SUBSYSTEMS OF SET THEORY AND ANALYSIS

by

Harvey Martin Friedman

SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF

PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF

TECHNOLOGY

August, 1967

Signature of Author...

Department of Mathematics
August, 1967

Certified by .............. Thesis Supervisor

Accepted by .............. Chairman, Departmental
Committee on Graduate
Students
Usual set theory is formulated in terms of closure conditions. A typical example is the power set axiom, which asserts closure under the operation of power set.

In Chapter I we consider set theory based on closure conditions applied only to definable sets. We formalize this set theory and call it ZF*. Our principal result of Chapter I is that, provably in first-order arithmetic, ZF is consistent if ZF* is. Our proof of this theorem uses a Skolem hull construction and a syntactic transformation.

In Chapter II, we consider three theories of hyperarithmetic analysis, $\Delta^1_1$-CA, $\Sigma^1_1$-AC, and $\Sigma^1_1$-DC. These are called theories of hyperarithmetic analysis primarily because the hyperarithmetic sets form a minimum $\omega$-model for each of them. We first show that $\Sigma^1_1$-DC is a conservative extension of $\Delta^1_1$-CA for purely $\Pi^1_2$ sentences. The proof is by means of an inner model construction. Careful attention has to be paid to limit the axioms we use to prove relevant sentences about hyperarithmetic sets. We then show that there are theorems of $\Sigma^1_1$-DC which are not theorems of $\Sigma^1_1$-AC. This is done by first finding a suitable sentence $S$ and considering the auxiliary
theories $\Sigma^1_1$-$AC + S$ and $\Sigma^1_1$-$DC + S$. We then obtain our independence result via Gödel's Theorem, by showing that

$\text{Con}(\Sigma^1_1$-$AC + S)$ is provable in $\Sigma^1_1$-$DC + S$. Last, we show that $\Sigma^1_1$-$AC$ is a conservative extension, for purely $\Pi^1_2$ sentences, of $T$, a natural subsystem of predicative analysis. The proof uses an inner model construction on certain auxiliary theories. Thus, a model for each finite subsystem of $\Sigma^1_1$-$AC$ is obtained as an inner model of a model of an extension, by the negation of an instance of induction, of a corresponding finite subsystem of $T$. Thus non-standard models are implicit in the construction.

Chapter III is concerned with hierarchies (based on the jump operator) on recursive linear orderings. Let $X$ be the set of recursive linear orderings which have no hyperarithmetic descending chains. Joseph Harrison showed that there are elements of $X$, which are not well-orderings, on which there are hierarchies. We first show that under certain weak conditions on a recursive linear ordering, that if there is a hierarchy on it, then it must be in $X$. Finally, we establish the existence of a recursive linear ordering which is in $X$, yet on which there are no hierarchies. The proofs of these assertions use certain Lemmas which are proved in the following indirect way: one assumes the Lemma is false, and then forms a theory consisting of the negation of the Lemma together with certain true sentences; then one shows that the resulting theory proves its own consistency.

Thesis Supervisor: Gerald E. Sacks

Title: Professor of Mathematics
# TABLE OF CONTENTS

Title Page ........................................................................ 1  
Abstract ........................................................................ 2  
Biographical Note .............................................................. 6  
Acknowledgments ............................................................... 7  
Bibliography .................................................................... 8  
Introduction .................................................................... 9  

Chapter I

1. General Situation .............................................................. 15  
2. Remarks on Terminology and Notation .............................. 16  
3. Axioms ........................................................................ 16  
4. Outline of Proof of Main Theorem ...................................... 18  
5. Development of Ordinals .................................................. 21  
6. Development of L .......................................................... 25  
7. The System ZF*' ........................................................... 32  
8. The System ZF' ............................................................ 37  
9. The Skolem Argument ...................................................... 38  
10. ZF, Parameterless ZF, and ZF* ......................................... 40
Chapter II

1. Definitions of Systems................................. 48
2. Preliminary Lemmas.................................... 52
3. Conservative Extension Result......................... 59
4. Independence Result.................................... 65
5. Relation between Predicative and Hyperarithmetic Analysis........................................... 68

Chapter III.......................................................... 77
BIOGRAPHICAL NOTE

ACKNOWLEDGMENTS

The author wishes to express thanks to Professors G. Sacks, S. Feferman, and G. Kreisel for their valuable conversations; to Professors Kreisel and Feferman for their valuable correspondence; and to Professor Sacks for his remarkable patience and encouragement.
BIBLIOGRAPHY


Set theory is usually formulated in terms of closure conditions. A typical example is the power set axiom, which asserts closure under the operation of power set. In Chapter I, we consider set theory based on closure conditions applied only to definable sets. We formalize this set theory and call it $ZF^*$, and we give a consistency proof of $ZF$ relative to $ZF^*$.

A direct method presents itself for obtaining this relative consistency result; namely, to use a constructible set construction, and prove within $ZF^*$ the relativized to the constructible sets of each instance of $ZF$. This is, of course, in analogy with the method of proof for the consistency of $ZF + \text{AxC}$ relative to $ZF$. However, an examination of the basic principles needed for such a constructible set construction to go through reveals the need for the least counterexample principle for ordinals to be provable in $ZF^*$. By the least counterexample principle for ordinals, we mean the schema

$$(\exists \alpha)P\alpha \rightarrow (\exists \mu \alpha)P\alpha,$$

where $P$ is any formula. It does not appear that this schema is derivable in $ZF^*$, even if $P$ is restricted to have only one free variable, $\alpha$. Of course, in $ZF$, the schema is derivable by means of a closure condition applied to all sets as follows: assume $(\exists \alpha)P\alpha$, and fix such an $\alpha$. Then form $\{\beta | \beta \in \alpha \& P\beta\}$, and use Foundation to obtain $(\mu \alpha)P\alpha$. This illustrates the basic difference between $ZF$ and $ZF^*$, in that the closure condition, $\exists \{\beta | \beta \in \alpha \& P\beta\}$, is necessarily provable in $ZF^*$ only when $\alpha$ is given a definition.
Such a direct attack seems hopeless. An outline of our proof can be found in Section 4 of Chapter I.

Towards the end of Chapter I, we show how to add elements "on top of" a model of ZF, to obtain nonstandard models of ZF*. By this means, we obtain results concerning independence from ZF*.

In Chapter II, we consider three theories of hyperarithmetic analysis. These are $\Delta^1_1$-CA, $\Sigma^1_1$-AC, and $\Sigma^1_1$-DC. Here CA refers to comprehension axiom; AC, to axiom of choice; DC to dependent choice. The hyperarithmetic sets form a minimum $\omega$-model for each of these theories. We first prove that $\Sigma^1_1$-DC is a conservative extension of $\Delta^1_1$-CA for purely $\Pi^1_2$ sentences. The proof is by means of an inner model construction. We show that given a model of $\Delta^1_1$-CA, if we then take the submodel of all sets hyperarithmetic in a fixed set, this submodel satisfies $\Sigma^1_1$-DC. Careful attention has to be paid to the way in which the notion of relative hyperarithmeticity is formalized. The formulation in terms of hierarchies seems to be the correct one here (not $\Delta^1_1$). The usual proof that the sets hyperarithmetic in a fixed set always form an $\omega$-model of $\Sigma^1_1$-DC is too crude for our purposes. It uses the comparability of all recursive well-orderings, which is a principle too strong to be provable in $\Delta^1_1$-CA. However, if we know, in $\Delta^1_1$-CA, that given orderings have hierarchies (based on the jump operator) on them, we can then conclude, in $\Delta^1_1$-CA, their comparability. Thus, the key point of our proof of the conservative extension result is the judicious use of
hierarchies on orderings.

Our inner model construction can, in the standard way, be transformed into a finitary consistency proof of $\Sigma^1_1$-DC relative to $\Delta^1_1$-CA.

The second result of Chapter II is the independence of $\Sigma^1_1$-DC from $\Sigma^1_1$-AC. It would seem to be the case that a Cohen type argument would be not only useful here, but perhaps necessary. We found that quite the contrary is true. Our proof does not use a Cohen type argument, and Cohen type arguments do not seem to be helpful here, since it seems difficult to find Cohen type models (starting with the standard model, the hyperarithmetic sets, of $\Sigma^1_1$-AC) which do not have the following property: any arithmetical predicate (in $x$) satisfied to have a solution in the new model has a solution hyperarithmetic (in $x$) in the new model. This property can be seen, from our Theorem 1 of Chapter II, to imply that the new model satisfies $\Sigma^1_1$-DC.

Instead, we use Gödel's theorem. We choose a sentence $S$ and consider the auxiliary theories $\Sigma^1_1$-AC + $S$ and $\Sigma^1_1$-DC + $S$. We show that $\Sigma^1_1$-DC + $S$ proves the consistency of $\Sigma^1_1$-AC + $S$. So if $\Sigma^1_1$-DC + $S = \Sigma^1_1$-AC + $S$, then $\Sigma^1_1$-DC + $S$ is inconsistent. But $S$ is chosen to be a true sentence; so $\Sigma^1_1$-DC $\neq \Sigma^1_1$-AC.

Notice that the assumption $\text{Con}(\Sigma^1_1$-DC + $S$) is needed for the independence. We do not know if there is a finitary independence proof (i.e., a finitary proof of consistency of $\Sigma^1_1$-AC + $\neg F$ relative to $\Sigma^1_1$-AC, for some $F$ that is provable in $\Sigma^1_1$-DC).
The key property of $\Sigma^1_1$-DC is that for any $\Pi^1_2$ sentence, $B$, one can construct, in $\Sigma^1_1$-DC + $B$, an $\omega$-model for $B$. The key property of the sentence $S$ is that the theory $\Sigma^1_1$-AC + $S$ is equivalent, in $\Sigma^1_1$-DC, to an extension of induction by a $\Pi^1_2$ sentence. $S$ intuitively says that for all sets $x$, every recursive well-ordering in $x$ has a hierarchy starting from $x$.

More information may be obtained than independence. We see that $\Sigma^1_1$-DC + $S$ proves $\text{Con}(\Sigma^1_1$-AC + $S$); hence $\Sigma^1_1$-DC proves $S \rightarrow \text{Con}(\Sigma^1_1$-AC + $S$), which is a purely $\Sigma^1_2$ sentence. Furthermore, an examination of our proof yields that $\Sigma^1_1$-AC together with only a finite number of instances of no parameter $\Sigma^1_1$-DC is needed to prove $S \rightarrow \text{Con}(\Sigma^1_1$-AC + $S$). Finally, $\Sigma^1_1$-DC + $S$ proves $\exists$ an $\omega$-model for $\Sigma^1_1$-AC + $S$. From all this we can conclude that there is a purely $\Sigma^1_2$ sentence which is provable in $\Sigma^1_1$-AC together with a finite number of instances of no parameter $\Sigma^1_1$-DC, but which is not an $\omega$-consequence of $\Sigma^1_1$-AC. (An instance of no parameter $\Sigma^1_1$-DC is the same as $\Sigma^1_1$-DC, except the hypothesis, $(f)(\exists g)A(f,g)$, must have $A$ arithmetical with no free variables other than $f$ and $g$.)

The last result in Chapter II is concerned with the relation between $\Sigma^1_1$-AC and a certain natural subsystem of predicative analysis, $T$. $T$ represents the first $\epsilon_0$ levels of predicative analysis. We show that $\Sigma^1_1$-AC is a conservative extension of $T$ for purely $\Pi^1_2$ sentences. This result, together with the known characterization of the provable ordinals of $T$, gives the provable ordinals of $\Sigma^1_1$-AC. Furthermore, our proof of conservative extension uses an inner
model construction, which can be transformed into a finitary proof of consistency of $\Sigma^1_1$-AC relative to $T$. Now $T$ is known to have a predicative consistency proof; and so, then, must $\Sigma^1_1$-AC.

The relative consistency result is somewhat surprising, since the minimum $\omega$-model of $\Sigma^1_1$-AC is so much larger than the minimum $\omega$-model of $T$. With this in mind, it is not surprising that nonstandard models (i.e., non-$\omega$-models) must be essential in our proof. In the proof, we obtain a model of each finite subsystem of $\Sigma^1_1$-AC as an inner model of a model of an extension, by the negation of an instance of induction, of a corresponding finite subsystem of $T$.

In Chapter III, we consider which elements of $W^*$ have hierarchies. We generalize $W^*$ to include recursive linear orderings whose field is not necessarily $\omega$. We also generalize hierarchies, so that at successor we merely have a set in which the jump of the set at the predecessor is recursive; at limits, we have a set in which the effective union of the previous sets is recursive. Harrison proved that $\exists n \in W^*$ on which there are hierarchies, in the less general sense $W^*$, $W$ and hierarchies. He left open whether every $n \in W^*$ has a hierarchy. We answer it in the negative for our general notion of hierarchy and the less general notion of $W^*$.

We also show, under weak conditions, that if $n$ has a hierarchy, then $n \in W^*$.

The proofs use certain Lemmas which are proved in the
following highly indirect way: we assume the Lemma is false, and form a theory by adding true sentences to the negation of the Lemma; we then show that the resulting theory proves its own consistency.
1. **General Situation.** Suppose we are given a comprehension axiom \((x_1)\ldots(x_n)(\exists y)Rx_1\ldots x_n y\). We are interested here in forming the derived schema consisting of the axioms

\[
[(\exists x_1)(\exists x_2)\ldots(\exists x_n)(F_1 x_1 \& \ldots \& F_n x_n)] \rightarrow \\
(\exists x_1)(\exists x_2)\ldots(\exists x_n)(F_1 x_1 \& F_2 x_2 \& \ldots \& F_n x_n \& (\exists y)Rx_1\ldots x_n y),
\]

where the \(F_i\) are formulae with only the free variable \(x_i\).

We are purposely vague about the general situation (what is a comprehension axiom?), since we have only looked at this derived schema when the original axiom is drawn from a natural set of axioms, such as ZF, or analysis, or other naturally occurring systems.

More specifically, we will look at the schema of schema formed by taking the union of all the schema defined above corresponding to each of the comprehension axioms of ZF. We will not perturb the other (non-comprehension) axioms of ZF, except in minor ways. We call this derived theory ZF*. (We inessentially modify the Replacement schema in ZF for convenience, so that each instance is appropriately placed in the form \((x_1)\ldots(x_n)(\exists y)Rxy\), so that we may pass to the derived schema in the manner above.) The axioms of ZF and ZF* are spelled out in detail, in an elegant form, in the next section.

We are interested in the relation between ZF and ZF*.
as axiomatic theories. Our main result is that
\[ \vdash_{\text{ENT}} \text{ConZF}^* \rightarrow \text{ConZF}. \]
It is obvious that \( \text{ZF}^* \) is a sub-system of \( \text{ZF} \).

2. Remarks on Terminology and Notation. The only (standard) symbols that can occur in a formula of \( \text{ZF} \) (or \( \text{ZF}^* \)) are the 2 2-ary relation symbols ",", ",\epsilon\"; the 2 quantifiers \((\exists x)\) and \((x)\); the propositional connectives; and variables \(x_i, y_i, z_i, u_i, v_i\), etc. Everything else is nonstandard; when nonstandard symbols occur in a formula, they are meant to be expanded out in such a way that the mere occurrence of a non-standard symbol implies existence of the corresponding set. For example, \( x = \cup y \) is an abbreviation for
\[ (\exists z)[(w)(w \epsilon z \equiv (\exists u)(w \epsilon u \& u \epsilon z)) \& x = z]. \]
Also, say, \( \emptyset \epsilon x \) would be \( (\exists y)[(z)(z \notin y) \& y \epsilon x]. \)

3. Axioms. For \( \text{ZF} \), we have

0. Axioms for predicate calculus with equality.
1. Extensionality. \( (x_0 = x_1) \equiv (x_2)(x_2 \epsilon x_0 \equiv x_2 \epsilon x_1). \)
2. Infinity. \( (\exists x_0)(\emptyset \epsilon x_0 \& (x_1)(x_1 \epsilon x_0 \rightarrow x_1 \cup \{x_1\} \epsilon x_0)). \)
3. Power set. \( (x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 = (x_3)(x_3 \epsilon x_2 \rightarrow x_3 \epsilon x_0)). \)
4. Sum set. \( (x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 = (\exists x_3)(x_3 \epsilon x_0 \& x_2 \epsilon x_3)). \)
5. Replacement schema. Let \( Axy \) be a formula with the free variables \( x \) and \( y \) and possibly more free variables \( x_1, \ldots, x_n \). Then
is an instance. (The domain is y and the axiom asserts the range of the partial function, $A'y_2 = Ay_2 \land (z) (Ay_2 z \rightarrow z = y_2)$, on the domain y, exists.)

6. Foundation. $(x_0)(x_0 \neq \emptyset \rightarrow (\exists x_1)(x_1 \epsilon x_0 \land (x_2)(x_2 \epsilon x_1 \rightarrow x_2 \epsilon x_0)))$.

It is clear that by the usual process of making partial functions into total functions axiom schema 5 is the same as the usual formulation in the present context.

Now let $Ax$ be a formula of 1 free variable. Then the formula $(\exists y)(x)(x \epsilon y = Ax)$ is abbreviated as $C_A$.

For ZF* we have

0. Same as ZF.
1. Same as ZF.
2. Infinity. $(\exists x_0)(\emptyset \epsilon x_0 \land (x_1)(x_1 \epsilon x_0 \rightarrow x_1 \cup \{x_1\} \epsilon x_0) \land (y)(y \epsilon x_0 = Fin(y))$. Fin(y) will be defined later. Intuitively it means $y$ is a finite ordinal.
3. Power set. The instances are $C_A \rightarrow (\exists x_0)(x_1) (x_1 \epsilon x_0 = (x_2)(x_2 \epsilon x_1 \rightarrow Ax_2))$, $A$ with 1 free variable.
4. Sum set. The instances are $C_A \rightarrow (\exists x_0)(x_1)(x_1 \epsilon x_0 = (\exists x_2)(x_1 \epsilon x_2 \land Ax_2))$, $A$ with 1 free variable.
5. Replacement schema. The instances are $C_A \rightarrow (\exists x_0)(x_1) (x_1 \epsilon x_0 = (\exists x_2)(Ax_2 \land Bx_2 x_1 \land (x_3)(Bx_2 x_3 \rightarrow x_3 = x_1)))$, for $A$ with 1 free variable, $B$ with exactly 2 free variables.
6. Same as ZF.
Remarks: Our formulation of Power set in ZF* is seen to be equivalent to the derived schema of Power set in ZF (as given in General Situation) by noticing 1) that if $(\exists y)Fy$, then $C_A$, where $A$ is $(\exists y)(Fy \& x\in y)$ and 2) that if $C_A$, then $(\exists x)(y)(y\in x \equiv Ay)$. These latter are obtained by axiom 1 of ZF*, Extensionality. The same remark applies to Sum Set.

