

Traveling Salesman Path Problems

by

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B.A., University of California, Berkeley, 2000

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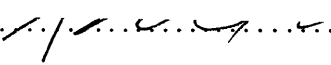
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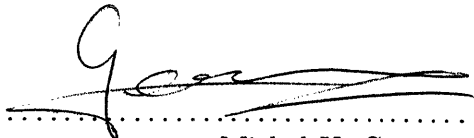
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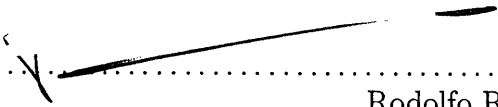
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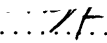
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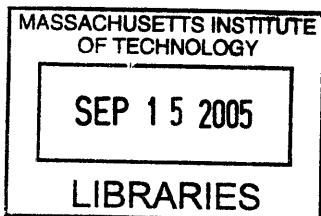
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Abstract

In the Traveling Salesman Path Problem, we are given a set of cities, traveling costs between city pairs and fixed source and destination cities. The objective is to find a minimum cost path from the source to destination visiting all cities exactly once. The problem is a generalization of the Traveling Salesman Problem with many important applications.

In this thesis, we study polyhedral and combinatorial properties of a variant we call the Traveling Salesman Walk Problem, in which the minimum cost walk from the source to destination visits all cities *at least* once. Using the approach of linear programming, we study properties of the polyhedron corresponding to a linear programming relaxation of the traveling salesman walk problem. Our results relate the structure of the underlying graph of the problem instance with polyhedral properties of the corresponding fractional walk polyhedron.

We first characterize traveling salesman walk perfect graphs, graphs for which the convex hull of incidence vectors of traveling salesman walks can be described by linear inequalities. We show these graphs have a description by way of forbidden minors and also characterize them constructively. We extend these results to relate the underlying graph structure to the integrality gap of the corresponding fractional walk polyhedron. We present several graph operations which preserve integrality gap; these operations allow us to find the integrality gap of graphs built from smaller bricks, whose integrality gaps can be found by computational or other methods.

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Contents

1	Introduction	13
1.1	Traveling Salesman Variants and Generalizations	15
1.2	Linear Programming Formulation	16
1.3	Traveling Salesman Paths	18
2	Walk-Perfection	23
2.1	C_6 Minor Free Graphs	25
2.2	Characterization of Walk-Perfect Graphs	29
2.3	Connection with TSP-Perfection	39
2.4	s - t Walk-Perfection	42
2.5	Asymmetric Traveling Salesman Path Problem	44
2.5.1	Path/Cycle Covers	45
2.5.2	$O(\sqrt{n})$ -Approximation	46
3	TSP-perfection	49
3.1	Bricks and TSP-Perfection Preserving Operations	49
3.2	TSP-Perfection of bricks	56
3.3	Operations Φ_1 and Φ_2	65
3.4	Exhaustive List of Bricks	70
3.4.1	Bricks from K_2 and K_3	73
3.4.2	Bricks from $K_4 (= W_3)$	73
3.4.3	Bricks from $W_4 (= K_5 \setminus \{e_1, e_2\})$	75
3.4.4	Bricks from K_5	78
3.4.5	Bricks from $K_5 \setminus e (= P_2)$	79
3.4.6	Bricks from wheels	82
3.4.7	Bricks from Propellers P_k for $k \geq 4$	82
3.4.8	Bricks from $P_3(0, 0, 0), P_3(0, 0, 1), P_3(0, 1, 1)$ and $P_3(1, 1, 1)$	83
4	Integrality Gap Preserving Operations	87
4.1	Integrality Gap Under Graph Minors	88
4.2	Integrality Gap under Degree-2 Vertex Duplication	90
4.3	Integrality Gap under Operations Φ_1 and Φ_2	101

5 Appendix	105
5.1 Graphs $K_4^{(1)}, K_4^{(2)}, W^{(1)}, K_5^{(1)}, K_5^{(2)}, P_3^{(1)}, P_3^{(2)}$, and $P_3^{(3)}$	106
5.2 s - t Walk-Perfect Labelings of M_1, M_2, M_3	130
5.3 TSP-perfection of $P_3(3, 3, 1)$	133
Bibliography	152

List of Figures

2.1	Excluded minors for TSP-perfect graphs.	24
2.2	Operation Φ_1	25
2.3	Vertex duplication of degree-2 vertex.	26
2.4	Examples of graphs in \mathcal{T}	27
2.5	Forbidden adjacencies in the ear operation.	28
2.6	Labeling on edges of K_5	31
2.7	u is duplicated to obtain s and t	35
2.8	Graph $G_{s,t}(3)$	39
2.9	Example showing an α -approximation algorithm for the ATSP does not give an α -approximation for the ATSPATH problem.	45
2.10	Attaching the cycles to the path.	47
3.1	Operation Φ_1	50
3.2	Operation Φ_2	50
3.3	Vertex duplication of degree-2 vertex.	51
3.4	An 8-wheel and a 3-propeller.	52
3.5	Forbidden minors M_1, M_2, M_3	54
3.6	$[M_1, M_2, M_3]$ minor free bricks not including wheels and propellers (i -paths are labeled by i in the figure).	55
3.7	Vertex and edge labels for $G' = W_{k-1}$ and $G = W_k$	57
3.8	Vertex and edge labels for W_k	58
3.9	Adding 2-paths.	70
3.10	Vertices s and t and paths indicating minor M_1	71
3.11	Shaded triangle and paths indicating minor M_2	71
3.12	$[M_1, M_2, M_3]$ minor free bricks not including wheels and propellers.	72
3.13	Wheel graph W_3^3	73
3.14	Bricks from K_4	73
3.15	(a) Case K4.III.i (b) Case K4.III.ii.	74
3.16	Bricks from W_4	75
3.17	Case W4.I (a) Graph $W_4^{(3)}$ (b) Graph $P_2(3, 1, 0)$. Rim edges are darkened and spoke vertices are indicated by black shading.	75
3.18	(a) Case W4.II.ii (b) Case W4.II.iii (graph $P_2(3, 3, 0)$). Rim edges are darkened and spoke vertices are indicated by black shading.	76
3.19	Case W4.II.v.	76
3.20	Case W4.III.ii. Minor M_2 is shown in figure to the right.	77

3.21	Case W4.III.iv.	77
3.22	Bricks from K_5	78
3.23	Case K5.II.ii.	78
3.24	Bricks from K_5	79
3.25	(a) Case P2.II.i (b) Case P2.II.ii.	80
3.26	Case P2.III.iii	80
3.27	Replacing spoke in W_k	82
3.28	Replacing spoke by 2-path in $P_4(0,0,0)$	82
3.29	Bricks from $P_3(0,0,0)$, $P_3(0,0,1)$, $P_3(0,1,1)$, and $P_3(1,1,1)$	83
3.30	(a) Case $P_3(0,0,0)$.I (b) Case $P_3(0,0,0)$.II.i. M_2 minor is obtained by contracting edge e	84
3.31	Case $P_3(0,0,1)$.I.iii	85
3.32	Case $P_3(0,0,1)$.I.iii. M_2 minor is obtained by contracting edge e	85
4.1	Duplicating degree-2 vertex.	91
4.2	Partition of tours.	94
4.3	Redistribution in Case I.	94
4.4	Modifying tours in Λ_{bu}	96
4.5	Redistribution in Case II.	99
4.6	Graphs G_1 and G_2	103
4.7	Counterexample showing Φ_2 does not preserve TSP integrality gap.	103
5.1	Edge labels for $[M_1, M_2, M_3]$ minor free bricks not including wheels and propellers.	107
5.2	Edge labels for graph $P_3(3,3,1)$	133

Chapter 1

Introduction

In this thesis, we study properties of the polyhedron corresponding to a linear programming relaxation of the traveling salesman walk problem. Our results relate the structure of the underlying graph of the problem instance with polyhedral properties of the corresponding fractional walk polyhedron. We first characterize the set of graphs for which the extreme points of the fractional walk polyhedron correspond to traveling salesman walks. For these graphs, the convex hull of solutions to the traveling salesman walk problem has a known complete description by linear inequalities. We extend these results to relate the structure of the underlying graph to the *integrality gap* of the corresponding fractional walk polyhedron. We present graph operations which preserve integrality gap; these operations allow us to find the integrality gap of graphs built from smaller *bricks*, whose integrality gaps can be found by computational or other methods.

The Traveling Salesman Problem (TSP) and its variants have occupied a central role in the field of combinatorial optimization. For a graph G , a *simple cycle* is a cycle with no repeated vertices and a *Hamiltonian cycle* is a simple cycle visiting all vertices of G . Given a cost function c on the edges of G , the traveling salesman problem is to find a Hamiltonian cycle in G of minimum cost. A great deal of research has been devoted to developing improved algorithms for solving large instances to optimality, designing heuristics, analyzing algorithms for random and online instances, and proving approximation guarantees. The books [23] and [29] provide a compendium of results and history on the problem.

An approach that has been extremely successful for this and many other combinato-

rial optimization problems is the method of *linear programming*. In the general linear programming (LP) problem, the objective is to minimize a linear function subject to a system of linear constraints. The feasible solutions of a linear programming problem form a polyhedron and if the objective function minimum is attained, it must include an extreme point of this polyhedron. Linear programming techniques such as the ellipsoid method and interior point methods apply if the polyhedron is given by linear inequalities ([22]). To cast the TSP in the framework of linear programming, we associate a polytope, called the *traveling salesman polytope*, to the set of TSP solutions by considering the convex hull of Hamiltonian cycles in G . The problem of minimizing a cost function over the traveling salesman polytope is equivalent to finding the minimal cost Hamiltonian circuit and hence, is NP-complete ([15]). Therefore, it is unlikely that a description of the traveling salesman polytope by linear inequalities exists for general graphs. This was formalized by Papadimitriou and Yannakakis in [34], who showed that if the problem of determining whether a given inequality is a facet of the traveling salesman polytope is in NP, then NP=co-NP.

Dantzig, Fulkerson, and Johnson developed methods to prove optimality of solutions by starting from the optimal solution to a relaxation of the problem and repeatedly adding inequalities or *cuts* ([10]). Since their foundational work on these *branch-and-cut* algorithms, there has been a significant increase in the size instances of the traveling salesman problem which can be solved exactly (see [10], [23], [30], [32]). In 1991, Gerhard Reinelt collected the TSP library (TSPLIB) of instances for benchmarking of TSP algorithms, with instances ranging from 17 to 85,900 cities. At the time, thirty TSPLIB instances were unsolved; in 1995, David Applegate, Bob Bixby, Bill Cook, and Vasek Chvatal gave solutions to all but ten of the instances. On the TSPLIB website, Reinelt writes

I had not expected when publishing this library (that) due to enormous algorithmic progress, all problems except for pla85900 are now solved to optimality!

Motivation for some problems in the library come from drilling holes in printed circuit boards and X-ray crystallography, where the cities are desired positions of holes and snapshot angles and the salesman tours are routes of the drill and diffractometer. Other applications include problems in routing, machine scheduling, clustering, computer wiring, and curve reconstruction.

Since the decision version of the traveling salesman problem is NP-complete, it is unlikely that a polynomial-time algorithm for the problem exists. Therefore, there has been increased focus on designing good *approximation algorithms* for the problem. An α -approximation algorithm for a minimization problem is a polynomial time algorithm which outputs a solution of value at most α times the optimal solution. The value α is the *approximation ratio* or *approximation factor* for the problem. In the case of general costs, there is no constant factor approximation algorithm for the TSP unless $P = NP$ ([35]). Therefore, we focus our attention on *metric* instances, with costs satisfying the triangle inequality.

While the metric case remains NP-hard, in 1976, Christofides showed a constant factor approximation algorithm exists, with approximation ratio $\frac{3}{2}$ ([8]). Despite many attempts to find a better approximation guarantee, improving this factor has remained an open problem for almost thirty years.

For the lower bound on approximability, Papadimitriou and Vempala have shown that unless $P = NP$, there is no polynomial time algorithm that finds a tour of length at most $220/219 - \epsilon$ times optimal for any $\epsilon > 0$ ([33]).

1.1 Traveling Salesman Variants and Generalizations

Much study has been devoted to variants of the traveling salesman problem, often obtained by modifying the objective function or underlying graph for the problem. Variants include the Asymmetric Traveling Salesman Problem (ATSP), Prize Collecting TSP, Maximum TSP and the Traveling Salesman Repairman Problem. Other variants which introduce constraints on the order in which cities can be visited include the TSP with precedence constraints ([2], [4]) and online TSP ([3], [27], [28]).

From the algorithmic perspective, there are special cases of the problem which are either polynomial time solvable ([5], [23]) or for which there are approximation algorithms with guarantees better than the factor $\frac{3}{2}$ of Christofides' algorithm. In the *Euclidean Traveling Salesman Problem*, we are given n vertices in \mathbb{R}^d and the edge cost between any two vertices is their Euclidean distance. In [1], Arora shows a polynomial time approximation scheme for the Euclidean TSP for fixed dimension d . For the class of 3-connected cubic graphs,

Gamarnik, Lewenstein, and Sviridenko show that the approximation guarantee is strictly better than $\frac{3}{2}$ by designing a $\frac{3}{2} - \frac{5}{389}$ algorithm for this set of graphs ([14]).

1.2 Linear Programming Formulation

Let $G = (V, E)$ be a graph on $n = |V|$ vertices. For any subset $S \subseteq V$, $G(S)$ will denote the induced subgraph on S . Let $\delta(S)$ denote the set of edges with exactly one endpoint in S and for subsets $S_1, S_2 \subseteq V$ let (S_1, S_2) denote the set of edges with one endpoint in S_1 and the other endpoint in S_2 . Given a Hamiltonian cycle, let x_e denote an indicator variable which takes value 1 for edges e in the Hamiltonian cycle and 0 for all other edges and for a subset $F \subseteq E$, let $x(F) = \sum_{e \in F} x_e$. Then the traveling salesman problem with symmetric costs can be captured by the following integer program formulation.

$$\min \sum_{e \in E} c_e x_e \tag{1.1}$$

$$\text{subject to } x(\delta(S)) \geq 2 \quad \text{for all } \emptyset \neq S \subset V, \tag{1.2}$$

$$x(\delta(v)) = 2 \quad \text{for all } v \in V \tag{1.3}$$

$$x_e \in \{0, 1\} \quad \text{for all } e \in E \tag{1.4}$$

By relaxing the integrality constraints, we obtain the following linear program.

$$\min \sum_{e \in E} c_e x_e \tag{1.5}$$

$$\text{subject to } x(\delta(S)) \geq 2 \quad \text{for all } \emptyset \neq S \subset V, \tag{1.6}$$

$$x(\delta(v)) = 2 \quad \text{for all } v \in V \tag{1.7}$$

$$x_e \geq 0 \quad \text{for all } e \in E \tag{1.8}$$

Constraints (1.6) are *cut constraints*, (1.7) are *degree constraints*, and (1.8) are *nonnegativity constraints*. The polytope defined by (1.6), (1.7), and (1.8) is the well-studied *subtour elimination polytope*. In [24] and [25], Held and Karp applied the method of Lagrangean relaxation to the TSP to show that the optimal linear program value over the subtour polytope is the value of the Lagrangean dual corresponding to the 1-tree relaxation (also

called Held-Karp bound). This value can be obtained by subgradient optimization and is the most efficient method known to solve the linear program. This heuristic also performs very well in practice, delivering solutions of near-optimal value for most instances of the symmetric TSP with triangle inequality. The best known bound on the ratio between the optimal tour and the Held-Karp lower bound is $\frac{3}{2}$, a result first proved by Wolsey ([41]). An independent proof is given by Shmoys and Williamson ([37], [40]), who also analyze the structure of the Held-Karp solutions.

One problem that arises from restricting the traveling salesman route to Hamiltonian cycles is that the shortest way to visit all the vertices of G may not be a simple cycle, i.e., may visit some vertices or edges multiple times. Another problem, arising from the linear programming relaxation for this formulation, is that the subtour elimination polytope is not full dimensional. We resolve these problems by considering the *Graphical Traveling Salesman Problem*. In this problem, given edge costs on a graph G , a *graphical traveling salesman tour* on G , or *tour* for short, is a cycle visiting all vertices at least once, with multiple visits to edges and vertices allowed. The Graphical TSP asks for a minimum cost tour of G . This is equivalent to the TSP on the *metric completion* of G , where the cost between any pair of cities is the cost of the shortest path connecting the cities.

Let $X_{TSP}(G)$ denote the set of tours of G . The *graphical traveling salesman polyhedron* of G is the convex hull of tours $\text{conv}(X_{TSP}(G))$. If T is a tour of G , then so is $T + (2e)$ for any edge e and therefore, the graphical TSP polyhedron is an unbounded polyhedron if G is connected. The *fractional traveling salesman polyhedron* of G , denoted $P(G)$, is defined by the cut constraints and nonnegativity constraints (1.6) and (1.8):

$$P(G) = \left\{ x \in \mathbb{R}^{|E|} : \begin{array}{l} x(\delta(S)) \geq 2 \quad \text{for } S \subsetneq V, S \neq \emptyset \\ x \geq 0 \end{array} \right\}.$$

Note that the face of $P(G)$ defined by the inequality $x(E) = n$ is precisely the subtour elimination polytope. In [20] and [31], it is shown that for cost functions satisfying the triangle inequality, minimizing the objective function (1.5) over the subtour elimination polytope is equivalent to minimizing the objective function over the fractional traveling salesman polyhedron.

Given polyhedra P and Q , P is a *relaxation* of Q if $P \supseteq Q$. For many combinatorial optimization problems, Q is an integral polyhedron obtained by taking the convex hull of solutions to the problem and P is a relaxation obtained by a set of valid inequalities for this solution set. In this case, the *integrality gap* of relaxation P is the smallest r such that for any cost function c ,

$$\min\{cx : x \in P\} \leq r \min\{cx : x \in Q\}.$$

For a polyhedron Q of blocking type, we need only consider nonnegative cost functions c . For the traveling salesman problem on a graph G , we consider the integrality gap between polytope $Q = \text{conv}(X_{TSP}(G))$ and its relaxation the fractional TSP polyhedron $P = P(G)$. We call the integrality gap of this relaxation the *TSP integrality gap*.

1.3 Traveling Salesman Paths

In this thesis, we study a generalization of the traveling salesman problem which has not received much attention, the *Traveling Salesman Path (TSPATH) Problem*. In this problem, we are given initial and final cities s and t as additional input; the goal is to find a Hamiltonian path from s to t visiting all cities exactly once. The problem arises naturally in many applications of the traveling salesman problem.

The integer program describing the problem is

$$\min \quad \sum_{e \in E} c_e x_e \quad (1.9)$$

$$\text{subject to} \quad x(\delta(S)) \geq 1 \quad \text{if } |\{s, t\} \cap S| = 1 \text{ for } S \subsetneq V, S \neq \emptyset \quad (1.10)$$

$$x(\delta(S)) \geq 2 \quad \text{if } |\{s, t\} \cap S| = 0 \text{ or } 2 \text{ for } S \subsetneq V, S \neq \emptyset \quad (1.11)$$

$$x(\delta(v)) = 2 \quad \text{for all } v \in V \setminus \{s, t\} \quad (1.12)$$

$$x(\delta(s)) = x(\delta(t)) = 1 \quad \text{if } s \neq t \quad (1.13)$$

$$x(\delta(s)) = x(\delta(t)) = 2 \quad \text{if } s = t \quad (1.14)$$

$$x_e \in \{0, 1\} \quad \text{for all } e \in E. \quad (1.15)$$

Relaxing the integrality constraints, we obtain the linear program

$$\min \quad \sum_{e \in E} c_e x_e \quad (1.16)$$

$$\text{subject to} \quad x(\delta(S)) \geq 1 \quad \text{if } |\{s, t\} \cap S| = 1 \text{ for } S \subsetneq V, S \neq \emptyset \quad (1.17)$$

$$x(\delta(S)) \geq 2 \quad \text{if } |\{s, t\} \cap S| = 0 \text{ or } 2 \text{ for } S \subsetneq V, S \neq \emptyset \quad (1.18)$$

$$x(\delta(v)) = 2 \quad \text{for all } v \in V \setminus \{s, t\} \quad (1.19)$$

$$x(\delta(s)) = x(\delta(t)) = 1 \quad \text{if } s \neq t \quad (1.20)$$

$$x(\delta(s)) = x(\delta(t)) = 2 \quad \text{if } s = t \quad (1.21)$$

$$x_e \geq 0 \quad \text{for all } e \in E. \quad (1.22)$$

As with the tour problem, we relax the condition of visiting every vertex exactly once and define an s - t *traveling salesman walk* as a walk from s to t visiting all vertices at least once with possibly multiple visits to edges or vertices. Given a graph with edge costs, the *traveling salesman walk (TSW) problem* asks for the minimum cost s - t traveling salesman walk.

The *fractional traveling salesman walk polyhedron* for a graph G with fixed vertices s and t is defined by

$$P(G, s, t) = \left\{ \begin{array}{l} x(\delta(S)) \geq 1 \quad \text{if } |\{s, t\} \cap S| = 1 \quad \text{for } S \subsetneq V, S \neq \emptyset \\ x \in \mathbb{R}^{|E|} : \quad x(\delta(S)) \geq 2 \quad \text{if } |\{s, t\} \cap S| = 0 \text{ or } 2 \quad \text{for } S \subsetneq V, S \neq \emptyset \\ x \geq 0 \end{array} \right\}$$

Let $X(G, s, t)$ denote the set of s - t traveling salesman walks. For the traveling salesman walk problem on a graph G , we consider the integrality gap between polytope $Q = \text{conv}(X(G, s, t))$ and its relaxation the fractional TSW polyhedron $P = P(G, s, t)$. We call the integrality gap of this relaxation the s - t *TSW integrality gap*. The maximum over all choices of s and t of the s - t TSW integrality gap is the *TSW integrality gap*. If the graph is disconnected, then the traveling salesman polytope and its fractional relaxation (in both the tour and walk problems) are empty; by convention, these graph will have integrality gap 1. Note that in the case $s = t$, the TSW problem reduces to the traveling salesman

tour problem and the TSW relaxation reduces to the TSP relaxation, implying the TSW integrality gap is at least that of the TSP.

In this thesis, we address two aspects of the traveling salesman walk problem. The first is inspired by the work of Fonlupt and Naddef which characterizes the set of graphs for which the extreme points of the fractional traveling salesman polyhedron are traveling salesman tours [12]. This family of graphs is called *TSP-perfect* and is characterized by a list of forbidden minors. For such graphs, the TSP polyhedron and fractional TSP polyhedron have the same extreme points, implying that the TSP polyhedron has a known description by linear inequalities and therefore, the optimal tour can be found in polynomial time.

We consider the analogous problem for the TSW problem and give a complete characterization of graphs for which the extreme points of the traveling salesman walk polyhedron correspond to traveling salesman walks. Our characterization of these *walk-perfect* graphs is also by forbidden minors. In Section 2.1, we give a constructive description for this set of graphs and in section 2.2, we use the description to prove our main theorem. In Section 2.3, we give a second proof of the characterization of these graphs based on the characterization of TSP-perfect graphs from [12].

Next, we address approximation algorithms for the TSW problem. In [26], Hoogeveen gives an approximation algorithm for the TSW problem for graphs with symmetric edge costs satisfying the triangle inequality. For fixed s and t , he gives a $5/3$ -approximation for the minimum cost s - t traveling salesman walk and for fixed s (and varying endpoint), he gives a $3/2$ -approximation for the minimum cost traveling salesman walk starting at s . An independent proof for the $5/3$ approximation algorithm for two fixed endpoints is due to Vempala ([39]). We address the asymmetric version of the traveling salesman walk problem (ATSW), in which edge costs satisfy the triangle inequality but may be asymmetric (i.e. $c_{ij} \neq c_{ji}$). For the asymmetric traveling salesman tour problem (ATSP), Frieze, Galbiati and Maffioli give a $\log n$ -approximation algorithm in [13]. Using similar methods, we give the first non-trivial algorithm for the ATSW problem, with approximation factor $O(\sqrt{n})$.

In Chapter 3, we give an independent proof for the characterization of TSP-perfect graphs using similar techniques to our characterization of walk-perfect graphs. We first use computational methods to show TSP-perfection of a set of 2-connected and 3-connected

graphs we call *TSP-perfect bricks*. We then show TSP-perfection is preserved under the two operations used to obtain all graphs without the stated excluded minors for TSP perfectness. This resolves the problem posed at the end of [12].

We extend these results in Chapter 4 to show operations which preserve integrality gap of the linear programming relaxations. The results in this chapter connect the integrality gap of the traveling salesman and traveling salesman walk problems with the excluded minor theory of Robertson, Seymour and Thomas ([21], [38]) and suggest new avenues for proving bounds on integrality gaps for these linear programming relaxations.

Chapter 2 of this thesis is based on joint work with Alantha Newman and Santosh Vempala.

Chapter 2

Walk-Perfection

In this chapter, we introduce the notion of walk-perfection of a graph and give a complete characterization of walk-perfect graphs. We first review previous work on TSP-perfect graphs and introduce notation from the literature. A *tour* of graph G is a connected multigraph with even degree at every vertex. Let $X_{TSP}(G)$ denote the set of traveling salesman tours of G and consider the fractional TSP polyhedron

$$P(G) = \left\{ \begin{array}{l} x \in \mathbb{R}^{|E|} : x(\delta(S)) \geq 2 \text{ for } S \subsetneq V, S \neq \emptyset \\ x \geq 0 \end{array} \right\}.$$

Clearly, $\text{conv}(X_{TSP}(G)) \subseteq P(G)$; however, there are graphs for which the inclusion is strict. A graph G is *TSP-perfect* if $\text{conv}(X_{TSP}(G)) = P(G)$, i.e., the vertices of the fractional traveling salesman polyhedron are traveling salesman tours. Note that the equality always holds for disconnected graphs G , since both the convex hull of tours and the fractional TSP polyhedron are the empty set. Therefore, all disconnected graphs are TSP-perfect.

A *minor* of a graph $G = (V, E)$ is a graph that can be obtained from G by a sequence of edge deletions (denoted $G \setminus \{e\}$) and edge contractions (denoted $G.e$). A family of graphs is *minor closed* if for any graph G in the family, every minor of G is also in the family. A graph G is *H minor free* if G does not contain H as a minor. For fixed graphs H_1, H_2, \dots, H_k , let $\mathcal{K}_{[H_1, H_2, \dots, H_k]}$ denote the set of graphs not containing any of H_1, H_2, \dots, H_k as a minor. Let M_1, M_2, M_3 be the three graphs shown in Figure 2.1. Fonlupt and Naddef show that there

is a forbidden minor characterization of TSP-perfect graphs using these graphs.

Theorem 2.0.1. [12] *A connected graph G is TSP-perfect if and only if G is $[M_1, M_2, M_3]$ minor free.*

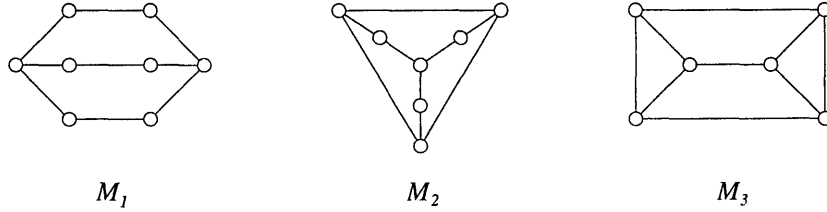


Figure 2.1: Excluded minors for TSP-perfect graphs.

We consider the analogous problem for the traveling salesman walk problem. An s - t *traveling salesman walk* is a connected multigraph with even degree at every vertex if $s = t$ and even degree at every vertex except s and t if $s \neq t$. Let $X(G, s, t)$ denote the set of s - t traveling salesman walks and consider the fractional traveling salesman walk polyhedron

$$P(G, s, t) = \left\{ \begin{array}{l} x \in \mathbb{R}^{|E|} : \\ \begin{array}{ll} x(\delta(S)) \geq 1 & \text{if } |\{s, t\} \cap S| = 1 \\ x(\delta(S)) \geq 2 & \text{if } |\{s, t\} \cap S| = 0 \text{ or } 2 \end{array} & \text{for } S \subsetneq V, S \neq \emptyset \\ x \geq 0 \end{array} \right\}.$$

Note that $X(G, s, t)$ is *not* necessarily the set of integral points in $P(G, s, t)$, as shown by the following example.

Example 2.0.2. *Consider the 6-cycle C_6 with s and t at distance 3. The assignment $x_e^* = 1$ for all edges e is an integral solution in $P(G, s, t)$, but does not correspond to an s - t traveling salesman walk.*

As with the traveling salesman problem, there are graphs for which the inclusion $\text{conv}(X(G, s, t)) \subseteq P(G, s, t)$ is strict. Our goal is to characterize graphs G for which equality holds for any choice of s and t .

Definition 2.0.3. *A graph G is s - t walk-perfect if $P(G, s, t) = \text{conv}(X(G, s, t))$ and G is walk-perfect if it is s - t walk-perfect for all choices of s and t .*

As in the case of TSP-perfection, any disconnected graph G satisfies $\text{conv}(X(G, s, t)) = P(G, s, t)$, since both the convex hull of s - t traveling salesman walks and the fractional TSW polytope are the empty set. Therefore, all disconnected graphs are walk-perfect and we focus our attention on characterizing the set of connected walk-perfect graphs.

In Example 2.0.2, if all edge costs in the six cycle are equal to a fixed positive value, x^* is an optimal solution over $P(G, s, t)$ that does not correspond to an s - t traveling salesman walk. This shows that C_6 with s and t at distance 3 is not s - t walk-perfect and therefore, C_6 is not walk-perfect. In the next two sections, we prove the following theorem.

Theorem 2.0.4. *A connected graph G is walk-perfect if and only if G is C_6 minor free.*

2.1 C_6 Minor Free Graphs

In this section, we give a constructive characterization of the set of C_6 minor free graphs. We will use this characterization to prove our main theorem.

We first show that we can reduce our problem to the characterization of 2-connected walk-perfect graphs. Suppose G_1 and G_2 are connected graphs with specified vertices $s_1, t_1 \in V(G_1)$ and $s_2, t_2 \in V(G_2)$. Let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ be chosen so that at least two of s_1, s_2, t_1, t_2 are equal to v_1 or v_2 . The operation Φ_1 identifies vertices v_1 and v_2 to obtain graph G (see Figure 2.2) with cut vertex v . If the set $\{s_1, s_2, t_1, t_2\} \setminus \{v_1, v_2\}$ has two vertices, then relabel these vertices by s and t . If it has one vertex, then relabel this vertex by s and let $t = v$; if it has no vertices, then let $s = v$ and $t = v$.

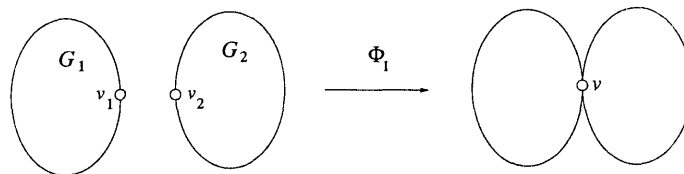


Figure 2.2: Operation Φ_1 .

Every 1-connected graph can be built by repeated applications of operation Φ_1 from *blocks* which are either 2-connected graphs or single edges. In Lemma 2.2.6, we will show that walk-perfection of a graph is preserved under operation Φ_1 and therefore, we can focus

our attention on the characterization of 2-connected walk-perfect graphs.

An *ear decomposition* $G_1, G_2, \dots, G_m = G$ of a graph G is a sequence of subgraphs starting from a simple graph G_1 (a vertex, edge or cycle) such that for each i , G_{i+1} is obtained from G_i by *adding an ear*. The operation of adding an ear is performed by choosing two vertices u and v (the *endpoints* of the ear) from G_i and adding a path from u to v using new vertices (or no vertices if the path is edge (u, v)). If $u \neq v$, the ear is *proper* and a proper ear decomposition is one in which every ear operation is proper. The following theorem is due to Robbins.

Theorem 2.1.1. [36] *G is 2-connected if and only if G has a proper ear decomposition starting from any cycle of G .*

One particular ear operation is duplication of a degree-2 vertex. In such an operation, for a vertex u of degree 2 in G_i with neighborhood $N(u) = \{a, b\}$, duplication of u results in graph G_{i+1} on vertices and edges

$$\begin{aligned} V(G_{i+1}) &= V(G_i) \cup \{u'\} \\ E(G_{i+1}) &= E(G_i) \cup \{(a, u'), (u', b)\}. \end{aligned}$$

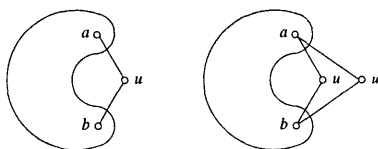
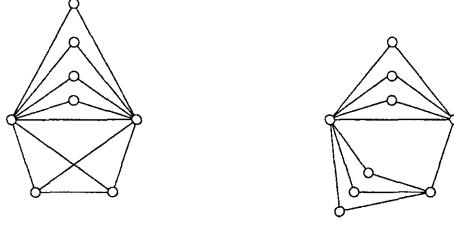


Figure 2.3: Vertex duplication of degree-2 vertex.

Let K_5 denote the complete graph on 5 vertices and consider the class \mathcal{T} of 2-connected graphs obtained from K_5 by repeated applications of the operations edge deletion, edge contraction, and duplication of degree-2 vertices. We show that this set of graphs is exactly the set of 2-connected graphs in $\mathcal{K}_{[C_6]}$.

Theorem 2.1.2. *A 2-connected graph G is C_6 minor free if and only if $G \in \mathcal{T}$.*

Figure 2.4: Examples of graphs in \mathcal{T} .

Proof. Since K_5 does not contain a 6-cycle and the size of the largest cycle cannot increase under edge deletion, contraction, or vertex duplication, no graph in \mathcal{T} contains a C_6 minor.

Conversely, suppose G is 2-connected and C_6 minor free. We will show $G \in \mathcal{T}$ by showing that there is an ear decomposition of G starting with a minor of K_5 such that each ear operation corresponds to edge addition or vertex duplication of a degree-2 vertex. By Theorem 2.1.1, G has a proper ear decomposition $G_1, G_2, \dots, G_m = G$ and we can choose the initial graph G_1 in the decomposition to be the largest cycle $C_k = \{v_1, v_2, \dots, v_k\}$ in G ($k \leq 5$ by assumption). The edges (v_i, v_{i+1}) for $i = 1, 2, \dots, k-1$ and (v_k, v_1) will be called *cycle edges* and the edges (v_i, v_j) with $j \neq i-1, i+1 \pmod{k}$ will be called *chords*. If there are $j-1$ induced chords in G between vertices v_1, v_2, \dots, v_k , let G_j denote the cycle v_1, v_2, \dots, v_k together with all induced chords and let $a, b \in \{v_1, \dots, v_k\}$ be the two vertices that are endpoints for the next ear operation. Because we have already included all chords, the next ear cannot be edge (a, b) . Also, note that the length of the longest path between a and b in G_j is at least $\lceil \frac{k}{2} \rceil$, so if the next ear is a path of length at least 3, then it would create a cycle of length at least $\lceil \frac{k}{2} \rceil + 3 > k$ (since $k \leq 5$), a contradiction to our choice of k . Therefore, it must be a path of length 2 which consists of an additional vertex u' and edges $(a, u'), (u', b)$. Now, if (a, b) is a cycle edge in C_k , then the longest path from a to b has length $k-1$, so adding an ear of length 2 would create a $k+1$ cycle, a contradiction. Therefore, (a, b) cannot be a cycle edge (but a and b may be connected by a chord). Since $k \leq 5$, a and b have a common neighbor, say u .

