Vector Valued Poisson Transforms on Riemannian Symmetric Spaces

by

An Yang

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1994

Author ....
Department of Mathematics
May 2, 1994

Certified by ....
Sigurdur Helgason
Professor of Mathematics
Thesis Supervisor

Accepted by ....
David Vogan
Chairman, Departmental Committee on Graduate Students

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Abstract

Let G be a connected real semisimple Lie group with finite center, and K a maximal compact subgroup of G. Let (τ, V) be an irreducible unitary representation of K, and $G \times_K V$ the associated vector bundle. In the algebra of invariant differential operators on $G \times_K V$ the center of the universal enveloping algebra of $Lie(G)$ induces a certain commutative subalgebra Z_τ. We are able to determine the characters of Z_τ. Given such a character we define a Poisson transform from certain principal series representations to the corresponding space of joint eigensections. We prove that for most of the characters this map is a bijection, in the spirit of a famous conjecture by Helgason which corresponds to τ the trivial representation. The main idea in the proof is an asymptotic expansion, generalizing the one developed by Ban and Schlichtkrull.

Thesis Supervisor: Sigurdur Helgason
Title: Professor of Mathematics

Acknowledgments

The author would like to thank Professor Sigurdur Helgason for his guidance. Without his help and support the thesis would be impossible. He also would like to thank Professor David Vogan for teaching him representation theory in classes and answering his questions. Last but not least He would like to thank his wife Jing to whom the thesis is dedicated.

Contents

§0 Introduction

Let G be a connected real semisimple Lie group with finite center, and K a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space of noncompact type. We fix an Iwasawa decomposition $G = KAN$. Let M be the centralizer of A in K. Let $\mathfrak{g}$ and $\mathfrak{a}$ be the Lie algebras of G and A, respectively, and $\Sigma(\mathfrak{g}, \mathfrak{a})$ the root system for $(\mathfrak{g}, \mathfrak{a})$. Let $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ be the positive roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$ for the ordering given by N. Let $D(G/K)$ be the algebra of invariant differential operators on G/K. It is well known that the characters of $D(G/K)$ are paramertrized by $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, the complex dual space of $\mathfrak{a}$. Let $\mathcal{E}_\lambda(G/K)$ denote the space of joint eigenfunctions corresponding to λ. For each $g \in G$ we write $g = k(g) \exp H(g) n(g)$ according to $G = KAN$. For each $\phi \in C^\infty(K/M)$ we define $P_\lambda \phi \in C^\infty(G/K)$ by

$$P_\lambda \phi(g) = \int_K \phi(k) e^{-(\lambda+\rho)H(g^{-1}k)} dk.$$

Here ρ is the half sum of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (including multiplicities). It turns out $P_\lambda \phi \in \mathcal{E}_\lambda(G/K)$. Also one can easily extend the definition of P_λ to the space $D'(K/M)$ (resp. $A'(K/M)$) of distributions (resp. analytic functionals) on K/M. In this paragraph we fix $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$ such that $\frac{2<\lambda,\alpha>}{<\alpha,\alpha>} \notin -\mathbf{N} - \{0\}$, for each $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. It was proved by Helgason in [Helg2] that P_λ defines a bijection from $C^\infty(K/M)_{K-finite}$ onto $\mathcal{E}_\lambda(G/K)_{K-finite}$. He also proved in the rank one case P_λ is a bijection from $A'(K/M)$ onto $\mathcal{E}_\lambda(G/K)$. He then conjectured this should be true for high rank case. The conjecture was eventually proved by six Japanese mathematicians in 1979. See [KKMOOT]. It should be mentioned a representation theoretic proof by Schmid, starting from the K-finite result, is indicated in [Sch]. Lewis, then a student of Helgason, made the following observation: Let $\mathcal{E}^*_\lambda(G/K)$ be the subspace of $\mathcal{E}_\lambda(G/K)$ where each element increases at most exponentially (See §2 for definition), then P_λ maps $D'(K/M)$ into $\mathcal{E}^*_\lambda(G/K)$. He was able to prove in the rank one case P_λ is a bijection from $D'(K/M)$ onto $\mathcal{E}^*_\lambda(G/K)$. This result has been generalized to high rank case by Oshima and Sekiguchi in [OS]. There is an alternative and independent proof by Wallach. By refining Wallach's idea Ban and Schlichtkrull have a third proof in [BS]. They define $\mathcal{E}_\lambda^\infty(G/K)$ as the subspace of $\mathcal{E}_\lambda(G/K)$ where each element and its derivatives increase at most exponentially (uniformly). Then they prove P_λ is a bijection from $C^\infty(K/M)$ onto $\mathcal{E}_\lambda^\infty(G/K)$. The bijectivity of P_λ from $D'(K/M)$ to $\mathcal{E}_\lambda^*(G/K)$ follows easily.

Let (τ, V) be an irreducible unitary representation of K. Let $G \times_K V$ be the associated vector bundle over G/K. The space of smooth sections of this vector bundle can be identified by

$$C^\infty Ind_K^G(\tau) = \{f \in C^\infty(G,V) \mid f(gk) = \tau(k^{-1})f(g), \forall g \in G, \forall k \in K\}.$$

Let D_τ denote the algebra of invariant differential operators on $C^\infty Ind_K^G(\tau)$. Notice when (τ, V) is the trivial representation we go back to the classical case. In the case where $\dim V = 1$, D_τ is commutative and its characters can be paramertrized by $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$. In [Shim] Shimeno is able to characterize the joint eigenspace of D_τ in terms a Poisson transform for most of λ. Gaillard's results about the eigenforms of the Laplacian on hyperbolic spaces are illuminating. They show considerable variety even for a simple space. See [Ga] for details. Ven in [Ven] considers vector valued Poisson transforms in the rank one case, extending Gaillard's results. His emphasis, however, is on the singular eigenvalues. Minemura in [Min] studies the properties of D_τ and obtains a result on the dimension of the spherical eigensections.

One of the difficulties people run into when trying to generalize the classical results is the complexity of D_τ, in particular its noncommutativity. The remedy used is either a condition on τ or a condition on G/K. We put a mild condition on $\mathfrak{g}$ (See beginning of §4) but no restriction on τ. We replace D_τ with a subalgebra Z_τ coming from $\mathfrak{Z}(\mathfrak{g})$, the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. Then we are able to determine the characters of Z_τ. It turns out they are given by $\lambda - \Lambda$, where $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, and Λ is given by the infinitesimal character of an irreducible representation of M contained in τ (See Proposition 1.11).

Let V be the representation space of τ, and

$$V = \bigoplus_{\sigma \in \hat{M}} V(\sigma)$$

the isotypic decomposition of V into M-isotypic parts. We say $\sigma \in \tau$ if $V(\sigma) \neq 0$. Define

$$V(\Lambda) = \bigoplus_{\sigma \in \tau, \Lambda_\sigma = \Lambda} V(\sigma).$$

Here Λ_σ is given by the infinitesimal character of σ. Let $\tau(\Lambda)$ be the restriction of τ to M with representation space $V(\Lambda)$. We define a Poisson transform (See §1 for definition)

$$P_\lambda : C^\infty Ind_{MAN}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) \to \mathcal{E}_{\lambda-\Lambda}^\infty Ind_K^G(\tau)$$

by

$$P_\lambda \phi(g) = \int_K \tau(k)\phi(gk)dk.$$

Here $C^\infty Ind_{MAN}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) = \{\phi \in C^\infty(G, V(\Lambda)) \mid \phi(gman) = a^{\lambda-\rho}\tau(m^{-1})\phi(g)\}$, and $\mathcal{E}_{\lambda-\Lambda}^\infty Ind_K^G(\tau)$ is the subspace of the total eigenspace where each element and its derivatives increase at most exponentially (uniformly). Let $C(\lambda)$ be the generalized Harish -Chandra's C-function corresponding to τ (See §8), and $\Sigma(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$ as defined after Remark 1.5.

Theorem *Let $\lambda - \Lambda \in \mathfrak{h}_\mathbf{C}^*$ satisfy the conditions*

$$\frac{2 < \lambda - \Lambda, \alpha >}{< \alpha, \alpha >} \notin \mathbf{Z}, \forall \alpha \in \Sigma(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C}), with \;\; \alpha|_\mathfrak{a} \neq 0; \;\; \frac{2 < \lambda, \beta >}{< \beta, \beta >} \notin -\mathbf{N}, \forall \beta \in \Sigma^+(\mathfrak{g}, \mathfrak{a}).$$

If in addition $\det C(\lambda) \neq 0$, *then P_λ is a bijection.*

This generalizes the result of Ban and Schlichtkrull mentioned above which corresponds to τ the trivial representation.

We have similar result about distributions and K-finite sections, Generalizing the above mentioned results for τ trivial.

The main idea in the proof is asymptotic expansion developed in [Ban] and [BS].

§1 Notations and preliminaries

Let G be a connected real semisimple Lie group with finite center and K a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space. We fix an Iwasawa decomposition $G = KAN$, and let M be the centralizer of A in K, M' the normalizer of A in K, $W = M'/M$ the Weyl group. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{m}$ be the corresponding Lie algebras of G, K, A, N, and M, respectively, and $U(\mathfrak{g})$, $U(\mathfrak{k})$, $U(\mathfrak{a})$, $U(\mathfrak{n})$, and $U(\mathfrak{m})$ the corresponding universal enveloping algebras of the complexified Lie algebras. Let $\Sigma(\mathfrak{g},\mathfrak{a})$ be the restricted root system for $(\mathfrak{g},\mathfrak{a})$, and $\Delta = \{\alpha_1, ..., \alpha_r\}$ the set of simple roots for the ordering of $\Sigma(\mathfrak{g},\mathfrak{a})$ given by N. Let $\mathfrak{z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. If $g \in G$ we write $g = k(g)\exp H(g)n(g)$ according to $G = KAN$.

Fix once and for all an irreducible unitary representation (τ, V) of K. Denote $G \times_K V$ the associated vector bundle. Then the space of its smooth sections may be identified with the following space:

$$C^\infty Ind_K^G(\tau) = \{f \in C^\infty(G,V) \mid f(gk) = \tau(k)^{-1}f(g), \ \ \forall g \in G, \ \ \forall k \in K\}.$$

Let D_τ denote the algebra of differential operators on $C^\infty Ind_K^G(\tau)$ that commute with the left translations by elements of G. The remaining section will be devoted to the study of this algebra. First for each $X \in \mathfrak{g}$ and $f \in C^\infty(G,V)$ we define L_X and R_X as follows:

$$L_X f(g) = (\frac{d}{dt} f(\exp(-tX)g))|_{t=0}, \ \ R_X f(g) = (\frac{d}{dt} f(g \exp tX))|_{t=0}, \forall g \in G.$$

Then L and R define two representations of $\mathfrak{g}$ which we extend to representations of $U(\mathfrak{g})$. Let $EndV$ denote the space of linear maps from V to itself. Then $U(\mathfrak{g}) \otimes EndV$ is an associative algebra with the natural multiplication. Let $I(\tau)$ be the left ideal of $U(\mathfrak{g}) \otimes EndV$ generated by $\{X \otimes 1 + 1 \otimes \tau(X) \mid X \in \mathfrak{k}\}$.

Proposition 1.1

$$U(\mathfrak{g}) \otimes EndV = (U(\mathfrak{a}) \otimes EndV) \oplus (\mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau)).$$

Proof: It suffices to show the left hand side is contained in the right hand side. Suppose $u \otimes T \in U(\mathfrak{g}) \otimes EndV$. By Poincaré-Birkhoff-Witt we can assume $u = u_1 u_2 u_3$, where $u_1 \in U(\mathfrak{n})$, $u_2 \in U(\mathfrak{a})$, and $u_3 \in U(\mathfrak{k})$. If $u_1 \in \mathfrak{n}U(\mathfrak{n})$ then $u \otimes T \in \mathfrak{n}U(\mathfrak{g}) \otimes EndV$. So we can assume $u = u_2 u_3$, where $u_2 \in U(\mathfrak{a})$, and $u_3 \in U(\mathfrak{k})$. Let $u_3 = X_1 ... X_j$, for $X_1, ..., X_j \in \mathfrak{k}$. It is easy to show $u_2 u_3 \otimes T \in (U(\mathfrak{a}) \otimes EndV) + I(\tau)$ by induction on j. This proves the proposition.

Define a K action on $U(\mathfrak{g}) \otimes EndV$ by

$$k.(X \otimes T) = Ad(k)X \otimes \tau(k) T \tau(k)^{-1},$$

for each $k \in K$.

Let $(U(\mathfrak{g}) \otimes EndV)^K$ be the fixed elements under the action.

Proposition 1.2 *Let Γ_1: $U(\mathfrak{g}) \otimes EndV \to U(\mathfrak{a}) \otimes EndV$ be the projection map according to the decomposition in Proposition 1.1. Then Γ_1 is a homomorphism from $(U(\mathfrak{g}) \otimes EndV)^K$ into $U(\mathfrak{a}) \otimes End_M V$, where*

$$End_M V = \{T \in EndV \mid \tau(m)T = T\tau(m), \forall m \in M\}.$$

Proof: Since M preserves $\mathfrak{n}$, it is easy to see Γ_1 maps $(U(\mathfrak{g}) \otimes EndV)^K$ into $U(\mathfrak{a}) \otimes End_M V$. We now check Γ_1 is a homomorphism.

Suppose D_1, $D_2 \in (U(\mathfrak{g}) \otimes EndV)^K$. Then

$D_1 - \Gamma_1(D_1) \in \mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau)$. Hence

$$D_1 D_2 - \Gamma_1(D_1) D_2 \in \mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau) D_2.$$

Assume $D_2 = \sum u_i \otimes T_i$, for $u_i \in U(\mathfrak{g})$, and $T_i \in EndV$. Then for any $X \in \mathfrak{k}$,

$$(X \otimes 1 + 1 \otimes \tau(X)) D_2 = \sum (X u_i \otimes T_i + u_i \otimes \tau(X) T_i)$$
$$= \sum (ad(X) u_i \otimes T_i + u_i \otimes [\tau(X), T_i]) + \sum (u_i X \otimes T_i + u_i \otimes T_i \tau(X)).$$

The first summation is zero since $D_2 \in (U(\mathfrak{g}) \otimes EndV)^K$. The second one is just $D_2(X \otimes 1 + 1 \otimes \tau(X))$. So we have proved $I(\tau) D_2 \subset I(\tau)$. Hence

$$D_1D_2 - \Gamma_1(D_1)D_2 \in \mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau).$$

However, $D_2 - \Gamma_1(D_2) \in \mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau)$, and

$$\Gamma_1(D_1)(\mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau)) \subset \mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau).$$

Therefore

$$D_1D_2 - \Gamma_1(D_1)\Gamma_1(D_2) \in \mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau).$$

This proves $\Gamma_1(D_1D_2) = \Gamma_1(D_1)\Gamma_1(D_2)$.

For $D = \sum u_i \otimes T_i \in U(\mathfrak{g}) \otimes EndV$, and $f \in C^\infty(G, V)$, we define

$$\mu_1(D)f = \sum T_i R_{u_i} f.$$

It is not difficult to show for each $D \in (U(\mathfrak{g}) \otimes EndV)^K$, and $f \in C^\infty Ind_K^G(\tau)$, $\mu_1(D)f$ remains in $C^\infty Ind_K^G(\tau)$. So $\mu_1(D) \in D_\tau$. In fact μ_1 is a surjective homomorphism from $(U(\mathfrak{g}) \otimes EndV)^K$ onto D_τ. For a proof see [Deit].

