).

<u>Lemma 1</u>: From Con(T' + B) we can conclude, by finitary means, that Con(Σ_1^1-AC + B).

Proof: It suffices to show (n)Con(Σ_1^1-AC with only induction I_n + B).

For each n, we get a model for "B + Σ_1^1-AC with only I_n" by applying the inner model technique to the theory

$T_{n+c^3}^{G(n+c^3)}$ + B + ~(the sentence "F($n+c^3$)").

Now F($n+c^3$) represents a certain instance of induction on $\bar{k}_{G(n+c^3)}$, and let this induction be applied to Qs, no parameters. Now B says $(\exists f)$(f satisfies some specific Π_1^1 property); we fix such an f. Then we wish to show the relativized of each instance of "B + Σ_1^1-AC with only induction I_n"

to the predicate, $Rx \equiv (\exists s)(\exists y)(H^{f}_{\bar{k}_s}(y)\ \&\ x \leq_T y\ \&\ (r)(r \leq_{\bar{k}} s$ $\longrightarrow Qr))$, is provable in $T^{G(n+c^3)}_{n+c^3} + B + \sim$(the sentence "$F(n+c^3)$"). This would give a consistency proof of "$B + \Sigma^1_1$-AC with only I_n" relative to $Con(T^{G(n+c^3)}_{n+c^3} + B + \sim$(the sentence "$F(n+c^3)$"). This in turn is immediately generalizable to a consistency proof of Σ^1_1-AC + B relative to $Con(T' + B)$.

Actually, it suffices to consider the case when $n \geq 5$. This lower bound will be convenient later.

So, certainly all instances of I_n provably relative to R since the predicate R has small complexity compared to c^3, and I_{n+c^3} is available in $T^{G(n+c^3)}_{n+c^3}$.

Certainly, B provably relativizes, since the numbers, and hence all the arithmetical relations, remain unchanged by taking this inner model, by fiat.

For similar reasons, clearly ReCA holds in this inner model. The inner model is non-empty, since $Q\bar{q}$, where $0_{\bar{k}}(\bar{q})$, since we assumed that induction on Q is false.

To verify the last, and most important axiom of "$B + \Sigma^1_1$-AC with only I_n", first note that since $n \geq 5$, we have that, in ReCA + I_n we can prove $W(\bar{k})$ by well-known techniques. Hence $(\ell)(\ell \leq_{\bar{k}} G(n+c^3) \longrightarrow (\exists ! y)H^{f}_{\bar{k}_\ell}(y))$ is provable in $T^{G(n+c^3)}_{n+c^3}$. From now on, whenever $\ell \leq_{\bar{k}} G(n+c^3)$, we denote the unique y above by $H^{f}_{\bar{k}_\ell}$. Then also

$(\ell)(p)([\ell \leq_{\bar{k}} p \ \& \ \ell \leq_{\bar{k}} G(n+c^3) \ \& \ p \leq_{\bar{k}} G(n+c^3)] \longrightarrow H^f_{\bar{k}_\ell} \leq_T H^f_{\bar{k}_p})$ is provable in $T^{G(n+c^3)}_{n+c^3}$.

Now suppose that $(p)(\exists g)A(p,g,x)$ holds relativized to the predicate R, where we also have Rx. We wish to conclude, in the theory $T^{G(n+c^3)}_{n+c^3} + B + \sim$"$F(n+c^3)$" that $\exists$ Skolem function for the above sentence, h, such that Rh.