Claim: $\deg_{G_j}(u) = 2$, i.e., the neighborhood of u in G_j is $N_{G_j}(u) = \{a, b\}$. Otherwise, let $w \in N_{G_j}(u) \setminus \{a, b\}$. Since $k \leq 5$, w must also be adjacent to either a or b , say a . Then the cycle formed by concatenating the path $(w, u), (u, a), (a, u'), (u', b)$ and the path from b to

w (along G_1 , but not through a) has length at least $k + 1$, which is a contradiction (see Figure 2.5).

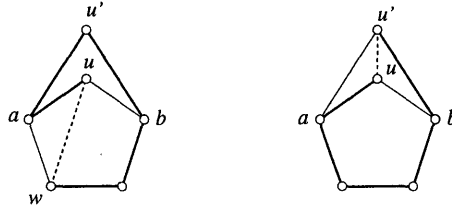


Figure 2.5: Forbidden adjacencies in the ear operation.

Therefore, u has degree 2 in G_j and the operation of adding vertex u' and edges (a, u') , (u', b) corresponds to vertex duplication of u . Note that since $(G_j \setminus \{u\}) \cup \{u'\} = G_j$, the same argument shows we cannot add a path p of any length from either u or u' to any other vertex in $G_j \setminus \{a, b\}$. Similarly, we cannot add a path p of any length between u and u' (denoted $u \xrightarrow{p} u'$), since the cycle formed by concatenating the path $(b, u), u \xrightarrow{p} u', (u', a)$ and the path of $k - 2$ cycle edges from a to b has length at least $k + 1$ (see Figure 2.5). Therefore, neither u nor u' can be chosen as endpoints of the next ear. This implies we must always use vertices among $\{v_1, v_2, \dots, v_k\}$ as ear endpoints and each ear operation corresponds to duplicating a vertex. Since G_1 is a minor of K_5 , it follows that $G \in \mathcal{T}$. \square

This theorem gives us a constructive characterization of the set of 2-connected C_6 minor free graphs. Note that the proof of Theorem 2.1.2 also shows the following.

Corollary 2.1.3. *Suppose $G \in \mathcal{T}$ is obtained from K_5 by a sequence of edge deletions, contractions and degree-2 vertex duplications. Then first performing all edge deletions and contractions followed by any permutation of the degree-2 vertex duplications also results in graph G .*

From this corollary, if graph $G \in \mathcal{T}$ has two specified vertices s and t which result from the duplication of a degree-2 vertex u , then we can reorder the vertex duplications so that the duplication of u to obtain s and t comes first in the ordering and all other vertex duplications follow. Otherwise, if s and t do not result from the duplication of a degree-2 vertex, we can assume that s and t are vertices in the initial subgraph of K_5 to which the

operations of edge deletion, edge contraction, and degree-2 vertex duplication are performed to obtain G .

2.2 Characterization of Walk-Perfect Graphs

In this section, we will show that C_6 is the only forbidden minor in the set of 2-connected traveling salesman walk-perfect graphs by showing that all graphs in $\mathcal{K}_{[C_6]}$ are walk-perfect. Since graph G has specified vertices s and t , we first define the notion of a *labeled minor* of a graph. The operation of edge deletion remains the same as for unlabeled graphs. For the operation of edge contraction, if an edge e is chosen for edge contraction, the resulting vertex from the contraction receives the labels of both endpoints of e , with possibly both labels s and t . In the case s and t label the same vertex in the resulting graph, an s - t traveling salesman walk is a traveling salesman tour.

We first show that walk-perfection is preserved under the labeled minor operations; the proof is modeled on Fonlupt and Naddef's proof that TSP-perfection is preserved under the minor operations [12].

Lemma 2.2.1. *Any connected labeled minor of a connected walk-perfect graph is walk-perfect.*

Proof. Suppose connected graph G has specified vertices $s, t \in V(G)$ and suppose G is s - t walk-perfect. We show that if deletion of an edge e results in a connected graph, then the minor $G \setminus \{e\}$ is s - t walk-perfect. Since $G \setminus \{e\}$ is connected, $P(G \setminus \{e\}, s, t)$ is nonempty. Then let y be an extreme point of $P(G \setminus \{e\}, s, t)$ and let

$$x_f = \begin{cases} y_f & \text{if } f \in E \setminus \{e\}, \\ 0 & \text{if } f = e. \end{cases}$$

Since y is an extreme point of $P(G \setminus \{e\}, s, t)$ and since x has one more variable and one more linearly independent tight constraint than y , x is an extreme point in $P(G, s, t)$. By s - t walk-perfection of G , x is an s - t traveling salesman walk in G , and since x does not use edge e , y is an s - t traveling salesman walk in $G \setminus \{e\}$. Thus, $G \setminus \{e\}$ is s - t walk-perfect.

Now, for the edge contraction operation, if G is connected, then $G.e$ is connected, so for any vertices s and t , $P(G.e, s, t)$ is nonempty. Let y be an extreme point of $P(G.e, s, t)$ and let

$$\bar{x}_f = \begin{cases} y_f & \text{if } f \in E \setminus \{e\}, \\ 0 & \text{if } f = e. \end{cases}$$

Consider cuts $\delta(W')$ of G containing e such that s and t are on the same side of the cut and let $\alpha = \min \bar{x}(\delta(W'))$. Similarly, consider cuts $\delta(W'')$ of G containing e such that s and t fall on different sides of the cut and let $\beta = \min \bar{x}(\delta(W''))$. Now, let

$$x_f = \begin{cases} y_f & \text{if } f \in E \setminus \{e\}, \\ \max\{0, 2 - \alpha, 1 - \beta\} & \text{if } f = e. \end{cases} \quad (2.1)$$

Note that $x \in P(G, s, t)$ since any cut $\delta(W')$ containing e that does not separate s and t satisfies $x(\delta(W')) \geq 2$, any cut $\delta(W'')$ containing e that separates s and t satisfies $x(\delta(W'')) \geq 1$, and any cut not containing e is also a cut in $G.e$.

Let $\theta(x)$ and $\theta(y)$ denote the set of tight constraints for x and y . By possibly taking complements, we can assume any tight constraint C in $\theta(y)$ does not contain the vertex resulting from contraction of edge e . Then C is also a tight constraint for x . Since any tight edge constraint for y is also tight for x , it follows that x is defined by $\theta(y)$ and

$$\begin{aligned} x_e &= 0 \text{ if } \alpha \geq 2 \text{ and } \beta \geq 1 \\ x(\delta(\bar{W}')) &= 2 \text{ if } \alpha < 2 \text{ and } 2 - \alpha \geq 1 - \beta \\ x(\delta(\bar{W}'')) &= 1 \text{ if } \beta < 1 \text{ and } 1 - \beta \geq 2 - \alpha, \end{aligned}$$

where $\bar{W}' = \arg \min \bar{x}(\delta(W'))$ and $\bar{W}'' = \arg \min \bar{x}(\delta(W''))$. Since x has one more variable and one more linearly independent tight constraint, it is an extreme point of $P(G, s, t)$ and therefore an s - t traveling salesman walk in G (by s - t walk-perfection of $P(G, s, t)$). Therefore, y is an s - t traveling salesman walk in $G.e$, implying $G.f$ is s - t walk-perfect. \square

K_5 EXTREME POINTS

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0011020100 00100011100 0000011201 0001020201
0000011021 0002020110 0001011110 0000020112
0000022110 0001020021 0002002110 0020011001
0020020110 0010020011 0011002100 0021020001
0210000011 0120010100 0110010001 0201020001
0101010011 0201000021 0200011001 0100012100
0100010102 0101001100 0102010100 0101101010
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0020120200 0010120101 0020120002 0020122000
0020120020 0021111000 0022120000 0011120010
0020102020 0002102020 0011102010 0002102200
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0200100202 02000002110 0200102200 0202000110
0201000201 0202100200 0200102002 0201002001
0100110012 0200120002 0100112010 0200122000
0102110010 0201111000 0202120000 0200100022
0200102020 0202100020 0202102000 0110001110
0110101200 0220000110 0210100101 0220100200
0110101002 0220100002 0120110010 0110121000
0220120000 0110101020 0220100020 0211000100
0111001001 0221000001 0222100000 0211100010
0112101000 0111110000 1100000012 1100100102
1200010002 2200100002 1100002010 1100102100
1200012000 1200101001 2200102000 1102000010
1101000101 2201000001 1201001000 1202010000
2202100000 1102100010 1111000000 2110101000
1120100100 2220100000 1110100001 1220010000
1110011000 1120000010 2011100010 2022100000
1011010010 1022010000 1021001000 2021000001
2011000100 2020100020 1020010020 2020102000
1020101001 1020012000 2020100002 1020010002
2020100200 2010100101 2020000110 1010010101
1020010200 2002100020 2000102020 2000100022
1000101021 1002010020 1000012020 1000010022
2002100200 2001000020 2002000110 2000102200
1000012200 2000002110 2000100202 1000010202
1001001200 1002010200 1000101201 2000000112
1000001111 1010001010 2010000011 1010101100
2001000021 1001001020

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is empty). The following theorem from [18] gives a condition for showing the extreme points of polyhedron $P(G, s, t)$ are integral.

Theorem 2.2.3. [18] *Let G be a connected graph and let $P = \{x : Ax \leq b\}$ be any polyhedron with $X(G, s, t) \subset P$. Then $P = \text{conv}(X(G, s, t))$ if for any non-zero cost function c , we can show that there exists an inequality in $\{Ax \leq b\}$ satisfied at equality by all optimal solutions to $\min\{cx : x \in X(G, s, t)\}$ whenever this minimum is finite.*

We use this theorem to show that walk-perfection is preserved under duplication of degree-2 vertices. Let $G \in \mathcal{K}_{[C_6]}$, $s, t \in V(G)$, and consider the ear decomposition of G in Theorem 2.1.2. If s and t are obtained by duplicating a vertex u , then by Corollary 2.1.3, we can reorder the vertex duplications so that the operation of duplicating u to obtain s and t comes first in the ordering and all other vertex duplications follow. In this case, the sequence of ear operations gives graphs $G_1, G_2, \dots, G_k = G$, where G_{i+1} is obtained from

G_i by an edge addition for $i < j$, G_{j+1} is obtained from G_j by duplicating u to obtain s and t and G_{i+1} is obtained from G_i by a degree-2 vertex duplication for $i > j$. Otherwise, if s and t are not obtained by duplicating the same vertex, we can choose the first graph G_1 to be the largest cycle containing s and t and no subsequent vertex duplication relabels a new vertex as s or t . We first show that for $i > j$ and for fixed $s, t \in V(G_i)$, if G_i is s - t walk-perfect, then G_{i+1} is also s - t walk-perfect.

Lemma 2.2.4. *For fixed $s, t \in V(G)$, suppose G is s - t walk-perfect and contains a vertex u of degree 2 with $N(u) = \{a, b\}$ (possibly $u = s$ or $u = t$). Then the graph $G' = (V \cup \{u'\}, E \cup \{(a, u'), (u', b)\})$ is also s - t walk-perfect.*

Proof. For any cost function c on G' , consider the set \mathcal{P} of minimum cost s - t traveling salesman walks in G' . If c is negative then the optimum is not finite, so we can assume that c is nonnegative. We show that there is an inequality of the fractional s - t walk polyhedron satisfied at equality by all s - t traveling salesman walks in \mathcal{P} . If c does not satisfy the triangle inequality, then there is an edge (i, j) such that $c_{ij} > c_{ik} + c_{kj}$ and in all optimal solutions, $x_{ij} \geq 0$ is a tight inequality.

Now, let c be a cost function satisfying the triangle inequality on G' , let $\mathcal{I} = N(a) \cap N(b)$ denote the set of vertices in G' adjacent to both a and b , and for any proper subset S , let $f(S) = 1$ if $|S \cap \{s, t\}| = 1$, $f(S) = 2$ otherwise. By abuse of notation, we will use $f(u)$ to denote $f(\{u\})$.

Case 1. $c_{aw} + c_{wb} > c_{av} + c_{vb}$ for some $v, w \in \mathcal{I}$.

If the inequality $x(\delta(w)) \geq f(w)$ is not tight for all optimal solutions $x \in \mathcal{P}$, there exists an optimal traveling salesman walk x^* such that $x^*(\delta(w)) > f(w)$. In this case, we show one of the edge constraints $x_{aw} \geq 0$ or $x_{wb} \geq 0$ is tight for all $x \in \mathcal{P}$. If $x_{aw}^* \geq 1$, $x_{wb}^* \geq 1$, decreasing both values by 1 and increasing x_{av}^* , x_{vb}^* by 1 results in a s - t traveling salesman walk of strictly smaller cost (since degree parity is preserved at every vertex and no vertex is disconnected), a contradiction to the optimality of x^* . Therefore, it must be the case that one of x_{aw}^* or x_{wb}^* is zero, say $x_{aw}^* = 0$. Then $x_{wb}^* \geq 3$ (since $x^*(\delta(w)) > f(w)$ and the degrees of s and t are odd). Since another traveling salesman walk is obtained by decreasing x_{wb}^* by 2, the optimality of x^* implies $c_{wb} = 0$. Now, $c_{aw} > c_{av} + c_{vb} = c_{av} + c_{vb} + c_{wb}$, so no

optimal s - t traveling salesman walk uses edge (a, w) , implying inequality $x_{aw} \geq 0$ is tight for all $x \in \mathcal{P}$.

Case 2 $c_{av} + c_{vb} = c_{aw} + c_{wb}$ for all $v, w \in \mathcal{I}$.

Case 2.1 c_{av} or $c_{vb} = 0$ for some $v \in \mathcal{I} \setminus \{s, t\}$.

Without loss of generality, let $c_{vb} = 0$. Then any s - t traveling salesman walk in $G = G' \setminus v$ can be extended by edge (v, b) (traversed twice) to an s - t traveling salesman walk in G' of the same cost. Conversely, since $c_{av} = c_{av} + c_{vb} = c_{aw} + c_{wb}$ for all $w \in \mathcal{I}, w \neq v$, any s - t traveling salesman walk x in G' can be converted into an s - t traveling salesman walk y in G of the same cost as follows. Choose some $w \in \mathcal{I} \setminus v$ and let $y_{aw} = x_{aw} + x_{av}$ and $y_{wb} = x_{wb} + x_{av}$. Since the parity of degrees at all vertices remain the same and the costs of solutions x and y are the same, the optimal s - t traveling salesman walks in G and the optimal s - t traveling salesman walks in G' have the same cost. Now, since $G = G' \setminus \{v\}$ is s - t walk-perfect, there exists some constraint that is tight for all optimal s - t traveling salesman walks in G . If this is an edge constraint $y_e \geq 0$, then constraint $x_e \geq 0$ is also tight for all optimal s - t traveling salesman walks x in G' . Otherwise, it is a cut constraint C and we can assume without loss of generality that $b \in C$. Then constraint $C' = C \cup \{v\}$ is tight for every $x \in \mathcal{P}$.

Case 2.2 $c_{av}, c_{vb} > 0$ for all $v \in \mathcal{I} \setminus \{s, t\}$.

We claim that for $v \in \mathcal{I} \setminus \{s, t\}$, any optimal integral solution x^* satisfies $x^*(\delta(v)) = f(v) = 2$. To prove this, assume $x^*(\delta(v)) \geq 3$. If x_{av}^* or $x_{vb}^* \geq 3$, decreasing x^* by 2 on this edge yields another integral solution of strictly smaller cost, contradicting minimality of x^* . Since $x^*(\delta(v))$ is even for $v \in \mathcal{I} \setminus \{s, t\}$, we must have $x_{av}^* = x_{vb}^* = 2$. For any other vertex $w \in \mathcal{I} \setminus v$, either $x_{aw}^* \geq 1$ or $x_{wb}^* \geq 1$, say $x_{aw}^* \geq 1$. Then by decreasing $x_{aw}^*, x_{av}^*, x_{vb}^*$ by 1 and increasing x_{wb}^* by 1, we obtain another s - t traveling salesman walk of strictly smaller cost, again a contradiction. Therefore, $x^*(\delta(v)) = f(v) = 2$ for all $v \in \mathcal{I} \setminus \{s, t\}$. \square

We have shown that performing vertex duplication on G to obtain a new vertex not labelled s or t preserves s - t walk-perfection of G . Now, we show walk-perfection is also

preserved under vertex duplication when the two resulting vertices are relabeled s and t . Consider the ear decomposition $G_1, G_2, \dots, G_k = G$ discussed above, where G_{i+1} is obtained from G_i by an edge addition for $i < j$ and G_{j+1} is obtained from G_j by duplicating u to obtain s and t .

Lemma 2.2.5. *If G_j is walk-perfect and G_{j+1} is obtained from G_j by duplicating vertex u to obtain s and t , then G_{j+1} is also walk-perfect.*

Proof. By construction of the ear decomposition, G_j is obtained from the cycle G_1 by edge additions and therefore, has no other vertex duplications (i.e., is a subgraph of the graph in Figure 2.7). Note that this graph is a subgraph of K_5 and is therefore walk-perfect.

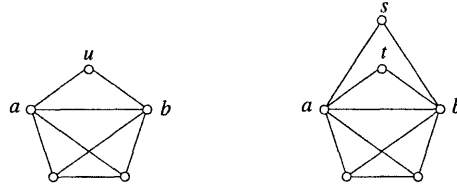


Figure 2.7: u is duplicated to obtain s and t .

Case 1. $c_{as} + c_{sb} > c_{at} + c_{tb}$ OR $c_{as} + c_{sb} < c_{at} + c_{tb}$

The analysis of Case 1 in Lemma 2.2.4 gives a tight constraint for this case.

Case 2. $c_{as} + c_{sb} = c_{at} + c_{tb}$.

Case 2.1. One of c_{as}, c_{sb}, c_{at} or c_{tb} equals 0.

Without loss of generality, let $c_{sb} = 0$. For $\hat{s} = b$ and $\hat{t} = t$, any \hat{s} - \hat{t} traveling salesman walk in $G = G' \setminus s$ can be extended by edge (s, b) to an s - t traveling salesman walk in G' of the same cost. Conversely, since $c_{as} = c_{as} + c_{sb} = c_{aw} + c_{wb}$ for all $w \neq s$, any s - t traveling salesman walk x in G' can be converted into an \hat{s} - \hat{t} traveling salesman walk y in G of the same cost as follows. Choose $w \in \mathcal{I} \setminus s$ and let $y_{aw} = x_{aw} + x_{as}$ and $y_{wb} = x_{wb} + x_{as}$. Since the parity of degrees at all vertices remain the same except at vertex $\hat{s} = b$ and the costs of solutions x and y are the same, the optimal \hat{s} - \hat{t} traveling salesman walks in G and the optimal s - t traveling salesman walks in G' have the same cost. Now, since

$G = G' \setminus \{s\}$ is walk-perfect, there exists some constraint that is tight for all optimal $\widehat{s-t}$ traveling salesman walks in G . If this is an edge constraint $y_e \geq 0$, then constraint $x_e \geq 0$ is also tight for all optimal $s-t$ traveling salesman walks x in G' . Otherwise, it is a cut constraint C and we can assume without loss of generality that $b \in C$. Then constraint $C' = C \cup \{s\}$ is tight for every $x \in \mathcal{P}$.

Case 2.2. $c_{as}, c_{sb}, c_{at}, c_{tb} > 0$.

If $c_{as} = c_{at}$ (and therefore $c_{sb} = c_{tb}$), then let F be the graph with vertices $V(F) = G' \setminus \{s, t\} \cup \{u\}$ and edges $E(F) = E(G') \cup \{(a, u), (u, b)\}$. Let $y_{au} = x_{as}^* + x_{at}^*$ and $y_{ub} = x_{sb}^* + x_{tb}^*$ and $y_e = x_e$ for all other edges e . Then y is a traveling salesman tour on F with cost at most the cost of x^* in G' . Also, any optimal traveling salesman tour on F can be converted to an $s-t$ traveling salesman walk x in F of smaller cost by letting $x_{as} = y_{au}, x_{at} = 0, x_{sb} = x_{tb} = y_{ub}/2$ if y_{au}, y_{ub} are both even (and therefore equal to 2, by optimality of y) and $x_{as} = y_{au}, x_{at} = x_{sb} = 0, x_{tb} = y_{ub}$ if y_{au}, y_{ub} are both odd. This shows minimum $s-t$ traveling salesman walks in G' and minimum traveling salesman tours in F have the same cost and since $F \simeq G$ is walk-perfect, there is a constraint that is tight for all optimal traveling salesman tours of F . If this is an edge constraint $y_e \geq 0$, then $x_e \geq 0$ is also tight for all $x \in \mathcal{P}$. Otherwise the tight constraint is a cut constraint C and we can assume without loss of generality that $u \in C$. Then $C' = C \setminus \{u\} \cup \{s, t\}$ is a tight constraint for all $x \in \mathcal{P}$.

Therefore, $c_{as} \neq c_{at}$ and $c_{sb} \neq c_{tb}$. If the inequality $x(\delta(s)) \geq f(s) = 1$ is not tight for all $x \in \mathcal{P}$, let x^* be an optimal solution with $x^*(\delta(s)) > 1$. Since $\deg(s)$ is odd and $x_{as}^*, x_{sb}^* < 3$ (by optimality of x^*), we can assume $x_{as}^* = 2, x_{sb}^* = 1$. Then $c_{as} < c_{at}$ and $c_{sb} > c_{tb}$ (otherwise, decreasing x_{as}^* by 2 and increasing x_{at}^* by 2 gives a solution of strictly smaller cost). If $\deg(t) = 3$, we have the following cases.

Case 2.2.i. $x_{at}^* \geq 1, x_{tb}^* \geq 1$. In this case, decreasing x_{as}^* by 2 gives an $s-t$ traveling salesman walk of strictly smaller cost, a contradiction.

Case 2.2.ii. One of x_{at}^*, x_{tb}^* is zero and the other is at least 3. Then subtracting

2 from the edge of value at least 3 gives an s - t traveling salesman walk of strictly smaller cost, again a contradiction.

Since $\deg(t)$ is odd, it must be the case that $x_{as}^* = 2, x_{sb}^* = 1$ and $\deg(t) = 1$. Now, consider the support graph $H = \{e \in E(V(G') \setminus \{s, t\}) : x_e^* > 0\}$ and let x_H^* denote the restriction of x^* to this graph. The remaining cases are the following.

Case 2.2.iii. $x_{at}^* = 1, x_{tb}^* = 0$. In this case, x_H^* contains an Eulerian walk from a to b in H since $x_H^*(w)$ is even for all $w \in H \setminus \{a, b\}$ and odd for $w = a$ or b . Therefore, H is connected and a traveling salesman walk of strictly smaller cost can be obtained from x^* by decreasing x_{as}^* by 2.

Case 2.2.iv. $x_{at}^* = 0, x_{tb}^* = 1$. If H is connected, the same argument in Case 2.2.iii gives a traveling salesman walk of strictly smaller cost, so we can assume H is not connected. Let C be the component of H containing a (note that $b \notin C$) and let $C' = C \cup \{s\}$. For any edge $e = (i, j) \in E(G)$ with $i \in C, j \notin C$, let q_{ij}^s (q_{ij}^t) denote the shortest path in x^* from i to a together with edges $(a, s), (s, b)$ (edges $(a, t), (t, b)$) and the shortest path in x^* from b to j . The cost of edge $e = (i, j)$ must be at least the cost of path q_{ij}^s (which is equal to the cost of path q_{ij}^t); otherwise, by replacing path q_{ij}^s by edge (i, j) , we do not disconnect any vertices of the graph (since G_j is a subset of the graph in Figure 2.7) while preserving the degree parity at every vertex, which yields an s - t traveling salesman walk of strictly smaller cost.

We claim $x(\delta(C')) = 1$ for every $x \in \mathcal{P}$. Otherwise, if $x^*(\delta(C')) \geq 2$ for some $x^* \in \mathcal{P}$, then $s \in C', t \notin C'$ implies $x^*(\delta(C')) \geq 3$. One of x_{sb}^*, x_{at}^* must be zero, say $x_{at}^* = 0$ (otherwise, if $x_{sb}^*, x_{at}^* \geq 1$, then decreasing both of these by 1 and increasing x_{as}^*, x_{tb}^* by 1 gives an s - t traveling salesman walk of strictly smaller cost). Now, consider edges $(k_1, l_1), (k_2, l_2), (k_3, l_3)$ (possibly including multiple copies of the same edge) crossing C' in path x^* . By rerouting $x_{k_1 l_1}^*, x_{k_2 l_2}^*$ and $x_{k_3 l_3}^*$ along the paths $q_{k_1 l_1}^s, q_{k_2 l_2}^s$ and $q_{k_3 l_3}^s$ (or keeping $x_{k_i l_i}^*$ if $(k_i, l_i) = (s, b)$), we obtain an s - t traveling salesman walk y of smaller or equal cost with either

$y_{as} \geq 3$ or $y_{sb} \geq 3$ (if $x_{sb}^* = 0$, then reroute along the paths $q_{k_1l_1}^t, q_{k_2l_2}^t$ and $q_{k_3l_3}^t$). Now, by decreasing this value by 2, we obtain an s - t traveling salesman walk of strictly smaller cost, a contradiction. Therefore, $x(\delta(C')) = 1$ for every $x \in \mathcal{P}$. \square

We now show that that walk perfection for any graph can be reduced to walk-perfection of its blocks.

Lemma 2.2.6. *s - t walk-perfection is preserved under operation Φ_1 .*

Proof. Suppose vertices v_1 and v_2 in connected graphs G_1 and G_2 are identified to obtain graph G and let $s, t \in V(G)$. Consider the labeled minor H_1 obtained by contracting G_2 to a single vertex in G . The result is graph G_1 where vertex v_1 has label s if $s \in V(G_2)$, label t if $t \in V(G_2)$, labels s and t if $s, t \in V(G_2)$ and is unlabeled if $s, t \in V(G_1) \setminus \{v_1\}$. Similarly, consider labeled minor H_2 obtained by contracting G_1 . Since s - t walk-perfection is preserved under connected labeled minors, if G is s - t walk-perfect, then so are H_1 and H_2 .

Conversely, suppose H_1 and H_2 are s - t walk-perfect, let $X(G, s, t)$ denote the set of optimal s - t traveling salesman walks in G , and let $x \in X(G, s, t)$. Then optimality and degree parity constraints imply that x is the union of two optimal s - t traveling salesman walks in labeled minors H_1 and H_2 . For any non-zero cost function c , the restriction of c to one of H_1 or H_2 must be non-zero; without loss of generality, assume c restricted to H_1 is non-zero. By Theorem 2.2.3, there is a constraint C in $P(H_1)$ which is tight for all optimal traveling salesman tours in H_1 . If constraint C is an edge constraint $x_e \geq 0$, then this edge constraint is tight for all $x \in X_{TSP}(H_1)$. Otherwise, we can assume constraint C is a cut constraint with $v_1 \notin C$; in this case, C is a tight constraint for all $x \in X_{TSP}(G)$. \square

Now, any 1-connected graph is C_6 minor free if and only if can be built by repeated applications of Φ_1 from blocks which are C_6 minor free. Therefore, our main theorem follows from Lemmas 2.2.2, 2.2.4, 2.2.5, and 2.2.6.

Theorem 2.0.4 A connected graph G is walk-perfect if and only if G has no C_6 minor.

2.3 Connection with TSP-Perfection

In this section, we establish a connection between walk-perfection and TSP-perfection. Using this connection, we give a second proof of the characterization of walk-perfect graphs based on the characterization of TSP-perfect graphs.

If graph G is walk-perfect, then it is also TSP-perfect, since by choosing s and t to be identical vertices, s - t walk-perfection corresponds to TSP-perfection. Thus, walk-perfection is a sufficient condition for TSP-perfection. We would like a condition in the reverse direction, i.e., a sufficient condition for walk-perfection based on TSP-perfection. For graph $G = (V, E)$ and vertices $s, t \in V$, let $G_{s,t}(3)$ denote the graph obtained by adding a path of length 3 from s to t (see Figure 2.8), that is, $G_{s,t}(3)$ is the graph with vertices and edges

$$V(G_{s,t}(3)) = V(G) \cup \{u, v\} \quad (u, v \notin V(G))$$

$$E(G_{s,t}(3)) = E \cup \{(s, u), (u, v), (v, t)\}.$$

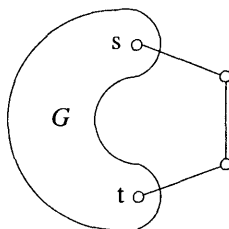


Figure 2.8: Graph $G_{s,t}(3)$.

Consider the fractional TSP polyhedron

$$P(G_{s,t}(3)) = \left\{ x \in \mathbb{R}^{|E|} : \begin{array}{ll} x(\delta(S)) \geq 2 & \text{for } S \subsetneq V(G_{s,t}(3)), S \neq \emptyset \\ x \geq 0 & \text{for all } e \in E(G_{s,t}(3)) \end{array} \right\}.$$

The following lemma relates the extreme points of the fractional traveling salesman walk polyhedron $P(G, s, t)$ with the extreme points of the fractional traveling salesman polyhedron $P(G_{s,t}(3))$.

Lemma 2.3.1. *If $x \in \mathbb{R}^{|E|}$ is an extreme point of $P(G, s, t)$, then $x' = (x, 1, 1, 1) \in \mathbb{R}^{|E(G_{s,t}(3))|}$ is an extreme point of $P(G_{s,t}(3))$, where the three additional variables correspond to edges (s, u) , (u, v) , and (v, t) .*

Proof. Let x be an extreme point of $P(G, s, t)$. Then it is tight for $m = |E|$ of the constraints in $P(G, s, t)$. We will show that the point $x' = (x, 1, 1, 1)$ is the unique solution to a set of $m + 3$ inequalities involving edges $E(G_{s,t}(3))$ and therefore is an extreme point $P(G_{s,t}(3))$.

First, we show the m tight constraints for x in $P(G, s, t)$ generate m tight constraints for x' in $P(G_{s,t}(3))$. Each tight constraint $x(\delta(S)) = f(S)$ in $P(G, s, t)$ gives rise to a tight constraint $x'(\delta(S')) = 2$ in $P(G_{s,t}(3))$ with

$$S' = \begin{cases} S \cup \{u, v\} & \text{if } s, t \in S \\ S' = S \cup \{u\} & \text{if } s \in S, t \in \bar{S} \\ S' = S \cup \{v\} & \text{if } s \in \bar{S}, t \in S \\ S' = S & \text{if } s, t \in \bar{S}. \end{cases}$$

This gives m tight constraints for x' in $P(G_{s,t}(3))$. Consider these constraints together with the following three inequalities:

$$\begin{aligned} x(\delta(V)) &= x_{su} + x_{vt} \geq 2 \\ x(\delta(V \cup \{u\})) &= x_{uv} + x_{vt} \geq 2 \\ x(\delta(V \cup \{v\})) &= x_{su} + x_{uv} \geq 2. \end{aligned}$$

The unique solution on edges (s, u) , (u, v) , (v, t) satisfying the last three inequalities at equality is $x_{su} = x_{uv} = x_{vt} = 1$. Furthermore, since x is the unique solution to the m tight constraints in $P(G, s, t)$, it follows that $x' = (x, 1, 1, 1)$ is the unique solution to the $m + 3$ tight constraints in $P(G_{s,t}(3))$ and therefore, x' is an extreme point of $P(G_{s,t}(3))$. \square

Lemma 2.3.2. *If $G_{s,t}(3)$ is TSP-perfect, then G is s - t walk-perfect. If $G_{s,t}(3)$ is TSP-perfect for every choice of s and t , then G is walk-perfect.*

Proof. By Lemma 2.3.1, if x is an extreme point of $P(G, s, t)$, then $(x, 1, 1, 1)$ is an extreme point of $P(G_{s,t}(3))$. Since $G_{s,t}(3)$ is TSP-perfect, the extreme point $(x, 1, 1, 1)$ is a tour of $G_{s,t}(3)$, which corresponds to an s - t traveling salesman walk in G together with the three edges (s, u) , (u, v) , and (v, t) . Thus, the extreme point x corresponds to an s - t traveling salesman walk, implying G is s - t walk-perfect. If this holds for every choice of s and t , G is walk-perfect. \square

Lemma 2.3.3. *For any $i \in \{1, 2, 3\}$ and any edge $e \in M_i$, $M_i \setminus \{e\}$ contains C_6 as a minor.*

Proof. This follows by inspection of Figure 2.1. \square

Theorem 2.3.4. *If G is C_6 minor free, then $G_{s,t}(3)$ is $[M_1, M_2, M_3]$ minor free for any choice of s and t .*

Proof. The theorem is clearly true if $s = t$, so we can assume $s \neq t$. Suppose $G_{s,t}(3)$ contains M_i ($i = 1, 2$, or 3) as a minor and label the edges of $G_{s,t}(3)$ according to whether they are contracted, deleted, or unchanged in the sequence of minor operations to obtain M_i . Consider the 3-path (s, u) , (u, v) , (v, t) . None of these edges can be marked for deletion, since this would imply G contains an M_i minor, and therefore a C_6 minor. If any of these edges is unchanged, then after performing the minor operations to obtain M_i , deleting this edge would leave a C_6 minor which must have been contained in G , a contradiction. Therefore, all 3 edges (s, u) , (u, v) , (v, t) must be marked for contraction.

Note that we can interpret minor M_i as a partition of the vertices of $G_{s,t}(3)$; two vertices a, b are in the same member of the partition if and only if there is a path of edges marked for contraction between a and b and two members A and B of the partition are connected by an edge if and only if there are vertices $a \in A$ and $b \in B$ such that edge $e = (a, b)$ is in graph $G_{s,t}(3)$ and e is unchanged under the minor operations. This implies that performing the edge contractions and deletions in any order results in the same graph minor.

