Gauge-fixing and Equivariant Cohomology

by

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Abstract

The problem of gauge-fixing in quantum field theory is discussed from a topological point of view. After some fundamentals of gauge theory are reviewed, the use of the Thom class to implement gauge-fixing is discussed. It is shown how the BRST transformations familiar to physicists can be used to find the Matthai-Quillen representative of the Thom class in equivariant cohomology.

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Chapter 1

Gauge theory

1.1 Introduction

In the 1950's, physicists developed the idea of gauge fields (first set forth by Yang and Mills [16]) while mathematicians independently developed the theory of connections on fiber-bundles. In the intervening years it was realized that these theories were dealing with the same basic objects and constructions, albeit under different names. Yang-Mills theory, or nonabelian gauge theory, has become an exciting point of contact for mathematics and physics. To cite just one example, in his celebrated work (see e.g. [6]), Simon Donaldson used ideas from gauge field theory to obtain new results on the diffeomorphism classification of four-dimensional manifolds, a fundamental problem of differential topology.

In the first chapter of this paper, we review the basic terminology and concepts of gauge field theory. Next we describe the problem of gauge-fixing which arises when we attempt to construct a quantum theory of gauge fields via the Feynman path-integral approach. A semiclassical limit to the full quantum theory is described.

The main mathematical subject matter of the paper is in the second chapter, in which the mathematical theory of equivariant cohomology is described, and a particular element of the equivariant cohomology group, the equivariant Thom class, is explicitly constructed, at least in the finite-dimensional case. We describe how the equivariant Thom class solves the gauge-fixing problem, and how it relates to the

so-called BRST transformations (introduced by the physicists Becci, Rouet, Stora, and Tyupkin [3]) which are commonly used in physics to implement gauge-fixing. It should be kept in mind that for rigorous results in quantum field theory, these results should be proven in an infinite-dimensional context. This problem is not taken up in the present paper.

1.2 Principal bundles

Suppose a Lie group G acts from the right on a space P, without fixed points. Let M denote the quotient P/G. The quotient mapping $\pi : P \to M$ gives P the structure of a principal fiber-bundle over M. For each point m of M, the inverse image $\pi^{-1}(m)$ (called the fiber over m) is homeomorphic to the group G.

Generally, fiber bundles are constructed the other way around, starting with the space M which is usually taken to be a differentiable manifold. The bundle P is constructed by gluing together sets of the form $U \times G$ where U is a small open set of M and G is the typical fiber. Although in a sufficiently small neighborhood of each point in M, the bundle P looks like a product space, it is not generally (except in the trivial case) a product space globally.

In physics, such models have been used, since as early as the 1920's (for instance in the works of Kaluza[8] and Klein[9]), to describe extended space-times. The base space M is taken to be a 4-manifold, which represents ordinary space-time. The fiber over each point is often called an "internal symmetry space" by physicists; it models extra degrees of freedom. For instance, if the group G is the circle group $U(1)$, then the space P is five-dimensional; the extra dimension is associated with the electromagnetic field. This extra degree of freedom is due to the fact that the source-free Maxwell equations are invariant under "rotation" of the plane spanned by the electric and magnetic field vectors at each point of space, for instance replacing $\vec{E}$ by $\vec{B}$ and $\vec{B}$ by $-\vec{E}$. This is perhaps the simplest example of a gauge transformation.

Higher dimensional, nonabelian Lie groups, in particular $SU(2)$ and $SU(3)$ are used to model isospin and chromodynamic degrees of freedom in elementary particle

physics. The "Standard Model" of elementary particle physics is based on such a model. (Note that mathematicians think of fibers as sitting "above" points of M while physicists tend to think of them as "inside" points of M).

1.3 Global and local gauge symmetries

From the point of view of gauge theory, the above rotation of the $\vec{E}$ and $\vec{B}$ is rather trivial, since the rotation angle is the same at all points of space-time. This type of transformation is called a *global gauge transformation.* It is simply the action of G on P. However, gauge theories possess a much larger symmetry group than G.

Since only the spacetime M and the global G-symmetry are physically observable, there are additional symmetries of the bundle P. Namely, any automorphism of the bundle P which is compatible with the G-action and the projection onto M is a gauge symmetry. That is, if $\phi : P \to P$ is a homeomorphism satisfying the conditions:

$$\pi \circ \phi = \pi \tag{1.1}$$

$$\phi \circ g = g \circ \phi \text{ for each } g \in G \tag{1.2}$$

then ϕ is regarded as a symmetry of the space P.

Such transformations are called *local gauge transformations.* The group of all such transformations is denoted $\mathcal{G}$. In terms of the electromagnetic example above, local gauge transformations amount to allowing the rotation angle in the $\vec{E} - \vec{B}$ plane to vary continuously over space-time. Since a local gauge transformation preserves fibers and commutes with the right G-action, its action on each fiber is equivalent to a left-action by some element of G. More generally, if P is a principal G-bundle over M and U is a neighborhood in M over which P is trivial, then the effect of the gauge transformation in the open set $\pi^{-1}(U)$ is described by a map $g : U \to G$, such that the fiber $\pi^{-1}(m)$ is transformed by left-multiplication by the group element $g(m)$. From this description we see that the local gauge group $\mathcal{G}$ is infinite-dimensional.

The difference between global and local gauge transformations is quite similar to

the difference between the types of coordinate transformations allowed in special and general relativity. (In fact, general relativity can be formulated as a gauge theory.) In special relativity, coordinate frames are related through a single Lorentz transform, which applies at all points of space-time. In general relativity, we allow the coordinate transformations to vary from point to point, bringing in a much larger group of symmetries. Of course we make such transformations at the expense of introducing a gravitational field, or potential. That is, to maintain invariance of the physical equations of motion, the group $\mathcal{G}$ acts not only on the bundle P but also on the space of potentials.

1.4 Gauge potentials

Physical fields may be defined by introducing potentials. In ordinary space-times, these tend to be scalar or vector functions (e.g. the gravitational or electromagnetic potential). In gauge theories, the mathematical objects which are used to describe potentials are called *connections*. Suppose P is a principal fiber bundle. At each point p of P, there is a distinguished linear subspace of the tangent space, namely the kernel of the tangent mapping induced by π. (For simplicity of notation we shall denote induced mappings on tangent spaces by the same symbol as the maps they are induced from). This space is called the *vertical* subspace and denoted V_p. If P is a trivial bundle, i.e. of the form $M \times G$, the vertical subspace consists of tangent vectors in the G direction, and we can consider a complimentary linear subspace at each point (the *horizontal space*), consisting of the tangent vectors pointing in the M direction. However, when P has no given trivialization, there is no canonical complement to the vertical subspace. However, we are free to choose such a horizontal space. We wish to do so in a way which is compatible with the G-action. This motivates the following definition.

Definitition 1 *A* connection *on the principal G-bundle P is a choice, for each point $p \in P$, of a linear subspace $H_p \subset T_p(P)$, satisfying the following three conditions:*

$$H_p \text{ varies smoothly with } p \tag{1.3}$$

$$H_p \oplus V_p = T_p(P) \text{ at each } p \tag{1.4}$$

$$\text{Equivariance: if } g \in G, \text{ then } H_{p \cdot g} = (H_p) \cdot g \tag{1.5}$$

In order to be able to explicitly write down and work with connections we introduce the notion of a *connection form.* This is a one-form on P whose nullspace defines the horizontal subspace. In order that this horizontal subspace be a connection, the one-form must satisfy certain conditions. Recall that since G acts on P, there is an induced action of the Lie algebra $\mathfrak{g}$ on P. If a is an element of $\mathfrak{g}$, let Ξ_a denote the vector field on P corresponding to the infinitesimal action of a. Ξ_a is called the vertical vector field generated by a. A connection form is then defined to be a $\mathfrak{g}$-valued one-form θ on P which satisfies:

$$\theta(\Xi_a) \equiv a \tag{1.6}$$

$$\text{If } g \in G \text{ and } v_p \in T_p(P), \text{ then } \theta_{p \cdot g}(v_p \cdot g) = Ad(g)(\theta_p(v_p)) \tag{1.7}$$

Condition 1.6 guarantees that $\ker(\theta)$ is complementary to the vertical space, and 1.7 guarantees that it satisfies the required equivariance property. Because of this equivariance, if we know the connection at one point of each fiber, it is determined everywhere. Thus, sometimes connections are considered as $\mathfrak{g}$-valued one-forms living on M. This is not quite correct since to obtain such a form requires a choice of a section of the bundle P, which is not canonical. However one can check that the difference of two connection forms is just the pullback of a $\mathfrak{g}$-valued one-form from M to P, and conversely, given a connection form, adding such a pullback form to it yields another connection form. This is summarized in the statement that the space of connections is an affine space modeled on $\Omega^1(M, \mathfrak{g})$.