So we have $(p)(\exists g)(A(p,g,x) \ \& \ Rg)$, A arithmetical. But consider an integer s satisfying the properties in the definition of Rg. Since induction fails on $\bar{k}_{G(n+c^3)}$ for Q, we see that $s <_{\bar{k}} G(n+c^3)$. It is straightforward to see, using $W(\bar{k})$ and ReCA, that 1) $(p)(\exists ! s)(s <_{\bar{k}} G(n+c^3) \ \& \ (\exists g \leq_T H^f_{\bar{k}_s})$ $(A(p,g,x)) \ \& \ (t)(t <_{\bar{k}} s \longrightarrow H^f_{\bar{k}_t}$ does not have a $g \leq_T H^f_{\bar{k}_t}$ with $A(p,g,x))$. It is also easy to see that the part of this sentence to the right of $(\exists ! s)$ can be written in a natural way in Σ^1_1 form (since $t <_{\bar{k}} s \longrightarrow H^f_{\bar{k}_t} \leq_T H^f_{\bar{k}_s}$). Hence consider the Σ^1_1 predicate Ps which holds iff $(\exists p)$(stuff to the right of $(\exists ! s)$ holds of p and s). It is clear that $(s)(Ps \longrightarrow [(s <_{\bar{k}} G(n+c^3) \ \& \ (r)(r \leq_{\bar{k}} s \longrightarrow Qr)])$. Applying the axiom $J_{G(n+c^3)}$ we get a l.u.b. in $\bar{k}$, call it k_0, on the s with Ps. Now $k_0 \leq_{\bar{k}} G(n+c^3)$. Then clearly $(r)(r <_{\bar{k}} k_0 \longrightarrow Qr)$. So if $k_0 = G(n+c^3)$, then $(r)(r <_{\bar{k}} G(n+c^3) \longrightarrow Qr)$, contradicting that induction fails on $G(n+c^3)$ with Q.

So $k_0 <_{\bar{k}} G(n+c^3)$. Hence, since $(r)(r <_{\bar{k}} k_0 \longrightarrow Qr)$, we have $(r)(r \leq_{\bar{k}} k_0 \longrightarrow Qr)$. A similar argument shows that $G(n+c^3)$ cannot be reached by a finite number of iterations of successor from k_0, in the ordering $\bar{k}$. But by sentence 1), it is clear that a Skolem function for our original sentence can be found recursive in at most a few jumps of $H^{f}_{\bar{k}_{k_0}}$. Hence a Skolem function can be found satisfying the predicate R.

In retrospect what we have shown is that for each sufficiently large n, "Σ^1_1-AC with only I_n + B" is consistent if $T^{G(n+c^3)}_{n+c^3}$ + B + $\sim(F(n+c^3))$" is. Consequently any Π^1_2 sentence provable in Σ^1_1-AC must also be provable in T'. We now observe

<u>Lemma 2</u>: T' = T.

Proof: It suffices to show that each J_p is provable in T. But this is clear, since it is well known that T (or even just I + ReCA) proves induction for any formula on $k_{\bar{p}}$, for each p.

<u>Lemma 3</u>: $T \subseteq \Sigma^1_1$-AC.

Proof: We have to show that for each p, the sentence $(y)(\exists x)H^y_{\bar{k}_p}(x)$ is provable in Σ^1_1-AC. Let A(q) be the predicate $(y)(\exists x)H^y_{\bar{k}_q}(x)$, and we apply induction to A on the ordering $\bar{k}_p$. (Remember, the full schema of induction on $\bar{k}_p$ is provable in Σ^1_1-AC). If $Suc_{\bar{k}}(r,q)$, then clearly $A(q) \longrightarrow$

$A(r)$, by taking jumps, using ReCA. Suppose $\lim_{\bar{k}}(q)$ & (r) $(r <_{\bar{k}} q \longrightarrow A(r))$. Then $(r)(r <_{\bar{k}} q \longrightarrow (\exists x)H^{y}_{\bar{k}_r}(x))$. We can form a Skolem function for this sentence, within Σ^1_1-AC. When we do, we can find, recursive in at most a few jumps of the Skolem function, a z with $H^{y}_{\bar{k}_q}(z)$. This is true of all y, and so we have $A(q)$. Hence by induction on $\bar{k}_p$, we have $(q)(q <_{\bar{k}} p \longrightarrow A(q))$. By a similar argument to the above, we obtain $(y)(\exists x)H^{y}_{\bar{k}_p}(x)$.

We have immediately

Theorem 3: Σ^1_1-AC is a conservative extension of T for Π^1_2 sentences.

CHAPTER III

In this chapter, we consider the question of just which recursive linear orderings can have certain structures placed on them; namely, hierarchies. It is convenient to consider more general notions of recursive linear orderings and hierarchies, than in Chapter II.

We say, in this chapter, $RLO^{+}(n)$ iff φ_e defines a recursive linear ordering (as in Chapter II) whose field is recursively enumerable. For any $p \in Field(n)$, we let n_p be the name for the subordering of n, whose field is all q with $q <_n p$. Thus n_p itself is an RLO^{+}.