Essentially the same idea yields that the union of the derived schema of the instances of Replacement in ZF is equivalent, in the present context, to the schema

$$(C_{A_1} \& \ldots C_{A_n} \& C_A) \longrightarrow (\exists x_0)(x_1 \in x_0 \equiv (\exists x_2)(Ax_2 \& B(x_2,x_1,x_0,A_{1x},A_{2x},\ldots,A_{nx},x_1\in x_0,A_{1x},\ldots,A_{nx})))$$

where $B$ has $n+2$ free variables, $A$ and the $A_i$ have 1 free variable. We want to show this schema is contained (in the present context) in Replacement in ZF*. But the above is easily seen to follow from that instance of 5 of ZF*, setting $A$ as $A$, $Bx$ as $(C_{A_1} \& \ldots \& C_{A_n}) \longrightarrow B(x_2,x_1,x_0,A_{1x},A_{2x},\ldots,A_{nx})$. (That the above schema contains Replacement in ZF* is obvious.)

NOTE: "In the present context" means "using the other axioms of ZF*.

4. Outline of Proof of Main Theorem. The main theorem is

$\vdash \text{ENT} \iff \text{ConZF} \longrightarrow \text{ConZF}$. The first step in proving this is developing in ZF* an adequate definition of ordinals, which
turns out to be a much more delicate matter than for ZF, due to the lack of certain key instances of Replacement in ZF*.

By an adequate definition of ordinals in ZF*, we mean a definition of ordinals such that provably in ZF*, members of ordinals are ordinals, and (for a natural definition in ZF* of $\omega$) $\omega$ is an ordinal, and ordinals are comparable by $\varepsilon$, and the $\varepsilon$-relation on the ordinals is transitive, and antisymmetric, and antireflexive, and every ordinal has a (natural) successor, except possibly the greatest ordinal. (It even turns out that in ZF* we can prove there is no greatest ordinal for our definition of ordinal given later.) The definition is made and Lemma 1 establishes the above properties for it in the next section; we even obtain more: that, provably in ZF*, the new definition of ordinal coincides with the usual definition given in ZF, on definable sets. (This is made precise in Lemma 1, f).

Next we develop an adequate definition of L within ZF*. Among the properties of the predicate $x \in L$ needed, we must have, provable in ZF*, every member of an $x \in L$ is $\in L$, $\omega \in L$, the new definition of $x \in L$ coincides with the usual definition of constructibility for definable $x$, and a definable well-ordering of $L$.

With this machinery, an apparently straightforward "proof" of our main result comes to mind. Namely, just to prove the relativized to $L$ of each instance of ZF in ZF* by taking least counterexamples of various things in ZF*, as Gödel established the relativization of the axioms of ZF to
L within ZF. But a moment's reflection will reveal that one can hardly expect that ZF* will prove any general least counterexample principles; i.e., one may well be able to prove in ZF* that \((\exists x)(x \in L \& Px)\), yet not be able to prove \((\exists \mu x)(x \in L \& Px)\) in ZF*, where \(\mu\) is defined in terms of the definable well-ordering of L. In fact, we do not even see how to prove each instance of the relativized of ZF* to L, within ZF*!

In order to get our main result, we form an auxiliary system \(ZF^{*'} \subseteq ZF^{*}\), whose definition depends on a certain crucial transformation on sentences, \(T\). This subsystem has the property that each instance of \(ZF^{*'}\) semi-relativized (semi-relativization is a certain modification of relativization) to \(L\), is provable in ZF*. We form another auxiliary theory \(ZF'\) which is related to \(ZF^{*'}\) about as \(ZF + V = L\) is to \(ZF^{*}\). It turns out that \(ZF' \supseteq ZF\). It also turns out that in the theory obtained by semi-relativizing each comprehension axiom of \(ZF^{*'}\) and retaining the other axioms, one can give a Skolem hull argument for each finite subsystem of \(ZF'\) that proves the existence of a (suitably definable) model of this finite subsystem, and hence its consistency.

Putting all this together we get a finitary proof that (n)

\[ \vdash_{ZF^*} \text{Con}(ZF_n). \quad \text{So} \quad \vdash_{\text{ENT}} \text{ConZF}^* \rightarrow \text{ConZF}. \]
5. **Development of Ordinals.** We define \( \text{Ord}_x = \text{Trans}(x) \& \varepsilon-\text{Conn}(x) \& (x \text{ is semi-closed under succession}) \& \text{there are no 3-chains in } x, \ i.e., \ \text{Ord}(x) = (y)(z)((y \varepsilon x \& z \varepsilon y) \rightarrow z \varepsilon x) \& (y)(z)((y \varepsilon x \& z \varepsilon y) \rightarrow y \varepsilon z \vee z \varepsilon y \vee y = z)) \& (y)(y \varepsilon x \rightarrow (\exists z) [(w)(w \varepsilon z \equiv (w \varepsilon y \& w = y)) \& (z \varepsilon x \vee z = x)]) \& (y)(z)(w)(\neg[y \varepsilon z \& z \varepsilon w \& (w \varepsilon y \& w = y)]). \) We define \( \text{Ord}'(x) = \text{Ord}_x \& (y)((\text{Ord}_y \& y \subseteq x) \rightarrow (y \varepsilon x \vee y = x)). \) \( \text{Ord}''(x) = \text{Ord}'(x) \& (y)(\text{Ord}'y \rightarrow (x \subseteq y \vee y \subseteq x)). \) \( \text{Ord}' \) will be our notion of ordinal in \( \text{ZF}^*. \)

It is obvious that \( (x)(y)(\text{Ord}'x \& \text{Ord}'y \rightarrow (x \varepsilon y \vee y \varepsilon x \vee y = x)). \)

For \( \text{Ord}(x) \), we define \( y \text{ is successor of } x \) if and only if \( y = x \cup \{x\}. \)

**Lemma 1:** The following are Theorems of \( \text{ZF}^* \), where \( \text{Ax} \) has 1 free variable:

a) If \( \text{Ord}_x \& y \varepsilon x \), then \( \text{Ord}_y \).

b) If \( \text{Ord}'x \) and \( y \varepsilon x \) then \( \text{Ord}'y \).

c) For \( x \) with \( \text{Ord}(x) \), and \( x \) not a successor, \( \cup x = x. \)

For \( \text{Ord}(x) \), \( x \) a successor, \( x = y \cup \{y\} \), we have \( \cup x = y. \)

d) If \( \text{Ord}''(x) \) and \( y \varepsilon x \) then \( \text{Ord}''(y). \)

e) If \( y = \{x|\text{Ax}\} \), \( \text{Ord}'(y) \), then \( \text{Ord}''(y). \)

f) If \( y = \{x|\text{Ax}\} \), \( \text{Ord}(y) \), then \( \text{Ord}''(y). \)

g) \( \text{Ord}''(w). \) (Explained below.)

h) If \( y = \{x|\text{Ax}\} \), and \( \text{Ax} \rightarrow \text{Ord}''(x) \), then \( \text{Ord}''(\cup y). \)

**Proof:** a) Claim \( \text{Trans}(y) \). Let \( z \varepsilon y, w \varepsilon z \). Then by \( \text{Trans}(x) \), we have \( w \varepsilon x \) and \( y \varepsilon x \). By \( \varepsilon-\text{Conn}(x) \), \( w \varepsilon y \vee y \varepsilon w \vee y = w \). We can't have \( y \varepsilon w \vee y = w \) because we would have a 3-chain. So \( w \varepsilon y \). To see \( \varepsilon-\text{Conn}(y) \), let \( z \varepsilon y, w \varepsilon y \). Then \( z \varepsilon x \) and \( w \varepsilon x \), and so \( z \varepsilon w \vee w \varepsilon z \vee w = z \) by \( \varepsilon-\text{Conn}(x) \).
Towards showing \( y \) semi-closed, let \( z \in y \). Hence \( z \in x \), and \( z \cup \{z\} \in x \) or \( z \cup \{z\} = x \). By \( \varepsilon\)-Conn\((x)\), we have \( z \cup \{z\} \in y \) or \( y \in z \cup \{z\} \). But if \( y \in z \cup \{z\} \), then \( y \in z \cup z = z \). The first yields the 3-chain \( z \in y, y \in z, z = x \); the second yields \( y \in y \), which yields the 3-chain \( y \in y, y \in y, y \in y \). So \( y \notin z \cup \{z\} \).

Now suppose \( a \in b, b \in c, (c \in a \lor c = a) \), \( a, b, c \in x \). Then \( a, b, c \in x \), and we have a 3-chain in \( x \).

b) By a), we have \( \text{Ord}(y) \). Towards showing \( \text{Ord}'(y) \), let \( z \subseteq y, \text{Ord}(z) \). By Trans\((x)\), \( z \subseteq x \). Hence \( z \in x \lor z = x \). But \( z \neq x \), for if \( z = x \), then \( x \subseteq y \), and hence \( y \in y \). So \( z \in x \). By \( \varepsilon\)-Conn\((x)\), \( z \in y \lor z = y \lor y \in z \). But \( y \notin z \), since if \( y \in z \), then \( y \in y \).

c) Let \( \text{Ord}(x) \), \( x \) with \( (y)(y \cup \{y\} \neq x) \). By Trans\((x)\), every member of a member of \( x \) is a member of \( x \). So \( \cup x \subseteq x \), if \( \cup x \) exists. Now let \( y \in x \). By semi-closure of \( x \), \( y \cup \{y\} \in x \). But \( y \in y \cup \{y\} \), and so \( x = \cup x \), since also every member of \( x \) is a member of a member of \( x \). If \( x = z \cup \{z\} \), then again \( \cup x \subseteq x \), if \( \cup x \) exists. But \( z \notin \cup x \), since \( x \) has no 3-chains. So \( \cup x \subseteq z \), if it exists. But every member of \( z \) is a member of \( x \), since if \( w \in z \), then \( w \) is a member of a member of \( x \). So \( \cup x \) exists and is \( z \).

d) By c), \( \text{Ord}'(y) \). Let \( \text{Ord}'(z) \). Then \( x \subseteq z \lor z \subseteq x \). If \( x \subseteq z \), then \( y \subseteq z \). If \( z \subseteq x \), then \( z \in x \lor z = x \). If \( z = x \), then \( y \subseteq x \). If \( z \in x \), then \( y \in z \lor z \in y \lor z = y \). Hence \( x \subseteq y \lor y \subseteq z \).
e) Let $\text{Ord}'(y), y = \{x | Ax\}$. Then either

1) All elements of $y$ are $\text{Ord}''$. Then let $z$ be any $\text{Ord}'$. For every $x \notin y$ we have $x \subseteq z \lor z \subseteq x$. Either all members of $y$ are $\subseteq z$ or some member of $y \supseteq z$. Suppose the first holds. Then if $y$ is not a successor, by c) we have $\cup y = y$, and so $y \subseteq z$. If $y = u \uparrow \{u\}$, then $u \subseteq z$. Since $\text{Ord}'(z)$, we have $u \in z \lor u = z$. If $u = z$, then $z \subseteq x$. If $u \in z$, then $y \subseteq z$.

Suppose the second holds, i.e., some $x \in y$ contains $z$. Then clearly $z \subseteq y$. So $y$ is $\text{ord}''$.

2) Some element of $y$ is not $\text{Ord}''$. We have just proved that any $\text{Ord}'$ such that every member is $\text{Ord}''$ is $\text{Ord}''$. The (unique) $\epsilon$-least element of $y$ which is not $\text{Ord}''$ is definable, and has every member an $\text{Ord}''$. (There is a least by suitable use of Replacement schema of ZF*, and Foundation.) Just apply 1) to obtain a contradiction.

f') We merely have to show all definable Ords are $\text{Ord}'$. As in e) we go down to a definable Ordinal all of whose members are $\text{Ord}''$. Let $y$ be such an ordinal, $y = \{x | Bx\}$. Let $z \subseteq y$, $\text{Ord}z$. We wish to show $z \in y \lor z = y$. Suppose $z \notin y \& y \notin z$. Then $\exists w \in y$ with $w \notin z$. But since $\text{Ord}'w$, we have $w \subseteq z \lor z \subseteq w$.

If $z \subseteq w$ then $z = w \lor z \in w$ and so $z \in y$. So $w \subseteq z$, $w \notin z$. Now either $w \cup \{w\} \in y \lor w \cup \{w\} = y$. If $w \cup \{w\} = y$ then since $z \subseteq y$, $z \in w$, and so $z \in y$. So $w \cup \{w\} \in y$. Now then $\text{Ord}'(w \cup \{w\})$. Hence $w \cup \{w\} \subseteq z \lor z \subseteq w \cup \{w\}$. The first is out, so $z \subseteq w \cup \{w\}$, and since $\text{Ord}'(w \cup \{w\})$ we have $z = w \cup \{w\} \lor z \in w \cup \{w\}$, either one implying $z \in y$. 
g) We now explain Axiom 3. \( \text{Fin}(x) = \text{Ordx} \land (\exists y)(x = y \cup \{y\}) \) and 
\((z)(z \in x \rightarrow (\exists w)(z = w \cup \{w\}))\). \(x\) is a successor \( \text{Ord} = \text{Ordx} \land (\exists y)(y = x \cup \{x\}) \). We let \( w \) be the \( x_0 \) in Ax. 2 of \( \text{ZF}^* \).

Now clearly \( w \) is definable, and so we merely have to show \( \text{Ord}(w) \).

1) \( \text{Trans}(w) \). Let \( n \in w \). Let \( x \in n \). Then \( x \) is a successor \( \text{Ord} \) or \( \emptyset \). Since \( n \) is transitive, \( x \) is a successor \( \text{Ord} \) or \( \emptyset \) and every member of \( x \) is a successor \( \text{Ord} \) or \( \emptyset \), and so \( x \) is a finite \( \text{Ord} \) (i.e., \( \text{Fin}(x) \)), and so \( x \in w \).

2) \( \varepsilon\text{-Conn}(w) \). Suppose \( \exists n \in w \) such that for some \( m \in w \), \( n \not\in m \land m \in n \land n \not\in m \). Take any \( \varepsilon\)-least such \( n \) and call it \( k \). There is an \( \varepsilon\)-least by Replacement in \( \text{ZF}^* \) and Foundation.

Now \( k \neq \emptyset \), since \( \emptyset \in m \lor \emptyset = m \), for any \( m \in w \), by Foundation and \( \text{Trans}(m) \). Hence \( k \) is a successor \( \text{Ord} \), and \( k = 1 \cup \{1\} \), and by \( \text{Trans}(w) \), \( 1 \in w \). We must have, for some \( m \in w \), \( k \in m \land m \not\in k \land k \not\in m \land (1 \in m \lor 1 \in m \land 1 = m) \). But if \( m = 1 \) then \( m \not\in k \). If \( m \neq 1 \), then \( m \not\in k \). If \( 1 \in m \) then \( 1 \cup \{1\} = m \lor 1 \cup \{1\} \in m \), i.e., \( k = m \lor k \in m \). Contradiction, by the lack of 3-chains in \( \text{Ords} \), and the comparability of \( 1 \) and \( m \).

3) Semi-closure under succession. This is insured directly from Axiom 2 of \( \text{ZF}^* \).

4) No 3-chains. Suppose \( n \in m \land m \in r \land r \notin n \), where \( n,m,r \in w \). By \( \text{Trans}(r) \), we have \( n \not\in r \). By \( \text{Trans}(r) \) again, we have \( r \not\in r \), if \( r \in n \). If \( r = n \), we also have \( r \not\in r \). These contradict \( \text{Ord}(r) \).
h) We have only to show \( \text{Ord}(\cup y) \).

1) Trans(\(\cup y\)). If \( z \in \cup y \), then \( z \in w \) for some \( \text{Ord}''(w) \), \( w \in y \); hence any \( u \in z \) has \( u \in w, w \in y \). So \( u \in \cup y \).

2) \( \varepsilon \)-Conn(\(\cup y\)). If \( z, w \in \cup y \), then \( z \in w', w \in w', w \in y \) both \( \text{Ord}'' \). So \( z \subseteq w' \) or \( w \subseteq z' \). Without loss of generality, assume \( z' \subseteq w' \). Then \( z \in w' \), \( w \in w' \). Hence \( z \in w \) and \( w \in z \).

3) Semi-closure. Let \( z \in \cup y \). Then \( z \in w \), \( \text{Ord}''(w) \). So
\[
\{ z \} \in \cup \{ z \} \in w \in \cup y.
\]
In the first case, \( z \in \cup \{ z \} \in \cup y \).
In the second suppose

A) some \( u \in y \) has \( w \subseteq u \), but \( w \notin u \). Hence \( w \in u \), and
so \( z \in \{ z \} = w \in \cup y \).

B) Every \( u \in y \) has either \( w \in u \) or \( u \subseteq w \). Then clearly
\( \cup y = w = z \cup \{ z \} \).

A) and B) are exhaustive.

4) No 3-chains. Suppose \( a \in b, b \in c, (c \in a \lor c = a) \), where
\( a, b, c \in \cup y \). Then \( a, b, c \) are \( \text{Ord}'' \), and \( a \in c \). If \( c \in a \), then
\( c \in c \). If \( c = a \), \( a \in c \), then \( c \in c \). Contradicts \( \text{Ord}(c) \).

6. Development of \( L \). We wish to define a class of sets, \( L \),
which has a definable well-ordering, and provably so in \( \text{ZF}^* \).
\( L \), of course, will not be an object. We are interested in
the predicate \( x \in L \).

We let \( n, m, r, p, q \) be special variables for elements of \( w \).
We let \( \alpha, \beta, \gamma, \ldots \) be special variables for sets \( x \) with
\( \text{Ord}''x \). We write \( \alpha + 1 \) for \( \alpha \cup \{ \alpha \} \). We let \( \lambda \) be a special
variable for limit (non-successor and non-null) \( \text{Ord}'' \)'s.
We say \( x = M(\alpha) = (\exists f)(\text{Dom} f = \alpha + 1 \land f(\emptyset) = \emptyset \land (\lambda)(\lambda \epsilon \alpha + 1 \rightarrow \bigcup f(x) = f(\lambda)) \land (\beta)(\beta \epsilon \alpha + 1 \rightarrow f(\beta + 1) = \text{Fodo}(f(\beta)) \land x = f(\alpha)), \) where \( \text{Fodo}(y) = \{ x \mid (\exists x_0)(\exists n)(x = [z \mid z \in y \land<y, e_y> \models n(z)[x_0]] \land \text{FinSeq}(x_0, y)) \}, \) where \( <y, e_y> \models n(z)[x_0] \) means the structure \( <y, e_y> \) satisfies the formula with Gödel number \( n \) at the sequence of elements of the domain \( y, (z, x_0), \) which is the sequence starting with \( z, \) followed by the sequence \( x_0. \)

Remark on formalization: We formalize the satisfaction relation in \( \text{ZF}^* \) the same way we do in \( \text{ZF}. \) Also see 2. Remarks on Terminology and Notation.