We need to understand how connection forms transform under the action of $\mathcal{G}$. This can be obtained directly by considering the action of $\mathcal{G}$ on the horizontal subspace determined by a connection form. If θ is a connection form and $g : U \to G$ a local gauge transformation, then θ is carried into

$$g\theta g^{-1} - (dg)g^{-1}. \tag{1.8}$$

The first term in 1.8 is the adjoint action of G on the values of θ, and is a linear action. The second term is nonlinear and shows that connection forms are not tensorial in nature. This term is the source of much of the subtlety of gauge theory.

To return to our simple electromagnetic example, the connection form can be identified with the four-vector consisting of the scalar and vector potentials. Since the group is abelian, the adjoint action is the identity. The nonlinear term in the gauge transformation equation corresponds to adding a four-gradient to the potential, which does not affect the electric and magnetic fields.

1.5 Curvature

Connections are of interest in differential geometry because they allow us to define the notion of *parallel transport.* Suppose γ is a curve in M (a smooth map $\gamma : [0,1] \to M$) and $m = \gamma(0)$ is its starting point. Given a point p in the fiber over m, we wish to define a curve $\tilde{\gamma}$ in P such that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}$ covers γ, i.e. $\pi \circ \tilde{\gamma} = \gamma$. There is no unique way of doing this. However, if P is endowed with a connection, then the curve $\tilde{\gamma}$ can be uniquely determined by the requirement that its tangent vector be everywhere horizontal. Given a curve γ in M, parallel transport defines a map from the fiber over the initial point of γ to the fiber over the terminal point. However this map depends on the curve chosen. The dependence of parallel transport on the base curve is measured by the *curvature form.* This is a $\mathfrak{g}$-valued 2-form Ω_θ on P defined by

$$\Omega_\theta = d\theta - \theta \wedge \theta \tag{1.9}$$

Usually, due to anticommutativity, the wedge product of a one-form with itself vanishes due to antisymmetry. However, for Lie algebra-valued forms, the definition of wedge product involves taking the Lie bracket of the values of the forms being multiplied, which is also anticommutative. Thus for $\mathfrak{g}$-valued one-forms, $\theta \wedge \theta$ does not automatically vanish.

Of course formula 1.9 can be derived from purely geometrical considerations. By parallel transport around small closed loops based at a point m in M, we can measure how far the horizontal distribution is from being involutive near m. Suppose v_1 and v_2 are tangent vectors at m. Consider a loop which leaves m in the v_1 direction and returns in the v_2 direction. Parallel transporting around this loop, we obtain a transformation of the fiber $\pi^{-1}(m)$, which is described by an element g of G. Shrinking this loop (and normalizing by its area) one obtains an infinitesimal action on the fiber (described by an element of $\mathfrak{g}$) which depends only on the vectors v_1 and v_2. Since exchanging these vectors reverses the orientation of the loop, which replaces the parallel-transport map by its inverse, we see that this assignment of Lie algebra elements to pairs of tangent vectors is in fact a $\mathfrak{g}$-valued two-form on M. The above curvature form Ω_θ is simply its pullback to M. The form of Ω_θ is such that under local gauge transformations, the nonlinear terms in the transformation law for θ cancel out. That is, Ω_θ is tensorial, transforming under the adjoint action of the gauge group.

If its curvature is everywhere zero, then the connection is called *flat*. In this case, the horizontal subspace is involutive, so that the parallel transport map depends only on the homotopy class of the base curve.

The physical interpretation is as follows: the connection form is a potential and the curvature form is the field strength tensor. The passage from potential to field always involves some sort of derivative (gradient, curl, etc.) and expression 1.9 above should be thought of as a generalized curl.

1.6 Path-integral quantization

1.6.1 The classical action principle

Classical dynamical systems are described by a *phase space*, the set of possible states of the system, and a functional S called the *action* which assigns to each path in phase space a real number. According to the classical "principle of least action" first set forth by deMaupertius in the 17th century, if we know that the system was in a certain initial state i at time t_i and in a final state f at time t_f we can state with certainty that the trajectory taken between these states in phase-space is one for which the action is extremized (typically minimized). For example, Newtonian mechanics of particles can be derived from an action functional which is given by the line-integral of the difference of potential and kinetic energies along the particle's trajectory.

1.6.2 The quantum action principle

According to the basic postulates of quantum mechanics, if no intermediate observations are made between times t_i and t_f, we cannot say with certainty which path was taken. Furthermore, quantum interference effects show that the system in some sense follows all available trajectories. To each trajectory is associated a complex number, the *probability amplitude*, which is given by $e^{iS/\hbar}$, where $\hbar$ is Planck's constant (an extremely small number).

Quantum mechanics tells us that given an initial state, we cannot predict with certainty the final state; we can only compute probabilities of different final states. The probability of transition from state i to state f is, according to the postulates of quantum mechanics, given by the norm-square of the complex probability amplitude, which is obtained by adding up the probability amplitudes for each possible trajectory from i to f. This is the motivation for the *Feynman path-integral*, which is an integral over all paths in phase-space connecting a given initial and final state. Note that this domain of integration is infinite-dimensional and it is not yet fully understood how to

make the Feynman integral completely rigorous. Nonetheless it is an important and useful tool in modern physics. According to the Feynman principle, the transition amplitude is given by

$$< f|i >= N \int_P e^{iS/\hbar} \tag{1.10}$$

where P is the set of paths in phase space connecting i and f, and N is a (somewhat troublesome) normalization constant which we ignore. The probability of a transition from state i to f is then given by $\| < f|i > \|^2$.

Expectation values of observables may also be obtained by the path-integral approach. If $\mathcal{O}$ is a function on state-space, its quantum expectation value is given by

$$< \mathcal{O} >= N \int e^{iS/\hbar} \mathcal{O}, \tag{1.11}$$

where the integral is over all of state-space.

1.6.3 The semiclassical limit

The path integral may help us gain intuition about the transition from quantum to classical mechanics. In the classical limit, the action S is very large compared to $\hbar$. Equivalently we may consider the limit $\hbar \downarrow 0$. If we consider a path which is away from any critical points of S, we see that the action will take on different values on nearby paths. Since we're integrating the exponential $e^{iS/\hbar}$, and $\hbar$ is small compared to S, the integrand will oscillate wildly and contributions from nearby paths will tend to cancel out. Only near stationary points of S will the contributions from nearby paths be in phase and reinforce. Thus, in the limit as $\hbar$ goes to zero, only classical paths survive.

The semiclassical limit may be viewed as a small-$\hbar$ approximation to the full quantum theory. It is constructed as follows. Let $\mathcal{P}$ denote the set of all paths and let $\mathcal{Z}$ denote the set of critical points of the action. Then let $N(\mathcal{Z})$ denote the normal bundle to $\mathcal{Z}$ in $\mathcal{P}$. Using a Riemannian metric, the exponential map provides a local homeomorphism of $N(\mathcal{Z})$ onto a neighborhood of $\mathcal{Z}$ in $\mathcal{P}$. We use this map to transfer the action functional to the normal bundle $N(\mathcal{Z})$, and retain terms up to and

including quadratic ones. The integration over $\mathcal{P}$ is approximated by an integration over $N(\mathcal{Z})$, which is done in two steps. The integration over the (linear) fibers is performed first; since we have a quadratic action, this is a Gaussian integral which may be computed exactly. The resulting function then must be integrated over $\mathcal{Z}$.

1.7 The Yang-Mills action

In Yang-Mills theory, the phase-space is the space $\mathcal{A}$ of connections on a principal bundle. (Since we are working over a space-time, we do not need to consider paths. The time component is already included in M). To describe the dynamics, we need an action functional. Given a connection θ, its action may be constructed as follows. First form its curvature tensor Ω_θ. This is the pullback to P of a form on M which we denote by Ω. This is a $\mathfrak{g}$-valued two-form. Applying the Hodge star operator we obtain a $\mathfrak{g}$-valued $(n-2)$-form, where n is the dimension of M. The wedge product is a $\mathfrak{g}$-valued n-form which we integrate over M to obtain an element of the Lie algebra $\mathfrak{g}$. Finally to obtain a real number we take the negative of the trace of this. (We take the negative trace to obtain a positive-semidefinite action). This action is called the Yang-Mills action:

$$S_{YM} = \int_M -\mathrm{Tr}\,\Omega \wedge (\star\Omega) \tag{1.12}$$

In order that the physical theory defined by this action be a gauge theory, we need that the action S_{YM} be invariant under the group $\mathcal{G}$ of local gauge transformations. This is easily seen to be the case, since the curvature tensor and its dual transform under the adjoint representation, and the trace on the Lie algebra is Ad-invariant.