We define $H_n^{+}(x)$ as 1) $(k)(k \epsilon x \longrightarrow (\exists m)(k \epsilon (x)_m \;\&\; m \in Field(n))$. 2) $(q)(q \in Field(n) \longrightarrow [(O_n(q) \;\&\; \emptyset \leq_T (x)_q) \vee (Suc_n(q,r) \;\&\; (x)_r^{(1)} \leq_T (x)_q) \vee (lim_n(q) \;\&\; \{P(r,s) | s <_n q \;\&\; r \in (x)_s\} \leq_T (x)_q)])$.

Note that in ReCA + I, we can define the satisfaction relation for (codings of) ω-models. We can do this, since we can prove in ReCA + I that $(x)(\ell)(\exists y)(y = x^{(\ell)})$. Given $H_e(x)$, we define the <u>corresponding</u> M_x (a coding of an ω-model) as (a coding of) the sets $\leq_T$ in some $(x)_m$, $m \in Field(n)$. We define $Reas^{+}(e)$ the same as in Chapter II, except replace $RLO(e)$ by $RLO^{+}(e)$. We define $lim(e)$ as $(p)(\exists q)(p \in Field(e) \longrightarrow p <_e q)$.

We define $W^{+}(e)$ and $W^{*+}(e)$ the same as in Chapter II, except replace $RLO(e)$ by $RLO^{+}(e)$.

We say that $NW(e,X)$ iff $RLO^{+}(e)$ & e is not well-founded with respect to X, i.e., 1) $(\exists n)(n \in X \ \& \ n \in Field(e))$ 2) there is no e-least member of X.

Lemma 1: Let x have $H_e^{+}(x)$, $Reas^{+}(e)$, $lim(e)$. Then for corresponding M_x, we have $M_x \models W^{*+}(e)$. In fact this Lemma is provable in $I + ReCA$.

Proof: We consider the theory $T = I + ReCA + (\exists x)(\exists e)$ (they satisfy hypotheses of the Lemma but not the conclusion). We will show that T proves its own consistency.

In the theory T, fix such an x, e and M = corresponding M_x. We will show, in T, that this $M \models T$.

It is clear that $M \models I$.

Also, since $lim(e)$, we have $M \models ReCA$.

It remains to show $M \models (\exists z)(\exists n)(H_n^{+}(z)$ & $Reas^{+}(n)$ & $lim(n)$ & corresponding $M_z \models \sim W^{*}(n))$.

Since $M \models \sim W^{*}(e)$, there is an r and an X with $H_r(X)$ & $M = W(r)$ & $\leq_T X$ there is a set Y with $NW(e,Y)$. Fix such an r. Now choose s such that 1) $Lim_e(s)$. (This is taken to imply implicitly $s \in Field(e)$). 2) $NW(Y, e_s)$. We can do this, using $Reas(e)$. Consider $(y)_s$. There is a set $z \leq_T (y)_s$ with $H_{e_s}^{+}(z)$, and with $(t)(t <_e s \longrightarrow (z)_t = (y)_t)$. We set $n = e_s$. We claim that for these values, our claim about satisfaction in M holds. For, since $M \models \sim W(e_s)$ & $Reas^{+}(e)$ & $(ReCA + I)$, and $M \models H_r(X)$ & $W(r)$, we see that X, and hence Y, must be in the model M_z, and M_z certainly $\models W(r)$. So $M_z \models \sim W^{*+}(e_s)$. So certainly $M \models (M_z \models \sim W^{*+}$

(e_s)). Note that $M_z \in M$, since lim(e).

This completes the proof that $M \models T$.

Hence T proves Con(T), since the Soundness Theorem can be formalized and proved in ReCA + I.

So, by Gödel's Theorem, T is inconsistent. So the Lemma has been established. In fact, in view of the inconsistency, the Lemma is provable in ReCA + I.

<u>Theorem 1</u>: If $\text{Reas}^+(e)$ and $\sim W^{*+}(e)$ (of course, in view of $\text{Reas}^+(e)$, W* and $W^{*+}(e)$ are identical notions, since $\text{RLO}^+(e)$)), then there is no x with $H_e^+(x)$.