Therefore, we can perform the contraction of edges (s, u) , (u, v) , (v, t) as the final three steps in the sequence of minor operations. Consider the graph at this stage, with only the three edge contractions remaining and let G' denote the subgraph of G with all minor operations on $E(G)$ carried out. At this stage, if edges (s, u) , (u, v) , (v, t) are contracted in graph $G' \cup \{(s, u), (u, v), (v, t)\}$, the result is graph M_i . If the contraction results in

any multi-edges, then arbitrarily mark all edges except one for deletion in G' (this does not change the final graph). Now, since all vertices in M_i have degree at most 3 and no multi-edges arise from the contraction of $(s, u), (u, v), (v, t)$, one of s or t (say t) satisfies $\deg_{G'}(t) \leq 1$ in G' . Let e be the edge adjacent to t in G' if $\deg_{G'}(t) = 1$, and let e be an arbitrary edge in G' if $\deg_{G'}(t) = 0$. Then vertex t has degree 0 in $G' \setminus \{e\}$ and degree 1 in graph $(G' \setminus \{e\}) \cup \{(s, u), (u, v), (v, t)\}$ and therefore, any C_6 minor in the graph $(G' \setminus \{e\}) \cup \{(s, u), (u, v), (v, t)\}$ cannot contain vertices t, u , or v . By Lemma 2.3.3, deleting edge e from G' results in a graph with a C_6 minor and since this C_6 minor does not contain any of t, u , or v , it is also a minor of graph G , a contradiction. \square

Note that since K_5 is C_6 minor free, this provides a second proof for the walk-perfection of K_5 , which was shown by computational methods in Section 2.2.

Corollary 2.3.5. *K_5 is walk-perfect.*

2.4 s - t Walk-Perfection

Now that we have characterized the set of walk-perfect graphs, a natural problem is to characterize connected graphs G which are s - t walk-perfect for *fixed* vertices s and t . For graph G and specified vertices s and t , we first show that the condition of TSP-perfection of $G_{s,t}(3)$ is *not* a necessary condition for s - t walk-perfection of G by giving the following counterexample to the converse of Lemma 2.3.2.

Example 2.4.1. *In Table 2.1, labels s and t are chosen in graphs M_1, M_2 , and M_3 . Using polymake, we have verified each graph G is s - t walk-perfect (see Appendix). However, each graph $G_{s,t}(3)$ contains one of M_1, M_2 , or M_3 as a minor and therefore, is not TSP-perfect.*

In fact, Table 2.1 is a *complete* list (up to isomorphism) of the choices of s and t resulting in s - t walk-perfect graphs. The remaining graphs, shown in Table 2.2, show the choices of s and t resulting in non s - t walk-perfect graphs. In each graph, two disjoint paths of length at least three from s to t are shown in bold. Note that in each of the s - t walk-perfect graphs, there are no such two disjoint paths of length at least three from s to t . This leads us to formulate the following conjecture.

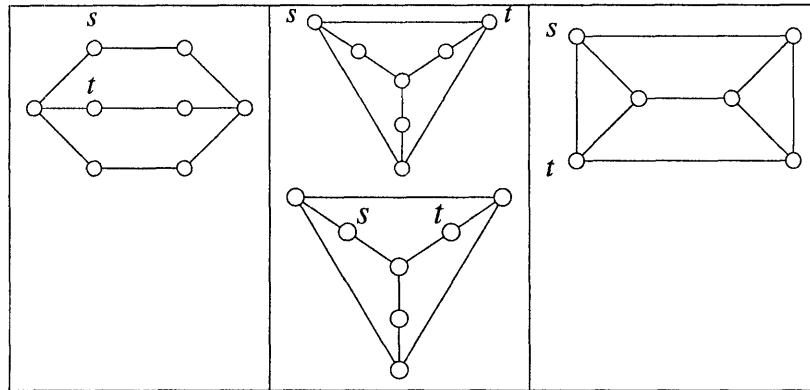


Table 2.1: Labelings of M_1, M_2, M_3 resulting in $s-t$ walk-perfect graphs.

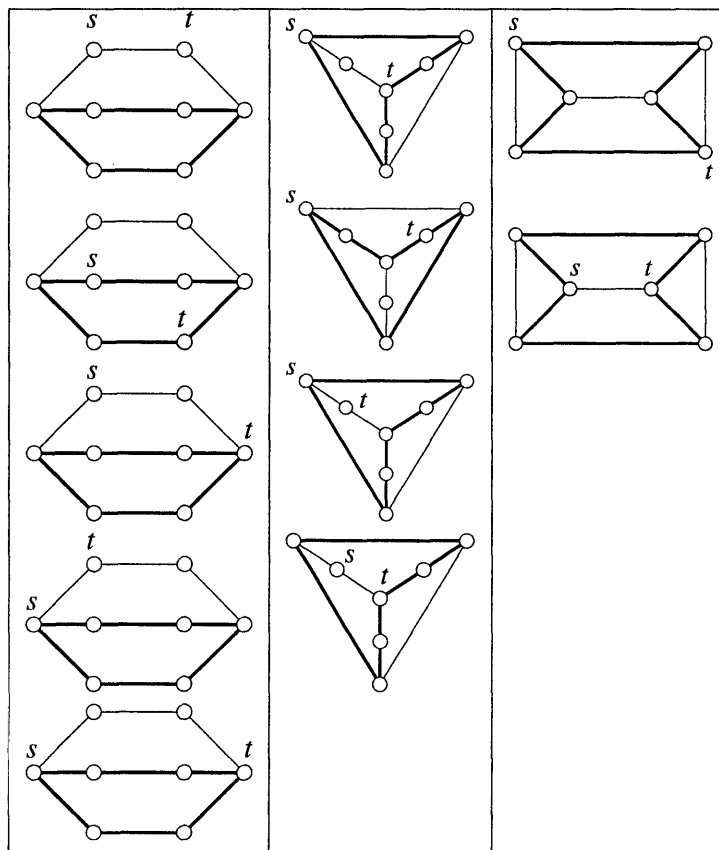


Table 2.2: Labelings of M_1, M_2, M_3 resulting in non- $s-t$ walk-perfect graphs.

Conjecture 2.4.2. *For a connected graph G with fixed vertices $s, t \in V(G)$, G is s - t walk-perfect if and only if G does not have as a labeled minor two vertex disjoint paths of length at least 3 between s and t .*

2.5 Asymmetric Traveling Salesman Path Problem

In this section, we give an approximation algorithm for the *asymmetric traveling salesman path* (ATSPATH) problem. In this problem, we have fixed vertices s and t in a graph $G = (V, A)$ with directed arcs and possibly asymmetric arc costs. The objective is to find a minimum cost directed Hamiltonian path from s to t . The *asymmetric traveling salesman walk* (ATSW) problem is to find a minimum cost directed walk from s to t that visits all vertices *at least* once. This problem is equivalent to finding a minimum cost directed Hamiltonian path from s to t in the metric completion of graph G . Therefore, we focus our attention on complete graphs satisfying the triangle inequality and assume we are given such an instance in our approximation algorithm. Our results are stated for the ATSPATH problem, but apply to the ATSW problem by replacing each arc (i, j) in the solution with a shortest directed path in the graph from i to j . The algorithm we present is similar to the $O(\log n)$ -approximation algorithm for the asymmetric traveling salesman problem due to Frieze, Galbiati, and Maffioli ([13]).

In the following example, we show that there are graphs for which the cost of the optimal asymmetric traveling salesman tour can be arbitrarily higher than that of the optimal asymmetric traveling salesman path. Thus, an α -approximation algorithm for the asymmetric traveling salesman tour problem does not immediately yield an α -approximation for the ATSPATH problem.

Example 2.5.1. *Figure 2.9 shows an instance for which the value of the minimum cost tour is arbitrarily higher than the value of the minimum cost s - t traveling salesman path. For this graph, arc (t, s) has arbitrarily high cost $c_{ts} = \alpha$, solid directed arcs have cost 1 and all remaining arcs have costs determined by metric completion. The minimum cost s - t path has value 10 and the minimum cost tour has value $\alpha + 10$.*

However, using a technique based on recursively building the asymmetric s - t traveling

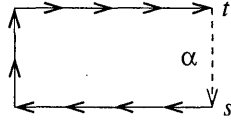


Figure 2.9: Example showing an α -approximation algorithm for the ATSP does not give an α -approximation for the ATSPATH problem.

salesman path, we prove that there is an $O(\sqrt{n})$ -approximation algorithm for the ATSPATH problem.

2.5.1 Path/Cycle Covers

An s - t -path/cycle cover in a directed graph G is a directed path from s to t together with a collection of directed cycles such that every vertex in V is contained in exactly one of these subgraphs. In particular, this implies the path and cycles must be disjoint and cover all vertices $V(G)$. Note that the value of the minimum s - t -path/cycle cover on G is a lower bound on the minimum cost asymmetric traveling salesman path in G . We first show that we can find a minimum s - t path/cycle cover for G efficiently via a reduction to the minimum cost perfect matching problem.

Construct bipartite graph G' by including two copies of each vertex $v \in V \setminus \{s, t\}$; call these copies v and v' . For each pair $i, j \in V \setminus \{s, t\}$, assign cost c_{ij} to arc (i, j') . Now, include vertices s and t' and for all $i \in V \setminus \{s, t\}$, assign cost c_{si} to arc (s, i') and c_{it} to arc (i, t') .

Lemma 2.5.2. *The cost of a minimum cost perfect matching in G' is equal to the cost of a minimum s - t -path/cycle cover in G .*

Proof. Let $d^-(v)$ and $d^+(v)$ denote the indegree and outdegree of vertex v respectively. An s - t -path/cycle cover is a subgraph of G in which vertices s and t satisfy $d^+(s) = d^-(t) = 1$ and $d^-(s) = d^+(t) = 0$, and every vertex $v \in V \setminus \{s, t\}$ satisfies $d^-(v) = d^+(v) = 1$. We first show that every s - t -path/cycle cover of G corresponds to a matching in G' with the same cost. For every directed arc (i, j) in the s - t -path/cycle cover, include arc (i, j') in the matching. Since every vertex $i \in V \setminus \{s, t\}$ has in-degree 1 and out-degree 1, both i and i' are matched in G' and since s has out-degree 1 and t has in-degree 1, s and t are also matched. Thus, there is a minimum cost perfect matching with the same cost

as the s - t -path/cycle cover. Conversely, a minimum cost perfect matching in G' yields a s - t -path/cycle cover in G with the same cost; for every arc (i, j') in the matching, include arc (i, j) in the path/cycle cover. \square

2.5.2 $O(\sqrt{n})$ -Approximation

The first step of the algorithm is to find a minimum cost s - t -path/cycle cover. If this subgraph contains at least \sqrt{n} cycles, then let $V' \subset V$ be the set of vertices in the path together with one vertex from each cycle and let G' be the graph induced by the vertices in V' (note that $|V'| \leq n - \sqrt{n}$). We then recurse on the graph G' . Such a recursion can occur at most \sqrt{n} times. When we reach a stage in which the path/cycle cover returns fewer than \sqrt{n} cycles, then we attach each cycle to the path resulting in a single s - t path.

This attachment operation proceeds as follows. For each cycle, pick an arbitrary vertex v in the cycle. The current s - t path contains an arc (a, b) such that in an optimal s - t traveling salesman path \vec{p} , vertex v falls after a and before b . To see why this is true, label all vertices in the current s - t path that appear after v in \vec{p} by 1 and label all vertices that appear before v in \vec{p} by 0. Then s has label 0 and t has label 1 and therefore, there is some arc (a, b) such that a has label 0 and b has label 1. Although we do not know which arc will satisfy the desired property, we can test all consecutive vertices along the s - t path and choose a and b to minimize the length of the sum of the two arcs (a, v) and (v, b) . Then by connecting vertex v to the s - t path by adding these two arcs, the cost incurred is at most OPT (see Figure 2.10). Since there are at most $k \leq \sqrt{n}$ cycles, the total cost of adding all these arcs is at most $\sqrt{n} \cdot OPT$. In the final step, we have an s - t -path on a subset of the vertices and we expand each vertex that represented a cycle at some stage of the algorithm by replacing the vertex with a complete traversal of that cycle. If a vertex v is visited multiple times in the result, then let (i, v) and (v, j) be two arcs in the solution. Since the graph is assumed to be a complete directed graph satisfying the triangle inequality, we can shortcut the solution by including arc (i, j) and deleting arcs (i, v) and (v, j) . Repeating this procedure until every vertex is visited exactly once results in a directed s - t traveling salesman path. \square

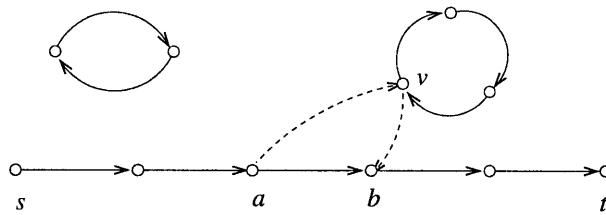


Figure 2.10: Attaching the cycles to the path.

ATSPATH-APPROX(G)

1. Find a minimum cost s - t path/cycle cover C for G .
 - (i) If C has less than \sqrt{n} cycles, then attach the cycles to the s - t path and let S be the resulting path.
 - (ii) Else if C has more than \sqrt{n} cycles, then let V' be the set of vertices in the s - t path plus one representative vertex from each cycle. Run ATSPATH-APPROX(G') for $G' = (V', A(V'))$.
2. For each vertex that represents a cycle in S , expand the cycle while traversing the path, shortcutting arcs through vertices which are visited multiple times.

Chapter 3

TSP-perfection

3.1 Bricks and TSP-Perfection Preserving Operations

As shown in Chapter 2, the families of TSP-perfect and walk-perfect graphs can be characterized by finite lists of forbidden minors. In general, giving a constructive description for a minor free family is a difficult problem. In the case of C_6 minor free graphs, Section 2.1 gave a constructive description starting from building block K_5 and repeatedly applying operations of edge deletion, edge contraction, and degree-2 vertex duplication. We used this construction to prove our characterization of walk-perfect graphs. In this chapter, we show we can extend this technique to TSP-perfect graphs, using a constructive description of $[M_1, M_2, M_3]$ minor free graphs to give an independent proof for the characterization of TSP-perfect graphs. We first give a constructive characterization of all $[M_1, M_2, M_3]$ minor free graphs, as being built from *bricks*, using operations to be defined together with the operations of edge contraction and deletion. We then prove that all bricks are TSP-perfect and that all operations used in the constructive characterization preserve TSP-perfection. This gives an alternate proof of the result of Fonlupt and Naddef and resolves the problem posed at the end of [12].

We first describe graph operations Φ_1 , Φ_2 , and degree-2 vertex duplication. Recall from Section 2.1 that for two connected graphs G_1 , G_2 , and vertices $v_1 \in G_1$, $v_2 \in G_2$, operation Φ_1 identifies vertices v_1 and v_2 to obtain graph G (see Figure 3.1). Note that since we are concerned with TSP-perfection rather than walk-perfection in this chapter, we do not need the specified vertices s_i or t_i for operation Φ_1 here.

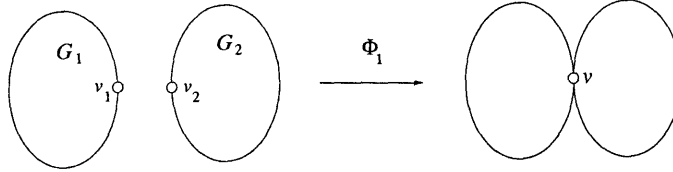


Figure 3.1: Operation Φ_1 .

To describe operation Φ_2 , suppose G has a 2-vertex disconnecting set $\{s, t\}$ and let k -path denote a path of k edges whose internal vertices have degree 2. Note that the graph $G \setminus \{s, t\}$ may have more than 2 components (for example, if there are any 2-paths from s to t). Let V_1 and V_2 be a partition of $V(G) \setminus \{s, t\}$ obtained by grouping connected components of $G \setminus \{s, t\}$ into two nonempty sets and for $i = 1, 2$, let H_i denote the graph $G(V_i \cup \{s, t\})$. If both graphs H_1 and H_2 have a path of length at least 3 from s to t , then let G_i be the graph H_i together with an additional 3-path from s to t (see Figure 3.2). For both operations Φ_1

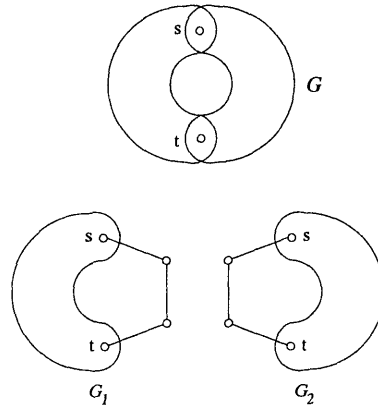


Figure 3.2: Operation Φ_2 .

and Φ_2 , we say that Φ_i decomposes G into G_1 and G_2 and that Φ_i composes G from G_1 and G_2 . We will denote the operations by $G = G_1 \circ_{\Phi_i} G_2$. If one of the graphs H_i does not

have a path of length at least 3 from s to t , then G cannot be decomposed by Φ_2 at $\{s, t\}$.

The operation of degree-2 vertex duplication (as introduced in Section 2.1) takes a graph G with a degree-2 vertex u and duplicates u to obtain graph $G_2(u)$ on vertices and edges

$$\begin{aligned} V(G_2(u)) &= V(G) \cup \{u'\} \\ E(G_2(u)) &= E(G) \cup \{(a, u'), (u', b)\}, \end{aligned}$$

where a and b are the two neighbors of u (see Figure 3.3).

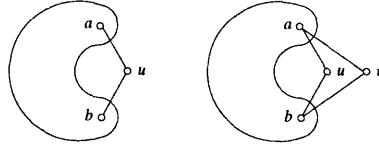


Figure 3.3: Vertex duplication of degree-2 vertex.

If Φ_i decomposes graph G into graphs G_1 and G_2 , each of which has strictly fewer than $|V(G)|$ vertices, then we say Φ_i produces a *strict decomposition* of G . We would like to characterize all the graphs that cannot be decomposed by any of the three operations described above. Consider the family of TSP-perfect graphs with no strict decomposition by Φ_1 or Φ_2 and with at most one 2-path between the same vertices s and t (as any duplicate 2-path can be removed by the inverse operation to the duplication of degree-2 vertices). Such a graph in the family must be 2-connected. Moreover, if the graph has a 2-vertex disconnecting set $\{s, t\}$ into nonempty components, then because there is no strict decomposition by Φ_2 , we claim that the only possibilities for the two graphs H_1 and H_2 are

- (i) one connected component and one 2-path between s and t or
- (ii) one connected component and one 3-path between s and t .

Otherwise, if either one of H_1 or H_2 contained more than one 3-path or a 3-path and a 2-path, operation Φ_2 would give a strict decomposition at $\{s, t\}$. If there were more than one 2-path between s and t , we could have eliminated all but one of them by the inverse operation to the duplication of degree-2 vertices. Thus, if we replace all 2-paths and 3-paths by 1-paths (edges) for any graph in this family, we obtain a simple graph which has

no vertex disconnecting set of size 2 and is therefore 3-connected. We would like to find a list of graphs from this family, which we will call *bricks*, such that every $[M_1, M_2, M_3]$ minor free graph can be obtained from the list of bricks by the operations of edge deletion, edge contraction, Φ_1 , Φ_2 , and duplication of degree-2 vertices.

To describe the list of bricks, we first define two families of graphs, *wheels* and *propellers*. The k -wheel W_k is the graph on vertices v_0, v_1, \dots, v_k such that $G(V \setminus v_0)$ is a cycle of length k and vertex v_0 is adjacent to v_1, v_2, \dots, v_k . The edges (v_0, v_i) will be called *spoke edges* and the edges $(v_i, v_{i+1}), (v_k, v_1)$ will be called *rim edges* (see Figure 3.4). The family of wheel graphs includes the k -wheels W_k as well as the graphs obtained from W_k by replacing rim edges in W_k by 2-paths and 3-paths. In particular, let $W_k^{(3)}$ denote the wheel graphs obtained from W_k by replacing every rim edge by a 3-path.

The k -propeller P_k is a graph on triangle a, b, c and k other vertices which are pairwise nonadjacent and adjacent to all of a, b, c . The edges $(a, b), (b, c), (a, c)$ will be called *rim edges* and all remaining edges will be called *spoke edges*; the vertices a, b, c will be called *rim vertices* and all remaining vertices will be called *spoke vertices*. For $0 \leq i_1, i_2, i_3 \leq 3$, $P_k(i_1, i_2, i_3)$ will denote the graph obtained from P_k by replacing the three rim edges of P_k with paths of i_1, i_2 , and i_3 edges (where a path of 0 edges means the rim edge is removed). The family of propeller graphs includes all graphs $P_k(i_1, i_2, i_3)$ for $k \geq 1, 0 \leq i_1, i_2, i_3 \leq 3$.

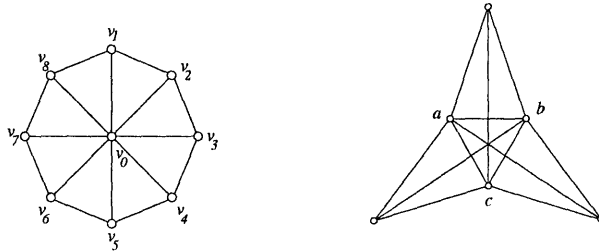


Figure 3.4: An 8-wheel and a 3-propeller.

As observed above, any brick must be 2-connected and if it has a 2-vertex disconnecting set $\{s, t\}$ into nonempty components, then one of the components must be an i -path for $i = 2$ or 3. Therefore, if we replace all such i -paths by edges, the resulting graph must be 3-connected. This shows that a complete set of bricks can be obtained from the set of 3-connected bricks by replacing certain edges with 2-paths and 3-paths.

The set of 3-connected M_3 minor free graphs is characterized by the following theorem.

Theorem 3.1.1. [11] *The following is a complete list of 3-connected M_3 minor free graphs:*

- (i) K_i for $i = 1, 2, \dots, 5$
- (ii) $K_5 \setminus \{e\}$ for any edge e
- (iii) W_k for $k \geq 3$
- (iv) $P_k(0, 0, 0), P_k(0, 0, 1), P_k(0, 1, 1), P_k(1, 1, 1)$ for $k \geq 3$.

Note that the graphs listed in Theorem 3.1.1 include *all* of their 3-connected subgraphs. For example, the only way to remove 2 disjoint edges from K_5 to obtain a 3-connected graph results in W_4 ; also, no proper subgraph of W_k is 3-connected, since every edge is adjacent to a vertex of degree 3. Furthermore, all graphs in Theorem 3.1.1 are also M_1 and M_2 minor free, so (i), (ii), (iii), and (iv) is in fact a complete list of 3-connected $[M_1, M_2, M_3]$ minor free graphs.

Let \mathcal{B} be the set of propellers $P_k(3, 3, 3)$ and wheels $W_k^{(3)}$, together with the eight graphs in Figure 3.6. In Section 3.4, we will use Theorem 3.1.1 to show that \mathcal{B} is an exhaustive list of bricks. This list appears in [12], except for graphs $K_5^{(2)}$, $P_3^{(2)}$, and $P_3^{(3)}$ (see Figure 3.6), which were omitted, and the second graph in Figure (17c) and Case 2 in the analysis of propeller graphs P_k of the paper, which are not $[M_1, M_2, M_3]$ minor free. In [12], Fonlupt and Naddef sketch a proof that this is an exhaustive list of bricks but omit the details, which we will fill in here.

Theorem 3.1.2. [12] *The family of $[M_1, M_2, M_3]$ minor free graphs is the family of graphs that can be obtained from \mathcal{B} by repeated applications of*

- (i) *edge deletion,*
- (ii) *edge contraction,*
- (iii) *operation Φ_1 ,*
- (iv) *operation Φ_2 , and*
- (v) *duplication of degree-2 vertices.*

Our proof for the characterization of TSP-perfect graphs proceeds by showing that the brick graphs \mathcal{B} are all TSP-perfect (Section 3.2), that operations Φ_1 , Φ_2 , and duplication

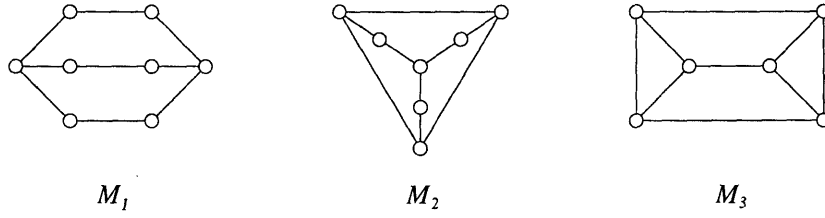


Figure 3.5: Forbidden minors M_1, M_2, M_3 .

of degree-2 vertices preserve TSP-perfection (Section 3.3), and that the list of bricks is exhaustive (Section 3.4).

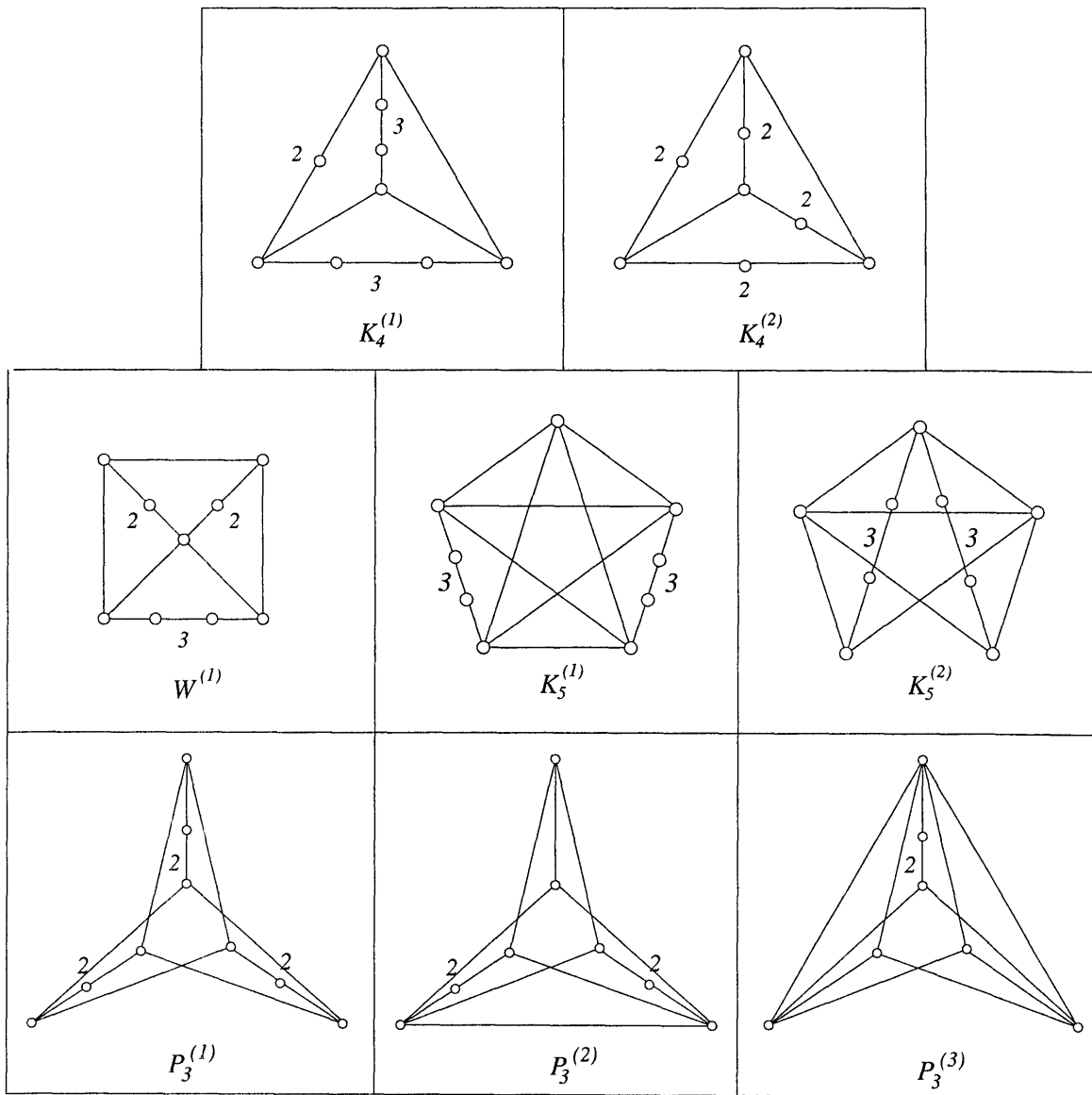


Figure 3.6: $[M_1, M_2, M_3]$ minor free bricks not including wheels and propellers (i -paths are labeled by i in the figure).

3.2 TSP-Perfection of bricks

We first show that all graphs in the list \mathcal{B} are TSP-perfect. For the eight finite graphs $K_4^{(1)}$, $K_4^{(2)}$, $W^{(1)}$, $K_5^{(1)}$, $K_5^{(2)}$, $P_3^{(1)}$, $P_3^{(2)}$, and $P_3^{(3)}$ in Figure 3.6, we use `polymake` to prove TSP-perfection. For each of these graphs, we input the cut and nonnegativity constraints of the fractional TSP polyhedron into `polymake` to generate the extreme points of the polyhedron and verify that all of the extreme points correspond to traveling salesman tours. We leave the details for the Appendix.

Lemma 3.2.1. *Graphs $K_4^{(1)}$, $K_4^{(2)}$, $W^{(1)}$, $K_5^{(1)}$, $K_5^{(2)}$, $P_3^{(1)}$, $P_3^{(2)}$, and $P_3^{(3)}$ are TSP-perfect.*

Proof. (See Appendix.)

We now consider the infinite members of \mathcal{B} and show TSP-perfection for these graphs.

Lemma 3.2.2. *The k -wheels W_k are TSP-perfect.*

Proof. For any non-zero cost function c , we will find a constraint in the fractional TSP polyhedron $P(W_k)$ satisfied at equality by all optimal integral tours. Theorem 2.2.3 then implies that $P(W_k)$ is equal to the convex hull of integer tours and therefore, W_k is TSP-perfect.

We proceed by induction with base case $W_2 = C_3$. In this case, any extreme point x satisfies three inequalities at equality among the cut and nonnegativity constraints. If all three tight constraints are cut constraints, i.e.,

$$x_{e_1} + x_{e_2} = x_{e_1} + x_{e_3} = x_{e_2} + x_{e_3} = 2,$$

then the unique solution is $x_e = 1$ for all edges e , which corresponds to a tour. Otherwise, x must satisfy two of the cut constraints and one of the nonnegativity constraints at equality. The unique solution to such a set of constraints has two edges of value 2 and one edge of value 0, which also corresponds to a tour.

Now, suppose W_{n-1} is TSP-perfect and consider $G = W_n$ with cost function c on the edges. If c does not satisfy the triangle inequality, then there is an edge (i, j) such that $c_{ij} > c_{ik} + c_{kj}$ and in all optimal solutions, $x_{ij} \geq 0$ is a tight constraint. Therefore, we

can restrict our attention to cost functions satisfying the triangle inequality. If there is a rim edge $e = \{u, w\}$ in $E(G)$ with cost zero, consider the graph $G' \simeq W_{n-1}$ obtained by contracting the endpoints of edge e into a single vertex v and let $f_u, f_w \in E(G)$, $f_v \in E(G')$ denote the spokes adjacent to u, w and v respectively. Then the triangle inequality and

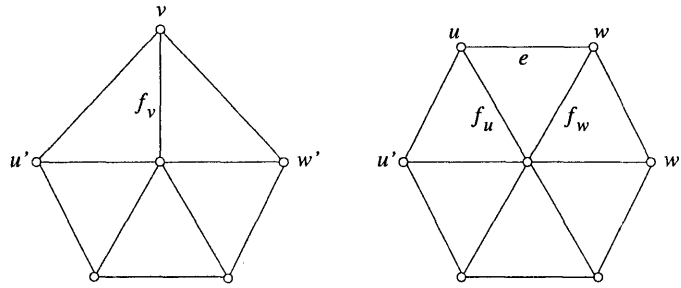


Figure 3.7: Vertex and edge labels for $G' = W_{k-1}$ and $G = W_k$.

$c_e = 0$ together imply $c_{f_u} = c_{f_w}$; assign this cost to edge f_v . We first show that the optimal traveling salesman tours in G and G' have the same cost. For any tour x in G , let $x'_f = x_f$ if $f \notin \{e, f_u, f_w\}$ and $x'_{f_v} = x_{f_u} + x_{f_w}$. Then x' is a tour in G' of the same cost as tour x in G . Also, we can extend any tour x' in G' to a tour x in G of the same cost in the following way. If x'_{f_v} is even, let

$$x_f = \begin{cases} \frac{x'_f}{2} & \text{for } f \in \{f_u, f_w\} \\ \frac{x'_f}{2} + 2 & \text{for } f = e \\ x'_f & \text{otherwise.} \end{cases}$$

Let u', w' denote the vertices adjacent to v on the rim of wheel G' (see Figure 3.7). If x'_{f_v} is odd, then exactly one of $x'_{v,u'}$ or $x'_{v,w'}$ is odd. By symmetry, assume $x'_{v,u'}$ is odd. Then x

defined as follows is a traveling salesman tour in G .

$$x_f = \begin{cases} 1 & \text{for } f = f_u \\ x'_{f_v} - 1 & \text{for } f = f_w \\ 2 & \text{for } f = e \\ x'_{v,u'} & \text{for } f = \{u, u'\} \\ x'_{v,w'} & \text{for } f = \{w, w'\} \\ x'_f & \text{otherwise.} \end{cases}$$

Therefore, the optimal traveling salesman tours in G and G' have the same cost. Since $G' = W_{n-1}$ is TSP-perfect, there is a constraint C that is tight for all optimal traveling salesman tours in G' . If this constraint is the edge constraint for edge f_v , then the edge constraints for f_u and f_v are satisfied at equality in G . If this constraint is any other edge constraint, then the same edge constraint is satisfied at equality for all optimal traveling salesman tours in G . If this constraint is a cut constraint, we can assume by possibly taking complements that vertex v is not in C . Then C is also a tight cut constraint for all optimal traveling salesman tours in G . Therefore, we can assume all rim edges have strictly positive

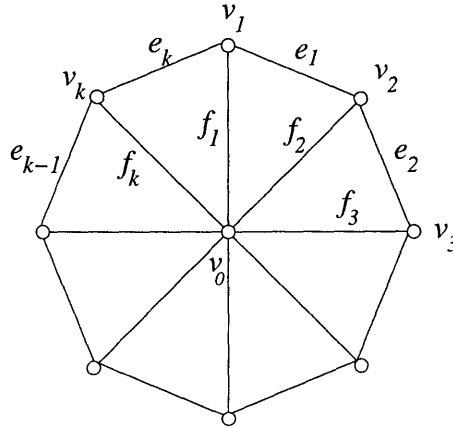


Figure 3.8: Vertex and edge labels for W_k .

cost, implying any optimal tour x satisfies $x_e \leq 2$ for all rim edges. Label the spokes of W_k by f_1, f_2, \dots, f_k such that $c_{f_1} = \max_i \{c_{f_i}\}$, label the rim vertex adjacent to spoke f_i by

v_i , and label the rim edges by e_1, e_2, \dots, e_k where $e_i = \{v_i, v_{i+1}\}$ for $i = 1, 2, \dots, k-1$ and $e_k = \{v_k, v_1\}$ (see Figure 3.8). Note that edge f_1 must have strictly positive cost (otherwise, all spokes would have cost zero, implying all rims have cost zero by the triangle inequality). Then $x_{f_1} \leq 2$ by optimality of x . Suppose $x(\delta(v_1)) > 2$. We first argue that it cannot be the case that $x_{f_1} \geq 1$ and $x_{e_1} = 2$. In such a case, consider x' defined by

$$\begin{aligned} x'_{f_2} &= x_{f_2} + 1 \\ x'_{e_1} &= x_{e_1} - 1 \\ x'_{f_1} &= x_{f_1} - 1 \\ x'_e &= x_e \quad \text{for all other edges } e. \end{aligned}$$

Since x' preserves parity at every vertex and does not disconnect any vertices, x' is also a traveling salesman tour and optimality of x implies $c_{f_2} \geq c_{f_1} + c_{e_1}$. However, $c_{e_1} > 0$ since all rim edges have strictly positive costs, contradicting the maximality of c_{f_1} . The same argument also shows the following cases cannot occur:

$$\begin{aligned} x_{f_1} \geq 1, x_{e_k} &= 2 \\ x_{e_1} \geq 1, x_{f_1} &= 2 \\ x_{e_k} \geq 1, x_{f_1} &= 2. \end{aligned}$$

The strictly positive costs on rim edges and edge f_1 together with optimality of x implies $x_e \leq 2$ for all rim edges and edge f_1 . If $x(\delta(v_1)) \geq 6$, then $x_{f_1} = x_{e_1} = x_{e_k} = 2$, which cannot happen by the argument above. If $x(\delta(v_1)) = 4$, then the only case for which the above argument does not apply is the case $x_{e_k} = x_{e_1} = 2, x_{f_1} = 0$. In this case, consider the smallest i for which $x_{f_i} > 0$ and the largest j for which $x_{f_j} > 0$. Since the solution must be connected and have even degree at every vertex, we must have either $x_{e_1} = x_{e_2} = \dots = x_{e_{i-1}} = 2$ or $x_{e_j} = x_{e_{j+1}} = \dots = x_{e_k} = 2$. By symmetry, we can assume

$x_{e_1} = x_{e_2} = \dots = x_{e_{i-1}} = 2$. Consider the tour x' with

$$\begin{aligned} x'_{f_1} &= x_{f_1} + 1 \\ x'_{e_j} &= x_{e_j} - 1 && \text{for } j \leq i - 1 \\ x'_{f_i} &= x_{f_i} - 1 \\ x'_e &= x_e && \text{for all other edges } e. \end{aligned}$$

Then x' is also a traveling salesman tour; by optimality of x , $c_{f_1} \geq c_{e_2} + \dots + c_{e_i} + c_{f_i}$ and by triangle inequality, equality holds. Now, consider tour x'' with $x''_{e_k} = x'_{e_k} - 1$, $x''_{f_1} = x'_{f_1} - 1$, $x''_{f_k} = x'_{f_k} + 1$ and $x''_e = x'_e$ for all other edges e . Since x'' preserves the parity of every vertex and no vertex is disconnected, optimality of x' implies $c_{f_k} \geq c_{e_k} + c_{f_1}$, a contradiction to the choice of f_1 (since $c_{e_k} > 0$). \square

As a corollary, we have the following.