Lemma 2: For each \( Ax \) with 1 free variable, and for each \( Cz_1 \ldots y_n, \) and \( Bz_1 \ldots w_z \) with only free variables shown, the following is provable in \( \text{ZF}^* \): If \( y = \{ x \mid Ax \} \) and if \( Bz_1 \ldots w_z \) is a well-ordering of \( y \lor \{ y \}, \) and \( y_1, \ldots, y_n \in y, \) then \( \{ z \mid z \in y \land<y, e_y> \models \bar{n}(z)[y_1, \ldots, y_n]\} = \{ z \mid z \in y \land Cz_1 \ldots y_n \}, \) where \( n \) is the Gödel number of \( C. \) Thus \( \text{Fodo}(x) \) means the set of all sets first-order definable over \( x, \) for definable sets \( x. \)

Proof: Suppose \( Cz_1 \) is \((\exists w)(w \in z \land w \notin y_1). \) We note that both \( \{ z \mid z \in y \land Cz_1 \} \) and \( \{ z \mid z \in y \land<y, e_y> \models \bar{n}(z)[y_1]\} \) exist by Replacement on the definable \( y. \) We want to show \( Cz_1 \models <y, e_y> \models \bar{n}(z, y_1), \) for \( z \in y. \) The proof in the case of \( \text{ZF} \) is routine. What complicates it in the case of \( \text{ZF}^* \) is that certain sets definable in terms of the members of \( y \) may not provably exist in \( \text{ZF}^* \), and also that the theory of Gödel numbering may not be formalizable in \( \text{ZF}^* \). The latter is not
the case, since \( \omega \) is definable, and hence by Replacement and Foundation, induction on \( \omega \) provably holds in \( \text{ZF}^* \), and also we may define + and \( x \), and prove the relevant properties. What comprehension axioms are needed to establish our equivalence? Apparently, what's involved is just that the theory of finitely hereditary sequences of elements of \( y \) and natural numbers provably in \( \text{ZF}^* \) have the intended interpretation.

For instance, we must verify provability in \( \text{ZF}^* \), for sentences like "for every \( y_1, y_2, y_3 \in y \), \( n, m \in \omega \), there exists the sequence \( \langle \{y_1, n\}, \{\{n, m\}\}, \{y_2, y_3, \{n\}\} \rangle \)." Such sentences can clearly be proved by suitable instances of Replacement in \( \text{ZF}^* \) for definable sets \( y_1, y_2, y_3 \), and \( n, m \). One assumes in \( \text{ZF}^* \) that such a sentence is false, and goes to definable counterexamples \( y_1, y_2, y_3 \), \( n, m \) via the definable well-ordering of \( y \cup \{y\} \), Bzw.

Lemma 3: Each instance of the following is provable in \( \text{ZF}^* \):

a) \[ \alpha = \{x | Ax\} \land (\beta = \alpha \lor \beta \in \alpha) \land (\exists x)(x = M(\beta)) \rightarrow \exists f \text{ satisfying the conditions given in the definition of } x = M(\beta). \]

b) \[ \alpha = \{x | Ax\} \rightarrow (\exists ! x_0)(x_0 = M(\alpha)). \text{ Also } \beta \in \alpha = \{x | Ax\} \rightarrow (\exists ! x_0)(x_0 = M(\beta)). \]

Proof: Assume \( \alpha = \{x | Ax\} \land (\exists x)(x = M(\alpha)) \). Suppose we have 2 functions \( f, g \) satisfying conclusion, and \( f \neq g \). We take, using Replacement and Foundation in \( \text{ZF}^* \) in the usual way, \( \beta \) to be the \( \varepsilon \)-least element of \( \alpha + 1 \) with the property that \( \exists ! f, g \) satisfying definition of \( x = M(\alpha) \), with \( f(\beta) \neq g(\beta) \). Then \( \beta \) is definable. Suppose \( \text{Lim}(\beta) \). Then use Replacement on \( \beta \) to get the \( \{y | (\exists \gamma)(y \in \beta \land \forall f \)
satisfying definition of \( x = M(a) \), \( f(\gamma) = y \}). We can apply sum set in \( ZF^* \) to get a union, \( U \). In any \( f \) satisfying the definition of \( x = M(a) \), clearly \( f(\beta) = U \). But this is contrary to hypothesis, \( f(\beta) \neq g(\beta) \).

Now suppose \( \beta = \delta + 1 \). By hypothesis, \( f(\delta) \) is fixed when \( f \) varies over the functions satisfying the definition of \( x = M(a) \). Clearly \( f(\delta + 1) = \text{Fodo}(f(\delta)) \) for any \( f \) satisfying the definition of \( x = M(a) \), and so \( f(\delta + 1) \) is also independent of \( f \), again contradicting the definition of \( \beta \).

Clearly \( \beta \neq \emptyset \).

So no such \( \beta \) exists.

Now suppose for some \( \beta \in \alpha \), part a) false. Take least such \( \beta \), and apply above, since least such \( \beta \) is definable. This concludes part a).

b) Now suppose \( (\exists x_0)(x_0 = M(\beta)) \) for all \( \beta \in \alpha \), but not for \( \beta = \alpha \). We may assume this without loss of generality, by taking least counterexamples. We conclude the proof of Lemma 3 by obtaining a contradiction. Suppose \( \text{Lim}(\alpha) \). Using Replacement in \( ZF^* \) on \( \alpha \) we can get the set of all \( M(\beta) \)'s, \( \beta \in \alpha \). (We write \( M(\beta) \) for that \( x_0 \) with \( x_0 = M(\beta) \) if it is unique.) Thus we may take the union by sum set in \( ZF^* \) and call this \( U \). Now it is not hard to see, under our hypothesis that using replacement there is an \( f \) consisting of only \( < \beta, M(\beta)> \)'s, \( \beta \in \alpha \), and \( <\alpha, U> \), and that this is the required \( f \) in the definition of \( U = M(\alpha) \). So \( (\exists x_0)(x_0 = M(\alpha)) \). Uniqueness comes from a). Suppose \( \alpha = \gamma + 1 \). It is easy to see that \( \gamma \) is definable and by our hypotheses, \( f(\gamma) \) is definable. It is
obvious that elements of Fodo(f(γ)) are elements of P(f(γ)) and that P(f(γ)) exists by power set in ZF*. Furthermore, Fodo(f(γ)) exists since it can be gotten by replacement on P(f(γ)). Proceed as above to get an appropriate f to give Fodo(f(γ)) = M(α). Uniqueness follows from a). The case α = ∅ is trivial.

We want to insure in ZF* there being a definable well-ordering of L (among other things). This insurance is easily obtained by a natural definition of L in ZF, but not in ZF*. We have no choice but to complicate the definition of ∈L by adding on conditions.

We define x ∈L, approximately as (∃α)(∃y)(y = M(α) & x ∈ y).

But this is not good enough for our purposes. We define 5 extra conditions on this α and y:

1) (β)(z)([z = M(β) & x ∈ z]) → (β ∈ α v β = α), and (z)(z = M(α) → z = y). Whenever x and α have such a y, we say O(x) = α.

2) For all β ∈ α, there is a unique corresponding y = M(β), and if β, γ ≤ α, then M(β) ⊆ M(γ).

3) The M(β)'s, β ∈ α, and M(α) are transitive sets.

4) For every z ∈ M(α), we have (∃β)(0(z) = β) and O(z) ≤ α. Also if z, w ∈ M(α), then [(z ∈ M(0(w)) → O(z) < O(w)].

5) Now, there is a usual definable mapping F in full set theory (identity this with the 2-ary relation F(x) = y) mapping the constructible sets 1-1 into ordinals. Of course, ZF* may well not be able to prove (x)(∃y)(F(x) = y). Condition 5) will be that (the 2-ary relation) F is a 1-1
function when restricted to domain \( M(\alpha) \), and

\((x)(x \in M(\alpha) \rightarrow \text{Ord}''(F(x))). \) (This is the \( F \) which, in full set theory, assigns (Gödel numbers in the form of ordinals) to each constructible set a sequence of ordinals, the first being the rank of the set in the constructible hierarchy, \( \alpha + 1 \); the rest of the sequence codes in, via \( F \) on the sets in \( M(\alpha) \), how the constructible set in question is first-order defined over \( M(\alpha) \).)

We define \( x \in L = (\exists \alpha)(\exists y)(y = M(\alpha) \land x \in y \land \alpha \text{ and } y \text{ satisfy conditions } 1) - 5 \) above).

We define \( x < y \equiv x \in L \land y \in L \land F(x) \in F(y), \) for \( F \) as in 5).

**Lemma 4**: The following are provable in \( ZF^* \): if \( x \in L \) then \( (y)(y \in M(O(x)) \rightarrow y \in L) \). Also \( (y)(y \notin x \rightarrow y \in L), \) if \( x \in L \).

**Proof**: Let \( y \in M(O(x)) \). By 4) in the definition of \( x \in L \), we have \( y, M(O(y)), O(y) \) satisfy 1), 2), and 3) in the definition of \( y \in L \), since \( O(y) \leq O(x) \). Towards verifying 4) in the definition of \( y \in L \), let \( z \in M(O(y)) \). Then \( z \in M(O(x)) \). Then \( (\exists \alpha)(O(z) = \alpha). O(z) \leq O(x), \) for suppose not. Then we get a contradiction via condition 2) in the definition of \( x \in L \). The rest of condition 4) follows for \( y \) because of condition 4) being satisfied for \( y \), and that \( M(O(y)) \subseteq M(O(x)) \).

\( y \) satisfies 5) (i.e., \( y \), together with \( O(y), M(O(y)) \) since \( x \) does, and \( M(O(y)) \subseteq M(O(x)) \).)

To show \( (y)(y \notin x \rightarrow y \in L) \), notice by \( \text{Trans}(M(O(x)) \) we have (assuming \( y \in x \)) that \( y \in M(O(x)) \), and so by the first part of Lemma 4, \( y \in L \).
Lemma 5: For each $Ax$, 1 free variable, $\beta = \{x | Ax\} \rightarrow (x)(x \in M(\beta) \rightarrow x \in L)$, is provable in $ZF^*$. (Note that $M(\beta)$ exists unambiguously by Lemma 3.) Also $\omega \in L$.

Proof of Lemma 5: Form $\{\alpha | \alpha \in \beta & M(\alpha) \text{ does not satisfy conclusion}\}$. Take $\varepsilon$-least member, and call it $\gamma$. Case 1. $\gamma$ is a limit. Now $\gamma$ is definable in $ZF^*$. Let $x \in M(\gamma)$. Then $x \in M(\alpha)$, for some $\alpha \in \gamma$, and hence by the definition of $\gamma$, $x \in L$. Case 2. $\gamma = \delta + 1$. Let $x \in M(\delta + 1)$. There is a definable well ordering on $M(\delta)$, $<$, and we may use this to definably well-order, in $ZF^*$, the finite sequences of elements of $M(\delta)$ in the natural way, proving in $ZF^*$ that it is a well-ordering. With this well-ordering of $M(\delta + 1)$, we take a least, in $M(\delta + 1)$, $x$ such that there is no $\alpha, M(\alpha)$ satisfying condition 1, assuming there is an $x$. This least $x$ is definable in $ZF^*$, and so we consequently can form $\{\alpha | \alpha \in \delta + 1 & x \in M(\alpha)\}$ and take the $\varepsilon$-least member, thereby obtaining a contradiction.

So every $x \in M(\delta + 1)$ possesses a (unique) $0(x)$.

Conditions 2)-4) are treated similarly, taking definable counterexamples and using definable well-orderings. The proof of 5), after taking least counterexamples, is much like our indication of construction of a definable well-ordering of $M(\delta + 1)$ on the basis of one for $M(\delta)$, above.

To show $\omega \in L$, it suffices to prove $\omega \in M(\omega + 1)$. The proof is like the proof of this fact in $ZF$.

Lemma 6: $ZF^*(x)(x = \omega = (x = \omega)'\prime)$, where $A'$ is $A$ relativized to the predicate $\varepsilon L$. 
Proof: Left to the reader.

7. **The System ZF*'.** We define a transformation mapping formula in prenex form in the standard notation (described in 2. Remarks on Terminology and Notation) into formulae which contain the '≤' symbol. If B is in prenex form define B- to be the usual prenex form for ∼B. Take T to be the identity on formulae with no quantifiers, and take T((∃x_i)Bx_i) to be (∃x_i)(T(Bx_i) & (x_j)(x_j < x_i → T(B-x_j))). T((x_i)Bx_i) is (x_i)(T(Bx_i) v (∃x_j)(x_j < x_i & T(B-x_j))). It is easily proved by induction that T(B-) and ∼T(B) are equivalent for any prenex B. Recall that the interpretation of x_j < x_i is x_j ∈ L & x_i ∈ L & F(x_j) ∈ F(x_i).

We form ZF*' as follows: Extensionality & Foundation & Infinity & Power Set & Sum Set & Modified Replacement. The latter is the only difference between ZF*' and ZF*. The other axioms are the same. Replacement in ZF*' is as follows: Any instance of Replacement in ZF* is an instance in ZF*' provided that the Axy be of the form T(Bxy) & (z)(z < y → T(Bxz)). Bxy having only 2 free variables. It is obvious that ZF*' ⊆ ZF*.

Lemma 7: Extensionality & Foundation & Infinity are theorems of ZF* when relativized to L.

Proof: For (Infinity)', take x_0 to be w. For (Extensionality)' and (Foundation)', just note from Lemma 4 that (x ∈ L & y ∈ x) → y ∈ L.
Lemma 8: The Power set and Sum set axioms (of $\text{ZF}^*$) are theorems of $\text{ZF}^*$ when relativized to L.

Proof: In power set, we have $x = \{y | Ay\}$. The relativized to L will be equivalent to $x = \{y | yeL & (Ay)'\} & xeL$. Now observe that the relation $(z \subseteq w)'$ is equivalent to $z \subseteq w & zeL & weL$. So we have to verify that if $xeL & x = \{y | yeL & (Ay)'\}$, then there is a set $x_0 eL$ with $x_0$ the set of all subsets $z$ of $x$ such that $zeL$. Now the hypothesis tells us that $x$ is definable, and so $x$ has a definable power set, $|P(x)|$. We use replacement on $|P(x)|$ to get the set of all $0(y)'s$ with $yeL$ and $y e|P(x)|$. This is a definable set of $\text{Ord}''$, and so it has a union, $U$. $U$ is definable, and so, all $y \subseteq x$ with $yeL$ have $yeM(U)$, because if $\beta e\alpha$, then $M(\beta) \subseteq m(\alpha)$. The required set $x_0$ of all subsets $y$ of $x$ with $yeL$ is in $M(U+1)$, by Lemma 2; hence $x_0 eL$, by Lemma 5.

The relativized of Sum Set is checked similarly.

Lemma 9: If $(\exists y)(y < z & Ayx_1...x_n)$, and $0(z), 0(x_1),...,0(x_n)$ all $e\alpha = \{x | Bx\}$, then $\exists \mu 0(y)$, with $y < z & Ayx_1...x_n$.
(That is, if $A$ any formula with the free variables shown, $B$ any formula with 1 free variable, the above is provable in $\text{ZF}^*$).

Proof: One just assumes there are counterexamples $z,x_1,...,x_n eM(\alpha)$ to this lemma, and then go to definable counterexamples. But we obtain a contradiction, since there is provably a $\mu 0(y)$ for these supposed definable counterexamples.

Lemma 10: The semi-relativized of each of Replacement in $\text{ZF}^*$ is a theorem of $\text{ZF}^*$. 
Proof: We take a particular instance of Replacement in ZF*, e.g., that one whose Axy is T((\exists z)(w)Czwx) & (u)(u < y \rightarrow \neg T((\exists z)(w)Czwxu)), where C is quantifier-free.
We let D be a definable domain, D\in L. We wish to show in ZF* that there is a set S\in L of all y\in L such that for some x\in L with x\in D, y is the unique y with (Axy)''. This is easily seen to be equivalent to finding a set S\in L of all y\in L such that for some x\in D,

1) \textstyle x \in L \land y \in L \land (\exists z)_L[(w)_L(Czwx \lor (\exists w')_L(w' < w \land \neg Czwx)] \land (z')_L(z' < z \rightarrow (\exists w)_L(\neg Czwx \lor (w')_L(w' < w \rightarrow Czwx)]) \land (u)_L[u < y \rightarrow (z)_L[(\exists w)_L(\neg Czwx \lor (w')_L(w' < w \rightarrow Czwx)]) \lor (\exists z')_L(z' < z \land (w)_L(Czwx \lor (\exists w')_L(w' < w \land \neg Czwx)])].

Convenient Notation: If X and Y are expressions occurring in 1), then let [X,Y] be the subformula of 1) beginning with X and ending with Y.

Let U = \text{union of the } O(y)'s such that 1) holds for some x\in D. We proceed to place definable bounds on the quantifiers above in such a way that the new formula is equivalent to 1) for x\in D, y\in M(U). (Note that for each x\in D, there is at most 1 y satisfying 1).)

Let \textstyle f_1(<x,y>) \text{ undefined if } 1 \text{ is false: be } 0(z) \text{ for } z \in L \text{ with } [(w)_L,\neg Czwx] \text{ otherwise. Define } U_1 = \text{union of the range of } f_1 \text{ on } D \times M(U).

Let \textstyle f_2(<x,y,z>) \text{ undefined if } [(w)_L,\neg Czwx]; \text{ be } 0(w) \text{ for } w \in L \text{ with } \neg[Czwx, \neg Czwx], \text{ otherwise. Define }
\[ \cup_2 = \text{union of the range of } f_2 \text{ on } D \times M(\cup) \times M(\cup_1). \]

Let \( f_3(<x,y,z,w>) \) be undefined if \( (w')_L(w' < w \rightarrow Czw'xy) \); otherwise be \( \mu O(w') \) for \( w' < w \) \& \( \neg Czw'xy \). (See Lemma 9.) Define \( \cup_3 = \text{union of the range of } f_3 \text{ on } D \times M(\cup) \times M(\cup_1) \times M(\cup_2). \)

Let \( f_4(<x,y,z>) \) be undefined if \( [(z')_L,Czw'xy] \); be \( \mu O(z') \) with \( \neg[z' < z, Czw'xy] \), otherwise. Define \( \cup_4 = \text{union of the range of } f_4 \text{ on } D \times M(\cup) \times M(\cup_1). \)

Let \( f_5(<x,y,z,z'>) \) be undefined if \( \neg[(\exists w)_L,Czw'xy] \); be \( O(w) \) for \( w \in L \) with \( \neg Czw'xy \), otherwise. Define \( \cup_5 = \text{union of the range of } f_5 \text{ on } D \times M(\cup) \times M(\cup_1) \times M(\cup_4). \)

Let \( f_6(<x,y,z,z',w>) \) be undefined if \( [(w')_L,Czw'xy] \); otherwise be \( \mu O(w') \) with \( w' < w \) \& \( Czw'xy \). Define \( \cup_6 = \text{union of the range of } f_6 \text{ on } D \times M(\cup) \times M(\cup_1) \times M(\cup_4) \times M(\cup_5). \)

Let \( f_7(<x,y>) \) be undefined if \( [(u)_L,\neg Czw'xu] \); otherwise be \( \mu O(u) \) with \( \neg[u < y, Czw'xu] \). Define \( \cup_7 = \text{union of the range of } f_7 \text{ on } D \times M(\cup). \)

Let \( f_8(<x,y,u>) \) be undefined if \( [(z)_L,\neg Czw'xu] \); otherwise be \( O(z) \) for \( z \in L \) with \( \neg(\exists w)_L,\neg Czw'xu \). Define \( \cup_8 = \text{union of the range of } f_8 \text{ on } D \times M(\cup) \times M(\cup_7). \)

Let \( f_9(<x,y,u,z>) \) be undefined if \( \neg[(\exists w)_L,Czw'xu] \); otherwise be \( O(w) \) with \( w \in L \) \& \( \neg Czw'xu \). Define \( \cup_9 = \text{union of the range of } f_9 \text{ on } D \times M(\cup) \times M(\cup_7) \times M(\cup_8). \)

Let \( f_{10}(<x,y,u,z,w>) \) be undefined if \( [(w')_L,Czw'xu] \); otherwise be \( \mu O(w') \) with \( w' < w \) \& \( \neg Czw'xu \). Define \( \cup_{10} = \text{union of the range of } f_{10} \text{ on } D \times M(\cup) \times M(\cup_7) \times M(\cup_8) \times M(\cup_9). \)
Let \( f_{11}(x, y, u, z) \) be undefined if \( \neg[\exists z'. \exists Cz'w'xu] \); otherwise be \( \mu 0(z') \) with \( z' < z, \neg Cz'w'xu \). Define \( U_{11} = \) union of the range of \( f_{11} \) on \( D \times M(\cup) \times M(U_7) \times M(U_8) \).