Proof: If lim(e), then if there was such an x, then corresponding M, by Lemma 1, has $M_x \models W^{*+}(e)$. However, since every hyperarithmetic set is in M (because $\sim W^+(e)$) we must have $M_x \models \sim W^{*+}(e)$, which is a contradiction.

If ~lim(e), then by Reas(e), there is an m with $\text{lim}_e(m)$ & there are only finitely many $n >_e m$. So $\sim W^{*+}(e_m)$. Then argue as above.

Let NTWO(n) be $\text{RLO}^+(n)$ & no tail of n is well-ordered, i.e., (p)(p $\in$ Field(n) ⟶ the subordering on $\{q \mid q \geq_n p\}$ is not well-founded).

<u>Lemma 2</u>: If $\text{Reas}^+(e)$ & $H_e^+(x)$ & corresponding $M_x \models$ NTWO(n), then $M_x \models$ I + ReCA + S, where S is the sentence of Chapter II.

Proof: Let $z \in M$. We wish to show $M_z \models$ ReCA + I + $(n)(W^z(n) \longrightarrow (\exists w)(H_n^z(w)))$. Clearly lim(e), since $M_z \models$ NTWO(n). Hence $M_z \models$ I + ReCA.

Clearly $(\exists s)(s \in \text{Field}(e)\ \&\ z \leq_T (x)_s)$. Let $n \in \text{Field}(e)$ such that the subordering of e defined on $\{q \mid s <_e q <_e n\}$ is satisfied not to be well-founded in the model M_x. Then M_x satisfies that there is a hierarchy y (in the generalized sense of this chapter) on a non-well-founded RLO^+, k, the subordering of e defined above, such that $(y)_p \geq_T z$, where $O_k(p)$. Then the obvious generalization of Lemma 5 in Chapter II gives $M_x \models (n)(W^z(n) \longrightarrow (\exists w)(H_n^z(w)\ \&\ w \leq_T y))$, in view of $M \models I + ReCA + (\exists X) \sim NW(k,X)$.

<u>Lemma 3</u>: The following is provable in I + ReCA + S: If $NTWO(n)\ \&\ W^{*+}(n)\ \&\ p \in \text{Field}(n)$, then $(\exists q)(q >_n p\ \&\ NTWO(n_q))$.

Proof: Choose $r \in \text{Field}(n)$ with $r >_n p$ and the subordering of n determined by $\{s \mid p \leq_n s <_n r\}$ is not well-founded. We can do this since $\sim W^+(n)$.

Now consider the Π_1^1 predicate, $Pt \equiv t <_n r$ & the subordering of n from t to r is well-founded. If P has no solutions, we are done, for then $NTWO(n_r)$.

Harrison (see [2]) has shown that every Π_1^1 predicate which has a solution in Field(n), where $W^{*+}(n)$, has an n-least solution. His proof uses only principles provable in I + ReCA, excepting the comparability of recursive well-orderings. In Chapter II, we showed this comparability Lemma is provable in I + ReCA + S, by Lemma 2 of Section 2.

So there is an n-least solution to the predicate Pt, call it q, and we can prove this in I + ReCA + S. By the way r was chosen, it is clear that $q >_n p$. And by the way

q is defined, it is clear that $NTWO(q)$.

<u>Lemma 4</u>: Let $Reas^+(e)$, $H_e^+(x)$, $lim(e)$. Then corresponding $M_x \models \sim NTWO(e)$.

Proof: Consider the theory $T = I + ReCA + (\exists e)(\exists x)$ (they satisfy hypotheses, but not conclusion). As in Lemma 1, we wish to show that T proves its own consistency.

So, we argue in T, that if e and x are chosen so that they violate this Lemma, then let $M = M_x$; and we will show, in T, that $M \models T$.

Clearly, as in Lemma 1, $M \models I + ReCA$.

It remains to show $M \models (\exists z)(\exists n)(H_n^+(z)$ & $Reas^+(n)$ & $lim(n)$ & corresponding $M_z \models NTWO(n))$.

By Lemma 2 of this chapter, $M \models S + I + ReCA$. Hence $M \models \Sigma_1^1\text{-}AC$. Now for each $p \in Field(e)$ with $M \models NTWO(e_p)$, we have $M \models (k)(k <_e p \longrightarrow (\exists X)NW(e_p,X))$. Hence $M \models (\exists Y)$ $(k)(k <_e p \longrightarrow NW(e_p,(Y)_k))$.