Corollary 3.2.3. *Graphs $W_k^{(3)}$ ($k \geq 2$) are TSP-perfect.*

Proof. This follows from Lemma 2.2.1 and Lemma 3.2.2 since $W_k^{(3)}$ is a minor of W_{3k} . \square

To prove propeller graphs are TSP-perfect, we first need the following theorems.

Theorem 3.2.4. [9] *If x is an extreme point of the fractional traveling salesman polyhedron $P(G)$, then x can be defined by a set of edge constraints $x_e = 0$ and cut constraints Θ such that $G(S)$ and $G(\bar{S})$ are both connected for any cut $S \in \Theta$.*

The next theorem follows from two claims in [12]; since it is not stated explicitly in the paper and is proved only for a special class of graphs, we include the proof here for completeness.

Theorem 3.2.5. [12, Claims 4.4, 4.5] *If x is an extreme point of $P(G)$ and e is an edge such that $x_e > 1$, then x restricted to graph $G.e$ is an extreme point of $P(G.e)$.*

Proof. Let $\Theta(x)$ denote the set of tight constraints satisfied by x . We first show that edge e belongs to at most one tight cut in $\Theta(x)$. Otherwise, if there are cuts C_1 and C_2 both containing e , then by possibly considering the complements of C_1 and C_2 , we can assume

$e \in (C_1, C_2)$ and $\overline{C_1} \cap \overline{C_2} \neq \emptyset$. Then we have

$$x(\delta(C_1)) = x(C_1, C_2 \cap \overline{C_1}) + x(C_1, \overline{C_2} \cap \overline{C_1}) = 2,$$

implying $x(C_1, \overline{C_1} \cap \overline{C_2}) < 1$. The same argument shows $x(C_2, \overline{C_2} \cap \overline{C_1}) < 1$. Since $\overline{C_1} \cap \overline{C_2} \neq \emptyset$, $C_1 \cup C_2$ is a cut of G and

$$x(\delta(C_1 \cup C_2)) \leq x(C_1, \overline{C_1} \cap \overline{C_2}) + x(C_2, \overline{C_2} \cap \overline{C_1}) < 2,$$

a contradiction.

Therefore, e is contained in at most one tight cut constraint C in $\Theta(x)$ and the constraints $\Theta(x) \setminus \{C\}$ are all tight for the solution $x.e$ induced by x in $G.e$. Since x is an extreme point in $P(G)$ and since at most one constraint for x becomes violated for $x.e$, this implies $x.e$ is an extreme point in $P(G.e)$. \square

For any $F \subseteq P(G)$, let

$$\begin{aligned} B_F &= \{j \in E(G) \mid x_j = 0 \text{ for all } x \in F\} \\ D_F &= \{S \subset V(G) \mid |S| \leq n - 2 \text{ and } x(\delta(S)) = 2 \text{ for all } x \in F\}. \end{aligned}$$

Two sets $C_1, C_2 \subseteq V(G)$ *cross* if $C_1 \cap C_2 \neq \emptyset$ and neither $C_1 \subseteq C_2$ nor $C_2 \subseteq C_1$. A *nested family* is a family of sets with no pair of crossing sets. In [9], Cornuejols, Fonlupt, and Naddef prove the following theorem.

Theorem 3.2.6. [9, Theorem 4.9] *For graph G , let x be an extreme point of the fractional TSP polyhedron $P(G)$. For any face F of $P(G)$, F can be defined by*

$$F = \{x \in P(G) \mid x_e = 0 \text{ for all } e \in B_F \text{ and } x(\delta(S)) = 2 \text{ for all } S \in D^*\}$$

for some $D^* \subseteq D_F$ such that D^* is a nested family.

For any $x \in P(G)$, let G_x denote the support graph of x . Note that for the nested family D^* in Theorem 3.2.6, we can assume that $V(G) \notin D^*$ and for any $S \in D^*$, $\overline{S} \notin D^*$. Then Theorem 3.2.6 implies the following bound on the number of edges in G_x for an extreme

point x .

Theorem 3.2.7. *If x is an extreme point of $P(G)$, then*

$$|E(G_x)| \leq 2|V(G)| - 3.$$

We now prove the TSP-perfection of propellers.

Lemma 3.2.8. *The propeller graphs $P_k(i_1, i_2, i_3)$ are TSP-perfect for $0 \leq i_1, i_2, i_3 \leq 3$.*

Proof. We proceed by induction. $P_1(3, 3, 3)$ is wheel graph $W_3^{(3)}$ and therefore the graphs $P_1(i_1, i_2, i_3)$ are all TSP-perfect.

Now, for some $k \geq 2$, suppose graphs $P_{k-1}(i_1, i_2, i_3)$ are TSP-perfect for all values of $i_1, i_2, i_3 \in \{0, 1, 2, 3\}$ and suppose there is an extreme point x of the fractional traveling salesman polyhedron of propeller $P_k(i_1, i_2, i_3)$ which does not correspond to a traveling salesman tour. Let G denote the support graph of x in $P_k(i_1, i_2, i_3)$ and let k_2 and k_3 denote the number of spoke vertices v with $\deg(v) = 2$ and $\deg(v) = 3$ respectively in graph G . Note that there are no spoke vertices of degree 1 in the support graph since the incident edge to this vertex would have value 2 and therefore could be contracted to obtain a counterexample to the TSP-perfection of $P_{k-1}(i_1, i_2, i_3)$ by Theorem 3.2.5.

Now, if $k_2 > 0$, let v be a spoke vertex of degree 2, let a and b be the neighbors of v , and suppose i_1 is the number of edges in the rim path between a and b . Consider the following cases.

Case I. $i_1 = 0, 1, 2$. Then consider the graph G' obtained from G by removing the rim path of i_1 edges between a and b and replacing the rim by edges (a, v) and (b, v) . Then G' is a subgraph of $P_{k-1}(2, i_2, i_3)$ and is TSP-perfect by induction. By Lemma 2.2.4, we can duplicate vertex v in G' to obtain a TSP-perfect graph G'' . Then graph G is a minor of G'' and is therefore TSP-perfect.

Case II. $i_1 = 3$. In this case, consider the subgraph G' of $P_{k-1}(i_1, i_2, i_3)$ obtained by removing vertex v and edges (a, v) and (b, v) . By induction, graph G' is TSP-perfect. If there are any other spoke vertices w of degree 2 adjacent to a and b , then by Lemma 2.2.4, we can duplicate w to obtain v while preserving TSP-perfection. Otherwise, graph G' contains at least two disjoint paths of length 3 from a to b and therefore, can be decomposed by operation Φ_2 at cutset $\{a, b\}$. Then by Corollary 3.3.5, we can add the 2-path $(a, v), (b, v)$ back to graph G' while preserving TSP-perfection.

Therefore, we can assume $k_2 = 0$ and $k_3 = k$. Assume our counterexample minimizes $i_1 + i_2 + i_3$; by Theorem 3.2.5, we can furthermore assume all edges satisfy $x_e \leq 1$. We first show that we can assume at least one of $i_1, i_2, i_3 \leq 1$. Otherwise, if all of the rim paths have two or more edges then by considering the cuts crossing only rim edges along the same rim path, the value of x along then entire path must be equal to 1. By Theorem 3.2.4, there is a set of tight cuts $\Theta(x)$ such that $G(S)$ and $G(\bar{S})$ are both connected for any cut $S \in \Theta(x)$. Any cut S separating rim vertices a, b, c cannot be tight since $\delta(S)$ must contain at least two edges of the paths of value 1 as well as some edge of strictly positive value adjacent to a spoke vertex of degree 3 (which exists since $k_3 > 0$). Thus, the only possible tight constraints are the sets of singleton spoke vertices $\{u_1\}, \{u_2\}, \dots, \{u_k\}$ and constraints containing only vertices along the same rim path (i.e., not separating vertices a, b, c). For rim paths of length i_1, i_2, i_3 , let $r(i_1, i_2, i_3)$ denote the number of tight constraints containing only rim edges along the same rim path. Since

$$r(i_1, i_2, i_3) = 2(i_1 + i_2 + i_3) - 9 \text{ for } i_1, i_2, i_3 \geq 2,$$

we have $i_1 + i_2 + i_3 \geq r(i_1, i_2, i_3)$ if $i_1, i_2, i_3 \geq 2$. Therefore

$$3k_3 + (i_1 + i_2 + i_3) > k_3 + r(i_1 + i_2 + i_3). \quad (3.1)$$

The left hand side of Equation (3.1) is the number of nonzero edges in our example and the right hand side is the maximum number of tight constraints. This shows that x cannot be an extreme point. Therefore, one of i_1, i_2 , or i_3 is at most 1, implying $i_1 + i_2 + i_3 \leq 7$.

Then by Theorem 3.2.7,

$$\begin{aligned}
 |E(G)| &\leq 2|V(G)| - 3 \\
 (i_1 + i_2 + i_3) + 3k_3 &\leq 2((i_1 + i_2 + i_3) + k_3) - 3 \\
 3k_3 &\leq (i_1 + i_2 + i_3) + 2k_3 - 3 \\
 k_3 &\leq (i_1 + i_2 + i_3) - 3 \\
 k_3 &\leq 7 - 3 = 4.
 \end{aligned}$$

Now, if $k_3 = 4$, then $i_1 + i_2 + i_3 = 7$. Furthermore, since one of i_1, i_2 , or i_3 is at most 1, it must be the case that two of i_1, i_2 , or i_3 are equal to 3 and the third is equal to 1. Without loss of generality, let $i_1 = i_2 = 3$ and $i_3 = 1$. In graph $P_4(3, 3, 1)$, the two rim paths of length three must have value 1 on all edges and therefore the cut containing the vertex adjacent to these two rim edges cannot be a tight cut (since it contains two edges of value 1 and at least one nonzero edge adjacent to a spoke vertex). This implies the maximum number of tight nested sets on these vertices is at most $2|V(P_4(3, 3, 1))| - 4 = 2(11) - 4 = 18$ while the number of edges is $|E(P_4(3, 3, 1))| = (i_1 + i_2 + i_3) + 3k_3 = 7 + 3(4) = 19$ and therefore, x cannot be an extreme point.

It follows that $k_3 \leq 3$. We show the graph $P_3(3, 3, 1)$ cannot be a counterexample by enumerating the extreme points of the corresponding fractional traveling salesman polytope in `polymake` and verifying all extreme points are traveling salesman tours (see Appendix).

□

3.3 Operations Φ_1 and Φ_2

Now that we have demonstrated the set of bricks are TSP-perfect, we turn to the two operations Φ_1 and Φ_2 , which give the remaining $[M_1, M_2, M_3]$ minor free graphs. We show that both operations preserve TSP-perfection.

Theorem 3.3.1. *TSP-perfection is preserved under Φ_1 .*

Proof. This follows from Lemma 2.2.6 by setting $s = t$. □

To prove TSP-perfection is also preserved under Φ_2 , we recall the following lemma. For graph $G = (V, E)$ and vertices $s, t \in V$, recall $G_{s,t}(3)$ is the graph obtained by adding a path of length 3 from s to t .

Lemma 2.3.2 If $G_{s,t}(3)$ is TSP-perfect, then G is s - t walk perfect.

In the next two theorems, let $\{s, t\}$ be a 2-vertex disconnecting set of G into nonempty components and suppose Φ_2 decomposes G into G_1 and G_2 at vertices s and t . Recall that H_i is the subgraph of G_i before the addition of the 3-path between s and t .

Theorem 3.3.2. *If G is TSP-perfect and can be decomposed by Φ_2 into graphs G_1 and G_2 (i.e., both H_1 and H_2 have paths of length at least 3 between s and t), then G_1 and G_2 are also TSP-perfect.*

Proof. Since H_1 and H_2 both have paths of length at least three between s and t , G_1 and G_2 are minors of G . Since TSP-perfection is preserved under minors, it follows that G_1 and G_2 are TSP-perfect. □

Theorem 3.3.3. *If G_1 and G_2 are TSP-perfect, then G is TSP-perfect.*

Proof. For a fixed cost function c on G , let $X_{TSP}(G)$ be the set of minimum cost traveling salesman tours on G and let $x \in X_{TSP}(G)$. An s - t 2-cycle cover will denote a union of two cycles, one containing s and the other containing t such that every vertex is covered by at least one of the cycles. Note that every traveling salesman tour in G is either the union of s - t traveling salesman walks in both H_1 and H_2 or the union of an s - t 2-cycle cover in one and a traveling salesman tour in the other. Let p_i, t_i, c_i denote the costs of the optimal s - t traveling salesman walk, optimal traveling salesman tour and optimal s - t 2-cycle cover in

H_i respectively. Since G_i is TSP-perfect, Lemma 2.3.2 implies H_i is s - t walk perfect. Since any tour in H_i is a feasible point in the fractional s - t walk polyhedron and any extreme point of $P(H_i, s, t)$ is an s - t traveling salesman walk by s - t walk-perfection, we have $p_i \leq t_i$.

Claim 3.3.4. $p_i - c_i \leq t_i - p_i$.

To see why this is true, let y_i be a minimal s - t 2-cycle cover in H_i and let z_i be a minimal tour in H_i . Then

$$\begin{aligned} y_i(C) + z_i(C) &\geq 2 \text{ for } |\{s, t\} \cap C| = 1 \\ y_i(C) + z_i(C) &\geq 4 \text{ for } |\{s, t\} \cap C| = 0 \text{ or } 2 \end{aligned}$$

Therefore, $\frac{y_i + z_i}{2}$ is a feasible point in the fractional s - t walk polyhedron of H_i and since H_i is s - t walk perfect, this implies the minimal s - t walk has cost

$$p_i \leq \frac{\text{cost}(y_i) + \text{cost}(z_i)}{2} = \frac{c_i + t_i}{2}.$$

This gives $2p_i \leq c_i + t_i$, or $p_i - c_i \leq t_i - p_i$ as claimed. \square

Without loss of generality, assume $p_1 - c_1 \leq p_2 - c_2$. Note that the optimal traveling salesman tour in G has cost

$$\text{cost}(x) = \min\{c_1 + t_2, p_1 + p_2, c_2 + t_1\}.$$

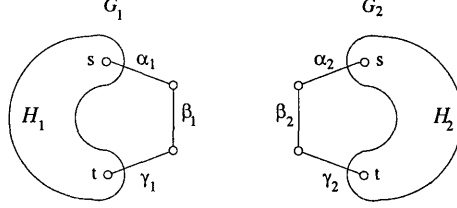
Case I. Suppose $c_1 + t_2 = p_1 + p_2 = c_2 + t_1$. In this case, $c_1 \leq p_1$ and $c_2 \leq p_2$ (otherwise $p_1 + p_2 < c_1 + t_2$ or $p_1 + p_2 < t_1 + c_2$). Furthermore, $t_2 - p_2 = p_1 - c_1$ implies the inequalities

$$p_1 - c_1 \leq p_2 - c_2 \leq t_2 - p_2$$

must all hold with equality and therefore

$$p_1 - c_1 = t_1 - p_1 = p_2 - c_2 = t_2 - p_2.$$

For $i = 1, 2$, let $\alpha_i, \beta_i, \gamma_i$ be the following edge costs on the 3-path from s to t :



$$\alpha_i = \gamma_i = 0$$

$$\beta_i = \frac{t_i - c_i}{2}.$$

For these edge costs (all of which are nonnegative), the optimal s - t 2-cycle covers, s - t traveling salesman walks and traveling salesman tours in H_i can all be extended to optimal traveling salesman tours in G_i with costs

$$\begin{aligned} c_i + 2(\alpha_i + \beta_i + \gamma_i) &= c_i + (t_i - c_i) \\ &= t_i \\ p_i + (\alpha_i + \beta_i + \gamma_i) &= p_i + \frac{t_i - c_i}{2} \\ &= t_i \\ t_i + 2(\alpha_i + \gamma_i) &= t_i. \end{aligned}$$

Furthermore, these are the only optimal traveling salesman tours of G_i . Since G_i is TSP-perfect, there is a constraint C_i in polyhedron $P(G_i)$ that is tight for all optimal traveling salesman tours. If one of the tight constraints is an edge constraint $x_e \geq 0$, then e cannot be an edge in $G_i \setminus H_i$ since there are optimal traveling salesman tours in G_i which use each edge of $G_i \setminus H_i$. Therefore, the edge constraint $x_e \geq 0$ is also tight for all $x \in X_{TSP}(G)$. If one of the constraints is a cut constraint C satisfying $s, t \notin C$ then the constraint C is tight for all $x \in T$. Otherwise, we can assume the cut constraints are C_1 and C_2 with $s \in C_i, t \notin C_i$. In this case, $C = C_1 \cup C_2$ satisfies $x(\delta(C)) = 2$ for all $x \in T$.

Case II. If $c_1 + t_2, p_1 + p_2$ and $c_2 + t_1$ are not all equal, we can assume without loss of

generality that $p_1 - c_1 < t_2 - p_2$. Then $p_1 + p_2 < c_1 + t_2$, so the optimal s - t 2-cycle cover in H_1 together with the optimal traveling salesman tour in H_2 is not optimal. Suppose $\min\{c_1 + t_2, p_1 + p_2, c_2 + t_1\}$ is achieved exactly twice with $c_2 + t_1 = p_1 + p_2 < c_1 + t_2$. Consider edge costs on the 3-path from s to t given by

$$\alpha_i = \gamma_i = 0$$

$$\beta_i = t_1 - p_1 (= p_2 - c_2).$$

For these edge costs (all of which are nonnegative), the optimal traveling salesman tours in H_1 , optimal s - t 2-cycle cover in H_2 , and optimal s - t traveling salesman walks in H_i can all be extended to optimal traveling salesman tours in G_1 and G_2 with costs

$$\begin{aligned} p_1 + (\alpha_1 + \beta_1 + \gamma_1) &= p_1 + (t_1 - p_1) & p_2 + (\alpha_2 + \beta_2 + \gamma_2) &= p_2 + (p_2 - c_2) \\ &= t_1 & &= 2p_2 - c_2 \\ t_1 + 2(\alpha_1 + \gamma_1) &= t_1 & c_2 + 2(\alpha_2 + \beta_2 + \gamma_2) &= c_2 + 2(p_2 - c_2) \\ & & &= 2p_2 - c_2. \end{aligned}$$

Note that the extension of the minimum 2-cycle cover in H_1 to a tour in G_1 has cost $c_1 + 2(t_1 - p_1) = c_1 + 2t_1 - 2p_1$, which is at least t_1 by Claim 3.3.4. Similarly, the extension of the minimum traveling salesman tour in H_2 to a tour in G_2 has cost t_2 which is at least $2p_2 - c_2$. Since G_1 and G_2 are TSP-perfect, there is a constraint C_i in each polyhedron $P(G_i)$ that is tight for all optimal traveling salesman tours. We argue as above to find a tight constraint C from C_i for all optimal traveling salesman tours $x \in X_{TSP}(G)$.

Case III. Finally, suppose $c_1 + t_2, p_1 + p_2$ and $c_2 + t_1$ are not all equal and the minimum of the three is achieved exactly once. We again assume without loss of generality that $p_1 - c_1 < t_2 - p_2$. The two cases are $c_2 + t_1 < p_1 + p_2$ and $p_1 + p_2 < c_2 + t_1$. In both cases, for edge costs

$$\alpha_i = \gamma_i = 0$$

$$\beta_i = \frac{t_i - c_i}{2},$$

the optimal s - t 2-cycle covers, optimal s - t traveling salesman walks and traveling salesman tours in H_i can be extended to optimal traveling salesman tours in G_i with costs

$$\begin{aligned}
c_i + 2(\alpha_i + \beta_i + \gamma_i) &= c_i + (t_i - c_i) \\
&= t_i \\
p_i + (\alpha_i + \beta_i + \gamma_i) &= p_i + \frac{t_i - c_i}{2} \\
&= p_i + \frac{(t_i - p_i) + (p_i - c_i)}{2} \\
&\leq p_i + \frac{2(t_i - p_i)}{2} = t_i \\
t_i + 2(\alpha_i + \gamma_i) &= t_i.
\end{aligned}$$

Therefore, in the case $c_2 + t_1 < p_1 + p_2$, the optimal tours in G_i are achieved by extending the optimal s - t 2-cycle covers in H_1 and the optimal traveling salesman tours in H_2 . Similarly, in the case $p_1 + p_2 < c_2 + t_1$, the optimal tours in G_i are achieved by extending the optimal traveling salesman walks in H_i to tours in G_i . Since G_1 and G_2 are TSP-perfect, there is a constraint C_i in each polyhedron $P(G_i)$ that is tight for all optimal traveling salesman tours. We argue as above to find a tight constraint C from C_i for all optimal traveling salesman tours $x \in X_{TSP}(G)$. \square

Corollary 3.3.5. *Let G be a graph with a 2-vertex disconnecting set $\{s, t\}$ which can be decomposed into graphs G_1 and G_2 by operation Φ_2 . Consider the graph G' obtained from G by adding a 2-path between s and t . If G is TSP-perfect, then G' is also TSP-perfect.*

Proof. First, suppose G has no 2-path between s and t and denote the added 2-path by $e_1 = (s, u), e_2 = (u, t)$ (for some vertex $u \notin V(G)$). For $G'_1 = G_1 \cup \{e_1, e_2\}$, $G'_2 = G_2$, and H as shown in Figure 3.9, we have

$$\begin{aligned}
G' &= G'_1 \circ_{\Phi_2} G'_2 \\
&= (G_1 \circ_{\Phi_2} H) \circ_{\Phi_2} G_2.
\end{aligned}$$

Since G is TSP-perfect and decomposes into G_1 and G_2 by operation Φ_2 , Theorem 3.3.2

implies graphs G_1 and G_2 are TSP-perfect. Also, H is TSP-perfect because it is a subset of wheel graph W_6 . Then applying Theorem 3.3.3 twice implies G' is TSP-perfect.

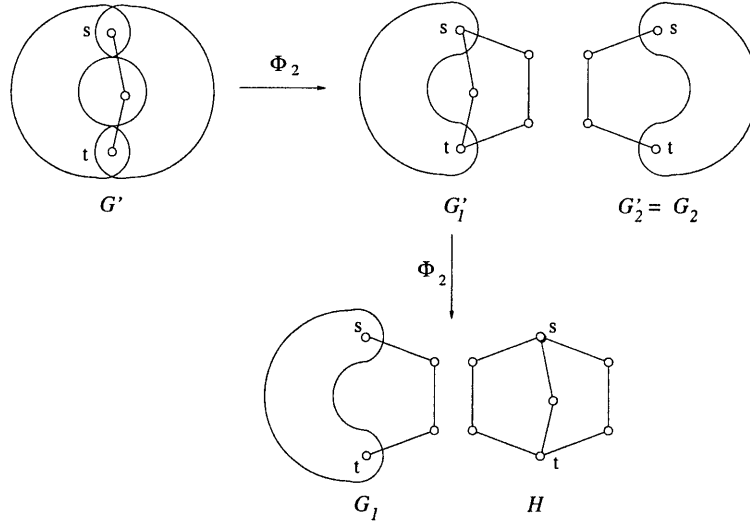


Figure 3.9: Adding 2-paths.

Now, if G has a 2-path $(s, u), (u, t)$ between s and t , then adding another 2-path between s and t is the same as duplicating vertex u , which preserves TSP-perfection by Lemma 2.2.4. □

3.4 Exhaustive List of Bricks

In this section, we show that the set \mathcal{B} of graphs in Figure 3.12 together with wheels and propellers is an exhaustive list of bricks. From Theorem 3.1.1, the only 3-connected M_3 minor free graphs are the following:

- (i) K_i for $i = 1, 2, \dots, 5$
- (ii) $K_5 \setminus \{e\}$ for any edge e
- (iii) W_k for $k \geq 3$
- (iv) $P_k(0, 0, 0), P_k(0, 0, 1), P_k(0, 1, 1), P_k(1, 1, 1)$ for $k \geq 3$.

For these graphs, we consider replacing each edge by a 2-path or 3-paths and show that the result either

- (a) is a minor of a wheel graph, propeller graph, or one of the graphs in Figure 3.12, or
- (b) contains one of M_1 , M_2 , or M_3 as a minor.

If the graph contains M_1 as a minor, we will indicate M_1 in the figure by two vertices with black shading together with three paths of length at least 3 between these two vertices (shown by dotted lines in Figure 3.10). If the graph contains M_2 as a minor, the cycle which

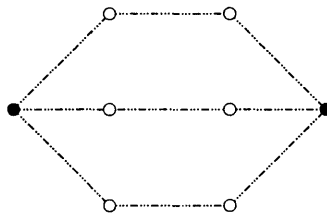


Figure 3.10: Vertices s and t and paths indicating minor M_1 .

corresponds to the triangle in M_2 after the minor contractions will be lightly shaded. Also, the vertex connected to the triangle by three paths of length at least 2 will be shaded black and the three paths will be dotted (see Figure 3.11).

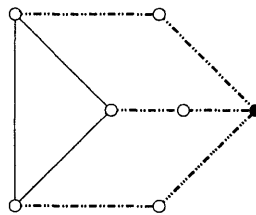


Figure 3.11: Shaded triangle and paths indicating minor M_2 .

Note that Lemmas 3.2.1, 3.2.2, and 3.2.8 together show that all graphs in \mathcal{B} are TSP-perfect and are therefore $[M_1, M_2, M_3]$ minor free.

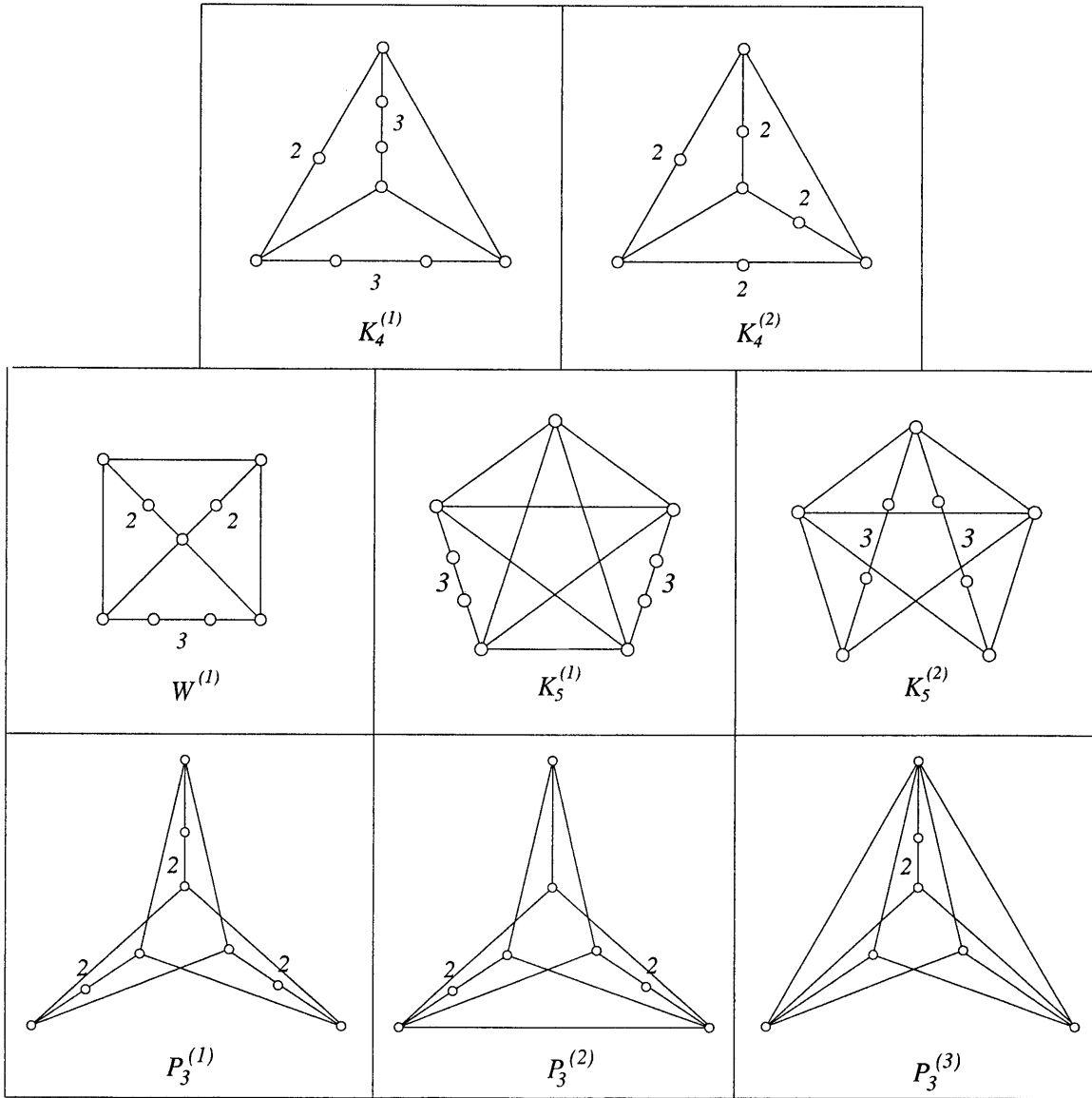


Figure 3.12: $[M_1, M_2, M_3]$ minor free bricks not including wheels and propellers.

3.4.1 Bricks from K_2 and K_3

In K_2 and K_3 , replacing all edges with 3-paths results in a minor of wheel graph $W_3^{(3)}$ (see Figure 3.13).

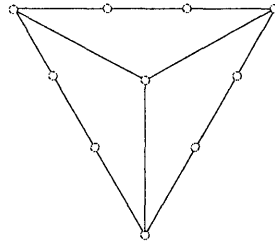


Figure 3.13: Wheel graph $W_3^{(3)}$.

3.4.2 Bricks from $K_4 (= W_3)$

In this section, we show the following graphs and their minors are the only $[M_1, M_2, M_3]$ minor free graphs obtained from K_4 by replacing edges by 2-paths and 3-paths.

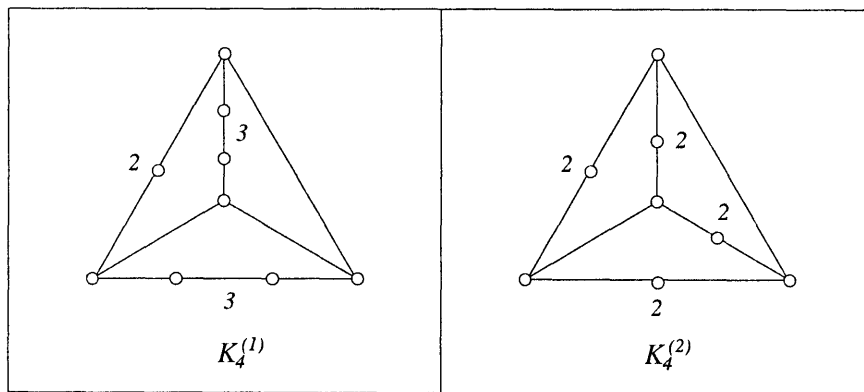


Figure 3.14: Bricks from K_4 .

Cases K4.I and K4.II If any edge or any two edges in K_4 are replaced by 3-paths, the resulting graph is a minor of $K_4^{(1)}$ or wheel graph $W_3^{(3)}$.

Case K4.III Replacing three edges:

Case K4.III.i If the three edges are adjacent to the same vertex, the result is graph M_2 (see Figure 3.15(a)).

Case K4.III.ii Suppose the three edges form a path with different initial and final vertices. If the path sequence replacing the edges is 2-path, 3-path, 2-path, the resulting graph contains an M_1 minor (see Figure 3.15(b)). For any other path sequence with different initial and final vertices, the resulting graph is either a minor of $K_4^{(1)}$ or contains a 2-path, 3-path, 2-path minor.