Let \( f_{12}(x, y, u, z, z') \) be undefined if \( \neg[(w)_L, \neg Cz'w'xu] \); otherwise be \( 0(w) \) for \( w \in L \) with \( \neg[Cz'wxu, \neg Cz'w'xu] \). Define \( U_{12} = \) union of the range of \( f_{12} \) on \( D \times M(U) \times M(U_7) \times M(U_8) \times M(U_{11}) \).

Let \( f_{13}(x, y, u, z, z', w) \) be undefined if \( \neg[\exists w'. \exists Cz'w'xu] \); otherwise be \( \mu 0(w') \) with \( w' < w \) & \( \neg Cz'w'xu \). Define \( U_{13} = \) union of the range of \( f_{13} \) on \( D \times M(U) \times M(U_7) \times M(U_8) \times M(U_{11}) \times M(U_{12}) \).

Note that by suitable instances of Replacement in ZF*, all of the above are provably well-defined. Note that each \( U_i \) is definable, so that each \( M(U_i) \subseteq L \). It is easily seen that for \( x \in D, y \in M(U) \), it is the case that \( Ax \) is equivalent to the predicate \( Bxy \) obtained by placing the bounds \( M(U_i), 1 \leq i \leq 13 \) on the appropriate quantifiers in \( l \).

Now each instance of the following is provable in ZF*:
If \( \alpha = \{x | Ax\} \), and \( \alpha \) a limit, then for \( x \) and \( y \in M(\alpha) \), \( x < y \) iff \( x < y \) holds when the quantifiers in the definition are relativized to \( M(\alpha) \), \( A \) of \( l \) free variable. The proof in ZF* of the schema is like the proof in ZF. Use the definable well-ordering of \( M(\alpha) \).

Now let \( V = \max(\cup, U_1, O(D)) \). Then relativizing the quantifiers occurring in the expansions of the "<"'s that occur in \( Bxy \), to \( M(V + \omega) \), we get the same predicate as \( Bxy \), for \( x \in D \). Hence we have shown our S we wanted to show.
originally $eL$, is first-order definable over $M(V+\omega)$, and hence $eM(V+\omega+1)$, $V + \omega + 1$ definable.

8. The System $ZF'$. Making use of the transformation $T$ defined in the previous section, we form $ZF'$ as follows:

first, Extensionality & Foundation & Infinity & Power Set and
Sum Set axioms of $ZF$. In addition, we have $(x)(x \epsilon L) \& (x)
(\text{Ord}x \rightarrow \text{Ord"}(x)) \& (x)(\exists y)(z)(y \epsilon x \& (z \epsilon x \rightarrow z \not< y)).$

Replacement in $ZF'$ will be the following: Let $Bx_1y_1...y_n$
be a formula in prenex form with only the free variables shown.
Then $(y_1)...(y_n)(x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 = (\exists x_2)(x_2 \epsilon x_0 &
T(Bx_2x_1y_1...y_n) \& (x_4)(x_4 < x_2 \rightarrow \neg T(Bx_3x_4y_1...y_n))))$, is an instance.

Lemma 11: ZF' $\supseteq ZF$.

Proof: First, we wish to show in $ZF'$ each instance of
$T(A) \equiv A$. This is trivial for $A$ with no quantifiers.

Suppose $T(A) \equiv A$ is provable in $ZF'$ for all $A$ in
prenex form with $n$ quantifiers. We then wish to show that
$T(A) \equiv A$ is provable with $A$ having $n + 1$ quantifiers in
prenex form. Then we will have shown $T(A) \equiv A$ provable for
any $A$ in prenex form in $ZF'$.

Let $B$ be in prenex form with $n + 1$ quantifiers.
Suppose $B$ is $(\exists x_1)(Ax_1)$. Then $T(B)$ is $(\exists x_1)(T(Ax_1) \&
(x_j)(\neg x_j < x_i v T(A-x_i)))$. Now $\vdash_{ZF'} T(Ax_1) = Ax_1$. Since
$T(A-x_j)$ is equivalent to $\neg T(Ax_j)$ we have $\vdash_{ZF'} T(A-x_j) = \neg Ax_j$.
We have to check that $(\exists x_1)(Ax_1) \equiv (\exists x_1)(Ax_1 \& (x_j)(\neg x_j < x_i v
\neg Ax_j))$ is provable in $ZF'$. 
Define $C_{xy}$ to be $y = x \& T(Ay)$. This is, of course, equivalent with $T(Ay \& y=x)$, a transformation on a wff of n quantifiers. Now $(x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 \equiv (\exists x)(x \epsilon x_0 \& Cxx_2 \& (x_4)(x_4 < x_2 \rightarrow \sim T(Ay \& y=x))))$, is (equivalent to) an axiom of $ZF'$. But $\vdash_{ZF'} T(Ax_2 \& x_2 = x_3) \equiv Ax_2 \& x_2 = x_3$. So $\vdash_{ZF'} (x_0)(\exists x_1)(x_2)(x_2 \epsilon x_1 \equiv Ax_2 \& x_2 \epsilon x_0)$. Now assume $(\exists x_1)Ax_1$. Choose any such $x_1$. Take $0(x_1)$, and set $x_0 = M(0(x_1))$, and use the above theorem of $ZF'$ to get the set of all elements of $M(0(x_1))$ having the property $A$. ($M(0(x_1))$ is defined and has required properties since $(x)(x \epsilon L)$ is an axiom of $ZF'$, and $x \epsilon L$ is formalized as in $ZF^*$, previously.) Hence by one of the axioms of $ZF'$, there is a $\prec$-least member. Hence we have shown by induction the equivalence between $T(A)$ and $A$, in $ZF'$. This has the effect of provably in $ZF'$ eliminating the $T$'s in the axioms of $ZF'$, and so $ZF' \supset ZF$.

9. The Skolem Argument. We wish to show $(n)\vdash_{ZF^*}\text{Con}(ZF'_{n})$, where $ZF'_{n}$ is the first $n$ axioms of $ZF'$ in some natural enumeration of them. If we succeed in showing this, then suppose $\sim\text{Con}(ZF)$. Then $\sim\text{Con}(ZF')$. Then $(\exists n)\sim\text{Con}(ZF'_{n})$. But then $(\exists n)\vdash_{\text{ENT}}\text{Con}(ZF'_{n})$. Since ENT is formalizable in $ZF^*$, $(\exists n)\vdash_{ZF^*}\text{Con}(ZF'_{n})$. Hence $(\exists n)(\vdash_{ZF^*}\text{Con}(ZF'_{n}) \& \vdash_{ZF^*}\sim\text{Con}(ZF'_{n}))$, and so $\sim\text{Con}(ZF^*)$. Hence $\text{ConZF^*} \rightarrow \text{ConZF}$.

We give, without loss of generality, a Skolem closure argument within $ZF^*$ to give, provably in $ZF^*$, a set which is a model for 1) Extensionality in $ZF$, 2) Foundation in $ZF$, 3) Infinity in $ZF$, 4) Power Set in $ZF$, 5) Sum Set in $ZF$, ...
6) \((x)(x \in L), 7) \((x)(\text{Ord} x \rightarrow \text{Ord}" x), 8) \((x)(\exists y)(z)(y \in x \& z \notin y)), 9) \text{Let } D(xyT) \text{ be the formula obtained from taking 1) in 7. The System } ZF^* \text{ and replacing the 4-place quantifier-free predicate } C, \text{ with some 5-place quantifier-free predicate } E(zwxyT). (D)(T)(\exists S)(y)(y \in S = (\exists x)(x \in D \& D(x,y,T)).

The construction, in } ZF^*, \text{ of the model of these 9 sentences will be much like a Skolem construction in which the initial model is } \emptyset. \text{ At each stage } n, \text{ we throw in some sets } x \in L, \text{ and we take the union as } n \text{ ranges over } w.

We simultaneously define } \alpha_n \text{ and } S_n. \text{ We are interested in } \bigcup_{n \in w} S_n.

\[ S_n = M(\alpha_n); S_0 = M(\emptyset) = \emptyset; \alpha_0 = \emptyset. \]

Consider, for each } x \in S_n, \text{ the } \langle \text{-} \text{least } y \in x \text{ with } (z)(z \in y \rightarrow z \notin x). \]

Consider, for each } x \in S_n, P_L(x) = \text{ set of all } y \subseteq x \text{ with } y \in L.

Consider, for each } x, y \in S_n, \text{ with } x \nsubseteq y, \text{ the } \langle \text{-} \text{least element of } x \text{ not in } y. \]

Consider, for each } x \in S_n, \cup_L(x) = \text{ set of all } y \in L \text{ such that } (\exists z)(z \in x \& y \in z). \]

Consider, for each } T \in S_n, \text{ the unique } S \in L \text{ satisfying the semi-relativized of 9) to } L. \text{ (Call this 9)").}

We continue "considering" through 9)"), closing } S_n, \text{ in effect, under the } "\text{Skolem functions}" \text{ for 9)"}, \text{ in such a way that, as in 7. The System } ZF^*, \text{ we have that the Skolem functions produce values definable in terms of the arguments.}
We take \( \alpha_{n+1} = (\text{union of the } 0(z)'s \text{ for the } z's \text{ considered above}) + \omega \). Take \( S_{n+1} = M(\alpha_{n+1}) \).

We can then use appropriate instances of Replacement in ZF* in combination with the definable well-ordering \(<\), to show that if the \( S_n \) and \( \alpha_n \) are not well-defined for each \( n \), then there are definable counterexamples to our construction in the following sense: for some specifically definable sets, the sets corresponding to them that we considered above do not exist. But this is impossible by Lemmas 7, 8, 9, and 10. So our construction is well-defined in ZF*.

Now our model \( \models S_n \) is an \( M(\alpha) \), \( \alpha \) definable, \( \alpha \) a new limit. In particular, it is transitive. It also contains \( \omega \).

Due to the absoluteness of the definition of \( L \) and of the definition of \( \omega \) in \( M(\alpha)'s \), \( \alpha \) a limit, it is easily seen in ZF*, putting all this together, that the sentences 1)-9) are true when the quantifiers range over \( M(\alpha) \). Furthermore, since \( M(\alpha) \) is definable, the definition of satisfaction and the induction on \( \alpha \) are easily developable in ZF*, in order to prove, in ZF*, that \( \text{Con(1)-9)} \).

From the remarks at the beginning of this section, we immediately have

**Theorem 1:** \( \vdash_{\text{ENT}} \text{ConZF*} \rightarrow \text{ConZF} \).

10. **ZF, Parameterless ZF, and ZF*.** This section is devoted partially to further consideration of the system ZF* of Chapter 1, and partially to some other subsystems of ZF.

We define a sentence of set theory to be **arithmetical if**
it is the relativized of some sentence of set theory to $\omega$.

**Corollary 1**: ZF is a conservative extension of ZF* for arithmetical sentences.

**Proof**: Let $A$ be arithmetical, and $\vdash_{ZF} A$. We can show $\text{Con}(ZF* + \neg A) \rightarrow \text{Con}(ZF + \neg A)$ by modifying the proof of Theorem 1 slightly; just redefine the systems $ZF^*, ZF'$ as $ZF^* + \neg A$, $ZF' + \neg A$, respectively. Due to Lemma 6, all of our Lemmas carry over. Now since $\neg \text{Con}(ZF + \neg A)$, we have $\neg \text{Con}(ZF* + \neg A)$, and so $\vdash_{ZF*} A$.

Our next Theorem concerns sentences of the form $(x)(\exists! y)Axy$, $A$ arbitrary, with only 2 free variables, that are provable in ZF*. Now in ZF there are many such sentences which define, provably, in ZF, a Skolem function which moves everything and which is 1-1. An example is $(x)(\exists y)(y = |P(x)|)$. Another is $(x)(\exists y)(y = \{x\})$. Not so in ZF*. Thus,

**Theorem 2**: Let $Axy$ be any formula with only free variables shown, and let $C = (x)(\exists y)(Axy \land y \neq x) \land (x)(y)(z)(((Axy \land Axz) \rightarrow y = z)$. Then $C$ is not provable in ZF*.

**Proof**: We let $C$ be a sentence of the above form, and we construct a model for $ZF* + \neg C$, given an arbitrary model for ZF, $\mathfrak{M} = \langle X, R \rangle$, where $R$ is a 2-ary relation on $X$, $X \neq \emptyset$. (All models are assumed to be equality models. Note that ZF* is a first-order theory with equality.)

We define $\mathfrak{B}$ as follows: The domain is to be $X \cup Q$, where $Q$ is the rationals. The 2-ary relation, $Sxy$, is defined as $Rxy$ if $x, y \in X$; $x < y$ if $x, y \in Q$; false, if $x \in Q$, $y \in X$; true if $x \in X, y \in Q$. 
We claim $B$ satisfies $ZF* + \neg C$.

First, we show that the elements of $Q$ in $B$ are indistinguishable in the sense that if $Ax_1\ldots x_n y_1\ldots y_m$ holds in $B$ for $x_i \in Q, y_j \in X$, then so does $Az_1\ldots z_n y_1\ldots y_m$ for $z_i \in Q$ if the two sequences of rationals, $x_i, y_i$ have the same order relations in $Q$, (i.e., there is a 1-1 order preserving map). To see this, it suffices to show that, given such a pair of similar sequences of rationals, there is an automorphism of $B$ which keeps the elements of $X$ fixed, and which maps, in an order-preserving way, the sequence $x_i$ onto $z_i$. And such an automorphism is easily given by any map which fixes the elements of $X$ and maps the rationals 1-1 onto itself, which maps the $x_i$ into the $y_j$.

Now suppose $B \models (x)(\exists y)(Axy)$. Then by indistinguishability, it is clear that for $x \in Q$, we have $B \models Axy$ for some $y \in X$, for otherwise we would have $Axy$ for $x, y \in Q$, and hence $Axz$ for $z = y + 1$. But now I claim that $B$ satisfies $A(x+1, y)$, since $y \in X$, by indistinguishability, assuming $x \in Q$. So $B$ does not satisfy $C$.

Clearly $B$ satisfies axiom 0 of $ZF*$.

To verify 1 of $ZF*$, suppose $B \models (x_2)(x_2 \varepsilon x_0 = x_2 \varepsilon x_1)$, for $x_2, x_1$ in the domain. Then either $x_1$ and $x_0 \in X$, or $x_1, x_0 \in Q$. In the first case, we can conclude that $\emptyset \models (x_2)(x_2 \varepsilon x_0 = x_2 \varepsilon x_1)$, and so since $\emptyset \models ZF$, we have $x_0 = x_1$. In the second case, we have $1/2(x_0 + x_1) \leq x_0$ iff $1/2(x_0 + x_1) \leq x_1$. But then $x_0 = x_1$. 

To verify 2 of ZF*, set \( x_0 = w \) of \( \mathcal{N} \). It is easy to see that \( \mathcal{B} \models \emptyset \vDash \), since \( \emptyset \) in \( \mathcal{B} \) is same as \( \emptyset \) in \( \mathcal{N} \). Also, \( x \cup \{x\} \) remains unchanged for \( x \in X \), when we pass from \( \mathcal{N} \) to \( \mathcal{B} \). Also, the members in \( \mathcal{N} \) of \( x_0 \) are the same as the members of \( x_0 \) in \( \mathcal{B} \). Also, the subsets in \( \mathcal{N} \) of \( x_0 \), or any of its members, are identical with the corresponding elements in \( \mathcal{B} \). Putting this together, we see that axiom 2 of ZF* is satisfied in \( \mathcal{N} \) "in the same way" as it is in \( \mathcal{B} \).

To see that 3 of ZF* is satisfied by \( \mathcal{B} \), suppose \( \mathcal{B} \) satisfies \( C_A \). Then the unique element defined in \( C_A \) must be \( eX \), by indistinguishability. We let this element be \( x \). What we are looking for is a power set of \( x \) in the model \( \mathcal{B} \). We claim that \( y = P(x) \) in the model \( \mathcal{N} \) does the trick. We have to show that \( \mathcal{B} \) satisfies \( y = P(x) \). But this is obvious, since the only members of \( y \) in \( \mathcal{B} \) are the members of \( y \) in \( \mathcal{N} \), and the only subsets of \( x \) in \( \mathcal{B} \) are the subsets of \( x \) in \( \mathcal{N} \).

Axiom 4 of ZF* is checked similarly.

To see that axiom 6 of ZF* is satisfied in \( \mathcal{B} \), let \( x \in X \cup Q \). If \( x \in X \), take \( x_0 \) in foundation as an R-least member of \( x \) in \( \mathcal{N} \). If \( x \in Q \), take \( x_0 \) to be \( \emptyset \) in \( \mathcal{N} \) (or \( \mathcal{B} \)). Obviously \( \mathcal{B} \models \emptyset \vDash \).

Axiom 5 of ZF* is the most complicated. The set that \( C_A \) defines in \( \mathcal{B} \) is again \( eX \). Call it \( D \). Then we are interested in the range in \( \mathcal{B} \) of the partial function in \( \mathcal{B} \),
Bx_2 x_1 & (x_3)(Bx_2 x_3 \rightarrow x_3=x_1), on the domain D. Now every
x_2 with S(x_2,D) has x_2 \in X, and so, by indistinguishability,
if S(x_2,D), and Bx_2 x_1 & (x_3)(Bx_2 x_3 \rightarrow x_3=x_1), then x_1 \in X.
Now suppose there is a formula Cx_2 x_1 such that, for x_2 \in X,
x_1 \in X, Cx_2 x_1 holds in \emptyset iff Cx_2 x_1 holds in B. Then by
Replacement in the model \emptyset, we would have (this instance of)
Replacement in B, and we would be done. It remains to show
that for each formula Ax_1...x_n, with the free variables shown,
there is a formula Bx_1...x_n which holds in \emptyset iff Ax_1...x_n
holds in B, when x_1 \in X.

It suffices to prove by induction that for any formula
Ax_1...x_n, and for any partial function f from \{i|1 \leq i \leq n\}
into Q, there is a formula Bx_1...x_n, \{i_1,...,i_k\} =
\{i|1 \leq i \leq n \& i \notin \text{Dom}(f)\}, such that for any sequence x_1...x_n
with x_i \in X iff i \notin \text{Dom}(f), x_i = f(i) if i \in \text{Dom}(f), we have
B \models Ax_1...x_n iff \emptyset \models Bx_1...x_n.