For the electromagnetic case $G = U(1)$, the Euler-Lagrange equations expressing stationarity of S_{YM} are just the source-free Maxwell equations. (The quantized theory describes photons, but not electrons.) The Yang-Mills fields discussed here are a direct generalization of electromagnetic fields to include nonabelian symmetries, such as $SU(3)$, which is postulated to describe color symmetry in the quark-gluon model of hadrons. Just as the Maxwell equations obtained from the action 1.12 do not include sources, the $SU(3)$ theory describes gluons but not quarks. The description of quarks

requires coupling the gluon fields with additional fields transforming under a different representation of the gauge group, which will not be discussed here.

1.7.1 Self-dual and antiself-dual connections

The zeroes of the Yang-Mills action are the flat connections. In the case where the base manifold M is four dimensional, one can describe a much wider class of critical points of the Yang-Mills action. When M is a four-manifold, the Hodge star operator is an involution on the space of two-forms on M. This applies as well to the case of $\mathfrak{g}$-valued two-forms. Since the square of the star operator on the space $\Omega^2(M;\mathfrak{g})$ is the identity, the star operator splits this space into eigenspaces with eigenvalues $+1$ and -1, which are called respectively the spaces of self-dual and antiself-dual curvatures. Connections with self-dual or antiself-dual curvatures are much-studied critical points of the Yang-Mills action.

1.8 Gauge-fixing

When we apply the path-integral quantization to gauge systems, the presence of the large symmetry group $\mathcal{G}$ creates problems. Transformations in $\mathcal{G}$ are supposed to take states into physically equivalent states, thus if we integrate over $\mathcal{A}$ we are overcounting states. The volume of $\mathcal{G}$ gives an indication of how many times each state is counted. If $\mathcal{G}$ were compact, we could simply divide out by its volume, however $\mathcal{G}$ is an infinite-dimensional group so the normalization factor N above is an "extremely infinite" quantity.

Gauge-fixing is a prescription for solving this overcounting problem. By fixing a gauge, we mean eliminating gauge freedom. We wish to pick a single point in each gauge orbit to integrate over. In other words we wish to integrate over $\mathcal{A}/\mathcal{G}$ by fixing a slice in $\mathcal{A}$ which is complementary to the $\mathcal{G}$-orbits. However, such a slice does not necessarily exist. In topological terms, the nontriviality of the bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ presents obstructions to finding a global section. However local sections exist and we may attempt to gauge-fix at least locally.

We shall consider in this paper the gauge-fixing problem in the classical limit, that is we shall restrict our attention to the space $\mathcal{A}_f$ of flat connections. Simple arguments show that $\mathcal{A}_f/\mathcal{G}$ (called the moduli space of flat connections) is homeomorphic to the space of conjugacy classes of G-representations of the fundamental group of M, which is finite-dimensional. We should expect integration over this space to be reasonable.

However, these moduli spaces are generally complicated nonlinear spaces (like orbifolds) and the domain of integration is not easy to parameterize. In the remainder of this paper, we use the idea of Poincaré duality to approach this problem. Recall that if $N \subset M$ is a submanifold, there exists a closed differential form u on M called the Poincaré dual of N, the degree of which is equal to the codimension of N in M. This form is determined by the condition that for any closed form η on N of degree equal to $\dim(N)$, the integral of η over N is equal to the integral of $\eta \wedge u$ over M. Thus Poincaré duality allows us to replace integration over subspaces by integration over larger spaces. If a gauge-slice exists, the Poincaré dual of this gauge-slice within $\mathcal{A}$ can be used to implement gauge-fixing. A few remarks are in order. First, the gauge-slice may only exist locally. Secondly, the problems of real physical interest are infinite-dimensional, but as remarked earlier we are only working in the finite-dimensional context. Finally, since we are interested in maintaining gauge symmetry, we wish to find representatives of Poincaré duals which are equivariant. This brings us to the topic of equivariant cohomology, the topic of the next chapter.

Chapter 2

Equivariant cohomology

2.1 Introduction

Throughout this chapter G shall denote a compact, connected Lie group and $\mathfrak{g}$ its Lie algebra.

Suppose X is a manifold on which G acts smoothly from the left. If this action is free, then the orbit space X/G is a manifold and its cohomology $H^*(X/G)$ can be used to classify free G-actions. However, for non-free actions this approach provides comparatively little information about the structure of the G-orbits on X. For example, consider the circle group $U(1)$ acting on the sphere S^2 by rotations about the north-south axis. Then $S^2/U(1)$ is homeomorphic to a closed interval, with trivial cohomology. This tells us nothing about the orbit structure, i.e. that there are two fixed points (the poles) and a one-parameter family of free orbits (the latitudes).

Equivariant cohomology is a generalized cohomology theory (i.e. it obeys all of the usual axioms for cohomology except the dimension axiom, so that a point may have nontrivial equivariant cohomology) which is adapted to the case of G-spaces (spaces with a group acting on them). We shall only consider the case of compact connected Lie groups acting on manifolds, although equivariant cohomology can be defined for more general groups and spaces. From the point of view of homotopy theory, the idea is simple: if a given G-action on X is not free, we replace X by a homotopically equivalent G-space X' for which the group action is free, then form the

quotient space X'/G and study its cohomology. Since we are interested in classifying G-actions, we require homotopy equivalence in the sense of G-spaces; not only must X' be of the same homotopy type as X, but the G-actions on the two spaces must be compatible. In particular, the maps $\phi : X \to X'$ and $\psi : X' \to X$ which provide the homotopy equivalence must be G-maps.

In order that equivariant cohomology be functorial, (it is in fact a functor from G-spaces to modules over a certain ring obtained from G, see below), we need to make this replacement of X by X' in a canonical way. This can be done using the notion of a universal G-bundle. For any Lie group G there exists a contractible space E on which G acts freely. (For convenience we shall assume that G acts on E from the *right*). Let B denote the orbit space E/G. B is called the classifying space of G, and is usually denoted BG to indicate its dependence on the group G. The bundle $E \to B$ is called the universal bundle for the group G. Note that in general E and B are infinite-dimensional, and can be constructed as CW-complexes but not as manifolds. For example, if $G = U(1)$, we can take $E = S^\infty$, the unit sphere in a complex separable Hilbert space. The $U(1)$-action of scalar multiplication is free, and then $B = \mathbf{C}P^\infty$. The construction of the space E for general topological groups is described in [12]. The conditions that E be contractible and possess a free G-action determine E up to homotopy equivalence. Since E is contractible, $E \times X$ has the same homotopy type as X, and the diagonal action, defined by

$$g \cdot (p, x) = (pg^{-1}, gx),$$

is free, and is compatible with the original action of G on X. Let X_G be the quotient of $E \times X$ by this diagonal action, or equivalently $E \times X/\sim$ where $(pg, x) \sim (p, gx)$. (This is where we use the fact that G acts on E from the right and on X from the left.) Then the equivariant cohomology $H^*_G(X)$ can be defined to be $H^*(X_G)$.

In the degenerate case where X is a single point with trivial G-action, the equivariant cohomology of X is the cohomology of the classifying space BG.

The following commutative diagram sums up the relationships between the various

spaces under consideration:

$$\begin{tikzcd}
E \arrow[d] & E \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
B & X_G \arrow[l, "\pi"'] \arrow[r, "\sigma"] & X/G
\end{tikzcd}$$

The maps $E \to B$ and $E \times X \to X_G$ are fibrations with fiber homeomorphic to G. The map $\pi : X_G \to B$ is a fibration with fiber homeomorphic to X and furthermore π induces a natural structure of $H^*(BG)$-module in equivariant cohomology. The maps $X \to X/G$ and $\sigma : X_G \to X/G$ are in general *not* fibrations. For any orbit $G \cdot x$, it is not hard to see that, up to homotopy equivalence, $\sigma^{-1}(G \cdot x)$ is a classifying space for the stabilizer group of m. This implies that for a free action, $\sigma^{-1}(G \cdot x)$ is contractible for each orbit $G \cdot x$. Under the hypotheses that G is compact and connected it can be shown that (as one might expect) $\sigma : X_G \to X/G$ is a homotopy equivalence when G acts freely on X, hence in the free case, $H_G^*(X) \cong H^*(X/G)$. In general however, the equivariant cohomology H_G^* contains more information about the orbit structure. This is because all orbits, whether free or not, map to single points in X/G. On the other hand, in X_G free orbits "look like" single points (that is, the inverse image under σ of a free orbit is a contractible space) but non-free orbits look like the classifying spaces of the respective stabilizer subgroups.