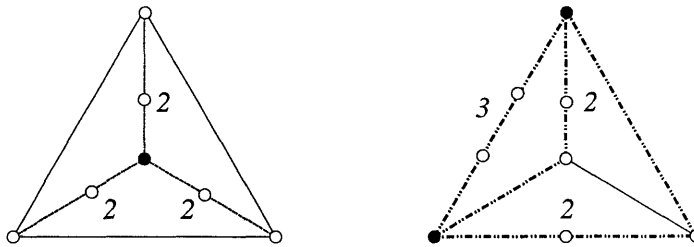


Figure 3.15: (a) Case K4.III.i (b) Case K4.III.ii.

Case K4.IV Replacing four edges:

By Case K4.III.i, if three of the edges replaced by paths are adjacent to the same vertex, then the resulting graph contains an M_2 minor. Otherwise, the four edges form a cycle. If any of the edges is a 3-path, then it contains the graph in Figure 3.15(b) (and therefore M_1) as a minor. Otherwise, all edges are replaced by 2-paths, resulting in graph $K_4^{(2)}$ (see Figure 3.14).

Case K4.V Replacing five edges:

Since there is no 5-cycle in K_4 , three of the edges replaced by paths must be adjacent to the same vertex, which is forbidden by Case K4.III.i.

3.4.3 Bricks from $W_4 (= K_5 \setminus \{e_1, e_2\})$

In this section, we show the following graph and its minors are the only $[M_1, M_2, M_3]$ minor free graphs obtained from W_4 by replacing edges by 2-paths and 3-paths.

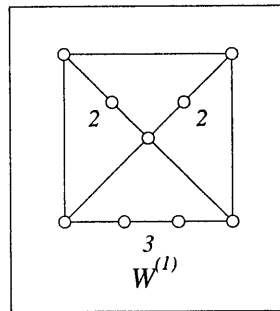


Figure 3.16: Bricks from W_4 .

Case W4.I Replacing one edge in W_4 results in a minor of wheel graph $W_4^{(3)}$ or propeller graph $P_2(3, 1, 0)$ (see Figure 3.17).

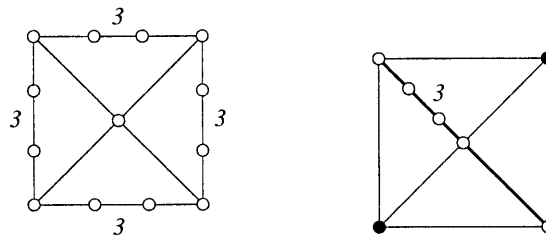


Figure 3.17: Case W4.I (a) Graph $W_4^{(3)}$ (b) Graph $P_2(3, 1, 0)$. Rim edges are darkened and spoke vertices are indicated by black shading.

Case W4.II Replacing two edges:

Case W4.II.i If the two edges are adjacent spokes and are both replaced by 2-paths, the resulting graph is a minor of $W^{(1)}$.

Case W4.II.ii If the two edges are adjacent spokes and are replaced by a 3-path and a 2-path, the result contains an M_1 minor (see Figure 3.18(a)).

Case W4.II.iii If the two edges are nonadjacent spokes, the result is a minor of propeller $P_2(3, 3, 0)$ (see Figure 3.18(b)).

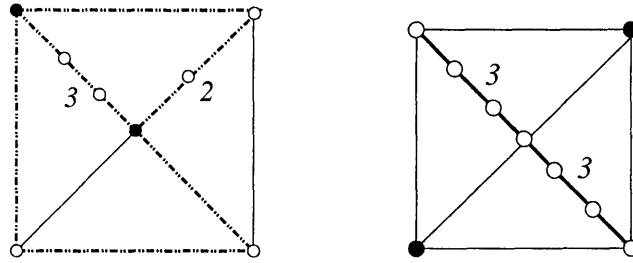


Figure 3.18: (a) Case W4.II.ii (b) Case W4.II.iii (graph $P_2(3, 3, 0)$). Rim edges are darkened and spoke vertices are indicated by black shading.

Case W4.II.iv If the two edges are a spoke and a nonadjacent rim, the result is a minor of $K_5^{(1)}$ (since W_4 is a subgraph of K_5 ; see Figure 3.22).

Case W4.II.v If the two edges are a spoke and an adjacent rim, the result contains an M_2 minor (see Figure 3.19).

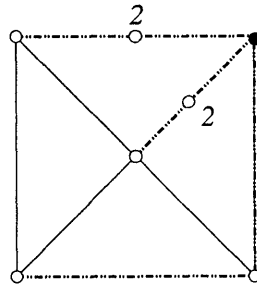


Figure 3.19: Case W.4.II.v.

Case W4.III Replacing three edges:

Case W4.III.i If all three edges are rim edges, then the resulting graph is a subgraph of $W_4^{(3)}$.

Case W4.III.ii If all three edges are spoke edges, the result contains an M_2 minor (see Figure 3.20).

Case W4.III.iii If the three edges are two spokes and a rim, then by Case W4.II.v, the rim cannot be adjacent to either spoke, implying the spokes are

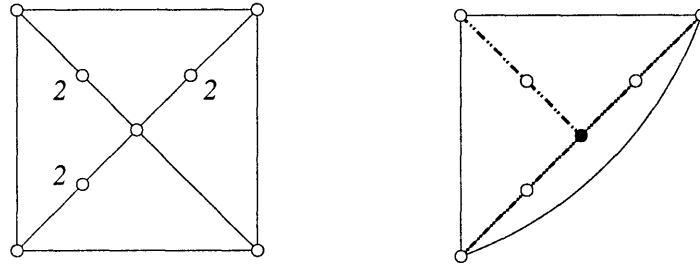


Figure 3.20: Case W4.III.ii. Minor M_2 is shown in figure to the right.

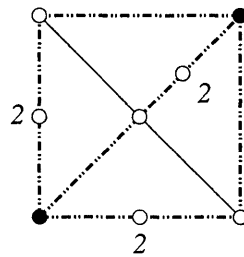


Figure 3.21: Case W4.III.iv.

adjacent. Then by Case W4.II.ii, the two spokes must both be replaced by 2-paths, which results in a minor of $W^{(1)}$ (see Figure 3.16).

Case W4.III.iv. If the three edges are two rims and a spoke, then by Case W4.II.v, the spoke must not be adjacent to either rim. Figure 3.21 shows that such a graph contains an M_1 minor.

Case W4.IV Replacing four edges:

Case W4.IV.i If the four edges are all rims, the result is a subgraph of $W_4^{(3)}$ (see Figure 3.17(a)).

Case W4.IV.ii The four edges cannot all be spokes by Case W4.III.ii.

Case W4.IV.iii If the four edges contain both rims and spokes, then it must contain an adjacent rim and spoke, which is forbidden by Case W4.II.v.

Case W4.V Any five edges must contain an adjacent rim and spoke, which is forbidden by Case W4.II.v.

3.4.4 Bricks from K_5

In this section, we show the following graph and its minors are the only $[M_1, M_2, M_3]$ minor free graphs obtained from K_5 by replacing edges by 2-paths and 3-paths.

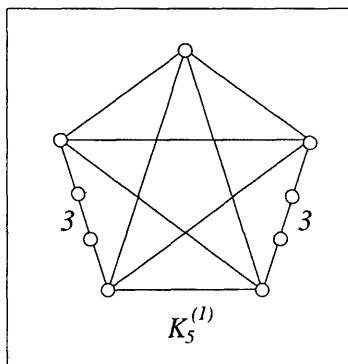


Figure 3.22: Bricks from K_5 .

Case K5.I Replacing one edge in K_5 results in a minor of $K_5^{(1)}$.

Case K5.II. Replacing two edges:

Case K5.II.i If the two edges are nonadjacent, the result is a subgraph of $K_5^{(1)}$.

Case K5.II.ii If the two edges are adjacent, the result contains an M_2 minor (see Figure 3.23).

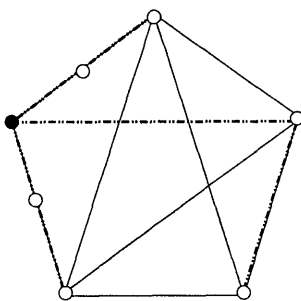


Figure 3.23: Case K5.II.ii.

Case K5.III Since there are two adjacent edges among any three edges in K_5 , Case K5.II.ii implies that no three or more edges can be replaced.

3.4.5 Bricks from $K_5 \setminus e$ ($= P_2$)

In this section, we show the following graph and its minors are the only $[M_1, M_2, M_3]$ minor free graphs obtained from $K_5 \setminus e$ by replacing edges by 2-paths and 3-paths.

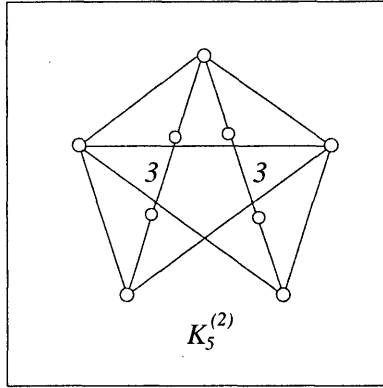


Figure 3.24: Bricks from K_5 .

Note that if we partition the vertices of $K_5 \setminus e$ into classes A and B , where B contains the two endpoints of e and $A = V \setminus B$, then all vertices within class A are isomorphic and all vertices within class B are isomorphic.

Case P2.I Replacing any single edge results in a subgraph of $K_5^{(1)}$ (see Figure 3.22).

Case P2.II Replacing two edges:

Case P2.II.i If the two edges e_1, e_2 are such that e_1 lies within $E(A)$, e_2 crosses cut (A, B) , and e_1 and e_2 are adjacent, the result contains an M_2 minor (see Figure 3.25(a)).

Case P2.II.ii If the two edges e_1, e_2 are such that e_1 lies within $E(A)$, e_2 crosses cut (A, B) , and e_1 and e_2 are nonadjacent, then the result is a minor of $K_5^{(1)}$.

Case P2.II.iii If both edges are adjacent to the same vertex in B , the result again contains an M_2 minor (see Figure 3.25(b)).

Case P2.II.iv If both edges cross cut (A, B) and are not both adjacent to the same vertex in B , the resulting graph is a minor of one of $K_5^{(1)}$ or $K_5^{(2)}$.

Case P2.II.v If both edges are in $E(A)$, then the result is a minor of propeller graph $P_2(3, 3, 3)$.

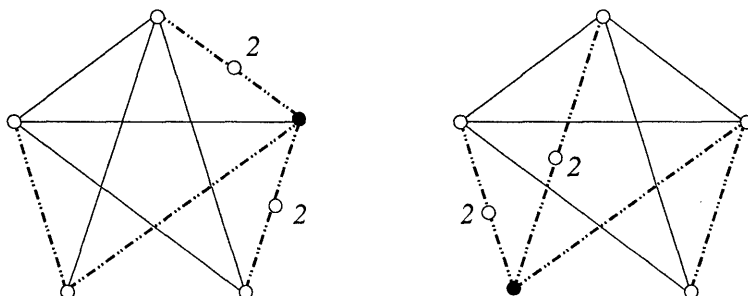


Figure 3.25: (a) Case P2.II.i (b) Case P2.II.ii.

Case P2.III Replacing three edges:

Case P2.III.i If all three edges are in $E(A)$, the graph is a minor of $P_2(3, 3, 3)$.

Case P2.III.ii If two edges e_1, e_2 are in $E(A)$ and the third e_3 is not, then e_3 must be adjacent to one of e_1 or e_2 , which is forbidden by Case P2.II.i.

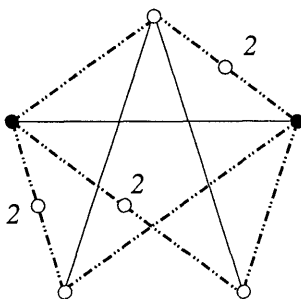


Figure 3.26: Case P2.III.iii

Case P2.III.iii If one edge is in $E(A)$ and the other two are not, then the only graphs not forbidden by Cases P2.II.i and P2.II.iii must have two edges crossing cut (A, B) which are adjacent to the same vertex in A and nonadjacent to the third edge. Any such graph contains the graph in Figure 3.26, which contains M_1 as a minor.

Case P2.III.iv If all three edges are in cut (A, B) , then two of these edges must be adjacent to the same vertex in B , which is forbidden by Case P2.II.iii.

Case P2.IV Among any four edges in $K_5 \setminus e$, there must be two edges in the pattern considered in either Case P2.II.i or Case P2.II.iii, so no replacement of four edges by 2-paths and 3-paths results in an $[M_1, M_2, M_3]$ minor free graph.

3.4.6 Bricks from wheels

In the wheel graphs W_k with $k \geq 5$, replacing all rim edges by 3-paths results in $[M_1, M_2, M_3]$ minor free wheel graph $W_k^{(3)}$. However, Figure 3.27 shows that replacing any spoke results in a graph containing an M_2 minor (see Figure 3.27).

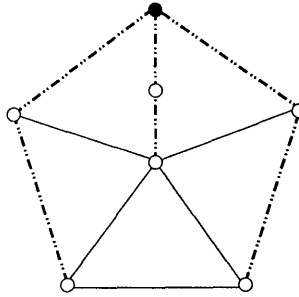


Figure 3.27: Replacing spoke in W_k .

3.4.7 Bricks from Propellers P_k for $k \geq 4$

In the propeller graphs P_k , replacing all rim edges results in a minor of $[M_1, M_2, M_3]$ minor free graph $P_k(3, 3, 3)$. However, Figure 3.28 shows that replacing any spoke edge in $P_k(0, 0, 0)$ for $k \geq 4$ (and hence replacing any spoke edge in $P_k(i_1, i_2, i_3)$ for $k \geq 4$) results in a graph containing an M_2 minor.

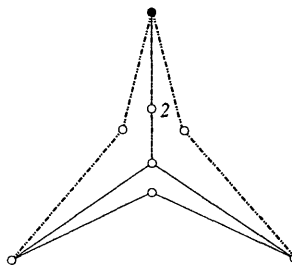


Figure 3.28: Replacing spoke by 2-path in $P_4(0, 0, 0)$.

3.4.8 Bricks from $P_3(0, 0, 0)$, $P_3(0, 0, 1)$, $P_3(0, 1, 1)$ and $P_3(1, 1, 1)$

In this section, we show the following graphs and their minors are the only $[M_1, M_2, M_3]$ minor free graphs obtained from $P_3(0, 0, 0)$, $P_3(0, 0, 1)$, $P_3(0, 1, 1)$ and $P_3(1, 1, 1)$ by replacing edges by 2-paths and 3-paths.

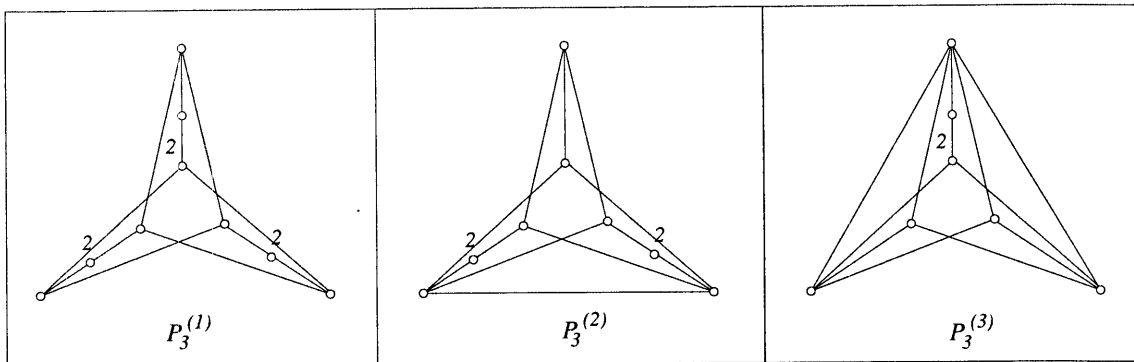


Figure 3.29: Bricks from $P_3(0, 0, 0)$, $P_3(0, 0, 1)$, $P_3(0, 1, 1)$, and $P_3(1, 1, 1)$.

In graphs $P_3(0, 0, 0)$, $P_3(0, 0, 1)$, $P_3(0, 1, 1)$, and $P_3(1, 1, 1)$, replacing all rim edges by 3-paths results in minors of propeller graphs $P_3(3, 3, 3)$. Therefore, we can consider replacements involving at least one spoke edge.

Case $P_3(0, 0, 0)$.I If we replace one spoke edge of $P_3(0, 0, 0)$ by a 2-path, the result is a minor of $P_3^{(1)}$. If we replace one spoke edge of $P_3(0, 0, 0)$ by a 3-path, the result contains an M_1 minor (see Figure 3.30(a)).

Case $P_3(0, 0, 0)$.II Since $P_3(0, 0, 0)$ is the complete bipartite graph $K(3, 3)$, the graph is vertex transitive and therefore, the pattern of edges replaced depends only on whether the edges are adjacent or not.

Case $P_3(0, 0, 0)$.II.i If the two edges are adjacent spokes, then the result contains an M_2 minor (see Figure 3.30(b)).

Case $P_3(0, 0, 0)$.II.ii If the two edges are non-adjacent spokes, then they must both be replaced by 2-paths by Case $P_3(0, 0, 0)$.I. The result is then a minor of $P_3^{(1)}$.

Case $P_3(0, 0, 0)$.III If we replace three edges in $P_3(0, 0, 0)$, then by Case $P_3(0, 0, 0)$.II.i, the

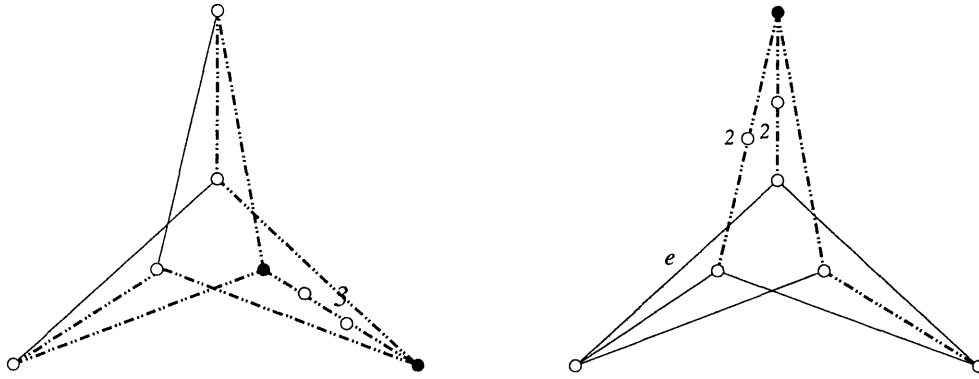


Figure 3.30: (a) Case $P_3(0,0,0)$.I (b) Case $P_3(0,0,0)$.II.i. M_2 minor is obtained by contracting edge e .

three edges must all be non-adjacent spokes and by Case $P_3(0,0,0)$.I, the three edges must all be replaced by 2-paths. The resulting graph is a minor of $P_3^{(1)}$.

Case $P_3(0,0,0)$.IV Any four edges in $P_3(0,0,0)$ must contain two adjacent spoke edges and therefore Case $P_3(0,0,0)$.II.i forbids replacing any four or more edges.

In graph $P_3(0,0,1)$, let v_1, v_2, v_3 denote the rim vertices and suppose the rim edge present in $P_3(0,0,1)$ is edge (v_2, v_3) .

Case $P_3(0,0,1)$.I Replacing one edge:

Case $P_3(0,0,1)$.I.i If any spoke edge of $P_3(0,0,1)$ is replaced by a 3-path, then by Case $P_3(0,0,0)$.I, the resulting graph contains an M_1 minor (see Figure 3.30(a)).

Case $P_3(0,0,1)$.I.ii If a spoke edge adjacent to rim edge (v_2, v_3) is replaced by 2-path, the resulting graph is a minor of graph $P_3^{(2)}$.

Case $P_3(0,0,1)$.I.iii If a spoke edge adjacent to the rim vertex v_1 is replaced by a 2-path, the resulting graph contains an M_2 minor (see Figure 3.31).

Case $P_3(0,0,1)$.II Replacing two edges:

Case $P_3(0,0,1)$.II.i If the two edges are adjacent spokes, the resulting graph contains an M_2 minor by case $P_3(0,0,0)$.II.i (see Figure 3.30(b)).

Case $P_3(0,0,1)$.II.ii If the two edges are non-adjacent spokes, then they must both be replaced by 2-paths by Case $P_3(0,0,0)$.I. Also, neither spoke can be

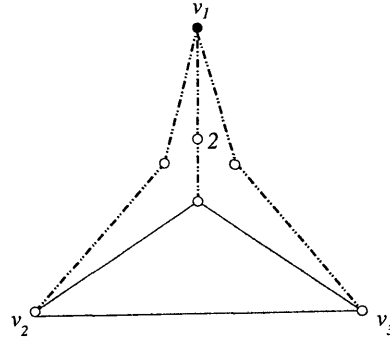


Figure 3.31: Case $P_3(0, 0, 1)$.I.iii

adjacent to rim vertex v_1 by Case $P_3(0, 0, 1)$.I.iii. Therefore, the resulting graph is a minor of $P_3^{(2)}$.

Case $P_3(0, 0, 1)$.II.iii If the two edges are an adjacent spoke edge and rim edge, then the result contains an M_2 minor (see Figure 3.32).

Case $P_3(0, 0, 1)$.II.iv If the two edges are a non-adjacent spoke edge and rim edge, then the spoke edge must be adjacent to rim vertex v_1 , which is forbidden by Case $P_3(0, 0, 1)$.I.iii (see Figure 3.31).

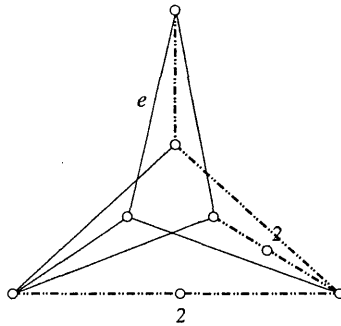


Figure 3.32: Case $P_3(0, 0, 1)$.I.iii. M_2 minor is obtained by contracting edge e .

Case $P_3(0, 0, 1)$.III If three edges of $P_3(0, 0, 1)$ are replaced, these edges cannot include a spoke adjacent to rim vertex v_1 by Case $P_3(0, 0, 1)$.I.iii. Therefore, the three edges must contain either two adjacent spoke edges or an adjacent spoke edge and rim edge, which are forbidden by Cases $P_3(0, 0, 0)$.II.i and $P_3(0, 0, 1)$.II.iii. Therefore, no three or more edges of $P_3(0, 0, 1)$ can be replaced by 2-paths or 3-paths.

In graph $P_3(0, 1, 1)$, let (v_1, v_2) and (v_2, v_3) be the two rim edges present in the graph.

Case $P_3(0, 1, 1)$.I If we replace one spoke edge in $P_3(0, 1, 1)$, then it must be a spoke edge adjacent to both rim edges, i.e., to vertex v_2 (otherwise if it is adjacent to v_1 or v_3 , the resulting graph contains an M_2 minor by Figure 3.31). Furthermore, the spoke edge must be replaced by a 2-path by Case $P_3(0, 0, 0)$.I. The resulting graph is a minor of $P_3^{(3)}$.

Case $P_3(0, 1, 1)$.II Replacing two edges:

Case $P_3(0, 1, 1)$.II.i If the two edges are both spoke edges, then since the spokes cannot be adjacent by Case $P_3(0, 0, 0)$.II.i, at least one of these spokes is adjacent to either v_1 or v_3 . Then there is a rim edge non-adjacent to this spoke and the resulting graph contains an M_2 minor (see Figure 3.31).

Case $P_3(0, 1, 1)$.II.ii If the two edges are a spoke edge and a rim edge, then the two edges cannot be adjacent by Case $P_3(0, 0, 1)$.II.iii. However, for non-adjacent spoke and rim edges, Case $P_3(0, 1, 1)$.I implies the resulting graph contains an M_2 minor (see Figure 3.31).

Therefore, no two or more edges other than rim edges can be replaced in by 2-paths and 3-paths in $P_3(0, 1, 1)$. Finally, we consider the graph $P_3(1, 1, 1)$.

Case $P_3(1, 1, 1)$.I If we replace any spoke edge in $P_3(1, 1, 1)$, then there is a rim edge non-adjacent to this spoke edge and the result contains an M_2 minor (see Figure 3.31). Therefore, no edge other than rim edges can be replaced by 2-paths and 3-paths in $P_3(1, 1, 1)$. \square

Chapter 4

Integrality Gap Preserving Operations

In Chapter 2 and Chapter 3, we considered several graph operations which preserve walk-perfection and TSP-perfection. We now study the relationship between these graph operations and the integrality gaps of the traveling salesman and traveling salesman walk relaxations. As defined in Chapter 1, the integrality gap of relaxation P for polyhedron Q is the smallest r such that for any cost function c ,

$$\min\{cx : x \in P\} \leq r \min\{cx : x \in Q\}.$$

The TSP integrality gap is the smallest such r for TSP polyhedron $Q = \text{conv}(X_{TSP}(G))$ and its fractional TSP relaxation $P = P(G)$. For fixed vertices s and t , the s - t TSW integrality gap is the smallest such r for TSW polyhedron $Q = \text{conv}(X(G, s, t))$ and its fractional TSW relaxation $P = P(G, s, t)$; the TSW integrality gap is the maximum s - t TSW integrality gap over all choices of s and t .

Since TSP-perfection corresponds to TSP integrality gap $r = 1$ and walk-perfection corresponds to TSW integrality gap $r = 1$, a natural question is whether the TSP-perfection and walk-perfection preserving operations are also integrality gap preserving. Note that if the initial and final vertices are identical, then the fractional TSW polyhedron and the fractional TSP polyhedron are the same, as are the the set of traveling salesman walks and

traveling salesman tours. Therefore, if an operation is TSW integrality gap preserving, then it is also TSP integrality gap preserving.

In this chapter, we show that some of the graph operations we have considered indeed preserve TSP and TSW integrality gap.

4.1 Integrality Gap Under Graph Minors

We first show that the TSW integrality gap of the fractional traveling salesman walk relaxation does not increase under the graph minor operations. This generalizes Lemma 2.2.1 which states that walk-perfection (TSW integrality gap $r = 1$) is preserved under the graph minor operations.

Theorem 4.1.1. *Suppose G is a graph with specified vertices $s, t \in V(G)$ and suppose G' is a labeled minor of G . If the s - t TSW integrality gap of $P(G', s, t)$ is r , then the s - t TSW integrality gap of $P(G, s, t)$ is at least r .*

Proof. Since G' can be obtained from G by a sequence of edge deletions and edge contractions, it suffices to show that the integrality gap does not increase under these edge operations. If labeled minor G' is disconnected, then the s - t TSW integrality gap of G' is 1 and the integrality gap of G is greater than or equal to 1, so the theorem follows. We can therefore assume G' is connected.

Consider any non-zero cost function c in $G \setminus \{e\}$ and consider the following cost function on the edges of G :

$$\tilde{c}_f = \begin{cases} c_f & \text{if } f \neq e \\ \infty & \text{if } f = e. \end{cases}$$

Then any s - t traveling salesman walk in G with minimum cost under \tilde{c} does not use edge e , implying it is also an s - t traveling salesman walk in $G \setminus \{e\}$ with minimum cost under c . This shows $\min\{cx : x \in X(G \setminus \{e\}, s, t)\} = \min\{\tilde{c}x : x \in X(G, s, t)\}$. Furthermore, any feasible solution in $P(G \setminus \{e\}, s, t)$ is also a feasible solution in $P(G, s, t)$, implying $\min\{cx : x \in P(G \setminus \{e\}, s, t)\} \geq \min\{\tilde{c}x : x \in P(G, s, t)\}$. Therefore,

$$\frac{\min\{cx : x \in X(G \setminus \{e\}, s, t)\}}{\min\{cx : x \in P(G \setminus \{e\}, s, t)\}} \leq \frac{\min\{\tilde{c}x : x \in X(G, s, t)\}}{\min\{\tilde{c}x : x \in P(G, s, t)\}}. \quad (4.1)$$

Since the s - t TSW integrality gap of G is the maximum over all cost functions of the expression on the right in Equation (4.1) and the s - t TSW integrality gap of $G \setminus e$ is the maximum over all cost functions of the expression on the left, the s - t TSW integrality gap of G is at least that of $G \setminus \{e\}$.

Similarly, consider any non-zero cost function c in $G.e$ and define the following cost function on the edges of G :

$$\tilde{c}_f = \begin{cases} c_f & \text{if } f \neq e \\ 0 & \text{if } f = e. \end{cases}$$

Consider an s - t traveling salesman walk p in G with minimum cost under \tilde{c} . Then the walk $p.e$ obtained by contracting edge e is an s - t traveling salesman walk in $G.e$ with minimum cost under cost c . This shows $\min\{cx : x \in X(G.e, s, t)\} = \min\{\tilde{c}x : x \in X(G, s, t)\}$. Furthermore, any feasible solution in $P(G.e, s, t)$ is also a feasible solution in $P(G, s, t)$, implying $\min\{cx : x \in P(G.e, s, t)\} \geq \min\{\tilde{c}x : x \in P(G, s, t)\}$. Therefore,

$$\frac{\min\{cx : x \in X(G.e, s, t)\}}{\min\{cx : x \in P(G.e, s, t)\}} \leq \frac{\min\{\tilde{c}x : x \in X(G, s, t)\}}{\min\{\tilde{c}x : x \in P(G, s, t)\}}. \quad (4.2)$$

Since the s - t TSW integrality gap of G is the maximum over all cost functions of the expression on the right in Equation (4.2) and the s - t TSW integrality gap of $G.e$ is the maximum over all cost functions of the expression on the left, the s - t TSW integrality gap of G is at least that of $G.e$ and the theorem follows. \square

We now have the following corollaries.

Corollary 4.1.2. *Suppose G' is a labeled minor of a graph G . If the TSW integrality gap of G' is r , then the TSW integrality gap of G is at least r .*

This gives a second proof that walk-perfection is preserved under graph minors (which

was proved independently in Lemma 2.2.1).

Corollary 4.1.3. *If graph G is walk-perfect, then G' is also walk-perfect for any labeled minor G' of G .*

By computational methods, we can verify that the TSP integrality gaps of M_1, M_2, M_3 are all equal to $\frac{10}{9}$ and the TSW integrality gap of C_6 is $\frac{7}{6}$. Then Theorem 4.1.1 implies the following integrality gap lower bounds for connected graphs that are not TSP-perfect and not walk-perfect.

Corollary 4.1.4. *For any connected graph G , either G is TSP-perfect or G has TSP integrality gap at least $\frac{10}{9}$.*

Corollary 4.1.5. *For any connected graph G , either G is walk-perfect or G has TSW integrality gap at least $\frac{7}{6}$.*

Theorem 4.1.1 also implies the following corollary.

Corollary 4.1.6. *For any $r \geq 1$, the set of graphs with TSP (or TSW) integrality gap at most r is a minor closed family.*

By the excluded minor theory of Robertson, Seymour, and Thomas, this implies that the set of graphs with TSP or TSW integrality gap at most r can be characterized by a finite list of excluded minors. From Chapter 2, the list of excluded minors for TSP integrality gap $r = 1$ is M_1, M_2, M_3 , and the list of excluded minors for TSW integrality gap $r = 1$ is C_6 . It is an interesting open problem to characterize the forbidden minors for TSP integrality gap at most r with $r > \frac{10}{9}$ and for TSW integrality at most r with $r > \frac{7}{6}$. Such a characterization for TSP integrality gap $r = \frac{4}{3}$ would resolve the Held-Karp conjecture, which can also be formulated as follows.

Conjecture 4.1.7. *The forbidden minors characterizing the set of graphs with TSP integrality gap at most $\frac{4}{3}$ is the empty set.*

4.2 Integrality Gap under Degree-2 Vertex Duplication

We consider the operation of degree-2 vertex duplication as introduced in Section 2.1. Suppose graph G has a degree-2 vertex u with neighbors $N(u) = \{a, b\}$. Let $G_2(u)$ denote

the graph obtained from G by duplicating u to obtain vertices v and w (see Figure 4.1). It was shown in Lemmas 2.2.4 and 2.2.5 that the operation of degree-2 vertex duplication

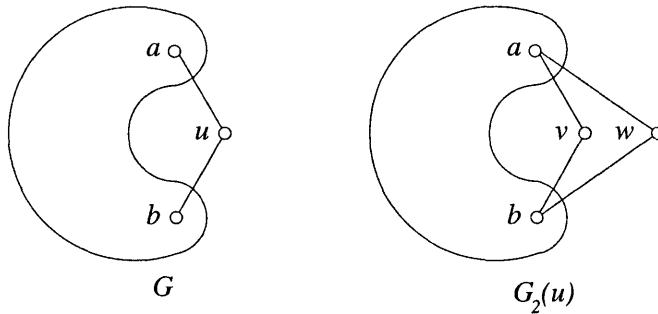


Figure 4.1: Duplicating degree-2 vertex.

preserves walk-perfection. We generalize this result to show that degree-2 vertex duplication also preserves integrality gap. To do this, we first define the *dominant* $\mathcal{D}(P)$ of a polyhedron P as the set of points which dominate some point in P , that is,

$$\mathcal{D}(P) = \{y \in \mathbb{R}^n : \exists x \in P \text{ such that } y_i \geq x_i \text{ for all } i = 1, 2, \dots, n\}.$$

A polyhedron is said to be of *blocking type* if $\mathcal{D}(P) = P$. The following theorem from [6] and [19] characterizes the integrality gap of a relaxation using the dominant.

Theorem 4.2.1. *The integrality gap of a relaxation P of an integral polyhedron Z in the positive orthant is r if and only if r is the smallest real number such that $rx \in \mathcal{D}(Z)$ for any $x \in P$.*

In particular, if the integrality gap of relaxation P is r , then rx dominates a convex combination of extreme points in integral polyhedron Z . In [7], Carr and Vempala present a polynomial algorithm to construct this convex combination. Note that if Z is a polyhedron of blocking type with integrality gap r , then $rx \in \mathcal{D}(Z) = Z$ for any feasible solution $x \in P$.

In the traveling salesman problem, the integral polyhedra we consider are the convex hulls of traveling salesman tours ($Z = \text{conv}(X_{TSP}(G))$) and traveling salesman walks ($Z = \text{conv}(X(G, s, t))$). The relaxations are the fractional TSP polyhedron ($P = P(G)$) and fractional TSW polyhedron ($P = P(G, s, t)$). Note that these polyhedra are all of blocking

type and therefore, if the TSP (TSW) integrality gap is r , then for any feasible solution $x \in P(G)$ ($x \in P(G, s, t)$), rx is equal to a convex combination of s - t traveling salesman tours (s - t traveling salesman walks). We use this theorem to prove that integrality gap is preserved under vertex duplication of degree-2 vertices.

Theorem 4.2.2. *Suppose the TSP integrality gap of G is r and suppose G has a vertex u of degree 2 with neighbors $N(u) = \{a, b\}$. Then $G_2(u)$ has TSP integrality gap r .*

Proof. Since G is a minor of $G_2(u)$, the TSP integrality gap of $G_2(u)$ is at least r by Lemma 2.2.1.