To see this for Ax_1...x_n quantifier-free, take B to
be the formula obtained from A by 1) replacing all instances
of x_i \in x_j, i \notin \text{Dom}(f), j \in \text{Dom}(f), by x_i = x_j, 2) replacing all
instances of x_i \in x_j, or x_i = x_j, or x_j = x_i, i \in \text{Dom}(f), j \notin \text{Dom}(f),
by x_j \notin x_j, 3) replacing all instances of x_i \in x_j, or x_i = x_j,
i,j \in \text{Dom}(f), by (\exists v)(v=v) if f(i) < f(j), (\exists v)(v=v) if not;
or (\exists v)(v=v) if f(i) = f(j), (\exists v)(v=v) if not, respectively.

Put Ax_1...x_n in prenex form, and suppose our claim is
true for all formulae with less quantifiers.

We may assume that Ax_1...x_n is (x_0)Cx_0 x_1...x_n,
since the existential case follows from this case by taking
negations. Let $f$ be a partial function from $\{i | 1 \leq i \leq n\}$ into $Q$. Now 2 finite partial functions $g, h$ from $w \rightarrow Q$ are said to be of the same type if 1) they have the same domain $D$, 2) $g(x) < g(y)$ iff $h(x) < h(y)$.

Consider the set of partial functions on $\{i | 0 \leq i \leq n\}$ which are identical to $f$ on $\{i | 1 \leq i \leq n\}$. There are only a finite number of types represented in this set. Pick a representative from each type, and call this set $\{f_1, f_2, \ldots, f_k\}$, $f_1 = f$. Let $D_i, 1 \leq i \leq k$, be the formula given by the inductive hypothesis for $C_{x_0x_1\ldots x_n}$ for $f_i$; i.e., each $D_i$ has exactly the free variables $x_l$ for $l \notin Dom(f_i)$, $0 \leq l \leq n$, and for any $x_0\ldots x_n$ with $x_l \notin X$ for $l \notin Dom(f_i)$, $x_l = f_i(l)$ for $l \in Dom(f_i)$, we have $\mathcal{B} \models C_{x_0x_1\ldots x_n}$ iff $\mathcal{B}$ \models $D_1 x_{p_1} \ldots x_{p_q}$, $\{p_1\ldots p_q\} = \{r | 0 \leq r \leq n \& r \notin Dom(f_i)\}$.

Then we take $B$ to be $(x_0)D_1 \& \wedge_{2 \leq i \leq k} D_i$. By indistinguishability, it is easily seen that $B$ and $A$ satisfies the conclusion of our claim for the function $f$. This concludes the proof of Theorem 2.

We now define some new subsystems of ZF. $\text{ZF}^n$ is to be the same as ZF except for the Replacement schema. $\text{ZF}^n$, instead, only allows the $\text{Axy}$ in the Replacement schema at most to have $n + 2$ free variables; $x$ and $y$ and possibly $n$ other ones. It is easy to see that $\text{ZF}^*$ is a subsystem of $\text{ZF}^0$. We also have

**Corollary 2:** $\text{ZF}^* \neq \text{ZF}^0$.

**Proof:** Consider the model of $\text{ZF}^*$ constructed in the
proof of Theorem 2. Consider the sentence \( A = (x)(\exists y)(\forall \xi x \rightarrow (\forall \xi y \land (z)(z \neq \emptyset \rightarrow (\xi x \quad \iff \quad z \in y))) \). A is obviously provable in \( \text{ZF}^0 \). But our model of \( \text{ZF}^* \) does not satisfy A.

We believe strongly that \( \text{ZF}^0 \neq \text{ZF} \), and in fact in the stronger conjecture that \( \text{ZF}^n \neq \text{ZF}^{n+1} \), for all n. Although the details of a proof of \( \text{ZF}^0 \neq \text{ZF} \) have not yet been carried out, we can give the definition of a very promising model of \( \text{ZF}^0 + (\exists x)(y)(\exists z)(\neg(z \in y \iff z = x)) \).

We let \( M(\alpha) \) be the minimal model of \( \text{ZF} \). We let \( S \) be any Cohen generic set of natural numbers over \( M(\alpha) \). We let \( M^S(\alpha) \) be the corresponding Cohen model for \( \text{ZF} + V \neq L \).

We let \( F_S \) be the set of all sets of natural numbers which are finitely different from \( S \) (i.e., whose symmetric difference from \( S \) is finite).

Recall that \( R(\beta + 1) = |P(R(\beta))|, R(\lambda) = \bigcup_{\beta < \lambda} R(\beta) \).

Consider the sets \( x \in M^S(\alpha) \) such that for any formula \( Ayzx \), we have, in \( M^S(\alpha), \{z | (\exists y)(\forall x \land A(x, y, F_S))\} \neq \{x\} \). We let \( X \) be the set of all such \( x \). We let \( Y \) be the set of all \( x \in X \) such that any finite combination of union and power set on \( x \) in \( M^S(\alpha) \) gives a set in \( X \).

For each \( \beta \), let \( R'(\beta) \) be the unique rank in the cumulative hierarchy up to \( \beta \) in the model \( M^S(\alpha) \).

Define, for each \( \beta \), a function \( f_\beta \) whose domain is \( R'(\beta) \cap Y \), and by the equations \( f_{\beta + 1}(x) = \{y | y \in Y \land (\exists z)(z \in x \land f_\beta(z) = y)\} \), \( f_\lambda = \bigcup_{\beta < \lambda} f_\beta \).

We define \( x \in Z \) as \( [(\exists \beta)(\beta < \alpha \land x \in \text{Range}(f_\beta)) \land x \in Y] \).
We conjecture that \(<Z,\varepsilon>\) is the desired model.

We feel that a detailed analysis of the relations between the theories \(ZF^n\), hopefully by finding natural sentences to distinguish each \(ZF^n\) from \(ZF^{n+1}\), would involve non-trivial applications of the notion of forcing. In particular, careful attention seems to be required as to the model-theoretic properties of models of theories obtained by forcing; e.g., the definability or indistinguishability of various elements of the models constructed.
CHAPTER II

1. Definitions of Systems. We will have for our language, number variables, n,m,p,q,r; set variables, x,y,z,w,u,v,...; the relation nex between numbers and sets; and the constant number "0" which can hold only between variables of the same type.

We introduce function variables in the usual way by defining them in terms of sets of natural numbers in the usual way. We will have full number theory at our disposal, since all systems considered here will have the unrestricted induction axiom schema (called I) \[ A_0 \& (n)(A_n \rightarrow A'_n) \rightarrow (n)A_n, \]
where A is any formula with possibly free variables of both kinds, and can have both number and set quantification.

Also, all systems considered here will have the axiom of extensionality (that any two sets with the same members are equal), and we will tacitly assume that the system we will call I includes this axiom. Thus I will be the unrestricted axiom schema of induction plus the axiom of extensionality.

In addition, all systems considered will have the recursively enumerable comprehension axiom schema (called ReCA),

\[(x_1)...(x_1)(e)(\exists y)(n)(n \in y \equiv (\exists k)T_e^k(e,n,k,x_1,...,x_1)).\]

The predicates \(T_e^k\) are understood to be written out in the usual way with bounded quantifiers. (Or, we may have instead introduced them as primitive, and defined them by adding axioms of primitive recursion.) The predicates \(T_e^k\)
above are what Kleene would call $T^{f_1,\ldots,f_l}$, where $f_1$ is the characteristic function for the set, $x_1$. See Kleene [3], p. 291.

Before we get into the mathematics of the systems we will be considering, we will state, informally, some propositions concerning what can be done in the system I + ReCA.

**Proposition 1:** We can, in I + ReCA, justify all uses of coding normally found in the development of hierarchy theory. Thus, we may justify the use of such symbols as $\langle x,y \rangle$ and $(x)_n$ (respectively $\{2^{n+1} \cdot 3^{m+1} \mid n \in \mathbb{N} \land m \in \mathbb{N} \}$, and $\{m \mid p_n \cdot \exists x \}$, where $p_n$ is the $n$th prime).

We define the relation between functions and sets alluded to above, as $f(n) = m$ iff $\langle n,m \rangle \in f$, where $\langle n,m \rangle = 2^{n+1} \cdot 3^{m+1}$.

**Proposition 2:** In ReCA + I, we may provably perform "collapse of like quantifiers". In other words, the usual way of collapsing 2 successive universal set quantifiers (or function quantifiers) into one can be completely justified on the basis of ReCA + I.

**Def. 1:** A predicate $A(n,x_1,\ldots,x_p)$ of $n$ is said to be essentially $\Pi_1$ if it is in prenex form followed by a matrix $T(e,n,m,x_1,\ldots,x_p,f_1,\ldots,f_l,n_1,\ldots,n_k)$, where there are no set quantifiers in $A$, and the $f_1$ occur as universal function quantifiers, in any order, mixed together with possibly number quantifiers $(n_1)$ and $(\exists n_j)$ and $(\exists m)$. (No existential function quantifiers in $A$.) An essentially $\Sigma_1$ formula is just the prenex form of the negation of an essentially $\Pi_1$
Proposition 3: For every arithmetical predicate (with parameters) there is a predicate \((f)(\exists m)T(e,n,m,x_1,\ldots,x_k, n_1,\ldots,n_p)\) which is, provably in \(I + ReCA\), equivalent. There is also a predicate \((\exists f)(m)\sim T(e,n,m,x_1,\ldots,x_k,n_1,\ldots,n_p)\) which is, provably in \(I + ReCA\), equivalent.

Proposition 4: Define \((x)^{(l)}\) as the \(l\)-th jump of \(x\). \(y = (x)^{(l)} \equiv \exists\) a sequence \((x_0,\ldots,x_k)\) such that \(x_0 = x\) and \(x_{i+1} = \text{jump of } x_i\). (Thus, \(y = (x)^{(l)}\) is a predicate of 3 variables, defined in the usual way.) Then in \(I + ReCA\) we may prove \((\exists y)(u = (x)^{(l)})\).

This is proved by induction on \(l\).

Consider
1) \((x)(p)(e)[(n)[(f)(\exists m)S_2(e,n,m,f,x) \equiv (\exists g)(r)\sim S_2(p,n,r,g,x)] \rightarrow (\exists y)(n)(n \equiv f)(\exists m)S_2(e,n,m,f,x)]\).

2) The schema, \((x)[(n)[Anx \equiv Bnx] \rightarrow (\exists y)(n)(n \equiv Anx)]\), where \(A\) is essentially \(\Pi^1_1\), \(B\) is essentially \(\Sigma^1_1\).

3) \((x)(e)(n)(\exists f)(m)\sim S_2, 1(e,0,m,n,f,x) \rightarrow (\exists y)(n)(y_n)\) is a function \& \((m)\sim S_2, 1(e,0,m,n,y_n,x))\).

4) \((x)(n)(\exists f)A(n,f,x) \rightarrow (\exists y)(n)(y_n)\) is a function \& \(A(n,y_n,x))\), where \(A\) is essentially \(\Sigma^1_1\).

5) \((x)(e)(f)(g)(h)(\exists m)\sim S_3(e,0,m,f,g,x) \rightarrow (f)(\exists y)(n)(y_0 = f \& (m)\sim S_3(e,0,m,y_n,y_{n+1},x)))\).

6) \((x)(f)(g)A(f,g,x) \rightarrow (f)(\exists y)(n)(y_0 = f \& A(y_n,y_{n+1},x)))\), where \(A\) is essentially \(\Sigma^1_1\) having only the free variables \(f,g,x\).

We call \(I + ReCA + 1\), the pure \(\Delta^1_1\)-CA; \(I + ReCA + 2\),
essentially $\Delta^1_1$-CA; $I + \text{ReCA} + 3$), pure $\Sigma^1_1$-AC; $I + \text{ReCA} + 4$), essentially $\Sigma^1_1$-AC; $I + \text{ReCA} + 5$), pure $\Sigma^1_1$-DC; $I + \text{ReCA} + 6$), essentially $\Sigma^1_1$-DC.

NOTE: "CA" is supposed to mean "comprehension axiom"; "AC", axiom of choice; "DC", dependent choices.

Proposition 5: In pure $\Sigma^1_1$-AC, for every essentially $\Pi^1_1$ predicate, there is a pure $\Pi^1_1$ predicate $(f)(\exists n)\mathcal{T}(\bar{e}, m, n, \ldots)$ which is provably equivalent.

To see this, first use Proposition 2 to collapse adjacent like quantifiers. Then use pure $\Sigma^1_1$-AC to interchange a number and function quantifier, and then use Proposition 2 and then pure $\Sigma^1_1$-AC, etc.

Proposition 6: Pure $\Sigma^1_1$-AC $\supset$ pure $\Delta^1_1$-CA.

The idea is that, in pure $\Delta^1_1$-CA, one has that for each $m$ there is a solution to one of two $\Pi^0_1$ predicates, and one uses pure $\Sigma^1_1$-AC to form a Skolem function. Then the required set in pure $\Delta^1_1$-CA is obtained, in pure $\Sigma^1_1$-AC, recursively in the jump of the Skolem function.

Proposition 7: Pure $\Sigma^1_1$-AC = essentially $\Sigma^1_1$-AC.

By Proposition 5, one has only to consider the case $(n)(\exists f)(\exists g)(\exists m)\mathcal{T} \rightarrow \exists$ Skolem function. But we may collapse the $f$ and $g$ quantifiers in the usual way, and apply pure $\Sigma^1_1$-AC to get a Skolem function, which will have recursive in it the Skolem function wanted in the implication above.

A similar argument shows

Proposition 8: Pure $\Sigma^1_1$-AC $\supset$ essentially $\Delta^1_1$-CA.
Conjecture: We do not know whether pure $\Delta^1_1$-CA = 
essentially $\Delta^1_1$-CA, or whether pure $\Delta^1_1$-CA = $\Sigma^1_1$-AC, or whether $\Sigma^1_1$-AC = essentially $\Delta^1_1$-CA, and we conjecture that none of the three statements is correct.

Proposition 9: Pure $\Sigma^1_1$-DC $\rightarrow$ $\Sigma^1_1$-AC. (By Prop. 5, we call essentially $\Sigma^1_1$-AC, and pure $\Sigma^1_1$-AC, just $\Sigma^1_1$-AC.)

Hence also

Proposition 10: Essentially $\Sigma^1_1$-DC = pure $\Sigma^1_1$-DC.

Proposition 11: We can prove in ReCA + I the recursion theorem (with parameters).

2. Preliminary Lemmas. We will eventually show that $\Sigma^1_1$-DC is a conservative extension for purely $\Pi^1_2$ sentences of pure $\Delta^1_1$-CA.

We define, in I + ReCA, several notions.

Def. 1: $P(n,m)$ is defined as the Gödel number (in the usual 1-1 onto Gödel numbering of pairs of natural numbers) of the pair $\langle n,m \rangle$.

Def. 2: $\varphi^x_n$ is defined in the usual way as the nth partial recursive function in x. Note that in I + ReCA we can prove $(n)(x)(\exists \varphi^x_n)$.

Def. 3: $\text{RLO}^x(n)$ is defined as "Range ($\varphi^x_n$) $\subseteq \{0,1\}$ & (m) ($m \in \text{Dom}(\varphi^x_n)$ & $\varphi^x_n$ defines a linear ordering," where $\varphi^x_n$ defines a linear ordering means that 1) $(m)(\varphi^x_n(P(m,m)) = 0)$, 2) $(p)(q)(r)$ ($[\varphi^x_n(P(p,q)) = 1 \& \varphi^x_n(P(q,r)) = 1] \rightarrow \varphi^x_n(P(p,r)) = 1)$), 3) $(p)(q)(\varphi^x_n(P(p,q)) = 1 \rightarrow \varphi^x_n(P(q,p)) = 0)$. 4) $(p)(q)(p = q \lor \varphi^x_n(P(p,q)) = 1 \lor \varphi^x_n(P(q,p)) = 1)$. We define $\text{RLO}(n) = \text{RLO}^0(n)$. 
Def. 4: We define \( p < n \cdot q = \text{RLO}^n(p, q) \). We define \( p \leq n \cdot q = \text{RLO}^n(p, q) \). When \( x = \emptyset \), we use the subscript \( n \) instead of \( n^\emptyset \). (\( \emptyset \) denotes the empty set.)

Def. 5: We define \( \text{WX}(n) = \text{RLO}^n(y)(\exists m)(y = \emptyset \lor (m \in y \land \forall r(r \in y \rightarrow m \leq n \cdot r))) \).

Def. 6: \( \text{Suc}^n_x(p, q) = \text{RLO}^n(n) \land q < n \cdot p \land (r < n \cdot p \rightarrow r \leq n \cdot q) \).

Def. 7: \( \text{HY}^n_x(z) = \text{RLO}(n^x) \land (p)(\text{Suc}^n_x(m, p) \rightarrow ((z)_p)^{(1)} = (z)_m \land (p)(\lim_{n \cdot x}(p) \rightarrow (x)_p = \{P(r, s)|s < n \cdot p \land r \in (z)_s\}) \land (p)(\lim_{n \cdot x}(p) \rightarrow (x)_p = y) \). Thus \( \text{HY}^n_x(z) \) is the predicate of 4 variables asserting that \( z \) is a hierarchy on the \( \text{RLO}, n^x \), starting from \( y \). It will be useful later on to include the condition \((k)(k \in z \rightarrow (\exists p)(k \in (z)_p))\), in the definition above.

Def. 8: \( \text{HY}^n_x(z) \) is the predicate of 5 variables asserting that \( z \) is a hierarchy on the \( \text{RLO}, n^x \), up to but not including \( l \), starting from \( y \). Thus if \( p \geq n^x \), then \( (z)_p = \emptyset \).

Def. 9: \( x \leq y \) is defined as "x is recursive in y" in the usual way.

Def. 10: \( |n^x| < |m^x| \) means \( \text{RLO}^n(n) \land \text{RLO}^m(m) \land (\exists f)(f \text{ is a proper imbedding of } n^x \text{ into } m^x, i.e., } (p)(q)(p \leq n^x q \leftrightarrow \ldots) \)
\[ f(p) \leq m^x f(q) \land ([p < q \land q \in \text{Range}(f)] \rightarrow p \in \text{Range}(f)). \]

|n^X| = |m^X| means that the range of the \( f \) above is \( \omega \).

|n^X| \leq |m^X| = |n^X| = |m^X| \lor |n^X| < |m^X|.

**Def. 11:** \( \text{Hyp}^X(y) = (\exists n)(\exists z)(\text{RLO}^X(n) \land (H^X_n(z)) \land W^X(n) \land y \leq T^z). \)

**Lemma 1:** \((x)(y)(z)([W^Z(n) \land H^Z_n(x) \land H^Z_n(z)]) \rightarrow y = x)\) is provable in \( I + \text{ReCA} \). Also, \([W^Z(n) \land H^Z_n(x) \land H^Z_n(y)] \rightarrow y = x \) is also provable.

**Proof:** Given \( n, x, y, z \) with \( W^Z(n), H^Z_n(y) \). Form \( \{p | (x)_p \neq (y)_p\} \) in \( I + \text{ReCA} \). By \( W^X(n) \), we have a \( <_{n^X} \)-least element \( q \). Clearly we must not have \( 0_{n^X}(q) \). If \( \lim_{n^X} (q) \), then, since \( (s)(s < n^X q \rightarrow (x)_s = (y)_s) \), we have \( (x)_q = (y)_q \). If \( \text{Suc}_{n^X}(q, s) \), then since \( (x)_s = (y)_s \), we have \( (x)_q = (y)_q \).