The above construction, is well-suited to proving that H_G^* satisfies the axioms of a generalized cohomology theory [14]. In the case of real coefficients, there is another construction of H_G^* more along the lines of deRham cohomology theory, which provides a more explicit way of writing down and computing with elements of the cohomology ring, by using differential forms [4]. Rather than constructing the space X_G as above, we construct an algebraic complex which computes its real cohomology. This construction is explained in the following sections.

2.2 The Weil algebra

We begin with the Lie algebra $\mathfrak{g}$ of the group G, and the deRham complex $\Omega(X)$ of exterior differential forms on X. Let $S(\mathfrak{g}^*)$ and $\Lambda(\mathfrak{g}^*)$ denote respectively the symmetric and exterior algebras constructed from the linear dual of the algebra $\mathfrak{g}$. They possess the structure of graded algebras. Let $\mathcal{W}(\mathfrak{g})$ be the graded tensor product $S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$. This algebra is called the *Weil algebra* of $\mathfrak{g}$. To explicitly describe its structure, we introduce the following notation for a set of generators: Fix a basis $\{T_a\}$ for $\mathfrak{g}$ and let $\{T^a\}$ denote the corresponding dual basis for $\mathfrak{g}^*$. Let ϕ^a denote the element $T^a \otimes 1$ in $S^1(\mathfrak{g}^*) \otimes \Lambda^0(\mathfrak{g}^*)$ and let ω^a denote the element $1 \otimes T^a$ in $S^0(\mathfrak{g}^*) \otimes \Lambda^1(\mathfrak{g}^*)$. The elements ϕ^a and ω^a generate $\mathcal{W}(\mathfrak{g})$ freely.

By assigning degree 2 to the ϕ^a and degree 1 to the ω^a we can make $\mathcal{W}(\mathfrak{g})$ into a graded-commutative algebra. Recall that a graded algebra is called graded-commutative if for any homogeneous elements a, b one has $ba = (-)^{a \cdot b} ba$. Here the notation $(-)^{a \cdot b}$ is shorthand for $(-1)^{\deg a \cdot \deg b}$.

2.2.1 Graded-commutative algebras

For convenience we collect some basic facts about graded-commutative algebras.

Definitition 2 *A* derivation of degree k *on a graded-commutative algebra A is a linear map $d : A \to A$ with the following two properties:*

Homogeneity: if $a \in A$ is homogeneous, then $\deg(da) = k + \deg(a)$, unless $da = 0$

Leibnitz property: for a, b homogeneous, $d(ab) = (da)b + (-)^{a \cdot k} a(db)$.

Let $Der^k(A)$ denote the set of derivations of degree k on A, and let $Der(A)$ be the direct sum $\bigoplus_k Der^k(A)$. Note that $Der(A)$ is a vector space - it is closed under finite linear combinations - but it is *not* closed under composition of operators. However, it is closed under the operation of graded commutator, or *superbracket*, defined on homogeneous derivations c, d as

$$[c, d] = cd - (-)^{c \cdot d} dc. \tag{2.1}$$

Under this operation, $Der(A)$ has the structure of a super-Lie algebra.

Note that if either c or d is of even degree, $[c, d]$ is the commutator $cd - dc$, and if they are both of odd degree, then $[c, d]$ is the anticommutator $cd + dc$.

If A and B are graded-commutative algebras the their tensor product $A \otimes B$ is defined in the usual way to be the algebra whose n-th graded part is $\bigoplus_{j+k=n} A^j \otimes B^k$, but in order that $A \otimes B$ be graded-commutative, the multiplication must be defined (for homogeneous elements) by

$$(a \otimes b)(a' \otimes b') = (-)^{b \cdot a'} aa' \otimes bb'. \tag{2.2}$$

If f is a homogeneous operator on A (i.e. $f : A \to A$ is a linear map, not necessarily an algebra homomorphism, satisfying $\deg(fa) = \deg(f) + \deg(a)$ whenever a is homogeneous and $fa \neq 0$) and g is a homogeneous operator on B, we define the operator $f \otimes g$ on $A \otimes B$ by

$$f \otimes g(a \otimes b) = (-)^{g \cdot a} fa \otimes gb \tag{2.3}$$

on homogeneous elements a, b.

If f is a derivation on A and g is a derivation on B, then in general $f \otimes g$ is *not* a derivation on the algebra $A \otimes B$. However, if the homogeneous operator f corresponds to multiplication by a fixed element of A and g is a derivation, then $f \otimes g$ is a derivation. The same holds if f is a derivation and g a multiplication operator.

Finally note that using the Leibnitz property, once we know the values of a derivation on a set of algebra generators, we can evaluate the derivation on products of generators, and by linearity, on the whole algebra. Thus to check equality of two derivations, it suffices to check that they have the same effect on generators. This simple principle is useful later.

2.2.2 Cohomology of the Weil algebra

With these generalities aside, let us return to the algebra $\mathcal{W}(\mathfrak{g})$. Relative to the chosen basis $\{T_a\}$ of $\mathfrak{g}$ the *structure constants* f^c_{ab} of $\mathfrak{g}$ are defined by $[T_a, T_b] = f^c_{ab} T_c$. Here,

the square braces denote the ordinary Lie bracket, and the summation convention applies, so that repeated indices are summed over. Define the derivation d of degree 1 by:

$$d\omega^a = -\frac{1}{2} f^a_{bc} \omega^b \omega^c + \phi^a \tag{2.4}$$

$$d\phi^a = -f^a_{bc} \omega^b \phi^c \tag{2.5}$$

The motivation for this definition comes from the definition of curvature, as described in section 1.5. Suppose P is a principal G-bundle (G acts on the right) and θ is a connection form on P. Recall that θ takes its values in $\mathfrak{g}$. By pairing the value of the connection form with a fixed element ξ of $\mathfrak{g}^*$, we get ordinary, $\mathbf{R}$-valued 1-form on P. That is, choosing a connection form on P determines a map $\mathfrak{g}^* \to \Omega^1(P)$, which extends naturally to a map $\Lambda(\mathfrak{g}^*) \to \Omega(P)$. Furthermore, since the space of connection forms is a convex space, a different choice of connection would determine a map in the same homotopy class. Similarly, once a connection form is chosen, the associated curvature form determines a map $\mathfrak{g}^* \to \Omega^2(P)$ which extends to a map $S(\mathfrak{g}^*) \to \Omega(P)$. Combining these maps, one has a map $\mathcal{W}(\mathfrak{g}) \to \otimes(\mathcal{P})$. Equations 2.4 and 2.5 define a differential on $\mathcal{W}(\mathfrak{g})$ such that this mapping is a chain map (the differential on $\Omega(P)$ being the exterior derivative). In particular, 2.4 is the definition of the "universal" curvature in terms of the (universal) connection, and 2.5 is the Bianchi identity. This is enough to show that the diagram

$$\begin{tikzcd}
\mathcal{W}(\mathfrak{g}) \arrow[r] \arrow[d, "d"] & \Omega(P) \arrow[d, "d"] \\
\mathcal{W}(\mathfrak{g}) \arrow[r] & \Omega(P)
\end{tikzcd}$$

commutes, for any choice of connection form on P. It remains to show that the derivation d on $\mathcal{W}(\mathfrak{g})$ is actually a differential, that is, $d^2 = 0$.

Proposition 1 *$d^2 = 0$ and $H^*(\mathcal{W}(\mathfrak{g}), d)$ is trivial (consists only of $\mathbf{R}$, in dimension zero).*

Proof: Let $\tilde{\phi}^a = \mathrm{d}\omega^a = \phi^a - \frac{1}{2} f^a_{bc}\omega^b\omega^c$. The elements ω^a of degree 1 and $\tilde{\phi}^a$ of degree 2 generate $\mathcal{W}(\mathfrak{g})$. Note that since d is a derivation of degree 1, the supercommutator $[d,d] = d^2 - (-)^1 d^2 = 2d^2$, so that $d^2 = \frac{1}{2}[d,d]$ is also a derivation. The zero map is a derivation, and to check for equality of two derivations, it suffices to check on a set of generators.