We define $\mu(D) = \mu_1(D \otimes 1)$, for each $D \in U(\mathfrak{g})^K$. By a theorem of Burnside which asserts that $\tau(U(\mathfrak{k})) = EndV$, one can prove μ is a surjective homomorphism from $U(\mathfrak{g})^K$ onto D_τ, using the surjectivity of μ_1. A proof can also be found in [Deit].

For each $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, we introduce an important function Ψ_λ on G with values in $EndV$ as follows:

$$\Psi_\lambda(nak) = a^{\lambda+\rho}\tau(k)^{-1},$$

for $n \in N$, $a \in A$, and $k \in K$. Here ρ is the half sum of the positive roots for $(\mathfrak{g}, \mathfrak{a})$.

Notice that for each $v \in V$, the function: $g \rightarrow \Psi_\lambda(g) \cdot v$ belongs to $C^\infty Ind_K^G(\tau)$.

Proposition 1.3 *For each $D \in U(\mathfrak{g})^K$, and $v \in V$,*

$$\mu(D)(\Psi_\lambda \cdot v) = \Psi_\lambda \cdot (\Gamma_1(D \otimes 1)(\lambda + \rho)v).$$

Proof: Since both sides are left N-invariant and behave in the same way under the right K-action, it is sufficient to show they are equal when restricted to A. By definition

$$D \otimes 1 = D_1 + \Gamma_1(D \otimes 1) + D_2,$$

where $D_1 \in \mathfrak{n}U(\mathfrak{g}) \otimes EndV$, and $D_2 \in I(\tau)$.

It is easy to see $\mu_1(D_1)(\Psi_\lambda \cdot v)|A = 0$, and $\mu_1(D_2)(\Psi_\lambda \cdot v) = 0$. So

$$\mu(D)(\Psi_\lambda \cdot v)|A = a^{\lambda+\rho}\Gamma_1(D \otimes 1)(\lambda + \rho)v.$$

Corollary 1.4 *There exists a homomorphism* Γ': $D_\tau \to U(\mathfrak{a}) \otimes End_M V$. *Moreover, for each* $D \in U(\mathfrak{g})^K$, $\Gamma'(\mu(D)) = \Gamma_1(D \otimes 1)$.

Remark 1.5 *It has been proved in section 3 in [Min] that* Γ' *is injective, using results from [Lep].*

In general D_τ is very complicated. For instance it is not abelian in most of the cases. For this reason we replace it by $\mu(\mathfrak{z}(\mathfrak{g}))$ which we denote by Z_τ.

Choose $\mathfrak{t}$ a maximal abelian subalgebra in $\mathfrak{m}$. Then $\mathfrak{h}_{\mathbf{C}} = (\mathfrak{t} + \mathfrak{a})_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Let $\Sigma(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ the root system for $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Let $\Sigma^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be the set of positive roots for some ordering, and $\mathfrak{g}_{\mathbf{C}}^+$ (resp. $\mathfrak{g}_{\mathbf{C}}^-$) the sum of positive (resp. negative) root spaces. Choose an ordering such that $\mathfrak{n} \subset \mathfrak{g}_{\mathbf{C}}^+$. We consider each $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ (resp. $\mathfrak{t}_{\mathbf{C}}^*$) an element of $\mathfrak{h}_{\mathbf{C}}^*$ by the requirement that λ be zero in $\mathfrak{t}$ (resp. $\mathfrak{a}$).

Let

$$P = \{\alpha \in \Sigma^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) \mid \alpha|\mathfrak{a} \neq 0\}, \quad P_0 = \{\alpha \in \Sigma^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) \mid \alpha|\mathfrak{a} = 0\}.$$

Define

$$\rho = \frac{1}{2}\sum_{\alpha \in P} \alpha, \quad \rho_0 = \frac{1}{2}\sum_{\alpha \in P_0} \alpha.$$

Let Θ be the Cartan involution of $\mathfrak{g}$ with fixed point set $\mathfrak{k}$ and extend it to an automorphism of $\mathfrak{g}_{\mathbf{C}}$. Then $\alpha \to -\Theta\alpha$ is a permutation of P, so $\rho|\mathfrak{t} = 0$. Hence ρ can be viewed as the half sum of positive roots for $(\mathfrak{g}, \mathfrak{a})$.

Let γ': $\mathfrak{z}(\mathfrak{g}) \to U(\mathfrak{h}_{\mathbf{C}})$ be defined by

$$Z - \gamma'(Z) \in \mathfrak{g}_{\mathbf{C}}^{-} U(\mathfrak{g}),$$

for $Z \in \mathfrak{Z}(\mathfrak{g})$.

Define $\gamma(Z)(\mu) = \gamma'(\lambda - \rho - \rho_0)$, for each $\mu \in \mathfrak{h}_{\mathbf{C}}^*$. This is the usual Harish-Chandra's homomorphism.

Let $V = \bigoplus_{\sigma \in \hat{M}} V(\sigma)$ be the decomposition into the M-isotypic parts. We say $\sigma \in \tau$ if $V(\sigma) \neq 0$.

For each irreducible representation (σ, V_σ) of M, we get a Lie algebra representation of $\mathfrak{m}$ by differentiation. We denote the representation by $d\sigma$. In general this is not irreducible. Fortunately it is a multiple of an irreducible representation of $\mathfrak{m}$. This fact can be seen in the following way.

Let M_0 be the identity component of M. By structure theory (See 1.1.3.8 in [War]) one can find $Z(A)$, a finite subgroup of M where each element commutes with every element of M_0.

Choose an irreducible representation (σ, V_1) of M_0 in (σ, V_σ). For each $z \in Z(A)$, $(\sigma, \sigma(z)V_1)$ gives an irreducible representation of M_0 in (σ, V_σ), which is equivalent to (σ, V_1). Since σ is irreducible, $V_\sigma = \sum_{z \in Z(A)} \sigma(z) V_1$.

So by Schur's lemma the center $\mathfrak{Z}(\mathfrak{m})$ of $U(\mathfrak{m})$ acts on V_σ by scalars. The action is determined by $\Lambda_\sigma \in \sqrt{-1}\mathfrak{t}^*$ as follows: For each $Z \in \mathfrak{Z}(\mathfrak{m})$, $d\sigma(Z) = \gamma(Z)(\Lambda_\sigma) I_{V_\sigma}$, where γ is the Harish-Chandra's homomorphism for $(\mathfrak{m}, \mathfrak{t})$, and I_{V_σ} the identity map of V_σ. We choose Λ_σ the highest weight of σ plus ρ_0.

Let Γ: $D_\tau \to U(\mathfrak{a}) \otimes End_M V$ be defined by

$$\Gamma(D)(\lambda) = \Gamma'(D)(\lambda + \rho).$$

Theorem 1.6 *For each* $Z \in \mathfrak{Z}(\mathfrak{g})$, *and* $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$,

$$\Gamma(\mu(Z))(\lambda) | V(\sigma) = \gamma(Z)(\lambda - \Lambda_\sigma) I_{V(\sigma)}.$$

We give a proof below using a well known proposition about $\mathfrak{Z}(\mathfrak{g})$. A more self contained proof is in [Wall].

First for the proof and later use we introduce the definition of Poisson transforms.

Let (δ, V_δ) be a finite dimensional representation of $B = MAN$, the minimal parabolic subgroup of G. Let

$$C^\infty Ind_B^G(\delta) = \{\phi \in C^\infty(G, V_\delta) \mid \phi(gman) = a^{-\rho}\delta^{-1}(man)\phi(g), \forall g \in G, \forall man \in B\}.$$

Let $C^\infty Ind_B^G(\delta)$ be endowed with the topology from $C^\infty(G, V_\delta)$. We will specify the topology on $C^\infty Ind_K^G(\tau)$ in the next section.

Definition 1.7 *A Poisson transform is a continuous, linear, and G-equivariant map from $C^\infty Ind_B^G(\delta)$ into $C^\infty Ind_K^G(\tau)$.*

Given $T \in Hom_M(V_\delta, V_\tau)$, and $\phi \in C^\infty Ind_B^G(\delta)$, we define

$$P_T(\phi)(g) = \int_K \tau(k)T(\phi(gk))dk.$$

One can easily check P_T is a Poisson transform.

Proposition 1.8 *The map $T \to P_T$ is a bijection from $Hom_M(V_\delta, V_\tau)$ onto the space of Poisson transforms.*

This result appears in [Ven]. We include a proof for the completeness.

Suppose P is a Poisson transform from $C^\infty Ind_B^G(\delta)$ into $C^\infty Ind_K^G(\tau)$. Define the Poisson kernel $p \in [C^\infty Ind_B^G(\delta)]' \otimes V$, the strong topological dual of $C^\infty Ind_B^G(\delta)$ tensored by V, by

$< p, \phi >= P\phi(e)$, for each $\phi \in C^\infty Ind_B^G(\delta)$.

By the G-equivariance of P the Poisson kernel completely determines P by

$P\phi(x) =< p, L_{x^{-1}}\phi >$, for any $\phi \in C^\infty Ind_B^G(\delta)$.

Here $L_{x^{-1}}\phi(g) = \phi(xg)$.

By Section 9 there is a K-equivariant isomorphism between $(C^\infty Ind_B^G(\delta))'$ and $C^{-\infty} Ind_M^K(\check{\delta}|M)$, where $C^{-\infty} Ind_M^K(\check{\delta}|M)$ denotes the space of vector-valued distributions f: $C^\infty(K, \mathbb{C}) \to V_\delta^*$, such that

$$R_m f = \check{\delta}(m)^{-1} f,$$

for any $m \in M$. Here $\check{\delta}$ is the dual representation of δ. And $R_m f(\phi) = f(R_{m^{-1}}\phi)$, where $(R_{m^{-1}}\phi)(k) = \phi(km^{-1})$.

So $p \in C^{-\infty}Ind_M^K(\check{\delta}|M) \otimes V$. However, for $\phi \in C^{\infty}Ind_B^G(\delta)$,

$$< p, L_k\phi >= P(L_k\phi)(e) = P\phi(k^{-1}) = \tau(k)(P\phi(e)) = \tau(k)(< p, \phi >).$$

Hence $p \in (C^{-\infty}Ind_M^K(\check{\delta}|M) \otimes V)^K$. Let π be the representation of K defined by $\pi(k)(v \otimes w) = v \otimes \tau(k)w$, for $v \in V_\delta^*$, and $w \in V$. Then $p \in C^{-\infty}(K, V_\delta^* \otimes W)$, and $L_k p = \pi(k^{-1})p$. By Lemma 9.3 p must be smooth. Its transformation properties imply that p is determined by $p(e)$, which belongs to $(V_\delta^* \otimes V)_M \cong Hom_M(V_\delta, V)$.

Proof of Proposition 1.8: From the definition of P_T, it is immediate that the Poisson kernel of P_T evaluated at the identity is T. This shows the map $T \to P_T$ is injective. On the other hand, let P be a Poisson transform, and let p be its Poisson kernel. Then

$$P\phi(x) =< p, L_{x^{-1}}\phi >= \textstyle\int_K < p(k), \phi(xk) > dk = \int_K \tau(k)p(e)\phi(xk)dk.$$

This proves $P = P_{p(e)}$, whence the surjectivity.

The following integration formula on K is due to Harish-Chandra. A simplified proof can be found on p.197 in [Helg1].

Lemma 1.9

$$\int_K F(k(g^{-1}k))dk = \int_K F(k)e^{-2\rho H(gk)}.$$

Let σ be a finite dimensional representation of M and $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$. Then $\sigma \otimes (-\lambda) \otimes 1$ defines a representation of B by $man \to a^{-\lambda}\sigma(m)$.

Corollary 1.10

$$P_T\phi(g) = \int_K \Psi_\lambda(k^{-1}g)T\phi(k)dk,$$

for each $\phi \in C^{\infty}Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)$.

Proof: $P_T\phi(g) = \int_K \tau(k)T\phi(gk)dk$

$$= \int_K \tau(k)T\phi(k(gk)\exp H(gk)n(gk))dk = \int_K e^{(\lambda-\rho)H(gk)}\tau(k)T\phi(k(gk))dk.$$

By Lemma 1.9,

$$\int_K e^{(\lambda-\rho)H(gk)}\tau(k)T\phi(k(gk))dk = \int_K e^{(\lambda+\rho)H(gk(g^{-1}k))}\tau(k(g^{-1}k))T\phi(k(gk(g^{-1}k)))dk$$
$$= \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\tau(k(g^{-1}k))T\phi(k)dk = \int_K \Psi_\lambda(k^{-1}g)T\phi(k)dk.$$

Proof of Theorem 1.6: Let δ be the restriction of τ to M with $V(\sigma)$ as the representation space. It is well known that for any $\phi \in C^\infty Ind_B^G(\delta \otimes (-\lambda) \otimes 1)$, and each $Z \in \mathfrak{z}(\mathfrak{g})$, $L_Z\phi = \gamma(Z)(\Lambda_\sigma - \lambda)\phi$. See [Vo]. Let $*$ denote adjoint. By Corollary 5.31 on p. 324 in [Helg1],

$$R_Z P_T\phi = L_{Z^*}P_T\phi = P_T L_{Z^*}\phi$$
$$= P_T(\gamma(Z^*)(\Lambda_\sigma - \lambda)\phi) = P_{\gamma(Z^*)(\Lambda_\sigma-\lambda)T}\phi = P_{\gamma(Z)(-\Lambda_\sigma+\lambda)T}\phi.$$

On the other hand, by Proposition 1.3 and Corollary 1.10, $R_Z P_T\phi = P_{\Gamma(\mu(Z))(\lambda)T}\phi$. So

$$P_{\gamma(Z)(-\Lambda_\sigma+\lambda)T} = P_{\Gamma(\mu(Z))(\lambda)T}.$$

By Proposition 1.8 we conclude

$$\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\lambda - \Lambda_\sigma)I_{V(\sigma)}.$$

By definition a character of Z_τ is a homomorphism from Z_τ to $\mathbf{C}$.

Proposition 1.11 *A character χ of Z_τ is given by $\lambda - \Lambda_\sigma$, where $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, and $\sigma \in \tau$. More specifically, $\chi(\mu(Z)) = \gamma(Z)(\lambda - \Lambda_\sigma)$, for each $Z \in \mathfrak{z}(\mathfrak{g})$.*

Lemma 1.12 *Let S be the common zeros of p_1, ..., p_m in $S(\mathfrak{h}_{\mathbf{C}})$. Assume in addition S is $\tilde{W}$-invariant, $\tilde{W}$ denoting the Weyl group for $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Then one can find q_1, ..., q_n in $I(\mathfrak{h}_{\mathbf{C}})$ such that S is the common zeros of q_1, ..., q_n.*

Proof: Define $R_i(X) = \Pi_{s\in\tilde{W}}(X - p_i^s)$. Then

$$R_i(X) = X^w + p_{i1}X^{w-1} + ... + p_{iw}.$$

Here $w = |\tilde{W}|$.

It is easy to see we can use p_{ij}'s as our $q_1, \ldots, q_n$.