To show the TSP integrality gap of $G_2(u)$ is at most r , let x be any feasible solution in $P(G_2(u))$. First, we show that we can assume $x_{av} \leq x_{aw} \leq x_{bw}$ and $x_{av} \leq x_{bv}$. Otherwise, if $x_{av} > x_{aw}$, the solution x' defined by switching v and w and thus defined by

$$\begin{aligned} x'_{av} &= x_{aw} & x'_{aw} &= x_{av} \\ x'_{bv} &= x_{bw} & x'_{bw} &= x_{bv} \\ x'_e &= x_e \text{ for all other edges } e \end{aligned}$$

is a feasible solution satisfying $x'_{av} \leq x'_{aw}$. Its feasibility follows from the observation that the set of cuts is unaffected by the switching of two vertices of degree 2 with the same neighbors. If $x_{aw} > x_{bw}$, then the solution x' defined by switching edges (a, w) and (b, w) and thus defined by

$$\begin{aligned} x'_{aw} &= x_{bw} & x'_{bw} &= x_{aw} \\ x'_e &= x_e \text{ for all other edges } e \end{aligned}$$

is a feasible solution satisfying $x'_{aw} \leq x'_{bw}$. Its feasibility follows from the observation that the set of cuts is unaffected by the switching of the two edges incident to a degree-2 vertex. We can therefore swap the edges adjacent to a and b if necessary so that the edges adjacent to b have larger value and then swap the vertices v and w if necessary in order to guarantee $x_{av} \leq x_{aw} \leq x_{bw}$ and $x_{av} \leq x_{bv}$. We use the feasible solution x in $P(G_2(u))$ to define

feasible solution \tilde{x} in $P(G)$ by

$$\tilde{x}_{au} = x_{av} + x_{aw}$$

$$\tilde{x}_{bu} = \max\{2 - (x_{av} + x_{aw}), x_{av} + x_{aw}\}.$$

The feasibility of \tilde{x} follows from the observation that any cut in G containing both edges (a, u) and (b, u) is valid and for any other cut in G , there is a corresponding valid cut in $G_2(u)$ of no larger value. Since the TSP integrality gap of G is r , Theorem 4.2.1 implies $r\tilde{x}$ dominates a convex combination of tours in G , i.e., there are tours \tilde{T}_i and values α_i satisfying $\alpha_i > 0$, $\sum_i \alpha_i = 1$, and

$$\sum_i \alpha_i \tilde{T}_i \leq r\tilde{x}. \quad (4.3)$$

Let $\tilde{y} = \sum_i \alpha_i \tilde{T}_i$. For simplicity, we assume without loss of generality that $x_e \leq 2$ and $(\tilde{T}_i)_e \leq 2$ for all edges $e \in E(G)$ (note that by imposing this assumption, Equation (4.3) may become a strict inequality for some edges). Now, partition the tours \tilde{T}_i into classes $\Lambda_T, \Lambda_P, \Lambda_{au}$ and Λ_{bu} according to whether \tilde{y} restricted to the two edges (a, u) and (b, u) is a tour, path, or 2-cycle cover (see Figure 4.2) and let

$$\begin{aligned} \alpha_T &= \sum_{i \in \Lambda_T} \alpha_i, \\ \alpha_P &= \sum_{i \in \Lambda_P} \alpha_i, \\ \alpha_{au} &= \sum_{i \in \Lambda_{au}} \alpha_i, \\ \alpha_{bu} &= \sum_{i \in \Lambda_{bu}} \alpha_i. \end{aligned}$$

The idea will be to use the tours \tilde{T}_i in G to construct tours T_i in $G_2(u)$ in such a way that after a suitable redistribution of the weights α_i , rx will dominate a convex combination of the modified tours T_i , i.e.,

$$rx \geq \sum_i \alpha'_i T_i. \quad (4.4)$$

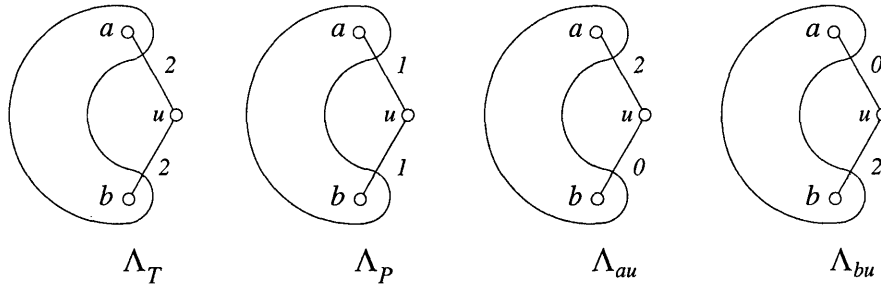


Figure 4.2: Partition of tours.

For the weights α as given by Equation (4.3), consider the following cases.

Case I. $rx_{av} \leq \alpha_T$.

In this case, consider redistributing the weights α_i for $i \in \Lambda_T, \Lambda_P, \Lambda_{au}, \Lambda_{bu}$ on tours of $G_2(u)$ as shown in Figure 4.3. The figure also labels the classes $\Lambda_T^{(1)}, \Lambda_T^{(2)}, \Lambda_P^{(1)}, \Lambda_{au}^{(1)}$, and $\Lambda_{bu}^{(1)}$ of modified tours in graph $G_2(u)$. Let $y = \sum_i \alpha'_i T_i$, where the α'_i are the redistributed weights

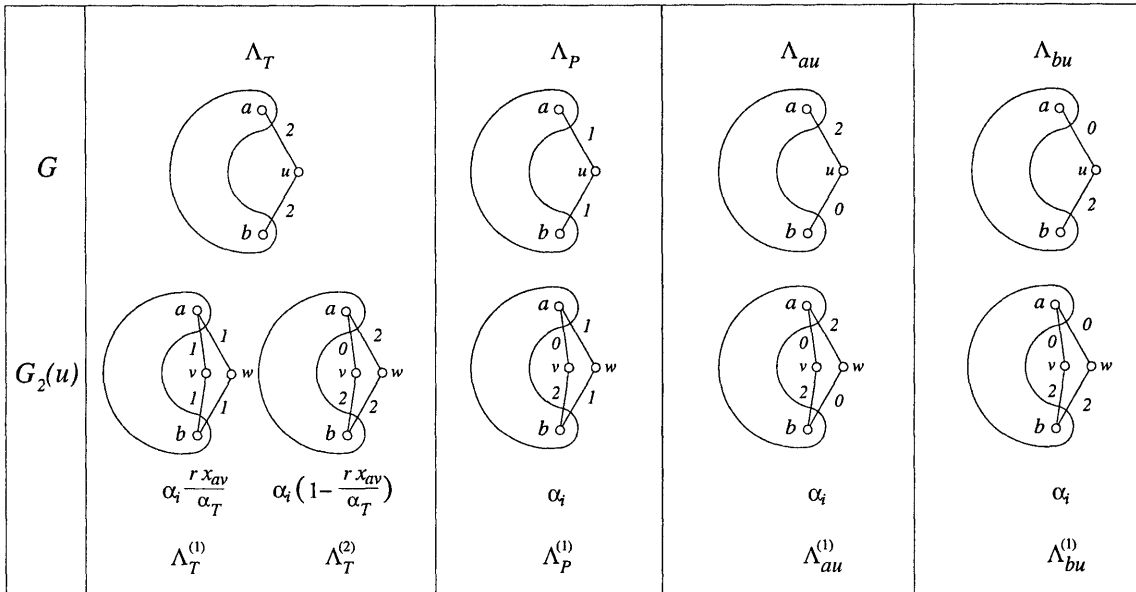


Figure 4.3: Redistribution in Case I.

on tours T_i in $G_2(u)$ as shown in Figure 4.3. We will show that either this redistribution or a slightly modified redistribution of weights gives $y_e \leq rx_e$ for all edges $e \in E(G_2(u))$. This will show rx dominates a convex combination of tours in $G_2(u)$.

Case I: Edge (a, v) . The weight y_{av} on edge (a, v) is

$$y_{av} = \sum_{\alpha_i \in \Lambda_T} \alpha_i \frac{rx_{av}}{\alpha_T} = rx_{av}.$$

Case I: Edge (b, v) . Note that each tour T_i in the decomposition satisfies $(T_i)_{bv} = 2 - (T_i)_{av}$, implying the weight y_{bv} on edge (b, v) is

$$\begin{aligned} y_{bv} &= 2 - y_{av} \\ &= 2 - rx_{av} \\ &\leq 2r - rx_{av} \\ &= r(2 - x_{av}) \\ &\leq rx_{bv}. \end{aligned}$$

Case I: Edge (a, w) . Since $2\alpha_T + \alpha_P + 2\alpha_{au} \leq r(x_{av} + x_{aw})$ by Equation (4.3), the weight y_{aw} on edge (a, w) is

$$y_{aw} = \sum_{\alpha_i \in \Lambda_T} \left(\alpha_i \frac{rx_{av}}{\alpha_T} + 2\alpha_i \left(1 - \frac{rx_{av}}{\alpha_T} \right) \right) + \sum_{\alpha_i \in \Lambda_P} \alpha_i + 2 \sum_{\alpha_i \in \Lambda_{au}} \alpha_i \quad (4.5)$$

$$= rx_{av} + 2\alpha_T \left(1 - \frac{rx_{av}}{\alpha_T} \right) + \alpha_P + 2\alpha_{au} \quad (4.6)$$

$$= rx_{av} + 2\alpha_T - 2rx_{av} + \alpha_P + 2\alpha_{au} \quad (4.7)$$

$$= (2\alpha_T + \alpha_P + 2\alpha_{au}) - rx_{av} \quad (4.8)$$

$$\leq r(x_{av} + x_{aw}) - rx_{av} \quad (4.9)$$

$$= rx_{aw}. \quad (4.10)$$

This completes the analysis for the edges (a, v) , (a, w) , and (b, v) . The final edge to consider is edge (b, w) .

Case I: Edge (b, w) . We can assume that either $\sum_{i \in \Lambda_{bu}^{(1)}} \alpha'_i = 0$ or the weight y_{aw} on edge (a, w) is equal to rx_{aw} . Otherwise, if there are tours in $\Lambda_{bu}^{(1)}$ with positive weight α'_i and $y_{aw} < rx_{aw}$, then we can modify tours T_i in $\Lambda_{bu}^{(1)}$ by exchanging the values $(T_i)_{aw}$ and

$(T_i)_{bw}$ as long as such modification does not violate the inequality (4.4). Note that after performing these modifications, the resulting tours belong to $\Lambda_{au}^{(1)}$ instead of $\Lambda_{bu}^{(1)}$ (see Figure 4.4) and the process terminates when either there are no more tours in $\Lambda_{bu}^{(1)}$ or the constraint $y_{aw} \leq rx_{aw}$ is satisfied at equality. Therefore, we consider the following two cases.

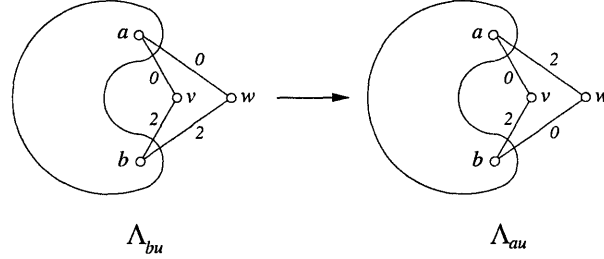


Figure 4.4: Modifying tours in Λ_{bu} .

Case I: Edge (b, w) (i) No tour in $\Lambda_{bu}^{(1)}$ has positive weight.

In this case, each tour T_i in $G_2(u)$ satisfies $(T_i)_{bw} \leq (T_i)_{aw}$ and therefore

$$y_{bw} \leq y_{aw} \leq rx_{aw} \leq rx_{bw}.$$

Case I: Edge (b, w) (ii) Edge (a, w) is saturated (i.e., $y_{aw} = rx_{aw}$) and some tour in $\Lambda_{bu}^{(1)}$ has positive weight.

To analyze this case, recall that $\tilde{x}_{bu} = \max\{2 - (x_{av} + x_{aw}), x_{av} + x_{aw}\}$. Then \tilde{x}_{bu} is equal to the first argument in the maximum if $x_{av} + x_{aw} \leq 1$ and the second if $x_{av} + x_{aw} \geq 1$. We again consider two separate cases.

(A) $\tilde{x}_{bu} = x_{av} + x_{aw}$. In this case, we have

$$2\alpha_T + \alpha_P + 2\alpha_{bu} \leq r\tilde{x}_{bu} = r(x_{av} + x_{aw})$$

and the weight on edge (b, w) is

$$\begin{aligned}
y_{bw} &= \sum_{\alpha_i \in \Lambda_T} \left(\alpha_i \frac{rx_{av}}{\alpha_T} + 2\alpha_i \left(1 - \frac{rx_{av}}{\alpha_T} \right) \right) + \sum_{\alpha_i \in \Lambda_P} \alpha_i + 2 \sum_{\alpha_i \in \Lambda_{bu}} \alpha_i \\
&= rx_{av} + 2\alpha_T \left(1 - \frac{rx_{av}}{\alpha_T} \right) + \alpha_P + 2\alpha_{bu} \\
&= rx_{av} + 2\alpha_T - 2rx_{av} + \alpha_P + 2\alpha_{bu} \\
&= (2\alpha_T + \alpha_P + 2\alpha_{bu}) - rx_{av} \\
&\leq r(x_{av} + x_{aw}) - rx_{av} \\
&= rx_{aw},
\end{aligned}$$

which is at most rx_{bw} by assumption.

(B) $\tilde{x}_{bu} = 2 - (x_{av} + x_{aw})$. In this case, $x_{av} + x_{aw} \leq 1$ and \tilde{x} satisfies

$$\begin{aligned}
r\tilde{x}(\delta(\{u\})) &= r(\tilde{x}_{au} + \tilde{x}_{bu}) \\
&= r(x_{av} + x_{aw} + 2 - (x_{av} + x_{aw})) \\
&= 2r.
\end{aligned}$$

By Equation (4.3), $\sum_i \alpha_i \tilde{T}_i \leq r\tilde{x}$, implying

$$\begin{aligned}
\tilde{y}(\delta(\{u\})) &\leq r\tilde{x}(\delta(\{u\})) \\
4\alpha_T + 2(\alpha_P + \alpha_{au} + \alpha_{bu}) &\leq 2r \\
4\alpha_T &\leq 2(r - (\alpha_P + \alpha_{au} + \alpha_{bu})) \\
2\alpha_T &\leq r - (1 - \alpha_T) \\
\alpha_T &\leq r - 1.
\end{aligned}$$

Since $x_{av} + x_{aw} \leq 1$, we have $x_{av} \leq 1$, $x_{aw} \leq 1$, and

$$x_{bv} \geq 2 - x_{av}, \quad (4.11)$$

$$x_{bw} \geq 2 - x_{aw}. \quad (4.12)$$

We now show we can assume constraints (4.11) and (4.12) are satisfied at equality. We first make the argument for constraint (4.11) and then the assumption for constraint (4.12) will follow by replacing v by w in the argument. We need to show that by letting $x_{bv} = 2 - x_{av}$, any cut involving edge (b, v) remains valid. First, cut $C = \{v\}$ is valid and therefore, any cut C such that $\delta(C)$ contains both edges (a, v) and (b, v) is valid. Otherwise, if $\delta(C)$ contains edge (b, v) but not (a, v) , we can assume by possibly taking complements that C contains vertices a and v but not b . Then $\delta(C \setminus \{v\}) = \delta(C) \setminus (b, v) \cup (a, v)$ is a cut not containing edge (b, v) and therefore $x(\delta(C \setminus \{v\})) \geq 2$. Now, since $x_{bv} \geq 1 \geq x_{av}$, we have $x(\delta(C)) \geq x(\delta(C) \setminus (b, v) \cup (a, v)) = x(\delta(C \setminus \{v\})) \geq 2$ and therefore, C is a valid cut. Now, we have

$$\alpha_T \leq r - 1$$

$$\alpha_T - rx_{av} \leq r - 1$$

$$x_{aw}(\alpha_T - rx_{av}) \leq (r - 1)x_{bw}$$

$$x_{aw}\alpha_T \left(1 - \frac{rx_{av}}{\alpha_T}\right) \leq (r - 1)x_{bw}$$

$$x_{bw} + x_{aw}\alpha_T \left(1 - \frac{rx_{av}}{\alpha_T}\right) \leq rx_{bw}$$

$$(2 - x_{aw}) + x_{aw}\alpha_T \left(1 - \frac{rx_{av}}{\alpha_T}\right) \leq rx_{bw}$$

$$(2 - x_{aw}) \left(1 - \alpha_T \left(1 - \frac{rx_{av}}{\alpha_T}\right)\right) + 2 \left(\alpha_T \left(1 - \frac{rx_{av}}{\alpha_T}\right)\right) \leq rx_{bw}. \quad (4.13)$$

Note that all tours T_i in $G_2(u)$ except those in $\Lambda_T^{(2)}$ satisfy $(T_i)_{bw} = 2 - (T_i)_{aw}$ and therefore, the left hand side of Equation (4.13) is precisely the weight y_{bw} on edge (b, w) .

Case I: All other edges. All other edges in $G_2(u)$ have the same weight after the redistribution as before, so rx dominates a convex combination of tours T_i .

Case II. $rx_{av} > \alpha_T$.

In this case, let

$$\begin{aligned} x'_{av} &= rx_{av} - \alpha_T \\ x'_{aw} &= rx_{aw} - \alpha_T \\ x'_{bv} &= rx_{bv} - \alpha_T \\ x'_{bw} &= rx_{bw} - \alpha_T. \end{aligned}$$

Note that x' is non-negative because $r \min\{x_{av}, x_{aw}, x_{bv}, x_{bw}\} = rx_{av} > \alpha_T$. Now, consider redistributing the weights α_i for $i \in \Lambda_P, \Lambda_{au}, \Lambda_{bu}$ on tours T_i of $G_2(u)$ as shown in Figure 4.5. The figure also labels the classes $\Lambda_T^{(1)}, \Lambda_P^{(1)}, \Lambda_P^{(2)}, \Lambda_{au}^{(1)}, \Lambda_{au}^{(2)}, \Lambda_{bu}^{(1)}$ of modified tours in $G_2(u)$. Note that from Equation (4.3),

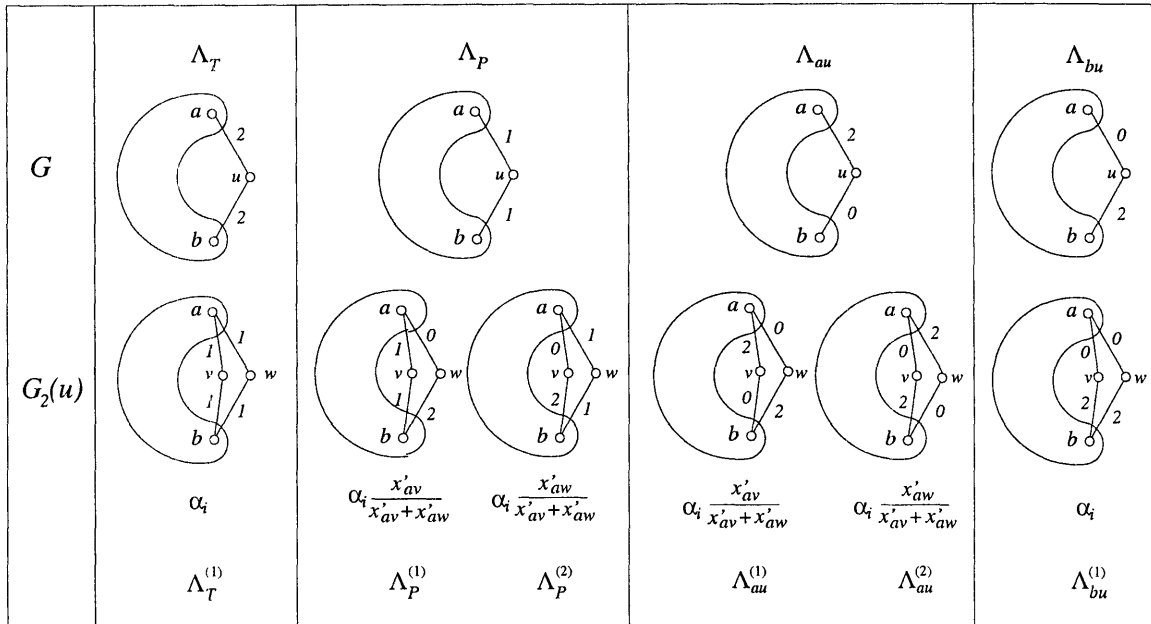


Figure 4.5: Redistribution in Case II.

$$2\alpha_T + \alpha_P + 2\alpha_{au} \leq r(x_{av} + x_{aw}),$$

implying

$$\alpha_P + 2\alpha_{au} \leq r(x_{av} + x_{aw}) - 2\alpha_T = x'_{av} + x'_{aw}.$$

Case II: Edge (a, v) . In the given redistribution, the weight on edge (a, v) is

$$\begin{aligned}
\sum_{\alpha_i \in \Lambda_T} \alpha_i + \sum_{\alpha_i \in \Lambda_P} \alpha_i \frac{x'_{av}}{x'_{av} + x'_{aw}} + 2 \sum_{\alpha_i \in \Lambda_{au}} \alpha_i \frac{x'_{av}}{x'_{av} + x'_{aw}} &= \alpha_T + (\alpha_P + 2\alpha_{au}) \frac{x'_{av}}{x'_{av} + x'_{aw}} \\
&\leq \alpha_T + (x'_{av} + x'_{aw}) \frac{x'_{av}}{x'_{av} + x'_{aw}} \\
&= \alpha_T + x'_{av} \\
&= rx_{av}.
\end{aligned}$$

Case II: Edge (a, w) . Note that the redistribution is symmetric with respect to vertices v and w . Therefore, the argument for the case of edge (a, v) with w in place of v shows that we can redistribute weights on tours T_i so that

$$y_{aw} \leq rx_{aw}.$$

Case II: Edge (b, v) . Suppose the inequality for edge (a, v) is tight, so the weight on (a, v) is equal to rx_{av} . Then each tour T_i in the decomposition satisfies $(T_i)_{bv} = 2 - (T_i)_{av}$, implying the weight on edge (b, v) is at most

$$\begin{aligned}
2 - rx_{av} &\leq 2r - rx_{av} \\
&= r(2 - x_{av}) \\
&\leq rx_{bv}.
\end{aligned}$$

If the inequality corresponding to edge (a, v) is strict, then in every tour T_i with $(T_i)_{bv} = 2$ and $(T_i)_{av} = 0$ (tours in $\Lambda_P^{(2)}$, $\Lambda_{au}^{(2)}$, and $\Lambda_{bu}^{(1)}$), consider the modified tour obtained by swapping the values of edges (a, v) and (b, v) . We perform this exchange of weight from edge (b, v) to edge (a, v) until either the inequality corresponding to edge (a, v) is tight or there is no remaining weight on edge (b, v) in these graphs to move.

If the inequality corresponding to edge (a, v) becomes tight, then we are in the case previously considered with the weight on edge (a, v) equal to rx_{av} . Otherwise, if there is no more remaining weight on edge (b, v) in these graphs to move, then in every tour T_i in

the decomposition, we now have $(T_i)_{bv} \leq (T_i)_{av}$, implying

$$y_{bv} \leq y_{av} \leq rx_{av} \leq rx_{bv}.$$

Case II: Edge (b, w) . Note that the redistribution is symmetric with respect to vertices v and w . Therefore, the argument for the case of edge (b, v) with w in place of v shows that we can redistribute weights on tours T_i so that

$$y_{bw} \leq rx_{bw}.$$

Case II: All other edges. All other edges in $G_2(u)$ have the same weight after the redistribution as before, so rx dominates the convex combination of tours T_i . \square

4.3 Integrality Gap under Operations Φ_1 and Φ_2

In this section, we consider the integrality gap of graphs under the operations Φ_1 and Φ_2 introduced in Section 2.1. We prove that operation Φ_1 preserves integrality gap while operation Φ_2 does not. Thus, all operations we have shown to preserve TSP-perfection and walk-perfection except Φ_2 also preserve integrality gap.

We first show operation Φ_1 preserves s - t TSW integrality gap. For vertices v_1 and v_2 in connected graphs G_1 and G_2 , operation Φ_1 identifies vertices v_1 and v_2 to obtain graph G . Let $s, t \in V(G)$. In the following theorem, H_1 will denote the labeled minor obtained by contracting G_2 to a single vertex in G . The result is graph G_1 where vertex v_1 has label s if $s \in V(G_2)$, label t if $t \in V(G_2)$, labels s and t if $s, t \in V(G_2)$ and is unlabeled if $s, t \in V(G_1) \setminus \{v_1\}$. Similarly, H_2 will denote the labeled minor obtained by contracting G_1 to a single vertex.

Theorem 4.3.1. *If the s - t TSW integrality gaps of H_1 and H_2 are r_1 and r_2 respectively, then the s - t TSW integrality gap of G is $\max\{r_1, r_2\}$.*

Proof. By Theorem 4.1.1, if G has s - t TSW integrality gap r , then both H_1 and H_2 have s - t TSW integrality gap at most r .

Conversely, suppose the s - t TSW integrality gaps of H_1 and H_2 are r_1 and r_2 respectively. Let x be any feasible point in the fractional walk polyhedron $P(G, s, t)$. Let x_1 and x_2 denote the restriction of x to graphs H_1 and H_2 ; then x_1 and x_2 are feasible solutions in $P(H_1, s, t)$ and $P(H_2, s, t)$ respectively. By Theorem 4.2.1 and since all polyhedra considered here are of blocking type, there are s - t traveling salesman walks $p_i^1 \in G_1, p_i^2 \in G_2$ and weights α_i^1, α_i^2 satisfying $\sum_i \alpha_i^1 = \sum_i \alpha_i^2 = 1, \alpha_i^1, \alpha_i^2 > 0$, and

$$\begin{aligned} \sum_i \alpha_i^1 p_i^1 &= r_1 x_1 \\ \sum_i \alpha_i^2 p_i^2 &= r_2 x_2. \end{aligned}$$

Let $p_i^1 \circ_{\Phi_1} p_j^2$ denote the result of combining s - t traveling salesman walks p_i^1 in H_1 and p_j^2 in H_2 . By the vertex degree constraints, $p_i^1 \circ_{\Phi_1} p_j^2$ is an s - t traveling salesman walk in G for any i, j . Similarly, for feasible solutions $x_1 \in P(H_1, s, t)$ and $x_2 \in P(H_2, s, t)$, let $x_1 \circ_{\Phi_1} x_2$ denote the result of combining the two feasible solutions to a feasible solution in G . Then for $r = \max\{r_1, r_2\}$,

$$\begin{aligned} rx &\geq r_1 x_1 \circ_{\Phi_1} r_2 x_2 \\ &= \left(\sum_i \alpha_i^1 p_i^1 \right) \circ_{\Phi_1} \left(\sum_j \alpha_j^2 p_j^2 \right) \\ &= \sum_i \sum_j \alpha_i^1 \alpha_j^2 (p_i^1 \circ_{\Phi_1} p_j^2), \end{aligned}$$

where $\sum_i \sum_j \alpha_i^1 \alpha_j^2 = \sum_i \alpha_i^1 \sum_j \alpha_j^2 = 1$. This shows rx dominates a convex combination of s - t traveling salesman walks in G and therefore, the s - t TSW integrality gap of G is at most r . \square

Corollary 4.3.2. *If the TSW integrality gaps of H_1 and H_2 are r_1 and r_2 respectively, then the TSW integrality gap of G is $\max\{r_1, r_2\}$.*

However, the operation Φ_2 does *not* preserve TSP integrality gap (and therefore does not preserve TSW integrality gap). We demonstrate this with the following example.

Example 4.3.3. Consider the graphs G_1 and G_2 in Figure 4.6, where the path labeled by

k denotes a path of k edges from s to t .

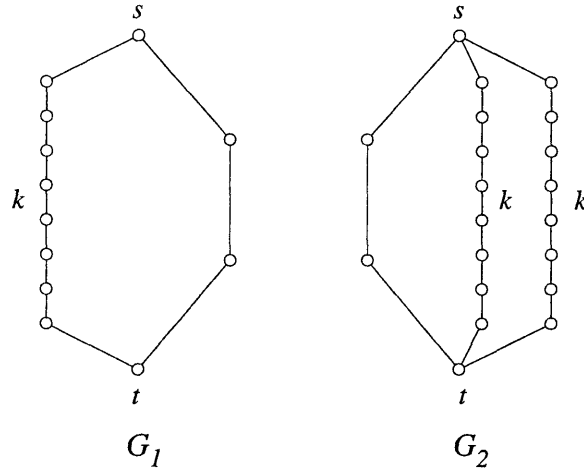


Figure 4.6: Graphs G_1 and G_2 .

Then Φ_2 composes graphs G_1 and G_2 at vertices s and t to obtain graph G in Figure 4.7.

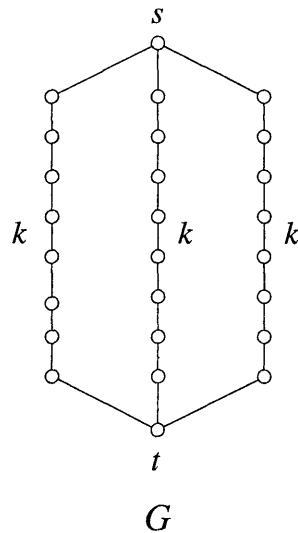


Figure 4.7: Counterexample showing Φ_2 does not preserve TSP integrality gap.

Note that G_1 is TSP-perfect because it is $[M_1, M_2, M_3]$ minor free.

Claim: G_2 has integrality gap $\frac{6}{5}$.

For any cost function on G_2 , let c_1 be the cost of the path of length 3, and let c_2 and

c_3 be the costs on the two other paths of length k . The solution $x_e = 1$ for all edges e is a feasible solution in the fractional TSP polyhedron. We will show that any other extreme point corresponds to a tour and therefore gives a ratio of 1. First, this is true for the case $k = 2$, which is a TSP-perfect graph. Now, suppose it is true for any path lengths k_1, k_2 with $k_1, k_2 \leq k, k_1 + k_2 < 2k$ and suppose x is any extreme point other than $x = 1$. Then $x_e > 1$ for some edge e and by Theorem 3.2.5, the solution $x.e$ (x restricted to the graph with edge e contracted) is an extreme point in $(G_2).e$. This extreme point must contain at least one edge that does not have value 1 (otherwise x would not have been an extreme point) and by assumption, corresponds to a tour. Now, in order for x to be an extreme point, there must be a tight cut constraint containing e and since $x_e > 1$, it follows that $x_e = 2$. Then x corresponds to the tour obtained by extending tour $x.e$ by traveling edge e twice. Therefore, all extreme points other than $x = 1$ correspond to tours. This shows that in order to find the integrality gap for G_2 , we can assume that the optimum LP solution is obtained at $x_e = 1$ for all edges e and has value $c_1 + c_2 + c_3$.

We can obtain a tour of G_2 by taking two of the paths together with twice all but one edge of the third path. By discarding the most expensive edge along this third path, this means that the minimum cost tour has value at most

$$\min \left(\frac{4}{3}c_1 + c_2 + c_3, c_1 + \frac{2(k-1)}{k}c_2 + c_3, c_1 + c_2 + \frac{2(k-1)}{k}c_3 \right).$$

The worst-case ratio between the optimum tour and the LP solution is obtained when the three quantities in the minimum are equal, i.e. when

$$\frac{c_1}{3} = \left(1 - \frac{2}{k}\right)c_2 = \left(1 - \frac{2}{k}\right)c_3 = c.$$

Therefore, $c_1 = 3c$, $c_2 = \frac{kc}{k-2}$ and $c_3 = \frac{kc}{k-2}$. The LP value is then $\left(5 + \frac{4}{k-2}\right)c$ while the minimum cost tour has value $\left(6 + \frac{4}{k-2}\right)c$, for a ratio that attains $\frac{6}{5}$ when k is arbitrarily large.

Therefore, both G_1 and G_2 have integrality gap at most $\frac{6}{5}$ while the integrality gap of G is arbitrarily close to $\frac{4}{3}$ for sufficiently large path length k .

Chapter 5

Appendix

We complete the proofs of TSP-perfection for several graphs by computational methods using the software package `polymake`.

`Polymake` is a versatile software system for computation on convex polyhedra and finite simplicial complexes. It was developed by the Discrete Geometry group at the Institute of Mathematics of Technische Universität Berlin by authors Evgenij Gawrilow and Michael Joswig ([16] and [17]). In addition to implementing many computational algorithms on polytopes, `polymake` includes a large array of interfaces to other software packages. More details on `polymake` can be found at the website

<http://www.math.tu-berlin.de/polymake/>

For each graph G under consideration, we first generate the cut and nonnegativity constraints defining the fractional traveling salesman polyhedron $P(G)$ or fractional traveling salesman walk polyhedron $P(G, s, t)$. In `polymake`, an inequality of the form $a_0 + a_1x_1 + \dots + a_dx_d \geq 0$ is input as vector (a_0, a_1, \dots, a_d) . We then use `polymake` to generate the extreme points of each polyhedron, which are also given in vector format. Thus, (x_1, x_2, \dots, x_d) is the solution with value x_1 on edge 1, x_2 on edge 2, etc. Finally, we verify that each extreme point x corresponds to a traveling salesman tour. In the case of the traveling salesman problem (or the traveling salesman walk problem with $s = t$), this means checking that x_e is integral for every edge e and every vertex has even degree in G_x . In the case of the traveling salesman walk problem with $s \neq t$, this means checking x_e is integral for every

edge e , that s and t have odd degree and every vertex $v \neq s, t$ has even degree in G_x .

5.1 Graphs $K_4^{(1)}, K_4^{(2)}, W^{(1)}, K_5^{(1)}, K_5^{(2)}, P_3^{(1)}, P_3^{(2)}$, and $P_3^{(3)}$

In this section, we give a proof of Lemma 3.2.1.

Lemma 3.2.1 Graphs $K_4^{(1)}, K_4^{(2)}, W^{(1)}, K_5^{(1)}, K_5^{(2)}, P_3^{(1)}, P_3^{(2)}$, and $P_3^{(3)}$ (see Figure 5.1) are TSP-perfect.