Hence we need only check $d^2 = 0$ on ω^a and $\tilde{\phi}^a$. Now,

$$\begin{aligned}
d\tilde{\phi}^a &= d\phi^a - \frac{1}{2} f^a_{bc} d(\omega^b\omega^c) \\
&= -f^a_{bc}\omega^b\phi^c - \frac{1}{2} f^a_{bc}\tilde{\phi}^b\omega^c + \frac{1}{2} f^a_{bc}\omega^b\tilde{\phi}^c \\
&= -f^a_{bc}\omega^b\phi^c + f^a_{bc}\omega^b\tilde{\phi}^c \text{ (because } \omega^b \text{ and } \tilde{\phi}^c \text{ commute, and } f^a_{bc} = -f^a_{cb}) \\
&= f^a_{bc}\omega^b(\tilde{\phi}^c - \phi^c) \\
&= -\frac{1}{2} f^a_{bc} f^c_{de}\omega^b\omega^d\omega^e \qquad (2.6)
\end{aligned}$$

By antisymmetry of the ω's, $\omega^b\omega^d\omega^e$ is zero unless b, d, e are *distinct* indices. For each triple of distinct indices there will be six terms in the sum, corresponding to permutations of the indices. Grouping these into odd and even permutations and using invariance of the product $\omega^b\omega^d\omega^e$ under cyclic permutations of b, d, e, one obtains two copies of the Jacobi identity. Thus $d^2 = 0$.

Since we've also shown that the generators $\tilde{\phi}^a$ of degree 2 are exact, and no linear combination of the generators ω^a of degree 1 is closed, the second part of the proposition, that $\mathcal{W}(\mathfrak{g})$ is acyclic, follows. □

2.3 Weil model for equivariant cohomology

Since the cohomology of the Weil algebra is trivial, we can think of this algebra as a model for the forms on the contractible space E. Recall that E has a free G-action on it, which must be reflected in our algebraic model, since our goal is to take a quotient by this action. The additional algebraic structure we need is an action of $\mathfrak{g}$ on $\mathcal{W}(\mathfrak{g})$ by derivations. For each basis element T_a of $\mathfrak{g}$, define a derivation Ia on $\mathcal{W}(\mathfrak{g})$ (called the contraction operator or interior product) by:

$$I_a\omega^b = \delta_a^b \tag{2.7}$$

$$I_a\phi^b = 0. \tag{2.8}$$

Then let the *Lie derivative* L_a be defined as the superbracket of this derivation with the derivation d defined above:

$$L_a = [I_a, d] = I_a d + dI_a. \tag{2.9}$$

Since I_a is of degree -1 and d is of degree 1, the Lie derivative is a derivation of degree 0. It corresponds to the coadjoint action of $\mathfrak{g}$ on the Weil algebra.

Since G acts on X there are corresponding derivations on $\Omega(X)$. Let ι_a denote the usual interior product of forms with the vertical vector field Ξ_a generated by X_a, and let $\mathcal{L}_a$ be the usual Lie derivative along Ξ_a, defined as $calL_a = d\iota_a + \iota_a d$.

We have indicated that the Weil algebra is to serve as an algebraic model for the deRham forms on the space E. To compute equivariant cohomology we are interested in $E \times X$. By the Künneth theorem, we may model forms on this product space by the graded tensor product of the differential algebras of forms on each factor. Thus, define the graded algebra A by:

$$A = \mathcal{W}(\mathfrak{g}) \otimes \Omega(X) \tag{2.10}$$

The algebra A can be equipped with a differential, which we continue to denote

by d, which is defined as $d \otimes 1 + 1 \otimes d$. (The derivation d on $\Omega(X)$ is just the exterior derivative.)

Since the Weil algebra is acyclic, the cohomology of A is just the deRham cohomology of X. We can get the cohomology of the homotopy quotient X_G, which is the equivariant cohomology, by restricting to a subalgebra of A called the algebra of basic forms. A form on a G-space is *basic* if it vanishes on vertical vectors (tangent vectors along G-orbits) and is G-invariant. The condition that a form be horizontal (vanish on vertical vectors) is equivalent to being annihilated by interior product with any vertical vector. Similarly, G-invariance is equivalent to being annihilated by all Lie derivatives in vertical directions. Thus, the basic subalgebra is

$$A_{bas} = (\bigcap_a \ker I_a \otimes 1 + 1 \otimes \iota_a) \cap (\bigcap_a \ker L_a \otimes 1 + 1 \otimes \mathcal{L}_a). \tag{2.11}$$

It is proven in [2] that when G is compact and connected, the differential algebra A_{bas} computes the desired equivariant cohomology, that is,

$$H^*(A_{bas}, d) \cong H^*_G(X). \tag{2.12}$$

This differential algebra is called the *Weil model* for equivariant cohomology.

2.4 Cartan model for equivariant cohomology

There is another model for equivariant cohomology which is more along the lines of the BRST theory we shall introduce later [7]. This construction begins with the same algebra $\mathcal{W}(\mathfrak{g}) \otimes \Omega(X)$ as above, but with a different differential. In order to avoid confusion we denote the algebra $\mathcal{W}(\mathfrak{g}) \otimes \Omega(X)$ by B in this section.

Define the differential δ on B by

$$\delta = 1 \otimes d + d \otimes 1 + \omega^a \otimes \mathcal{L}_a - \phi^b \otimes \iota_b. \tag{2.13}$$

(The summation convention applies to the indices a, b above.) It follows from

a long calculation that δ^2 is zero, but this also is a consequence of the following proposition. We now define an algebra homomorphism $\Psi : B \to A$.

The map Ψ is defined as $\exp(-\omega^a \otimes \iota_a)$, where the exponential is defined by the customary power series. However, depending on the degree of the form it is acting on, only finitely many terms will be nonzero, since a form of degree n can only be contracted n times before vanishing. In the following, the summation convention apples to roman but not to greek indices. Since $\omega^\alpha \otimes \iota_\alpha$ and $\omega^\alpha \otimes \iota_\alpha$ commute,

$$\Psi = \prod_\alpha (1 - \omega^\alpha \otimes \iota_\alpha), \tag{2.14}$$

and Ψ can be seen to be a gradation-preserving isomorphism of algebras. (Its inverse is given by $\exp(\omega^a \otimes \iota_a)$).

Proposition 2 *The diagram*

$$\begin{tikzcd} B \arrow[r, "\Psi"] \arrow[d, "\delta"'] & A \arrow[d, "d"] \\ B \arrow[r, "\Psi"] & B \end{tikzcd}$$

commutes.

Proof: By our general principle, to show $d \circ \Psi = \Psi \circ \delta$ it is enough to check equality on algebra generators.

On elements of the form $(u \otimes 1)$, we have $\delta(u \otimes 1) = du \otimes 1$ since all derivations vanish on the constant 1. Similarly, Ψ acts as the identity on $du \otimes 1$ since $\iota_\alpha 1$ vanishes. Thus,

$$\Psi \circ \delta(u \otimes 1) = du \otimes 1 = d \circ \Psi(u \otimes 1). \tag{2.15}$$

On elements of the form $(1 \otimes f)$ where f is a function (zero-form) on X we have $\Psi(1 \otimes f) = (1 \otimes f)$ since the interior product ι_α vanishes on functions. Then $d \circ \Psi(1 \otimes f) = (1 \otimes df)$. On the other hand,

$$\begin{aligned} \Psi \circ \delta(1 \otimes f) &= \exp(-\omega^a \otimes \iota_a)(1 \otimes df + \omega^a \otimes \mathcal{L}_a f) \\ &= 1 \otimes df + \omega^a \otimes \mathcal{L}_a f - \omega^a \otimes \iota_a df \end{aligned}$$

$$= \quad 1 \otimes df. \tag{2.16}$$

Finally we check elements of the form $(1 \otimes \eta)$ where η is a one-form on X. First we compute

$$\begin{aligned} d \circ \Psi(1 \otimes \eta) &= d(1 \otimes \eta - \omega^a \otimes \iota_a \eta) \\ &= 1 \otimes d\eta - d\omega^a \otimes \iota_a \eta + \omega^a \otimes d\iota_a \eta \\ &= 1 \otimes d\eta + \omega^a \otimes d\iota_a \eta - (\phi^a + \frac{1}{2} f^a_{bc} \omega^b \omega^c) \otimes \iota_a \eta. \end{aligned} \tag{2.17}$$

We need to compare this with expression 2.16 for $\Psi \circ \delta(1 \otimes \eta)$. Computing,

$$\begin{aligned} \Psi \circ \delta(1 \otimes \eta) &= \exp(-\omega^a \otimes \iota_a)(1 \otimes d\eta + \omega^a \otimes \mathcal{L}_a \eta - \phi^b \otimes \iota_b \eta) \\ &= 1 \otimes d\eta + \omega^a \otimes \mathcal{L}_a \eta - \phi^b \otimes \iota_b \eta - \omega^a \otimes \iota_a d\eta \\ &\quad + \omega^a \omega^b \otimes \iota_a \mathcal{L}_b \eta - \frac{1}{2} \omega^a \omega^b \otimes \iota_a \iota_b \eta. \end{aligned} \tag{2.18}$$

