Classification and Uniqueness of Invariant Geometric Flows [1]

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Abstract – In this note we classify geometric flows invariant to subgroups of the projective group. We proof that the geometric heat flow is the simplest of all possible flows. These results are based on the theory of differential invariants and symmetry groups.

Classification et Unicité des Flux Géométriques Invariants

Résumé – Dans cet article on classifie des flux géométriques, qui sont invariants aux sousgroupes du groupe projectif. On prouve que l'équation de la chaleur géométrique est la plus simple possible. Les résultats se basent sur la théorie des invariants differentiels et des groupes de symmétrie.

Version francaise abrégée

Etant donné $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$, une courbe plane simple, et r, la longueur d'arc d'un groupe G, le flux de la chaleur géométrique invariant au groupe G est donné par l'équation

$$\frac{\partial \mathcal{C}}{\partial t} = \frac{\partial^2 \mathcal{C}}{\partial r^2}. \tag{1}$$

Un flux géométrique plus général est obtenu en incorporant la courbure du group

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \Psi(\chi, \frac{\partial \chi}{\partial r}, \ldots, \frac{\partial^n \chi}{\partial r^n}) \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p). \end{aligned} \tag{2}$$

Pour ces flux on prouvera les résultats suivants:

Lemme 1 – *Localement on peut répresenter la solution de (1) comme $y = u(x,t)$, et l'évolution est donnée par*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2},$$

quand $g = dr/dx$.

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Theoreme 1 – *Pour tout sousgroupe du group projectif on maintient:*

1 Tout invariant différentiel est une fonction

$$I\left(\chi, \frac{d\chi}{dr}, \frac{d^2\chi}{dr^2}, \ldots, \frac{d^n\chi}{dr^n}\right).$$

2 Toute évolution invariante est donnée par

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2}\, I. \tag{3}$$

Corollaire 1 – *Si G est un des groupes suivants: Euclidien, affine, affine spécial, similaire et projectif, donc le flux suivant est le seul invariant d'ordre minimal:*

$$\frac{\partial u}{\partial t} = \frac{c}{g^2}\frac{\partial^2 u}{\partial x^2},$$

où c est une constante.

La prémière partie du Theoreme 1 ne necessite pas que G soit un sousgroupe de $\mathrm{SL}(\mathbf{R}, 3)$. Cette condition est nécessaire pour la déuxième partie. La classification peut être construite par d'autres groupes, mais les relations seront plus compliquées.

L'unicité de l'équation de la chaleur géométrique pour le groupe Euclidien et le groupe affine, est demonstrée en [1] avec une autre méthodologie et sous certaines conditions. Le résultat presenté ici est plus général, parce qu'il n'implique pas une analyse independante pour chaque sousgroupe (au contraire du résultat en [1]). On considère aussi que l'interprétation géométrique presenteé ici est claire, intuitive, et décrit naturellment des flux invariants en fonction des invariants primaires de la géométrie differentielle: la longueur d'arc et la courbure.

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1 Introduction

Let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ ($\mathcal{C} = [x,y]^T$) be a family of simple planar curves, where p parametrizes the curve and t the family (p and t are independent). Assume that we want to formulate an intrinsic *geometric heat flow* for plane curves which is invariant under a certain transformation group G. This type of flows replace the classical heat flow, which is equivalent to Gaussian smoothing, and frequently used in image analysis. Let r denote the group arc-length, i.e., the simplest invariant parametrization of the group [4, 5]. Then, the *invariant geometric heat flow* is given by [7]

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p). \end{aligned} \tag{4}$$

If G acts linearly, it is easy to see that since dr is an invariant of the group, so is $\mathcal{C}_{rr}$. $\mathcal{C}_{rr}$ is called the *group normal.* For nonlinear actions, the flow (4) is still G-invariant, since $\frac{\partial}{\partial r}$ is the unique *invariant derivative* [4, 5].