By Lemma 3 of this chapter, $M \models (p)(p \in Field(e) \longrightarrow (\exists q)$ $(q >_e p$ & $(\exists Y)(k)(k <_e q \longrightarrow NW(e_q, (Y)_k)))$, since $M \models I +$ $ReCA + S + W*^+(e)$. So again, by $M \models \Sigma_1^1\text{-}AC$, we obtain a $Z \in M$ with $M \models (p)(p \in Field(e) \longrightarrow (Z)_p$ is (Y,q) with $q >_e p$ & $(k)(<_e q \longrightarrow NW(e_q,(Y)_k)))$.

Fix such a Z, and let r have 1) $r \in Field(e)$, 2) $Z \leq_T (x)_r$. Let $s >_e r$ with $M = NTWO(s)$. Consider $(x)_s$. There is a natural $y \leq_T (x)_s$ such that $H_{e_s}^+(y)$, and $(t)(t <_e s \longrightarrow (y)_t = (x)_t)$. We set $z = y$, $n = e_s$, in our claim about satisfaction in M.

Clearly $\lim(e_s)$ & $\mathrm{Reas}^+(e_s)$. It remains to show corresponding $M_y \models \mathrm{NTWO}(e_s)$.

We have $M \models \mathrm{NTWO}(s)$, and $Z \in M_y$. So every set $\leq_T Z$ is in M_y. Hence $M_y \models \mathrm{NTWO}(e_s)$.

This completes the proof of the self-consistency proving of T, and hence the inconsistency of T. Hence Lemma 4 must be true.

<u>Lemma 5</u>: There is an e with $W^{*+}(e)$ & $\mathrm{Field}(e) = \{n|n$ is even$\}$ such that 1) $(\exists x)H_e^+(x)$, 2) $(y)(H_e^+(y) \longrightarrow (\exists X)(X \leq_T y^{(10)}$ & $NW(e,X)))$.

Proof: We define a total recursive function F on indices of the partial recursive functions. We define G(n) to be the Gödel number of the RLO (field ω) associated with the Π_1^1 sentence "$\sim\mathrm{RLO}(n) \vee (x)(\sim H_n^+(x))$." So for every n, G(n) is the Gödel number of some RLO. Let F(n) be the RLO^+ with domain $\{n|n$ is even$\}$ defined by $p <_{G(n)} q$ iff $2p <_{F(n)} 2q$.

By the recursion theorem, there is an e with $\varphi_e = \varphi_{F(e)}$. Fix such an e. Then e is the Gödel number of an RLO^+ whose field is $\{n|n$ is even$\}$. It is clear that $(X)(NW(e,X) \longrightarrow (\exists y)(H_e^+(y)$ & $y \leq_T X^{(10)}))$ & $(y)(H_e^+(y) \longrightarrow (\exists x)(NW(e,X)$ & $X \leq_T y^{(10)}))$. Hence $W^{*+}(e)$. For, if not, then $\sim W^+(e)$, and $(\exists x)(NW(e,X)$ & $\mathrm{Hyp}(X))$, contradicting the 1st conjunct of the above conjunction. We claim $\sim W(e)$. For, if not, then $(\exists y)(H_e^+(y))$, contradicting the 2nd conjunct of the above conjunction. So by the 1st conjunct, we have $(\exists y)(H_e^+(y))$. So e has all the properties stated in this Lemma.

<u>Theorem 2</u>: There are n with $W^*(n)$ (hence $\text{Field}(n) = \omega$) such that $(x)(\sim H_n^+(x))$.

Proof: Take e as in Lemma 5. Take n to be the natural RLO with 1) the ordering e is an initial segment, 2) $\text{Field}(n) = \omega$, 3) the ordering n corresponds to $e \times \omega$ (i.e., ω copies of e).

Now suppose $H_n^+(x)$. Then $M_x \models \text{NTWO}(n)$ by Lemma 4. But $(\exists y)(\text{NW}(e,y) \ \& \ y \in M_x)$. Hence there is a z which can be found recursively in such a y, with the property that no tail of n is well-founded with respect to z. And $z \in M_x$. But this contradicts $M_x \models \sim\text{NTWO}(n)$.