Proof of Proposition 1.11: Let $A = \mu \circ \gamma^{-1}$: $I(\mathfrak{h}_{\mathbf{C}}) \to Z_\tau$. By Theorem 1.6 $\ker(A) = \{p \in I(\mathfrak{h}_{\mathbf{C}}) \mid p|(-\Lambda_\sigma + \mathfrak{a}^*_{\mathbf{C}}) = 0$, for all $\sigma \in \tau$ $\}$. Here we use Remark 1.5 which asserts that Γ is injective. Suppose χ: $Z_\tau \to \mathbf{C}$ is a character of Z_τ. Then there exists $\mu \in \mathfrak{h}^*_{\mathbf{C}}$, such that $\chi \circ A = \chi_\mu$, where χ_μ is the homomorphism defined by evaluation at μ. Obviously $p(\mu) = 0$, for all $p \in \ker(A)$. Let

$$S = \cup_{\sigma \in \tau, w \in \tilde{W}} w(-\Lambda_\sigma + \mathfrak{a}^*_{\mathbf{C}}) \subset \mathfrak{h}^*_{\mathbf{C}}.$$

One can find $p_1, \ldots, p_m$ in $S(\mathfrak{h}_{\mathbf{C}})$ such that S is the common zeros of $p_1, \ldots, p_m$. Then by Lemma 1.12 we can find $q_1, \ldots, q_n$ in $I(\mathfrak{h}_{\mathbf{C}})$ such that S is the common zeros of $q_1, \ldots, q_n$. This shows $q_1, \ldots, q_n$ are in $\ker(A)$. So $q_1(\mu) = \ldots = q_n(\mu)$. Therefore $\mu \in S$, i.e. $\mu = w(\lambda - \Lambda_\sigma)$, for some $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, $\sigma \in \tau$, and $w \in \tilde{W}$.

The next proposition is about a property of the generalized Harish-Chandra's homomorphism. It is a weak version of a conjecture by Lepowsky.

For $s \in M'$, define $s.(X \otimes T) = Ad(s)X \otimes \tau(s)T\tau(s^{-1})$, for $X \in U(\mathfrak{a})$, and $T \in EndV$.

Proposition 1.13 *For each $s \in M'/M$, $s.\Gamma(D) = \Gamma(D)$, for each $D \in Z_\tau$.*

For the proof of this result we need more facts about Weyl groups. Let $\tilde{W}_1 \subset \tilde{W}$ be the subgroup where every element stablizes $\mathfrak{a}$. It is well known there is a surjective homomorphism $\tilde{W}_1 \to M'/M$. The kernel $\tilde{W}_0$ is the Weyl group for $(\mathfrak{m}, \mathfrak{t})$.

Lemma 1.14 *For each $s \in M'/M$, choose $w(s)$ in $\tilde{W}_1$ in the preimage of s under the homomorphism above. Then $\Lambda_{\sigma^s} = w(s)\Lambda_\sigma$.*

Proof (by Vogan): Take a maximal torus T of M_0. sTs^{-1} is another maximal torus. So there is $m \in M_0$, such that $msTs^{-1}m^{-1} = T$. To avoid cumbersome notations we assume $sTs^{-1} = T$. It is easy to see that $Ad(s)^*$, the transpose of $Ad(s)$, preserves $\Sigma(\mathfrak{m}, \mathfrak{t})$. We can also assume $Ad(s)^*$ preserves $\Sigma^+(\mathfrak{m}, \mathfrak{t})$. For $Z \in \mathfrak{z}(\mathfrak{m})$,

$Z - \gamma'(Z) \in \mathfrak{m}^- U(\mathfrak{m})$. Hence

$Ad(s)Z - Ad(s)\gamma'(Z) \in \mathfrak{m}^- U(\mathfrak{m})$. So

$$\sigma^s(Z) = \sigma(Ad(s)Z) = Ad(s)\gamma'(Z)(\Lambda_\sigma - \rho_0)$$
$$= \gamma'(Z)(Ad(s)^*\Lambda_\sigma - \rho_0) = \gamma(Ad(s)^*\Lambda_\sigma).$$

Hence $\Lambda_{\sigma^s} = Ad(s)^*\Lambda_\sigma = w(s)\Lambda_\sigma$.

Proof of Proposition 1.13: Take $Z \in \mathfrak{Z}(\mathfrak{g})$ such that $D = \mu(Z)$. Then for each $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, and $s \in M'$,

$$s.\Gamma(D)(\lambda)|V(\sigma) = s.\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(Ad(s)^*\lambda - \Lambda_{\sigma^s})I_{V(\sigma)}.$$

By Lemma 1.14 $\Lambda_{\sigma^s} = w(s)\Lambda_\sigma$. So

$$s.\Gamma(D)(\lambda)|V(\sigma) = \gamma(Z)(Ad(s)^*\lambda - w(s)\Lambda_\sigma)I_{V(\sigma)}$$
$$= \gamma(Z)(\lambda - \Lambda_\sigma)I_{V(\sigma)} = \Gamma(\mu(Z))(\lambda)|V(\sigma) = \Gamma(D)(\lambda)|V(\sigma).$$

Now let $\bar{\mathfrak{n}} = \Theta\mathfrak{n}$. Similarly as in Proposition 1.10 we get

$$U(\mathfrak{g}) \otimes EndV = U(\mathfrak{a}) \otimes EndV \oplus [\bar{\mathfrak{n}}U(\mathfrak{g}) \otimes EndV + I(\tau)].$$

Then we define $\tilde{\Gamma}_1$: $U(\mathfrak{g}) \otimes EndV \rightarrow U(\mathfrak{a}) \otimes EndV$ as the projection according to this decomposition.

Corollary 1.15 *For each $Z \in \mathfrak{Z}(\mathfrak{g})$, and $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$,*

$$\tilde{\Gamma}_1(Z \otimes 1)(\lambda) = \Gamma(\mu(Z))(\lambda + \rho).$$

Proof: Take $s \in M'$, such that $Ad(s)^*\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \Sigma^-(\mathfrak{g}, \mathfrak{a})$.

By definition $Z \otimes 1 - \Gamma_1(Z \otimes 1) \in \mathfrak{n}U(\mathfrak{g}) \otimes EndV + I(\tau)$. Hence $s.(Z \otimes 1) - s.\Gamma_1(Z \otimes 1) \in \bar{\mathfrak{n}}U(\mathfrak{g}) \otimes EndV + I(\tau)$. So $\tilde{\Gamma}_1(Z \otimes 1) = s.\Gamma_1(Z \otimes 1)$. Hence

$$\tilde{\Gamma}_1(Z \otimes 1)(\lambda) = \tau(s)\Gamma_1(Z \otimes 1)(Ad(s)^*\lambda)\tau(s^{-1}) = \tau(s)\Gamma(\mu(Z))(Ad(s)^*\lambda - \rho)\tau(s^{-1})$$
$$= \tau(s)\Gamma(\mu(Z))(Ad(s)^*(\lambda + \rho))\tau(s^{-1}) = \Gamma(\mu(Z))(\lambda + \rho).$$

§2 Some function spaces on G

In this section we introduce a certain growth condition on a function on G with values in V. It turns out the condition is satisfied by $P_T\phi$ for any $\phi \in C^\infty Ind_B^G(\delta)$, where δ is a certain finite dimensional representation of B.

For each $g \in G$, we denote $\|g\|$ the operator norm of $Ad(g)$ on $\mathfrak{g}$, which is equipped with the inner product $< X, Y >_\Theta = -K(X, \Theta Y)$. Here K is the Killing form on $\mathfrak{g}$.

Lemma 2.1 *(i)* $\|g\| = \|\Theta g\| = \|g^{-1}\| \geq 1$,

(ii) $\|g_1 g_2\| \leq \|g_1\|\|g_2\|$,

(iii) if $g = k_1 a k_2$ *with* k_1, $k_2 \in K$, $a \in A$, *then*

$$\|g\| = \exp(\max_{\alpha\in\Sigma(\mathfrak{g},\mathfrak{a})} |\alpha(\log a)|),$$

(iv) there are constants C_1, $C_2 > 0$, *such that if* $x = \exp X$ *with* $X \in \mathfrak{p}$, *then* $e^{C_1|X|} \leq \|x\| \leq e^{C_2|X|}$. *Here* $\mathfrak{p}$ *is the* -1 *eigenspace of* Θ, *and* $|X| = \sqrt{< X, X >_\Theta}$,

(v) $\|a\| \leq \|an\|$, *for* $a \in A$, *and* $n \in N$.

Proof: See [BS].

For any function f: $G \to V$ and $r \in \mathbf{R}$, we define

$$\|f\|_r = \sup_{g\in G} \|g\|^{-r} |f(g)|.$$

We say f increases at most exponentially if $\|f\|_r < \infty$. Let $C_r(G, V)$ denote the Banach space of continuous functions f on G with values in V with $\|f\|_r \leq \infty$

Example 2.2 *Let* $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, *and* σ *a finite dimensional representation of* M. *Let* $C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1) = \{\phi \in C^\infty(G, V_\sigma) \mid \phi(gman) = a^{\lambda-\rho}\sigma(m^{-1})\phi(g)\}$. *Let* $r(\lambda) = C_1^{-1}|Re\lambda - \rho|$, *where* C_1 *is the constant in Lemma 2.1 (v). Then for any* $\phi \in C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)$, $P_T\phi \in C_{r(\lambda)}(G, V)$, *where* $T \in Hom_M(V_\sigma, V)$. *This is in [BS] when* τ *is trivial and* τ *general does not offer additional difficulties.*

Define

$$C_r^\infty(G, V) = \{f \in C^\infty(G, V) \mid L_u f \in C_r(G, V), \ \forall u \in U(\mathfrak{g})\}.$$

We endow $C_r(G,V)$ with its standard topology: Let X_1, ..., X_p be a basis of $\mathfrak{g}$, and $X^I = X^{i_1}...X^{i_p} \in U(\mathfrak{g})$ for $I = (i_1, ..., i_p) \in \mathbf{N}^p$. For $q \in \mathbf{N}$ and $f \in C^q(G,V)$, a q times continuously differentiable function from G to V, we define

$$\|f\|_{q,r} = \sum_{|I|\leq q} \|L_{X^I} f\|_r.$$

Endowed with this norm the space

$$C^q_r(G,V) = \{f \in C^q(G,V) \mid \|f\|_{q,r} < \infty\}$$

is a Banach space. Obviously $C^q_r \subset C^{q'}_r$ if $q' \leq q$, and $C^\infty_r(G,V) = \cap_q C^q_r(G,V)$. The topology on $C^\infty_r(G,V)$ is given by the family of norms $\|\cdot\|_{q,r}$, $q \in \mathbf{N}$ on $C^\infty_r(G,V)$. We now consider for each $q \in \mathbf{N}$ the action of L and R on $C^\infty_r(G,V)$. Recall for g, $x \in G$, and $f \in C^q(G,V)$, $L_x f(g) = f(x^{-1}g)$, and $R_x f(g) = f(gx)$. Obviously L_x leaves $C^q_r(G,V)$ invariant. In fact $\|L_x f\|_{q,r} \leq C\|x\|^{r+s}\|f\|_{q,r}$, for each $f \in C^q_r(G,V)$, and $x \in G$. Here C and s are constants.

On the other hand, $\|R_x f\|_{q,r} \leq \|x\|^r \|f\|_{q,r}$.

From Example 2.2, we see P_T maps $C^\infty Ind^G_B(\sigma \otimes (-\lambda) \otimes 1)$ into $C^\infty_{r(\lambda)}(G,V)$ continuously.

Recall from Proposition 1.11 a character of Z_τ is given by $\lambda - \Lambda$, where $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, and Λ is the infinitesimal character of an irreducible representation of M in τ. Let $\mathcal{E}_{\lambda-\Lambda} Ind^G_K(\tau)$ denote the corresponding eigenspace of Z_τ. Let

$$\mathcal{E}^\infty_{\lambda-\Lambda,r} Ind^G_K(\tau) = \mathcal{E}_{\lambda-\Lambda} Ind^G_K(\tau) \cap C^\infty_r(G,V),$$

$$\mathcal{E}^\infty_{\lambda-\Lambda} Ind^G_K(\tau) = \cup_{r\in\mathbf{R}} \mathcal{E}^\infty_{\lambda-\Lambda,r} Ind^G_K(\tau).$$

Our goal is to describe $\mathcal{E}^\infty_{\lambda-\Lambda} Ind^G_K(\tau)$ in terms of a Poisson transform, at least for the "generic" $\lambda - \Lambda$. The following well known result is very important to us.

Proposition 2.3 $C(\lambda) = \int_{\overline{N}} \tau(k(\overline{n}))e^{-(\lambda+\rho)H(\overline{n})}d\overline{n}$ *is holomorphic on*

$$\{\lambda \in \mathfrak{a}^*_{\mathbf{C}} \mid Re < \lambda, \alpha >> 0, \text{ for each } \alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})\}.$$

Moreover there exists a meromorphic continuation to $\mathfrak{a}^*_{\mathbf{C}}$.

Proposition 2.4 *Let $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$ such that $Re < \lambda, \alpha >> 0$, for $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. Then*

$$\lim_{t \to \infty} e^{(-\lambda+\rho)(H)} P_T \phi(g \exp tH) = C(\lambda) T\phi.$$

for each $H \in \mathfrak{a}^+$, $T \in Hom_M(V_\sigma, V)$, and $\phi \in C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)$. Here $\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \forall \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}$.

Proof: First we observe $k \to \tau(k) T\phi(g \exp tHk)$ is a function on K/M. By Theorem 5.20 in Chapter I in [Helg1]

$$\begin{aligned} P_T\phi(g \exp tH) &= \textstyle\int_{\overline{N}} \tau(k(\overline{n})) T\phi(g \exp tHk(\overline{n})) e^{-2\rho H(\overline{n})} d\overline{n} \\ &= \textstyle\int_{\overline{N}} e^{-(\lambda+\rho)H(\overline{n})} \tau(k(\overline{n})) T\phi(g(\exp tH)\overline{n}) d\overline{n} = \\ & e^{(\lambda-\rho)tH} \textstyle\int_{\overline{N}} e^{-(\lambda+\rho)H(\overline{n})} \tau(k(\overline{n})) T\phi(g a_t \overline{n} a_t^{-1}) d\overline{n} \end{aligned}$$

Here $a_t = \exp tH$. So

$$e^{-(\lambda-\rho)tH} P_T\phi(g \exp tH) = \textstyle\int_{\overline{N}} e^{-(\lambda+\rho)H(\overline{n})} \tau(k(\overline{n})) T\phi(g a_t \overline{n} a_t^{-1}) d\overline{n}.$$

Since $a_t \overline{n} a_t^{-1} \to e$, as $t \to \infty$. Formally we have $P_T\phi(g \exp tH) \to C(\lambda) T\phi(g)$, as $t \to \infty$. To justify the exchange of two limits we use an argument due to Helgason.

Let $\lambda = \xi + \sqrt{-1}\eta$, for ξ, $\eta \in \mathfrak{a}^*$. Our assumption on λ amounts to $A_\xi \in \mathfrak{a}^+$, where A_ξ is given by $< \mu, A_\xi >= K(\xi, \mu)$, for each $\mu \in \mathfrak{a}^*$.

It was proved by Harish-Chandra that

$B(H, H(\overline{n})) \geq 0$, $B(H, H(\overline{n}) - H(a_t \overline{n} a_t^{-1})) \geq 0$, for each $H \in \mathfrak{a}^+$.