The flow given by (4) is non-linear, since the group arc-length r is a time-dependent parametrization. This flow gives the invariant geometric heat-type flow of the group, and provides the invariant direction of the deformation. For subgroups of the full projective group $\mathrm{SL}(\mathbf{R}, 3)$, we show in Theorem 1 below that the most general invariant evolutions are obtained if the group curvature χ, i.e., the simplest non-trivial differential invariant of the group, and its derivatives (with respect to arc-length) are incorporated into the flow:

$$\begin{aligned}\frac{\partial \mathcal{C}(p,t)}{\partial t} &= \Psi(\chi, \frac{\partial \chi}{\partial r}, \ldots, \frac{\partial^n \chi}{\partial r^n}) \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) &= \mathcal{C}_0(p),\end{aligned} \tag{5}$$

where $\Psi(\cdot)$ is a given function. (We discuss the existence of possible solutions of (5) in [7].) Since the group arc-length and group curvature are the basic differential invariants of the group transformations, it is natural to formulate (5) as the most general geometric invariant flow.

2 Uniqueness of Invariant Heat Flows

In this section, we give a result, which elucidates in what sense our invariant heat-type equations (4) are unique. We use here the action of the projective group $\mathrm{SL}(\mathbf{R}, 3)$ on $\mathbf{R}^2$. The proofs are based on the theory of Lie groups, prolongations, and symmetry groups. See [3, 4, 5] for this corresponding background. We will first note that locally, we may express a solution of (4) as the graph of $y = u(x,t)$.

Lemma 1 *Locally, the evolution (4) is equivalent to*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2} \frac{\partial^2 u}{\partial x^2},$$

where g is the G–invariant metric ($g = dr/dx$).

Proof. Indeed, locally the equation

$$\mathcal{C}_t = \mathcal{C}_{rr},$$

becomes

$$x_t = x_{rr}, \quad y_t = y_{rr}.$$

Now $y(r,t) = u(x(r,t),t)$, so

$$y_t = u_x x_t + u_t, \;\; y_{rr} = u_{xx} x_r^2 + u_x x_{rr}.$$

Thus,

$$u_t = y_t - u_x x_t = y_{rr} - u_x x_{rr} = x_r^2 u_{xx}.$$

Therefore the evolution equation (4) reduces to

$$u_t = g^{-2} u_{xx},$$

since $dr = g dx$. □

We can now state the following fundamental result:

Theorem 1 *Let G be a subgroup of the projective group $SL(\mathbf{R},3)$. Let $dr = gdp$ denote the G-invariant arc-length and χ the G-invariant curvature. Then*

1 . *Every differential invariant of G is a function*

$$I\left(\chi, \frac{d\chi}{dr}, \frac{d^2\chi}{dr^2}, \ldots, \frac{d^n\chi}{dr^n}\right)$$

of χ and its derivatives with respect to arc length.

2 . *Every G-invariant evolution equation has the form*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2} I, \tag{6}$$

where I is a differential invariant for G.

We are particularly interested in the following subgroups of the full projective group: Euclidean, similarity, special affine, affine, full projective.

Corollary 1 *Let G be one of the listed subgroups of the projective group $SL(\mathbf{R},3)$. Then there is, up to a constant factor, a unique G-invariant evolution equation of lowest order, namely*

$$\frac{\partial u}{\partial t} = \frac{c}{g^2}\frac{\partial^2 u}{\partial x^2},$$

where c is a constant.

Proof of Theorem (Outline).
Part 1 follows immediately from classical results in the theory of differential invariants [2, 3, 4, 5] (see Theorem 7 in [5]), and the definitions of dr and χ. (Note also that for a subgroup of $SL(\mathbf{R},3)$ acting on $\mathbf{R}^2$, we have each differential invariant of order k is in fact unique, see [3, 4] and Theorem 6 in [5].)

As for part 2, let

$$\mathbf{v} = \xi(x,u)\partial_x + \varphi(x,u)\partial_u$$

be an infinitesimal generator of G, and Let pr $\mathbf{v}$ denote its prolongation to the jet space. Since dr is (by definition) an invariant one-form, we have

$$\mathbf{v}(dr) = [\,\mathrm{pr}\,\mathbf{v}(g) + gD_x\xi\,]dx,$$

which vanishes if and only if

$$\mathrm{pr}\,\mathbf{v}(g) = -gD_x\xi = -g(\xi_x + u_x\xi_u). \tag{7}$$

Applying pr $\mathbf{v}$ to the evolution equation (6), and using condition (7), we have (since ξ and φ do not depend on t)

$$\begin{aligned}\mathrm{pr}\,\mathbf{v}[u_t - g^{-2}u_{xx}I] &= (\varphi_u - u_x\xi_u)u_t - 2g^{-2}(\xi_x + u_x\xi_u)u_{xx}I - \\ &- g^{-2}\mathrm{pr}\,\mathbf{v}[u_{xx}]I - g^{-2}u_{xx}\mathrm{pr}\,\mathbf{v}[I].\end{aligned} \tag{8}$$