Proof of Lemma 3.2.1. The inequalities and extreme points for each of $K_4^{(1)}, K_4^{(2)}, W^{(1)}, K_5^{(1)}, K_5^{(2)}, P_3^{(1)}, P_3^{(2)}$, and $P_3^{(3)}$ are enumerated in the following tables. We verify that each extreme point corresponds to a traveling salesman tour, proving the lemma. \square

$K_4^{(1)}$ INEQUALITIES

-2000000000111	-2000000001100	-2000000001011	-2000000011000	-2000000011111
-2000000010100	-2000000010011	-2000001100001	-2000001100110	-2000001101101
-2000001101010	-2000001111001	-2000001111110	-2000001110101	-2000001110010
-2000010000000	-2000010000111	-2000010001100	-2000010001011	-2000010110000
-2000010111111	-2000011010100	-2000011010011	-2000010100001	-2000010100110
-2000010101101	-2000010101010	-2000010111001	-2000010111110	-2000010110101
-2000010110010	-2001100000000	-2001100000111	-2001100001100	-2001100001011
-2001100011000	-2001100011111	-2001100010100	-2001100010011	-2001111000001
-2001111100110	-2001111101101	-2001111101010	-2001111110001	-2001111111110
-2001111101011	-2001111100100	-2001010000000	-2001010000111	-2001010001100
-2001010001011	-2001010110000	-2001010111111	-2001010101000	-2001010100011
-2001001000001	-2001001000110	-2001001010101	-2001001010101	-2001001111001
-2001000111110	-2001000110101	-2001000110010	-2001000000010	-2001000000101
-2011000001110	-2011000001001	-2011000011010	-2011000011101	-2011000010110
-2011000010001	-2011011000011	-2011011001000	-2011011010111	-2011011010000
-2011011110101	-2011011111000	-2011011101011	-2011011100000	-2011011100010
-2011110000101	-2011110001110	-2011110001001	-2011110110101	-2011110110101
-2011110101010	-2011110100011	-2011101000011	-2011101000100	-2011101010111
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-2110010100011	-2110010100001	-2110010100110	-2110010101101	-2110010101010
-2110010101001	-2110010100010	-2110010101010	-2110010101101	-2110010101010
-2110010101101	-2110010111110	-2110010110101	-2110010110010	-2110010000000
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-2111100010100	-2111100010011	-2111111000001	-2111111000110	-2111111001010
-2111111010101	-2111111110001	-2111111111010	-2111111101010	-2111111100010
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-2111010011111	-2111010010100	-2111010010011	-2111001000001	-2111001000110
-2111000101101	-2111000101010	-2111000111001	-2111000111110	-2111000110101
-2111000110010	-2101000000010	-2101000000101	-2101000001110	-2101000001001
-2101000011010	-2101000011101	-2101000010110	-2101000010001	-2101011000011
-2101011000100	-2101011010111	-2101011010000	-2101011110101	-2101011110010
-2101011101011	-2101011100000	-2101110000010	-2101110000101	-2101110001110
-2101110001001	-2101110101010	-2101110101101	-2101110101010	-2101110100001
-2101101000011	-2101101000100	-2101101001011	-2101101001000	-2101101001010
-2101101011100	-2101101010111	-2101101010000	-2100100000010	-2100100000101
-2100100001110	-2100100001001	-2100100010101	-2100100011101	-2100100010100
-2100100010001	-2100111000011	-2100111000100	-2100111001011	-2100111010000
-2100011110101	-2100011111100	-2100011110101	-2100011100000	-2100010000010
-2100001000101	-2100001000110	-2100001000001	-2100001010101	-2100001011010
-2100001010100	-2100001010001	-2100001000011	-2100001000100	-2100001010111
-2100000101010	-2100000111011	-2100000111000	-2100000110111	-2100000110000
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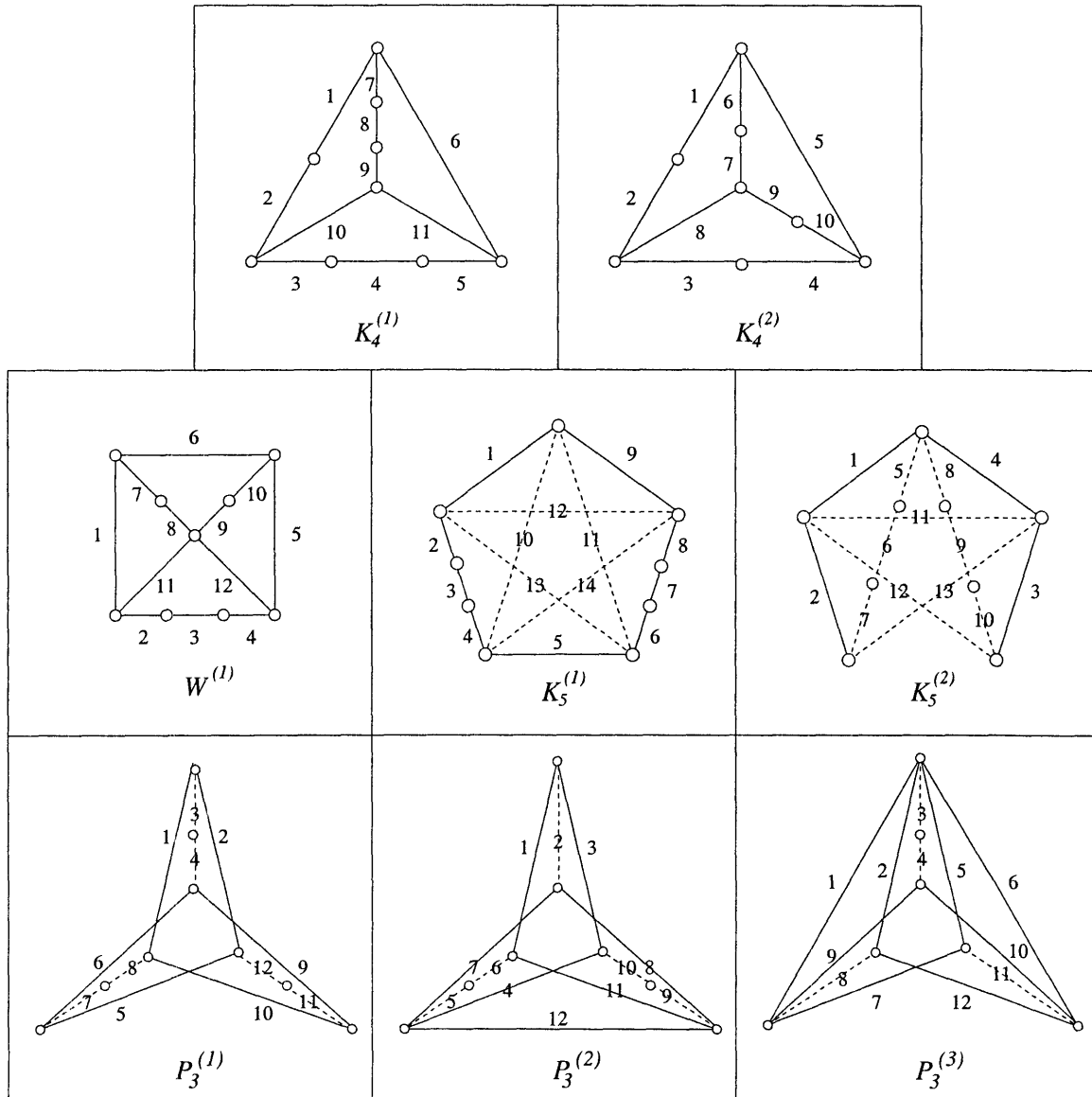


Figure 5.1: Edge labels for $[M_1, M_2, M_3]$ minor free bricks not including wheels and propellers.

$K_4^{(1)}$ EXTREME POINTS

02202111121	02111220211	02111111110	02111222011	02111202211
02220111121	02022111121	02022222220	02022022222	02022220222
02022222022	02022202222	02202222220	02202022222	02202202222
02202222022	02202220222	02220222220	02220022222	02220220222
02220222022	02220202222	02222022202	02222022220	02111022211
02222202220	02222202202	02222220220	02222220202	02222220220
02222222002	02222222200	02222111101	11022120211	11022211110
11022122011	11022102211	22022022202	22022002222	22022022022
11022011112	22022020222	22022202220	22022202202	22022220220
22022220202	22022222020	22022222002	22022222200	22022111101
11220120211	11220211110	11220122011	11220102211	22220022202
22220002222	22220022022	11220011112	22220020222	22220202220
22220202202	22220220220	22220220202	22220222020	22220222002
22220222200	22220111101	11111102220	11111102202	11111122020
11111122002	11111122200	11111120202	11111120220	22222022200
22222022002	22222022020	22222020202	22222020220	22222002202
22222002220	22111020211	11111011101	11222011110	22111022011
22111002211	22202111101	22202222200	22202222002	22202222020
22202220202	22202220220	22202202202	22202202220	22202020222
11202011112	22202022022	22202002222	22202022202	11202120211
11202211110	11202122011	11202102211	20222111101	20222222200
20222222002	20222222020	20222220202	20222220220	20222202202
20222202220	20111022211	20222202220	20222202202	20220202222
20220222022	20220220222	20220022222	20220222220	20202220222
20202222022	20202202222	20202022222	20202222220	20022202222
20022222022	20022220222	20022022222	20022222220	20202111121
20111220211	20111111110	20111222011	20111202211	20220111121

$K_4^{(2)}$ INEQUALITIES

-200000000011	-2000000011110	-200000001101	-200000011000	-200000011011
-200000010110	-200000010101	-200011000001	-200011000010	-200011011111
-20001101100	-20001111001	-20001111010	-20001110111	-20001110100
-20011000000	-20011000011	-20011001110	-20011001101	-20011011000
-20011011011	-20011010110	-20011010101	-200101000001	-20010100010
-20010101111	-20010101100	-20010111001	-20010111010	-20010111011
-20010110100	-20110000100	-20110000111	-20110001010	-20110001001
-20110011100	-20110011111	-20110010010	-20110010001	-20111100101
-20111100110	-20111101011	-20111101000	-20111111101	-20111111110
-20111110011	-20111110000	-20101000100	-20101000111	-20101001010
-20101001001	-20101011100	-20101011111	-20101010010	-20101010001
-20100100101	-20100100110	-20100101011	-20100101000	-20100111101
-20100111110	-20100110011	-20100110000	-21100000000	-21100000011
-21100001110	-21100001101	-21100011000	-21100011011	-21100010110
-21100010101	-21101100001	-21101100010	-21101101111	-21101101100
-21101111001	-21101111010	-21101111011	-21101111010	-21111000000
-21111000011	-21111001110	-21111001101	-21111011100	-21111011011
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-21010011111	-21010010010	-21010010001	-210111000101	-21011100110
-21011101011	-21011101000	-21011111101	-21011111110	-21011110011
-21011110000	-21001000100	-21001000111	-21001001010	-21001001001
-21000101110	-21000101111	-21001011010	-21000101000	-21000100101
-21000100110	-21000101011	-21000101000	-21000111101	-21000111110
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$K_4^{(2)}$ EXTREME POINTS

0220111211	0211202111	0211111120	0211111102	0211220111	0202111211
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0220222220	0220022222	0220220222	0220202222	0222022022	0211022111
0222022220	0222022202	0222202202	0222202220	0222202022	0222220220
0222202022	0222220022	0222222002	0222111011	0222222020	1102102111
1102211120	1102211102	1102120111	2202022022	2202002222	1102011122
2202020222	2202202202	2202202220	2202202022	2202220220	2202220202
2202220022	2202222002	2202111011	2202222020	2211002111	1111011011
1122011120	1122011102	2211020111	2222002202	2222002220	2222002022
2222020220	2222020202	2222020022	2222022002	2222022020	1111120220
1111120202	1111120022	1111122020	1111122002	1111102022	1111102220
1111102202	2220222020	2220111011	2220222002	2220220022	2220220202
2220220220	2220202022	2220202220	2220202202	2220020222	1120011122
2220002222	2220022022	1120102111	1120211120	1120211102	1120120111
2022222020	2022111011	2022222002	2022220022	2022220202	2022220220
2022202022	2022202220	2022202202	2022022202	2022022220	2011022111
2022022022	2020202222	2020220222	2020022222	2020222220	2020222202
2002220222	2002202222	2002022222	2002222202	2002222220	2020111211
2011202111	2011111120	2011111102	2011220111	2002111211	

$K_5^{(1)}$ INEQUALITIES

```

-2000000001100101 -2000000011000000 -200000010100101 -2000000110000000
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-200001011101111 -200001001001010 -200001000101111 -200001000010001
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```

$K_5^{(1)}$ EXTREME POINTS

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 01110022011012 01110022101011 02021111110020 02021111200021
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 0022122011020 00221111011110 00221220011220 00221220011022
 00221220101021 00221220211020 00221220002111 00221022011220
 00221022011022 00221022101021 00221022211020 00221022002111
 00220022002222 00220220002222 00220111002112 00220202002222
 00220022112021 00220022022022 00220022022220 00220022222020
 00220220112021 00220220022022 00220220022220 00220220222020
 00220111121020 00220222022020 00220111022110 00220202222020
 002202022220 00220202022022 00220202112021 00220022202022
 0022020202022 00220202202022 00220222002022 00222022002220
 00222022002022 00222022202020 00222202002220 00222202002022
 0022220202020 002222220002220 002222111101020 00220111020112
 00221111002021 00222222002020 00220202022202 0022020202222
 0022022022202 00220220020222 00220222020202 00221202011202
 0022202020202 00221111020201 00222220020202 00221220011202
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$K_5^{(1)}$ EXTREME POINTS (continued)

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022202020222000	02220222020002	022202220220000	02220222110001
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02220220200022	02220220202002	02220111101002	02220202202002
02220202200022	02220222200002	01110022200121	01110022202101
01110220200121	01110220202101	02220111101200	01110111101101
01110202202101	01110202200121	02220022200220	02220022202200
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01111022200210	02220220022002	0222022002002	02220222020000
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02222111101000	02221111110000	12021022001101	22021022000111
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120202002120120	22020202112001	22020202020220	12020202020210
22020202020022	12020202021012	12020202111011	22020202110021
22020022022200	12020022122100	22020022022002	22020022222000
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12022022201010	22022022000202	12022022100102	22022022002200
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22022022200002	12022202001210	22022202000022	12022202001012
22022202000220	12022202100120	22022202200020	12022202201010
22022202000202	12022202100102	12022202102100	22022202002200
22022202002002	22022202202000	12021202111100	12022202120100
22022202020200	22021202011200	22021202011002	22022202020002
22022202110001	22021202211000	22022202220000	22022202200002
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12022220100120	22022220200020	12022220201010	22022220000202
12022220100102	22022220002200	12022220102100	22022220002002
22022220202000	22021220011200	22022220020200	12022220120100
12021220111100	22021220011002	22022220020002	22022220110001
22021220211000	22022220220000	22022220200002	22021111000021
12021111001011	22022111000110	22022222000020	12022222001010
2202222000200	22021111000201	22021111002001	22022222002000
2202122201000	22022222020000	22022222020000	22022222200000
12022111100010	12022222100100	120211111000101	22021111200001
22022111101000	22021111110000	11110022100021	11110022010022
211100222000121	11110022001111	11110022010220	21110022002101

$K_5^{(1)}$ EXTREME POINTS (continued)

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11110202001111	21110202000121	11110202010022	11110202100021
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12220202001012	22220202000220	12220202100120	22220202200020
12220202201010	12220202102100	22220202002200	22220202002002
22220202202000	11110202012200	11110202012002	11110202212000
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11110220012002	11110220212000	12220202001210	22220202000022
12220202001012	22220202000220	12220202100120	22220202200020
12220202201010	12220202102100	22220202002200	22220202002002
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12220220100120	22220220200020	12220220201010	22220220002200
12220220102100	22220220002002	22220220202000	11110220012200
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11110222010002	22220111000110	22220222000020	12220222001010
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22222220000002	211112220000101	11111220001002	21111220000012
11112220010200	11111220001200	11112022210000	11111022201000
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22202220200002	22202220220000	22201202211000	22202202110001
222022020200002	22201202011002	22201202011200	22202202020200
12202202120100	12201202111100	22202202202000	22202202002002
22202202002200	12202202102100	12202202100102	22202202000202
12202202201010	22202202200020	12202202100120	22202202000220
12202202001012	22202202000022	12202202001210	22202022200002
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22200022202220	12200022020210	22200022020022	12200022021012
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12201022210010	22201022101001	12201022100011	12202022010101

$K_5^{(1)}$ EXTREME POINTS (continued)

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12201202100011	12200222010101	12200220010121	12200220012101
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20222222002000	20221111002001	20221111000201	20222222000200
10222222001010	20222222000020	2022111000110	10221111001011
20221111000021	202222220200002	20222222020000	20221220211000
202222220110001	202222220020002	20221220011002	10221220111100
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202222220000202	102222220102100	202222220002200	102222220100102
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202222020002200	10222202102100	10222202100102	20222202000202
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20221022011200	20222022202000	20222022002002	10222022102100
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20222022200020	10222022100120	20222022000220	10222022001012
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20220222200002	20220202200022	10220202201012	20220202202002
20220111101002	20220220202002	20220220200022	10220220201012
20220022202002	20220022200022	10220022201012	20220111121000
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20220202220020	10220202221010	202201111011001	10220111021100
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10220022221010	20220022220020	10220022120120	20220022112001
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10221202100011	10220222010101	10220220010121	10220220012101
10220202010121	10220202012101	10220022010121	10220022012101
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$K_5^{(2)}$ INEQUALITIES

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$K_5^{(2)}$ EXTREME POINTS

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00112221111020	02101111111001	0120111111010	01101111111100	0112202022011
0112202202011	0112202220011	02112021111020	01111111022010	02211111022001
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1201022220120	1101022222001	2202022222000	1201022222100	2202202022020
2202202202020	2202202220020	120120222120	1201202202120	1201202220120
1101202222001	2202202222000	1201202222100	2202220022020	2202220202020

$K_5^{(2)}$ EXTREME POINTS (continued)

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1121220220001	11212202020001	1211220022010	2222220022000	1221220022100
1211220202010	22222202020000	1221220202100	1211220220010	2222220220000
1221220220100	22112201110000	1122111202000	1111111111000	1122111220000
11221110220000	22112021110000	1221202220100	2222202220000	1211202220010
1221202202100	22222022020000	1211202202010	12212022022100	2222202202200
1211202022010	11212020222001	1121202220001	1121202202001	1120111222000
1120111220200	1110111220110	1120111220020	1120111202200	1110111202110
1120111202020	1120111022200	1110111022110	1120111022020	2220220222000
2220220220200	2210220220110	2220220220200	2210220202110	2220220022200
2210220022110	21202202020101	2120220220101	2120220022101	2220220220020
2220220202020	2220220022020	2220220202002	2220220220002	2220220022002
2220202222000	2220202220200	2210202220110	2220202202200	2210202202110
2220202022200	22102020222110	2120202022101	2120202220101	2120202202101
2220202220020	2220202202020	2220202022020	2220202022002	2220202220002
2220202220002	2220022222000	2220022220200	2210022220110	2220022220101
2210022202110	2220022202200	2220022202020	2220022202020	2220022202002
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1021222022100	2022222022000	1011222022010	1021222022102	10212220220102
1021220022102	2022220202002	2011220111002	2022220220002	2022220220002
1021202022102	1021202220102	1021202202102	2022202022002	2022202220002
2011202111002	2022202202002	1021022202102	1021022220102	1021022220102
2022022202002	2011022111002	2022022220002	2022022022002	2020202222002
2020220222002	2020022222002	2020222220200	2020222220200	2010222220110
2020222202020	2020222202200	2010222202110	2020222202020	2020222022200
2010222022110	2020222022020	2020222022002	2020222220002	1010222111100
1020222111010	2020222202002	2020220202202	2020220220202	2020220022202
2020202022202	2020202220202	2020202202202	2020022202202	2020022220202
2020022022202	1010202111102	2010202022112	2010202202112	2010202220112
2010220220112	2010220202112	2010220022112	10102220111102	2010022220112
2010022202112	2010022022112	1010022111102	1020202111012	2020202022022
2020202202022	2020202220022	2020220220022	2020220202022	2020220022022
1020220111012	2020022220022	2020022202022	2020022022022	1020022111012
2002220222002	2002202222002	2002022222002	2002022202202	2002022220022
1002022111012	2002022202022	2002220022022	2002220220022	1002220111012
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2002222202020	1001222022120	2002222022020	2000222222200	2000222220220
2000222202220	2000222022220	1000222111210	2000222022022	2000222220022
2000222202022	1002222111010	2000222222002	1000222111012	1001220222102
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2000022022222	1000022111212	1001022202122	1001022220122	1001022202122
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1011202220012	2021111022001	2021111220001	2021111202001	1011220220012
1011220202012	1011220022012	1011022220012	10110222202012	1011022202012
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1000111220121	1000111202121	1001111111011	1000111222101	

$W^{(1)}$ EXTREME POINTS

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002221111120 002201111122 002212021121 002212201121 002211110221
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022011110221 022011112021 011112201110 011122200211 011222202011
022211112001 011111112010 022211110201 011111110210 011122022011
011122020211 011112021110 022201111102 011101111111 022201111120
020211112021 020211110221 020212201121 020212021121 020201111122
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02022220220 02022222020 02022202220 02022202220 02022022022
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02202022202 02202022220 02202022220 02202222020 02202222020
02202220220 02200220222 02200202222 02202022022 02202202022
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12022010212 120210111111 120220112012 220210201121 220220202022
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220222020220 220222022020 22022202200 220222000202 220212201101
220222202002 220222200220 220222202020 220222202200 120222110210
22022220200 120222112010 21022200211 211102200211 111101201101
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211100022211 111100112201 211100202211 222200022202 22220022202
222200022220 122200112210 222200202220 222202022002 22220202022
22220202220 222202022020 222202022200 222202200202 22220220022
222202200220 222202202020 222202202200 222202110210 22220222020
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111111202200 111111202020 111111200220 111111202002 111111200202
222220222000 122220112010 222220220200 122220110210 222220220220
222220202020 222220200220 222220202002 222210201101 222220200202
222220022200 222220022020 222220020220 22222002022 222210021101
222220022002 211120200211 211120202011 211110201110 111120112001
111110111100 111120110201 211110021110 211120022011 211120020211
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222010201121 122020110212 122010111111 122020110212 222010021121
222020022022 222020020222 222020222200 222002022202 22200222022
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102210111111 102220110212 202210021121 202220022022 202220022022
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202202222002 202200202222 102200112212 202200022222 102201021112
202202020222 202202022022 102202110212 102202112012 20220202222
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$P_3^{(1)}$ EXTREME POINTS

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011111220202	011111220022	011111022220	011111022202	022201111120
022201111102	011102021211	011101110111	011102221011	011102201211
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002202202222	002222022220	002222022202	002211021211	002222022220
002222202202	002211201211	002220202222	002220022222	002210112111
002220222202	002220222220	002202222022	002222202220	002222202202
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020222202222	020202220222	020222202202	020222202220	020222202202
020222022220	020222222002	020222222020	020222220022	020222220202
020222220220	020211221011	020202222022	020202222022	020202220222
022002222222	022002202222	022001111122	022002022222	022002022222
022020022222	022002220222	022022022222	022022200222	022002220222
022022022220	022022022202	022022200222	022022220202	022011221011
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022020222202	022020222220	022020222022	022020022022	022020202222
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022202220220	022202220022	022220020222	022220200222	022202002220
022220022220	022220202220	022220222022	022220220022	022220220202
022220220220	022220222020	022220222002	011120221011	011111020222
011111200222	110220022111	110220202111	110211201102	110211201120
220211201011	110212112002	110212112020	110212110022	110212110202
110211102202	220211021011	110211021102	110211021120	110202022111
220201111102	220201111120	110201111011	110202020211	110200222111
220202202202	220202202220	220202022202	220202022220	220200222202
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220222202002	112200220111	121102201102	121102201120	211102201011
121101112002	121101112020	1211011110202	121101110220	121101110022
121102021102	121102021120	211102021011	222200220022	222200220202
222200220220	121100221102	121100221120	222200222020	222200222002
211100221011	222202022002	222202022020	222202020202	112202020111
222202020220	222202020022	222202202002	222202200220	112202200111
222202020022	211111020220	211111020022	211111202002	211111022020
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211120021011	222022202002	222022202020	222022200202	112022200111
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222022020202	222022202020	222022202002	222020202022	112010112022
222020022022	222020202202	222020202220	112010112220	112010112202
222020022220	222020022202	222002220022	222002220220	112002220111
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222002202022	222000222220	222000222202	222002022220	222002022202
222002202220	222002202202	112020022111	112020202111	112011201102
112011201120	222011201011	112012112002	112012112020	112012110022
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202222022002	202220222002	202220222020	101120221120	101120221102
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202220200222	202220022222	202220022022	202202022022	202220220222
202220200222	202220202222	101101110222	202200220222	202020222022
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202022220202	202022220220	202011221011	202022220220	202022202220
202022202202	202022022220	202002222022	202022200222	202022020222
202002220222	202020022222	202020202222	202002022222	202002022222
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200202220222	200201111122	200202022222	200220202222	200220022222
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101111200111	1011222201102	1011222201120	202211201011	101121112002
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101111020111	101122221102	1011222201120	101101112022	101102201122
101102021122	1011002221122	200211201211	200211021211	200210112111
200212110111	200221111102	200221111120		

$P_3^{(2)}$ INEQUALITIES

```

-20000000001100 -200000110000000 -200000110001100 -20011000000100
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-2100001001110 -21000100000010 -2100010001110 -2101101000110
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-20100000111100 -20100111110000 -2010011111100 -20111000110100
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-2011010000010 -2011001001110 -2011001000010 -2110111001100
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0000010000000 0000001000000 0000000100000 0000000100000
000000001000 0000000000100 0000000000001 0000000000001

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$P_3^{(2)}$ EXTREME POINTS (continued)

211120012001	121120002011	121120000211	112022010210	222022020200
211022011100	222022022000	112022012010	222022002200	222022002020
222022002002	222022000220	222022000202	222020202200	121020201110
222020200202	222020200220	222020202002	222020202020	222020222000
112020212010	222020220200	112020210210	222020110201	222020112001
222020020202	222020022002	222020002202	222020002022	222020000222
222002112001	222002110201	112002210210	222002220200	112002212010
222002222000	222002202020	222002202002	222002200220	222002200202
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200222202020	200222200220	200022202202	200022112201	200222202002
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200202022022	200102021121	200202020222	200202022220	200211022011
200111021110	200211020211	200220022220	200220020222	200120021121
200220022022	200220110221	200120111120	200220112021	200022022220
200011022211	200020222220	200020022222	200020112221	200002222220
200002022222	200002112221	200222022200	101002021112	101002201112
101002221110	101002111111	101011021101	101022021110	202011110210
101011111100	202011112010	101011201101	101020111111	101020221110
101020201112	101020021112	202011200211	202011202011	101220021110
101120020211	101120022011	101120110210	101120112010	101120202011
101120200211	202120021101	202120111100	101111200220	101111202020
101111200202	101111202002	101111202200	101111222000	101111120011
101111110201	101111220200	101111022002	101111020202	202102111100
202102021101	101202021110	101102020211	101102022011	101102110210
101102112010	101102202011	101102200211	200211202011	200211200211
200011202211	200011112210	200211110210	200211112010	200111201110

$P_3^{(3)}$ INEQUALITIES

-20011000000000	-20000101000010	-20011101000010	-20100000100001
-20111000100001	-20100101100011	-20111101100011	-20001000001100
-20010000001100	-20001101011110	-20010101011110	-20101000011101
-20110000011101	-20101101111111	-20110101111111	-20000010000111
-20011010000111	-2000011100101	-2001111100101	-2010001010110
-201110101010110	-20100111110100	-20111111110100	-2000101001011
-200100100101011	-2000111101001	-2001011101001	-2010101011010
-201100101101010	-2010111111000	-2011011111000	-2100000111000
-2101100111000	-2100010011010	-2101110011010	-2110000101001
-2111100101001	-2110010001011	-2111110001011	-2100100110100
-21010001110100	-2100110010110	-2101010010110	-2110100100101
-2111000100101	-2110110000111	-2111010000111	-2100001111111
-21011011111111	-2100011011101	-2101111011101	-2110001101110
-211110110110110	-2110011001100	-2111111001100	-2100101110011
-2101001110011	-2100111010001	-2101011010001	-2110101100010
-2111001100010	-2110111000000	-2111011000000	-2111011000000
01000000000000	00100000000000	00010000000000	00001000000000
00000010000000	00000001000000	00000000100000	00000000010000
00000000001000	00000000000100	00000000000010	00000000000001

$P_3^{(3)}$ EXTREME POINTS

000211110201	000211112001	00110101011021	001101211001	001121011001
002211110001	001110101202	001110002112	001110110101	001110011011
002011112001	002011110201	000211121100	000202121110	000211011111
000211101102	000222011101	000202211101	000202011121	000202112011
000202110211	000202101112	000220011121	002011121100	002002121110
002002011121	002002211101	002022011101	002011101102	002011011111
002002112011	002002110211	002002101112	002020011121	001101121010
001101022120	001101222100	002200110211	001101110111	001101101012
001101002122	002202110011	001101202102	001110022110	001110121200
001110121002	001110121020	001110101022	001110020112	001121002102
001121022100	001112101002	001112121000	012001011120	012001211100
012021011100	022011101100	012010101101	012010011110	012001112010
022000101112	012001110210	012001101111	022002101110	012010112200
012010112002	012010112020	012010110202	012010110220	012010002211
0120121110200	012012112000	0211100002110	011120002101	011120011200
021110101200	011120011002	011120011020	021110101002	021110101020
012210110002	012210110020	012210110200	011111101001	021110101000
012212110000	011111110100	011111101010	011122011000	021121002100
011102211000	021101202100	011102011020	021101002120	011100200121
011100011022	011100211020	0111000211002	0111000202101	011100211200
012201110010	021101101010	011100011220	011100002121	011100101011
011100110110	010212112000	010212110200	010210002211	010210110220
010210110202	010210112020	010210112002	010210112200	020202101110
010201101111	010201110210	020200101112	010201112010	010201011120
010201211100	010221011100	020211101100	010210101101	010210011110
000211002212	000211002212	000211022210	000222022200	000222222000
000202022220	000222002202	000202202202	000202002222	002002022220
002002222200	002022022200	002011022210	002002002222	002002202202
002022002202	001101200122	000202222020	000202220220	000202200222
000202202022	002002222020	002002220220	002002200222	002002202022
001101220120	002202220020	002222000222	000220002222	002020022222
000220022220	002020022220	000220101112	000220110211	000220112011
001110200112	002220110011	002020112011	002020101112	002020110211
000220121110	000220211101	000211222010	000211220210	000220222000
000220222002	000220222020	000220222020	000220222020	000222220200
000222222000	000220202202	000211200212	000211202012	000222200202
000222202002	000220200222	000220202022	002020211101	002020121110
002011222010	002011220210	002020222000	002020222002	002020222020
002020220202	002020220220	002022220200	002022220000	002020202202
002011200212	002011202012	002022200202	002022202002	002020200222
002020202022	001110220110	002211220010	002211200012	002220220002
002220220020	002220220200	002220200022	002220200202	001121200102
002222200002	001121220100	002222220000	002200011121	002201011112
002200211101	002200202202	002200002222	002200022202	002220011101
002200121110	002200112011	002200022220	002200222200	002200222002
002200222020	002200022022	002202022020	002202222000	002202002202
002220220002	002200202022	002220222000	002220022002	002220022020
002220002022	002211002012	002220002002	002220202000	002211022010
002200200222	002200220220	000211020212	000202022022	000202222002
000202020222	000202220202	000211022012	000222020202	000222022002
001101020122	001101220102	002202020022	002202220002	002222020002
001121020102	002211020012	002022022002	002022020202	002011022012
002002220202	002002020222	002002220002	002002022022	002020202222
000220020222	000220022022	002200202022	020220011101	010210020211
002200020222	002200110211	020200011121	020200121110	020200112011
020200211101	020200222200	020200222002	020200222020	020200022022
020200022220	020202022020	010201222001	020202222000	020200022022
010201022021	020202022020	010201220201	020202020220	020202220200
020200220202	010201020221	020220022200	020220022002	020220022020
020200220220	010210022011	020211020210	020222020200	010221020201
020220020202	020220020220	020211022010	020220002202	020220022020
010221022001	020221002210	010221002201	020222002200	020202022020
020200002222	010201202201	010201002221	020200200222	020200202022
010201200221	010201202021	020202202020	020202200220	020220002220
020220101110	010210200211	010210202011	020211202010	020211200210
020220202200	020220202002	020220202020	020220200202	020220200220
020222200200	010221200201	010221202021	020222202000	022000011121
022000110211	022000211101	012010020211	022020011101	022000121110
022000112011	022000022220	022000222200	022000222002	022000222020
022000022022	012001022021	022001022020	012001222001	022002220000
022000020222	022000220202	012001020221	012001220201	022002020200
022002220200	022000220220	012010022011	022020022200	022020022002
022020022020	022020020202	022020020220	022011020210	022020202000
012021020201	012021022001	022022022000	022011022010	022000002222
022000202202	022020002202	012001202201	012001202201	022000202220
022002202200	022022002200	012021002201	022011002210	022000200222
022000202022	012001200221	012001202021	022002202020	022001220020
022020002220	012010200211	022020101110	012010202011	022011200210
022020002220	022020202200	022020202002	022020202020	022020200202

$P_3^{(3)}$ EXTREME POINTS (continued)