Thus if we choose ϵ such that $0 < \epsilon < 1$, $A_\rho - \epsilon A_\xi \in \mathfrak{a}^+$, and put

$$C = \sup_{\overline{n}, t} |\tau(k(\overline{n})) T\phi(gk(a_t \overline{n} a_t^{-1}))| < \infty,$$

then

$$\begin{aligned} |e^{-(\lambda+\rho)H(\overline{n})} \tau(k(\overline{n})) T\phi(g a_t \overline{n} a_t^{-1})| &= |e^{-(\lambda+\rho)H(\overline{n})} e^{(\lambda-\rho)H(a_t \overline{n} a_t^{-1})} \tau(k(\overline{n})) T\phi(gk(a_t \overline{n} a_t^{-1}))| \\ &\leq C|e^{-(\xi+\rho)H(\overline{n})} e^{(\xi-\rho)H(a_t \overline{n} a_t^{-1})}| \leq C|e^{-(\xi+\rho)H(\overline{n})} e^{(\xi-\epsilon\xi)H(a_t \overline{n} a_t^{-1})}| \\ &\leq C|e^{-(\xi+\rho)H(\overline{n})} e^{(\xi-\epsilon\xi)H(\overline{n})} \leq C|e^{(-\epsilon\xi-\rho)H(\overline{n})}|. \end{aligned}$$

This being integrable over $\overline{N}$ justifies letting $t \to \infty$ under the integral sign and proves Proposition 2.4.

§3 Asymptotics

By a formal expansion at a point $H_0 \in \mathfrak{a}^+$, we mean a formal sum

$$\sum_{\xi \in X} p_\xi(H,t) e^{t\xi(H)},$$

where X is a subset of $\mathfrak{a}^*_{\mathbf{C}}$ such that the subset $X(N)$ given by

$$X(N) = \{\xi \in X \mid Re\xi(H_0) \geq N\}$$

is a finite set for each $N \in \mathbf{R}$, where p_ξ is a continuous function defined in a neighborhood of $\{H_0\} \times \mathbf{R}$ and polynomial in the last variable.

Let f be a function $\mathfrak{a}^+ \to V$. If $N \in \mathbf{R}$ we say the formal sum is asymptotic to f of order N at H_0, if there exist a neighborhood of H_0 in $\mathfrak{a}^+$, say U, and constants $\epsilon \geq 0$, $C \geq 0$, such that

$$|f(tH) - \sum_{\xi \in X(N)} p_\xi(H,t) e^{t\xi(H)}| \leq Ce^{(N-\epsilon)t},$$

for each $H \in U$, $t \geq 0$.

Moreover, we say the formal expansion is an asymptotic expansion for f at H_0 if for every $N \in \mathbf{R}$ it is asymptotic to f of order N at H_0. We write this as

$$f(tH) \sim \sum_{\xi \in X} p_\xi(H,t) e^{t\xi(H)} \qquad (t \to \infty)$$

The following result shows that the p_ξ's are essentially unique.

Proposition 3.1 *Let $X \subset \mathfrak{a}^*_{\mathbf{C}}$, and let $\sum_{\xi \in X} p_\xi(H,t)e^{t\xi(H)}$ and $\sum_{\xi \in X} q_\xi(H,t)e^{t\xi(H)}$ be formal expansions at H_0, both assumed to be asymptotic to $f \colon \mathfrak{a}^+ \to V$. Then for each $\xi \in X$, there is a neighborhood U of H_0, such that $p_\xi = q_\xi$ on $U \times \mathbf{R}$.*

Proof: See Proposition 3.1 in [BS].

Let $\lambda - \Lambda$ be a character of Z_τ in the sense of Proposition 1.11, where $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, and Λ is given by the infinitesimal character of an irreducible representation of M. Let

$X(\lambda, \Lambda)$ be the subset of $\mathfrak{a}_{\mathbf{C}}^*$ defined by

$$X(\lambda, \Lambda) = \{w(\lambda - \Lambda) + \Lambda_\sigma - \rho - \mathbf{N} \cdot \Delta \mid w \in \tilde{W}, \ \sigma \in \tau, \ (w(\lambda - \Lambda) + \Lambda_\sigma)|\mathfrak{t} = 0\}$$

Theorem 3.2 *(i) For each* $f \in \mathcal{E}^\infty_{\lambda-\Lambda} Ind_K^G(\tau)$, $x \in G$, *and* $\xi \in X(\lambda, \Lambda)$, *there exists a unique polynomial* $p_{\lambda,\xi}(f, x, \cdot)$ *on* $\mathfrak{a}$ *with values in* V, *such that*

$$f(tH) \sim \sum_{\xi \in X(\lambda,\Lambda)} p_{\lambda,\xi}(f, x, tH) e^{t\xi(H)} \qquad (t \to \infty)$$

at every $H_0 \in \mathfrak{a}^+$, *and the polynomials have degree* $\leq d$, *where* d *is the number of elements in* $\Sigma^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$,

(ii) let $r \in \mathbf{R}$ *and* $\xi \in X(\lambda, \Lambda)$, *there exists* $r' \in \mathbf{R}$ *such that* $f \to p_{\lambda,\xi}(f, \cdot, \cdot)$ *is a continuous map of* $\mathcal{E}^\infty_{\lambda-\Lambda,r} Ind_K^G(\tau)$ *into* $C^\infty_{r'}(G, V) \otimes P_d(\mathfrak{a})$, *equivariant for the left action of* G *on* $\mathcal{E}^\infty_{\lambda-\Lambda,r} Ind_K^G(\tau)$ *to* $C^\infty_{r'}(G, V) \otimes P_d(\mathfrak{a})$.

Theorem 3.3 *Let* Ω *be an open set in* $\mathfrak{a}_{\mathbf{C}}^*$. *Let* $\{f_\lambda\}_{\lambda \in \Omega}$ *be a holomorphic family in* $C_r^\infty Ind_K^G(\tau)$ *such that* $f_\lambda \in \mathcal{E}^\infty_{\lambda-\Lambda,r} Ind_K^G(\tau)$ *for each* $\lambda \in \Omega$. *Fix* $\lambda_0 \in \Omega$ *and* $\xi_0 \in X(\lambda_0, \Lambda)$. *Let*

$$\Xi(\lambda) = \{w(\lambda - \Lambda) + \Lambda_\sigma - \rho - \mu \in X(\lambda, \Lambda) \mid w(\lambda_0 - \Lambda) + \Lambda_\sigma - \rho - \mu = \xi_0\}.$$

There exist an open neighborhood $\Omega_0 \subset \Omega$ *of* λ_0 *and a constant* $r' \in \mathbf{R}$ *such that the map* $(\lambda, H) \to \sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(f_\lambda, \cdot, H) e^{\xi(H)}$ *is continuous from* $\Omega \times \mathfrak{a}^+$ *into* $C^\infty_{r'}(G, V)$ *and in addition holomorphic in* λ.

§4 Some algebraic results

This section is a necessary preparation for the proof of the theorems stated in last section. It is strongly influenced by [Ban] and [BS].

Let E be the set of W-harmonic polynomials on $\mathfrak{a}^*$. It is well known that j: $E \otimes I(\mathfrak{a}) \to S(\mathfrak{a})$ is bijective, where $j(e \otimes h) = eh$.

Now let r: $I(\mathfrak{h}_{\mathbf{C}}) \to I(\mathfrak{a})$ be the restriction map. *We assume r is surjective for the rest of the thesis.* According to [Helg3] if G/K is irreducible there are just four exceptions, and they only occur among symmetric spaces of exceptional groups.

Pick a set of algebraically independent homogeneous generators of $I(\mathfrak{a})$, say, p_1, ..., p_m. Choose homogeneous elements q_1, ..., q_m in $I(\mathfrak{h}_{\mathbf{C}})$, such that $r(q_i) = p_i$, for $i = 1, ..., m$. Define $I_1(\mathfrak{h}_{\mathbf{C}}) = \{P(q_1, ... q_m) \mid P \text{ is any polynomial } \}$.

For any $\mu \in \mathfrak{h}^*_{\mathbf{C}}$, let

$$I^-_{1,\mu} = \{(T_\mu p)^- \mid p \in I_1(\mathfrak{h}_{\mathbf{C}})\}$$

Here $T_\mu p(\nu) = p(\mu + \nu)$, for each $\nu \in \mathfrak{h}^*_{\mathbf{C}}$, and $(T_\mu p)^-(\lambda) = p(\mu + \lambda)$, for each $\lambda \in \mathfrak{a}^*$.

Proposition 4.1 *The map j_μ: $E \otimes I^-_{1,\mu} \to S(\mathfrak{a})$ is bijective, where $j_\mu(e \otimes h) = eh$.*

Proof: Observe $(T_\mu q_i)^- = p_i + r_i$, with $degr_i < degp_i$. Using the fact that j is bijective and by induction we are done.

Let $\mathfrak{z}_1(\mathfrak{g}) = \gamma^{-1}(I_1(\mathfrak{h}))$. Here γ is the Harish-Chandra's homomorphism. For each $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, $\Lambda = \Lambda_\sigma$ for some $\sigma \in \tau$, let

$$I(\lambda, \Lambda) = \{Z \in \mathfrak{z}_1(\mathfrak{g}) \mid \gamma(Z)(\lambda - \Lambda) = 0\}.$$

Recall $I(\tau)$ is the left ideal of $U(\mathfrak{g}) \otimes EndV$ generated by $X \otimes 1 + 1 \otimes \tau(X)$, for all $X \in \mathfrak{k}$. Let $J(\lambda, \Lambda)$ be the left ideal generated by $I(\lambda, \Lambda)$ and $I(\tau)$. Let

$$\mathfrak{Y}_{\lambda,\Lambda} = U(\mathfrak{g}) \otimes EndV / J(\lambda, \Lambda).$$

Our interest in $\mathfrak{Y}_{\lambda,\Lambda}$ comes from the fact that for $f \in \mathcal{E}_{\lambda-\Lambda} Ind^G_K(\tau)$, the map $u \otimes T \to TR_u f$ factors through $\mathfrak{Y}_{\lambda,\Lambda}$ since f is killed by $J(\lambda, \Lambda)$. We shall find below an underlying vector space for $\mathfrak{Y}_{\lambda,\Lambda}$ independent of λ.

Define $\mathfrak{Y} = U(\bar{\mathfrak{n}}) \otimes E \otimes EndV$. We shall construct a linear bijection of $\mathfrak{Y}$ with $\mathfrak{Y}_{\lambda,\Lambda}$. For this purpose we need the following proposition.

First we identify $\mathfrak{Y}$ with a subspace of $U(\mathfrak{g}) \otimes EndV$ as follows: $u \otimes e \otimes T \to (u \cdot e) \otimes T$, for $u \in U(\bar{\mathfrak{n}})$, $e \in E$, and $T \in EndV$. Here $\cdot$ denotes the multiplication in $U(\mathfrak{a} + \bar{\mathfrak{n}})$.

Let $\Psi : \mathfrak{Y} \otimes \mathfrak{Z}_1(\mathfrak{g}) \to U(\mathfrak{g}) \otimes EndV/I(\tau)$ be the map defined by

$$\Psi(y \otimes Z) = y \cdot (Z \otimes 1) + I(\tau),$$

for $y \in \mathfrak{Y}$, $Z \in \mathfrak{Z}_1(\mathfrak{g})$. Here $\cdot$ means the multiplication in $U(\mathfrak{g}) \otimes EndV$.

Proposition 4.2 *Ψ is bijective.*

Proof: By the Iwasawa decomposition $U(\mathfrak{g}) \otimes EndV/I(\tau) \cong U(\bar{\mathfrak{n}}) \otimes U(\mathfrak{a}) \otimes EndV$. Via this isomorphism the degree on $U(\mathfrak{a})$ induces a degree on $U(\mathfrak{g}) \otimes EndV/I(\tau)$, denoted by $deg_{\mathfrak{a}}$. Let $\mathfrak{Y} \otimes \mathfrak{Z}_1(\mathfrak{g})$ be filtered by the total degree on $E \otimes \mathfrak{Z}_1(\mathfrak{g})$. Notice

$$deg_{\mathfrak{a}}(Z \otimes 1 - (T_{\rho-\Lambda_\sigma}\gamma(Z))^{-} \otimes 1 + I(\tau)) < deg(Z \otimes 1),$$

for $Z \in \mathfrak{Z}_1(\mathfrak{g})$, and each $\sigma \in \tau$.

So Ψ preserves the filtrations. It also follows that the graded map

$$gr\Psi : U(\bar{\mathfrak{n}}) \otimes gr(E \otimes \mathfrak{Z}_1(\mathfrak{g})) \otimes EndV \to U(\bar{\mathfrak{n}}) \otimes U(\mathfrak{a}) \otimes EndV$$

associated to Ψ, is given by

$$u \otimes e \otimes Z \otimes T \to u \cdot e \cdot (T_{\rho-\Lambda_\sigma}\gamma(Z))^{-} \otimes T,$$

for $u \in U(\bar{\mathfrak{n}})$, $e \in E$, $Z \in \mathfrak{Z}_1(\mathfrak{g})$, and $T \in Hom(V_\sigma, V)$. Here we use Proposition 1.15.

This is bijective because of Proposition 4.1. So the proposition follows.

Corollary 4.3 *(i) Ψ maps $\mathfrak{Y} \otimes I(\lambda, \Lambda)$ onto $J(\lambda, \Lambda)$ modulo $I(\tau)$, (ii) for each $u \in U(\mathfrak{g}) \otimes EndV$ there exists a unique $y \in \mathfrak{Y}$, such that $u - y \in J(\lambda, \Lambda)$.*

Proof: See Corollary 5.2 in [BS].

From the corollary we obtain a linear bijection b_λ of $\mathfrak{Y}(\lambda, \Lambda)$ onto $\mathfrak{Y}$, defined by $u - b_\lambda(u + J(\lambda, \Lambda)) \subset J(\lambda, \Lambda)$. Through this bijection $\mathfrak{Y}$ is equipped with a $(\mathfrak{g}, K)$ module structure from $\mathfrak{Y}(\lambda, \Lambda)$, by making b_λ a morphism of modules. Recall the $\mathfrak{g}$ action on $\mathfrak{Y}(\lambda, \Lambda)$ is induced from the left multiplication in $U(\mathfrak{g})$, and the K action is induced from the following K action on $U(\mathfrak{g}) \otimes EndV$.

$$k.(u \otimes T) = Ad(k)u \otimes T\tau(k^{-1}),$$

for each $k \in K$, $u \in U(\mathfrak{g})$, and $T \in EndV$. Notice the difference from the action we use to define $U(\mathfrak{g})^K$.

Let τ_λ denote the resulting $\mathfrak{g}$ action on $\mathfrak{Y}$. Notice the action of $\bar{\mathfrak{n}}$ on $\mathfrak{Y}$ is just the left multiplication. The action of $\mathfrak{a}$ can be determined as follows: Let $y \in \mathfrak{Y} \subset U(\mathfrak{g}) \otimes EndV$, $H \in \mathfrak{a}$, then $H \cdot y$ can be written (modulo $I(\tau)$) as $\Psi(\sum y_i \otimes Z_i)$ according to Proposition 4.2. Then by the definition of τ_λ we have

$$(*) \qquad \tau_\lambda(H)y = \sum \gamma(Z_i)(\lambda - \Lambda)y_i.$$

For each $k \in \mathbf{N}$, let $\bar{\mathfrak{n}}^k$ be the linear span of k times product of $\bar{\mathfrak{n}}$ in $U(\bar{\mathfrak{n}})$. Then τ_λ induces a representation τ_λ^k of $\mathfrak{a} + \mathfrak{m}$ on the finite dimensional space $\mathfrak{Y}/\bar{\mathfrak{n}}^k\mathfrak{Y}$. In particular τ_λ^1 is a representation of $\mathfrak{a} + \mathfrak{m}$ on $\mathfrak{Y}/\bar{\mathfrak{n}}\mathfrak{Y} \cong E \otimes EndV$. By $(*)$ we know τ_λ and τ_λ^k are holomorphic in λ.