If G is to be a symmetry group, this must vanish on solutions of the equation; thus, in the first term, we replace u_t by $g^{-2}u_{xx}I$. Now, since G was assumed to be a subgroup of the projective group, which is the symmetry group of the second order ordinary differential equation $u_{xx} = 0$, we have pr $\mathbf{v}[u_{xx}]$ is a multiple of u_{xx}; in fact, inspection of the general prolongation formula for pr $\mathbf{v}$ (see [4] or Theorem 5 in [5]) shows that in this case

$$\text{pr } \mathbf{v}[u_{xx}] = (\varphi_u - 2\xi_x - 3\xi_u u_x)u_{xx}. \tag{9}$$

(The terms in pr $\mathbf{v}[u_{xx}]$ which do not depend on u_{xx} must add up to zero, owing to our assumption on $\mathbf{v}$.) Substituting (9) into (8) and combining terms, we find

$$\text{pr } \mathbf{v}[u_t - g^{-2}u_{xx}I] = g^{-2}u_{xx}\text{pr } \mathbf{v}[I],$$

which vanishes if and only if pr $\mathbf{v}[I] = 0$, a condition which must hold for each infinitesimal generator of G. But this is just the infinitesimal condition that I be a differential invariant of G, and the theorem follows. □

The Corollary follows from the fact that, for the listed subgroups, the invariant arc length r depends on lower order derivatives of u than the invariant curvature χ. (This fact holds for most (but *not* all) subgroups of the projective group; one exception is the group consisting of translations in x, u, and scalings $(x, u) \mapsto (\lambda x, \lambda u)$.) The orders are as follows:

Group	*Arc Length*	*Curvature*
Euclidean	1	2
Similarity	2	3
Special Affine	2	4
Affine	4	5
Projective	5	7

The explicit formulas are given in the following table:

Group	*ArcLength*	*Curvature*
Euclidean	$\sqrt{1+u_x^2}dx$	$\dfrac{u_{xx}}{(1+u_x^2)^{3/2}}$
Similarity	$\dfrac{u_{xx}dx}{(1+u_x^2)^2}$	$\dfrac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{u_{xx}^2}$
Special Affine	$(u_{xx})^{1/3}dx$	$\dfrac{P_4}{(u_{xx})^{8/3}}$
Affine	$\dfrac{\sqrt{P_4}}{u_{xx}}dx$	$\dfrac{P_5}{(P_4)^{3/2}}$
Projective	$\dfrac{(P_5)^{1/3}}{u_{xx}}dx$	$\dfrac{P_7}{(P_5)^{8/3}}$

Here

$$\begin{aligned} P_4 &= 3u_{xx}u_{xxxx} - 5u_{xxx}^2, \\ P_5 &= 9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3, \\ P_7 &= \frac{1}{3}u_{xx}^2[6P_5D_x^2P_5 - 7(D_xP_5)^2] + 2u_{xx}u_{xxx}P_5D_xP_5 - (9u_{xx}u_{xxxx} - 7u_{xxx}^2)P_5^2. \end{aligned}$$

Part 1 of the Theorem 1 (suitably interpreted) does not require G to be a subgroup of the projective group; however for part 2 and the corollary this is essential. One can, of course, classify the differential invariants, invariant arc-lengths, invariant evolution equations, etc., for any group of transformations in the plane, but the interconnections are more complicated. See Lie [2] and Olver [4] for the details of the complete classification of all groups in the plane and their differential invariants.

The uniqueness of the Euclidean and affine flows (see [6, 7]), was also proven in [1], using a completely different approach. In contrast with the results here presented, the ones in [1] were proven independently for each group, and when considering a new group, a new analysis must be carried out. Our result is a general one, can be applied to any sub-group. Also, with the geometric approach here presented, we believe that the result is clear and intuitive. As it is well know that all differential invariants can be computed based on the arc-length and curvature, it was expected from the invariant flows to hold this property as well, and exactly this is the result from the theorems above.

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