**Lemma 2:** The following is provable in pure \( \Delta^1_1-\text{CA} \): if \( W^Z(n) \land W^Z(m) \land (\exists x)(H^Z_n(x)) \), then exactly one of the following three holds: a) there is a unique imbedding of the ordering \( n^Z \) onto a proper initial segment of the ordering \( m^Z \), and
there is no imbedding of the ording $m^z$ onto an initial segment of the ordering $n^z$.

b) there is a unique imbedding of the ordering $n^z$ onto the whole ordering $m^z$, and vice versa, and there are no imbeddings onto proper initial segments in either direction.

c) the interchanging of $n$ with $m$ in a).

Furthermore, denote $x^{(1)}$ for the result of applying $l$ iterations of the jump operator to $x$, $l \in \omega$. Then all the above maps may be found $\leq x^{(l)}$, for some $l$ depending on $n$ and $m$.

Proof: In a), if there is an imbedding, $f$, of $n^z$ onto a proper initial segment of $m^z$, then it must be unique, since if $g$ is another, form, in $I + \text{ReCA}$, $\{ k \mid f(k) \neq g(k) \}$, and take an $n$-least member, $p$. Running through the 3 cases, $0^z_n(p)$, $\text{Suc}_n^z(p)$, and $\lim^z_n(p)$ in a straightforward way, using the linearity of $n^z$ and $m^z$, we get a contradiction. Same with c).

If there is an imbedding, $g$, of $m$ onto an initial segment of $n$, take $g^{-1}$, and note that $g^{-1}$ must disagree with $f$ somewhere. Then follow the procedure above.

If there is an imbedding of $n$ onto the whole ordering $m$, then there is one from $m$ to $n$ by taking inverse. (Note that we can form inverses by ReCA.) And there are no proper imbeddings, using least counterexample argument above.

So the main thing is the existence of these imbeddings.

We define a function $f^z_n(p) = \text{greatest } k \text{ such that}$
\((\exists q)(\text{Suck}_n^k(p,q))\), where \(\text{Suck}_n^k\) is \(k\) iterations of \text{Suc}. (This is well defined, since \((p)(q)(r)\left( [\text{Suck}_n^k(p,r) \& \text{Suck}_n^k(p,q)] \rightarrow r = q \right)\)). There is always a greatest \(k\), since otherwise we would have an arithmetically defined chain through \(n^z\), and finitely many iterations of \(\text{ReCA}(+1)\) would realize this claim as an object, and would contradict \(W^Z(n)\).

Fix \(n,m\), with \(W^Z(n), W^Z(m), (\exists x)(H^Z_n^z(x))\). Let \(f_n^z(p)\) be abbreviated \(f(p)\). We claim that, for each \(p\), either there is an imbedding of the ordering \(m\) onto an initial segment \(S\) of the ordering \(n\), so that \(k \in S \rightarrow k < n^z p\), or there is an imbedding of \(\{k | k < n^z p\}\) of \(n\), onto an initial segment of \(m\), and these imbeddings are to be found \(\leq_T(x)p^{(10(f(p)+1))}\). (This is the result of applying \(10(f(p)+1)\) iterations of the jump operator to \((x)p\).)

We assume this is false, and, as usual, form \(\{p|\text{claim is false}\}\). We take an \(n\)-least member, \(q\). If \(0^Z_n(q)\), then clearly claim is true for \(q\). If \(\text{Suck}_n^z(q,r)\), then certainly \(f(r) + 1 = f(q)\). So there exists a comparison mapping, \(\leq_T(x)_r^{(10(f(r)+1))}\), between the segment of \(n\) up to \(r\), and \(m\). It is then easy to see that there is a (unique) comparison map between the segment of \(n\) up to \(q\), and \(m\), \(\leq_T(x)_q^{(10(f(r)+1))}\). The limit case is similarly easy; the uniqueness of these comparison maps is used heavily, and that the property of being a comparison map is low arithmetical, and that the function \(f\) is low arithmetical.
Lemma 3: The following is provable in I + ReCA: If $W^y(n)$ and $W^y(m)$, and $|n^y| < |m^y|$, and $(\exists x)(H^y_n(x))$, then

$(\exists z)(z \leq_T t(x)p (10(f(f)+1)))$ \& $H^y_{m^y}(z))$, where $p$ is the l.u.b. in $n^y$, of the range of the imbedding of $m^y$ in $n^y$, and $f$ is $f_{n^y}$, defined in the proof of Lemma 2.

Proof: Similar to the proof of Lemma 2. One notes, e.g., that if $H^y_{n^y}$ has a solution, it must be unique.

Lemma 4: If $|m^z| < |n^z|$, and $p$ is the l.u.b. in $n^z$ of the image of the imbedding, and $(\exists y)(H^z_y(y))$, then

$(\exists x)(H^z_x(x) \& y \leq_T t(x)(10(f(p)+1)))$, where $f$ is $f_{n^z}$. This is provable in I + ReCA.

Proof: One can prove first, as in Lemma 2, that

$(\exists x)(H^z_x(x))$. Then, again like Lemma 2, starting with this $x_{n^z}$ with $H^z_x(x)$, one can prove that $y \leq_T t(x)(10(f(p)+1))$. (Of course, it is provable in ReCA + I that $(\exists x)(H^z_x(x) \rightarrow (\exists x) H^z_x(x))$.)

Def. 12: Reas $(n^x) = RLO^x(n) \& (p)(0^{\chi^x}(p) \lor \lim_{n^x}(p) \lor (\exists k)\Sigma^k(n^x(p,q) \& \lim_{n^x}(q)) \lor (\exists r)(0^{\chi^x}(r)) \& (p)[\sim(\exists q)(\Sigma^k(n^x(q,p) \rightarrow (q)(q \leq_{n^x} p)])$. (Reas(n) reads "n is reasonable.")

Lemma 5: $(z)(n)(x)(y)(m)([\text{Reas}(m^z) \& H^z_{n^z}(x) \& H^z_{m^z}(y) \& \sim W^z(m)] \rightarrow (\exists p)(x \leq_T t(y)p \& there are infinitely many q >_{m^z} p))$, is provable in I + ReCA.
Proof: Assume $\text{Reas}(m^z), H^z_n(x), H^z_n(y), \sim W^z(n)$. Then there is a set, $w$, which has no $m^z$-least member. It is then easy to see, using $I + \text{ReCA}$, that $(\exists g)(k)(g(k+1) < \sim m^z g(k))$. We will show that $(p)(k)((x)_p \leq T(y)_g(k))$. By $\text{Reas}(m^z)$, this is sufficient.

For, we take, in $I + \text{ReCA}$, $\{p| (\exists k)(\sim ((x)_p \leq T(y)_g(k)))\}$. We can do this, since the predicate we are taking the extension of is arithmetical in $x,y$. And we take an $n^z$-least such $p$, call it $q$, by $W^z(n)$. We then obtain a contradiction.

1) $n^z(q)$. To get a contradiction for this case, it suffices to show that $(k)(z \leq T(y)_g(k))$. If $\lim m^z(g(k))$, then clearly $z \leq T(y)_g(k)$. If $\lim m(g(k))$, also easy. If $\text{Suc}(m^z(g(k)))$, then by $\text{Reas}(m^z)$, we have $\text{Suc}^r_m(g(k),s)$, some $r,s$ with $\lim(s)$, and so $z \leq T(y)_s$. So clearly $(y)_g(k) = (y)_s^{(r)}$, and so $z \leq T(y)_g(k)$.

2) $\text{Suc}(q,r)$. Then $(x)_r \leq T(y)_g(k)$, all $k$. But then $(x)_r^{(1)} \leq T(y)_g(k-1)$, all $k > 0$, and hence $(x)_q \leq T(y)_g(k)$, all $k$ for the following general reason: if $p < \sim m^z q$, then $(y)_p^{(1)} \leq T(y)_q$. The case when $\lim m^z(q)$ is easy. When $\text{Suc}(m^z(q))$, use $\text{Reas}(m^z)$ as in Case 1) above.

3) $\lim n^z(q)$. Let $k$ be arbitrary. We know $(l) (l < n^z q$ $\rightarrow (x)_l \leq T(y)_g(k+10))$. Furthermore, the question of whether a set is $(x)_1$, is a question low arithmetical in $l$, due to the uniqueness of Hierarchies on $RLO's$, and even hierarchies
on initial segments of RLO's. Hence \((x)_q\) is low arithmetical in \((y)_g(k+10)\), and hence \((x)_q\) will be recursive in \((y)_{g(k+10)}\). Hence, by reasonableness of \(m^2\), we have \((x)_q \leq T(y)_g(k)\).

**Lemma 6:** "The predicate \(W^X(n)\) is not \(\Sigma^1_1\) in \(x\)" is provable in \(I + ReCA\).

**Proof:** We have, in \(ReCA + I\), the Kleene normal form for predicates \(\Sigma^1_1\) in \(x\), and we use that to formalize the Lemma. So assume A) \((n)(W^X(n) = (3f)(m)\neg T(e,n,m,f,x))\). Now there is an explicit recursive function \(g\), for which we can prove in \(ReCA + I\), that B) \((k)(n)[(f)(3m)T(k,n,m,f,x) = W^X(g(k,n))]\). Hence \(\exists \tilde{e}\) such that \((n)[(f)(3m)T(n,0,m,f,x) = (3f)(m)\neg T(\tilde{e},n,f,x)]\), is provable in \(I + ReCA\), since the predicate \((f)(3m)T(m,0,m,f,x)\) of \(n\) is \(\Sigma^1_1\) in \(x\), by A) and B), and Kleene's normal form Theorem is provable. Now substitute \(\tilde{e}\) for \(n\) to get a contradiction.

3. **Conservative Extension Result.**

**Theorem 1:** Given any model \(M\) of pure \(\Delta^1_1\)-CA + \(S\), where \(S\) is any purely \(\Sigma^1_2\) sentence, there exists a model \(M'\) of \(\Sigma^1_1\)-DC + \(S\), where \(S\) is \((3y)(g)(3n)T(\tilde{e},0,n,y,z)\), for some \(e\).

**Proof:** A model of pure \(\Delta^1_1\)-CA + \(S\) consists of an interpretation \(J\) of the natural numbers, and +,x,0,,' together with a set of objects, \(X\), and a binary relation \(R(j,x)\), \(j \in J\), \(x \in X\). We define a new model \(\text{Hyp}^Y(M) = M'\), as 1) having same \(J,+,x,0,,'; 2) having same \(R(j,x)\) but restricted to those
j \in J, x \in X \text{ with } M \models \text{Hyp}^Y(x). \text{ Now since } M \models S, \text{ choose } y \text{ such that } M \models (g)(\exists n)T(\bar{e},0,n,y,g). \text{ Then clearly } M' \models (g)(\exists n)T(\bar{e},0,n,y,g). \text{ So } M' \models S.

We claim } M' \models \Sigma^1_1-\text{DC}. \text{ It is sufficient to show that the relativized of each axiom of } \Sigma^1_1-\text{DC} \text{ to the predicate } \text{Hyp}^Y \text{ is a theorem of pure } \Delta^1_1-\text{CA}. \text{ (We define the relativized, } T(A), \text{ of a formula } A, \text{ to the predicate } \text{Hyp}(x) \text{ by } T(A \lor B) = T(A) \lor T(B), T(A \land B) = T(A) \land T(B), T(\neg A) = \neg T(A), T(\exists x A) = (\exists x)(T(A) \land \text{Hyp}^Y(x)), T((x)A) = (x)(\text{Hyp}^Y(x) \rightarrow T(A)), T(Q) = Q, Q \text{ quantifier free, } T(\exists n A) = (\exists n)T(A), T((n)A) = (n)(T(A))).

In other words, the universal closure, obtained by inserting the universal quantifier (y), of the relativized of each axiom of } \Sigma^1_1-\text{DC} \text{ to } \text{Hyp}^Y \text{ is a Theorem of pure } \Delta^1_1-\text{CA}.

1) \text{ Induction. It is clear that the relativized of each instance of induction to } \text{Hyp}^Y \text{ is again an instance of induction, and so is provable in pure } \Delta^1_1-\text{CA}.

2) \text{ ReCA. To prove the relativized of ReCA, it suffices to prove, in pure } \Delta^1_1-\text{CA}, \text{ that } (x)(\text{Hyp}^Y(x) \rightarrow \text{Hyp}^Y(x^{(1)})). \text{ To see this, let } H^Y_{\nu}(z), W^Y(n), x \leq Tz. \text{ So } x^{(1)} \leq Tz^{(1)}. \text{ If } n^Y \text{ has a greatest element, } i, \text{ set } k = f^Y_{n^Y}(1) + 2. \text{ If not, set } k = 2. \text{ Define a new ordering } m^X \text{ by adding on the first } 10 \cdot k \text{ integers on top of } n, \text{ and make, in the trivial way, the ordering } m^Y \text{ total, so that } \text{RLo}^Y(m) \text{ and } |n^Y| < |m^Y|. \text{ Now clearly by Lemma 4, } (\exists w)(H^Y_{m^Y}(w)), \text{ and so clearly } m^Y_p (\exists w)(H^Y_{m^Y}(w)), \text{ by } I + \text{ReCA}. \text{ But, also by Lemma 4, } z^{(1)}
3) Suppose \((f)(\text{Hyp}_Y^V(f) \rightarrow (\exists g)(\text{Hyp}_Y^V(g)) \& (m)\sim_T^3 (e,0,m,f,g,x)))\), where \(\text{Hyp}_Y^V(x)\). Let \(\text{Hyp}_Y^V(f)\). We consider the predicate \(P_n\) is a member of a finite sequence \(n_0,n_1,\ldots,n_k, k \geq 0\), such that I) \((i)W^V(n_i)\). II) For any sequence of sets \(x_0,x_1,\ldots,x_j, j \leq k\), such that \((i)(H^V_y(x_i))\), we have that \(\exists\) another sequence \(z_i\) such that each \(z_i \leq_T x_i\), and \(z_0 = f\) and \((i < j)(m)\sim_T^3 (e,0,m,z_{i+1},x)\). III) For any sequence \(x_0,x_1,\ldots,x_j, j < k\), with \((i \leq j)(H^V_y(x_i))\), it is not the case that \((\exists z)(\exists p)[H^V_y(z) \& p\text{ has infinitely many } q > (n_j+1)]\).

Also consider \(Q_n\) is a member of a finite sequence \(n_0,\ldots,n_k, k \geq 0\), \(\text{Reas}(n_i^Y)\), such that \(\exists\) sequence of sets \(x_0,x_1,\ldots,x_k\), with I) \((i)H^V_y(x_i)\). II) \(\exists\) a sequence \(z_i\) such that each \(z_i \leq_T x_i\) and \(z_0 = f\) and \((i)(m)\sim_T^3 (e,0,m,z_{i+1},x)\). III) Let \(-1 \leq j < k\). It is not the case that \((\exists z)(\exists p) [z \leq_T x_{j+1} \& H^V_y(z) \& p\text{ has infinitely many } q > (n_j+1)]\).

We first note that \(P_n\) is a \(\Pi^1_1\) predicate in \(f,x,y\) and that \(Q_n\) is a \(\Sigma^1_1\) predicate in \(f,x,y\). We wish to show, in I + ReCA, that \((n)(P_n = Q_n)\).
Suppose $P_n$. Assume that $n_0, n_1, \ldots, n_k$ have the properties mentioned in the definition of $P_n$. We claim that 

(1) $(\exists x_i) H^y_i (x_i)$. For, if for some $i$, $\neg (\exists x_i) H^y_i (x_i)$, then take $i$ to be least such (induction), and then by (1)$W^y_i (n_i)$ 

we have that $(q) ([ W^y(q) \land (\exists z)(H^y_q (z))] \rightarrow |q^y| < |n^y_i|)$, 

and, in fact, $q^y$ is imbedded into $n^y_i$ with l.u.b. $l$ having infinitely many points $p > l$. Hence the $z$ with $H^y_q (z)$ must be $\leq_T$ some $w$ with $H^y_{n_i^y}(w)$, where $r$ has infinitely many points $s > r$. We claim that this violates condition III) of $P_n$ when $j = i-1$. (We allow $j$ to be negative.) For just apply condition 2) of $P_n$, and that $(f)(\exists g)(m) \sim_T$ relativized to $\text{Hyp}^y$, and that $\text{Hyp}^y(f)$. In condition II) of $P_n$, use that $(s)(s < i \rightarrow (\exists x_i) H^y_i (x_i))$, 

and use the $x_i$.

So we want to show $Q_n$, and we may choose $x_0, x_1, \ldots, x_k$ with $H^y_i (x_i)$. II) in $Q_n$ follows from 2) in the definition of $P_n$. III) in $Q_n$ follows from III) in $P_n$ plus the observation that the predicates $H^y_i$ have unique solutions $(n^y_j)_p$ 

$\leq_T$ in $(x_j)_p$.

So $P_n \rightarrow Q_n$.

Suppose $Q_n$. We first to show (1)$W^y(n_i)$. Using Lemma 5, and taking $i$ least with $\neg W^y(n_i)$, we contradict III) of $Q_n$ in the same way as the argument above for $P_n \rightarrow Q_n$. For, one uses II) of $Q_n$, and the relativization of $(f)(\exists g)(m) \sim_T$.
So \((i)\mathcal{W}^V(n_1)\).

Now II) of \(P_n\) follows from II) of \(Q_n\) by uniqueness of hierarchies on \(n_1\) because of \(\mathcal{W}^V(n_1)\).

III) of \(P_n\) follows from III) of \(Q_n\) for the same reason.

Next, we show that we can eliminate the parameters \(f\) and \(x\) in \(P_n\) and \(Q_n\), and have the same meaning and still be, respectively, \(\Pi^1_1\) and \(\Sigma^1_1\). For, since \(\text{Hyp}^V(x)\) and \(\text{Hyp}^V(f)\), let \(e_1, e_2\) have \(\mathcal{W}^V(e_1)\) and \(\mathcal{W}^V(e_2) \& (\exists z)(\mathcal{H}^V_y(z) \& x \leq_t z)\) with Gödel number \(k\) and \((\exists w)(H^V_y(w) \& f \leq_t w)\) with Gödel number 1). Then take \(P'n\) to be \((x)(f)(w)(z)(\mathcal{H}^V_y(z) \& e_1^V(w) \& x \leq_t w)\) with Gödel number 1] \(\rightarrow P(n,f,x))\). Take \(Q'n\) to be \((\exists x)(\exists f)(\exists w)(\exists z)\) \((Q(n,x,f) \& H^V_y(w) \& x \leq_t z)\) with Gödel number \(k\) and \(f \leq_t w\) with Gödel number 1). Here, \(e_1,e_2,k\), and \(l\) are constants, not variables.