Let $\{\lambda_1, ..., \lambda_l\}$ be the set of weights of τ_λ^1 restricted to $\mathfrak{a}$, and $\Lambda_k \subset -\mathbf{N} \cdot \Delta$ an enumeration of the weights of the $\mathfrak{a}$-module $U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}})$.

Proposition 4.4 *For each $k \in \mathbf{N}$, $k \geq 1$, the set of weights of $(\tau_\lambda^k, \mathfrak{a})$ is*

$$\{\lambda_i + \mu \ \mid \ i = 1, ..., l, \mu \in \Lambda_k\}.$$

Proof: By induction on k. It is trivial for $k = 1$. For $k > 1$, the induction step is a consequence of the following two exact sequences of $\mathfrak{a}$-modules.

$$0 \to \bar{\mathfrak{n}}^{k-1}U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}}) \otimes \mathfrak{Y}(\lambda, \Lambda)/\bar{\mathfrak{n}}\mathfrak{Y}(\lambda, \Lambda) \to \mathfrak{Y}(\lambda, \Lambda)/\bar{\mathfrak{n}}^k\mathfrak{Y}(\lambda, \Lambda) \to \mathfrak{Y}(\lambda, \Lambda)/\bar{\mathfrak{n}}^{k-1}\mathfrak{Y}(\lambda, \Lambda) \to 0$$

$$0 \to \bar{\mathfrak{n}}^{k-1}U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}}) \to \bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}}) \to \bar{\mathfrak{n}}^{k-1}U(\bar{\mathfrak{n}}) \to 0$$

Let $\overline{V}_k = \mathfrak{Y}/\bar{\mathfrak{n}}^k\mathfrak{Y}$, and $\tilde{V}_k$ be a finite dimensional subspace of $\mathfrak{Y}$ mapped bijectively onto $\overline{V}_k$ by the canonical projection. Let π: $\tilde{V}_k \to \overline{V}_k$ be the restriction of the canonical projection. Define m: $\mathfrak{Y} \to U(\mathfrak{g}) \otimes EndV$ by

$$m(u \otimes e \otimes T) = (u \cdot e) \otimes T$$

for $u \in U(\bar{\mathfrak{n}})$, $e \in E$, and $T \in EndV$.

Let V_k be the image of $\tilde{V}_k$ under m. Let η: $V_k \to \tilde{V}_k$ be the inverse of $m|\tilde{V}_k$.

Proposition 4.5 *For $k \in \mathbf{N}$, $k \geq 1$, there exist*

(i) an algebra homomorphism $b_k(\lambda, \cdot)$: $\mathfrak{Z}(\mathfrak{a}+\mathfrak{m}) \to EndV_k$,

(ii) a linear map y_λ: $\mathfrak{Z}(\mathfrak{a}+\mathfrak{m}) \otimes V_k \to \bar{\mathfrak{n}}^k U(\mathfrak{a}+\bar{\mathfrak{n}}) \otimes EndV$, both depending polynomially on λ, such that for all $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, $D \in \mathfrak{Z}(\mathfrak{a}+\mathfrak{m})$, and $v \in V_k$,

$$Dv - b_k(\lambda, D)v - y_\lambda(D, v) \in J(\lambda, \Lambda).$$

Proof: Let p_λ: $U(\mathfrak{g}) \otimes EndV \to \mathfrak{Y}$ be the map defined by

$$p_\lambda(u \otimes T) = \tau_\lambda(u)(1 \otimes 1 \otimes T)$$

for $u \in U(\mathfrak{g})$, and $T \in EndV$.

Define for $D \in \mathfrak{Z}(\mathfrak{a}+\mathfrak{m})$, $\tilde{v} \in \tilde{V}_k$ the maps

$$\tilde{b}_k(\lambda, D) = \pi^{-1} \circ \tau^k_\lambda \circ \pi \in End\tilde{V}_k,$$

$$\tilde{y}_\lambda(D, \tilde{v}) = p_\lambda((D \otimes 1) \cdot m(\tilde{v})) - m(\tilde{b}_k(\lambda, D)\tilde{v}) \in \mathfrak{Y}.$$

Then $b_k(\lambda, \cdot)$ and y_λ are defined by

$$b_k(\lambda, D) = m \circ \tilde{b}_k(\lambda, D) \circ \eta,$$

$$y_\lambda(D, v) = m(\tilde{y}_\lambda(D, \eta(v))),$$

for $D \in \mathfrak{Z}(\mathfrak{a} + \mathfrak{m})$, $v \in V_k$.

Corollary 4.6 *As a representation of* $\mathfrak{a}$, $b_k(\lambda, \cdot)$ *has the same weights as* $(\tau_\lambda^k, \mathfrak{a})$, *i.e.* $\{\lambda_i + \mu \mid i = 1, ..., l, \mu \in \Lambda_k\}$.

Proof: By definition $b_k(\lambda, D) = m \circ \tilde{b}_k(\lambda, D) \circ \eta$, and $\eta = (m|\tilde{V}_k)^{-1}$. So $b_k(\lambda, \cdot)$ has the same weights as $\tilde{b}_k(\lambda, \cdot)$. Since $\tilde{b}_k(\lambda, \cdot) = \pi^{-1} \circ \tau_\lambda^k \circ \pi$ the proof is complete.

Let V_k^* be the dual space of V_k, and $b_k^*(\lambda, \cdot)$ be the transpose of $b_k(\lambda, \cdot)$. For each weight ξ of $b_k^*(\lambda, \cdot)$ we denote $P_{\lambda,\xi}$ the projection map from V_k^* onto the generalized weight space of ξ, along the remaining generalized weight spaces. We now consider the holomorphic dependence of $P_{\lambda,\xi}$ on λ.

Proposition 4.7 *There exists for each* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, *and each weight* ξ *a unique polynomial* $q_{\lambda,\xi}$ *on* $\mathfrak{a}$ *with values in* $End V_k^*$, *such that*

$$P_{\lambda,\xi} q_{\lambda,\xi}(H) P_{\lambda,\xi} = q_{\lambda,\xi}(H),$$

$$\exp b_k^*(\lambda, H) = \sum_\xi e^{\xi(H)} q_{\lambda,\xi}(H),$$

for $H \in \mathfrak{a}$.

Proof: Let $V_k^*(\xi)$ be the generalized weight space of ξ. Then the restriction of $b_k^*(\lambda, \cdot)$ to $V_k^*(\xi)$ gives a representation of $\mathfrak{a}$. $\mathfrak{a}$ is abelian so in particular solvable. Hence by Lie's theorem one can find a basis such that $b_k^*(\lambda, H)|V_k^*(\xi)$ corresponds to an upper triangular matrix, for each $H \in \mathfrak{a}$. The diagonal entries are $\xi(H)$. So there exists a unique polynomial $q_{\lambda,\xi}(H)$ on $\mathfrak{a}$ with values in $End V_k^*$, such that

$$\exp b_k^*(\lambda, H)|V_k^*(\xi) = e^{\xi(H)} q_{\lambda,\xi}(H),$$

Fix $\lambda_0 \in \mathfrak{a}_{\mathbf{C}}^*$, and ξ_0 a weight of $b_k^*(\lambda_0, \cdot)$. For each $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, let

$$\Xi(\lambda) = \{w(\lambda - \Lambda) + \Lambda_\sigma - \rho - \mu \in X(\lambda, \Lambda) \mid w(\lambda_0 - \Lambda) + \Lambda_\sigma - \rho - \mu = \xi_0\}$$

Proposition 4.8 *There exist a neighborhood $\Omega_0(\lambda_0)$ of λ_0 and a neighborhood $V(\xi_0)$ of ξ_0, such that*

$$P(\lambda) = \sum_{\xi \in V(\xi_0)} P_{\lambda,\xi} \in End V_k^*$$

is holomorphic in $\Omega_0(\lambda_0)$, and $\{\xi \in V(\xi_0) \mid \xi$ is a weight of $b_k^*(\lambda, \cdot)\} \cap X(\lambda, \Lambda) \subset \Xi(\lambda)$.

Proof: It follows at once from Lemma 4.9 below.

Let F be an N-dimensional complex vector space, and τ_z a family of representations of $\mathfrak{a}$ in F, depending on a parameter $z \in \mathbf{C}^n$. For each weight ξ of τ_z let $P_{z,\xi}$ be the projection map from F onto the generalized weight space $V(\xi)$, along the remaining generalized weight spaces. Fix $z_0 \in \mathbf{C}^n$, and ξ_0 a weight of τ_{z_0}

Lemma 4.9 *Given any neighborhood $N(\xi_0)$ of ξ_0 there exist a neighborhood $V(\xi_0)$ of ξ_0 in $N(\xi_0)$, and a neighborhood $\Omega(z_0)$ of z_0, such that*

$$P(z) = \sum_{\xi \in V(\xi_0)} P_{z,\xi} \in End F$$

is holomorphic in z in $\Omega(z_0)$.

Proof: We use the argument in Chapter II in [Kato]. First let us consider the case where $\dim \mathfrak{a} = 1$.

Pick a nonzero element $H_0 \in \mathfrak{a}$. Let

$$T(z) = \tau_z(H_0) \in End F.$$

Then $\lambda_0 = \xi_0(H_0)$ is an eigenvalue of $T(z_0) = \tau_{z_0}(H_0)$. Define

$$R(z, \lambda) = (T(z) - \lambda)^{-1},$$

for $z \in \mathbf{C}^n$, and $\lambda \in \mathbf{C}$. By Theorem 1.5 in Section 3 of Chapter II in [Kato], $R(z, \lambda)$ is holomorphic in the two variables z and λ in each domain where λ is not an eigenvalue of $T(z)$. Moreover, for each (z_1, λ) in such a domain,

$$R(z, \lambda) = R(z_1, \lambda) + \sum_{I \in \mathbf{N}^n} R_I(\lambda)(z - z_1)^I,$$

where $R_I(\lambda)$ are determined by $R(z_1, \lambda)$, and they are holomorphic in λ.

This is called the second Neumann series for the resolvent. It is uniformly convergent for sufficiently small $z - z_1$ and $\lambda \in \Gamma$ if Γ is a compact subset of the resolvent set of $T(z_1)$.

Let Γ be a closed positively oriented curve in the resolvent set of $T(z_0)$ enclosing λ_0 but no other eigenvalues of $T(z_0)$. Then

$$P(z) = -\frac{1}{2i\pi} \int_\Gamma R(z, \lambda) d\lambda$$

is holomorphic in z, for $z - z_0$ sufficiently small.

It is easy to see $P(z)$ is equal to the sum of the eigenprojections for all eigenvalues of $T(z)$ lying inside Γ. This basically takes care of the case $\dim \mathfrak{a} = 1$. In general we choose a basis e_1, ..., e_m for $\mathfrak{a}$. We can duplicate the above process to $T_i(z) = \tau_z(e_i)$, for $i = 1$, ..., m. Thus we get $P_i(z)$, $i = 1$, ..., m. Then the composition of P_i's is our $P(z)$.

§5 Existence of asymptotic expansion

The methods we use in this section are similar to those used in [Ban], Section 12. Also see [BS], Section 6.

Fix $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, $H_0 \in \mathfrak{a}^+$ and $r \in \mathbf{R}$. If A_1, A_2 are Banach spaces we denote $B(A_1, A_2)$ the Banach space of bounded linear operators from A_1 to A_2.

Proposition 5.1 *There exist, for each $N \in \mathbf{R}$,*

(a) open neighborhoods Ω of $\lambda_0 \in \mathfrak{a}_{\mathbf{C}}^*$ and U of $H_0 \in \mathfrak{a}^+$,

(b) constants k, $q \in \mathbf{N}$, $r' \geq r$, and C, $\epsilon > 0$,

(c) a continuous map

$$\Phi : \Omega \times U \to B(C_r^q(G,V), V_k^* \otimes C_{r'}(G,V)),$$

holomorphic in the first variable, and

(d) a linear form $\eta \in (V_k^*)^*$, such that

(i) $\Phi(\lambda, H)$ intertwines the left actions of G on $C_r^q(G,V)$ and $C_{r'}(G,V)$, for all $(\lambda, H) \in \Omega \times U$, and

(ii)

$$\|R_{\exp tH} f - (\eta \circ \exp b_k^*(\lambda, tH) \otimes 1)\Phi(\lambda, H) f\|_{r'} \leq C \|f\|_{q,r} e^{(N-\epsilon)t}$$

for $f \in \mathcal{E}_{\lambda-\Lambda} Ind_K^G(\tau) \cap C_r^q(G,V)$, $\lambda \in \Omega$, $H \in U$, $t \geq 0$.

Proof: In the same way as for Proposition 12.6 in [Ban].

We now begin the proof of Theorem 3.3. Using Proposition 4.7 we can write

$$(\eta \circ \exp b_k^*(\lambda, tH) \otimes 1)\Phi(\lambda, H) = \sum_{\xi} p_{\lambda,\xi}(H,t) e^{t\xi(H)},$$

for $\lambda \in \Omega$, $H \in U$, $t \geq 0$, where the summation extends to the weights ξ of $b_k^*(\lambda, \cdot)$ which by Corollary 4.6 is the set

$$\{\lambda_i + \mu \mid i = 1, ..., l, \mu \in \Lambda_k\},$$

and where $p_{\lambda,\xi}(H,t) = (\eta \circ q_{\lambda,\xi}(tH) \otimes 1)\Phi(\lambda, H) \in B(C_r^q, C_{r'})$, which is continuous in H and polynomial in t.

From (d) (ii) of Proposition 5.1 we have

$$\|R_{\exp tH} f - \sum_{\xi} e^{t\xi(H)} p_{\lambda,\xi}(H,t) f\|_{r'} \leq C\|f\|_{q,r} e^{t(N-\epsilon)}$$

for $f \in \mathcal{E}_{\lambda-\Lambda} Ind_K^G(\tau) \cap C_r^q(G, V)$.

Since N is arbitrary we have for each $g \in G$,

$$f(g \exp tH) \sim \sum_{\xi \in \tilde{X}(\lambda,\Lambda)} (p_{\lambda,\xi}(H,t)f)(g) e^{t\xi(H)}, \qquad (t \to \infty)$$

Here $\tilde{X}(\lambda, \Lambda) = \{\lambda_i + \mu \mid i = 1, ..., l, \mu \in -\mathbf{N} \cdot \Delta\}$

Lemma 5.2 *Let $X \subset \mathfrak{a}_{\mathbf{C}}^*$ and $f\colon \mathfrak{a}^+ \to V$. Assume that for each $H_0 \in \mathfrak{a}^+$ there is a given formal expansion*

$$\sum_{\xi \in X} p_{\xi,H_0}(H,t) e^{t\xi(H)}$$

which is an asymptotic expansion for f at H_0. Then for each $\xi \in X$ there exists a unique continuous function $p_\xi\colon \mathfrak{a}^+ \to V$ such that for each $H_0 \in \mathfrak{a}^+$ there is a neighborhood U with

$$p_{\xi,H_0}(H,t) = p_\xi(tH),$$

for $H \in U$, and $t > 0$.

Proof: See Corollary 3.4 in [BS].