(Note that we have used the fact that $(\omega^a \otimes \iota_a)(\omega^b \otimes \iota_b) = -\omega^a \omega^b \otimes \iota_a \iota_b$ consistent with the rules of superalgebra.) To show that this is equal to expression 2.17, we isolate the last two terms, $\omega^a \omega^b \otimes \iota_a \mathcal{L}_b \eta - \frac{1}{2} \omega^a \omega^b \otimes \iota_a \iota_b \eta$. Because the ω's and ι's are anticommuting variables (odd degree), one has that $\omega^\alpha \omega^\beta \otimes \iota_\alpha \iota_\beta = \omega^\beta \omega^\alpha \otimes \iota_\beta \iota_\alpha$, and this vanishes when $\alpha = \beta$. Thus we can rewrite the last two terms of expression 2.18 as:

$$\sum_{a<b} \omega^a \omega^b \otimes [\iota_a, \mathcal{L}_b] \eta - \sum_{a<b} \omega^a \omega^b \otimes \iota_a \iota_b d\eta. \tag{2.19}$$

Recall that if X, Y are vector fields, then

$$d\eta(X, Y) = X \cdot \eta(Y) - Y \cdot \eta(X) - \eta([X, Y]). \tag{2.20}$$

Since $\iota_a \iota_b d\eta = d\eta(\Xi_b, \Xi_a) = -d\eta(\Xi_a, \Xi_b)$, it follows that

$$-\iota_a \iota_b d\eta = \iota_a d\iota_b \eta - \iota_b d\iota_a \eta - f^c_{ab} \iota_c \eta, \tag{2.21}$$

as required to complete the proof that $\Psi \circ \delta(1 \otimes \eta) = d \circ \Psi(1 \otimes \eta)$.

Since the functions and one-forms generate $\Omega(X)$, we have checked a complete set of generators for the algebra A. Thus the above diagram commutes. $\square$

Since $\delta = \Psi^{-1} \circ d \circ \Psi$, and $d^2 = 0$, it follows that $\delta^2 = 0$ as well. The map Ψ induces a cohomology isomorphism from $H^*(B, \delta)$ to $H^*(A, d)$ which is in turn isomorphic to the deRham cohomology of X. Furthermore, if we define B' as the inverse image subalgebra $\Psi^{-1}(A_{bas})$, then $H^*(B') \cong H^*_G(X)$. To identify the algebra B', we need to know which operators on B correspond to the operators $I_\alpha \otimes 1 + 1 \otimes \iota_\alpha$ and $L_\alpha \otimes 1 + 1 \otimes \mathcal{L}_\alpha$ on A (the kernels of which define the basic subalgebra A_{bas}). It is easy to check that the required operators are simply $I_\alpha \otimes 1$ and $L_\alpha \otimes 1 + 1 \otimes \mathcal{L}_\alpha$. That is, the Lie derivative operators are the same, and the contraction operators differ in that they act only on the Weil algebra. Any element of B which contains factors ω^a will not give zero when acted on by the contraction operator $I_a \otimes 1$. Thus the algebra B' consists of all ω-free, G-invariant elements, or symbolically,

$$B' = (S(g^*) \otimes \Omega(X))^G. \tag{2.22}$$

The restriction of the differential δ to B' is simply $1 \otimes d - \phi^a \otimes \iota_a$. The algebra B' equipped with this differential is known as the Cartan model for equivariant cohomology. It was first introduced in [5]. Kalkman [7] notes that by introducing a parameter t and replacing the map Ψ with

$$\Psi_t = \exp(-t\omega^a \otimes \iota_a), \tag{2.23}$$

we obtain a deformation of (B, δ) onto (A, d). By taking $\Psi^{-1}(A_{bas})$ we obtain a one-parameter family of models connecting the Weil and Cartan models.

2.5 The Thom form

Suppose $\mathcal{V}$ is a complex rank-n vector bundle over X with projection map π. Since the fiber is a contractible space, the spaces X and $\mathcal{V}$ have the same cohomology.

However, on $\mathcal{V}$ we may introduce another cohomology theory, namely cohomology with compact supports in the vertical direction, denoted $H_c^*(\mathcal{V})$. In this theory, rather than considering the full deRham complex $\Omega(\mathcal{V})$, we only allow forms for which the restriction to any fiber vanishes outside some compact subset of that fiber. By an excision argument, $H_c^*(\mathcal{V})$ is isomorphic to the relative cohomology $H^*(\mathcal{V}, \mathcal{V} - X)$. The Thom isomorphism theorem allows us to compare this to the cohomology of X. In particular, it states that

$$H^*(X) \cong H_c^{*+n}(\mathcal{V}), \tag{2.24}$$

and that this isomorphism is given by sending $\eta \in H^*(X)$ to $\pi^*(\eta) \wedge \tau$, where τ is a certain element of$H_c^n(\mathcal{V})$ known as the *Thom class.* (For a proof of this theorem, see [4]). A closed differential form representing this cohomology class is called a Thom form, and we shall also denote this form by τ. For example, if $\mathcal{V}$ is the trivial bundle $X \times V$ (where V is a vector space), let ν be a closed differential form representing a generator of $H_c^n(V)$. Then $\tau = 1 \otimes \nu$ is a Thom form. For nontrivial bundles, the Thom form is not as easy to construct explicitly. Our interest in the Thom form is that the Thom form of a vector bundle is the Poincaré dual of the zero-section.

There is an equivariant version of the Thom isomorphism theorem. It applies to bundles with G-action, i.e. a group acts on the spaces $\mathcal{V}$ and X, and the group action is compatible with the bundle projection. In this case, we may form a homotopy-equivalent bundle $E \times \mathcal{V} \to E \times X$, on which the diagonal G-action is free. Taking the quotient by this diagonal action we arrive at the vector bundle $\mathcal{V}_G \to X_G$. Recall that $H^*(X_G) = H_G^*(X)$, and similarly for $\mathcal{V}_G$. The equivariant Thom isomorphism theorem states that

$$H_G^*(X) \cong H_{G,c}^{*+n}(\mathcal{V}), \tag{2.25}$$

and, as before, the isomorphism is provided by a particular cohomology class, now lying in $H_{G,c}^n(\mathcal{V})$ (equivariant cohomology with compact supports). We wish to find an explicit representative for this class. For simplicity, we shall assume that $\mathcal{V}$ is the trivial bundle $X \times V$.

First, as noted in [1] and [11], we may with impunity replace the complex of

compactly supported forms with the complex of forms which satisfy the Schwartz condition of rapid decay at infinity (in the vertical direction). This complex will be denoted by $\Omega_s(\mathcal{V})$ or for simplicity of notation we may omit the $\mathcal{V}$ where no confusion will result. Similarly we denote the n-th graded part of the symmetric algebra $S(\mathfrak{g}^*)$ simply by S^n. The algebra $B' = (S \otimes \Omega_s)^G$ may be assigned a bigrading as follows: let p be the polynomial degree (i.e. degree in the symmetric algebra part) and let q be the polynomial degree plus the form degree. On B', the differential is $d - \phi^a \otimes \iota_a$. For simplicity of notation, let ϕ denote the operator $\phi^a \otimes \iota_a$. The operator d raises form degree while leaving polynomial degree fixed, hence it increments q. The operator ϕ increases polynomial degree by 1 while lowering form degree by 1, hence it increases p by 1 without changing q. We may display the bigrading as follows:

$$
\begin{tikzcd}
0 \arrow[r, "\phi"] & (S^1 \times \Omega_c^n)^G \arrow[r, "\phi"] & (S^2 \times \Omega_c^{n-2})^G \arrow[r, "\phi"] & \cdot \\
(S^0 \times \Omega_c^n)^G \arrow[u, "d"] \arrow[r, "\phi"] & (S^1 \times \Omega_c^{n-1})^G \arrow[u, "d"] \arrow[r, "\phi"] & (S^2 \times \Omega_c^{n-2})^G \arrow[u, "d"] \arrow[r, "\phi"] & \cdot \arrow[u, "d"] \\
(S^0 \times \Omega_c^{n-1})^G \arrow[u, "d"] \arrow[r, "\phi"] & (S^1 \times \Omega_c^{n-2})^G \arrow[u, "d"] \arrow[r, "\phi"] & (S^2 \times \Omega_c^{n-3})^G \arrow[u, "d"] \arrow[r, "\phi"] & \cdot \arrow[u, "d"] \\
(S^0 \times \Omega_c^{n-2})^G \arrow[u, "d"] \arrow[r, "\phi"] & (S^1 \times \Omega_c^{n-3})^G \arrow[u, "d"] \arrow[r, "\phi"] & (S^2 \times \Omega_c^{n-4})^G \arrow[u, "d"] \arrow[r, "\phi"] & \cdot \arrow[u, "d"] \\
\cdot \arrow[u, "d"] \arrow[r, "\phi"] & \cdot \arrow[u, "d"] \arrow[r, "\phi"] & \cdot \arrow[u, "d"] \arrow[r, "\phi"] & \cdot \arrow[u, "d"] \\
(S^0 \times \Omega_c^1)^G \arrow[u, "d"] \arrow[r, "\phi"] & (S^1 \times \Omega_c^0)^G \arrow[u, "d"] & & \\
(S^0 \times \Omega_c^0)^G \arrow[u, "d"] \arrow[r, "\phi"] & 0 \arrow[u] & &
\end{tikzcd}
$$