Now, \(Q'n\) is \(\Sigma^1_1\). Also, \((n)(Q'n \rightarrow \mathcal{W}^V(n))\), since \(Q'n = P'n\). So define \(Rm = (\exists n)(Q'n \& |m^V| \leq |n^V|)\). So clearly \(\mu_1(m)(Rm \rightarrow \mathcal{W}^V(x))\), and \(R\) is \(\Sigma^1_1\). So by Lemma 6, \((\exists p)(\mathcal{W}^V(p) \& ~\mathcal{R}p)\). Fix such a \(p\). So \(\mathcal{W}^V(p) \& (n)(Q'n \rightarrow |n^V| \leq |p^V|)\). We wish to form \(\langle r, s |\mathcal{Q}'s \& |s| \leq \text{the segment of } p^V \text{ up to } r |\rangle\). We can, using pure \(\Delta^1_1\)-CA, since this is also equivalent to \(\langle r, s |\mathcal{P}'s \& ~\langle\text{the segment of } p^V \text{ up to } r |\rangle |s| \rangle\), form this (these) set(s). Call this set \(S\). Let \(T = \{ r |(\exists s)(\langle r, s \rangle \in S) \}. \) By \(\mathcal{W}^V(p)\), let \(p_0\) be the \(<_p^V\)-least
upper bound of this set. Let \( q \) have \( W^q(q) \), with \( |q| = |p_0| \). If there is no upper bound for \( T \), take \( q = p \). In any case, clearly \((n)(q'n \rightarrow |n^q| < |q^q|)\). Also, clearly \((\ell)(\exists n)(q'n \rightarrow |n^q| \geq |\text{the segment of } q \text{ up to } \ell|)\). Hence, clearly \((\ell)(\exists z)(R^q_q(z))\).

We wish to show \((\exists w)(H^q_q(w))\). The only hard case is when \( q^q \) has no greatest point. But we can form, in pure \( \Delta^1_1-CA \), \([\langle \ell, j \rangle | (\exists z)(R^q_q(z) \& j \in z)]\), since it, besides this \( \Sigma^1_1 \) definition, has the \( \Pi^1_1 \) definition \([\langle \ell, j \rangle | (z)(R^q_q(z) \rightarrow j \in z)]\), and we may, arithmetically in this set, get a \( w \) with \( H^q_q(w) \).

Finally, we claim that the sequence (not necessarily unique) needed to verify the relativized \( \Sigma^1_1-DC \) to Hyp\( ^q \), can be defined arithmetically in \( w \) with \( H^q_q(w) \), and hence would have what we wanted all along, namely satisfying Hyp\( ^q \). To see this, it suffices to show that any finite sequence \( n_0, \ldots, n_k \) satisfying the conditions in the definition of \( P_n \) can be extended to \( n_1, n_1, \ldots, n_{k+1} \), satisfying \( P_n \), and also that there are sequences satisfying the definition of \( P_n \) to begin with. To see the latter, let \( f \leq_T z \) with \( H^e_q(z), W^e_q(e) \). Take \([k | f \leq_T (z)_k]\), and let \( k \) be a l.u.b. Take \( s \) with \( |g^q| = |\text{the segment of } e^q \text{ up to } k \) with an appropriate finite number of points added on top so as to make, by the Lemmas, \((\exists w)(R^s_q(w) \& z \leq_T w)\). If there is no l.u.b., take \( s = e \). Take \( n_0 = s \). Then the sequence \( <n_0> \) satisfies the
the definition of $P_n$. The same trick allows one to see how to extend a sequence $n_0, n_1, \ldots, n_k$ satisfying $P_n$ to an $n_0, n_1, \ldots, n_k, n_{k+1}$ satisfying $P_n$, for one uses the fact that $(f)(\exists g)(m)\models T$ relativizes, and use that $\langle n_0, n_1, \ldots, n_k \rangle$ satisfies (the definition of) $Q_n$.

4. **Independence Result.** We wish to show here that $\Sigma^1_1$-$\text{DC}$ is independent of $\Sigma^1_1$-$\text{AC}$. It suffices to show that $\Sigma^1_1$-$\text{AC} \not\subseteq \Sigma^1_1$-$\text{DC}$. We do this by showing that, for a suitably chosen sentence $S$ with $\text{Con}(S + \Sigma^1_1$-$\text{DC})$, we can prove in $S + \Sigma^1_1$-$\text{DC}$ that $\text{Con}(S + \Sigma^1_1$-$\text{AC})$. For then, if $\Sigma^1_1$-$\text{DC} = \Sigma^1_1$-$\text{AC}$, then $S + \Sigma^1_1$-$\text{DC}$ proves $\text{Con}(S + \Sigma^1_1$-$\text{DC})$, and hence by G"odel's theorem, $S + \Sigma^1_1$-$\text{DC}$ would be inconsistent.

**Lemma 1:** If $S$ is any sentence in prenex form starting with universal number and set quantifiers from left, followed by existential number and set quantifiers, followed by a matrix containing only number quantifiers, then $\Pr(\Sigma^1_1$-$\text{DC} + S$, "$\exists$ a (coding into natural numbers) $\mathcal{M}$-model satisfying $S$").

Proof: Collapse all the universal number quantifiers in the left part of $S$, and collapse all the existential number quantifiers on the right part of $S$ (before the matrix) so that $S$ becomes $S' = (n)(x)(\exists m)(\exists y)A(n,x,m,y)$. Then push $(\exists m)$ into the matrix, to get $(n)(x)(\exists y)B(n,x,y)$.

Now in set theory, given two $\mathcal{M}$-models $\mathcal{M}$ and $\mathcal{N}$, we can talk of $\mathcal{B}$ being a closure of $\mathcal{N}$ under the sentence $(z)(\exists w)C(z,w)$, i.e., $\mathcal{N} \subseteq \mathcal{B}$ and $(z)(z \in \text{Dom}(\mathcal{N}) \rightarrow (\exists w) (w \in \text{Dom}(\mathcal{B}) \& C(z,w)))$. 
Instead of talking of real w-models, talk of codings of them into a set of natural numbers in a natural (arithmetical) way. Then we can prove in $\Sigma^1_1$-DC + $S$ that for any $n$ and for any finite w-model, $\exists$ a finite closure of this w-model under the $n$ sentences $(x)(\exists y)B(k,x,y), k < n$. This depends heavily on that $(n)(x)(\exists y)B(n,x,y)$ is provable in $\Sigma^1_1$-DC + $S$.

Now, consider the arithmetical predicate $D(z,w) = "z$ is a (coding of) pair $(k,x)$, $w$ is a (coding of) pair $(k+1,y)$ where $y$ is a (coding of) finite w-model which is a closure of the (coding of) finite w-model $x$ under the $k$ sentences $(x)(\exists y)B(p,x,y), p < k+1."$ Now applying $\Sigma^1_1$-DC to this predicate $D$, we obtain a sequence of dependent choices in which the desired w-model of $S$ can be obtained recursively.

Now suppose we found a true sentence $S$ in the form for Lemma 1, such that $I + S$ proves $\Sigma^1_1$-AC. Then we would be done, since $I + S$ would be provably consistent in $\Sigma^1_1$-DC + $S$ (remember all w-models provably satisfy $I$) and $\Sigma^1_1$-DC + $S$ is consistent, since $S$ is true.

By well-known techniques, ReCA is finitely axiomatizable, since we need only consider the case of two parameters. We take $S$ to be the conjunction of this finite axiomatization with the sentence $(x)(n)(W^X(n) \rightarrow (\exists y)(H_{n^X}^X(y))))$. Then $S$ is clearly of the proper form.

We define $(W^*)^X(n) = RLO^X(n) \& (y)(Hyp^X(y) \rightarrow \exists$ an $n^X$-least element of $y$, provided $y \not\in \emptyset$).

We know that $S + I \supset ReCA + I$, so we can use Lemmas of the previous chapter about provability in ReCA + I.
Lemma 2: The following is provable in $S + I$: for every $n$ with $W^X(n)$, there is a coding of a function $f: w \rightarrow P(w)$ such that whenever $(W^*)(X(m) \& \sim W^X(m))$, we have $f(m)$ is an imbedding of $n^X$ onto an initial segment of $m^X$.

Proof: We can prove, like Lemma 2 of the previous section, in $I + ReCA$, that if $(W^*)(X(n) \& (\exists z)(H^X_x(z)))$, then the conclusion of the Lemma is true. But $S$ guarantees that $(\exists z)(H^X_x(z))$ just on the basis of $W^X(n)$.

Now suppose $(n)(\exists f)(m) \sim T(e, 0, m, f, n, x)$. The predicate of $p$, $(\exists z)(H^X_x(z)) \& Reas^X(p)$ is $\Sigma^1_1$, and so, since every $p$ with $W^X(p)$ satisfies it, $\exists k$ such that $(\exists z)(H^X_x(z)) \& Reas^X(k) \& \sim W^X(k)$. Then by Lemma 5, $(z)(H^X_x(z) \rightarrow \sim Hyp^X(z))$. Clearly we have $(n)(\exists f)(\exists z)[H^X_x(z) \& (m) \sim T(e, 0, m, f, n, x)]$. Collapsing the quantifiers in the usual way, and putting the result in Kleene Normal Form, we end up with $(n)(\exists h)(m) \sim T(q, 0, h, n, x)$. Furthermore, any Skolem function for this sentence would have, recursive in it, a Skolem function for $(n)(\exists f)(m) \sim T(e, 0, m, f, n, x)$. So it remains to show, in $S + I$, that $(n)(\exists h)(m) \sim T(q, 0, m, h, n, x)$ has a Skolem function.

Now, there is a standard recursive function $F$ such that $(n)(RLO^X_F(n))$ & any set for which there is no $F(n)^X$-least element, has recursive in it a solution $h$ to $(m) \sim T(q, 0, m, h, n, x)$ & any solution $h$ has recursive in it a set for which there is no $F(n)^X$-least element. So $(n)((W^*)(X(F(n))) \& \sim W^X(F(n)))$.

Now consider the $\Sigma^1_1$-predicate $P_F = RLO^X(k) \& (\exists f)(n)(f$ a coding of a function from $w$ into $P(w)$ such that $f(n)$ is
an imbedding of $k^X$ into $F(n)^X$). By Lemma 2 of this section, 
this $\Sigma^1_1$ predicate holds of all $k$ with $W^X(k)$, and so holds 
for some $r$ with $\neg W^X(r)$, by Lemma 6. Then we have a function 
f such that for each $n$, $f(n)$ is an imbedding of $r^X$ into 
$F(n)^X$. Let $X$ be any non-empty set for which there is no 
r$^X$-least member. Then recursively in $X$, $f$, we can find a 
coding $g$ of a function $\omega \rightarrow P(\omega)$, such that, for each $n$, 
g(n) is a non-empty set for which there is no $F(n)^X$-least 
element. Hence, by the special property of the recursive 
function $F$, we may obtain the desired Skolem function for 
$(n)(\exists h)(m)\& (q, 0, m, h, n, x)$ recursively in $g$.

Thus we have verified that $I + S$ proves $\Sigma^1_1$-AC, and we
immediately have

Theorem 2: $\Sigma^1_1$-DC is independent of $\Sigma^1_1$-AC.

5. Relation Between Predicative and Hyperarithmetic Analysis.

$\Sigma^1_1$-AC (or $\Delta^1_1$-CA or $\Sigma^1_1$-DC) are considered reasonable 
formulations of so-called hyperarithmetic analysis, in view of 
the fact that they are natural systems whose minimum $\omega$-model 
consists of exactly the hyperarithmetic sets of natural numbers.

In this section we compare $\Sigma^1_1$-AC with a system $T$ which 
represents the formalization of a small part of predicative 
analysis (see Feferman, [1]).

The system $T$ is $I + ReCA +$ the infinite list of axioms 
$(y)(\exists x)(H^Y_{k_n}(x))$, where $n$ varies, and $k$ is fixed, and $k$ 
is the Gödel number of a natural well-ordering of type $\varepsilon_0$. 
Thus the infinite list of sentences is obtained by changing \( n \).

We will show that \( \Sigma^1_1\)-AC is a conservative extension of \( T \) with respect to all purely \( \Pi^1_2 \) sentences.

(ReCA is formalized as one sentence, in the standard way, i.e., using only 2 parameters \( x_0, x_1 \).)

It is easily seen by well-known techniques that the proof of conservative extension can be made finitary. So we obtain a finitary proof of \( \text{Con}(\Sigma^1_1\text{-AC}) \) relative to \( \text{Con}(T) \). Finitary generally means here, in PRA (primitive recursive arithmetic).

A widely used index of complexity of an axiomatic theory is how large is the least upper bound of its provable ordinals. In the present context, an ordinal \( \alpha \) is said to be a provable ordinal of a given fixed theory, if there is an \( n \) with \( \text{RLO}(n) \) and \( n \) has order type \( \alpha \) and the theory proves \( W(n) \). In view of the conservative extension for \( \Pi^1_2 \) that we will prove, it is clear that the provable ordinals of \( \Sigma^1_1\text{-AC} \) are exactly the provable ordinals of \( T \).

It follows from work of Feferman [1] and Tait [5] that the least upper bound on the provable ordinals of \( T \) is the ordinal represented by the \( \varepsilon_0 \)-th critical function at 0. (See Feferman [1], pp. 14-16). Furthermore, it also follows from their work that in PRA + rule of primitive recursive induction on the natural RLO corresponding to the \( \varepsilon_0 \)-th critical function at 0, a consistency proof may be given for \( T \). So, by our results, such a consistency proof may be given for \( \Sigma^1_1\text{-AC} \).

By an instance of induction on \( k_n \), we mean a statement of
the form \( \{ A(q) \land (p)([p < k n \land (m)(m < k p \rightarrow Am)] \rightarrow Ap) \} \rightarrow (p)(p < k n \rightarrow Ap) \), where \( \tilde{q} \) has \( O_k(q) \), and \( A \) is any formula (in the language of Analysis, described at the beginning of this chapter), with possibly free variables, and \( A \) is in prenex form.

By the complexity of a formula in prenex form, we will mean here the total number of quantifiers occurring. Let \( \text{Comp}(A) \) be the complexity of \( A \).

For each integer \( m \), there is a natural predicate \( T_m(n) \) on Gödel numbers of formulae of complexity \( \leq n \), such that \( T_m(n) \) says that the formula with Gödel number \( n \) is true. The formalization of these truth predicates involve placing formulae in a weak Kleene normal form (i.e., no attention is paid to the form of the quantifiers, but only that the matrix be the T-predicate of the appropriate variables).

By the reflection principle, for a theory \( S \), (of this language) of complexity \( \leq m \), we mean the single sentence

\[
(n)([n \text{ Gödel number of a sentence of complexity } \leq \bar{m} \land \text{Pr}(S,n)] \rightarrow T_m(n)). \quad \text{We call this } R_m(S).
\]

We need some facts from proof theory which follow from work of Tait (see [5]), Feferman (see [1], p. 23) and Kreisel (see [4]).

**Fact 1:** ∃ primitive recursive functions \( F, G, \) and \( H \) such that the following are provable in PRA:

1) For each \( m,n > 0 \), \( F(m,n) \) gives the Gödel number of an instance of induction on \( \tilde{K}_G(n) \) applied to a formula of complexity \( \leq H(m) \), with no parameters.
2) For each $m,n$ we have that $R_m(I_n)$ is provable in the theory $I_n +$ the formula represented by $F(m,n)$. $I_n$ is defined as the subsystem of $I$ consisting of induction applied only to formulae of complexity $\leq n$.

Let $J_p$ be the sentence "for every purely $\Sigma_1$ predicate $P_n$ (possibly with parameters), $(\exists l)(q)((q < l \rightarrow P_q) & (l = p \lor (\exists k (l \leq k \land (r)(r \geq k \rightarrow \neg P_r)))))." We let $T' = T +$ the infinite list $J_p$.

In the theory $T$, the axioms $(\exists x)H(x)$, as $n$ varies, have bounded complexity. Choose $c'$ such that the conjunction of any sentence $(\exists x)(H_n(x))$ with ReCA and any purely $\Sigma_2$ sentence and any $J_p$, has complexity $\leq c'$, some fixed constant $c'$. We are interested in Fact 1, when $m = c'$. We let $c = H(c')$. We define new theories $T'_n = I_n + (\exists x)H_n(x) +$ ReCA $+ J_n$. Applying Fact 1 for $m = c'$, we get

**Fact 2:** $\exists$ primitive recursive functions $F$ and $G$ such that the following are provable in PRA:

1) For each $n > 0$, $F(n)$ gives the G"odel number of an instance of induction on $\bar{k}_n G(n)$ applied to a formula with no parameters of complexity $\leq c$.

2) For each $n > 0$, we have that $R_m(I_n)$ is provable in the theory $I_n +$ the formula represented by $F(n)$.

Now, since $\text{Comp}(A) = \text{Comp}(\neg A)$, we see that $R_m(I_n)$ formally implies $\text{Con}(T'_n + B)$ within $T'_n + B$, where $p$ is any
integer, and $B$ is any purely $\Sigma_2^1$ sentence. So hence the formula represented by $F(n)$ must not be provable in $T_n^P + B$ if $T_n^P + B$ is consistent, by Gödel's Theorem. This is so, no matter what $n$ and $p$ are.

Let $T' = \bigcup_{n \in \omega} T_n^P$. Thus

Fact 3: $\exists$ primitive recursive functions $F, G$, such that, in PRA, under the assumption $\text{Con}(T' + B)$, $B$ some fixed $\Sigma_2^1$ sentence, we can derive, for some constant $c$,

$$(n)(p)\text{Con}(T_n^P + B + \neg(\text{the sentence } "F(n)")) \land (n)(F(n) \text{ represents an instance of induction on } k_{G(n)} \text{ applied to a formula with no parameters of complexity } \leq c).$$

Lemma 1: From $\text{Con}(T' + B)$ we can conclude, by finitary means, that $\text{Con}(\Sigma_1^1 - \text{AC} + B)$.

Proof: It suffices to show $(n)\text{Con}(\Sigma_1^1 - \text{AC with only induction } I_n + B)$.

For each $n$, we get a model for "$B + \Sigma_1^1 - \text{AC with only } I_n$" by applying the inner model technique to the theory $T_G(n + c^3) + B + \neg(\text{the sentence } "F(n + c^3)"))$.

Now $F(n + c^3)$ represents a certain instance of induction on $k_{G(n + c^3)}$, and let this induction be applied to $Q_s$, no parameters. Now $B$ says $(\exists f)\text{ satisfies some specific } \Pi_1^1 \text{ property});$ we fix such an $f$. Then we wish to show the relativized of each instance of "$B + \Sigma_1^1 - \text{AC with only induction } I_n$"
to the predicate, $Rx = (\exists s)(\exists y)(H^f_{\frac{k}{k}}(y) \land x \leq \mathfrak{y}_k \land (r)(r \leq \frac{s}{k} \rightarrow \mathfrak{q}_r))$, is provable in $T^G(n+c^3) + B + \sim$(the sentence 
"$F(n+c^3)$"). This would give a consistency proof of "$B + \Sigma^1_1$-AC with only $I_n$" relative to $\text{Con}(T^G(n+c^3) + B + \sim$(the sentence 
"$F(n+c^3)$")). This in turn is immediately generalizable to a
consistency proof of $\Sigma^1_1$-AC + B relative to $\text{Con}(T' + B)$.

Actually, it suffices to consider the case when $n \geq 5$. This lower bound will be convenient later.

So, certainly all instances of $I_n$ provably relative to $R$ since the predicate $R$ has small complexity compared to $c^3$, and $I$ is available in $T^G(n+c^3)$.