As can be seen in the proof of Proposition 12.6 in [Ban], $\Phi(\lambda, tH) = \Phi(\lambda, H)$, for $t > 0$, $H \in U$ with $tH \in U$. Thus $(p_{\lambda,\xi}(H,t)f)(g) = (p_{\lambda,\xi}(tH,1)f)(g)$, for $t > 0$, $H \in U$ with $tH \in U$. By Lemma 5.2, for each $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, $\xi \in \tilde{X}(\lambda, \Lambda)$, and $r \in \mathbf{R}$, there exist constants $r' \in \mathbf{R}$, $q \in \mathbf{N}$, and a unique continuous map $p_{\lambda,\xi}(\cdot,\cdot,\cdot)$: $\mathfrak{a}^+ \to B(\mathcal{E}_{\lambda-\Lambda} Ind_K^G(\tau) \cap C_r^q(G,V), C_{r'}(G,V))$, such that

$$f(g\exp tH) \sim \sum_{\xi\in\tilde{X}(\lambda,\Lambda)} p_{\lambda,\xi}(f,g,tH)e^{t\xi(H)}, \qquad (t\to\infty)$$

at every $H_0 \in \mathfrak{a}$, for $f \in \mathcal{E}_{\lambda-\Lambda}Ind_K^G(\tau)\cap C_r^q(G,V)$.

To complete the proof of Theorem 3.3 it remains to show

(1) we can replace $\tilde{X}(\lambda,\Lambda)$ by $X(\lambda,\Lambda)$,

(2) $p_{\lambda,\xi}(f,g,H)$ is a polynomial in H with order $\leq d$.

We shall finish the proof in the next section. We now consider the holomorphic dependence in λ in order to prove Theorem 3.4.

Let $r \in \mathbf{R}$ and Ω be an open set in $\mathfrak{a}_{\mathbf{C}}^*$. Let $\{f_\lambda\}_{\lambda\in\Omega}$ be a holomorphic family in $C_r^\infty(G,V)$, and $f_\lambda \in \mathcal{E}_{\lambda-\Lambda}^\infty Ind_K^G(\tau)$, for each $\lambda\in\Omega$. We now study the asymptotic expansion of f_λ. Fix $\lambda_0\in\Omega$, and $\xi_0\in\tilde{X}(\lambda_0,\Lambda)$

Proposition 5.3 *There exist a neighborhood $\Omega(\lambda_0)$ of λ_0 in Ω and a neighborhood $V(\xi_0)$ of ξ_0 in $\mathfrak{a}_{\mathbf{C}}^*$, such that*

$$(\lambda, H) \to \sum_{\xi\in V(\xi_0)} p_{\lambda,\xi}(f_\lambda,\cdot,H)e^{\xi(H)}$$

is continuous from $\Omega(\lambda_0)\times U$ to $C_{r'}^{q'}(G,V)$ for some $q'\in\mathbf{N}$, $r'\in\mathbf{R}$, and in addition holomorphic in λ. Moreover, we can choose $V(\xi_0)$ such that $V(\xi_0)\cap X(\lambda,\Lambda)\subset\Xi(\lambda)$.

Proof: It follows from Proposition 4.8.

§6 Differential equations for the coefficients

In this section we derive certain differential equations for the vector-valued functions $p_{\lambda,\xi}(f,g,\cdot)$ on $\mathfrak{a}^+$, where $f \in \mathcal{E}^{\infty}_{\lambda-\Lambda} Ind_K^G(\tau)$, and $g \in G$.

Fix $Z \in \mathfrak{Z}(\mathfrak{g})$, and $D = \mu(Z) \in Z_\tau$. We can choose finitely many $x_i \in \bar{\mathfrak{n}}U(\bar{\mathfrak{n}})$, and $v_i \in U(\mathfrak{a}) \otimes EndV$, such that

$$Z - \tilde{\Gamma}_1(Z \otimes 1) - \sum x_i v_i \in I(\tau),$$

and $ad(\mathfrak{a})$ acts on x_i by a weight $-\eta_i \neq 0$, where $\eta_i \in \mathbf{N}\cdot\Delta$. v_i, $\tilde{\Gamma}_1(Z\otimes 1) \in U(\mathfrak{a})\otimes EndV$ can be interpreted as differential operators with constant coefficients on $C^\infty(\mathfrak{a}, V)$.

Proposition 6.1 *Let $f \in \mathcal{E}^{\infty}_{\lambda-\Lambda} Ind_K^G(\tau)$. Then the functions $p_{\lambda,\xi}(f,\cdot,\cdot)e^\xi$ from $G \times \mathfrak{a}^+$ to V satisfy the following recursive equations*

$$1\otimes\partial(\tilde{\Gamma}_1(Z\otimes 1)-\gamma(Z)(\lambda-\Lambda))(p_{\lambda,\xi}(f,\cdot,\cdot)e^\xi) = -\sum_{i,\xi+\eta_i\in\tilde{X}(\lambda,\Lambda)} R_{x^i}\otimes e^{-\eta_i}\partial(v^i)(p_{\lambda,\xi+\eta_i}(f,\cdot,\cdot)e^{\xi+\eta_i}),$$

for all $\xi \in \tilde{X}(\lambda,\Lambda)$.

The proof is the same as for Proposition 7.1 in [BS].

Proof of Theorem 3.3: Let

$$V = \oplus_{\Lambda_1 \in \mathfrak{t}^*} V(\Lambda_1),$$

where $V(\Lambda_1) = \oplus_{\sigma\in\tau,\Lambda_\sigma=\Lambda_1} V(\sigma)$.

Let $P(\Lambda_1)$ be the projection from V to $V(\Lambda_1)$. By Corollary 1.15 $\tilde{\Gamma}_1(Z\otimes 1)|V(\Lambda_1) = (T_{\rho-\Lambda_1}\gamma(Z))^- \otimes I_{V(\Lambda_1)}$.

For ξ_1, $\xi_2 \in \mathfrak{a}^*_{\mathbf{C}}$, we say $\xi_1 \prec \xi_2$ if there exists $\eta \in \mathbf{N}\cdot\Delta$ such that $\xi_2 = \xi_1 + \eta$. This defines a partial order on $\mathfrak{a}^*_{\mathbf{C}}$.

For each $f \in \mathcal{E}^{\infty}_{\lambda-\Lambda} Ind_K^G(\tau)$, define $E(\lambda,\Lambda,f)$ by

$$E(\lambda,\Lambda,f) = \{\xi \in \tilde{X}(\lambda,\Lambda) \mid p_{\lambda,\xi}(f,\cdot,\cdot) \not\equiv 0\}$$

Let $E_L(\lambda,\Lambda,f)$ denote the set of maximal elements in $E(\lambda,\Lambda,f)$. Suppose $\xi \in$

$E_L(\lambda, \Lambda, f)$. Then $p_{\lambda,\xi}(f, \cdot, \cdot) \not\equiv 0$. So one can find $g \in G$, $\Lambda_1 \in \mathfrak{t}^*$, such that $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot) \not\equiv 0$.

Since the right hand side of the equation in Proposition 6.1 is zero because ξ is maximal in $E(\lambda, \Lambda, f)$,

$$\partial(\tilde{\Gamma}_1(Z \otimes 1) - \gamma(Z)(\lambda - \Lambda))(p_{\lambda,\xi}(f, g, \cdot)e^{\xi}) = 0.$$

So

$$\partial((T_{-\Lambda_1+\rho}\gamma(Z))^{-} - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)e^{\xi}) = 0.$$

We extend $p_{\lambda,\xi}(f, g, \cdot)e^{\xi}$ to a function on $\mathfrak{a}^+ + \sqrt{-1}\mathfrak{t} \subset \mathfrak{h} = \mathfrak{a} + \sqrt{-1}\mathfrak{t}$, by abuse of notation still denoted by $p_{\lambda,\xi}(f, g, \cdot)e^{\xi}$, by the requirement that it be constant in the $\mathfrak{t}$ direction. Hence

$$\partial((T_{-\Lambda_1+\rho}\gamma(Z)) - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)e^{\xi}) = 0.$$

So

$$\partial(\gamma(Z)) - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)e^{\xi-\Lambda_1+\rho}) = 0.$$

By Theorem 3.13, Chapter III in [Helg1], $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)e^{\xi-\Lambda_1+\rho} = \sum q_i e^{\mu_i}$, where q_i are polynomials on $\mathfrak{h}$, $\mu_i \in \mathfrak{h}^*_{\mathbf{C}}$. Recall that $p_{\lambda,\xi}(f, g, tH)$ is a polynomial in t. We conclude $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)$ is a polynomial on $\mathfrak{h}$, and

$$\xi - \Lambda_1 + \rho = w(\lambda - \Lambda),$$

for some $w \in \tilde{W}$. Also $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)$ is a $\tilde{W}(w(\lambda - \Lambda))$-harmonic, where $\tilde{W}(\mu) = \{w \in \tilde{W} \mid w\mu = \mu\}$, for each $\mu \in \mathfrak{h}^*_{\mathbf{C}}$. So

$$deg(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)) \leq d.$$

Here d is the number of elements in $\Sigma^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$.

It follows that we can replace $\tilde{X}(\lambda,\Lambda)$ by $X(\lambda,\Lambda)$ since $E_L(\lambda,\Lambda,f)\subset X(\lambda,\Lambda)$.

By induction on ξ using Proposition 6.1 one can easily show $p_{\lambda,\xi}(f,g,\cdot)$ is a polynomial with degree $\leq d$. Note we only need to show it for $g=e$. So this completes the proof of Theorem 3.3.

The proof of Theorem 3.4 follows from Proposition 5.3.

§7 Leading exponents

We further consider the properties of a leading term in the asymptotic expansion of $f \in \mathcal{E}^{\infty}_{\lambda-\Lambda} Ind_K^G(\tau)$.

Proposition 7.1 *For each $\xi \in E_L(\lambda, \Lambda, f)$, $man \in B$, $H \in \mathfrak{a}$, and $g \in G$,*

$$p_{\lambda,\xi}(f, gman, H) = e^{\xi(\log a)}\tau(m)^{-1} p_{\lambda,\xi}(f, g, H + \log a)$$

Proof: The same as for Theorem 8.4 in [BS].

Let $\alpha^{\vee} = \frac{2\alpha}{<\alpha,\alpha>}$. We introduce conditions on $\lambda - \Lambda$ and λ as follows:

$$\mathfrak{A}_1 = \{\lambda - \Lambda \mid \lambda \in \mathfrak{a}^*_{\mathbf{C}}, \Lambda \in \mathfrak{t}^*_{\mathbf{C}}, < \lambda - \Lambda, \alpha^{\vee} > \notin \mathbf{Z}, \forall \alpha \in \Sigma(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}), \alpha|\mathfrak{a} \neq 0\}.$$

$$\mathfrak{A}_2 = \{\lambda \in \mathfrak{a}^*_{\mathbf{C}} \mid < \lambda, \beta^{\vee} > \notin -\mathbf{N}, \forall \beta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}.$$

Let $\tilde{W}_0 = \{w \in \tilde{W} \mid w|\mathfrak{a} = id\}$, and $\tilde{W}_1 = \{w \in \tilde{W} \mid w\mathfrak{a} = \mathfrak{a}\}$.

Proposition 7.2 *Suppose $\lambda - \Lambda \in \mathfrak{A}_1$.*

(i) If $w(\lambda - \Lambda) = \lambda - \Lambda$, for some $w \in \tilde{W}$, then $w \in \tilde{W}_0$,

(ii) if there exist $w \in \tilde{W}$, $\sigma \in \tau$ such that

$$(w(\lambda - \Lambda) + \Lambda_{\sigma})|\mathfrak{t} = 0,$$

then $w \in \tilde{W}_1$, and $\Lambda_{\sigma} = w\Lambda$.

Proof: (i) Since $w(\lambda - \Lambda) = \lambda - \Lambda$, $w = w_{\alpha_1} \ldots w_{\alpha_s}$, where $\alpha_j \in \Sigma(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, and $< \lambda - \Lambda, \alpha_j >= 0$. Then we conclude $\alpha_j|\mathfrak{a} = 0$ from $\mathfrak{A}_1$. So $w \in \tilde{W}_0$. (ii) For any $\beta \in \Sigma(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ with $\beta|\mathfrak{a} = 0$, we have $< w(\lambda - \Lambda) + \Lambda_{\sigma}, \beta >= 0$ since $(w(\lambda - \Lambda) + \Lambda_{\sigma})|\mathfrak{t} = 0$. Hence

$$\frac{2 < \lambda - \Lambda, w^{-1}\beta >}{< \beta, \beta >} = -\frac{2 < \Lambda_{\sigma}, \beta >}{< \beta, \beta >}.$$

$$\frac{2 < \lambda - \Lambda, w^{-1}\beta >}{< w^{-1}\beta, w^{-1}\beta >} = -\frac{2 < \Lambda_\sigma, \beta >}{< \beta, \beta >}.$$

The right hand side being integral forces $w^{-1}\beta|\mathfrak{a} = 0$. This shows w preserves $\mathfrak{t}$. Therefore w preserves $\mathfrak{a}$. Trivially $\Lambda_\sigma = w\Lambda$.

Proposition 7.3 *Let $f \in \mathcal{E}^\infty_{\lambda-\Lambda} Ind^G_K(\tau)$. Suppose $\lambda - \Lambda \in \mathfrak{A}_1$, and $\xi \in E_L(\lambda, \Lambda, f)$. Then $\xi \in W\lambda - \rho$, and $p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$ for each $g \in G$.*

Proof: In the last section we showed if $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot) \not\equiv 0$, then there exists $w \in \tilde{W}$, such that $\xi - \Lambda_\sigma + \rho = w(\lambda - \Lambda)$. So

$$(w(\lambda - \Lambda) + \Lambda_\sigma)|\mathfrak{t} = 0.$$

By Proposition 7.2 (ii) $w \in \tilde{W}_1$. So $\xi + \rho = w\lambda$. Hence $\xi \in W\lambda - \rho$.

We also showed $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot)$ is $\tilde{W}(w(\lambda - \Lambda))$-harmonic. Since $w \in \tilde{W}_1$, $w(\lambda - \Lambda) \in \mathfrak{A}_1$. By Proposition 7.2 (i) $\tilde{W}(w(\lambda - \Lambda)) \subset \tilde{W}_0$ We conclude $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$ This shows $p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$ since $\sigma \in \tau$ is arbitrary. In this case we denote it by $p_{\lambda,\xi}(f, g)$.

Corollary 7.4 *If $\lambda - \rho \in E_L(\lambda, \Lambda, f)$, and in addition λ is regular, i.e., $W(\lambda) = \{w \in W \mid w\lambda = \lambda\} = e$, then*

$$p_{\lambda,\lambda-\rho}(f, g) = P(\Lambda)p_{\lambda,\lambda-\rho}(f, g).$$

Proof: If for some $\sigma \in \tau$, such that $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g) \not\equiv 0$, then there exists $w \in \tilde{W}_1$, with

$$w\lambda = (\lambda - \rho) + \rho, w\Lambda_\sigma = \Lambda.$$

λ being regular implies $w \in \tilde{W}_0$. But then $P(\Lambda) = P(\Lambda_\sigma)$ by definition.

By Appendix II in [KKMOOT] if $\lambda \in \mathfrak{A}_2$, then $\lambda - \rho$ is always maximal in $W\lambda - \rho$, hence always in $E_L(\lambda, \Lambda, f)$. So we have the following definition.