2.5.1 The zig-zag construction

Starting with an element a in $(S^0 \otimes \Omega_c^n)^G$ which generates $H_c^n(V)$, we may use the "zig-zag" construction to construct an equivariant Thom form. This relies on the facts that d and ϕ commute up to sign on the G-invariant elements of B, and that $H_c^*(V)$ is trivial except in degree n. Because a is already of top form degree, da is zero. We say that a is d-closed. We wish to construct from a an element which is $(d-\phi)-closed$. Although ϕa may be nonzero, the sign-commutativity of the upper left square in the diagram implies that $d(\phi a)$ is zero. Then since the second column is exact at the term $(S^1 \times \Omega_c^{n-1})^G$, we may choose an element b in $(S^1 \times \Omega_c^{n-2})^G$ such that $db = \phi a$. Then $(d-\phi)(a+b) = db - \phi a - \phi b = -\phi b$. Now we may proceed in analogous fashion to add a term c in $(S^2 \times \Omega_c^{n-4})^G$ so that dc cancels ϕb, at the expense of introducing a term ϕc in $(d-\phi)(a+b+c)$. However, iterating this construction, we reach a region of the diagram where all groups are zero for dimensional reasons. At this stage we are left with a string of elements along the antidiagonal, and the sum of these elements is $(d-\phi)$-closed.

If $z^1, \ldots, z^n$ is an orthonormal coordinate system on V, then we may apply the zig-zag construction to the element $e^{-|z|^2} dz^1 \wedge \ldots \wedge dz^n$ of $(S^0 \otimes \Omega_c^n)^G$ to obtain an equivariant Thom form. This is carried out in [11]. However, this construction is difficult to apply to non-trivial bundles and the generalization to infinite-dimensional bundles is far from obvious. Hence we introduce an alternative construction via Fourier transform of differential forms.

2.6 A Fourier transform for differential forms

Let V be a vector space with orthonormal coordinates $z^1, \ldots, z^n$, and let $\Lambda(V)$ denote its exterior algebra over the complex numbers.

Definitition 3 *The* Berezin integral *is the linear map*

$$\int_V : \Lambda(V) \to \mathbf{C} \tag{2.26}$$

which simply extracts the coefficient of the volume form $dz^1 \wedge \ldots \wedge dz^n$ from elements of the exterior algebra.

By tensoring with the identity on $\Lambda(V^*)$ we extend the Berezin integral to a map

$$\int_V : \Lambda(V^*) \otimes \Lambda(V) \to \Lambda(V^*). \tag{2.27}$$

Let $\mathcal{S}(V)$ denote the Schartz functions (rapidly decreasing at infinity) on V. Now, $\Omega_s(V) = \mathcal{S}(V) \otimes \Lambda(V)$. The ordinary Fourier transform may be interpreted as a linear map $\mathcal{F} : \mathcal{S}(V) \to \mathcal{S}(V^*)$ defined as follows: If $f \in \mathcal{S}(V)$ and $b \in V^*$, then

$$(\mathcal{F}f)(b) = N \int_V f(z) e^{i<b|z>} dz^1 \wedge \ldots \wedge dz^n. \tag{2.28}$$

Here, the integral over V is an ordinary (not Berezin) integral, and $< b|z >$ denotes the pairing of an element of V with an element of its dualspace (no inner product on V is required, only a volume form). N is a normalization constant which we shall ignore from now on.

Next we define a Fourier transform (also denoted F from $\Lambda(V)$ to $\Lambda(V^*)$. Let $\bar{z}_1, \ldots, \bar{z}_n$ denote the duals to $z^1, \ldots, z^n$. Starting with an element η in $\Lambda(V)$, we first multiply it by $e^{id\bar{z}_j \otimes dz^j}$ to obtain an element of $\Lambda(V^*) \otimes \Lambda(V)$. Then we apply the extended Berezin integral to obtain an element of $\Lambda(V^*)$. Symbolically,

$$\mathcal{F}(\eta) = \int_V \eta \wedge e^{id\bar{z}_j \otimes dz^j}. \tag{2.29}$$

Since $\Omega_s(V) = \mathcal{S}(V) \otimes \Lambda(V)$, we may combine the Fourier transforms 2.28 and 2.29 defined on $\mathcal{S}(V)$ and $\Lambda(V)$ to obtain a Fourier transform

$$\mathcal{F} : \Omega_s(V) \to \Omega_s(V^*), \tag{2.30}$$

which is defined by

$$\mathcal{F}(f \otimes \eta) = \int_V f(\cdot) e^{i<b|\cdot>} \otimes \eta \wedge e^{id\bar{z}_j \otimes dz^j}. \tag{2.31}$$

(The normalization constant has been omitted.) Here, the symbol $\int_V$ serves double duty: it denotes an ordinary integral of functions together with a Berezin integral of forms. Note that this transform takes k-forms to $(n-k)$-forms.

This Fourier transform enjoys many of the properties of the classical Fourier transform. In particular, it is easy to check that if n is even, $\mathcal{F} \circ \mathcal{F}$ is the identity. (To see this, it is enough to verify that $\mathcal{F}(dz^1 \wedge \ldots \wedge dz^k = (i)^{n^2-k^2} d\bar{z}_{k+1} \wedge \ldots \wedge d\bar{z}_n$.)

Since $\mathcal{F}$ is an isomorphism, we may construct a differential k on the complex $\Omega_s(V^*)$ such that the diagram

$$\begin{array}{ccc} \Omega^q(V) & \xrightarrow{\mathcal{F}} & \Omega^{n-q}(V) \\ \Big\downarrow d & & \Big\downarrow k \\ \Omega^{q+1}(V) & \xrightarrow{\mathcal{F}} & \Omega^{n-q-1}(V) \end{array}$$

commutes for each q.

To describe the operator k explicitly, we compute its action on algebra generators. Note that $\Omega_s(V^*) \cong \mathcal{S}(V^*) \otimes \Lambda(V^*)$. The generators can be taken to be elements of the form $f \otimes d\bar{z}^j$ where f is a Schwartz function on V^* and $d\bar{z}^1, \ldots, d\bar{z}^n$ are the generators of $\Lambda(V^*)$. We compute:

$$\begin{aligned} k(f \otimes d\bar{z}^j) &= \mathcal{F} d \mathcal{F}(f \otimes d\bar{z}^j) \\ &= \mathcal{F} d\{i^{n-1}(-)^{\frac{1}{2}n(n-1)}(-)^{j-1}\hat{f} dz^1 \wedge \ldots \wedge dz^{j-1} \wedge dz^{j+1} \wedge \ldots \wedge dz^n\} \\ &= \mathcal{F}(i^{n-1}(-)^{\frac{1}{2}n(n-1)} \frac{\partial \hat{f}}{\partial z^j} dz^1 \wedge \ldots \wedge dz^n \\ &= i^{n-2}(-)^{\frac{1}{2}n(n-1)} f z_j \\ &= (-)^{\frac{1}{2}(n-1)}(-)^{\frac{1}{2}(n)(n-1)} f z_j \\ &= -f z_j. \end{aligned}$$

Thus, the operator k is none other than the usual Koszul differential on the complex $\mathcal{S}(V^*) \otimes \Lambda(V^*)$.

2.7 Poincaré duals via Fourier transforms

We use the above construction to obtain representatives for the Poincaré duals of particular submanifolds. Suppose $F : X \to V$ a submersive map of the manifold X to a vector space V. Then the zero-locus $F^{-1}(0)$ is a submanifold Z of X. Let $\mathcal{V}$ denote the trivial bundle $X \times V$; then F is a section of $\mathcal{V}$ and the pullback via F of the Thom form τ of $\mathcal{V}$ is the Poincaré dual of Z.

2.7.1 The BRST algebra

To find the form τ, we introduce the BRST algebra $\Omega(X) \otimes \Omega_s(V) \otimes \Omega_s(V^*)$. As before, let $z^1, \ldots, z^n$ be an orthonormal basis for V and $\bar{z}_1, \ldots, \bar{z}_n$ the associated dual basis for V^*. This algebra is given a grading, by assigning degree 1 to elements dz^i and degree -1 to elements $d\bar{z}_i$.