022020200220	022022200200	012021200201	012021202001	022022202000
022200101110	022200002220	022200202200	022200202002	022200202020
022200002022	0122201002021	022202002020	0122201202001	022202202000
022200110011	011100020121	011100220101	022200220002	022200220020
022200220200	022200020022	022200020220	0122201020021	021101020120
022202020020	021101220100	0122201220001	022202220000	022200200022
022200200220	021101200120	0122201200021	022202200020	012210002011
022220002200	022220002002	022220002020	012210020011	021110020110
011120020101	022220020002	022220020020	022220020200	011120200101
021110200110	012210200011	022220200002	022220200020	022220200200
012221002001	022222002000	022211002010	021121020100	012221020001
022222020000	022211020010	021121200100	012221200001	022222000000
022211200010	120210001110	110220001101	210210000211	120221001100
120201201100	120201001120	110200201101	110200100211	110200001121
110200111110	220200101110	110200102011	220200002220	110200012220
120201102010	220200202200	220200202002	220200202020	110200212200
110200212002	110200212020	110200012022	220200002022	210201002021
220202002020	110202012020	210201202001	220202202000	110202212000
110200010222	220200002222	110200210202	220200200202	120201100210
210201000221	210201200201	220202000220	110202010220	220202200200
110202210200	110200210220	220200200220	210210002011	220220002200
110220012200	120210102200	220220002002	110220012002	220220002020
110220012020	120210102002	120210102020	220220000202	110220010202
120210100202	220220000220	110220010220	120210100220	220211000210
220222000200	210221000201	110211010210	110222010200	210221000201
220222002000	220211002010	110222012000	110211012010	120212100200
110211100201	110211102001	120212102000	121100100110	112200100011
121100001220	211100000121	122201100010	121100201200	211100200101
222200200002	121100201002	222200200020	121100201020	222200200200
112200210002	112200210020	112200210200	112200010022	222200000222
121100001022	222200000220	112200010220	212201000021	111101001021
221101000120	222202000020	121102001020	111101010120	112202010020
221101200100	212201200001	222202200000	111101201001	121102201000
111101210100	112202210000	221121000100	212221000001	222220000000
222211000010	111121001001	121122001000	121111001010	111121010100
112222010000	112211010010	122212100000	112211100001	121111000100
122210100200	122210100020	122210100002	112220010200	222200002000
112220010020	121120001020	222220000020	112220010002	121120001002
222220000002	111110100101	111110001010	121120001200	211120000101
221110000110	111110001011	212210000011	122012102000	112011102001
112011100201	122012100200	112011012010	112022012000	222011002010
222022002000	212021002001	112022010200	112011010210	212021000201
222022000200	222011000210	122010100220	112020010220	222020000220
122010100202	112020010202	222020000202	122010102020	122010102002
112020012020	222020002020	112020012002	222020002002	122010102200
112020012200	222020002200	212010002011	222000200220	112000210220
112002210200	222002200200	112002010220	222002000220	212001200201
212001000221	122001100210	222000200202	112000210202	222000002222
112000010222	112002212000	222002202000	212001202001	112002012020
222002002020	212001002021	222000002022	112000012022	112000212020
112000212002	112000212200	222000202020	222000202002	222000222000
122001102010	112000012220	222000002220	112000102011	222000101110
112000111110	122010001110	112020001101	212010000211	122021001100
122001201100	122001001120	112000201101	112000100211	112000001121
102212120000	101111120100	102212100002	101111100102	101111021010
101122021000	202211020010	202222020200	201121020100	101122001002
202222000002	201121000102	101111001012	202211000012	102210100202
202220000202	102210100022	101120001022	202220000022	102210120200
102210120020	102210120002	202220020200	101120021020	202220020020
101120021002	202220020002	101120021200	201110020110	202200200022
101100201022	101102201002	202202200002	101102001022	202202000022
201101200102	201101000122	102201100012	202200200202	202200000222
101102221000	202202220000	101101220100	101102021020	202202002020
201101020120	202200020220	202200020022	101100021022	101100221020
101100221002	101100221200	202200220200	202200220020	202200220002
102201120010	101100021220	202200110011	101100111011	101100120110
202020002022	102010102022	202020000222	102010100222	202000202022
202000200222	102012102002	102012100202	202022002002	202022000202
202011002012	202002200202	202002000222	202002202002	202002002022
102001102012	102021001102	202011000212	102001201102	102001100212
102001001122	202020002202	102010102202	202000202202	202000002222
102012122000	102012120200	202011022010	202022022000	202022020200
202011020210	102010120220	202020002220	102010120202	202020020202
102010122020	102010122002	102010122200	202020022020	202020022002
202020022200	202000220220	202002220200	202002020220	102001120210
202000220202	202000020222	102001122010	202002220000	202002022020
202000022022	202000222020	202000222002	202000222000	202000222020
202000112011	202000121110	102010021110	202000222000	202000222020
102021021100	102001221100	102001021120	202000211101	102010111101
202000011121	200220002022	100210102022	100210100222	200220000222

$P_3^{(3)}$ EXTREME POINTS (continued)

2002000202022	2002000200222	100212102002	100212100202	200222002002
200222000202	200211002012	200202200202	200202000222	200202202002
2002020002022	100201102012	100201001122	100201100212	100201201102
200211000212	100221001102	200200002222	200200202202	100210102202
200220002202	100212122000	100212120200	200211022010	200222022000
200222020200	200211020210	100210120220	200220020220	100210120202
200220020202	100210122020	100210122002	100210122200	200220022020
200220022002	200220022200	200200220220	200202220200	200202020220
100201120210	200200220202	200200020222	100201122010	200202222000
200202022020	200200022022	200200222020	200200222002	200200222200
200200022220	200200112011	200200121110	200200011121	200200110211
200200211101	100201021120	100201221100	100221021100	100210111101
200220011101	100210021110	102010010211	102010001112	202000101112
102001012021	102001212001	102001010221	102001210201	102010012011
102021010201	102021012001	102210010011	101120010101	201110000112
101120001202	102221010001	102201210001	102201010021	101100001222
101100201202	101100210101	101100100112	101100010121	100221012001
100221010201	100210012011	100201210201	100201010221	100201212001
100201012021	200200101112	100210010211	100210001112	

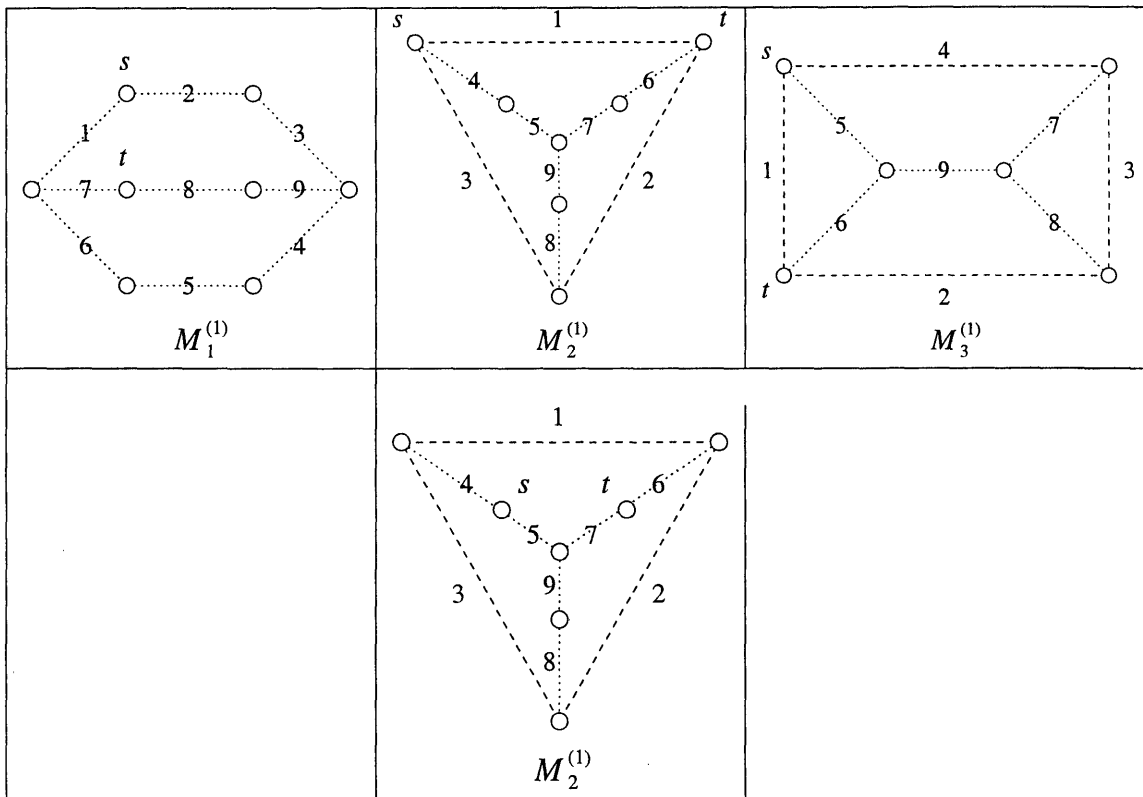


Table 5.1: Labelings of M_1, M_2, M_3 resulting in $s-t$ walk-perfect graphs.

5.2 $s-t$ Walk-Perfect Labelings of M_1, M_2, M_3

We prove the $s-t$ labelings of graphs M_1, M_2 , and M_3 in Table 5.1 are $s-t$ walk perfect. The inequalities and extreme points for each of the polyhedra are enumerated in the following tables. We verify that each extreme point corresponds to an $s-t$ traveling salesman walk, showing these graphs are $s-t$ walk-perfect.

$M_1^{(1)}$ INEQUALITIES

-20000000011	-10000000110	-10000000101	-20000011000	-20000011011	-10000011110
-1000011101	-2000110000	-2000110011	-1000110110	-1000110101	-2000101000
-2000101011	-1000101110	-1000101101	-2001100001	-2001100010	-1001100111
-1001100100	-2001111001	-2001111010	-1001111111	-1001111100	-2001010001
-2001010010	-1001010111	-1001010100	-2001001001	-2001001010	-1001001111
-1001001100	-2011000000	-2011000011	-1011000010	-1011000101	-2011011000
-2011011011	-1011011110	-1011011101	-2011110000	-2011110011	-1011110110
-1011110101	-2011110100	-2011110111	-1011101110	-1011101101	-2010100001
-2010100010	-1010100111	-1010100100	-2010111001	-2010111010	-1010111111
-1010111100	-2010010001	-2010010010	-1010010111	-1010010100	-2010001001
-2010001010	-1010001111	-1010001100	-1110000000	-1110000011	-2110000110
-2110000101	-1110011000	-1110011011	-2110011110	-2110011101	-1110110000
-1110110011	-2110110110	-2110110101	-1110101000	-1110101011	-2110101110
-2110101101	-1111100001	-1111100010	-2111100111	-2111100100	-1111111001
-1111111010	-2111111111	-2111111100	-1111010001	-1111010010	-2111010111
-2111010100	-1111001001	-1111001010	-2111001111	-2111001100	-1101000000
-1101000011	-2101000110	-2101000101	-1101011000	-1101011011	-2101011110
-2101011101	-1101110000	-1101110011	-2101110110	-2101110101	-1101101000
-1101101011	-2101101110	-2101101101	-1100100001	-1100100010	-2100100111
-2100100100	-1100111001	-1100111010	-2100111111	-2100111100	-1100010001
-1100010010	-2100010111	-2100010100	-1100001001	-1100001010	-2100001111
-2100001100	0100000000	0010000000	0001000000	0000100000	0000010000
0000001000	0000000100	0000000010	0000000001		

$M_1^{(1)}$ EXTREME POINTS

011022211	011220211	011111102	011111120	011222011	011202211
120022122	120220122	120111011	120202122	120222102	120222120
211202011	122202102	122202120	122220120	122220102	211220011
122022120	122022102	211022011	102222120	102222102	102022122
102220122	102111011	102202122			

$M_2^{(1)}$ INEQUALITIES

-2000010101	-2000000011	-2000010110	-2000001100	-2000011001	-2000001111
-2000011010	-2000110000	-2000100101	-2000110011	-2000100110	-2000111100
-2000101001	-2000111111	-2000101010	-2011000010	-2011010111	-2011000001
-2011010100	-2011001110	-2011011011	-2011001101	-2011011000	-2011110010
-2011100111	-2011110001	-2011100100	-2011111110	-2011101011	-2011111101
-2011110100	-1110001000	-1110011101	-1110001011	-1110011110	-1110000100
-1110010001	-1110000111	-1110010010	-1110111000	-1110101101	-1110111011
-1110101110	-1110110100	-1110100001	-1110110111	-1110100010	-1101001010
-1101011111	-1101001001	-1101011100	-1101000110	-1101010011	-1101000101
-1101010000	-1101111010	-1101101111	-1101111001	-1101101100	-1101110110
-1101100011	-1101110101	-1101100000	0100000000	0010000000	0001000000
0000100000	0000010000	0000001000	0000000100	0000000010	0000000001

$M_2^{(1)}$ EXTREME POINTS

000111122	001201111	001021111	002111102	002111120	011020222
011022022	011022202	011022220	011220202	011220220	011222020
011222002	011202220	011202202	011202022	011200222	020111120
020111102	010112011	010110211	110021111	120020222	120022022
120022202	120022220	110201111	120200222	120202022	120202202
120202220	120220202	120220220	120222002	120222020	102222020
102222002	101112011	102220220	102220202	101110211	102202220
102202202	102202022	102200222	102022220	102022202	102022022
102020222	100222022	100220222	100202222	100022222	

 $M_2^{(2)}$ INEQUALITIES

-2000010101	-2000000011	-2000010110	-1000001100	-1000011001	-1000001111
-1000011010	-1000110000	-1000100101	-1000110011	-1000100110	-2000111100
-2000101001	-2000111111	-2000101010	-2011000010	-2011010111	-2011000001
-2011010100	-1011001110	-1011011011	-1011001101	-1011011000	-1011110010
-1011100111	-1011110001	-1011100100	-2011111110	-2011101011	-2011111101
-2011101000	-2110001000	-2110011101	-2110001011	-2110011110	-1110000100
-1110010001	-1110000111	-1110010010	-1110111000	-1110101101	-1110111011
-1110101110	-2110110100	-2110100001	-2110110111	-2110100010	-2110100101
-2101011111	-2101001001	-2101011100	-1101000110	-1101010011	-1101000101
-1101010000	-1101111010	-1101101111	-1101111001	-1101101100	-2101110110
-2101100011	-2101110101	-2101100000	0100000000	0010000000	0010000000
0000100000	0000010000	0000001000	0000000100	0000000010	0000000001

 $M_2^{(2)}$ EXTREME POINTS

001102111	000212122	002212102	002212120	011121020	011121002
011101022	011101220	011101202	020212120	020212102	010211011
002012122	020210122	012011011	022012120	022012102	022010122
021100111	022210120	022210102	110100111	210011011	220012120
220012102	220010122	120101202	120101220	120101022	220210120
220210102	120121002	120121020	111210102	111210120	111010122
111012102	111012120	102121020	102121002	202210102	202210120
102101022	102101220	102101202	201100111	202010122	202012102
202012120	200210122	200012122	100101222	101011011	100121022

 $M_3^{(1)}$ INEQUALITIES

-2000000111	-2000011001	-2000011110	-2001100010	-2001100101	-2001111011
-2001111100	-2011000100	-2011000011	-2011011101	-2011011010	-2010100110
-2010100001	-2010111111	-2010111000	-1110001000	-1110001111	-1110010001
-1110010110	-1111101010	-1111101101	-1111110011	-1111110100	-1101001100
-1101001011	-1101010101	-1101010010	-1100101110	-1100101001	-1100110111
-1100110000	0100000000	0010000000	0001000000	0000100000	0000010000
0000001000	0000000100	0000000010	0000000001		

 $M_3^{(1)}$ EXTREME POINTS

000101211	001101101	002101011	000211202	000011222	000211220
002011202	001011112	002011022	002211002	002211020	002211200
001211110	012010101	011010011	022011002	022011020	022011200
021011110	011120002	011120020	011120200	011102200	011102020
011102002	011100022	011100202	020211200	020211020	020211002
020011022	020011220	010010121	010100112	020101011	010102110
010120110	010210101	120020022	110001121	120000222	120002220
120020220	120002022	120200202	110201101	120200022	120202002
120202020	120202200	120110011	120220002	120220020	120220200
121020110	122020200	122020020	122020002	121002110	122002200
122002020	122002002	122000022	112001101	122000202	121000112
111001011	101220110	102220200	102220020	102220020	102210011
101202110	102202200	102202020	102202002	102200022	102200202
101200112	101020112	102020202	102002022	102002202	101002112
102020022	100200222	100202220	100220220	100020222	100220202
100202202	100002222	101110101	100110211		

5.3 TSP-perfection of $P_3(3, 3, 1)$

We now prove the final step of Lemma 3.2.8.

Final Claim in Lemma 3.2.8: Propeller graph $P_3(3, 3, 1)$ is TSP-perfect.

Proof. Figure 5.2 shows the edge labels for graph $P_3(3, 3, 1)$ (the rim edges in the propeller are shown in bold). The inequalities and extreme points of $P(P_3(3, 3, 1))$ generated by `polymake` are enumerated in the following tables. We verify each extreme point corresponds to a traveling salesman tour, which completes the proof of Lemma 3.2.8. \square

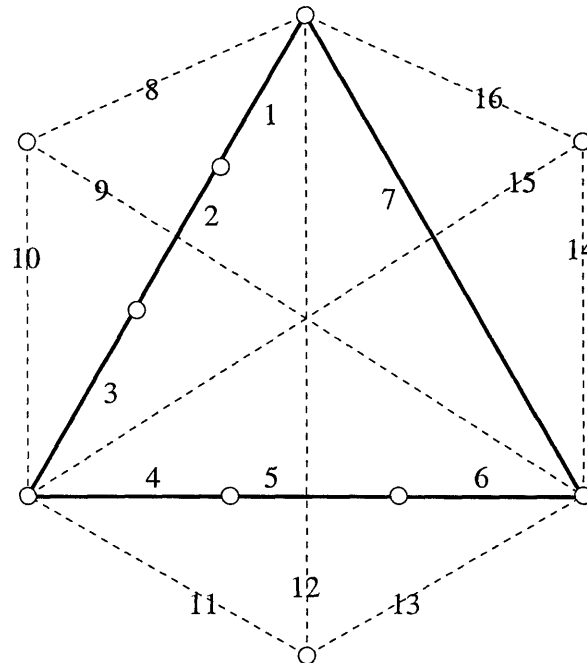


Figure 5.2: Edge labels for graph $P_3(3, 3, 1)$.

$P_3(3, 3, 1)$ INEQUALITIES (continued)

```

-201000100001100101 -201000100001011010 -201000100001011101
-20100010110100010 -201000101101000101 -20100010110001010
-20100010110011101 -20100001011101110 -20100001011101001
-20100001011010110 -20100001011010001 -20100001100101110
-20100001100101001 -20100001100010110 -20100001100010001
-21100000000000000 -21100000000000111 -21100000000111000
-21100000000111111 -21100000111000000 -21100000111000111
-21100000111111000 -21100000111111111 -21100011010001100
-21100011010001011 -21100011010110100 -21100011010110011
-21100011101001100 -21100011101001011 -21100011101110100
-21100011101110011 -21100110000000000 -21100110000000111
-21100110000111000 -21100110000111111 -21100110111000000
-21100110111000111 -21100110111110000 -21100110111111111
-21100101010001100 -21100101010001011 -21100101010101000
-21100101010110011 -21100101101001100 -21100101101000101
-21100101101101100 -21100101101110011 -21101100000000000
-21101100000000111 -21101100000111000 -21101100000111111
-21101100111000000 -21101100111000111 -21101100111111000
-21101100111101000 -2110111010001100 -21101111010001011
-21101111010110100 -21101111010110011 -21101111101001100
-21101111101001011 -21101111101110100 -21101111101110011
-21101010000000000 -211010101000000111 -2110101010000111000
-21101010111111000 -21101010111100000 -21101010111000111
-21101001010001011 -21101001010110100 -21101001010110011
-21101001101001100 -21101001101001011 -21101001101101000
-211010001101110011 -21111000001100010 -21111000011001101
-211110000110100101 -21111000110011010 -21111000110011101
-2111101101101101110 -21111011011101001 -21111011011010110
-21111011011010001 -21111011100101110 -21111011100101001
-21111011100010110 -21111011100010001 -21111110001100010
-21111110001100101 -21111110001011010 -21111110001011101
-21111110110100010 -21111110110100101 -21111110110011010
-21111110110011101 -21111110101110110 -211111101011101001
-211111101011010110 -211111101011010001 -21111110110010110
-211111101100101001 -211111101100010110 -211111101100010001
-211110100001100010 -21110100001100101 -21110100001011010
-21110100001011101 -21110100110100010 -21110100110100101
-21110100110011001101 -21110100110011001101 -21110111011011100
-211101110110101001 -21110111011010110 -21110111011010001
-211101111000101110 -211101111000101001 -21110111100010110
-21110111100010001 -21110010001100010 -21110010001100101
-21110010001011010 -21110010001011101 -21110010110100010
-21110010110100101 -21110010110011010 -21110010110100010
-21110001011101110 -21110001011101001 -21110001011010110
-21110001011010001 -21110001100101110 -21110001100101001
-21110001100010110 -21110001100010001 -211100010001100101
-21010000000000111 -21010000000111000 -21010000000111111
-21010000111000000 -21010000111000111 -21010000111110000
-21010000111111111 -21010011010001100 -21010011010001011
-21010011010110100 -21010011010110011 -21010011101001100
-21010011101001011 -21010011101101010 -21010011101110011
-21010110000000000 -21010110000000111 -21010110000111000
-21010110001111111 -21010110111111111 -21010101010001100
-21010101010001011 -2101010101010110100 -21010101010110011
-21010101101001100 -21010101101001011 -21010101101110100
-21010101101110011 -21011100000000000 -21011100000000111
-21011100000111000 -21011100000111111 -21011100011100000

```

$P_3(3, 3, 1)$ INEQUALITIES (continued)

```

-210111001110001111 -210111001111111000 -210111001111111111
-210111110100011100 -210111110100010111 -210111110101101010
-210111110101100111 -210111111010011100 -210111111010010111
-210111111011101000 -210111111011100111 -210110100000000000
-210110100000000111 -210110100001110000 -210110100001111111
-2101101011110000000 -210110101110001111 -210110101111110000
-210110101111111111 -210110010100001100 -210110010100010111
-2101100101011010100 -210110010101100111 -210110011010011000
-210110011010001011 -210110011011101010 -210110011011100111
-210010000011000101 -210010000011001011 -210010000010110101
-210010000010111101 -210010001101000101 -210010001101001011
-210010001100110101 -210010001100111101 -210010110110110111
-210010110111010100 -210010110110101110 -210010110110100011
-210010111001011110 -210010111001010011 -210010111000101110
-210010111000100011 -210011100011000101 -210011100011001011
-210011100010110101 -210011100010111101 -210011101101000101
-210011101101001011 -210011101100110101 -210011101100111101
-210011010111011110 -210011010111010011 -210011010110101110
-210011010110100011 -210011011001011110 -210011011001010011
-210011011000101110 -210011011000100011 -210001000011000101
-210001000011001011 -210001000010111010 -210001000010111101
-210001001101000101 -210001001101001011 -210001001100111010
-210001100110011101 -210001110111011110 -210001110111010011
-210001110110101110 -210001110110100011 -210001111001011110
-210001111001010011 -210001111000101110 -210001111000100011
-210000100011000101 -210000100011001011 -210000100010110101
-210000100010111011 -210000101101000101 -210000101101001011
-210000101100110101 -210000101100111101 -210000010111011110
-210000010111010011 -210000010110101110 -210000010110100011
-210000011001011110 -210000011001010011 -210000011000101110
-210000011000100011 -210000011000100011 -210000000000000000
001000000000000000 000100000000000000 000010000000000000
000001000000000000 000000100000000000 000000010000000000
000000001000000000 000000000100000000 000000000010000000
0000000000100000 000000000000100000 000000000000001000
00000000000000100 00000000000000010000 0000000000000001000
00000000000000100 000000000000000101 00000000000000001

```


$P_3(3, 3, 1)$ EXTREME POINTS

```

0220220011011011011 022022001111010101 0220220110110110110 022022011010101011
02202201010111110 0220220101101101101 0220221101101200 022022110110101020
02202211011010002 0220221101200110 0220221101020110 0220221101002110
0220221200110110 0220221200101011 0220221020101011 0220221020110110
0220221002110110 0220221002101011 0220221011200011 022022101110200
0220221011100020 0220221011100002 022022101020011 0220221011002011
0220220002110211 02202200020110211 0220220110011220 0220220101110220
022022000110211 0220220101002211 022022010102021 0220220101200211
02202200202112011 0220220020112011 0220220110202101 0220220110202101
0220220200112011 0220220101112020 0220220101112002 0220220101112200
0220220022011101 0220220022110011 0220220121002011 0220220121020011
0220220121110020 0220220121110002 0220220121110200 0220220121110200
0220220011022110 0220220011101202 0220220220101110 0220220101110202
0220220101022011 0220220220110011 022022002101110 022022002101110
0220222011020110 0220222011002110 0220222011101002 0220222011101020
0220222011101200 0220222011101101 0220222011200110 0220222011101020
02202220201110110 0220222020110011 022022200101110 0220222001101011
0220222020101110 0220222101020011 0220221101011011 0220222101200011
0220222101110002 0220222101110002 0220221101110101 0220222101102001
0220221110101110 0220221110110011 0222200011011011 0222200011110101
0222200110110110 0222200110101011 022220010101110 0222200101101101
0222201101101200 0222201101101020 0222201101101002 0222201101200110
0222201101020110 0222201020110110 0222201002110110 022220102011002110
0222202002101110 0222202002110110 0222202001101020 02222020011002110
0222202011101002 0222202011101020 0222202002110011 0222202020101101
0222202011200110 0222202002110011 0222202020101110 0222202020101011
0222202200101110 0222202200110011 0222202101002011 0222202101020011
0222201101011011 0222202101200011 022220210110002 0222202101110002
0222201101110101 0222202101110200 0222201110101110 0222201110110011
0221111002200011 0221111002110200 0221111002110020 0221111002110020
0221111002020011 0221111002002011 0221111002101101 0221111002101101
0221112002200110 0221112002200110 0221112002200110 0221112002200110
0221111020110002 0221111020110002 0221111020110020 0221111020110020
022111202002110 0221112020020110 0221112020020110 0221112020020110
0221111020101101 0221112020101200 0221112020101020 0221112020101002
0221112011002020 0221112011002200 0221112011002200 0221112011002200
0221111011011020 0221111011011002 0221111011011200 0221111011011200
0221112011020002 0221112011020200 0221111011020101 0221111011020101
0221112011200020 0221112011200200 0221111011200101 0221111011200101
022111110002110 022111110200110 022111110101200 022111110101200
022111110101002 0221112200101002 0221112200101020 0221112200101020
0221111200101101 0221112200200110 0221112200002110 0221112200002110
0221111200011110 0221111200200011 0221111200110200 0221111200110200
0221111200110020 0221111200020011 0221111200002011 0221111200002011
022111101002002 0221111101002200 0221111101020020 0221111101020020
022111101020200 0221111101200200 0221111101200020 0221111101200020
0222220110200101 0222220110020101 0222220110011200 0222220110011002
0222220110011020 0222220110002101 0221110220101002 0221110220101002
0221110220101200 0221110220200110 0221110220002110 0221110220002110
0222220101110200 0222220101110002 0222220101110020 0222220101110020
0222220101020011 0222220101002011 0222220200110011 0222220200110011
0221110200022110 0221110200101202 0222220020110011 0222220020110011
0222220002110011 0222220002011101 0221110011022200 0221110011022200
0221110011022020 0221110020022110 0221110002022110 0221110002022110
0221110011020202 0221110011002202 0221110020101202 0221110020101202
0221110101200101 0221110101020101 0221110101011200 0221110101011200
0221110101011020 0221110101002101 0221110200011011 0221110200011011
0221110101101011 0221110101110110 0221110101010101 0221110101010101
0221110110200011 0221110110110200 0221110110110002 0221110110110002
0221110110020011 0221110110002011 0221110011110011 0221110011110011
0221110020110101 0221110020011011 0221110002011011 0221110002011011
0222021110110011 0222021110101110 0222022101110200 0222021101110101
0222022101110020 0222022101110002 0222022101200011 0222021101010101
0222022101020011 0222022101002011 0222022200110011 0222022200101110
0222022020110011 0222022020101110 0222022002110011 0222022011200110
0222021011101101 0222022011101200 0222022011101020 0222022011101002
0222022011002110 0222022011020110 022202101101110 0222022002101110
0222020220110011 0222020201102011 0222020101110202 0222020220101110

```


$P_3(3, 3, 1)$ EXTREME POINTS

2202021020220101 2202022020220200 2202021110220020 2202021110220002
 2202021110220200 2202022200220200 2202021200220101 2202022200220002
 2202022200220020 2202021200211200 2202021200211020 2202021200211002
 2202020220220200 2202020220220002 2202020220220020 2202020200220202
 2202020020220202 2202020002220202 2202020200121110 2202020200211101
 2202020110220101 2202020110211200 2202020110211020 2202020110211002
 2202020011220110 2202020020211101 2202020020121110 2202020011121200
 2202020011121020 2202020011121002 2202020002211101 2202020002121110
 2202020002220220 2202020020220220 2202020200220220 2202020002222200
 2202020002222020 2202020002222002 2202020020222020 2202020020222002
 2202020020222000 2202020020222000 2202020200222002 2202020200222020
 22020200202220002 2202020022220020 2202020022220200 2202021112020200
 2202021112020020 2202021112020002 2202021112020002 2202021112020200
 2202021112002200 2202021112200020 2202021112200020 2202021112200002
 2202022202200200 2202021202200101 2202022202200020 2202022202200002
 2202022202002200 2202021202002101 2202022202002020 2202022202002002
 2202022202020002 2202022202020020 2202022202020200 2202021202020101
 22020212020002 2202022202020020 2202022202020200 2202021202020101
 2202021202011200 2202021202011020 2202021202011002 2202020202002220
 2202020202020220 2202020202200220 2202020202202200 2202020202202020
 2202020202202002 2202020222002020 2202020222002002 2202020222002200
 2202020222020200 2202020222020002 2202020222020020 2202020222020020
 220202022200020 220202022200002 220202020202202 220202020202202
 2202020002202202 2202020022200202 2202020022200220 220202002220202
 22020200022220 2202020020022220 2202020002022220 220202002022200
 2202020022020200 2202020022020020 2202020022020020 2202020022002220
 2202020220202020 220202022020020 220202022020002 220202022020002
 2202020220200200 2202020220200200 2202020220200200 2202020220200020
 2202022022200002 2202021022200101 2202021022002101 2202021022020101
 2202021022011200 2202021022011020 2202021022011002 2202022002202200
 2202022002202020 2202022002202002 2202022020202020 2202022020202020
 2202022020202200 2202022200202200 2202022200202020 2202022200202002
 2202021200202101 2202021110202200 2202021110202020 2202021110202002
 2202021020202101 2202021002202101 2202022002002220 2202022002020220
 2202022002200220 2202022020020220 2202022020002220 2202022020002220
 22020222000220 2202022200002220 2202022200020220 2202021200011220
 2202021110200220 2202021110002220 2202021110020220 2202021020011220
 2202021002011220 2202200002011121 2202200002101112 22022000011020112
 22022000011002112 22022000011200112 22022000020011121 22022000020101112
 22022000011101022 22022000110020121 22022000110002121 22022000110200121
 2202200110011022 2202200200101112 2202200200011121 2202200002022022
 2202200020022022 2202200200022022 2202200220002022 2202200220020022
 2202201200020121 2202201200020121 2202201200002121 2202201200200121
 2202202200002022 2202201200011022 2202202200020022 2202202200200022
 2202201110200022 2202201110020022 2202201110002022 2202202020200022
 2202202020020022 2202201020011022 2202202020002022 2202201020200121
 2202201020002121 2202201020020121 2202201002011022 2202202002002022
 2202202002200022 2202202002200022 2202201002200121 2202201002020121
 2202201002002121 2202200002200222 2202200002020222 2202200002002222
 220220002002222 2202200020020222 2202200020200222 2202200200200222
 220220020020222 2202200020002222 2202200002202022 22022000202022
 2202200200202022 220220002200222 2202200022020022 2202200022200022
 22022000021110 2202200002211101 22022000011121002 22022000011121020

$P_3(3, 3, 1)$ EXTREME POINTS

```

2221110002002110 1111110020002101 1111110020011020 1111110020011002
1111110020011200 1111110020020101 1112220020002011 2221110020002110
2221110020020110 1112220020020011 1111110020110110 1112220020110020
1112220020110002 1112220020110200 2221110020101200 2221110020101002
2221110020101020 1112220020200011 2221110020200110 1111110020101011
111110020200101 2221110011002020 2221110011002002 2221110011002200
2221110011020200 2221110011020002 2221110011020020 2221110011200002
2221110011200020 2221110011200200 1111110110002020 1111110110002002
1111110110002200 1111110110020020 1111110110020002 1112220101200200
1112220101200020 1112220101200002 1112220101020020 1112220101020002
1112220101020200 1112220101002200 1112220101002002 1112220101002020
1111110200200101 1111110200101011 2221110200200110 1112220200200011
2221110200101020 2221110200101002 2221110200101200 1112220200110200
1112220200110002 1112220200110020 1111110200110110 1112220200020011
2221110200020110 2221110200002110 1112220200002011 1111110200020101
1111110200011200 1111110200011002 1111110200011020 1111110200020101
1111110101002110 1111110101020110 1111110101101200 1111110101101020
1111110101101002 1111110101200110 2222220002200002 2222220002200020
2222220002200200 2222220002020200 2222220002020002 2222220002020020
2222220002002200 2222220002002020 2222220002002002 2222220002002020
2222220020002002 2222220020002200 2222220020020200 2222220020020002
2222220020002020 22222200200002002 22222200200002200 22222200200002020
2222220200020002 2222220200002002 2222220200002200 2222220200002020
2222220200200002 1111111200002020 1111111200002002 1111111200002200
1111111200020200 1111111200020002 1111111200020002 1111111200020020
1111111200200020 111111120020002 111111120020002 1111111200200200
1111111020002200 1111111020002002 1111111020002020 1111111020002002
1111111002002020 1111111002002200 1111111002020020 1111111002020002
1111111002202020 1111111002200200 1111111002200020 1111111002200002
2222201110200002 2222201110200020 2222201110200020 2222201110020020
2222201110002020 2222201120000201 2222201120000201 22222011200011200
2222201200011002 2222201200011020 2222201200020101 2222201200020002
2222202000200020 2222202000200200 222220200020020 222220200020002
2222202000202020 222220200002200 222220200002200 222220200002200
2222201020200101 22222010200020101 2222201020002100 22222010200011002
2222201020011020 2222201020002101 2222201020002200 2222201020002020
2222202020200200 2222202020002002 2222202020002002 2222202020002020
2222202020002101 2222202020002200 2222202020002200 2222202020002200
2222202002200020 2222202002200020 2222202002200020 2222202002200020
2222202002200200 2222202002200200 2222202002200200 2222202002200200
2222202002200220 2222202002200220 2222202002200220 2222202002200220
2222200020020202 2222200020002202 2222200020002202 2222200020002202
2222200020020202 2222200020002202 2222200020002202 2222200020002202
222220002200202 2222200022002202 2222200022002202 2222200022002202
222220002200202 222220002200202 222220002200202 222220002200202
1112200121020020 1112200121020002 1112200121020200 1112200121002200
1112200121002002 1112200121002020 2222200022202020 2222200022202002
2222200022202002 2222200022202002 2222200022202020 2222200022202020
1112200022110200 1112200022110020 1112200022110002 1112200022200011
1112200022002011 11122000101202002 11122000101202020 11122000101202200
1112200200112020 1112200200112002 1112200200112200 2222200200202200
2222200200202002 2222200200202020 1112200200202011 1112200200202011
2222200020202020 2222200020202002 2222200020202200 1112200020112200
1112200020112002 11122000020112020 2222200002202200 2222200002202200
222220002202002 1112200002202011 11122000022112002 1112200002112020
1112200022112200 11122001012020220 1112200101020220 11122000101002220
1112200200002211 11122002000020211 2222200200002220 2222200200020220
1112200200110220 2222200200200220 1112200200200211 1112200200200211
2222200022002211 1112200020002211 2222200002002220 2222200002002220
2222200022002220 1112200002110220 1112200002002211 1112200002002211
1112200022002110 1112201002002110 1112201002020110 1112201002020110
1112201002101002 1112201002101002 1112201002200110 1112201002200110
1112201011020020 1112201011020002 1112201011002002 1112201011002020
1112201011002200 1112201011200020 1112201011200020 1112201011200002
1112202002200011 1112202002110101 1112202002110200 1112202002110020
1112202002110002 1112202002020011 1112202002002011 1112202002002011
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