Definition 7.5 *Let $\lambda - \Lambda \in \mathfrak{A}_1$, and $\lambda \in \mathfrak{A}_2$. For $f \in \mathcal{E}^{\infty}_{\lambda-\Lambda} Ind^G_K(\tau)$, $\beta_\lambda(f)$ is defined by*

$$\beta_\lambda(f) = p_{\lambda,\lambda-\rho}(f,\cdot).$$

we call β_λ the boundary value map.

Theorem 7.6 *Let $\lambda - \Lambda \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$. Then*

(i) β_λ maps $\mathcal{E}^{\infty}_{\lambda-\Lambda,r} Ind^G_K(\tau)$ linearly, continuously, and G-equivariantly into $C^\infty Ind^G_B(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$ for each $r \in \mathbf{R}$, where $\tau(\Lambda)$ is the restriction of τ to M with representation space $V(\Lambda)$,

(ii) let $\Omega \subset \mathfrak{a}^*_{\mathbf{C}}$ be open, and $\{f_\lambda\}_{\lambda\in\Omega}$ be a holomorphic family in $\mathcal{E}^{\infty}_{\lambda-\Lambda} Ind^G_K(\tau)$, then $\lambda \to \beta_\lambda(f_\lambda)$ is holomorphic in $\Omega \cap \mathfrak{A}_2$.

Proof: (i) comes from Theorem 3.3; (ii) is a result of Theorem 3.4.

Finally we notice for certain λ we can obtain the boundary value map by a simple limit procedure.

Lemma 7.7 *Let $\lambda - \Lambda \in \mathfrak{A}_1$. If $Re < \lambda, \alpha >> 0$, for each $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, then*

$$\beta_\lambda f(g) = \lim_{t\to\infty} e^{(-\lambda+\rho)(tH)} f(g \exp tH),$$

for $f \in \mathcal{E}^{\infty}_{\lambda-\Lambda} Ind^G_K(\tau)$, and $H \in \mathfrak{a}^+$.

Proof: The condition on λ implies that $Re\xi(H) < Re(\lambda-\rho)(H)$ for all $\xi \in X(\lambda, \Lambda)$ with $\xi \neq \lambda - \rho$. Then the result follows from Theorem 3.3 and the very definition of asymptotic expansion.

For each $\phi \in C^\infty Ind^G_B(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$, we define $P_\lambda\phi$ by

$$P_\lambda\phi(g) = \int_K \tau(k)\phi(gk)dk.$$

From the proof of Theorem 1.6 we conclude $P_\lambda\phi \in \mathcal{E}_{\lambda-\Lambda,r} Ind^G_K(\tau)$. By Example 2.2 $P_\lambda\phi \in \mathcal{E}^{\infty}_{\lambda-\Lambda,r} Ind^G_K(\tau)$.

Corollary 7.8 *Under the same condition as in Lemma 7.7, $\beta_\lambda P_\lambda \phi = C(\lambda)\phi$, for each $\phi \in C^\infty Ind^G_B(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.*

Proof: By Proposition 2.4 and Lemma 7.7.

§8 The inversion of the Poisson transform

Let $C(\lambda)$ be the generalized Harish-Chandra's C-function given by

$$C(\lambda) = \int_{\overline{N}} e^{-(\lambda+\rho)H(\overline{n})}\tau(k(\overline{n}))d\overline{n}.$$

Recall P_λ: $C^\infty Ind_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) \to \mathcal{E}^\infty_{\lambda-\Lambda} Ind_K^G(\tau)$ is defined by

$$P_\lambda \phi(g) = \int_K \tau(k)\phi(gk)dk$$

Theorem 8.1 *Let $\lambda - \Lambda \in \mathfrak{A}_1$, and $\lambda \in \mathfrak{A}_2$. Then*

$$\beta_\lambda P_\lambda \phi = C(\lambda)\phi,$$

for each $\phi \in C^\infty Ind_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.

Proof: If $Re < \lambda, \alpha >> 0$, for all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$, then by Corollary 7.8

$$\beta_\lambda P_\lambda \phi = C(\lambda)\phi.$$

Since $P_\lambda \phi$ is a holomorphic family in $\mathcal{E}^\infty_{\lambda-\Lambda} Ind_K^G(\tau)$, by Theorem 7.6 the left hand side is holomorphic . The right hand side is meromorphic on $\mathfrak{a}^*_{\mathbf{C}}$. Hence two sides must coincide.

Corollary 8.2 *If in addition we assume* $\det C(\lambda) \neq 0$, *then β_λ is surjective. Hence P_λ is injective.*

Theorem 8.3 *Let $\lambda - \Lambda \in \mathfrak{A}_1$, and $\lambda \in \mathfrak{A}_2$, and* $\det C(\lambda) \neq 0$. *Then P_λ is bijective, and the inverse of P_λ is given by $C(\lambda)^{-1}\beta_\lambda$.*

For the proof we introduce a definition which can be found in [Wall], Section 11.6. Let $\mathfrak{V}$ be a finitely generated $(\mathfrak{g}, K)$-module.

Definition 8.4 *$\mathfrak{V}^*_{mod}$ denotes the set of all $\mu \in \mathfrak{V}^*$, such that there exists $d_\mu \in \mathbf{R}$ and for each $\nu \in \mathfrak{V}$ there exist an analytic function $f_{\mu,\nu}$ and a constant $C_{\mu,\nu} > 0$ with the following properties:*

(i) $L_u f_{\mu,\nu}(k) = \mu(k^{-1}.(u.\nu))$, for $u \in U(\mathfrak{g})$, $k \in K$,

(ii) $|f_{\mu,\nu}(g)| \leq C_{\mu,\nu}\|g\|^{d_\mu}$, *for each* $g \in G$.

Recall $(C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1))'$ is the strong topological dual of $C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)$. The following result can also be found in [Wall], Section 11.7.

Proposition 8.5 $[(C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1))_{K-finite}]^*_{mod} = (C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1))'$. *Here* $(C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1))_{K-finite}$ *denotes the space of* K*-finite elements in* $C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)$*, and* σ *is any finite dimensional representation of* M.

Before we go ahead with the proof of Theorem 8.3, we mention the following result about the irreducibility of the principal series representations. Let $\sigma \in \hat{M}$.

Lemma 8.6 *As a* $(\mathfrak{g}, K)$ *module* $C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K-finite}$ *is irreducible if* $\lambda - \Lambda \in \mathfrak{A}_1$.

Proof: This is a direct consequence of Theorem 1.1 in [SV].

Proof of Theorem 8.3: It suffices to show β_λ is injective. Assume the opposite. Then there exists $f_0 \in \mathcal{E}^\infty_{\lambda-\Lambda} Ind_K^G(\tau)$, such that $\beta_\lambda f_0 = 0$, and $f_0 \not\equiv 0$. We can assume $f_0(e) \neq 0$ since β is G-equivariant.

Define f_K by

$$f_K(g) = \int_K Tr\tau(k) f_0(kg) dk.$$

Then f_K is K-finite, and $f_K(e) = \frac{1}{\dim(\tau)} f_0(e) \neq 0$. Let

$$\mathfrak{W} = L_{U(\mathfrak{g})} L_K f_K.$$

Then $\mathfrak{W}$ is a finitely generated $(\mathfrak{g}, K)$-module. Let $\mathfrak{W}_1$ be an irreducible submodule of $\mathfrak{W}$. By the subrepresentation theorem and Lemma 8.4 there exists $\sigma \in \hat{M}$, such that $\mathfrak{W}_1 \cong C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K-finite}$. So there is a $(\mathfrak{g}, K)$ map P_σ: $C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K-finite} \to \mathfrak{W}$. It is easy to see $\Lambda = \Lambda_\sigma$.

Define $\mu \in \mathfrak{W}^* \otimes V$ by

$$\mu(\nu) = \nu(e),$$

for each $\nu \in \mathfrak{W}$.

Taking $f_{\mu,\nu} = \nu \in \mathcal{E}^{\infty}_{\lambda-\Lambda} Ind_K^G(\tau)$ in Definition 8.4, we can verify that (i) and (ii) are satisfied. So $\mu \in \mathfrak{W}^*_{mod} \otimes V$. Hence $\mu^\sharp = \mu \circ P_\sigma \in [(C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1))_{K-finite}]^*_{mod} \otimes V$. Then by Proposition 8.5 $\mu^\sharp \in (C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1))' \otimes V$.

Now define $P_\sigma^\sharp$: $C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1) \to C^\infty Ind_K^G(\tau)$

$$P_\sigma^\sharp \phi(g) = \mu^\sharp(L_{g^{-1}}\phi).$$

Since P_σ is a $\mathfrak{g}$ map and eigensections are analytic we can show $P_\sigma \phi = P_\sigma^\sharp \phi$, for $\phi \in C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K-finite}$, by showing they are identical at e along with their derivatives.

We observe $P_\sigma^\sharp$ is a linear, continuous, and G-equivariant map from $C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)$ to $C^\infty Ind_K^G(\tau)$. By Proposition 1.8 we conclude $\sigma \in \tau$, and there exists $T \in Hom_M(V_\sigma, V)$, such that $P_\sigma^\sharp = P_T$. Hence

$$P_\sigma = P_T : C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K-finite} \to \mathfrak{W}.$$

Pick any $\phi \in C^\infty Ind_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K-finite}$ such that $0 \neq f = P_T\phi$. Then $f = P_\lambda(T\phi)$. Notice $T\phi \in C^\infty Ind_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)_{K-finite}$. So

$$\beta_\lambda f = \beta_\lambda P_\lambda(T\phi) = C(\lambda)T\phi \neq 0.$$

This contradicts to $f \in \mathfrak{W} \subset \ker(\beta_\lambda)$.

§9 Vector-valued distributions

Suppose K is a Lie group and V a finite dimensional space over $\mathbf{C}$. Let $C^{-\infty}(K,V)$ denote all continuous $\mathbf{C}$-linear maps from $C_c^{\infty}(K,\mathbf{C})$ to V.

Let M be a compact subgroup of K, and (π, V) a finite dimensional representation of M. Let

$$C^{-\infty}Ind_M^K(\pi) = \{f \in C^{-\infty}(K,V) \mid R_m f(\phi) = \pi(m^{-1})f(\phi), \forall \phi \in C_c^{\infty}(K,\mathbf{C}), \forall m \in M.\}$$

Here $R_m f(\phi) = f(R_{m^{-1}}\phi)$, where $R_{m^{-1}}\phi(k) = \phi(km^{-1})$.

Let $(\check{\pi}, V^*)$ be the dual representation of (π, V), and $<,>$ the nondegenerate bilinear form on $V \times V^*$. Let $(C_c^{\infty}Ind_M^K(\pi))'$ the strong dual of $C_c^{\infty}Ind_M^K(\pi)$. For each $T \in (C_c^{\infty}Ind_M^K(\pi))'$, $\phi \in C_c^{\infty}(K,\mathbf{C})$, and $v \in V$, we define $\xi_1(T)(\phi) \in V^*$ by

$$< v, \xi_1(T)(\phi) >= T(\xi_1(\phi, v)),$$

where $\xi_1(\phi,v)(k) = \int_M \phi(km)\pi(m)v dm$. It is easy to show $\xi_1(T) \in C^{-\infty}Ind_M^K(\check{\pi})$.

Proposition 9.1 *The map* ξ_1: $(C_c^{\infty}Ind_M^K(\pi))' \to C^{-\infty}Ind_M^K(\check{\pi})$ *is bijective.*

Proof: Define η_1: $C^{-\infty}Ind_M^K(\check{\pi}) \to (C_c^{\infty}Ind_M^K(\pi))'$ as follows: for each $f \in C^{-\infty}Ind_M^K(\check{\pi})$, and $\phi \in C_c^{\infty}Ind_M^K(\pi)$, the map

$$f_\phi: \quad u \to f(< \phi, u >)$$

is a linear map from V^* to V^*. Then we define

$$\eta_1(f) = Tr(f_\phi).$$

It is a long but rather straightforward calculation to show ξ_1 and η_1 are inverses to each other.

Now let $G = KAN$, and (δ, V_δ) a finite dimensional representation of $B = MAN$. Let

$$C^{\infty}Ind_B^G(\delta) = \{f \in C^{\infty}(G, V_\delta) \mid R_{man}f = a^{-\rho}\delta^{-1}(man)f, \forall man \in B.\}.$$

$$C^{-\infty}Ind_B^G(\delta) = \{f \in C^{-\infty}(G, V_\delta) \mid R_{man}f = a^{-\rho}\delta^{-1}(man)f, \forall man \in B.\}.$$

For $T \in (C^{\infty}Ind_B^G(\delta))'$, $\xi(T)$ is defined by

$$< v, \xi(T)(\phi) >= T(\xi(\phi, v)),$$

for each $v \in V_\delta$, and $\phi \in C_c^{\infty}(G, \mathbf{C})$. Here $\xi(\phi, v)(g) = \int_{MAN} \phi(gman)a^{\rho}\delta(man)vdmdadn$.

Next we show $\xi(T) \in C^{-\infty}Ind_B^G(\check{\delta})$. By definition, $< v, \xi(T)(R_{(man)^{-1}}\phi) >= T(\xi(R_{(man)^{-1}}\phi, v))$. However, it is a simple calculation to see $\xi(R_{(man)^{-1}}\phi, v) = \xi(\phi, a^{-\rho}\delta(man)v)$. Hence

$$\begin{aligned} < v, R_{man}\xi(T)(\phi) > &=< v, \xi(T)(R_{(man)^{-1}}\phi) >= T(\xi(\phi, a^{-\rho}\delta(man)v)) \\ &=< a^{-\rho}\delta(man)v, \xi(T)(\phi) >=< v, a^{-\rho}\check{\delta}((man)^{-1})T(\phi) >. \end{aligned}$$

This proves $\xi(T) \in C^{-\infty}Ind_B^G(\check{\delta})$.

Theorem 9.2 *Let ξ be defined as above. Then ξ is G-equivariant bijection from $(C^{\infty}Ind_B^G(\delta))'$ to $C^{-\infty}Ind_B^G(\check{\delta})$.*

Lemma 9.3 *Let L be a Lie group and (π, V) a finite dimensional representation of L on V. Suppose $f \in C^{-\infty}(L, V)$, satisfying*

$$R_l f = \pi(l^{-1})f,$$

for each $l \in L$. Let dl be the right invariant Haar measure on L. Then there exists a unique vector $v \in V$, such that

$$f(\phi) = \int_L \phi(l)\pi(l^{-1})vdl,$$

for each $\phi \in C^{\infty}(L, \mathbf{C})$.

Proof: We use an argument due to Helgason. For ϕ and ψ in $C_c^\infty(L, \mathbf{C})$, we define $\phi * \psi$ in $C_c^\infty(L, \mathbf{C})$ by

$$\phi * \psi(x) = \int_L \phi(l)\psi(xl^{-1}).$$

Then

$$f(\phi * \psi) = \int_L \phi(l) f(R_{l^{-1}}\psi) dl = \int_L \phi(l)\pi(l^{-1}) f(\psi) dl.$$

Choose a sequence ψ_n such that $\check{\psi}_n \to \delta$, the delta function, as $n \to +\infty$. Here $\check{\psi}_n(l) = \psi_n(l^{-1})$. Let $v_n = f(\psi_n)$. Then

$$(*) \qquad f(\phi * \psi_n) = \int_L \phi(l)\pi(l^{-1}) v_n dl.$$

We can choose an appropriate ϕ (e.g. close to δ), such that $\int_L \phi(l)\pi(l^{-1})$ is invertible. Since $\phi * \psi_n \to \phi$, by letting $n \to +\infty$ in $(*)$, we conclude there exists $v \in V$, such that $v_n \to v$, and

$$f(\phi) = \int_L \phi(l)\pi(l^{-1}) v dl.$$

the uniqueness follows from the fact that there is ϕ such that $\int_L \phi(l)\pi(l^{-1})$ is invertible.