We next introduce the BRST operator, usually denoted by s. This is a derivation of degree 1, which acts on the BRST algebra as follows:

$$\begin{aligned} s(\eta) &= d\eta \text{ if } \eta \text{ is a form on } X \\ s(z^i) &= dz^i \\ s(dz^i) &= 0 \\ s(\bar{z}_i) &= 0 \\ s(d\bar{z}_i) &= -\bar{z}_i \end{aligned} \tag{2.32}$$

It is routine to check that s^2 is zero, so that s is a coboundary operator.

Proposition 3 *The element*

$$\int_{V^*} e^{is(z^i d\bar{z}_i - i\sum_j \bar{z}_j d\bar{z}_j)} \tag{2.33}$$

represents the Thom class of $\mathcal{V}$.

Proof: The above integral is just the Fourier transform of $e^{-\bar{z}\cdot\bar{z}}$, which gives a

generator of $H_c^n(V)$. As stated in section 2.5.1, this provides a Thom form for the trivial bundle $X \times V$.

2.7.2 The equivariant case

Now suppose that a group G acts on both X and V, and the map F is equivariant, that is, $g \cdot F(m) = F(g \cdot m))$ for each g in G and m in X. The case we have in mind is where $\mathcal{G}$ is a gauge group, X is a space of connections, V is the space of curvature tensors ($\mathfrak{g}$-valued 2-forms), and F is the map taking a connection to its curvature. However, we are going to work only on the finite-dimensional analogue.

The action of G on V induces an action of $\mathfrak{g}$ on V, which extends naturally to an action on $\Lambda(V)$. An inner product on V allows us to transfer this action to V^* and $\Lambda(V^*)$. Then the contraction operators ι_a can be defined on these algebras as well.

The equivariant BRST algebra is defined to be

$$[S(\mathfrak{g}^*) \otimes \Omega(X) \otimes \Omega_s(V) \otimes \Omega_s(V^*)]^G, \tag{2.34}$$

where the exponent stands for the G-invariant subalgebra. Note that in the case where V is one-dimensional, the algebra 2.34 coincides with the Cartan algebra defined in 2.22. The BRST algebra is an extension of the Cartan model.

Elements of form degree q in $\Omega_s(V^*)$ are assigned grading $-q$ in the BRST algebra. We extend the operator s to this algebra. The equivariant BRST operator (still denoted by s) is the derivation which restricts to the equivariant differential $d - \phi^a \otimes \iota_a$ on the subalgebra $(S(\mathfrak{g}^*) \otimes \Omega(X) \otimes \Omega_s(V))^G$ and restricts to the Fourier image of $d - \phi^a \otimes \iota_a$ on the subalgebra $(S(\mathfrak{g}^*) \otimes \Omega(X) \otimes \Omega_s(V^*))^G$. This operator may be described explicitly by computing its effect on generators. The details are omitted here, the computation may be found in [7]. The action of s on the algebra generators is:

$$\begin{aligned} s(d\bar{z}_j) &= -\bar{z}_j \\ s(\bar{z}_j) &= \phi^a \otimes \iota_a d\bar{z}_j \end{aligned}$$

$$
\begin{aligned}
s(\phi^a) &= 0 \\
s(z^j) &= dz^j - \phi^a \otimes \iota_a dz^j \\
s(dz^j) &= -\phi^a \otimes \iota_a dz^j
\end{aligned} \tag{2.35}
$$

We are now in a position to construct the equivariant Thom form.

Proposition 4 *The element*

$$\int_{V^*} e^{is(<z-i\bar{z}|d\bar{z}>)} \tag{2.36}$$

is a representative of the equivariant Thom class for G acting on V. It is equal to the Thom form constructed in section 2.5.1.

Proof: Since the integrand is equal to $e^{is<z|d\bar{z}>}e^{s<\bar{z}|d\bar{z}>}$, the integral may be regarded as a Fourier transform of the element $e^{s<\bar{z}|d\bar{z}>}$. Since this form is s-closed and s is the Fourier image of $d - \phi^a \otimes \iota_a$, the integral is $(d - \phi^a \otimes \iota_a)$-closed, as required to represent a cohomology class. Furthermore its component in $(S^0(\mathfrak{g}^*) \otimes \Omega^n_s(V))^G$ is (up to normalization) equal to $e^{-|z|^2} dz^1 \wedge \ldots \wedge dz^n$ as can be seen by setting $\phi = 0$. □

2.8 Applications and further directions

This section follows references [7] and [1] rather closely.

As in section 2.7.2, suppose F is a G-equivariant map from a manifold X to a vector space V. An infinite dimensional example of physical interest is provided by considering the "curvature map" from $\mathcal{A}$ (the space of connection forms on some principal bundle) to the vector space of $\mathfrak{g}$-valued 2-forms, which assigns to any connection its curvature. The group $\mathcal{G}$ of local gauge transformations acts on these spaces, and the curvature map is $\mathcal{G}$-equivariant. The zero-locus of the curvature map is the space $\mathcal{A}_f$ of flat connections. An even more interesting special case, when M is a four-manifold, is the map which assigns to a connection the self-dual component of its curvature tensor, (i.e. the map F as above, followed by linear projection onto the space of self-dual 2-forms, as defined in section 1.7.1). The zero-locus of this map

is the space of antiself-dual connections, which is crucial for the Donaldson theory. However since only the finite-dimensional case has been treated here, we will approach these examples only by analogy. Thus we retain the notation G for the symmetry group acting on X and V.

Since the map F is assumed to be G-equivariant, it descends to a map on the quotients, which we denote $\bar{F} : X/G \to V/G$. Let $Z \subset X/G$ be the zero-locus $\bar{F}^{-1}(0)$. The above construction for the equivariant Thom class may be used to obtain a representative for the Poincaré dual of Z in X/G. The classical approximation involves integration over Z. Using the Poincaré dual of Z, this integration may be replaced by an integration over X/G. We would like to go further and replace this with an integration over X. (In the physical example of the previous paragraph, this would be the affine space $\mathcal{A}$. Integrations over infinite-dimensional spaces are not well-defined in general, but for the particularly nice case of functions which are Gaussians times polynomials, these integrals can be made sense of.)

We have as yet not used the full power of the BRST theory; we have only considered a subalgebra of what physicists call the BRST complex. In order to reduce the problem to an integration over X, we extend the algebra of section 2.7.2 by introducing a dual Weil algebra, $\mathcal{W}(\mathfrak{g}^*) = S(\mathfrak{g}) \otimes \Lambda(\mathfrak{g})$ with symmetric algebra generators $\bar{\phi}_a$ in degree -2 and exterior algebra generators $\bar{\omega}_a$ in degree -1.

The differential s is extended by setting

$$\begin{aligned} s\bar{\phi}_a &= \bar{\omega}_a \\ s\bar{\omega}_a &= -f^c_{ab}\phi^b \otimes \bar{\phi}_c. \end{aligned} \tag{2.37}$$

This differential coincides with the BRST operator as discussed in [10] and [13]. The BRST cohomology can be interpreted geometrically as the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the module of functions on X/G. Elements of the zero-dimensional cohomology correspond to gauge-invariant functions.

The differential of the G-action on X gives a map from $\mathfrak{g}$to the space $Vec(X)$ of vector fields on X. Using an ad-invariant inner product on $\mathfrak{g}$ and a Riemannian

structure on X, this map may be dualized to a map $\nu : \mathfrak{g}^* \to \Omega^1(X)$. Recall that we have fixed a basis $\{T^a\}$ of $\mathfrak{g}^*$. Let ν^a denote the one-form $\nu(T^a)$ on X. Then, according to arguments which can be found in [1], multiplying the integrand in equation 2.36 by

$$\int_{\mathcal{W}(\mathfrak{g}^*)} e^{is(\bar{\phi}_a \nu^a)} \tag{2.38}$$

provides a "gauge-fixed" action, so that we may perform all integrations over X. It is claimed in [7] that the action so obtained reproduces the action for the topological Yang-Mills theory obtained by Witten in [15].

2.9 Closing remarks

The analysis of gauge-fixing in this paper is in many ways unsatisfactory. Much more can be said about the geometrical interpretation of the BRST construction. Inasmuch as we have discussed BRST theory, we have stressed its algebraic structure. For a more complete discussion, see [10]

As remarked several times throughout this paper, a fully rigorous quantum gauge field theory requires generalizations of these constructions to an infinite-dimensional setting.

Despite the fact that the mathematical foundations are still partially missing, physicists have had great success using the BRST transformations, for example to prove renormalizability of nonabelian gauge theories.

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