Certainly, $B$ provably relativizes, since the numbers, and hence all the arithmetical relations, remain unchanged by taking this inner model, by fiat.

For similar reasons, clearly ReCA holds in this inner model. The inner model is non-empty, since $\mathbb{Q}_k$, where $\mathbb{Q}_k$, since we assumed that induction on $Q$ is false.

To verify the last, and most important axiom of "$B + \Sigma^1_1$-AC with only $I_n$", first note that since $n \geq 5$, we have that, in ReCA + $I_n$ we can prove $W(\mathbb{E})$ by well-known techniques. Hence $(\ell)(\ell \leq \mathfrak{g}(n+c^3) \rightarrow (\exists y)H^f_{\frac{k}{k}}(y))$ is provable in
$T^G(n+c^3)$. From now on, whenever $\ell \leq \mathfrak{g}(n+c^3)$, we denote the unique $y$ above by $H^f_{\frac{k}{k}}$. Then also
(\ell)(p)((l \leq p \& l \leq G(n+c^3) & p \leq G(n+c^3)) \rightarrow H^f\leq T^f_{k\ell})

is provable in $TG(n+c^3)$.

Now suppose that \((p)(\exists g)A(p,g,x)\) holds relativized to the predicate $R$, where we also have $Rx$. We wish to conclude, in the theory $TG(n+c^3) + B + \sim"F(n+c^3)"$ that \(\exists\) Skolem function for the above sentence, $h$, such that $Rh$.

So we have $(p)(\exists g)(A(p,g,x) \& Rg)$, $A$ arithmetical. But consider an integer $s$ satisfying the properties in the definition of $Rg$. Since induction fails on $G(n+c^3)$ for $Q$, we see that $s < G(n+c^3)$. It is straightforward to see, using $W(k)$ and $ReCA$, that $1) (p)(\exists s)(s < G(n+c^3) \& (\exists g \leq T^{f}_{k}s)^{(A(p,g,x)) \& (t)(t < s \rightarrow H^{f}_{k} \leq T^{f}_{k}t)}$

$(A(p,g,x)) \& (t)(t < s \rightarrow H^{f}_{k} \leq T^{f}_{k}t)$ does not have a $g \leq T^{f}_{k}s$ with $A(p,g,x))$. It is also easy to see that the part of this sentence to the right of $(\exists s)$ can be written in a natural way in $\Sigma^1_1$ form (since $t < s \rightarrow H^{f}_{k} \leq T^{f}_{k}t$). Hence consider the $\Sigma^1_1$ predicate $Ps$ which holds iff $(\exists p)(\text{stuff to the right of } (\exists s))$ holds of $p$ and $s$. It is clear that $(s)(Ps \rightarrow [(s < G(n+c^3) \& (r)(r < s \rightarrow Qr)]$. Applying the axiom $J_{G(n+c^3)}$ we get a l.u.b. in $k$, call it $k_0$, on the $s$ with $Ps$. Now $k_0 < G(n+c^3)$. Then clearly $(r)(r < k_0 \rightarrow Qr)$. So if $k_0 = G(n+c^3)$, then $(r)(r < G(n+c^3) \rightarrow Qr)$, contradicting
that induction fails on $G(n+c^3)$ with $Q$.

So $k_0 < \overline{G(n+c^3)}$. Hence, since $(r)(r < k_0 \rightarrow qr)$, we have $(r)(r < k_0 \rightarrow qr)$. A similar argument shows that $G(n+c^3)$ cannot be reached by a finite number of iterations of successor from $k_0$, in the ordering $\overline{k}$. But by sentence 1), it is clear that a Skolem function for our original sentence can be found recursive in at most a few jumps of $H_{k_0}^k$. Hence a Skolem function can be found satisfying the predicate $R$.

In retrospect what we have shown is that for each sufficiently large $n$, "$\Sigma^1_1$-AC with only $I_n + B$" is consistent if

$$T_{G(n+c^3)} + B + \neg(F(n+c^3))$$

is. Consequently any $\Pi^1_2$ sentence provable in $\Sigma^1_1$-AC must also be provable in $T'$. We now observe

**Lemma 2**: $T' = T$.

**Proof**: It suffices to show that each $J_p$ is provable in $T$. But this is clear, since it is well known that $T$ (or even just $I + ReCA$) proves induction for any formula on $k_p$ for each $p$.

**Lemma 3**: $T \subseteq \Sigma^1_1$-AC.

**Proof**: We have to show that for each $p$, the sentence

$$(y)(\exists x)H^Y_{k_p}(x)$$

is provable in $\Sigma^1_1$-AC. Let $A(q)$ be the predicate $(y)(\exists x)H^Y_{k_q}(x)$, and we apply induction to $A$ on the ordering $\overline{k_p}$. (Remember, the full schema of induction on $\overline{k_p}$ is provable in $\Sigma^1_1$-AC). If $Suc_1(r,q)$, then clearly $A(q) \rightarrow$
A(r), by taking jumps, using ReCA. Suppose \( \lim_{k \to \infty} (q) \& (r) \)

\( (r < q \to A(r)) \). Then \( (r)(r < q \to (\exists x) H^y_k(x)) \). We can form a Skolem function for this sentence, within \( \Sigma^l_1 \)-AC. When we do, we can find, recursive in at most a few jumps of the Skolem function, a \( z \) with \( H^y_k(z) \). This is true of all \( y \), and so we have \( A(q) \). Hence by induction on \( k \), we have

\( (q)(q < p \to A(q)) \). By a similar argument to the above, we obtain \( (y)(\exists x) H^y_k(x) \).

We have immediately

**Theorem 3:** \( \Sigma^l_1 \)-AC is a conservative extension of \( T \) for \( \Pi^l_2 \) sentences.
CHAPTER III

In this chapter, we consider the question of just which recursive linear orderings can have certain structures placed on them; namely, hierarchies. It is convenient to consider more general notions of recursive linear orderings and hierarchies, than in Chapter II.

We say, in this chapter, $RLO^+(n)$ iff $\varphi_e$ defines a recursive linear ordering (as in Chapter II) whose field is recursively enumerable. For any $p \in \text{Field}(n)$, we let $n_p$ be the name for the subordering of $n$, whose field is all $q$ with $q < n^p_p$. Thus $n_p$ itself is an $RLO^+$.

We define $H^+(x)$ as 1) $(k)(k \in x \rightarrow (\exists m)(k \in (x)_m \land m \in \text{Field}(n))$. 2) $(q)(q \in \text{Field}(n) \rightarrow [(0_n(q) \land \emptyset \leq_T (x)_q) \lor (\text{Suc}_n(q,r) \land (x)_r^{(1)} \leq_T (x)_q) \lor (\text{lim}_n(q) \land [P(r,s) \land s < nq \land r \in (x)_s] < (x)_q)]).

Note that in ReCA + I, we can define the satisfaction relation for (codings of) $\omega$-models. We can do this, since we can prove in ReCA + I that $(x)(\exists y)(y = x^{(\ell)})$. Given $H_e(x)$, we define the corresponding $M_x$ (a coding of an $\omega$-model) as (a coding of) the sets $\leq_T$ in some $(x)_m$, $m \in \text{Field}(n)$. We define $\text{Reas}^+(e)$ the same as in Chapter II, except replace $RLO(e)$ by $RLO^+(e)$. We define $\text{lim}(e)$ as $(p)(\exists q)(p \in \text{Field}(e) \rightarrow p < e_q)$.

We define $W^+(e)$ and $W^*+(e)$ the same as in Chapter II, except replace $RLO(e)$ by $RLO^+(e)$. 
We say that NW(e,X) iff RLO\(^+(e)\) & e is not well-founded with respect to X, i.e., 1) \((\exists n)(n \in X & n \in \text{Field}(e))\) 2) there is no e-least member of X.

**Lemma 1:** Let x have \(H_e^+(x), \text{Reas}^+(e), \text{lim}(e)\). Then for corresponding \(M_x\), we have \(M_x \models W^+(e)\). In fact this Lemma is provable in I + ReCA.

**Proof:** We consider the theory \(T = I + \text{ReCA} + (\exists x)(\exists e)\) (they satisfy hypotheses of the Lemma but not the conclusion).

We will show that T proves its own consistency.

In the theory T, fix such an x, e and \(M = \) corresponding \(M_x\). We will show, in T, that this \(M \models T\).

It is clear that \(M \models I\).

Also, since \(\text{lim}(e)\), we have \(M \models \text{ReCA}\).

It remains to show \(M \models (\exists z)(\exists n)(H_n^+(z) & \text{Reas}^+(n) & \text{lim}(n) & \text{corresponding } M_z \models \neg W^+(n))\).

Since \(M \models \neg W^+(e)\), there is an r and an X with \(H_r(X) & M = W(r) \leq_T X\) there is a set Y with NW(e,Y).

Fix such an r. Now choose s such that 1) \(\text{Lim}(s)\). (This is taken to imply implicitly \(s \in \text{Field}(e)\). 2) NW(Y, e_s).

We can do this, using Reas(e). Consider \((y)_s\). There is a set \(z \leq_T (y)_s\) with \(H_{e_s}^+(z)\), and with \((t)(t < e_s \rightarrow (z)_t = (y)_t)\). We set \(n = e_s\). We claim that for these values, our claim about satisfaction in \(M\) holds. For, since \(M \models \neg W(e_s)\) & \(\text{Reas}^+(e) & (\text{ReCA} + I)\), and \(M \models H_r(X) & W(r)\), we see that X, and hence Y, must be in the model \(M_z\), and \(M_z\) certainly \(\models W(r)\). So \(M_z \models \neg W^+(e_s)\). So certainly \(M \models (M_z \models \neg W^+)\).
Note that $M_z \in M$, since $\lim(e)$.

This completes the proof that $M \models T$.

Hence $T$ proves $\text{Con}(T)$, since the Soundness Theorem can be formalized and proved in ReCA + I.

So, by Gödel's Theorem, $T$ is inconsistent. So the Lemma has been established. In fact, in view of the inconsistency, the Lemma is provable in ReCA + I.

**Theorem 1:** If $\text{Reas}^+(e)$ and $\neg W^+(e)$ (of course, in view of $\text{Reas}^+(e)$, $W^*$ and $W^+(e)$ are identical notions, since $\text{RLO}^+(e)$), then there is no $x$ with $H^e(x)$.

**Proof:** If $\lim(e)$, then if there was such an $x$, then corresponding $M$, by Lemma 1, has $M_x \not\models W^+(e)$. However, since every hyperarithmetic set is in $M$ (because $\neg W^+(e)$) we must have $M_x \models \neg W^+(e)$, which is a contradiction.

If $\neg \lim(e)$, then by $\text{Reas}(e)$, there is an $m$ with $\lim_e(m)$ & there are only finitely many $n > e_m$. So $\neg W^+(e_m)$. Then argue as above.

Let $\text{NTWO}(n)$ be $\text{RLO}^+(n)$ & no tail of $n$ is well-ordered, i.e., $(p)(p \in \text{Field}(n) \rightarrow \text{the subordering on } \{q|q \geq p\} \text{ is not well-founded}).$

**Lemma 2:** If $\text{Reas}^+(e)$ & $H^e(x)$ & corresponding $M_x \models \text{NTWO}(n)$, then $M_x \models I + \text{ReCA} + S$, where $S$ is the sentence of Chapter II.

**Proof:** Let $z \in M$. We wish to show $M_z \models \text{ReCA} + I + (n)(W^Z(n) \rightarrow (\exists w)(H^Z_n(w)))$. Clearly $\lim(e)$, since $M_z \models \text{NTWO}(n)$. Hence $M_z \models I + \text{ReCA}$. 
Clearly \((\exists s)(s \in \text{Field}(e) \& z \leq_T (x)_s)\). Let \(n \in \text{Field}(e)\) such that the subordering of \(e\) defined on \([q|s <_e q < e n]\) is satisfied not to be well-founded in the model \(M_x\). Then \(M_x\) satisfies that there is a hierarchy \(y\) (in the generalized sense of this chapter) on a non-well-founded \(RLO^+\), \(k\), the subordering of \(e\) defined above, such that \((y)_p \geq_T z\), where \(O_k(p)\). Then the obvious generalization of Lemma 5 in Chapter II gives \(M_x \models (n)(w^Z(n) \rightarrow (\exists w)(H^Z_n(w) \& w \leq_T y))\), in view of \(M \models I + \text{ReCA} + (\exists x) \sim \text{NW}(k, x)\).

**Lemma 3:** The following is provable in \(I + \text{ReCA} + S\): If \(\text{NTWO}(n) \& W^+(n) \& p \in \text{Field}(n)\), then \((\exists q)(q > n p \& \text{NTWO}(nq))\).

**Proof:** Choose \(r \in \text{Field}(n)\) with \(r > n p\) and the subordering of \(n\) determined by \([s|p \leq_n s < n r]\) is not well-founded. We can do this since \(\sim W^+(n)\).

Now consider the \(\Pi^1_1\) predicate, \(P_t = t < n r\) & the subordering of \(n\) from \(t\) to \(r\) is well-founded. If \(P\) has no solutions, we are done, for then \(\text{NTWO}(n_r)\).

Harrison (see [2]) has shown that every \(\Pi^1_1\) predicate which has a solution in \(\text{Field}(n)\), where \(W^+(n)\), has an \(n\)-least solution. His proof uses only principles provable in \(I + \text{ReCA}\), excepting the comparability of recursive well-orderings. In Chapter II, we showed this comparability Lemma is provable in \(I + \text{ReCA} + S\), by Lemma 2 of Section 2.

So there is an \(n\)-least solution to the predicate \(P_t\), call it \(q\), and we can prove this in \(I + \text{ReCA} + S\). By the way \(r\) was chosen, it is clear that \(q > n p\). And by the way
q is defined, it is clear that $\text{NTWO}(q)$.

**Lemma 4:** Let $\text{Reas}^+(e)$, $H^+_e(x)$, $\lim(e)$. Then corresponding $M_x \models \neg\text{NTWO}(e)$.

**Proof:** Consider the theory $T = I + \text{ReCA} + (\exists e)(\exists x)$ (they satisfy hypotheses, but not conclusion). As in Lemma 1, we wish to show that $T$ proves its own consistency.

So, we argue in $T$, that if $e$ and $x$ are chosen so that they violate this Lemma, then let $M = M_x$, and we will show, in $T$, that $M \models T$.

Clearly, as in Lemma 1, $M \models I + \text{ReCA}$.

It remains to show $M \models (\exists z)(\exists n)(H^+_n(z) & \text{Reas}^+(n) & \lim(n) & \text{corresponding } M_z \models \text{NTWO}(n))$.

By Lemma 2 of this chapter, $M \models S + I + \text{ReCA}$. Hence $M \models \Sigma^1 - \text{AC}$. Now for each $p \in \text{Field}(e)$ with $M \models \text{NTWO}(e_p)$, we have $M \models (k)(k < e_p \rightarrow (\exists X)\text{NW}(e_p, X))$. Hence $M \models (\exists Y)(k)(k < e_p \rightarrow \text{NW}(e_p, (Y)_k))$.

By Lemma 3 of this chapter, $M \models (p)(p \in \text{Field}(e) \rightarrow (\exists q)(q > e_p \& (\exists Y)(k)(k < e_q \rightarrow \text{NW}(e_q', (Y)_k))))$, since $M \models I + \text{ReCA} + S + W^*_+(e)$. So again, by $M \models \Sigma^1 - \text{AC}$, we obtain a $Z \in M$ with $M \models (p)(p \in \text{Field}(e) \rightarrow (Z)_p$ is $(Y,q)$ with $q > e_p \& (k)(< e_q \rightarrow \text{NW}(e_q', (Y)_k)))$.

Fix such a $Z$, and let $r$ have 1) $r \in \text{Field}(e)$, 2) $z \leq_T (x)_r$. Let $s > e_r$ with $M = \text{NTWO}(s)$. Consider $(x)_s$. There is a natural $y \leq_T (x)_s$ such that $H^+_e(y)$, and $(t)(t < e_s \rightarrow (y)_t = (x)_t)$. We set $z = y$, $n = e_s$, in our claim about satisfaction in $M$. 
Clearly $\lim(e_s) & \text{Reas}^+(e_s)$. It remains to show corresponding $M_y \models \text{NTWO}(e_s)$.

We have $M \models \text{NTWO}(s)$, and $Z \in M_y$. So every set $\leq_T Z$ is in $M_y$. Hence $M_y \models \text{NTWO}(e_s)$.

This completes the proof of the self-consistency proving of $T$, and hence the inconsistency of $T$. Hence Lemma 4 must be true.

Lemma 5: There is an $e$ with $W^*(e) & \text{Field}(e) = \{n|n$ is even} such that 1) $(\exists x)H^+_e(x)$, 2) $(y)(H^+_e(y) \rightarrow (\exists x)(x \leq_T y^{(10)} & \text{NW}(e,x)))$.

Proof: We define a total recursive function $F$ on indices of the partial recursive functions. We define $G(n)$ to be the Gödel number of the RLO (field $w$) associated with the $\Pi^1_1$ sentence "$\neg \text{RLO}(n) \vee (x)(\neg H^+_n(x))$." So for every $n$, $G(n)$ is the Gödel number of some RLO. Let $F(n)$ be the RLO$^+$ with domain $\{n|n$ is even$\}$ defined by $p < G(n)q$ iff $2p < F(n)^2q$.

By the recursion theorem, there is an $e$ with $\varphi_e = \varphi_{F(e)}$. Fix such an $e$. Then $e$ is the Gödel number of an RLO$^+$ whose field is $\{n|n$ is even$\}$. It is clear that $(x)(\text{NW}(e,x) \rightarrow (\exists y)(H^+_e(y) & y \leq_T x^{(10)}))$ & $(y)(H^+_e(y) \rightarrow (\exists x)(\text{NW}(e,x) & x \leq_T y^{(10)}))$. Hence $W^*(e)$. For, if not, then $\neg W^*(e)$, and $(\exists x)(\text{NW}(e,x) & \text{Hyp}(x))$, contradicting the 1st conjunct of the above conjunction. We claim $\neg W(e)$. For, if not, then $(\exists y)(H^+_e(y))$, contradicting the 2nd conjunct of the above conjunction. So by the 1st conjunct, we have $(\exists y)(H^+_e(y))$. So $e$ has all the properties stated in this Lemma.
Theorem 2: There are \( n \) with \( W^*(n) \) (hence \( \text{Field}(n) = \omega \)) such that \( (x)(\neg H^+_n(x)) \).

Proof: Take \( e \) as in Lemma 5. Take \( n \) to be the natural RLO with 1) the ordering \( e \) is an initial segment, 2) \( \text{Field}(n) = \omega \), 3) the ordering \( n \) corresponds to \( e \times \omega \) (i.e., \( \omega \) copies of \( e \)).

Now suppose \( H^+_n(x) \). Then \( M_x \models \text{NTWO}(n) \) by Lemma 4. But \( (\exists y)(NW(e,y) \land y \in M_x) \). Hence there is a \( z \) which can be found recursively in such a \( y \), with the property that no tail of \( n \) is well-founded with respect to \( z \). And \( z \in M_x \). But this contradicts \( M_x \models \neg \text{NTWO}(n) \).