Proof of Theorem 9.2: First we construct the inverse η of ξ as follows:

Take $f \in C^{-\infty} Ind_B^G(\check{\delta})$, and $\psi \in C^\infty(K, \mathbf{C})$. Then $\phi \to f(\psi \otimes \phi)$ defines a continuous linear map from $C_c^\infty(A \times N, \mathbf{C})$ to V_δ^*, where

$$(\psi \otimes \phi)(kan) = \psi(k)\phi(an).$$

It is easy to check this map satisfies all the conditions as in Lemma 9.3 if we take $L = AN$, $\pi(an) = a^\rho \check{\delta}(an)$. So there exists a unique element in V_δ^*, which we denote by $f^-(\psi)$, such that

$$f(\psi \otimes \phi) = \int_{A \times N} \phi(an) a^\rho \check{\delta}^{-1}(an) f^-(\psi) da dn.$$

Notice $a^{2\rho}dadn$ gives a right invariant Haar measure on AN.

It is fairly easy to see $f^- \in C^{-\infty}Ind_M^K(\check{\delta}|M)$. Then by Proposition 9.1 $\eta_1(f^-)$ gives an element in $(C^\infty Ind_M^K(\delta|M))'$. Since $C^\infty Ind_M^K(\delta|M) \cong C^\infty Ind_B^G(\delta)$, one can view $\eta_1(f^-)$ as an element in $(C^\infty Ind_B^G(\delta))'$. Finally we define $\eta(f)$ by

$$\eta(f) = \eta_1(f^-).$$

The final step of the proof is to show $\eta \circ \xi = id$, and $\eta \circ \xi = id$.

For each $T \in (C^\infty Ind_B^G(\delta))'$, $\psi \in C^\infty(K, \mathbf{C})$, and $\phi \in C_c^\infty(A \times N, \mathbf{C})$,

$$\xi(T)(\psi \otimes \phi) = \int_{A\times N} \phi(an)a^\rho \check{\delta}^{-1}(an)(\xi(T))^- dadn.$$

So for each $v \in V$,

$$(**) \qquad < v, \xi(T)(\psi \otimes \phi) >=< v, \int_{A\times N} \phi(an)a^\rho \check{\delta}^{-1}(an)(\xi(T))^-(\psi)dadn > .$$

By definition

$$\begin{aligned}\xi(\psi \otimes \phi, v)(k) &= \textstyle\int_{MAN}(\psi \otimes \phi)(kman)a^\rho \delta(man)vdmdadn \\ &= \textstyle\int_{MAN} \psi(km)\delta(m)\phi(an)a^\rho\delta(an)vdmdadn = \xi_1(\psi, v_1),\end{aligned}$$

where $v_1 = \int_{A\times N} a^\rho \phi(an)\delta(an)vdadn$. So by (**)

$$\begin{aligned}< v, \xi(T)(\psi \otimes \phi) >&= T(\xi_1(\psi, v_1)) =< v_1, \xi_1(T)(\psi) > \\ &=< v, \textstyle\int_{A\times N} \phi(an)a^\rho\check{\delta}^{-1}(an)\xi_1(T)dadn >\end{aligned}$$

By comparing both sides of (**) we have $\xi_1(T) = (\xi(T))^-$. So

$$T = {\xi_1}^{-1}((\xi(T))^-) = \eta_1((\xi(T))^-) = \eta(\xi(T)).$$

Similarly we can verify $\xi \circ \eta = id$, Note it is enough to check on functions of the form $\psi \otimes \phi$. So this completes the proof.

Now suppose V_δ is a Hilbert space. Let δ^* be the representation defined as follows: for each $g \in G$, $w, v \in V_\delta$, we have $< \delta(g)v, w >=< v, \delta(g)^t w >$, then $\delta^*(g) = \delta(g^{-1})^t$.

Let $C^{-\infty}Ind_B^G(\delta^*)$ be the space of conjugate linear maps f from $C_c^\infty(G,\mathbf{C})$ to V_δ, such that

$$R_{man}f = a^{-\rho}\delta^*((man)^{-1})f.$$

For each $T \in (C^\infty Ind_B^G(\delta))'$, and $\phi \in C_c^\infty(G,\mathbf{R})$, $\xi(T)(\phi)$ is defined by $< v, \xi(T)(\phi) >= T(\xi(\phi, v))$, for each $v \in V_\delta$. Here

$$\xi(\phi, v)(g) = \int_{MAN} \phi(gman)a^\rho\delta(man)vdmdadn.$$

Corollary 9.4 *ξ is a bijection from $(C^\infty Ind_B^G(\delta))'$ to $C^{-\infty}Ind_B^G(\delta^*)$.*

Let σ be a unitary representation of M and $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$. $\sigma\otimes\overline{\lambda}\otimes 1$ is the representation of B defined by $man \to a^{\overline{\lambda}}\sigma(m)$. Then $(\sigma\otimes\overline{\lambda}\otimes 1)^* = \sigma\otimes(-\lambda)\otimes 1$

Corollary 9.5 *The map*

$$\xi : \ (C^\infty Ind_B^G(\sigma\otimes\overline{\lambda}\otimes 1))' \to C^{-\infty}Ind_B^G(\sigma\otimes(-\lambda)\otimes 1)$$

is a bijection.

§10 Distribution boundary values

In this section we introduce a weak growth condition in the eigenspace $\mathcal{E}_{\lambda-\Lambda} Ind_K^G(\tau)$. Recall from Section 2 we have

$$C_r^q(G,V) = \{f \in C^q(G,V) \mid \|f\|_{q,r} < \infty\},$$

$q \in \mathbf{N}$ and $r \in \mathbf{R}$. $C_r^\infty(G,V) = \cap_q C_r^q(G,V)$. We define the $\mathfrak{F}$ to be the space

$$\mathfrak{F} = \cap_r C_r^\infty(G,V) = \cap_{q,r} C_r^q(G,V).$$

endowed with the projective limit topology for the intersection over q and r (i.e., the topology given by the family of forms $\|\cdot\|_{q,r}$).

Using the same argument as on p.142 in [BS] we conclude $\mathfrak{F}$ is a Fréchet space. It follows from Section 2 that L and R act smoothly on $\mathfrak{F}$.

Let $\mathfrak{F}'$ be the space dual to $\mathfrak{F}$, equipped with the strong dual topology. For each $T \in \mathfrak{F}'$, $q \in \mathbf{N}$, and $r \in \mathbf{R}$, we define

$$\|T\|'_{q,r} = \sup\{T(\varphi) \mid \varphi \in \mathfrak{F}, \|\varphi\|_{q,r} \leq 1\}$$

The space $C_r^q(G,V)' = \{T \in \mathfrak{F}' \mid \|T\|_{q,r} \leq \infty\}$ with this norm is the dual space of $C_r^q(G,V)$. Moreover, we have $\mathfrak{F}' = \cup_{q,r} C_r^q(G,V)'$. By duality $\mathfrak{F}'$ is the inductive limit of these spaces.

Using Lemma 2.1 we can prove that for some $b \in \mathbf{R}$, $\int_G \|g\|^b dg < \infty$. It follows that there is a continuous injection of $C_r^0(G,V)$ into $C_{b-r}^0(G,V)'$ defined by integration over G. Hence there is a continuous injection of $C_r^0(G,V)$ into $\mathfrak{F}'$.

Let $q' \geq q$, and $r \in \mathbf{R}$. For each $T \in C_r^q(G,V)'$, and $\varphi \in C_r^{q'}(G,\mathbf{R})$, we define an element $L^\vee(\varphi)T$ in $C_r^{q'-q}(G,V)$ by

$$< v, L^\vee(\varphi)T(x) >= T(R_{x^{-1}}\varphi \cdot v).$$

Note if $f \in C_r^0(G,V)$, and $\varphi \in C_{b-r}^0(G,\mathbf{C}))$, then

$$L^\vee(\varphi)f(x) = \int_G \varphi(g)F(gx)dg.$$

Lemma 10.1 *Let q, $q' \in \mathbf{N}$ with $q \le q'$. There exist $s \ge 0$ and $C \ge 0$ such that*

$$\|L^\vee(\varphi)T\|_{q'-q,r} \le C\|T\|'_{q',r}\|\varphi\|_{q',r-s},$$

for all $r \in \mathbf{R}$, $T \in C^q_r(G,V)'$, and $\varphi \in C^{q'}_{r-s}(G,\mathbf{R})$.

Proof: See Lemma 11.1 in [BS].

Let $\mathcal{E}^*_{\lambda-\Lambda}Ind^G_K(\tau)$ denote the closed subspace $\mathcal{E}_{\lambda-\Lambda}Ind^G_K(\tau) \cap \mathfrak{F}'$. We call the elements of $\mathcal{E}^*_{\lambda-\Lambda}Ind^G_K(\tau)$ eigensections of weak moderate growth. Notice if $f \in \mathcal{E}^*_{\lambda-\Lambda}Ind^G_K(\tau)$, and $\varphi \in C^\infty_c(G,\mathbf{R})$, then $L^\vee(\varphi)f \in \mathcal{E}^\infty_{\lambda-\Lambda}Ind^G_K(\tau)$ by Lemma 10.1.

For $\lambda - \Lambda \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$, and $f \in \mathcal{E}^*_{\lambda-\Lambda}Ind^G_K(\tau)$, we define a vector-valued distribution $\overline{\beta}_\lambda f$ on G by

$$\overline{\beta}_\lambda f(\varphi) = \beta_\lambda(L^\vee(\varphi)f)(e),$$

for each $\varphi \in C^\infty_c(G,\mathbf{R})$.

Proposition 10.2 *$\overline{\beta}_\lambda f$ is a linear , continuous, and G-equivariant map from $\mathcal{E}^*_{\lambda-\Lambda}Ind^G_K(\tau)$ to $C^{-\infty}Ind^G_B(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$.*

Proof: It suffices to show $\overline{\beta}_\lambda f \in C^{-\infty}Ind^G_B(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$. By definition,

$$\begin{aligned} L^\vee(R_{(man)^{-1}}\varphi)f(x) &= f(R_{x^{-1}}R_{(man)^{-1}}\varphi) \\ &= f(R_{x^{-1}}R_{(manx)^{-1}}\varphi) = L^\vee(\varphi)(manx). \end{aligned}$$

However, β_λ is G-equivariant, so

$$\begin{aligned} &\beta_\lambda(L^\vee(R_{(man)^{-1}}\varphi)f) \\ &= \beta_\lambda(L^\vee(\varphi)f)(man) = \tau(\Lambda)(m^{-1})a^{\lambda-\rho}\beta_\lambda(L^\vee(\varphi)f)(e). \end{aligned}$$

This proves $\overline{\beta}_\lambda f \in C^{-\infty}Ind^G_B(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$.

For each $T \in (C^\infty Ind^G_B(\tau(\Lambda)\otimes\overline{\lambda}\otimes 1))'$, we define $\overline{P}_\lambda T$ as follows:

$$< v, \overline{P}_\lambda T(g) >= T(P(\Lambda)L_g\Phi_\lambda \cdot v) >,$$

for each $v \in V$. Here $\Phi_\lambda(x)$ is the transpose of $\Psi_\lambda(x^{-1})$, and $P(\Lambda)$ the projection from V to $V(\Lambda)$. The motivation of this definition is from Corollary 1.10.

Proposition 10.3 *$\overline{P}_\lambda T \in \mathcal{E}^*_{\overline{\lambda}-\Lambda} Ind_K^G(\tau)$, for all $T \in (C^\infty Ind_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$. And $\overline{P}_\lambda$ is linear, continuous, and G-equivariant.*

Proof: Similar to the proof for Corollary 11.3 in [BS].

Lemma 10.4 *Let $T \in (C^\infty Ind_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$, and $\varphi \in C_c^\infty(G, \mathbf{R})$. Then $L^\vee(\varphi)\overline{P}_\lambda T = P_\lambda(L^\vee(\varphi)\xi(T))$. Here ξ is the isomorphism in Corollary 9.5, and $L^\vee(\varphi)\xi(T)(x) = \xi(T)(R_{x^{-1}}\varphi)$.*

Proof: $L^\vee(\varphi)$, $\overline{P}_\lambda$, and P_λ are continuous. So it is enough to check for $T \in C^\infty Ind_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$. The proof follows from the G-equivariance of P_λ.

By a similar argument we get

Lemma 10.5 *Let $f \in \mathcal{E}^*_{\overline{\lambda}-\Lambda} Ind_K^G(\tau)$, and $\varphi \in C_c^\infty(G, \mathbf{R})$. Then*

$$L^\vee(\varphi)\overline{\beta}_\lambda f = \beta_\lambda(L^\vee(\varphi)f).$$

Theorem 10.6 *Under the same condition as in Theorem 8.3, $\overline{P}_\lambda$ is a G-equivariant topological isomorphism from $(C^\infty Ind_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$ to $\mathcal{E}^*_{\overline{\lambda}-\Lambda} Ind_K^G(\tau)$. And $\eta \circ C(\lambda)^{-1} \circ \overline{\beta}_\lambda$ gives the inverse of $\overline{P}_\lambda$.*

Proof: By Theorem 8.1 and Lemma 10.4, 10.5, for $T \in (C^\infty Ind_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$

$$L^\vee(\varphi)\overline{\beta}_\lambda \overline{P}_\lambda T = \beta_\lambda P_\lambda L^\vee(\varphi)\xi(T) = C(\lambda)L^\vee(\varphi)\xi(T).$$

Similarly for each $f \in \mathcal{E}^*_{\overline{\lambda}-\Lambda} Ind_K^G(\tau)$

$$L^\vee(\varphi)\overline{P}_\lambda \eta(C(\lambda)^{-1}\overline{\beta}_\lambda f) = P_\lambda C(\lambda)^{-1}\beta_\lambda L^\vee(\varphi)f = L^\vee(\varphi)f.$$

So we have

$$\overline{\beta}_\lambda \circ \overline{P}_\lambda = C(\lambda) \circ \xi, \quad \overline{P}_\lambda \circ \eta \circ C(\lambda)^{-1} \overline{\beta}_\lambda = id.$$

Remark 10.7 *Let $\mathcal{E}_{\lambda-\Lambda,r} = \mathcal{E}_{\lambda-\Lambda} \cup C_r(G,V)$ be equipped with the Banach space topology inherited from $C_r(G,V)$. Then $\mathcal{E}^*_{\lambda-\Lambda}$ is identical with the inductive limit topology for the union $\mathcal{E}^*_{\lambda-\Lambda} = \cup_r \mathcal{E}_{\lambda-\Lambda,r}$. See Page 146 in [BS].*

By a classical result the left K-finite elements in $\mathcal{E}_{\lambda-\Lambda} Ind_K^G(\tau)$ increase at most exponentially. So by the remark above we easily get

Corollary 10.8 *Under the same condition as in Theorem 8.3, P_λ is a bijection from $C^\infty Ind_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)_{K-finite}$ to $\mathcal{E}_{\lambda-\Lambda} Ind_K^G(\tau)_{K-finite}$.*

Remark 10.9 *I think by Schmid's method which is indicated in [Sch] one should be able to get a bijection on the level of hyperfunctions from Corollary 10.8.*

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